

Discrete versions of the Li-Yau gradient estimate

DOMINIK DIER, MORITZ KASSMANN AND RICO ZACHER

Abstract. We study positive solutions to the heat equation on graphs. We prove variants of the Li-Yau gradient estimate and the differential Harnack inequality. For some graphs, we can show the estimates to be sharp. We establish new computation rules for differential operators on discrete spaces and introduce a relaxation function that governs the time dependency in the differential Harnack estimate.

Mathematics Subject Classification (2010): 35R02 (primary); 35K05, 35K10, 05C10, 05C81 (secondary).

1. Introduction

The heat equation plays a fundamental role in several fields of Mathematics and provides a link between Analysis, Stochastics and Geometry. It has been intensively studied on different state spaces, *e.g.*, the Euclidean space, Riemannian manifolds, and general metric measure spaces. In this work, we study pointwise estimates for positive solutions to the heat equation on graphs. We aim at precise results whenever this is possible. If the graph under consideration is small, *i.e.*, if it contains only few vertices, then we check our estimates by explicit computation. For the sequence of graphs given by $(\tau\mathbb{Z}^d)_{\tau>0}$ we try to trace the influence of the parameter $\tau \rightarrow 0+$. This allows us to compare the estimates with well-known results for the limit space \mathbb{R}^d .

Before we explain the framework of our study in greater detail, let us review some fundamental results with regard to the heat equation on Riemannian manifolds. The classical gradient estimate given by Li-Yau [13] holds true for positive solutions $u : [0, \infty) \times M \rightarrow (0, \infty)$ of the heat equation $\partial_t u - \Delta u = 0$ on a complete d -dimensional Riemannian manifold M with $\text{Ric}(M) \geq 0$:

$$\frac{|\nabla u(t, x)|^2}{u^2(t, x)} - \frac{\partial_t u(t, x)}{u(t, x)} \leq \frac{d}{2t} \quad (t > 0, x \in M), \quad (1.1)$$

Rico Zacher was supported by the DFG (Project 355354916, GZ ZA-547/4-1). Support by the DFG (CRC 1283) is gratefully acknowledged.

Received July 12, 2018; accepted in revised form February 03, 2020.

Published online June 2021.

or, equivalently,

$$|\nabla \log u(t, x)|^2 - \partial_t(\log u)(t, x) \leq \frac{d}{2t} \quad (t > 0, x \in M). \tag{1.2}$$

An important consequence of this estimate is a pointwise bound on the solution itself, which can be obtained from integration over a path that connects two given points (t_1, x_1) and (t_2, x_2) with $t_2 > t_1 > 0$:

$$u(t_1, x_1) \leq u(t_2, x_2) \left(\frac{t_2}{t_1}\right)^{d/2} \exp\left(\frac{\rho^2(x_1, x_2)}{4(t_2 - t_1)}\right). \tag{1.3}$$

Note that estimates (1.1), (1.2), and (1.3) are sharp in the sense that corresponding equalities hold true for the fundamental solution to the heat equation on \mathbb{R}^d , *i.e.*, if $u(t, x)$ equals $(4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right)$.

The aim of the current project is to study estimates of the type of inequalities (1.1), (1.2), and (1.3) for positive solutions to the heat equation on graphs. In order to establish a corresponding theory, we establish new computation rules for functions defined on discrete spaces.

Let $G = (V, E)$ be a graph. All graphs appearing in this work are assumed to be undirected. For two vertices $x, y \in V$ we write $x \sim y$ if there is an edge between x and y , that is, $xy \in E$. We allow for edge weights; the weight of the edge xy from x to y is denoted by w_{xy} and is always assumed to be positive. Moreover, we assume that the graph is locally finite, *i.e.*, for every $x \in V$ the set of all $y \in V$ with $y \sim x$ is finite.

Set $\mathbb{R}^V := \{u : V \rightarrow \mathbb{R}\}$ and assume $\mu : V \rightarrow (0, \infty)$. We consider the generalized Laplacian on G , which is the operator $\Delta : \mathbb{R}^V \rightarrow \mathbb{R}^V$ defined by

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x)) \quad (x \in V). \tag{1.4}$$

We will also use the operator $L := -\Delta$. We say that a function $u : [0, \infty) \times V \rightarrow \mathbb{R}$ solves the heat equation on G if $\partial_t u - \Delta u = 0$ on $[0, \infty) \times V$. We recall the definition of $\Gamma, \Gamma_2 : \mathbb{R}^V \times \mathbb{R}^V \rightarrow \mathbb{R}^V$:

$$\begin{aligned} 2\Gamma(v, w) &= \Delta(vw) - v\Delta w - w\Delta v, \\ 2\Gamma_2(v, w) &= \Delta(\Gamma(v, w)) - \Gamma(v, \Delta w) - \Gamma(w, \Delta v). \end{aligned}$$

As it is usual, we write $\Gamma(v)$ instead of $\Gamma(v, v)$ and analogously $\Gamma_2(v)$ instead of $\Gamma_2(v, v)$. A crucial identity in the classical approach to Li-Yau estimates is

$$\Delta(\log u) = \frac{\Delta u}{u} - |\nabla \log u|^2 \tag{1.5}$$

for positive functions $u : M \rightarrow \mathbb{R}$. The equality follows directly from the chain rule. One way to compensate the lack of the chain rule for differences is provided in [2]. Instead of (1.5), the authors invoke the identity

$$2\sqrt{u}\Delta(\sqrt{u}) = \Delta u - 2\Gamma(\sqrt{u}), \tag{1.6}$$

which holds true on graphs, too. This equality allows to derive estimates of $\Gamma(\sqrt{u})$ if u is a positive solution to the heat equation. In the present work, we suggest to follow another path. We provide a discrete version of (1.5) and show for positive functions $u : V \rightarrow \mathbb{R}$

$$\Delta(\log u) = \frac{\Delta u}{u} - \Psi_\Upsilon(\log u), \tag{1.7}$$

where $\Psi_\Upsilon(v)(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} \Upsilon(v(y) - v(x))$ and $\Upsilon(z) = e^z - z - 1$. Note $\Psi_\Upsilon(\log u)$ equals $\Gamma^{\log u}$ as in [17, Section 3]. In Section 2 we provide more general computation rules. Note that $v \mapsto \Psi_\Upsilon(v)$ is a replacement of the quadratic function $v \mapsto \Gamma(v)$ and $z \mapsto \Upsilon(z) = e^z - z - 1$ replaces the square function in the expression $|\nabla \log u|^2$.

One of our main results is a Li-Yau type inequality for positive solutions u to the heat equation on a finite connected undirected graph $G = (V, E)$:

Theorem 1.1. *Assume that G satisfies $CD(F; 0)$ and let φ be the relaxation function associated with the CD-function F . Suppose that $u : [0, \infty) \times V \rightarrow (0, \infty)$ is a solution of the heat equation on G . Then*

$$-\Delta(\log u)(t, x) \leq \varphi(t) \quad \text{in } (0, \infty) \times V, \tag{1.8}$$

and thus

$$\Psi_\Upsilon(\log u)(t, x) - \partial_t(\log u)(t, x) \leq \varphi(t) \quad \text{in } (0, \infty) \times V. \tag{1.9}$$

The condition $CD(F;0)$ is formulated locally at each point $x \in V$ and involves only neighbors of second order, cf. Definition 3.8. We use the abbreviation CD as in “curvature dimension” although the relation to classical CD-conditions like $\Gamma_2(f) \geq \frac{1}{d}(\Delta f)^2$ for all f or more recent related conditions from [2, 17] (like, e.g., the so-called exponential curvature dimension inequality $CDE(n,0)$) has not yet been established. The examples from Section 3 suggest that there is a close relation between $CD(F;0)$ and other conditions from the literature. Note that $F : [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $F(0) = 0$, $F(x)/x$ is strictly increasing, and $1/F$ is integrable at $+\infty$. The relaxation function φ is the unique positive solution to $\dot{\varphi}(t) + F(\varphi(t)) = 0$ on $(0, \infty)$ with $\varphi(0+) = \infty$, cf. Lemma 3.5. In some examples, we can compute the relaxation function φ explicitly. For example, $\varphi(t) = -\log(\tanh t)$ for the unweighted two-point graph.

The differential Harnack inequality (1.9) implies pointwise bounds on the function u itself by a chaining argument, cf. [13] and [2] in the case of graphs. We apply the same strategy. In the case of finite graphs, our Harnack inequality then reads as follows:

Theorem 1.2. *Let G be a finite connected graph satisfying the assumption of Theorem 1.1. Assume $u : [0, \infty) \times V \rightarrow (0, \infty)$ is a solution of the heat equation on G and $0 < t_1 < t_2$ and $x_1, x_2 \in V$. Then*

$$u(t_1, x_1) \leq u(t_2, x_2) \exp\left(\int_{t_1}^{t_2} \varphi(t) dt\right) \exp\left(\frac{2\mu_{\max}d(x_1, x_2)^2}{w_{\min}(t_2 - t_1)}\right). \tag{1.10}$$

Remark 1.3. For the sake of simplicity, in this introduction, we choose to present our results, Theorem 1.1 and Theorem 1.2, for the case of finite graphs. Versions for general locally finite connected graphs are given in Section 5 and Section 6.

Remark 1.4. Note that in all examples studied in this work, the relaxation function φ turns out to be integrable at $t = 0$. Thus, it is possible to consider the case $t_1 = 0$ in (1.10). This is in contrast to the Harnack inequality on manifolds.

Let us comment on related results in the literature. One approach to Li-Yau type estimates on graphs is given in [2] and several subsequent works. Since the results of the present work are closely related, let us explain the approach of [2]. The authors establish the following estimate

$$\frac{\Gamma(\sqrt{u})(t, x)}{u(t, x)} - \frac{\partial_t(\sqrt{u})(t, x)}{\sqrt{u}(t, x)} \leq \frac{n}{2t} \quad (t > 0, x \in V) \tag{1.11}$$

for positive solutions u to the heat equation on G , which should be contrasted with (1.2) and (1.9). A significant difference between this result and our estimate is that we estimate the term $\Psi_\gamma(\log u)$, which, in some sense, is the correct discrete replacement for $|\nabla \log u|^2$. As a consequence of (1.11), the authors obtain a Harnack inequality

$$u(t_1, x_1) \leq u(t_2, x_2) \left(\frac{t_2}{t_1}\right)^n \exp\left(\frac{4D\rho^2(x_1, x_2)}{t_2 - t_1}\right) \quad (0 < t_1 < t_2, x_i \in V), \tag{1.12}$$

where D equals the maximal degree of a vertex in G . Note that \mathbb{Z}^d satisfies CDE($n,0$) with $n = 2d$. Thus, the exponent n in $\left(\frac{t_2}{t_1}\right)^n$ is off by a factor 4 from what one would expect, based on the corresponding estimates in the Euclidean space. In [2], the authors study graphs which satisfy the exponential curvature dimension inequality CDE($n,0$).

Computation rules and estimates for the logarithm of positive solutions appear also in [17]. The main aim of [17] is to establish generalized curvature dimension inequalities and to prove a Li-Yau inequality on finite graphs. In this way, [17] enhances some of the results of [2], *e.g.*, the estimate (1.12). The relation between the conditions (curvature dimension inequalities) of [17] and [2] is studied in [18].

The main difference between the present work and the approach in [2, 17] and other existing works is that we do not restrict ourselves to expressions respectively functions of the form $t \mapsto ct^{-1}$ in the differential Harnack inequality. In this respect, (1.9) and (1.11) are rather different. As can be seen from (1.10), the function φ plays an important role in the pointwise estimate for the positive solution u . In light of (1.3) the estimate (1.12) looks natural but the behavior for $t_1 \rightarrow 0+$ seems far from being optimal. Note that the Laplace operator, when defined on a graph with bounded degree, is a bounded operator. Thus, one should expect a robust estimate for all $t_1 > 0$. We believe that an optimal result requires the time-dependence

to be captured by a function φ depending on the graph under consideration. This is why, in our approach, φ is linked to the graph via the CD-function F from the condition $\text{CD}(F;0)$.

Another difference between the present work and [2] concerns the analysis on infinite graphs. Infinite graphs are not studied in [17]. As in the case of Riemannian manifolds, it is necessary to decompose $\Psi_\gamma(\log u)$ into two parts in order to apply successfully cut-off functions. We develop a systematic approach for this procedure, which we call α -calculus, where $\alpha \in [0, 1)$. The special case $\alpha = \frac{1}{2}$ is strongly connected to the methods of [2].

It is worth mentioning that, in general, Ricci curvature bounds play an important role. If the Ricci curvature of a Riemannian manifold is bounded from below by a strictly positive number, then, in addition to the Harnack inequality, several properties can be established. Isoperimetric inequalities follow as well as lower bounds for the eigenvalues of the Laplacian. There have been several attempts to develop a notion of Ricci curvature bounds for discrete or, more generally, for non-smooth spaces starting from the theory of Bakry and Emery [1], which is based on properties of the corresponding semigroup. For recent developments in this direction, we refer to [4, 8–10, 12], and [11]. Note that the last mentioned work contains several concrete examples and computations. Following the theory of Lott, Villani, and Sturm for metric measure spaces, techniques from optimal transport have been applied, *cf.* [3, 6, 7, 15, 16, 19], or the nice survey in [20]. Since, in the present work, neither semigroups nor optimal transport are used, we omit a further discussion here.

The article is organized as follows: in Section 2 we study computation rules for difference operators, in particular a discrete version of the chain rule. It turns out, that it is possible to obtain nice formulas for expressions of the form of $\Delta(\log u)$. In Section 3 we introduce a new notion of curvature inequality, which is parametrized by a CD-function F . This function is computed explicitly for several examples of graphs in Section 3. Section 4 contains the proof of the Li-Yau estimate on finite graphs. In Section 5 we explain how to obtain a similar result on infinite Ricci-flat graphs. In the special case of the lattice \mathbb{Z} respectively the sequence $(\tau\mathbb{Z})_{\tau>0}$, we show in Subsection 5.2 how to recover the classical sharp Li-Yau estimate on \mathbb{R} in the limit $\tau \rightarrow 0+$. Finally, in Section 6 we apply the chaining argument from [2] and derive a Harnack inequality from the Li-Yau estimate. We prove the result for locally finite graphs thus establishing Theorem 1.2.

2. Fundamental identities

This section is concerned with a basic identity, which can be viewed as a kind of chain rule for the operator Δ . We refer to it as the *fundamental identity*. Given a function $H : \mathbb{R} \rightarrow \mathbb{R}$, we also define the operator $\Psi_H : \mathbb{R}^V \rightarrow \mathbb{R}^V$ by

$$\Psi_H(u)(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} H(u(y) - u(x)), \quad x \in V, u \in \mathbb{R}^V. \quad (2.1)$$

Observe that in case of the function $H(y) = \frac{1}{2}y^2$ we have $\Psi_H(u) = \Gamma(u)$.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}$ be an open set and $u \in \mathbb{R}^V$ such that the range of u is contained in Ω . Let further $H \in C^1(\Omega; \mathbb{R})$. Then there holds for every $x \in V$*

$$\Delta(H(u))(x) = H'(u(x))\Delta u(x) + \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} \Lambda_H(u(y), u(x)), \tag{2.2}$$

where we set

$$\Lambda_H(w, z) := H(w) - H(z) - H'(z)(w - z), \quad w, z \in \mathbb{R}. \tag{2.3}$$

Proof. For each neighbor y of x we have

$$\begin{aligned} H(u(y)) - H(u(x)) &= H'(u(x))(u(y) - u(x)) \\ &+ \left(H(u(y)) - H(u(x)) - H'(u(x))[u(y) - u(x)] \right). \end{aligned} \tag{2.4}$$

Multiplying (2.4) by the weight $w_{xy}/\mu(x)$ and summing over all $y \sim x$ yields the assertion. □

Note that the quantity $\Lambda_H(w, z)$ resembles the Bregman distance from convex analysis. Identity (2.2) is the analogue in the graph setting of the classical rule

$$\Delta H(u) = H'(u)\Delta u + H''(u)|\nabla u|^2 \quad (u \in C^2(\mathbb{R}^d)).$$

See also [21] for an application of a similar identity in the context of evolution equations with fractional time derivatives. Note that in case of a convex function H we obtain $\Lambda_H \geq 0$ and thus identity (2.2) yields the inequality $\Delta H(u) \geq H'(u)\Delta u$. Let us look at some examples.

Example 2.2. Take $\Omega = \mathbb{R}$ and $H(y) = \frac{1}{2}y^2$. Then

$$\Lambda_H(w, z) = \frac{1}{2}w^2 - \frac{1}{2}z^2 - z(w - z) = \frac{1}{2}(w - z)^2$$

and thus we get for any $u \in \mathbb{R}^V$ and $x \in V$

$$\frac{1}{2}\Delta(u^2)(x) = u(x)\Delta u(x) + \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy} (u(y) - u(x))^2.$$

Hence

$$\Delta(u^2) = 2u\Delta u + 2\Gamma(u). \tag{2.5}$$

Example 2.3. Take $\Omega = (0, \infty)$ and consider the function $H(y) = \sqrt{y}$, $y > 0$. Then

$$\Lambda_H(w, z) = \sqrt{w} - \sqrt{z} - \frac{1}{2\sqrt{z}}(w - z) = -\frac{1}{2\sqrt{z}}(\sqrt{w} - \sqrt{z})^2.$$

Assuming that $u \in \mathbb{R}^V$ is positive, the fundamental identity then gives

$$\Delta(\sqrt{u})(x) = \frac{1}{2\sqrt{u(x)}} \Delta u(x) - \frac{1}{2\mu(x)\sqrt{u(x)}} \sum_{y \sim x} w_{xy} (\sqrt{u(y)} - \sqrt{u(x)})^2.$$

Multiplying by $2\sqrt{u}$ we obtain

$$2\sqrt{u} \Delta \sqrt{u} = \Delta u - 2\Gamma(\sqrt{u}). \quad (2.6)$$

Relation (2.6) is the key identity for the square root approach used in [2]. Observe that (2.6) is also an immediate consequence of formula (2.5); just substitute $v = \sqrt{u}$ in (2.6) to see this.

Example 2.4. Take $\Omega = (0, \infty)$ and $H(y) = -\log y$, $y > 0$. Then

$$\begin{aligned} \Lambda_H(w, z) &= -\log w + \log z + \frac{1}{z}(w - z) \\ &= \log\left(\frac{z}{w}\right) + \frac{w}{z} - 1 \\ &= \Upsilon(\log w - \log z), \end{aligned}$$

where

$$\Upsilon(y) := e^y - 1 - y = \sum_{j=2}^{\infty} \frac{y^j}{j!}, \quad y \in \mathbb{R}.$$

Assuming that $u \in \mathbb{R}^V$ is positive, the fundamental identity yields

$$-\Delta(\log u)(x) = -\frac{1}{u(x)} \Delta u(x) + \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} \Upsilon(\log u(y) - \log u(x)).$$

This shows the important relation

$$\frac{1}{u} \Delta u = \Delta(\log u) + \Psi_{\Upsilon}(\log u), \quad (2.7)$$

which is remarkable since the right-hand side is formulated using only terms involving the function $\log u$. Replacing the positive function u in (2.7) by u^α with $\alpha > 0$ yields the identity

$$\frac{\Delta(u^\alpha)}{\alpha u^\alpha} = \Delta(\log u) + \frac{1}{\alpha} \Psi_{\Upsilon_\alpha}(\log u), \quad (2.8)$$

where we set $\Upsilon_\alpha(y) = \Upsilon(\alpha y)$.

Lemma 2.5. *Let $\alpha \in (0, 1)$. The function $g_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ defined by*

$$g_\alpha(z) = \Upsilon(z) - \frac{1}{\alpha} \Upsilon(\alpha z), \quad z \in \mathbb{R},$$

is non-negative on \mathbb{R} and satisfies

$$g_\alpha(z) \geq \frac{1-\alpha}{2} z^2, \quad z \geq 0. \quad (2.9)$$

Moreover, we have the representation

$$g_\alpha(z) = h_\alpha(e^z), \quad z \in \mathbb{R},$$

where

$$h_\alpha(z) = z - \frac{1}{\alpha} z^\alpha + \frac{1-\alpha}{\alpha}, \quad z \geq 0.$$

In particular, in case $\alpha = \frac{1}{2}$, there holds

$$g_{1/2}(z) = (e^{z/2} - 1)^2, \quad z \in \mathbb{R}.$$

Proof. By definition of Υ we have

$$g_\alpha(z) = e^z - 1 - z - \frac{1}{\alpha} (e^{\alpha z} - 1 - \alpha z) = e^z - \frac{1}{\alpha} e^{\alpha z} + \frac{1}{\alpha} - 1 = h_\alpha(e^z),$$

and thus

$$g'_\alpha(z) = e^z - e^{\alpha z},$$

which shows that g_α is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$, with $g_\alpha(0) = 0$ being the global minimum. For any $z > 0$, Taylor's theorem gives

$$g_\alpha(z) = \frac{1}{2} g''_\alpha(\xi) z^2 \quad (2.10)$$

with some $\xi \in (0, z)$. Clearly, the function $g''_\alpha(z) = e^z - \alpha e^{\alpha z}$ is strictly increasing on $[0, \infty)$ and $g''_\alpha(0) = 1 - \alpha$, and so (2.10) implies the inequality (2.9). The last assertion follows from the identity

$$h_{1/2}(z) = (\sqrt{z} - 1)^2, \quad z \geq 0. \quad \square$$

Note that Lemma 2.5 also shows that in case $\alpha = 1/2$ we have for any positive $u \in \mathbb{R}^V$ and $x \in V$ that

$$\begin{aligned} \Psi_\Upsilon(\log u)(x) - \frac{1}{\alpha} \Psi_{\Upsilon_\alpha}(\log u)(x) &= \Psi_{(\exp(\cdot/2)-1)^2}(\log u)(x) \\ &= \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} \left(e^{(\log u(y) - \log u(x))/2} - 1 \right)^2 \\ &= \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} \left(\frac{\sqrt{u(y)}}{\sqrt{u(x)}} - 1 \right)^2 \\ &= \frac{2\Gamma(\sqrt{u})(x)}{u(x)}. \end{aligned} \tag{2.11}$$

The aforementioned computation rules directly apply to more general nonlocal operators. We formulate this result for the Euclidean space. In the following lemma, $B_\varepsilon(0)$ denotes the open ball in \mathbb{R}^d with radius ε and center at 0.

Lemma 2.6. *Assume $(\mu(x, dy))_{x \in \mathbb{R}^d}$ is a family of measures on the Borel sets of \mathbb{R}^d satisfying $\mu(x, \{x\}) = 0$ and*

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (1 \wedge |x - y|^2) \mu(x, dy) < \infty.$$

Assume $H \in C^2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $u : \mathbb{R}^d \rightarrow \mathbb{R}$ are such that

$$\mathcal{L}u(x) := \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d \setminus B_\varepsilon(0)} (u(y) - u(x)) \mu(x, dy)$$

and $\mathcal{L}(H \circ u)(x)$ exist. Then

$$\mathcal{L}(H \circ u)(x) = H'(u(x)) \mathcal{L}u(x) + \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d \setminus B_\varepsilon(0)} \left(\Lambda_H(u(y), u(x)) \right) \mu(x, dy), \tag{2.12}$$

with Λ_H as in (2.3).

The proof of this result is as simple as the proof of Lemma 2.1. Repeating Example 2.2, Example 2.3 and Example 2.4 for the case of the fractional Laplace operator $\Delta^s = -(-\Delta)^s$ with $0 < s < 1$ in \mathbb{R}^d , we obtain the following computation rules for sufficiently regular functions u :

$$\Delta^s(u^2) = 2u \Delta u + 2\Gamma(u), \tag{2.13}$$

$$\Delta^s \sqrt{u} = \frac{\Delta^s u}{2\sqrt{u}} - \frac{2\Gamma(\sqrt{u})}{2\sqrt{u}}, \tag{2.14}$$

$$\Delta^s \log u = \frac{\Delta^s u}{u} - c_{d,s} \int_{\mathbb{R}^d} \frac{\Upsilon(\log u(y) - \log u(\cdot))}{|y - \cdot|^{d+2s}} dy. \tag{2.15}$$

Here, Γ denotes the carré du champ operator that corresponds with Δ^s , and Υ is as above. The constant $c_{d,s}$ is the normalizing constant that appears in the representation of Δ^s as an integrodifferential operator. It satisfies $c_{d,s} \asymp (1-s)s$ for $0 < s < 1$. For sufficiently regular functions v , the following observation holds:

$$c_{d,s} \int_{\mathbb{R}^d} \frac{\Upsilon(v(y) - v(\cdot))}{|y - \cdot|^{d+2s}} dy \rightarrow |\nabla v|^2 \quad \text{as } s \rightarrow 1 - .$$

3. Conditions related to curvature-dimension inequalities

In this section, we introduce a family of conditions $CD_\alpha(F;0)$ on graphs. Here $\alpha \in [0, 1)$ is a parameter and $F : [0, \infty) \rightarrow [0, \infty)$ is a function with $F(0) = 0$ and some additional properties. As we will show, condition $CD_0(F;0)$ ¹ ensures that positive solutions to the heat equation satisfy a Li-Yau type estimate. We provide examples of graphs that satisfy $CD_0(F;0)$ and examples that do not have this property. The case $\alpha \in (0, 1)$ is of particular interest for infinite graphs, cf. Section 5.

3.1. A new version of the CD-inequality

Definition 3.1. A continuous function $F : [0, \infty) \rightarrow [0, \infty)$ is called CD-function, if $F(0) = 0$, $F(x)/x$ is strictly increasing on $(0, \infty)$, and $1/F$ is integrable at ∞ .

Note that for any CD-function F we have $F(x) > 0$ for all $x \in (0, \infty)$ and F is strictly increasing on $[0, \infty)$. An example of a CD-function is given by $F(x) = cx^2$ with $c > 0$.

Proposition 3.2. If F_1, F_2 are CD-functions, then the functions $F_1 + F_2$ and $\min(F_1, F_2)$ are CD-functions, as well. In addition, $\alpha F_1(\beta \cdot)$ is a CD-function for every $\alpha, \beta \in (0, \infty)$.

Proof. The argument for $F_1 + F_2$ is straightforward. As to the minimum $F := \min(F_1, F_2)$, note that

$$H(x) := \frac{F(x)}{x} = \min(H_1(x), H_2(x)) \quad \text{with } H_i(x) := \frac{F_i(x)}{x}, \quad x \in (0, \infty).$$

It follows from the intermediate value theorem that the minimum of two strictly increasing and continuous functions is again strictly increasing. Thus H is strictly increasing on $(0, \infty)$. Further, we have for $x_1, x_2 > 0$ that

$$\frac{1}{\min(x_1, x_2)} < \frac{1}{x_1} + \frac{1}{x_2},$$

and so it is evident that the integrability of $1/F_i$ at ∞ , $i = 1, 2$, implies the same property for $1/F$. This shows that F is a CD-function. The last assertion is obvious. \square

¹ In the sequel, we will write $CD(F;0)$ instead of $CD_0(F;0)$.

Remark 3.3. Let $g : [0, \infty) \rightarrow [0, \infty)$ be a strictly convex function with $g(0) = 0$. Then the function $g(x)/x$ is strictly increasing on $(0, \infty)$. In fact, strict convexity of g implies that the difference quotients of g are strictly increasing, and thus $g(x)/x = (g(x) - g(0))/x$ is strictly increasing. Note that a CD-function need not be convex as the example $F(x) = \min(x^2, x^3)$ shows.

The following family of CD-functions plays a central role in the context of Ricci-flat graphs.

Proposition 3.4. Let $\lambda \in (0, 1)$ and the function $F : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$F(x) = e^{-\frac{1-\lambda}{2}x} \left(\lambda e^{(1-\lambda)x} + (1-\lambda)e^{-\lambda x} - 1 \right), \quad x \in \mathbb{R}. \quad (3.1)$$

Then F is non-negative and strictly convex on \mathbb{R} , and the restriction of F to $[0, \infty)$ is a CD-function. Moreover, there exists an $x_* > 0$ such that the function $F(x)/x$ is convex in $[x_*, \infty)$. Furthermore, for any $\tilde{x} > 0$ we have

$$\frac{d}{dx} \left(\frac{F(x)}{x} \right) \geq \frac{1}{2} \lambda (1-\lambda) e^{-\frac{1+\lambda}{2}\tilde{x}}, \quad x \in (0, \tilde{x}]. \quad (3.2)$$

Proof. Let $S(x)$ denote the term in brackets in (3.1) and set $\beta = \frac{1-\lambda}{2}$. By the convexity of the exponential function, we have

$$1 = e^0 = e^{\lambda[(1-\lambda)x] + (1-\lambda)[- \lambda x]} \leq \lambda e^{(1-\lambda)x} + (1-\lambda)e^{-\lambda x}.$$

This shows non-negativity of S , and thus $F(x) \geq 0$ for all $x \in \mathbb{R}$. Evidently, $F(0) = 0$. Further,

$$\begin{aligned} F''(x) &= e^{-\beta x} (\beta^2 S(x) - 2\beta S'(x) + S''(x)) \\ &= e^{-\beta x} \left(\beta^2 S(x) - 2\beta [\lambda(1-\lambda)(e^{(1-\lambda)x} - e^{-\lambda x})] \right. \\ &\quad \left. + [\lambda(1-\lambda)^2 e^{(1-\lambda)x} + (1-\lambda)\lambda^2 e^{-\lambda x}] \right) \\ &= e^{-\beta x} (\beta^2 S(x) + \lambda(1-\lambda)e^{-\lambda x}) \\ &\geq \lambda(1-\lambda)e^{-(\beta+\lambda)x}. \end{aligned} \quad (3.3)$$

This implies strict convexity of F on \mathbb{R} , and thus by Remark 3.3 that $F(x)/x$ is strictly increasing on $(0, \infty)$. Since $F(x)$ is exponentially increasing as $x \rightarrow \infty$, $1/F$ is integrable at ∞ . Hence F is a CD-function.

As to (3.2), we have for $x > 0$

$$\frac{d}{dx} \left(\frac{F(x)}{x} \right) = \frac{x F'(x) - F(x)}{x^2},$$

and by Taylor's theorem

$$0 = F(0) = F(x) + F'(x)(-x) + \frac{1}{2} F''(\xi)x^2,$$

for some $\xi \in (0, x)$. Using (3.3), it follows that for $x \in (0, \tilde{x}]$

$$\begin{aligned} \frac{d}{dx} \left(\frac{F(x)}{x} \right) &= \frac{1}{2} F''(\xi) \geq \frac{1}{2} \lambda(1 - \lambda) e^{-(\beta+\lambda)\xi} \\ &\geq \frac{1}{2} \lambda(1 - \lambda) e^{-(\frac{1-\lambda}{2} + \lambda)x} \geq \frac{1}{2} \lambda(1 - \lambda) e^{-\frac{1+\lambda}{2}\tilde{x}}. \end{aligned}$$

Turning to the convexity of $F(x)/x$, we have for $x > 0$

$$\begin{aligned} \frac{d^2}{dx^2} \left(\frac{F(x)}{x} \right) &= \frac{x^2 F''(x) - 2x F'(x) + 2F(x)}{x^3} \\ &= \frac{e^{-\beta x}}{x^3} \left(x^2 (\beta^2 S(x) + \lambda(1 - \lambda) e^{-\lambda x}) \right. \\ &\quad \left. - 2x (-\beta S(x) + S'(x)) + 2S(x) \right) \\ &\geq \frac{e^{-\beta x}}{x^3} \left(\frac{(1 - \lambda)^2}{4} x^2 [\lambda e^{(1-\lambda)x} - 1] - 2x \lambda(1 - \lambda) e^{(1-\lambda)x} \right). \end{aligned}$$

Inside the brackets, the term with x^2 clearly dominates the other term for large x , and thus one finds an $x_* > 0$ such that $(F(x)/x)'' \geq 0$ for all $x \in [x_*, \infty)$. \square

Lemma 3.5. *Let $F : [0, \infty) \rightarrow [0, \infty)$ be a CD-function. Then there is a unique strictly positive solution φ of the ODE*

$$\dot{\varphi}(t) + F(\varphi(t)) = 0, \quad t > 0, \tag{3.4}$$

with $\varphi(0+) = \infty$. The function φ is strictly decreasing and log-convex, and it satisfies $\varphi(\infty) = 0$.

Proof. We define $G(x) = \int_x^\infty dr/F(r)$, $x > 0$. Then $G'(x) = -1/F(x) < 0$, that is, G is strictly decreasing. Since $F(x)/x$ is increasing on $(0, \infty)$, we have $F(x) \leq F(1)x$ for all $x \in (0, 1]$, and thus

$$G(x) = \int_x^1 \frac{dr}{F(r)} + \int_1^\infty \frac{dr}{F(r)} \geq \frac{1}{F(1)} \int_x^1 \frac{dr}{r} + G(1), \quad x \in (0, 1],$$

which shows that $G(0+) = \infty$. Observe also that $G(\infty) = 0$.

Suppose φ is a strictly positive solution of the ODE (3.4) on $(0, \infty)$ with $\varphi(0+) = \infty$. Then for $t, t_1 \in (0, \infty)$ we have

$$t - t_1 = \int_t^{t_1} \frac{\dot{\varphi}(\tau)}{F(\varphi(\tau))} d\tau = \int_{\varphi(t)}^{\varphi(t_1)} \frac{dr}{F(r)}.$$

Sending $t_1 \rightarrow 0+$ yields $t = G(\varphi(t))$, that is

$$\varphi(t) = G^{-1}(t), \quad t > 0, \tag{3.5}$$

which shows uniqueness. On the other hand, it is easy to verify that (3.5) defines a strictly positive solution φ of the ODE (3.4) with $(0, \infty)$ as its maximal interval of existence. Evidently, $\varphi(0+) = \infty$, $\varphi(\infty) = 0$, and $\dot{\varphi}(t) < 0$ for all $t \in (0, \infty)$.

Finally, since φ is strictly decreasing and $F(x)/x$ is strictly increasing, the function

$$\eta(t) := \frac{d}{dt} (\log \varphi(t)) = \frac{\dot{\varphi}(t)}{\varphi(t)} = -\frac{F(\varphi(t))}{\varphi(t)}, \quad t > 0,$$

is strictly increasing, which in turn implies that φ is log-convex. □

Definition 3.6. Let $F : [0, \infty) \rightarrow [0, \infty)$ be a CD-function. The positive function φ that solves (3.4) with $(0, \infty)$ as maximal interval of existence is called relaxation function associated with F .

We now discuss the asymptotic properties of the relaxation function. Here and in the sequel, we write $f(r) \sim g(r)$ ($r \rightarrow a$) for $a \in \{0, +\infty\}$ and two functions f and g , if the ratio $f(r)/g(r)$ tends to 1 as $r \rightarrow a$. Note that we use the same symbol to describe that two vertices $x, y \in V$ are neighbors, i.e., $x \sim y$.

Lemma 3.7. Let F be a CD-function and φ the corresponding relaxation function. Then the following statements hold:

- (i) Let $\tilde{x} \in [0, \infty)$ and $\tilde{F} : [\tilde{x}, \infty) \rightarrow (0, \infty)$ be continuous. Assume further that $F(r) \sim \tilde{F}(r)$ as $r \rightarrow \infty$ and define $\tilde{G} : [\tilde{x}, \infty) \rightarrow (0, a]$ with $a = \int_{\tilde{x}}^{\infty} dr/\tilde{F}(r)$ by $\tilde{G}(x) = \int_x^{\infty} dr/\tilde{F}(r)$, $x \geq \tilde{x}$. Let $\tilde{\varphi} : (0, a] \rightarrow (0, \infty)$ be defined by $\tilde{\varphi}(t) = \tilde{G}^{-1}(t)$. Then

$$\tilde{\varphi}(t) \sim \varphi(t) \quad \text{as } t \rightarrow 0+.$$

In particular, if $F(r) \sim c e^{\gamma r}$ as $r \rightarrow \infty$ with $c, \gamma > 0$ the relaxation function has a logarithmic singularity at $0+$,

$$\varphi(t) \sim -\frac{1}{\gamma} \log t \quad \text{as } t \rightarrow 0+;$$

- (ii) Suppose that $F(r) \sim \nu r^2$ as $r \rightarrow 0+$ with some constant $\nu > 0$, and assume that there exists $\nu_0 > 0$ such that $F(r) \geq \nu_0 r^2$ for all $r \geq 0$. Then

$$\varphi(t) \sim \frac{1}{\nu t} \quad \text{as } t \rightarrow \infty.$$

Proof. (i) The first assertion follows directly from the representation formula for φ ,

$$\varphi(t) = G^{-1}(t) \quad \text{with } G(x) = \int_x^{\infty} \frac{dr}{F(r)}.$$

Recall that as $t \rightarrow 0+$ we have that $\varphi(t) \rightarrow \infty$ and thus the formula for G shows that the behavior of F at ∞ determines the behavior of φ at $0+$. In the case $\tilde{F}(r) = c e^{\gamma r}$ we find that

$$\tilde{G}(x) = \frac{1}{c} \int_x^\infty e^{-\gamma r} dr = \frac{1}{c\gamma} e^{-\gamma x}, \quad x \geq 0,$$

which yields

$$\tilde{\varphi}(t) = -\frac{1}{\gamma} \log(c\gamma t) \sim -\frac{1}{\gamma} \log t \quad \text{as } t \rightarrow 0+.$$

(ii) Let $F_0(r) = \nu r^2, r \geq 0$. We set

$$F_\tau(r) = \frac{1}{\tau^2} F(\tau r), \quad \tau > 0, r \geq 0,$$

and

$$G_\tau(x) = \int_x^\infty \frac{dr}{F_\tau(r)}, \quad \tau \geq 0, x > 0.$$

We first claim that $F_\tau \rightarrow F_0$ uniformly on any interval $(0, r_1]$ as $\tau \rightarrow 0+$. In fact, letting $r_1 > 0$ the assumption that $F(r) \sim \nu r^2$ as $r \rightarrow 0+$ implies that given $\varepsilon > 0$ there is $\delta > 0$ such that $|F(s)/(\nu s^2) - 1| \leq \varepsilon/(\nu r_1^2)$ for all $s \in (0, \delta]$. Suppose now that $\tau \in (0, \delta/r_1]$. Then we have for $r \in (0, r_1]$ that $\tau r \leq \delta$ and thus

$$|F_\tau(r) - F_0(r)| \leq \nu r^2 \left| \frac{F(\tau r)}{\nu(\tau r)^2} - 1 \right| \leq \varepsilon.$$

Next, it follows from the previous property and the lower bound for F that $G_\tau \rightarrow G_0$ uniformly on any interval $[x_0, x_1] \subset (0, \infty)$ as $\tau \rightarrow 0+$. This can be seen by writing

$$G_\tau(x) = \int_1^\infty \frac{dr}{F_\tau(r)} + \int_x^1 \frac{dr}{F_\tau(r)}, \quad x \in [x_0, x_1],$$

where the convergence of the first integral to $\int_1^\infty \frac{dr}{F_0(r)}$ follows from the dominated convergence theorem.

Using the property that $G_\tau \rightarrow G_0$ on any compact subinterval of $(0, \infty)$ as $\tau \rightarrow 0+$ it is not difficult to check that then for each $t \in (0, \infty)$ we have

$$\varphi_\tau(t) := G_\tau^{-1}(t) \rightarrow G_0^{-1}(t) = \frac{1}{\nu t} =: \varphi_0(t) \quad (3.6)$$

as $\tau \rightarrow 0+$. Observe that by the definitions of F_τ and G_τ ,

$$\dot{\varphi}_\tau(t) = -F_\tau(\varphi_\tau(t)) = -\frac{1}{\tau^2} F(\tau \varphi_\tau(t)), \quad t \in (0, \infty),$$

as well as $\varphi_\tau(0+) = \infty$. Invoking Lemma 3.5, this shows that

$$\varphi_\tau(t) = \frac{1}{\tau} \varphi\left(\frac{t}{\tau}\right), \quad t, \tau > 0,$$

which together with (3.6) gives for any fixed $t > 0$

$$\frac{t}{\tau} \varphi\left(\frac{t}{\tau}\right) = t\varphi_\tau(t) \rightarrow \frac{1}{\nu} \quad \text{as } \tau \rightarrow 0+.$$

Hence $s\varphi(s) \rightarrow 1/\nu$ as $s \rightarrow \infty$. This proves (ii). □

For $\alpha \in [0, 1)$, $v \in \mathbb{R}^V$ and $x \in V$ we define (with $L = -\Delta$)

$$\begin{aligned} \mathcal{L}_0(v)(x) &= Lv(x), \\ \mathcal{L}_\alpha(v)(x) &= -\frac{1}{\alpha} \Psi_{\Upsilon'}(\alpha v)(x), \quad \text{if } \alpha \in (0, 1) \end{aligned} \tag{3.7}$$

and

$$\mathcal{C}_\alpha(v)(x) := \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} e^{\alpha(v(y)-v(x))} (\Psi_{\Upsilon'}(v)(y) - \Psi_{\Upsilon'}(v)(x)). \tag{3.8}$$

Here we recall that $\Upsilon(y) = e^y - 1 - y$ and thus $\Upsilon'(y) = e^y - 1 = \Upsilon(y) + y$, cf. Example 2.4. Observe that

$$\mathcal{C}_0(v)(x) = \Delta \Psi_{\Upsilon'}(v)(x)$$

and

$$\mathcal{L}_\alpha(v)(x) \rightarrow Lv(x) \quad \text{as } \alpha \rightarrow 0+.$$

Notice as well that

$$\mathcal{L}_\alpha(v)(x) = Lv(x) - \frac{1}{\alpha} \Psi_\Upsilon(\alpha v)(x),$$

which in particular shows that positivity of $\mathcal{L}_\alpha(v)(x)$ implies the same property for $Lv(x)$. Note $\mathcal{L}_\alpha(\log u)$ equals $-\Delta^\psi u$ of [17] for the choice $\psi(s) = (s^\alpha - 1)/\alpha$. The following condition is of great importance throughout this paper.

Definition 3.8. Let $\alpha \in [0, 1)$, F be a CD-function, $G = (V, E)$ an undirected graph, and $x \in V$. We say that the graph G satisfies condition $\text{CD}_\alpha(F;0)$ at $x \in V$ for the generalized Laplace operator Δ given by (1.4), if for every function $v : V \rightarrow \mathbb{R}$ satisfying

$$\mathcal{L}_\alpha(v)(x) > 0 \quad \text{and} \quad \mathcal{L}_\alpha(v)(x) \geq \mathcal{L}_\alpha(v)(y) \quad \text{for all } y \sim x,$$

there holds

$$\mathcal{C}_\alpha(v)(x) \geq F(Lv(x)). \tag{3.9}$$

We say that G satisfies $\text{CD}_\alpha(F;0)$ if it satisfies $\text{CD}_\alpha(F;0)$ at every $x \in V$. In the case $\alpha = 0$, we drop the subscript '0' in the notation and simply speak of the CD-inequality $\text{CD}(F;0)$.

Remark 3.9. (i) The notion $CD_\alpha(F;0)$ suggests that there is a more general condition $CD_\alpha(F;K)$, where $K \in \mathbb{R}$ denotes some curvature bound. So far, we do not allow for terms that measure the curvature as it is the case in the classical curvature dimension inequality. This will be subject to further research. (ii) The condition $CD_0(F;0)$ relates to the classical curvature dimension inequality in a natural way. Note that, in the case $\alpha = 0$, \mathcal{L}_α equals $-\Delta$. Now, let us look at the Euclidean case. Assume $x \in \mathbb{R}^d$ and $v : \mathbb{R}^d \rightarrow \mathbb{R}$ is a smooth function such that the function $-\Delta v$ has a local, strictly positive maximum in x . Then

$$\underbrace{\Delta \Delta v(x)}_{\geq 0} + \underbrace{2(\nabla \Delta v(x), \nabla v(x))}_{=0} + 2 \sum_{i,j=1}^d (\partial_i \partial_j v(x))^2 \geq \frac{2}{d} (-\Delta v(x))^2, \quad (3.10)$$

which is the classical curvature dimension inequality. Note that the left-hand side of (3.10) corresponds to $\Delta \Psi_{\Upsilon'}(v)(x)$. In this sense, the condition $CD_0(F;0)$ is consistent with the classical curvature dimension inequality.

Using Proposition 3.2 we immediately obtain the following:

Proposition 3.10. *Let $\alpha \in [0, 1)$, $G = (V, E)$ be a graph, F_i be CD-functions for $i = 1, \dots, l$ and assume that for any $x \in V$ the graph satisfies $CD_\alpha(F_i, 0)$ at x for some $i \in \{1, \dots, l\}$. Set $F := \min(F_1, \dots, F_l)$. Then the graph satisfies $CD_\alpha(F; 0)$.*

3.2. Some simple illustrating examples

Example 3.11. We first consider the connected graph that only consists of two different vertices, say x_1 and x_2 . For the Laplace operator, we take the most simple form (without weight), that is,

$$\Delta u(x) = u(\tilde{x}) - u(x), \quad x \in V = \{x_1, x_2\},$$

where $\tilde{x}_1 = x_2$ and vice versa. Let $v \in \mathbb{R}^V$ and $x \in V$. Then we have

$$\begin{aligned} \Delta \Psi_{\Upsilon'}(v)(x) &= \Psi_{\Upsilon'}(v)(\tilde{x}) - \Psi_{\Upsilon'}(v)(x) \\ &= \Upsilon'(v(x) - v(\tilde{x})) - \Upsilon'(v(\tilde{x}) - v(x)) \\ &= e^{Lv(x)} - e^{-Lv(x)} \\ &= F(Lv(x)), \end{aligned}$$

where $F(a) = 2 \sinh a$, which is easily verified to be a CD-function. Thus, condition $CD(2 \sinh; 0)$ is satisfied. A straight-forward computation shows that the relax-

ation function corresponding to F is given by

$$\varphi(t) = \log \left(\frac{1 + e^{-2t}}{1 - e^{-2t}} \right) = -\log (\tanh t), \quad t > 0. \tag{3.11}$$

In the case $\alpha \in (0, 1)$ one obtains

$$\mathcal{C}_\alpha(v)(x) = e^{-\alpha Lv(x)} (e^{Lv(x)} - e^{-Lv(x)}) \geq e^{(1-\alpha)Lv(x)} - e^{-(1-\alpha)Lv(x)}, \tag{3.12}$$

that is, the CD-inequality $CD_\alpha(F_\alpha; 0)$ holds with

$$F_\alpha(y) = 2 \sinh ((1 - \alpha)y).$$

Note that $\tilde{F}(y) = e^{-\alpha y}(e^y - e^{-y})$, $y \geq 0$ is not a CD-function, since $\tilde{F}(y)/y$ is decreasing near 0. Note also that in (3.12) we used that $Lv(x) = v(x) - v(\tilde{x}) > 0$, which follows from $\mathcal{L}_\alpha(v)(x) > 0$.

Example 3.12. We next consider the case of a triangle, *i.e.*, $V = \{x_*, x_1, x_2\}$ and $E = \{x_*x_1, x_*x_2, x_1x_2\}$. Again, we look at the most simple case without weights and with $\mu \equiv 1$. Let $v \in \mathbb{R}^V$ and set $z_j = v(x_j)$ for $j \in \{*, 1, 2\}$ and $a_j = z_* - z_j$ for $j \in \{1, 2\}$.

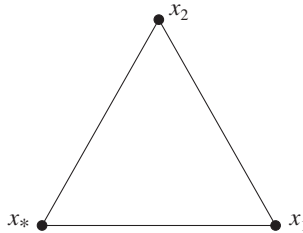


Figure 3.1. A triangle.

Now,

$$\begin{aligned} \Delta \Psi_{\Upsilon'}(v)(x_*) &= \Psi_{\Upsilon'}(v)(x_1) + \Psi_{\Upsilon'}(v)(x_2) - 2\Psi_{\Upsilon'}(v)(x_*) \\ &= \Upsilon'(z_* - z_1) + \Upsilon'(z_2 - z_1) + \Upsilon'(z_* - z_2) + \Upsilon'(z_1 - z_2) \\ &\quad - 2(\Upsilon'(z_1 - z_*) + \Upsilon'(z_2 - z_*)) \\ &= e^{a_1} + e^{a_1 - a_2} + e^{a_2} + e^{a_2 - a_1} - 2e^{-a_1} - 2e^{-a_2} =: f(a_1, a_2). \end{aligned}$$

Further,

$$\begin{aligned} Lv(x_*) &= (z_* - z_1) + (z_* - z_2) = a_1 + a_2, \\ Lv(x_1) &= (z_1 - z_*) + (z_1 - z_2) = a_2 - 2a_1, \\ Lv(x_2) &= (z_2 - z_*) + (z_2 - z_1) = a_1 - 2a_2. \end{aligned}$$

We see that Lv has a positive maximum at x_* if and only if $a_j \geq 0$ for $j = 1, 2$ and $a_1 + a_2 > 0$. By convexity of $\Upsilon'(y) = e^y - 1$ on \mathbb{R} , we have

$$\begin{aligned} f(a_1, a_2) &= (e^{a_1} + e^{a_2} - 2)(e^{-a_1} + e^{-a_2} + 1) \\ &= (\Upsilon'(a_1) + \Upsilon'(a_2))(e^{-a_1} + e^{-a_2} + 1) \\ &\geq 2\Upsilon'\left(\frac{a_1 + a_2}{2}\right)(e^{-a_1} + e^{-a_2} + 1). \end{aligned}$$

Assuming that $a_1 + a_2 > 0$ we see that $\Upsilon'\left(\frac{a_1+a_2}{2}\right) > 0$ and thus we may estimate further by using the convexity of the exponential function, thereby obtaining

$$f(a_1, a_2) \geq 2\Upsilon'\left(\frac{a_1 + a_2}{2}\right)\left(2e^{-\frac{a_1+a_2}{2}} + 1\right)$$

which yields that

$$\Delta\Psi_{\Upsilon'}(v)(x_*) \geq F(Lv(x_*)),$$

where

$$F(a) = 2(e^{\frac{a}{2}} - 1)(2e^{-\frac{a}{2}} + 1) = 2\left(e^{\frac{a}{2}} + 1 - 2e^{-\frac{a}{2}}\right).$$

Observe that F is not a CD-function, since $F(a) = 3a - a^2/2 + O(a^3)$ as $a \rightarrow 0+$, which implies that $F(x)/x$ is not increasing near 0. However, one can find many CD-functions \tilde{F} with $F \geq \tilde{F}$ on $[0, \infty)$, e.g., $\tilde{F}(a) = 4 \sinh(a/2)$, and so $\text{CD}(\tilde{F}; 0)$ holds for any such function.

The aforementioned examples are special cases of the class of complete graphs. Next, let us treat complete graphs in general.

Example 3.13. Let $G = (V, E)$ be a complete graph with $D + 1$ vertices, $D \in \mathbb{N}$. That is, for every pair of vertices $x, y \in V$ with $x \neq y$ we have $x \sim y$. Let $V = \{x_0, x_1, \dots, x_D\}$ and $\alpha \in [0, \frac{1}{2}]$. We consider the case without edge weights and with $\mu(y) = \mu_0 > 0$ for all $y \in V$. Suppose that $v : V \rightarrow \mathbb{R}$ is such that

$$\mathcal{L}_\alpha(v)(x_0) > 0 \quad \text{and} \quad \mathcal{L}_\alpha(v)(x_0) \geq \mathcal{L}_\alpha(v)(x_j) \quad \text{for all } j = 1, \dots, D.$$

Setting

$$F_\alpha(a) := \frac{D}{\mu_0^2} e^{-\frac{\alpha\mu_0}{D}a} (e^{\frac{\mu_0}{D}a} - 1) (De^{-\frac{\mu_0}{D}a} + 1), \quad a \geq 0,$$

we claim that

$$\mathcal{C}_\alpha(v)(x_0) \geq F_\alpha(Lv(x_0)). \tag{3.13}$$

To prove this, we put $z_k = v(x_k)$ for $k = 0, 1, \dots, D$ and $a_k = z_0 - z_k$, $k = 1, \dots, D$. First we show that the maximum property of $\mathcal{L}_\alpha(v)$ at x_0 implies that

$a_k \geq 0$ for all $k = 1, \dots, D$. By symmetry, it suffices to prove this for $k = 1$. If $\alpha > 0$ we have by definition of $\mathcal{L}_\alpha(v)$ that

$$\mathcal{L}_\alpha(v)(x_0) = -\frac{1}{\alpha\mu_0} \sum_{j=1}^D \Upsilon'(\alpha(z_j - z_0))$$

and

$$\mathcal{L}_\alpha(v)(x_1) = -\frac{1}{\alpha\mu_0} \sum_{j=2}^D \Upsilon'(\alpha(z_j - z_1)) - \frac{1}{\alpha\mu_0} \Upsilon'(\alpha(z_0 - z_1)).$$

From $\mathcal{L}_\alpha(v)(x_0) \geq \mathcal{L}_\alpha(v)(x_1)$ we may thus deduce that

$$\sum_{j=1}^D e^{-\alpha a_j} \leq \sum_{j=2}^D e^{\alpha(a_1 - a_j)} + e^{\alpha a_1},$$

which is equivalent to

$$e^{-\alpha a_1} - e^{\alpha a_1} \leq (e^{\alpha a_1} - 1) \sum_{j=2}^D e^{-\alpha a_j}. \tag{3.14}$$

Now, if $a_1 < 0$, then the left-hand side in (3.14) is greater than zero while the right-hand side becomes negative, a contradiction. Hence $a_1 \geq 0$.

In the case $\alpha = 0$ we have

$$\mathcal{L}_0(v)(x_0) = Lv(x_0) = \frac{1}{\mu_0} \sum_{j=1}^D (z_0 - z_j) = \frac{1}{\mu_0} \sum_{j=1}^D a_j$$

and

$$\begin{aligned} \mathcal{L}_0(v)(x_1) &= Lv(x_1) = \frac{1}{\mu_0} \left(z_1 - z_0 + \sum_{j=2}^D (z_1 - z_j) \right) \\ &= \frac{1}{\mu_0} \left(-a_1 + \sum_{j=2}^D (a_j - a_1) \right). \end{aligned}$$

By means of these formulas it is easy to see that the condition $\mathcal{L}_0(v)(x_0) \geq \mathcal{L}_0(v)(x_1)$ is equivalent to $a_1 \geq 0$.

Next, we introduce the function

$$\psi(a) = e^{-\alpha a}(e^a - 1) = e^{(1-\alpha)a} - e^{-\alpha a}, \quad a \in \mathbb{R}.$$

We claim that ψ is convex on $[0, \infty)$ for all $\alpha \in [0, \frac{1}{2}]$. Indeed,

$$\psi''(a) = (1 - \alpha)^2 e^{(1-\alpha)a} - \alpha^2 e^{-\alpha a} \geq 0, \quad a \in [0, \infty),$$

since $(1 - \alpha)^2 \geq \alpha^2$ and $e^{(1-\alpha)a} \geq e^{-\alpha a}$.

Using that $a_k = z_0 - z_k \geq 0$ for all $k = 1, \dots, D$ and employing the convexity of ψ on $[0, \infty)$ we may now estimate as follows:

$$\begin{aligned}
 \mathcal{C}_\alpha(v)(x_0) &= \frac{1}{\mu_0} \sum_{k=1}^D e^{\alpha(z_k - z_0)} \left(\Psi_{\Gamma'}(v)(x_k) - \Psi_{\Gamma'}(v)(x_0) \right) \\
 &= \frac{1}{\mu_0^2} \sum_{k=1}^D e^{\alpha(z_k - z_0)} \left(\sum_{j=0, j \neq k}^D e^{z_j - z_k} - \sum_{j=1}^D e^{z_j - z_0} \right) \\
 &= \frac{1}{\mu_0^2} \sum_{k=1}^D e^{\alpha(z_k - z_0)} \left(\sum_{j=1}^D (e^{z_j - z_k} - e^{z_j - z_0}) + e^{z_0 - z_k} - 1 \right) \\
 &= \frac{1}{\mu_0^2} \sum_{k=1}^D e^{-\alpha(z_0 - z_k)} (e^{z_0 - z_k} - 1) \left(\sum_{j=1}^D e^{z_j - z_0} + 1 \right) \\
 &= \frac{1}{\mu_0^2} \sum_{k=1}^D \psi(z_0 - z_k) \left(\sum_{j=1}^D e^{z_j - z_0} + 1 \right) \\
 &\geq \frac{D}{\mu_0^2} \psi \left(\frac{1}{D} \sum_{k=1}^D (z_0 - z_k) \right) \left(\sum_{j=1}^D e^{z_j - z_0} + 1 \right) \\
 &= \frac{D}{\mu_0^2} \psi \left(\frac{\mu_0}{D} Lv(x_0) \right) \left(\sum_{j=1}^D e^{z_j - z_0} + 1 \right).
 \end{aligned}$$

Since ψ is positive on $(0, \infty)$ and $Lv(x_0) \geq \mathcal{L}_\alpha(v)(x_0) > 0$, the ψ -term in the last line is positive. Using the convexity of the exponential function we may thus further deduce that

$$\begin{aligned}
 \mathcal{C}_\alpha(v)(x_0) &\geq \frac{D}{\mu_0^2} \psi \left(\frac{\mu_0}{D} Lv(x_0) \right) \left(D \exp \left(\frac{1}{D} \sum_{j=1}^D (z_j - z_0) \right) + 1 \right) \\
 &= \frac{D}{\mu_0^2} \psi \left(\frac{\mu_0}{D} Lv(x_0) \right) \left(D \exp \left(-\frac{\mu_0}{D} Lv(x_0) \right) + 1 \right) = F_\alpha(Lv(x_0)).
 \end{aligned}$$

Note that F_α is not a CD-function in general. Note that in the case $\alpha = 0$, $\mu_0 = 1$, we obtain

$$F_0(a) = D(e^{\frac{a}{D}} - De^{-\frac{a}{D}} + (D-1)),$$

which reduces to $F_0(a) = e^a - e^{-a} = 2 \sinh(a)$ in the case $D = 1$ and to $F_0(a) = 2(e^{\frac{a}{2}} - 2e^{-\frac{a}{2}} + 1)$ in the case of $D = 2$. Thus, the case of general complete graphs is consistent with Example 3.11 and Example 3.12.

Example 3.14. The next example is a path consisting of three vertices. Let $V = \{x_*, x_1, x_2\}$ and $E = \{x_*x_1, x_*x_2\}$. We consider the case without weights and with



Figure 3.2. A chain-like graph.

$\mu(x_i) = 1, i = 1, 2$ and $\mu(x_*) = 2$, so μ coincides at every vertex with its degree. Letting $v \in \mathbb{R}^V$ we use the same notation as in Example 3.12.

Then, we have for the vertex x_*

$$\begin{aligned} &\Delta \Psi_{\Upsilon'}(v)(x_*) \\ &= \frac{1}{2} \left(\Psi_{\Upsilon'}(v)(x_1) + \Psi_{\Upsilon'}(v)(x_2) - 2\Psi_{\Upsilon'}(v)(x_*) \right) \\ &= \frac{1}{2} \left(\Upsilon'(z_* - z_1) + \Upsilon'(z_* - z_2) - 2 \cdot \frac{1}{2} (\Upsilon'(z_1 - z_*) + \Upsilon'(z_2 - z_*)) \right) \\ &= \frac{1}{2} \left(e^{a_1} + e^{a_2} - e^{-a_1} - e^{-a_2} \right) = \frac{1}{2} (\phi(a_1) + \phi(a_2)), \end{aligned}$$

where $\phi(y) = e^y - e^{-y}$. Further,

$$Lv(x_*) = \frac{1}{2}(a_1 + a_2), \quad Lv(x_1) = -a_1, \quad Lv(x_2) = -a_2.$$

Lv has a positive maximum at x_* if and only if $3a_1 + a_2 \geq 0, 3a_2 + a_1 \geq 0$ and $a_1 + a_2 > 0$. Assuming this, by symmetry, we may assume that $a_1 \leq a_2$. Then $3a_1 + a_2 \geq 0$ implies that $3a_2 + a_1 \geq 0$, so the first condition is the stronger one and will be assumed.

Suppose first that $0 \leq a_1 (\leq a_2)$. Then, by convexity of ϕ on $[0, \infty)$, we find that

$$\Delta \Psi_{\Upsilon'}(v)(x_*) = \frac{1}{2} (\phi(a_1) + \phi(a_2)) \geq \phi \left(\frac{a_1 + a_2}{2} \right).$$

In the case $a_1 < 0 < a_2$ we know that $a_1 \geq -\frac{a_2}{3}$. Since ϕ is odd and strictly increasing on \mathbb{R} , we may thus estimate as

$$\phi(a_1) + \phi(a_2) \geq \phi \left(-\frac{a_2}{3} \right) + \phi(a_2) = -\phi \left(\frac{a_2}{3} \right) + \phi(a_2).$$

By convexity and monotonicity of ϕ on $[0, \infty)$,

$$\phi(a_2) = \phi(a_2) + \phi(0) \geq 2\phi \left(\frac{a_2}{2} \right) \geq \phi \left(\frac{a_2}{2} \right) + \phi \left(\frac{a_1 + a_2}{2} \right)$$

Combining the last two formulas yields

$$\phi(a_1) + \phi(a_2) \geq \phi \left(\frac{a_1 + a_2}{2} \right).$$

All in all, this shows that

$$\Delta \Psi_{\Upsilon'}(v)(x_*) \geq F_1(Lv(x_*))$$

with $F_1(y) = \frac{1}{2}\phi(y) = \sinh(y)$. Hence $CD(\sinh;0)$ holds at x_* .

Let us now study an endpoint of the path. At the vertex x_1 , we have

$$\begin{aligned} \Delta \Psi_{\Upsilon'}(v)(x_1) &= \Psi_{\Upsilon'}(v)(x_*) - \Psi_{\Upsilon'}(v)(x_1) \\ &= \frac{1}{2}(\Upsilon'(z_1 - z_*) + \Upsilon'(z_2 - z_*)) - \Psi_{\Upsilon'}(v)(z_* - z_1) \\ &= \frac{1}{2}(e^{-a_1} + e^{-a_2}) - e^{a_1} =: f(a_1, a_2). \end{aligned}$$

The condition $Lv(x_1) \geq Lv(x_*)$ is equivalent to $3a_1 + a_2 \leq 0$, and $Lv(x_1) > 0$ means that $a_1 = -Lv(x_1) < 0$. Since f is strictly decreasing w.r.t. a_2 , we obtain $f(a_1, a_2) \geq f(a_1, -3a_1)$ (by increasing a_2 for fixed $a_1 < 0$). This gives

$$\Delta \Psi_{\Upsilon'}(v)(x_1) \geq F_2(Lv(x_1)) \quad \text{with } F_2(a) = \frac{1}{2}(e^a + e^{-3a}) - e^{-a}.$$

Observing that

$$F_2(a) = \frac{1}{2}e^{-a}(e^a - e^{-a})^2 \quad \text{and } F_2''(a) = F_2(a) + 4e^{-3a} > 0$$

we easily see that F_2 is a CD-function. Hence, for $i = 1, 2$, the condition $CD(F_2;0)$ holds at x_i .

Concerning the entire graph, it follows from Proposition 3.10 that the condition $CD(F;0)$ holds with $F = \min(F_1, F_2)$. One can easily check that $F_1(a) \geq F_2(a)$ for all $a \geq 0$. Hence $F = F_2$.

3.3. Ricci-flat graphs

Next, we show that Ricci-flat graphs satisfy the condition $CD(F;0)$ with a CD-function F that we can compute explicitly. The notion of Ricci-flat graphs has been introduced in [5] as a notion of graphs with non-negative curvature.

Definition 3.15. Let $G = (V, E)$ be a D -regular graph with $D \in \mathbb{N}$, let $x \in V$ and $N(x) = \{x\} \cup \{y \in V \mid y \sim x\}$. G is called Ricci-flat at x if, there exist maps $\eta_1, \dots, \eta_D : N(x) \rightarrow V$ such that the following conditions are satisfied:

- (i) $\eta_i(y) \sim y$ for all $i \in \{1, \dots, D\}$ and all $y \in N(x)$;
- (ii) $\eta_i(y) \neq \eta_j(y)$ for $y \in N(x)$ and $i \neq j$;
- (iii) $\bigcup_{j=1}^D \eta_i(\eta_j(x)) = \bigcup_{j=1}^D \eta_j(\eta_i(x))$ for all $i \in \{1, \dots, D\}$.

The graph G is called Ricci-flat if it is Ricci-flat at every vertex $x \in V$.

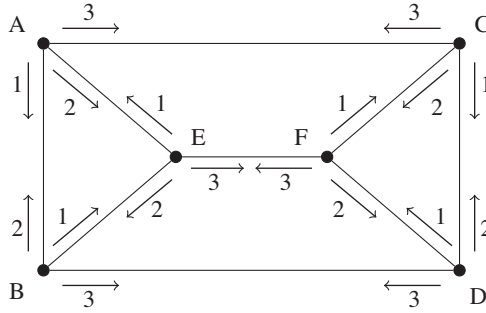


Figure 3.3. A Ricci-flat graph together with the maps $\eta_i, i \in \{1, 2, 3\}$.

The graph \mathbb{Z}^d with $x \sim y \Leftrightarrow |x - y|_1 = 1$ is Ricci-flat. Any Cayley graph of a finitely generated group is Ricci-flat if the generating system is closed under conjugation.

It is proved in [14] that Ricci-flat graphs satisfy $\Gamma_2(f, f) \geq 0$ for every $f \in \mathbb{R}^V$, thus it is reasonable to think of Ricci-flat graphs as graphs with non-negative curvature. One can also think of Ricci-flat graphs as generalizations of Cayley graphs of Abelian groups. We will make use of the following property of Ricci-flat graphs, which is proved in [17].

Lemma 3.16. *Let $G = (V, E)$ be a D -regular graph which is Ricci-flat at the vertex $x \in V$. Let η_1, \dots, η_D be the maps as in Definition 3.15.*

(i) *For any function $u : V \rightarrow \mathbb{R}$ and for all $i \in \{1, \dots, D\}$ one has*

$$\sum_{j=1}^D u(\eta_i(\eta_j(x))) = \sum_{j=1}^D u(\eta_j(\eta_i(x))); \tag{3.15}$$

(ii) *For every $i \in \{1, \dots, D\}$ there exists a unique $i^* \in \{1, \dots, D\}$ such that $\eta_i(\eta_{i^*}(x)) = x$. Moreover, the map $i \mapsto i^*$ is a permutation of $\{1, \dots, D\}$.*

As mentioned above, we can show that Ricci-flat graphs satisfy the condition $CD(F;0)$ for a function F that we can compute explicitly. This is the content of the next result. Note that, in several examples, it is possible to prove the condition $CD(F;0)$ with a larger function \tilde{F} , i.e., the function F given in Theorem 3.17 is not best possible.

Theorem 3.17. *Let $G = (V, E)$ be a D -regular unweighted Ricci-flat graph with $D \geq 2$. Assume that $\mu(y) = \mu_0 > 0$ for all $y \in V$. Then $CD(F; 0)$ holds with*

$$F(a) = \frac{D}{\mu_0^2} \exp\left(-\frac{\mu_0}{D}a\right) \left[\Upsilon\left(\frac{2\mu_0}{D}a\right) + (D-1)\Upsilon\left(-\frac{2\mu_0}{D(D-1)}a\right) \right]. \tag{3.16}$$

Proof. We first verify that F is a CD-function. Setting $\eta = \frac{\mu_0}{D}$ and $\lambda = \frac{1}{D}$ we can write

$$\begin{aligned} F(a) &= \frac{1}{\mu_0 \eta} e^{-\eta a} \left[e^{2\eta a} + (D - 1)e^{-\frac{2\eta}{D-1}a} - D \right] \\ &= \frac{1}{\eta^2} e^{-\eta a} \left[\lambda e^{2\eta a} + (1 - \lambda)e^{-\frac{2\eta}{D-1}a} - 1 \right]. \end{aligned}$$

We scale the argument by putting $\tilde{a} = \frac{2\eta}{1-\lambda}a = \frac{2\mu_0}{D-1}a$ and introduce the function \tilde{F} by means of $\tilde{F}(\tilde{a}) = F(a)$. This gives

$$\tilde{F}(\tilde{a}) = \frac{1}{\eta^2} e^{-\frac{1-\lambda}{2}\tilde{a}} \left[\lambda e^{(1-\lambda)\tilde{a}} + (1 - \lambda)e^{-\lambda\tilde{a}} - 1 \right].$$

Proposition 3.4 and Proposition 3.2 now imply that \tilde{F} , and thus also F , are strictly convex CD-functions. From Proposition 3.4 we also know that both functions are non-negative and strictly convex on all of \mathbb{R} .

Let now $x \in V$ and $v \in \mathbb{R}^V$ such that

$$Lv(x) > 0 \quad \text{and} \quad Lv(x) \geq Lv(\eta_j(x)) \quad \text{for all } j = 1, \dots, D.$$

Set $z = v(x)$, $z_i = v(\eta_i(x))$ and $z_{ij} = v(\eta_j(\eta_i(x)))$ for $i, j = 1, \dots, D$. We have

$$\begin{aligned} \Delta \Psi_{\Upsilon'}(v)(x) &= \frac{1}{\mu_0} \sum_{i=1}^D [\Psi_{\Upsilon'}(v)(\eta_i(x)) - \Psi_{\Upsilon'}(v)(x)] \\ &= \frac{1}{\mu_0^2} \sum_{i=1}^D \sum_{j=1}^D [\Upsilon'(z_{ij} - z_i) - \Upsilon'(z_j - z)] \\ &= \frac{1}{\mu_0^2} \sum_{i=1}^D \sum_{j=1}^D [e^{z_{ij} - z_i} - e^{z_j - z}] \\ &= \frac{1}{\mu_0^2} \sum_{j=1}^D e^{z_j - z} \sum_{i=1}^D (e^{z_{ij} - z_i - z_j + z} - 1). \end{aligned} \tag{3.17}$$

Setting $w_j = z - \frac{1}{2}z_j - \frac{1}{2}z_{j^*}$ and recalling that $z_{j^*j} = z$, the inner sum can be written as

$$\sum_{i=1}^D (e^{z_{ji} - z_i - z_j + z} - 1) = e^{2w_j} - 1 + \sum_{i=1, i \neq j^*}^D (e^{z_{ij} - z_i - z_j + z} - 1).$$

By convexity of the exponential function, Lemma 3.16 (i) and the local maximum property of Lv at x we may now estimate as follows:

$$\begin{aligned} & \sum_{i=1, i \neq j^*}^D (e^{z_{ij} - z_i - z_j + z} - 1) \\ & \geq (D - 1) \left[\exp \left(\frac{1}{D - 1} \sum_{i=1, i \neq j^*}^D [z_{ij} - z_i - z_j + z] \right) - 1 \right] \\ & = (D - 1) \left[\exp \left(\frac{1}{D - 1} \sum_{i=1}^D [z_{ij} - z_i - z_j + z] - \frac{2w_j}{D - 1} \right) - 1 \right] \\ & = (D - 1) \left[\exp \left(\frac{1}{D - 1} \sum_{i=1}^D [z - z_i + z_{ji} - z_j] - \frac{2w_j}{D - 1} \right) - 1 \right] \\ & = (D - 1) \left[\exp \left(\frac{\mu_0}{D - 1} [Lv(x) - Lv(\eta_j(x))] - \frac{2w_j}{D - 1} \right) - 1 \right] \\ & \geq (D - 1) \left[\exp \left(-\frac{2w_j}{D - 1} \right) - 1 \right]. \end{aligned}$$

Combining this and the previous identities yields

$$\Delta \Psi_{\Gamma'(v)}(x) \geq \frac{1}{\mu_0^2} \sum_{j=1}^D e^{z_j - z} \left(e^{2w_j} - 1 + (D - 1) \left[\exp \left(-\frac{2w_j}{D - 1} \right) - 1 \right] \right). \tag{3.18}$$

The next step consists in symmetrizing the sum. Since we do not have $(j^*)^* = j$ in general, we use the rearrangement inequality, which says that for all permutations π on $\{1, \dots, D\}$ and all $0 \leq a_1 \leq a_2 \leq \dots \leq a_D$ and all $0 \leq b_1 \leq b_2 \leq \dots \leq b_D$, one has

$$\sum_{j=1}^D a_{\pi(j)} b_j \geq \sum_{j=1}^D a_{D+1-j} b_j. \tag{3.19}$$

Without restriction of generality, we may assume that $z_1 \leq z_2 \leq \dots \leq z_D$. We set $j' := D + 1 - j$ and $\tilde{w}_j = z - \frac{1}{2}z_j - \frac{1}{2}z_{j'}$. Using (3.19) we then have

$$\begin{aligned} \sum_{j=1}^D e^{z_j - z} \exp \left(-\frac{2w_j}{D - 1} \right) &= \sum_{j=1}^D \exp \left(z_j - z + \frac{1}{D - 1} (z_j - 2z) \right) \exp \left(\frac{1}{D - 1} z_{j^*} \right) \\ &\geq \sum_{j=1}^D \exp \left(z_j - z + \frac{1}{D - 1} (z_j - 2z) \right) \exp \left(\frac{1}{D - 1} z_{j'} \right) \\ &= \sum_{j=1}^D e^{z_j - z} \exp \left(-\frac{2\tilde{w}_j}{D - 1} \right). \end{aligned}$$

Furthermore,

$$\sum_{j=1}^D e^{\tilde{z}_j - \tilde{z}} e^{2w_j} = \sum_{j=1}^D e^{\tilde{z} - \tilde{z}_{j^*}} = \sum_{j=1}^D e^{\tilde{z} - \tilde{z}_{j'}} = \sum_{j=1}^D e^{\tilde{z}_j - \tilde{z}} e^{2\tilde{w}_j}.$$

These relations and (3.18) imply that

$$\Delta \Psi_{\Upsilon'}(v)(x) \geq \frac{1}{\mu_0^2} \sum_{j=1}^D e^{\tilde{z}_j - \tilde{z}} \left(e^{2\tilde{w}_j} - 1 + (D-1) \left[\exp\left(-\frac{2\tilde{w}_j}{D-1}\right) - 1 \right] \right). \tag{3.20}$$

Compared to (3.18), inequality (3.20) has the advantage that $\tilde{w}_j = \tilde{w}_{j'}$, since $(j')' = j$. Employing this and the convexity of the exponential function as well as the non-negativity and convexity of F on \mathbb{R} we have

$$\begin{aligned} & \Delta \Psi_{\Upsilon'}(v)(x) \\ & \geq \frac{1}{2\mu_0^2} \sum_{j=1}^D [e^{\tilde{z}_j - \tilde{z}} + e^{\tilde{z}_{j'} - \tilde{z}}] \left(e^{2\tilde{w}_j} - 1 + (D-1) \left[\exp\left(-\frac{2\tilde{w}_j}{D-1}\right) - 1 \right] \right) \\ & \geq \frac{1}{\mu_0^2} \sum_{j=1}^D e^{-\tilde{w}_j} \left(e^{2\tilde{w}_j} - 1 + (D-1) \left[\exp\left(-\frac{2\tilde{w}_j}{D-1}\right) - 1 \right] \right) \\ & = \frac{1}{D} \sum_{j=1}^D F\left(\frac{D\tilde{w}_j}{\mu_0}\right) \geq F\left(\sum_{j=1}^D \frac{\tilde{w}_j}{\mu_0}\right) = F(Lv(x)). \end{aligned}$$

This proves the asserted inequality. □

It turns out that for Ricci-flat graphs with constant μ , in general, the CD-function F provided by Theorem 3.17 is optimal, at least if D is an even number. This can be seen by looking at the lattice \mathbb{Z}^d . More precisely, we have the following result.

Theorem 3.18. *Let $G = (V, E)$ be the lattice \mathbb{Z}^d and consider the case without weights and with the Laplace operator given by*

$$\Delta u(x) = \frac{1}{\mu_0} \sum_{y \sim x} (u(y) - u(x)), \quad x \in \mathbb{Z}^d,$$

where $\mu_0 > 0$ is a constant. Then for any $a > 0$, there exists a function $v \in \mathbb{R}^V$ satisfying

$$Lv(0) = a > 0 \quad \text{and} \quad Lv(0) \geq Lv(y) \quad \text{for all } y \sim 0,$$

and such that

$$\Delta \Psi_{\Upsilon'}(v)(0) = F(Lv(0)) = F(a),$$

where F is the CD-function given by (3.16) with $D = 2d$.

Proof. Let e_j be the j th unit vector in \mathbb{R}^d and set $\eta_j(x) = x + e_j$ and $\eta_{j+d}(x) = x - e_j$ for $j = 1, \dots, d$ and $x \in \mathbb{Z}^d$. For any vertex, the mapping $j \rightarrow j^*$ from Lemma 3.16(ii) is then given by $j^* = j + d$ for $j = 1, \dots, d$ and $j^* = j - d$ for $j = d + 1, \dots, 2d$.

Let $\alpha > \beta$ and define $v(0) = \alpha$ and $v(\eta_j(0)) = \beta$ for $j = 1, \dots, 2d$. For $i, j \in \{1, \dots, 2d\}$ with $i \neq j^*$ we further set $v(\eta_j(\eta_i(0))) = \gamma$. We put $v(x) = 0$ elsewhere. We then have

$$Lv(0) = \frac{2d(\alpha - \beta)}{\mu_0} > 0.$$

The idea is now to choose $\gamma \in \mathbb{R}$ such that $Lv(\eta_j(0)) = Lv(0)$ for all $j = 1, \dots, 2d$. Note that, by symmetry, Lv then assumes the same value at all neighbors of 0. We have

$$Lv(\eta_j(0)) = \frac{1}{\mu_0}(2d\beta - \alpha - (2d - 1)\gamma), \quad j = 1, \dots, 2d,$$

and so the condition for γ becomes

$$2d(\alpha - \beta) = 2d\beta - \alpha - (2d - 1)\gamma. \tag{3.21}$$

Selecting $\gamma = \gamma(\alpha, \beta, d)$ such that (3.21) is satisfied, we have by (3.17), using the same notation as above,

$$\begin{aligned} & \Delta \Psi_{\gamma'}(v)(0) \\ &= \frac{2d}{\mu_0^2} e^{\beta - \alpha} \left((2d - 1)[e^{\gamma - 2\beta + \alpha} - 1] + e^{2\alpha - 2\beta} - 1 \right) \\ &= \frac{2d}{\mu_0^2} e^{\beta - \alpha} \left(e^{2\alpha - 2\beta} - 1 - (2\alpha - 2\beta) + (2d - 1)[e^{\gamma - 2\beta + \alpha} - 1 - (\gamma - 2\beta + \alpha)] \right) \\ &= \frac{2d}{\mu_0^2} e^{\beta - \alpha} \left(\Upsilon(2\alpha - 2\beta) + (2d - 1)\Upsilon(\gamma - 2\beta + \alpha) \right) \\ &= \frac{2d}{\mu_0^2} \exp\left(-\frac{\mu_0}{2d}Lv(0)\right) \left(\Upsilon\left(\frac{\mu_0}{d}Lv(0)\right) + (2d - 1)\Upsilon\left(-\frac{\mu_0}{d(2d - 1)}Lv(0)\right) \right) \\ &= F(Lv(0)), \end{aligned}$$

since

$$(2d - 1)(\gamma - 2\beta + \alpha) = -2(\alpha - \beta).$$

This proves the assertion as for any given $a > 0$, we can clearly choose α and β such that $Lv(0) = a$. □

Example 3.19. We consider the scaled d -dimensional integer lattice $(\tau\mathbb{Z})^d$ with scaling parameter $\tau > 0$. So V is the set of all points (x_1, \dots, x_d) where every x_i

is an integral multiple of τ . Let us assume that all the weights are equal to 1 and that $\mu(x) = \tau^2$ for all $x \in V$. That is, we have

$$\Delta u(x) = \frac{1}{\tau^2} \sum_{i=1}^d (u(x + \tau e_i) - 2u(x) + u(x - \tau e_i)),$$

where e_i denotes the i th unit vector. The graph is $2d$ -regular and Ricci-flat. By Theorem 3.17, the condition $CD(F_\tau; 0)$ holds with

$$F_\tau(a) = \frac{2d}{\tau^4} \exp\left(-\frac{\tau^2}{2d}a\right) \left[\Upsilon\left(\frac{\tau^2}{d}a\right) + (2d - 1)\Upsilon\left(-\frac{\tau^2}{d(2d - 1)}a\right) \right]. \tag{3.22}$$

Since $\Upsilon(y) \sim \frac{1}{2}y^2$ as $y \rightarrow 0$, we obtain that as $a \rightarrow 0+$

$$F_\tau(a) \sim \frac{2d}{\tau^4} \left(\frac{1}{2} \frac{\tau^4}{d^2} a^2 + \frac{2d - 1}{2} \frac{\tau^4}{d^2(2d - 1)^2} a^2 \right) = \frac{2d}{d(2d - 1)} a^2.$$

In the same way we see that for fixed $a \geq 0$

$$F_\tau(a) \rightarrow \frac{2d}{d(2d - 1)} a^2 \quad \text{as } \tau \rightarrow 0+.$$

In particular, we obtain for $d = 1$ that $F_\tau(a)$ tends as $\tau \rightarrow 0+$ to the quadratic function $2a^2$, which appears in the classical (continuous) case in one dimension!

Theorem 3.20. *Let $G = (V, E)$ be a D -regular unweighted Ricci-flat graph with $D \geq 2$. Assume that $\mu(y) = \mu_0 > 0$ for all $y \in V$. Then for any $\alpha \in (0, 1)$, $CD_\alpha(F_\alpha; 0)$ holds with*

$$F_\alpha(a) = \frac{D}{\mu_0^2} \exp\left(-\frac{\mu_0(1 - \alpha)}{D}a\right) \left[\exp\left(\frac{2(1 - \alpha)\mu_0}{D}a\right) + \frac{1 - \alpha}{\alpha} \exp\left(-\frac{2\alpha\mu_0}{D}a\right) - \frac{1}{\alpha} \right]. \tag{3.23}$$

Proof. We first show that F_α is a strictly convex CD-function. Setting $\eta = \frac{\mu_0}{D}$ we can write

$$F_\alpha(a) = \frac{1}{\alpha\eta\mu_0} e^{-\eta(1-\alpha)a} [\alpha e^{2(1-\alpha)\eta a} + (1 - \alpha)e^{-2\alpha\eta a} - 1].$$

Scaling the argument by putting $\tilde{a} = 2\eta a$ and introducing the function \tilde{F} via $\tilde{F}(\tilde{a}) = F_\alpha(a)$ we obtain

$$\tilde{F}(\tilde{a}) = \frac{1}{\alpha\eta\mu_0} e^{-\frac{1-\alpha}{2}\tilde{a}} [\alpha e^{(1-\alpha)\tilde{a}} + (1 - \alpha)e^{-\alpha\tilde{a}} - 1].$$

It now follows from Proposition 3.4 and Proposition 3.2 that \tilde{F} , and thus also F_α , are strictly convex CD-functions. Proposition 3.4 also ensures that both functions are non-negative and strictly convex on all of \mathbb{R} .

In what follows, we use the same notation as in the proof of Theorem 3.17. Let $x \in V$ and suppose that $v \in \mathbb{R}^V$ is such that

$$\mathcal{L}_\alpha(v)(x) > 0 \quad \text{and} \quad \mathcal{L}_\alpha(v)(x) \geq \mathcal{L}_\alpha(v)(\eta_j(x)) \quad \text{for all } j = 1, \dots, D. \quad (3.24)$$

Recall that we defined

$$\mathcal{L}_\alpha(v)(x) = -\frac{1}{\alpha} \Psi_{\Upsilon'}(\alpha v)(x).$$

We have

$$\begin{aligned} \mathcal{C}_\alpha(v)(x) &= \frac{1}{\mu_0} \sum_{i=1}^D e^{\alpha(z_i-z)} [\Psi_{\Upsilon'}(v)(\eta_i(x)) - \Psi_{\Upsilon'}(v)(x)] \\ &= \frac{1}{\mu_0^2} \sum_{i=1}^D e^{\alpha(z_i-z)} \sum_{j=1}^D [\Upsilon'(z_{ij} - z_i) - \Upsilon'(z_j - z)] \\ &= \frac{1}{\mu_0^2} \sum_{i=1}^D e^{\alpha(z_i-z)} \sum_{j=1}^D [e^{z_{ij}-z_i} - e^{z_j-z}] \\ &= \frac{1}{\mu_0^2} \sum_{j=1}^D e^{z_j-z} \sum_{i=1}^D (e^{z_{ij}-z_j-(1-\alpha)(z_i-z)} - e^{\alpha(z_i-z)}). \end{aligned} \quad (3.25)$$

Let $i, j \in \{1, \dots, D\}$. Then, by Young's inequality, we have for $a = e^{z_{ij}-z_j} > 0$ and $b = e^{z_i-z} > 0$

$$a^\alpha = \left(\frac{a}{b^{1-\alpha}}\right)^\alpha b^{\alpha(1-\alpha)} \leq \alpha \frac{a}{b^{1-\alpha}} + (1-\alpha)b^\alpha,$$

and thus

$$\frac{a}{b^{1-\alpha}} \geq \frac{1}{\alpha} a^\alpha - \frac{1-\alpha}{\alpha} b^\alpha,$$

which gives

$$\begin{aligned} e^{z_{ij}-z_j-(1-\alpha)(z_i-z)} - e^{\alpha(z_i-z)} &\geq \frac{1}{\alpha} e^{\alpha(z_{ij}-z_j)} - \frac{1-\alpha}{\alpha} e^{\alpha(z_i-z)} - e^{\alpha(z_i-z)} \\ &= \frac{1}{\alpha} \Upsilon'(\alpha(z_{ij} - z_j)) - \frac{1}{\alpha} \Upsilon'(\alpha(z_i - z)). \end{aligned} \quad (3.26)$$

Using $z_{j^*j} = z$, inequality (3.26) for all $i \neq j^*$, Lemma 3.16 (i) as well as the local maximum property (3.24), we may now estimate as follows.

$$\begin{aligned}
& \sum_{i=1}^D \left(e^{z_{ij}-z_j-(1-\alpha)(z_i-z)} - e^{\alpha(z_i-z)} \right) \\
& \geq e^{z-z_j-(1-\alpha)(z_{j^*}-z)} - e^{\alpha(z_{j^*}-z)} \\
& \quad + \frac{1}{\alpha} \sum_{i=1, i \neq j^*}^D \left(\Upsilon'(\alpha(z_{ij}-z_j)) - \Upsilon'(\alpha(z_i-z)) \right) \\
& \geq e^{z-z_j-(1-\alpha)(z_{j^*}-z)} - e^{\alpha(z_{j^*}-z)} + \mu_0 [\mathcal{L}_\alpha(v)(x) - \mathcal{L}_\alpha(v)(\eta_j(x))] \\
& \quad - \frac{1}{\alpha} e^{\alpha(z-z_j)} + \frac{1}{\alpha} e^{\alpha(z_{j^*}-z)} \\
& \geq e^{z-z_j-(1-\alpha)(z_{j^*}-z)} - \frac{1}{\alpha} e^{\alpha(z-z_j)} + \frac{1-\alpha}{\alpha} e^{\alpha(z_{j^*}-z)}.
\end{aligned}$$

Combining the last inequality and (3.25) yields

$$\begin{aligned}
\mathcal{C}_\alpha(v)(x) & \geq \frac{1}{\mu_0^2} \sum_{j=1}^D e^{z_j-z} \left(e^{z-z_j-(1-\alpha)(z_{j^*}-z)} - \frac{1}{\alpha} e^{\alpha(z-z_j)} + \frac{1-\alpha}{\alpha} e^{\alpha(z_{j^*}-z)} \right) \\
& = \frac{1}{\mu_0^2} \sum_{j=1}^D e^{(1-\alpha)(z_j-z)} \left(e^{(1-\alpha)(2z-z_j-z_{j^*})} + \frac{1-\alpha}{\alpha} e^{\alpha(z_j-2z+z_{j^*})} - \frac{1}{\alpha} \right) \\
& = \frac{1}{\mu_0^2} \sum_{j=1}^D e^{(1-\alpha)(z_j-z)} \left(e^{2(1-\alpha)w_j} + \frac{1-\alpha}{\alpha} e^{-2\alpha w_j} - \frac{1}{\alpha} \right).
\end{aligned}$$

Assuming without restriction of generality that $z_1 \leq z_2 \leq \dots \leq z_D$, we can argue as in the proof of Theorem 3.17 invoking the rearrangement inequality (3.19). Thereby we obtain that

$$\mathcal{C}_\alpha(v)(x) \geq \frac{1}{\mu_0^2} \sum_{j=1}^D e^{(1-\alpha)(z_j-z)} \left(e^{2(1-\alpha)\tilde{w}_j} + \frac{1-\alpha}{\alpha} e^{-2\alpha\tilde{w}_j} - \frac{1}{\alpha} \right). \quad (3.27)$$

Note that the term inside the brackets in (3.27) is non-negative, since F_α is non-negative on \mathbb{R} . So we can symmetrize the exponential factor in front of it and then

use the convexity of the exponential function and F_α on \mathbb{R} to get that

$$\begin{aligned} & C_\alpha(v)(x) \\ & \geq \frac{1}{2\mu_0^2} \sum_{j=1}^D [e^{(1-\alpha)(z_j-z)} + e^{(1-\alpha)(z_{j'}-z)}] \left(e^{2(1-\alpha)\tilde{w}_j} + \frac{1-\alpha}{\alpha} e^{-2\alpha\tilde{w}_j} - \frac{1}{\alpha} \right) \\ & \geq \frac{1}{\mu_0^2} \sum_{j=1}^D e^{-(1-\alpha)\tilde{w}_j} \left(e^{2(1-\alpha)\tilde{w}_j} + \frac{1-\alpha}{\alpha} e^{-2\alpha\tilde{w}_j} - \frac{1}{\alpha} \right) \\ & = \frac{1}{D} \sum_{j=1}^D F_\alpha \left(\frac{D\tilde{w}_j}{\mu_0} \right) \geq F_\alpha \left(\sum_{j=1}^D \frac{\tilde{w}_j}{\mu_0} \right) = F_\alpha(Lv(x)). \end{aligned} \quad \square$$

3.4. Examples of graphs that do not satisfy the condition

In this section we provide examples of graphs for which the condition $CD(F;0)$ does not hold. For the graphs under consideration, we construct a family of functions v such that $C_0(v) = \Delta\Psi_{\Gamma'}(v)$ becomes arbitrarily negative at a point x_* . Thus, there cannot be a CD-function F with $C_0(v)(x_*) \geq F(Lv(x_*))$.

Example 3.21. We consider the unweighted graph $G = (V, E)$ with $V = \{x_*, x_1, x_2, x_3\}$, $E = \{x_*x_j : j = 1, 2, 3\}$ with $\mu \equiv 1$ on V . Let $v \in \mathbb{R}^V$ and set $z_j = v(x_j)$ for $j \in \{*, 1, 2, 3\}$.

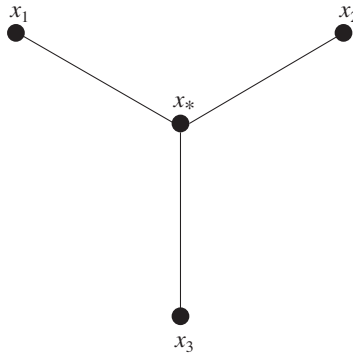


Figure 3.4. A star-like graph.

At the vertex x_* , we have

$$\begin{aligned} \Delta\Psi_{\Gamma'}(v)(x_*) &= \Psi_{\Gamma'}(v)(x_1) + \Psi_{\Gamma'}(v)(x_2) + \Psi_{\Gamma'}(v)(x_3) - 3\Psi_{\Gamma'}(v)(x_*) \\ &= \Upsilon'(z_* - z_1) + \Upsilon'(z_* - z_2) + \Upsilon'(z_* - z_3) \\ &\quad - 3[\Upsilon'(z_1 - z_*) + \Upsilon'(z_2 - z_*) + \Upsilon'(z_3 - z_*)] \\ &= e^{z_*-z_1} + e^{z_*-z_2} + e^{z_*-z_3} - 3 - 3[e^{z_1-z_*} + e^{z_2-z_*} + e^{z_3-z_*} - 3] \\ &= e^{a_1} + e^{a_2} + e^{a_3} - 3 - 3[e^{-a_1} + e^{-a_2} + e^{-a_3} - 3], \end{aligned}$$

where we set $a_j = z_* - z_j$. We choose v such that $z_* = 0$, $-z_1 = a_1 = -t$ and $-z_j = a_j = t$ for $j = 2, 3$, where $t > 0$ is a parameter. Then

$$\begin{aligned} Lv(x_*) &= a_1 + a_2 + a_3 = t > 0, \\ Lv(x_1) &= -a_1 = t, \\ Lv(x_j) &= -a_j = -t, \quad j = 2, 3. \end{aligned}$$

So we see that Lv has a positive maximum at x_* . On the other hand, inserting the values of a_j , gives

$$\Delta\Psi_{\Upsilon'}(v)(x_*) = e^{-t} + 2e^t + 6 - 3e^t - 6e^{-t} = 6 - e^t - 5e^{-t},$$

which shows that

$$\Delta\Psi_{\Upsilon'}(v)(x_*) \rightarrow -\infty \quad \text{as } t \rightarrow \infty.$$

Example 3.22. We consider the graph from the previous example and add two edges at each of three ends so that the resulting graph becomes a tree. More precisely, we have

$$\begin{aligned} V &= \{x_*, x_1, x_2, x_3, x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}\}, \\ E &= \{x_*x_j : j = 1, 2, 3\} \cup \{x_jx_{jk} : j = 1, 2, 3, k = 1, 2\}. \end{aligned}$$

We consider the case without weights and with $\mu \equiv 1$ on V . Let $v \in \mathbb{R}^V$ and set $z_j = v(x_j)$ for $j \in \{*, 1, 2, 3\}$ and $z_{jk} = v(x_{jk})$ for $j \in \{1, 2, 3\}$ and $k \in \{1, 2\}$. As before, we put $a_j = z_* - z_j$ for $j = 1, 2, 3$ and choose v such that $z_* = 0$, $-z_1 = a_1 = -t$, $-z_j = a_j = t$ for $j = 2, 3$, with $t > 0$. As to the new vertices, we put $z_{jk} := z_j$ for all $j = 1, 2, 3$ and $k = 1, 2$.

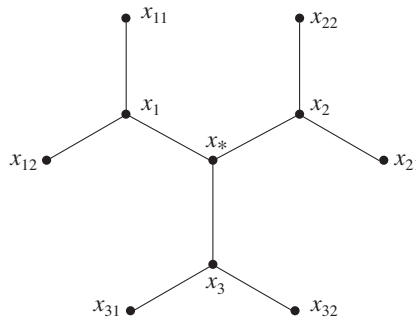


Figure 3.5. Part of a hexagonal tiling.

At the vertex x_* , we now obtain the same expression as above, since $\Upsilon'(0) = 0$.

Indeed,

$$\begin{aligned}
 \Delta \Psi_{\Upsilon'}(v)(x_*) &= \Psi_{\Upsilon'}(v)(x_1) + \Psi_{\Upsilon'}(v)(x_2) + \Psi_{\Upsilon'}(v)(x_3) - 3\Psi_{\Upsilon'}(v)(x_*) \\
 &= \Upsilon'(z_* - z_1) + \Upsilon'(z_{11} - z_1) + \Upsilon'(z_{12} - z_1) \\
 &\quad + \Upsilon'(z_* - z_2) + \Upsilon'(z_{21} - z_2) + \Upsilon'(z_{22} - z_2) \\
 &\quad + \Upsilon'(z_* - z_3) + \Upsilon'(z_{31} - z_3) + \Upsilon'(z_{32} - z_3) \\
 &\quad - 3[\Upsilon'(z_1 - z_*) + \Upsilon'(z_2 - z_*) + \Upsilon'(z_3 - z_*)] \\
 &= \Upsilon'(z_* - z_1) + \Upsilon'(z_* - z_2) + \Upsilon'(z_* - z_3) \\
 &\quad - 3[\Upsilon'(z_1 - z_*) + \Upsilon'(z_2 - z_*) + \Upsilon'(z_3 - z_*)] \\
 &= 6 - e^t - 5e^{-t}.
 \end{aligned}$$

The values of Lv on the set $\{x_*, x_1, x_2, x_3\}$ remain unchanged, since

$$Lv(x_j) = 3z_j - z_* - z_{j1} - z_{j2} = z_j - z_* = -a_j, \quad j = 1, 2, 3.$$

Thus Lv has a positive local maximum at x_* and as before $\Delta \Psi_{\Upsilon'}(v)(x_*) \rightarrow -\infty$ as the parameter $t \rightarrow \infty$.

Example 3.23. Let us consider the graph, which is given by a hexagonal tiling of the plane. The graph shown in Figure 3.5 obviously is a subgraph of this tiling. It follows from the previous example that there is no CD-function F for which $CD(F;0)$ is satisfied.

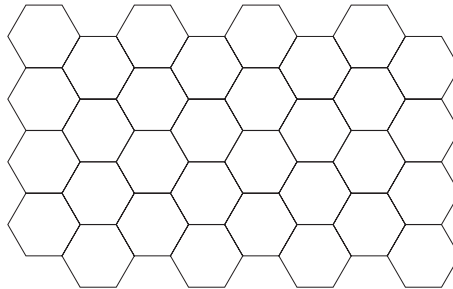


Figure 3.6. An hexagonal tiling.

4. Li-Yau inequalities on finite graphs

4.1. A new Li-Yau inequality

In this section we provide a proof of Theorem 1.1. Let $G = (V, E)$ be a finite connected graph. Suppose that $u : [0, \infty) \times V \rightarrow (0, \infty)$ is a solution of the heat equation on G , that is

$$\partial_t u - \Delta u = 0 \quad \text{in } [0, \infty) \times V. \tag{4.1}$$

Multiplying (4.1) by u^{-1} and using identity (2.7) we obtain for $v := \log u$ the equation

$$\partial_t v - \Delta v = \Psi_\Upsilon(v) \quad \text{in } (0, \infty) \times V. \tag{4.2}$$

Since

$$\Upsilon(z) + z = e^z - 1 = \Upsilon'(z),$$

we can rewrite equation (4.2) as

$$\partial_t v = \Psi_{\Upsilon'}(v) \quad \text{in } (0, \infty) \times V. \tag{4.3}$$

For the readers' convenience, let us repeat Theorem 1.1.

Theorem. *Assume that G satisfies $\text{CD}(F; 0)$ and let φ be the relaxation function associated with the CD-function F . Suppose that $u : [0, \infty) \times V \rightarrow (0, \infty)$ is a solution of the heat equation on G (equation (4.1)). Then*

$$-\Delta \log u \leq \varphi(t) \quad \text{in } (0, \infty) \times V, \tag{4.4}$$

and thus

$$\partial_t (\log u) \geq \Psi_\Upsilon(\log u) - \varphi(t) \quad \text{in } (0, \infty) \times V. \tag{4.5}$$

Proof. We define on $[0, \infty) \times V$ the function G by setting

$$G(t, x) = -\frac{1}{\varphi(t)} \Delta v(t, x) = \frac{1}{\varphi(t)} Lv(t, x), \quad t > 0, x \in V,$$

and $G(0, x) = 0, x \in V$. Observe that G is continuous in time, since $\varphi(t) \rightarrow \infty$ as $t \rightarrow 0+$.

Let $t_1 > 0$ be arbitrarily fixed. Suppose that G (restricted to the set $[0, t_1] \times V$) assumes the global maximum at $(t_*, x_*) \in [0, t_1] \times V$ and that $G(t_*, x_*) > 0$. By definition of G it is clear that $t_* > 0$, and thus $(\partial_t G)(t_*, x_*) \geq 0$.

Now, equation (4.3) implies that

$$\partial_t \Delta v = \Delta \Psi_{\Upsilon'}(v) \quad \text{in } (0, \infty) \times V,$$

which in turn entails that

$$\partial_t G = -\varphi(t)^{-1} \Delta \Psi_{\Upsilon'}(v) - \dot{\varphi}(t) \varphi(t)^{-1} G.$$

It follows that at the maximum point (t_*, x_*) we have that

$$0 \leq -\varphi(t)^{-1} \Delta \Psi_{\Upsilon'}(v) - \dot{\varphi}(t) \varphi(t)^{-1} G,$$

which is equivalent to

$$\Delta \Psi_{\Upsilon'}(v) \leq -\dot{\varphi}(t) \varphi(t)^{-1} Lv \tag{4.6}$$

at (t_*, x_*) . Since $Lv(t_*, x_*) > 0$ is the global maximum of $Lv(t_*, x)$ over $x \in V$, we may apply condition $CD(F;0)$, which gives

$$F(Lv(t_*, x_*)) \leq (\Delta \Psi_{\Gamma'}(v))(t_*, x_*). \tag{4.7}$$

Setting $a = Lv(t_*, x_*) (> 0)$, we infer from (4.6) (at (t_*, x_*)) and (4.7) that

$$\frac{F(a)}{a} \leq \frac{-\dot{\varphi}(t_*)}{\varphi(t_*)}.$$

Since φ satisfies the differential equation

$$-\dot{\varphi}(t) = F(\varphi(t)), \quad t > 0,$$

it follows that

$$\frac{F(a)}{a} \leq \frac{-\dot{\varphi}(t_*)}{\varphi(t_*)} = \frac{F(\varphi(t_*))}{\varphi(t_*)},$$

and thus, by strict monotonicity of $F(x)/x$,

$$Lv(t_*, x_*) = a \leq \varphi(t_*).$$

This in turn gives

$$G(t_*, x_*) \leq 1.$$

Since (t_*, x_*) was a global maximum point of G restricted to the set $[0, t_1] \times V$ with $t_1 > 0$ arbitrarily chosen, we obtain

$$G(t_1, x) \leq G(t_*, x_*) \leq 1, \quad t_1 \in (0, \infty), x \in V.$$

This shows inequality (4.4), which together with (4.2) implies the inequality (4.5). □

Example 4.1. We consider the most simple case, *i.e.*, the two point graph from Example 3.11. So $V = \{x_1, x_2\}$ and $\Delta u(x) = u(\tilde{x}) - u(x)$, where \tilde{x} denotes the only neighbor of $x \in V$. Let $u : [0, \infty) \times V \rightarrow (0, \infty)$ be a solution of the heat equation on the graph. Then Theorem 1.1 yields the estimate $-\Delta \log u \leq \varphi(t)$ where φ is given by

$$\varphi(t) = \log \left(\frac{1 + e^{-2t}}{1 - e^{-2t}} \right) = -\log(\tanh t), \quad t > 0,$$

cf. Example 3.11. Let us show that this estimate is optimal. Setting $u_i(t) = u(t, x_i)$ for $i = 1, 2$, the functions u_1, u_2 solve the ODE system

$$\dot{u}_1 = u_2 - u_1, \quad \dot{u}_2 = u_1 - u_2, \quad t \geq 0.$$

Adding the initial conditions $u_i(0) = u_i^0 > 0, i = 1, 2$, the solution is given by

$$\begin{aligned} u_1(t) &= \frac{1}{2}(u_1^0 - u_2^0)e^{-2t} + \frac{1}{2}(u_1^0 + u_2^0), \\ u_2(t) &= \frac{1}{2}(u_2^0 - u_1^0)e^{-2t} + \frac{1}{2}(u_1^0 + u_2^0). \end{aligned}$$

By symmetry, we may assume without loss of generality that $u_1^0 \geq u_2^0$. This implies $u_1(t) \geq u_2(t)$ for all $t \geq 0$, and thus

$$-\Delta \log u(t, x_2) = \log \left(\frac{u_2(t)}{u_1(t)} \right) \leq 0 \leq \log \left(\frac{u_1(t)}{u_2(t)} \right) = -\Delta \log u(t, x_1) =: w(t).$$

So we have to examine the function $w(t)$. Setting $\alpha = u_1^0/(u_1^0 + u_2^0) > 0$ and $\beta = u_2^0/(u_1^0 + u_2^0) > 0$ we have $\alpha + \beta = 1$, and for $t > 0$ there holds

$$\begin{aligned} w(t) &= \log \left(\frac{(\alpha - \beta)e^{-2t} + 1}{(\beta - \alpha)e^{-2t} + 1} \right) \\ &= \log \left(\frac{(2\alpha - 1)e^{-2t} + 1}{(1 - 2\alpha)e^{-2t} + 1} \right) \leq \log \left(\frac{1 + e^{-2t}}{1 - e^{-2t}} \right) = \varphi(t), \end{aligned}$$

here the upper estimate corresponds to the limiting and extreme case where $\alpha = 1$ and $\beta = 0$. We can be arbitrarily close to this case by choosing u_1^0 and u_2^0 such that u_1^0/u_2^0 is sufficiently large, which shows that $-\Delta \log u \leq \varphi(t)$ is a sharp estimate.

Note that as $t \rightarrow \infty$,

$$\varphi(t) = \log \left(1 + \frac{2e^{-2t}}{1 - e^{-2t}} \right) \sim \frac{2e^{-2t}}{1 - e^{-2t}} \sim 2e^{-2t}.$$

As $t \rightarrow 0$, we have

$$\varphi(t) = -\log \left(\frac{1 - e^{-2t}}{1 + e^{-2t}} \right) \sim -\log(t).$$

4.2. α -calculus and comparison with the square root approach

Let $G = (V, E)$ be a finite connected graph. Suppose that $u : [0, \infty) \times V \rightarrow (0, \infty)$ is a solution of the heat equation on G . We have shown that $v = \log u$ satisfies the equation

$$\partial_t v - \Delta v = \Psi_\Upsilon(v) \quad \text{in } (0, \infty) \times V. \tag{4.8}$$

Letting $\alpha \in (0, 1)$, (4.8) implies that

$$\partial_t v - \Delta v - \frac{1}{\alpha} \Psi_{\Gamma_\alpha}(v) = \Psi_\Gamma(v) - \frac{1}{\alpha} \Psi_{\Gamma_\alpha}(v) \quad \text{in } (0, \infty) \times V.$$

In view of the identity (2.8) we have

$$\frac{\Delta(u^\alpha)}{\alpha u^\alpha} = \Delta v + \frac{1}{\alpha} \Psi_{\Gamma_\alpha}(v) = \frac{1}{\alpha} \Psi_{\Gamma'}(\alpha v),$$

and thus

$$\partial_t v - \frac{\Delta(u^\alpha)}{\alpha u^\alpha} = \Psi_\Gamma(v) - \frac{1}{\alpha} \Psi_{\Gamma_\alpha}(v) \quad \text{in } (0, \infty) \times V. \tag{4.9}$$

Note that in case $\alpha = \frac{1}{2}$, identity (2.11) shows that equation (4.9) then takes the form

$$\partial_t v - 2 \frac{\Delta(\sqrt{u})}{\sqrt{u}} = 2 \frac{\Gamma(\sqrt{u})}{u} \quad \text{in } (0, \infty) \times V.$$

We are interested in an estimate of the form

$$-\frac{\Delta(u^\alpha)}{\alpha u^\alpha} = -\frac{1}{\alpha} \Psi_{\Gamma'}(\alpha v) = \mathcal{L}_\alpha(v) \leq \eta(t) \quad \text{in } (0, \infty) \times V,$$

for some appropriate positive function η . To achieve this, we first derive an equation for the temporal derivative of the quantity to be estimated. Using the equation for v we obtain

$$\begin{aligned} \partial_t \left(\frac{1}{\alpha} \Psi_{\Gamma'}(\alpha v) \right)(t, x) &= \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} e^{\alpha(v(t,y) - v(t,x))} (\partial_t v(t, y) - \partial_t v(t, x)) \\ &= \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} e^{\alpha(v(t,y) - v(t,x))} (\Psi_{\Gamma'}(v)(t, y) - \Psi_{\Gamma'}(v)(t, x)) \\ &= \mathcal{C}_\alpha(v)(t, x), \end{aligned}$$

cf. (3.8).

Theorem 4.2. *Let $\alpha \in (0, 1)$, $G = (V, E)$ be a finite connected graph which satisfies $CD_\alpha(F; 0)$ and let φ be the relaxation function corresponding to the CD-function F . Suppose that $u : [0, \infty) \times V \rightarrow (0, \infty)$ is a solution of the heat equation on G (equation (4.1)). Then*

$$-\frac{\Delta(u^\alpha)}{\alpha u^\alpha} = -\frac{1}{\alpha} \Psi_{\Gamma'}(\alpha \log u) = \mathcal{L}_\alpha(\log u) \leq \varphi(t) \quad \text{in } (0, \infty) \times V, \tag{4.10}$$

and consequently

$$\partial_t(\log u) \geq \Psi_\Gamma(\log u) - \frac{1}{\alpha} \Psi_{\Gamma_\alpha}(\log u) - \varphi(t) \quad \text{in } (0, \infty) \times V. \tag{4.11}$$

Proof. The proof is almost entirely analogous to the one of Theorem [theo:finite-graphs](#), the only difference being that in addition one uses the inequality

$$F(\mathcal{L}_\alpha(v)(t_*, x_*)) \leq F(Lv(t_*, x_*)),$$

which holds, since F is strictly increasing and

$$\mathcal{L}_\alpha(v)(t_*, x_*) = Lv(t_*, x_*) - \frac{1}{\alpha} \Psi_\Upsilon(\alpha v)(t_*, x_*) \leq Lv(t_*, x_*). \quad \square$$

5. Local Li-Yau inequalities and estimates on infinite graphs

The approach of Section 4 to Li-Yau type estimates is restricted to finite graphs. Proofs of similar results in the case of infinite graphs are more involved. They additionally require the use of cut-off functions. The same difficulty arises when one aims at local Li-Yau inequalities for positive functions that solve the heat equation only on a part of the graph, *e.g.*, in a ball. It turns out that the α -calculus from Subsection 4.2 is sufficiently robust to obtain the desired estimates. Throughout this section we confine ourselves to Ricci-flat graphs, which are introduced in Subsection 3.3.

5.1. General Ricci-flat graphs

Throughout this subsection we assume that

$$\begin{aligned} G = (V, E) \text{ is a } D\text{-regular unweighted Ricci-flat graph with} \\ D \geq 2 \text{ and } \mu(y) = \mu_0 > 0 \text{ for all } y \in V. \end{aligned} \tag{R}$$

We first need a slight generalization of the $CD_\alpha(F_\alpha; 0)$ property provided by Theorem 3.20.

Corollary 5.1. *Assume (R) and let $\alpha \in (0, 1)$. Let $x_* \in V$ and $\psi \in \mathbb{R}^V$ such that $\psi(x_*) > 0$ and $\psi(y) > 0$ for all $y \sim x_*$. Let $v \in \mathbb{R}^V$ and suppose that the function*

$$M(x) := \psi(x)\mathcal{L}_\alpha(v)(x), \quad x \in V,$$

has a positive local maximum at x_ , that is,*

$$M(x_*) > 0 \quad \text{and} \quad M(x_*) \geq M(y) \quad \text{for all } y \sim x_*.$$

Then the following inequality holds true:

$$C_\alpha(v)(x_*) \geq F_\alpha(Lv(x_*)) - \frac{1}{\mu_0} \mathcal{L}_\alpha(v)(x_*) \sum_{y \sim x_*} e^{v(y)-v(x_*)} \frac{|\psi(x_*) - \psi(y)|}{\psi(y)}, \tag{5.1}$$

where F_α is the function given by (3.23).

Proof. We follow the lines of the proof of Theorem 3.20 with x replaced by x_* . When estimating the inner sum we now have by the local maximum property of M

$$\begin{aligned} & \mathcal{L}_\alpha(v)(x_*) - \mathcal{L}_\alpha(v)(\eta_j(x_*)) \\ &= \frac{1}{\psi(x_*)} (M(x_*) - M(\eta_j(x_*))) + \frac{\psi(\eta_j(x_*)) - \psi(x_*)}{\psi(\eta_j(x_*))\psi(x_*)} M(\eta_j(x_*)) \\ &\geq -\frac{|\psi(\eta_j(x_*)) - \psi(x_*)|}{\psi(\eta_j(x_*))\psi(x_*)} M(x_*) = -\mathcal{L}_\alpha(v)(x_*) \frac{|\psi(x_*) - \psi(\eta_j(x_*))|}{\psi(\eta_j(x_*))}. \end{aligned}$$

Arguing as in the proof of Theorem 3.20, the last term leads to the second term on the right of inequality (5.1). □

Lemma 5.2. *Assume (R). Let $\alpha \in (0, 1)$, $x_* \in V$ and $u : V \rightarrow (0, \infty)$ such that $\Delta(u^\alpha)(x_*) < 0$. Then one has*

$$\sum_{y \sim x_*} u(y) \leq D^{1/\alpha} u(x_*). \tag{5.2}$$

Proof. For positive numbers $a_j, j = 1, \dots, D$ we have the inequality

$$\left(\sum_{j=1}^D a_j \right)^\alpha \leq \sum_{j=1}^D a_j^\alpha,$$

and thus

$$\begin{aligned} \sum_{y \sim x_*} \frac{u(y)}{u(x_*)} &\leq \left(\sum_{y \sim x_*} \frac{u(y)^\alpha}{u(x_*)^\alpha} \right)^{1/\alpha} \\ &= \left(\frac{1}{u(x_*)^\alpha} \sum_{y \sim x_*} (u(y)^\alpha - u(x_*)^\alpha) + D \right)^{1/\alpha} \leq D^{1/\alpha}. \quad \square \end{aligned}$$

The following lemma is a very useful auxiliary result when proving estimates involving CD-functions.

Lemma 5.3. *Let $g : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing C^1 function with $g(0) = 0$. Assume that there exists $a_* > 0$ such that $c_0 := \min_{[0, a_*]} g' > 0$ and $g|_{[a_*, \infty)}$ is convex. Let $c_1 = \max_{[0, a_*]} g'$ and set $\gamma = c_1/c_0$. Then there holds*

$$g(x) + g(y) \leq g(x + \gamma y), \quad x, y \in [0, \infty). \tag{5.3}$$

Proof. Let $x, y \in [0, \infty)$. It suffices to show (5.3) in the case $x \geq y$. In fact, if $x < y$, we have (since $\gamma \geq 1$)

$$y + \gamma x \leq x + \gamma y,$$

and thus $g(y + \gamma x) \leq g(x + \gamma y)$.

So let us assume that $x \geq y$. We consider two cases. Suppose first that $y \leq a_*$. Set $\xi = x + \gamma y$. Then

$$g(x) + g(y) \leq g(x) + c_1 y = g(x) + c_0(\xi - x) \leq g(\xi),$$

since $g' \geq c_0$ in $[0, \infty)$. So the desired inequality holds.

Now suppose that $a_* < y (\leq x)$. By convexity of g in $[a_*, \infty)$ and since $g(a_*) \leq c_1 a_*$, we have

$$\begin{aligned} g(y) &\leq g(a_*) + g'(y)(y - a_*) \leq c_1 a_* + g'(y)(y - a_*) \\ &= \gamma g'(y)y + c_1 a_* - g'(y)((\gamma - 1)y + a_*) \\ &\leq \gamma g'(y)y + c_1 a_* - c_0((\gamma - 1)y + a_*) \\ &= \gamma g'(y)y - (c_1 - c_0)(y - a_*) \leq \gamma g'(y)y. \end{aligned}$$

Using this inequality and the convexity of g in $[a_*, \infty)$, it follows that

$$\begin{aligned} g(x) + g(y) &\leq g(x) + \gamma g'(y)y \leq g(x) + \gamma g'(x)y \\ &= g(x) + ((x + \gamma y) - x)g'(x) \leq g(x + \gamma y). \end{aligned}$$

This proves the lemma. □

The main result in this subsection is the following.

Theorem 5.4. *Assume (R). Let $x_0 \in V$ and $r \in \mathbb{N}$. Let the function $u : [0, \infty) \times V \rightarrow (0, \infty)$ be a solution of the heat equation on the ball $\bar{B}_{2r}(x_0) = \{y \in V : d(y, x_0) \leq 2r\}$, that is,*

$$\partial_t u(t, x) - \Delta u(t, x) = 0, \quad t \geq 0, x \in \bar{B}_{2r}(x_0).$$

Then for any $\alpha \in (0, 1)$ there exists a constant $C = C(\alpha, \mu_0, D) > 0$ such that

$$\partial_t(\log u) \geq \Psi_\gamma(\log u) - \frac{1}{\alpha} \Psi_{\gamma_\alpha}(\log u) - \varphi_\alpha(t) - \frac{C}{r} \quad \text{in } (0, \infty) \times \bar{B}_r(x_0), \quad (5.4)$$

where φ_α is the relaxation function corresponding to the CD-function F_α given in (3.23).

Proof. Setting $v = \log u$ we know from Subsection 4.2 that

$$\partial_t v + \mathcal{L}_\alpha(v) = \Psi_\gamma(v) - \frac{1}{\alpha} \Psi_{\gamma_\alpha}(v) \quad \text{in } (0, \infty) \times \bar{B}_{2r}(x_0), \quad (5.5)$$

and

$$\partial_t \mathcal{L}_\alpha(v) = -\mathcal{C}_\alpha(v) \quad \text{in } (0, \infty) \times \bar{B}_{2r}(x_0). \quad (5.6)$$

We define a cut-off function $\psi : V \rightarrow [0, \infty)$ by

$$\psi(x) = \begin{cases} 0 & \text{for } 2r < d(x, x_0) \\ \frac{2r - d(x, x_0)}{r} & \text{for } r \leq d(x, x_0) \leq 2r \\ 1 & \text{for } d(x, x_0) < r. \end{cases} \tag{5.7}$$

Let $\alpha \in (0, 1)$ be fixed and set $\varphi(t) = \varphi_\alpha(t)$. We consider the quantities G and \tilde{G} defined on $[0, \infty) \times V$ by

$$G(t, x) = \frac{\mathcal{L}_\alpha(v)(t, x)}{\varphi(t)} \quad \text{and} \quad \tilde{G}(t, x) = \psi(x)G(t, x), \quad t > 0, x \in V,$$

and $G(0, x) = \tilde{G}(0, x) = 0, x \in V$. Note that G and \tilde{G} are continuous in time, since $\varphi(t) \rightarrow \infty$ as $t \rightarrow 0+$.

Multiplying (5.6) by $\psi(x)\varphi(t)^{-1}$ we obtain

$$\partial_t \tilde{G}(t, x) = -\dot{\varphi}(t)^{-1} \psi(x) \mathcal{C}_\alpha(v)(t, x) - \dot{\varphi}(t) \varphi(t)^{-1} \tilde{G}(t, x) \quad \text{in } (0, \infty) \times \bar{B}_{2r}(x_0). \tag{5.8}$$

Let $t_1 > 0$ be arbitrarily fixed. Suppose that \tilde{G} (restricted to the set $[0, t_1] \times \bar{B}_{2r}(x_0)$) assumes the global maximum at $(t_*, x_*) \in [0, t_1] \times \bar{B}_{2r}(x_0)$ and that $\tilde{G}(t_*, x_*) > 0$. By definition of \tilde{G} it is clear that $t_* > 0$ and $x_* \in B_{2r}(x_0) = \{y \in V : d(y, x_0) < 2r\}$. In particular, we have $(\partial_t \tilde{G})(t_*, x_*) \geq 0$.

We now distinguish three cases.

Case 1: $x_* \in B_r(x_0)$. Then $\psi(x_*) = 1$ and also $\psi(y) = 1$ for all $y \sim x_*$. Thus $G(t_*, x_*) = \tilde{G}(t_*, x_*) \geq \tilde{G}(t, y) = G(t, y)$ for all $t \in [0, t_1]$ and all $y \sim x_*$. In particular, $\mathcal{L}_\alpha(v)(t_*, x_*) \geq \mathcal{L}_\alpha(v)(t_*, y)$ for all $y \sim x_*$. So we can apply $\text{CD}_\alpha(F_\alpha; 0)$ from Theorem 3.20 (with local maximum of $\mathcal{L}_\alpha(v)$), thereby obtaining at the maximum point that

$$\psi(x_*)\varphi^{-1}(t_*)F_\alpha(\mathcal{L}_\alpha(v)(t_*, x_*)) \leq -\psi(x_*)\dot{\varphi}(t_*)\varphi^{-1}(t_*)G(t_*, x_*).$$

We can then argue as in the proof of Theorem 4.2 (see also Theorem 1.1) to find that $G(t_*, x_*) \leq 1$, which implies

$$\tilde{G}(t_1, x) \leq \tilde{G}(t_*, x_*) = G(t_*, x_*) \leq 1, \quad x \in \bar{B}_{2r}(x_0).$$

Case 2: $d(x_*, x_0) = 2r - 1$. In this case $\psi(x_*) = \frac{1}{r}$. Here we estimate \tilde{G} directly without using (5.8). We have for arbitrary $t > 0$ and $x \in V$

$$\mathcal{L}_\alpha(v)(t, x) = -\frac{\Delta(u^\alpha)(t, x)}{\alpha u(t, x)^\alpha} = \frac{1}{\alpha\mu_0} \sum_{y \sim x} \left(1 - \frac{u(t, y)^\alpha}{u(t, x)^\alpha}\right) \leq \frac{D}{\alpha\mu_0},$$

by positivity of u . Hence

$$\begin{aligned} \tilde{G}(t_1, x) &\leq \tilde{G}(t_*, x_*) = \frac{\psi(x_*)\mathcal{L}_\alpha(v)(t_*, x_*)}{\varphi(t_*)} \\ &\leq \frac{D}{\alpha\mu_0r\varphi(t_*)} \leq \frac{D}{\alpha\mu_0r\varphi(t_1)}, \quad x \in \bar{B}_{2r}(x_0), \end{aligned}$$

since φ is non-increasing.

Case 3: $\psi(x_*) = s/r$ with $s \in \{2, \dots, r\}$. Here $\psi(y) > 0$ for all $y \sim x_*$, and thus we may apply Theorem 5.1 at the time level t_* . From (5.8) we then obtain at the maximum point (t_*, x_*)

$$\begin{aligned} &F_\alpha(Lv(t_*, x_*)) \\ &\leq \mathcal{L}_\alpha(v)(t_*, x_*) \left(\frac{1}{\mu_0} \sum_{y \sim x_*} e^{v(t_*, y) - v(t_*, x_*)} \frac{|\psi(x_*) - \psi(y)|}{\psi(y)} - \dot{\varphi}(t_*)\varphi(t_*)^{-1} \right). \end{aligned} \tag{5.9}$$

By definition of ψ , we have for all $y \sim x_*$

$$\frac{|\psi(x_*) - \psi(y)|}{\psi(y)} \leq \frac{1/r}{(s-1)/r} = \frac{1}{s-1}.$$

Further, we know that $\frac{\Delta(u^\alpha)(t_*, x_*)}{\alpha u(t_*, x_*)^\alpha} < 0$, which implies $\Delta(u^\alpha)(t_*, x_*) < 0$ as well, by positivity of u . Lemma 5.2 then yields

$$\sum_{y \sim x_*} u(t_*, y) \leq D^{1/\alpha} u(t_*, x_*),$$

and thus

$$\begin{aligned} \sum_{y \sim x_*} e^{v(t_*, y) - v(t_*, x_*)} \frac{|\psi(x_*) - \psi(y)|}{\psi(y)} &\leq \frac{1}{s-1} \sum_{y \sim x_*} e^{v(t_*, y) - v(t_*, x_*)} \\ &= \frac{1}{s-1} \sum_{y \sim x_*} \frac{u(t_*, y)}{u(t_*, x_*)} \leq \frac{D^{1/\alpha}}{s-1}. \end{aligned}$$

Using this estimate, together with the ODE for the relaxation function φ and

$$F_\alpha(\mathcal{L}_\alpha(v)(t_*, x_*)) \leq F_\alpha(Lv(t_*, x_*)),$$

it follows from (5.9) that

$$F_\alpha(\mathcal{L}_\alpha(v)(t_*, x_*)) \leq \mathcal{L}_\alpha(v)(t_*, x_*) \left(\frac{D^{1/\alpha}}{\mu_0(s-1)} + \frac{F_\alpha(\varphi(t_*))}{\varphi(t_*)} \right).$$

We define the function $H : [0, \infty) \rightarrow [0, \infty)$ by $H(0) = 0$ and $H(x) = F_\alpha(x)/x$ for $x > 0$. We also put $\omega_s = \frac{D^{1/\alpha}}{\mu_0(s-1)}$. Suppressing the arguments, the last inequality is then equivalent to

$$H(\mathcal{L}_\alpha(v)) \leq H(\varphi) + \omega_s. \tag{5.10}$$

By Proposition 3.4, we may apply Lemma 5.3 to the function $g = H$. Let $\gamma = \gamma(\alpha, D, \mu_0) > 0$ be the corresponding constant. Then Lemma 5.3 gives

$$H(\varphi) + \omega_s = H(\varphi) + H(H^{-1}(\omega_s)) \leq H(\varphi + \gamma H^{-1}(\omega_s)),$$

which when combined with (5.10) yields

$$H(\mathcal{L}_\alpha(v)) \leq H(\varphi + \gamma H^{-1}(\omega_s)).$$

Since H is strictly increasing, we deduce that

$$\mathcal{L}_\alpha(v)(t_*, x_*) \leq \varphi(t_*) + \gamma H^{-1}(\omega_s),$$

that is

$$G(t_*, x_*) \leq 1 + \varphi(t_*)^{-1} \gamma H^{-1}(\omega_s).$$

We now find that

$$\begin{aligned} \tilde{G}(t_1, x) &\leq \tilde{G}(t_*, x_*) = \frac{s}{r} G(t_*, x_*) \\ &\leq \frac{s}{r} (1 + \varphi(t_*)^{-1} \gamma H^{-1}(\omega_s)) \\ &\leq 1 + \frac{\gamma C_s}{r} \varphi(t_1)^{-1}, \end{aligned}$$

where $C_s := sH^{-1}(\omega_s)$. It is now not difficult to check that there exists a number $M = M(\alpha, D, \mu_0) > 0$ such that $C_s \leq M$ for all $s \geq 2$ (recall that $H(x)$ behaves as a linear function as $x \rightarrow 0$). It follows that

$$\tilde{G}(t_1, x) \leq 1 + \frac{\gamma M}{r} \varphi(t_1)^{-1}, \quad x \in \bar{B}_{2r}(x_0).$$

Combining all three cases we see that for arbitrary $t_1 > 0$

$$\tilde{G}(t_1, x) \leq 1 + \max \left\{ \gamma M, \frac{D}{\alpha \mu_0} \right\} \frac{\varphi(t_1)^{-1}}{r}, \quad x \in \bar{B}_{2r}(x_0),$$

which implies that

$$\mathcal{L}_\alpha(v)(t, x) \leq \varphi(t) + \max \left\{ \gamma M, \frac{D}{\alpha \mu_0} \right\} \frac{1}{r}, \quad t > 0, x \in \bar{B}_r(x_0).$$

This together with (5.5) proves the theorem. □

We remark that

$$\varphi_\alpha(t) \sim \frac{D}{2(1-\alpha)t} \quad \text{as } t \rightarrow \infty.$$

This follows from Lemma 3.7 and the fact that $F_\alpha(a) \sim \frac{2(1-\alpha)}{D}a^2$ as $a \rightarrow 0+$. Using Lemma 3.7 we also see that

$$\varphi_\alpha(t) \sim -\frac{D}{\mu_0(1-\alpha)} \log t \quad \text{as } t \rightarrow 0+.$$

5.2. The example \mathbb{Z}

In the special case where the graph is given by the lattice \mathbb{Z} we can improve the estimate of Theorem 5.4 in two ways. On the one hand, we are able to treat the limit case $\alpha = 0$. On the other hand, we obtain an estimate with the relaxation function φ associated to the full CD-function F given by (3.16) with $D = 2$ and $\mu_0 = 1$. This is possible due to the special structure of the term $\Delta\Psi_{\Upsilon'}(v)$. Note that

$$\varphi(t) \sim \frac{1}{2t} \quad \text{as } t \rightarrow \infty,$$

and

$$\varphi(t) \sim -2 \log t \quad \text{as } t \rightarrow 0+,$$

by Lemma 3.7.

Lemma 5.5. *Let $G = (V, E)$ be the lattice \mathbb{Z} without weights and with $\mu \equiv 1$ on $V = \mathbb{Z}$. Let $\eta_j : \mathbb{Z} \rightarrow \mathbb{Z}$, $j = 1, 2$ be defined by $\eta_1(x) = x - 1$ and $\eta_2(x) = x + 1$. Then for any $v \in \mathbb{R}^V$ and $x \in \mathbb{Z}$ there holds*

$$\Delta\Psi_{\Upsilon'}(v)(x) \geq 2e^{-\frac{Lv(x)}{2}} \Upsilon(Lv(x)) + \Theta(v)(x),$$

where

$$\Theta(v)(x) = \sum_{j=1}^2 e^{v(\eta_j(x))-v(x)} \left(e^{-Lv(\eta_j(x))} - 1 + Lv(x) \right). \tag{5.11}$$

Moreover, if $Lv(x) \geq 1$, we have

$$\Theta(v)(x) \geq 2e^{-\frac{Lv(x)}{2}} (Lv(x) - 1).$$

Proof. We use the same notation as in the proof of Theorem 3.17. Following the first lines from there and observing that

$$2w_j = 2z - z_j - z_{j^*} = Lv(x), \quad j = 1, 2,$$

we see that

$$\begin{aligned} \Delta \Psi_{\Upsilon'}(v)(x) &= \sum_{j=1}^2 e^{z_j - z} (e^{2w_j} - 1 + e^{z_j j - 2z_j + z} - 1) \\ &= \sum_{j=1}^2 e^{z_j - z} (\Upsilon(Lv(x)) + e^{-Lv(\eta_j(x))} - 1 + Lv(x)) \\ &= \Upsilon(Lv(x)) \sum_{j=1}^2 e^{z_j - z} + \Theta(v)(x) \\ &\geq 2e^{-\frac{Lv(x)}{2}} \Upsilon(Lv(x)) + \Theta(v)(x), \end{aligned}$$

by convexity of the exponential function. The last assertion follows from the definition of $\Theta(v)$ and the same convexity inequality we used before. \square

The following simple fact will be needed in our argument.

Lemma 5.6. *Let $\eta > 1$ and $f : [0, \infty) \rightarrow \mathbb{R}$ be given by*

$$f(a) = e^{-\eta a} - 1 + a.$$

Then f has exactly two zeros: $a = 0$ and $a = a_(\eta) > 0$, where*

$$a_*(\eta) \sim 2(\eta - 1) \quad \text{as } \eta \rightarrow 1. \tag{5.12}$$

Proof. It is clear that $a_*(\eta) \rightarrow 0$ as $\eta \rightarrow 1$. Thus, as $\eta \rightarrow 1$ we have by expanding the exponential function around 0,

$$\begin{aligned} 0 &= e^{-\eta a_*(\eta)} - 1 + a_*(\eta) \\ &= \left(1 - \eta a_*(\eta) + \frac{1}{2} \eta^2 a_*(\eta)^2 \right) - 1 + a_*(\eta) + O([-\eta a_*(\eta)]^3) \\ &= a_*(\eta) \left(\frac{1}{2} \eta^2 a_*(\eta) - (\eta - 1) \right) + O([-a_*(\eta)]^3). \end{aligned}$$

This implies that the term inside the big brackets tends to 0 as $\eta \rightarrow 1$, which in turn entails (5.12). \square

The main result of this subsection reads as follows.

Theorem 5.7. *Let $G = (V, E)$ be the lattice \mathbb{Z} without weights and with $\mu \equiv 1$ on $V = \mathbb{Z}$. Let $x_0 \in \mathbb{Z}$ and $r \in \mathbb{N}$. Let the function $u : [0, \infty) \times \mathbb{Z} \rightarrow (0, \infty)$ be a solution of the heat equation on the ball $\bar{B}_{2r}(x_0)$. Then there exists a constant $C > 0$ such that*

$$\partial_t(\log u) \geq \Psi_{\Upsilon}(\log u) - \varphi(t) - \frac{C}{r} \quad \text{in } (0, \infty) \times \bar{B}_r(x_0), \tag{5.13}$$

where φ is the relaxation function corresponding to the CD-function F given in (3.16) with $D = 2$ and $\mu_0 = 1$.

Proof. Setting $v = \log u$ we know from Subsection 4.1 that

$$\partial_t v + Lv = \Psi_\Upsilon(v) \quad \text{in } (0, \infty) \times \bar{B}_{2r}(x_0), \tag{5.14}$$

and

$$\partial_t Lv = -\Delta \Psi_{\Upsilon'}(v) = -\mathcal{C}_0(v) \quad \text{in } (0, \infty) \times \bar{B}_{2r}(x_0). \tag{5.15}$$

Let ψ be the cut-off function from (5.7) and define the functions G and \tilde{G} on $[0, \infty) \times \mathbb{Z}$ by

$$G(t, x) = \frac{Lv(t, x)}{\varphi(t)} \quad \text{and} \quad \tilde{G}(t, x) = \psi(x)G(t, x), \quad t > 0, x \in \mathbb{Z},$$

and $G(0, x) = \tilde{G}(0, x) = 0, x \in \mathbb{Z}$. Multiplying (5.15) by $\psi(x)\varphi(t)^{-1}$ we obtain

$$\partial_t \tilde{G} = -\varphi^{-1}\psi\mathcal{C}_0(v) - \dot{\varphi}\varphi^{-1}\tilde{G} \quad \text{in } (0, \infty) \times \bar{B}_{2r}(x_0). \tag{5.16}$$

Let $t_1 > 0$ be arbitrarily fixed. Suppose that \tilde{G} (restricted to the set $[0, t_1] \times \bar{B}_{2r}(x_0)$) attains the global maximum at $(t_*, x_*) \in [0, t_1] \times \bar{B}_{2r}(x_0)$ and that $\tilde{G}(t_*, x_*) > 0$. Then $t_* > 0, x_* \in B_{2r}(x_0)$, and in particular we have $(\partial_t \tilde{G})(t_*, x_*) \geq 0$.

We now distinguish three cases.

Case 1: $x_* \in B_r(x_0)$. Then $\psi(x_*) = 1$ and $\psi(y) = 1$ for all $y \sim x_*$. Consequently, $G(t_*, x_*) = \tilde{G}(t_*, x_*) \geq \tilde{G}(t, y) = G(t, y)$ for all $t \in [0, t_1]$ and all $y \sim x_*$. In particular, $Lv(t_*, x_*) \geq Lv(t_*, y)$ for all $y \sim x_*$. By the CD-inequality from Theorem 3.17 we obtain at the maximum point that

$$\psi\varphi^{-1}F(Lv) \leq -\psi\dot{\varphi}\varphi^{-1}G.$$

We can then argue as in the proof of Theorem 3.17, thereby getting that $G(t_*, x_*) \leq 1$, which implies that

$$\tilde{G}(t_1, x) \leq \tilde{G}(t_*, x_*) = G(t_*, x_*) \leq 1, \quad x \in \bar{B}_{2r}(x_0).$$

Case 2: $d(x_0, x_*) = 2r - 1$, that is, $\psi(x_*) = \frac{1}{R}$. From Lemma 5.5 we know that

$$\mathcal{C}_0(v) \geq 2e^{-\frac{Lv}{2}}\Upsilon(Lv) + \Theta(v)$$

where $\Theta(v)$ is given by (5.11). Note that $\Theta(v) \geq 0$ if $Lv(x) \geq 1$. If this is the case we even have an estimate of the form

$$\Theta(v) \geq 2e^{-\frac{Lv}{2}}(Lv - 1). \tag{5.17}$$

Case 2a: Suppose that $Lv(t_*, x_*) \leq 1$. Then

$$\begin{aligned} \tilde{G}(t_1, x) &\leq \tilde{G}(t_*, x_*) = \psi(x_*)\varphi(t_*)^{-1}Lv(t_*, x_*) \\ &\leq \frac{1}{r}\varphi(t_*)^{-1} \leq \frac{1}{r}\varphi(t_1)^{-1}, \quad x \in \bar{B}_{2r}(x_0), \end{aligned}$$

since φ is decreasing.

Case 2b: Suppose now that $Lv(t_*, x_*) > 1$. Now we use (5.16). At the maximum point, we can bound $\Theta(v)$ from below by the bound given in (5.17), and so we obtain

$$\psi\varphi^{-1} \cdot 2e^{-\frac{Lv}{2}} \left(\Upsilon(Lv) + Lv - 1 \right) \leq -\psi\dot{\varphi}\varphi^{-1}G.$$

This implies at the maximum point

$$2e^{-\frac{Lv}{2}} \left(\Upsilon(Lv) + Lv - 1 \right) \leq -\dot{\varphi}\varphi^{-1}Lv,$$

which is equivalent to

$$F(Lv) = 2e^{-\frac{Lv}{2}} \left(\Upsilon(Lv) + \Upsilon(-Lv) \right) \leq -\dot{\varphi}\varphi^{-1}Lv + 2e^{-\frac{Lv}{2}} e^{-Lv}.$$

Since $Lv(t_*, x_*) > 1$, the last inequality implies

$$F(Lv) \leq -\dot{\varphi}\varphi^{-1}Lv + 2e^{-\frac{3}{2}}Lv.$$

Setting $H(0) = 0$ and $H(x) = F(x)/x, x > 0$, and $\omega = 2/\sqrt{e^3}$ this can be rewritten as

$$H(Lv) \leq -\dot{\varphi}\varphi^{-1} + \omega.$$

Since $-\dot{\varphi} = F(\varphi)$, we thus get (still at the maximum point)

$$\begin{aligned} H(Lv) &\leq H(\varphi) + \omega = H(\varphi) + H(H^{-1}(\omega)) \\ &\leq H(\varphi + \gamma H^{-1}(\omega)), \end{aligned}$$

where we used Lemma 5.3 with corresponding constant $\gamma > 0$; this lemma applies to H thanks to Proposition 3.4. Since H is strictly increasing, the last inequality implies that

$$Lv(t_*, x_*) \leq \varphi(t_*) + \gamma H^{-1}(\omega),$$

that is,

$$G(t_*, x_*) \leq 1 + \varphi(t_*)^{-1}\gamma H^{-1}(\omega).$$

We now obtain

$$\begin{aligned} \tilde{G}(t_1, x) &\leq \tilde{G}(t_*, x_*) = \frac{1}{r} G(t_*, x_*) \\ &\leq \frac{1}{r} \left(1 + \varphi(t_*)^{-1}\gamma H^{-1}(\omega) \right) \\ &\leq \frac{1}{r} \left(1 + \varphi(t_1)^{-1}\gamma H^{-1}(2/\sqrt{e^3}) \right), \end{aligned}$$

where in the last step we used the fact that φ is decreasing.

Case 3: $\psi(x_*) = s/r$ with $s \in \{2, \dots, r\}$. Since \tilde{G} has a maximum at (t_*, x_*) we have for both neighbors of x_*

$$\psi(\eta_j(x_*))Lv(\eta_j(x_*)) \leq \psi(x_*)Lv(x_*),$$

that is

$$Lv(t_*, \eta_j(x_*)) \leq \frac{\psi(x_*)}{\psi(\eta_j(x_*))}Lv(t_*, x_*) \leq \frac{s/r}{(s-1)/r}Lv(t_*, x_*) = \eta Lv(t_*, x_*),$$

with $\eta = s/(s-1) \in (1, 2]$. By Lemma 5.6, the second zero $a_*(\eta) > 0$ of the function $f(a) = e^{-\eta a} - 1 + a$ behaves as $2(\eta - 1)$ as $\eta \rightarrow 1$. This implies that there exists $C_* > 0$ independent of r such that

$$sa_*(\eta) = \frac{\eta}{\eta - 1}a_*(\eta) \leq C_*, \tag{5.18}$$

for all $s \in \{2, \dots, r\}$. We now distinguish two cases.

Case 3a: Suppose that $Lv(t_*, x_*) \leq a_*(\eta)$. Then, by (5.18),

$$\begin{aligned} \tilde{G}(t_1, x) &\leq \tilde{G}(t_*, x_*) = \psi(x_*)\varphi(t_*)^{-1}Lv(t_*, x_*) \\ &\leq \frac{s}{r}\varphi(t_*)^{-1}a_*(\eta) \leq \frac{C_*}{r}\varphi(t_1)^{-1}, \quad x \in \bar{B}_{2r}(x_0). \end{aligned}$$

Case 3b: Suppose that $Lv(t_*, x_*) > a_*(\eta)$. Then

$$e^{-\eta Lv(t_*, x_*)} - 1 + Lv(t_*, x_*) > 0.$$

Now we use (5.16). At the maximum point, we can bound $\Theta(v)$ from below as follows.

$$\begin{aligned} \Theta(v)(t_*, x_*) &\geq \sum_{j=1}^2 e^{v(t_*, \eta_j(x_*)) - v(t_*, x_*)} \left(e^{-\eta Lv(t_*, x_*)} - 1 + Lv(t_*, x_*) \right) \\ &\geq 2e^{-\frac{Lv}{2}} \left(e^{-\eta Lv} - 1 + Lv \right) \\ &= 2e^{-\frac{Lv}{2}} \left(e^{-Lv} - 1 + Lv \right) + 2e^{-\frac{Lv}{2}} \left(e^{-\eta Lv} - e^{-Lv} \right) \\ &= 2e^{-\frac{Lv}{2}} \Upsilon(-Lv) + 2e^{-\frac{Lv}{2}} \left(e^{-\eta Lv} - e^{-Lv} \right). \end{aligned}$$

By convexity of the exponential function we have (at the maximum point)

$$e^{-Lv} - e^{-\eta Lv} \leq e^{-Lv}(-Lv + \eta Lv) = (\eta - 1)Lve^{-Lv}.$$

Now,

$$\eta - 1 = \frac{s}{s-1} - 1 = \frac{1}{s-1},$$

and thus we obtain at the maximum point (using (5.16))

$$F(Lv) = 2e^{-\frac{Lv}{2}} \left(\Upsilon(Lv) + \Upsilon(-Lv) \right) \leq -\dot{\phi}\varphi^{-1}Lv + \frac{2}{s-1}e^{-\frac{Lv}{2}}Lve^{-Lv}.$$

Dividing by Lv and using that $Lv > a_*(\eta)$ it follows that

$$H(Lv) \leq H(\varphi) + \frac{2}{s-1}e^{-\frac{3}{2}a_*(\eta)}.$$

Setting

$$\omega_s = \frac{2}{s-1}e^{-\frac{3}{2}a_*(\eta)}$$

and applying Lemma 5.3 then gives

$$H(Lv) \leq H(\varphi) + \omega_s = H(\varphi) + H(H^{-1}(\omega_s)) \leq H(\varphi + \gamma H^{-1}(\omega_s)),$$

and thus

$$Lv(t_*, x_*) \leq \varphi(t_*) + \gamma H^{-1}(\omega_s),$$

that is

$$G(t_*, x_*) \leq 1 + \varphi(t_*)^{-1}\gamma H^{-1}(\omega_s).$$

We now find that

$$\begin{aligned} \tilde{G}(t_1, x) &\leq \tilde{G}(t_*, x_*) = \frac{s}{r}G(t_*, x_*) \\ &\leq \frac{s}{r}(1 + \varphi(t_*)^{-1}\gamma H^{-1}(\omega_s)) \\ &\leq 1 + \frac{\gamma C_s}{r}\varphi(t_1)^{-1}, \end{aligned}$$

where $C_s := sH^{-1}(\omega_s)$. It is now not difficult to see that there exists a number $M > 0$ such that $C_s \leq M$ for all $s \geq 2$ (recall that $H(x)$ behaves as a linear function as $x \rightarrow 0$). It follows that

$$\tilde{G}(t_1, x) \leq 1 + \frac{\gamma M}{r}\varphi(t_1)^{-1}.$$

Collecting the estimates from all cases we see that for arbitrary $t_1 > 0$

$$\tilde{G}(t_1, x) \leq 1 + \max \{1, \gamma H^{-1}(2/\sqrt{e^3}), C_*, \gamma M\} \frac{\varphi(t_1)^{-1}}{r}, \quad x \in \bar{B}_{2r}(x_0),$$

which implies that

$$Lv(t, x) \leq \varphi(t) + \max \{1, \gamma H^{-1}(2/\sqrt{e^3}), C_*, \gamma M\} \frac{1}{r}, \quad t > 0, x \in \bar{B}_r(x_0).$$

This together with (5.14) proves the theorem. □

From Theorem 5.7 we obtain the following result for global positive solutions of the heat equation on the grid $\tau\mathbb{Z}$.

Corollary 5.8. *Let $\tau > 0$ and G be the grid $\tau\mathbb{Z}$. Consider the Laplace operator given by*

$$\Delta_\tau u(x) = \frac{1}{\tau^2} (u(x + \tau) - 2u(x) + u(x - \tau)), \quad x \in \tau\mathbb{Z}.$$

Suppose that $u : [0, \infty) \times \tau\mathbb{Z} \rightarrow (0, \infty)$ solves the heat equation on $\tau\mathbb{Z}$. Then

$$-\Delta_\tau(\log u)(t, x) \leq \varphi_\tau(t) \quad \text{on } (0, \infty) \times \tau\mathbb{Z}, \tag{5.19}$$

where

$$\varphi_\tau(t) = \frac{1}{\tau} \varphi\left(\frac{t}{\tau}\right)$$

is the relaxation function corresponding to the CD-function F given in (3.16) with $D = 2$ and $\mu_0 = \tau$, and φ is as in Theorem 5.7.

Remark 5.9. If one considers the limit $\tau \rightarrow 0+$ in (5.19), one recovers the classical sharp Li-Yau inequality

$$-\frac{\partial^2}{\partial x^2}(\log u)(t, x) \leq \frac{1}{2t} \quad \text{on } (0, \infty) \times \mathbb{R}. \tag{5.20}$$

This follows from the fact that $\varphi(s) \sim \frac{1}{2s}$ as $s \rightarrow \infty$, cf. Lemma 3.7.

Proof of Corollary 5.8. The case $\tau = 1$ follows directly from Theorem 5.7 by sending $R \rightarrow \infty$. The case of arbitrary $\tau > 0$ is reduced to the case $\tau = 1$ by means of a scaling argument. In fact, putting $t = s\tau^2$, $x = \tau y$ and $w(s, y) = u(t, x)$, the function w solves the equation with Laplace operator $\Delta = \Delta_1$ on \mathbb{Z} , and thus

$$-\Delta(\log w)(s, y) \leq \varphi(s) \quad \text{on } (0, \infty) \times \mathbb{Z}.$$

Scaling back to the original variables yields the result. □

6. Harnack inequalities

The aim of this section is to provide a proof of the Harnack inequality. The case of finite graphs, Theorem 1.2, follows from the more general case, which we formulate here.

Theorem 6.1. *Let $G = (V, E)$ be a connected and locally finite graph and $\mu : V \rightarrow (0, \infty)$ be bounded above by μ_{\max} . Let further $w_{\min} > 0$ be a lower bound for all weights w_{xy} with $xy \in E$.*

Suppose that $u : (0, \infty) \times V \rightarrow (0, \infty)$ is C^1 in time and satisfies the differential Harnack estimate

$$\partial_t(\log u) \geq \Psi_\Upsilon(\log u) - \eta(t) \quad \text{on } (0, \infty) \times V, \tag{6.1}$$

where $\eta : (0, \infty) \rightarrow [0, \infty)$ is continuous. Then for any $0 < t_1 < t_2$ and $x_1, x_2 \in V$ we have

$$u(t_1, x_1) \leq u(t_2, x_2) \exp \left(\int_{t_1}^{t_2} \eta(t) dt + \frac{2\mu_{\max}d(x_1, x_2)^2}{w_{\min}(t_2 - t_1)} \right). \tag{6.2}$$

In the proof, we closely follow the strategy of [2, Theorem 5.2].

Proof. We first consider the situation where $x_1 \sim x_2$. Let $0 < t_1 < t_2$ and $s \in J := [t_1, t_2]$. Then we have by assumption (6.1) that

$$\begin{aligned} \log \frac{u(t_1, x_1)}{u(t_2, x_2)} &= \log \frac{u(t_1, x_1)}{u(s, x_1)} + \log \frac{u(s, x_1)}{u(s, x_2)} + \log \frac{u(s, x_2)}{u(t_2, x_2)} \\ &= - \int_{t_1}^s \partial_t \log u(t, x_1) dt + \log \frac{u(s, x_1)}{u(s, x_2)} - \int_s^{t_2} \partial_t \log u(t, x_2) dt \\ &\leq \int_{t_1}^s (\eta(t) - \Psi_\Upsilon(\log u)(t, x_1)) dt + \log \frac{u(s, x_1)}{u(s, x_2)} \\ &\quad + \int_s^{t_2} (\eta(t) - \Psi_\Upsilon(\log u)(t, x_2)) dt \\ &\leq \int_{t_1}^{t_2} \eta(t) dt + \log \frac{u(s, x_1)}{u(s, x_2)} - \int_s^{t_2} \Psi_\Upsilon(\log u)(t, x_2) dt \\ &\leq \int_{t_1}^{t_2} \eta(t) dt + \log \frac{u(s, x_1)}{u(s, x_2)} \\ &\quad - \frac{w_{\min}}{\mu_{\max}} \int_s^{t_2} \Upsilon(\log u(t, x_1) - \log u(t, x_2)) dt \\ &= \int_{t_1}^{t_2} \eta(t) dt + \delta(s) - \gamma \int_s^{t_2} \Upsilon(\delta(t)) dt, \end{aligned} \tag{6.3}$$

where we set $\gamma = w_{\min}/\mu_{\max}$ and

$$\delta(t) = \log u(t, x_1) - \log u(t, x_2), \quad t \in J.$$

We choose $s \in J$ in such a way that the continuous function ω defined by

$$\omega(t) := \delta(t) - \gamma \int_t^{t_2} \Upsilon(\delta(\tau)) d\tau, \quad t \in J,$$

attains its minimum at s .

Suppose that $\omega(s) \geq 0$. Then the positivity of Υ implies that $\delta \geq 0$ in J , and thus

$$\Upsilon(\delta(t)) \geq \frac{1}{2} \delta(t)^2, \quad t \in J,$$

since $\Upsilon(z) \geq z^2/2$ for all $z \geq 0$. This means that

$$\omega(t) \leq \delta(t) - \frac{\gamma}{2} \int_t^{t_2} \delta(\tau)^2 d\tau =: \tilde{\omega}(t).$$

It follows that $\omega(s) \leq \min_{t \in J} \tilde{\omega}(t)$. From Lemma 5.5 in [2] we now know that

$$\min_{t \in J} \tilde{\omega}(t) \leq \frac{2}{\gamma(t_2 - t_1)}.$$

Hence

$$\omega(s) = \min_{t \in J} \omega(t) \leq \frac{2}{\gamma(t_2 - t_1)}. \tag{6.4}$$

If $\min_{t \in J} \omega(t) < 0$, this conclusion is obvious. Combining (6.3) and (6.4) yields

$$\log \frac{u(t_1, x_1)}{u(t_2, x_2)} \leq \int_{t_1}^{t_2} \eta(t) dt + \frac{2}{\gamma(t_2 - t_1)}. \tag{6.5}$$

Now we consider the case when x_1 and x_2 are not adjacent. Set $l = d(x_1, x_2)$. Since G is connected, there is a path $x_1 = y_0 \sim y_1 \sim \dots \sim y_l = x_2$ of length l . Define the numbers $\tau_i, i = 0, \dots, l$ by $\tau_i = t_1 + i(t_2 - t_1)/l$. Employing (6.5) we may estimate as follows.

$$\begin{aligned} \log \frac{u(t_1, x_1)}{u(t_2, x_2)} &= \sum_{i=1}^l \log \frac{u(\tau_{i-1}, y_{i-1})}{u(\tau_i, y_i)} \\ &\leq \sum_{i=1}^l \left(\int_{\tau_{i-1}}^{\tau_i} \eta(t) dt + \frac{2}{\gamma(\tau_i - \tau_{i-1})} \right) \\ &= \int_{t_1}^{t_2} \eta(t) dt + \frac{2l^2}{\gamma(t_2 - t_1)}. \end{aligned}$$

This implies (6.2). □

Recall the definition $\Upsilon_\alpha(y) = \Upsilon(\alpha y)$.

Theorem 6.2. *Let $G = (V, E)$, μ_{\max} and w_{\min} as in Theorem 6.1, and let $\alpha \in (0, 1)$.*

Suppose that $u : (0, \infty) \times V \rightarrow (0, \infty)$ is C^1 in time and satisfies the differential Harnack inequality

$$\partial_t(\log u) \geq \Psi_\Upsilon(\log u) - \frac{1}{\alpha} \Psi_{\Upsilon_\alpha}(\log u) - \eta_\alpha(t) \quad \text{on } (0, \infty) \times V,$$

where $\eta_\alpha : (0, \infty) \rightarrow [0, \infty)$ is continuous. Then for any $0 < t_1 < t_2$ and $x_1, x_2 \in V$ we have

$$u(t_1, x_1) \leq u(t_2, x_2) \exp \left(\int_{t_1}^{t_2} \eta_\alpha(t) dt + \frac{2\mu_{\max}d(x_1, x_2)^2}{w_{\min}(1-\alpha)(t_2-t_1)} \right).$$

Proof. The arguments are the same as in the proof of Theorem 6.1. Instead of (6.3), we obtain in the first step the estimate

$$\log \frac{u(t_1, x_1)}{u(t_2, x_2)} \leq \int_{t_1}^{t_2} \eta_\alpha(t) dt + \delta(s) - \gamma \int_s^{t_2} \left(\Upsilon(\delta(t)) - \frac{1}{\alpha} \Upsilon(\alpha\delta(t)) \right) dt.$$

From Lemma 2.5, we know that the function $g(z) := \Upsilon(z) - \frac{1}{\alpha} \Upsilon(\alpha z)$ is non-negative on \mathbb{R} and satisfies $g(z) \geq (1-\alpha)z^2/2$ on $[0, \infty)$. Therefore, we can argue as above, replacing γ by $\gamma(1-\alpha)$. \square

References

- [1] D. BAKRY and M. ÉMERY, *Diffusions hypercontractives*, In: “Séminaire de Probabilités”, XIX, 1983/84, Lecture Notes in Math., Vol. 1123, 1985, 177–206.
- [2] F. BAUER, P. HORN, Y. LIN, G. LIPPNER, D. MANGOUBI and S.-T. YAU, *Li-Yau inequality on graphs*, J. Differential Geom. **99** (2015), 359–405.
- [3] A.-I. BONCIOCAT and K.-T. STURM, *Mass transportation and rough curvature bounds for discrete spaces*, J. Funct. Anal. **256** (2009), 2944–2966.
- [4] D. CUSHING, S. LIU and N. PEYERIMHOFF, *Bakry-Émery curvature functions of graphs*, Canad. J. Math. **72** (2020), 89–143.
- [5] F. R. K. CHUNG and S.-T. YAU, *Logarithmic Harnack inequalities*, Math. Res. Lett. **3** (1996), 793–812.
- [6] M. ERBAR and J. MAAS, *Ricci curvature of finite Markov chains via convexity of the entropy*, Arch. Ration. Mech. Anal. **206** (2012), 997–1038.
- [7] M. ERBAR, J. MAAS and P. TETALI, *Discrete Ricci curvature bounds for Bernoulli-Laplace and random transposition models*, Ann. Fac. Sci. Toulouse Math. (6) **24** (2016), 781–800.
- [8] B. HUA, J. JOST and S. LIU, *Geometric analysis aspects of infinite semiplanar graphs with non-negative curvature*, J. Reine Angew. Math. **700** (2015), 1–36.
- [9] P. HORN, Y. LIN, S. LIU and S.-T. YAU, *Volume doubling, Poincaré inequality and Gaussian heat kernel estimate for non-negatively curved graphs*, J. Reine Angew. Math. **757** (2019), 89–130.
- [10] J. JOST and S. LIU, *Ollivier’s Ricci curvature, local clustering and curvature-dimension inequalities on graphs*, Discrete Comput. Geom. **51** (2014), 300–322.
- [11] B. KLARTAG, G. KOZMA, P. RALLI and P. TETALI, *Discrete curvature and Abelian groups*, Canad. J. Math. **68** (2016), 655–674.
- [12] S. LAKZIAN and Z. MCGUIRK, *A global Poincaré inequality on graphs via a conical curvature-dimension condition*, Geom. Metr. Spaces **6** (2018), 32–47.
- [13] P. LI and S.-T. YAU, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 153–201.
- [14] Y. LIN and S.-T. YAU, *Ricci curvature and eigenvalue estimate on locally finite graphs*, Math. Res. Lett. **17** (2010), 343–356.

- [15] J. MAAS, *Gradient flows of the entropy for finite Markov chains*, J. Funct. Anal. **261** (2011), 2250–2292.
- [16] A. MIELKE, *Geodesic convexity of the relative entropy in reversible Markov chains*, Calc. Var. Partial Differential Equations **48** (2013), 1–31.
- [17] F. MÜNCH, *Li-Yau inequality on finite graphs via non-linear curvature dimension conditions*, J. Math. Pures Appl. (9) **120** (2018), 130–164.
- [18] F. MÜNCH, *Remarks on curvature dimension conditions on graphs*, Calc. Var. Partial Differential Equations **56** (2017), Art. 11, 8 pp.
- [19] Y. OLLIVIER, *Ricci curvature of Markov chains on metric spaces*, J. Funct. Anal. **256** (2009), 810–864.
- [20] Y. OLLIVIER, *A survey of Ricci curvature for metric spaces and Markov chains*, In: “Probabilistic Approach to Geometry”, Adv. Stud. Pure Math., Vol. 57, Math. Soc. Japan, Tokyo, 2010, 343–381.
- [21] R. ZACHER, *A weak Harnack inequality for fractional evolution equations with discontinuous coefficients*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **12** (2013), 903–940.

Institut für Angewandte Analysis
Universität Ulm
Helmholtzstrasse 18
D-89081 Ulm, Germany
dominik.dier@uni-ulm.de
rico.zacher@uni-ulm.de

Fakultät für Mathematik
Postfach 100131
D-33501 Bielefeld, Germany
moritz.kassmann@uni-bielefeld.de