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Genericity of infinite entropy for maps with low regularity

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Abstract. For bi-Lipschitz homeomorphisms of a compact manifold it is known that topological entropy is always finite. For compact manifolds of dimension two or greater, we show that in the closure of the space of bi-Lipschitz homeomorphisms, with respect to either the Hölder or the Sobolev topologies, topological entropy is generically infinite. We also prove versions of the C^1-Closing Lemma in either of these spaces. Finally, we give examples of homeomorphisms with infinite topological entropy which are Hölder and/or Sobolev of every exponent.

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Contents

1. Introduction

1.1. Background

In 1974 Palis, Pugh, Shub and Sullivan [34] published a list of dynamical properties satisfied by a generic homeomorphism acting on an arbitrary compact manifold. Six

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years later, Yano [44] submitted an extra striking property in relation to *topological entropy*. Recall that topological entropy is a non-negative extended real number defined for any continuous self-map of a compact space, and that this number is an invariant of topological conjugacy. It was first introduced by Adler, Konheim, and McAndrew [2] as an analogue, in the topological category, of the Kolmogorov-Sinai entropy for measure-preserving transformations. Topological entropy, whose precise definition will be given below, is a useful way of quantifying topological aspects of chaos. Yano proved the following result.

Yano's theorem. *For homeomorphisms of compact manifolds of dimension greater than one topological entropy is generically infinite.*

Here, the space of homeomorphisms is endowed with the uniform topology. Yano also states the same result in the case of endomorphisms on compact manifolds of dimension one or greater. However, in this article we will focus on the homeomorphism case. Note that, in Yano's result, the fact that the space being acted upon is a manifold matters: there are compact metric spaces for which a generic homeomorphism has zero topological entropy. To be specific, a generic homeomorphism of the Cantor space has zero topological entropy: this was first proved by Glasner and Weiss in [19]. It has also been shown by D'Aniello and Darji [5] that a generic *endomorphism* of the Cantor space also has zero topological entropy; Bernardes and Darji [7] have proved a significantly stronger result, namely that, generically, no map of the Cantor space has a Li-Yorke pair. In this zero-dimensional setting even stronger results are possible. For example, as proved by Kechris and Rosendal [28], then by Akin, Glasner and Weiss [3] and then by Bernardes and Darji [7], the space of homeomorphisms of the Cantor space has a *comeager conjugacy class*. See also the work of Hochman [24] concerning transitive homeomorphisms of the Cantor space. By contrast, it is know that in higher dimensions there are no large conjugacy classes and hence one relies on genericity properties rather than sizes of conjugacy classes.

For Lipschitz or smooth maps on compact manifolds it had already been demonstrated by Kushnirenko [29], then by Itō [25] for homeomorphisms and soon afterwards by Bowen [9] for general endomorphisms, that the topological entropy is always finite. Subsequently, in [44], Yano also obtained the following result.

Corollary. *A generic homeomorphism of a compact manifold of dimension two or greater is not topologically conjugate to any diffeomorphism.*

Let us now recall the definition of topological entropy [2]. As we will be working only in compact metric spaces, we give the reformulation in this setting due to Bowen[1]. Let f be a continuous self-mapping of a compact metric space (X, d). A subset E of X is (n, ϵ)-separated for f if for all distinct points $x, y \in E$ there exists a non-negative integer $k < n$ such that $d(f^k(x), f^k(y)) > \epsilon$. Let $S_f(n, \epsilon)$ denote

[1] Michel Hénon pointed out to one of the authors (C.T.) that this definition has a significant advantage over the original one when trying to compute entropy for specific systems: only the forward iterates of the map must be considered, rather than the backward iterates!

the maximal cardinality of an (n,ϵ)-separated set. Then the *topological entropy* of f is given by

$$h_{\text{top}}(f) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log S_f(n,\epsilon) .$$

That is, topological entropy is the growth rate of the maximal size of (n,ϵ)-separated sets at arbitrarily small scales ϵ. Note that in some applications, one prefers to compute the metric entropy $h_\mu(f)$ of f with respect to some f-invariant measure μ. However, by virtue of the variational principle [43, Theorem 8.6], topological entropy is always the supremum (and in some important cases the maximum) of the metric entropies $h_\mu(f)$, where μ varies over all f-invariant Borel probability measures.

As was already stated, for smooth maps on manifolds the topological entropy is finite. In fact, if X is a compact metric space of Hausdorff dimension $\dim_H(X)$ and f is a self-map of X with Lipschitz constant L, then

$$h_{\text{top}}(f) \le \dim_H(X) \cdot \log^+(L) .$$

See, for instance, [27, Theorem 3.2.9]. We note that, in the case of smooth maps acting on smooth manifolds, this bound was already implicit in the work of Itō [25] and Bowen [9].

The above discussion thus shows that the existence of bounds on $h_{\text{top}}(f)$ changes dramatically when the regularity goes from just continuity to Lipschitz continuity. However, the notion of "going from" continuity to Lipschitz continuity must be treated with care. In this paper we make an initial foray into the problem of determining what occurs between these two cases by considering mappings in Hölder and Sobolev classes. These are perhaps two of the most classical ways of interpolating between C^0- and Lipschitz-regularity. Homeomorphisms with Hölder or Sobolev regularity have been of interest recently in the study of certain PDE's, such as the Ball-Evans Problem in nonlinear elasticity (*cf.* [26] and references therein). However, the study of the dynamics of maps in either of these classes has essentially remained untouched. With this work we hope to remedy this situation while also laying the groundwork for further dynamical investigations.

One possible reason for why the dynamics of maps in these spaces has not been considered before is that α-Hölder and $W^{1,p}$-Sobolev classes are not closed under composition. This being said, spaces of such maps are closed under pre- and post-composition by Lipschitz maps, and the union of such spaces, under α and p respectively, *is* closed under composition. This allows us to still make local perturbations of these maps. Also note that C^1, or even Lipschitz, is not in general dense in any Hölder or Sobolev class. Thus results concerning such classes cannot be derived from direct approximation arguments.

Below we will show that, in the closure of the space of bi-Lipschitz maps in either of these topologies, for suitable parameters of regularity, infinite entropy is a generic property. It also follows from our results that there is no "barrier" separating infinite entropy maps from the space of Lipschitz maps (we even give explicit examples of homeomorphisms with infinite entropy which are Hölder or Sobolev of every exponent).

1.2. Summary of our results

The results stated below were announced in [13]. To topologise the space of Hölder or Sobolev homeomorphisms on a smooth manifold one requires (in principle) additional structure: a distance function in the first case and a Riemannian structure in the second. We take a different approach by defining topologies on function spaces which are analogous to the Whitney topology. More specifically, for $0 \leq \alpha < 1$, let $\mathcal{H}^\alpha(M)$ denote the space of homeomorphisms on M which are bi-α-Hölder continuous in all local charts. We also denote by $\mathcal{H}^1(M)$ the space of homeomorphisms which are bi-Lipschitz in all local charts. In Section 3.1 we define a topology on $\mathcal{H}^\alpha(M)$ which we call the *α-Hölder-Whitney topology*. For $0 \leq \alpha < \beta \leq 1$, we denote by $\mathcal{H}^\beta_\alpha(M)$ the closure of $\mathcal{H}^\beta(M)$ with respect to the α-Hölder-Whitney topology. Recall that a property is *generic* in a Baire space if the set of points satisfying this property contains a residual subset (*i.e.*, a countable intersection of open and dense subsets). We show the following.

Theorem A (Generic infinite entropy for Hölder classes). *Let M be a smooth compact manifold of dimension d greater than or equal to two. For $0 \leq \alpha < 1$, the following holds. In $\mathcal{H}^1_\alpha(M)$, infinite topological entropy is a generic property.*

Similarly, for $1 \leq p, p^* < \infty$, let $\mathcal{S}^{p,p^*}(M)$ denote the space of homeomorphisms on M which in all local charts are of Sobolev class $W^{1,p}$ and whose inverse is of Sobolev class W^{1,p^*}. In Section 4.1 we define a topology on $\mathcal{S}^{p,p^*}(M)$ which we call the *(p, p^*)-Sobolev-Whitney topology*.

Theorem B (Generic infinite entropy for Sobolev classes). *Let M be a smooth compact manifold of dimension d.*

(a) *If $d = 2$ and $1 \leq p, p^* < \infty$ then, in $\mathcal{S}^{p,p^*}(M)$, infinite topological entropy is a generic property;*
(b) *If $d > 2$ and $d - 1 < p, p^* < \infty$, then, in $\mathcal{S}^{p,p^*}(M)$, infinite topological entropy is a generic property.*

Additionally, we give an alternative proof of (a) in the case when $p^* = 1$. This proof uses a variant of the Radó-Kneser-Choquet theorem for p-harmonic mappings [4,26]. We do not know whether this approach extends to higher dimensions, though we suspect not, as there exists a counterexample to the classical Radó-Kneser-Choquet theorem in dimension three (see, *e.g.*, [15, Section 3.7]).

Let us also note that we prove two versions of the Closing Lemma along the way. Namely, for both the spaces of bi-Hölder homeomorphisms and the space of bi-Sobolev homeomorphisms given above, we show that the analogue of Pugh's C^1-Closing Lemma holds. We remark that it would be interesting to determine whether there is another, more direct, approach using Pugh's C^1-result and an approximation argument, demonstrating that a homeomorphism of bi-Hölder or bi-Sobolev type is approximable by C^1-diffeomorphisms.

1.3. Structure of the paper

In order to better motivate our main results, stated above, we begin with some examples. More precisely, in Section 2, we give an explicit construction of homeomorphisms in dimension two with infinite topological entropy which lie in *all* Hölder or Sobolev classes. These examples arise as perturbations of the identity transformation, each of them containing an infinite sequence of small horseshoes with an increasing number of branches.

Following Section 2, we show that such examples are *typical* (in the Baire category sense) in the various universes of bi-Hölder or bi-Sobolev maps (for all dimensions ≥ 2). Such work is divided into two main parts, which we proceed to describe.

In Section 3 we investigate some properties of bi-Hölder homeomorphisms. After the preliminary Section 3.1, where a suitable Hölder-Whitney topology is given on the space of bi-Hölder homeomorphisms between manifolds, the Closing Lemma for this class of maps is proved in Section 3.2. Following this the genericity of infinite topological entropy is shown in Section 3.3.

Section 4 investigates bi-Sobolev homeomorphisms. The structure of Section 4 mirrors that of Section 3, with the exception that we also give another proof of the genericity of infinite entropy in the special case of compact surfaces. Specifically, in Section 4.1 we introduce the space of bi-Sobolev homeomorphisms together with the Sobolev-Whitney topology. We prove a Closing Lemma for maps in this class in Section 4.2. Two different proofs of the genericity of infinite topological entropy, one specific to dimension two and another for dimensions two and greater, are given in Section 4.3.

Finally, the perturbative tools used throughout the paper are collected in the Appendices A and B.

1.4. Notation and terminology

Throughout this article, we use the following notation. We denote the Euclidean norm in $\mathbb{R}^d$ by $|\cdot|_{\mathbb{R}^d}$. We denote the Euclidean distance by $d(\cdot,\cdot)$. Denote the open r-ball about the point x in $\mathbb{R}^d$ by $B^d(x,r)$. When the dimension is clear we will write this as $B(x,r)$. In the special case of the unit ball in $\mathbb{R}^d$ centred at the origin we denote this by B^d.

Given a manifold M endowed with distance function $d_M(\cdot,\cdot)$ denote the open r-ball about ξ in M, with respect to d_M, by $B_M(\xi,r)$. Given points $a, b \in \mathbb{R}^d$ and $r > 0$ define

$$E(a,b;r) = \left\{x \in \mathbb{R}^d : d\left(x, tp + (1-t)q\right) < r, \text{ some } t \in [0,1]\right\}$$

We call such a set an *elongated neighbourhood*. Given subsets Ω_0 and Ω_1 in some metric space we denote the Hausdorff distance between Ω_0 and Ω_1 by $\mathrm{dist}_H(\Omega_0,\Omega_1)$, *i.e.*,

$$\mathrm{dist}_H(\Omega_0, \Omega_1) = \max\left\{ \sup_{x_0\in\Omega_0} \inf_{x_1\in\Omega_1} d(x_0, x_1),\ \sup_{x_1\in\Omega_1} \inf_{x_0\in\Omega_0} d(x_0, x_1) \right\}$$

and the diameter of Ω_0 by $\mathrm{diam}(\Omega_0)$.

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2. Planar homeomorphisms with infinite topological entropy

In this section we build a family of examples of orientation-preserving homeomorphisms of the plane having compact support and infinite topological entropy which are bi-Hölder of every exponent. More precisely, we construct for each $\beta > 0$ an orientation-preserving homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$, with compact support and $h_{\mathrm{top}}(f) = \infty$, such that $f, f^{-1} \in C^{\Lambda_\beta}(\mathbb{R}^2)$, where Λ_β is the modulus of continuity $\Lambda_\beta(t) = t|\log t|^\beta$. In particular, f and f^{-1} are α-Hölder continuous for all $0 < \alpha < 1$. The construction will also show that both f and f^{-1} are in the Sobolev class $W^{1,p}$ for all $1 \leq p < \infty$.

Terminology

Given a domain $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$, we denote by $W^{1,p}_{\mathrm{loc}}(\Omega)$ the Sobolev space of maps $\phi : \Omega \to \mathbb{R}^2 \equiv \mathbb{C}$ such that the partial derivatives $\phi_z, \phi_{\bar{z}}$ exist Lebesgue almost everywhere in Ω and are locally in L^p. The *distortion* (or *dilatation*) of a map $\phi : \Omega \to \mathbb{R}^2$ is the function K_ϕ given by

$$K_\phi = \frac{|\phi_z| + |\phi_{\bar{z}}|}{|\phi_z| - |\phi_{\bar{z}}|}\,.$$

The map ϕ is said to have *finite distortion* if $K_\phi(z) < \infty$ for Lebesgue a.e. $z \in \Omega$. We say that ϕ is a *regular map with finite distortion* if $\phi \in W^{1,2}_{\mathrm{loc}}(\Omega)$, its Jacobian $J_\phi = |\phi_z|^2 - |\phi_{\bar{z}}|^2$ is locally integrable and ϕ has finite distortion. Such a map is differentiable almost everywhere, and the inequality $|D\phi(z)|^2 \leq K_\phi(z)J_\phi(z)$ holds for Lebesgue a.e. $z \in \Omega$. Examples of regular maps with finite distortion include local diffeomorphisms, as well as quasiconformal homeomorphisms of (some portion of) the plane. (See, *e.g.*, [22].)

Tool

The main tool in the construction below is the following result due to Goldstein and Voldop'yanov, a proof of which can be found in [6, pages 530–534].

Proposition 2.1. *Let $f\colon \Omega \to \mathbb{R}^2$ be a regular map of finite distortion. Then f is continuous, and for all $z, w \in \Omega$ we have*

$$|f(z) - f(w)|^2 \;\le\; \frac{2\pi \displaystyle\int_{2D} |Df|^2}{\log\left(e + \dfrac{\operatorname{diam}(D)}{|z-w|}\right)} \tag{2.1}$$

for every disk D such that $z, w \in D \subset 2D \subset \Omega$. [2]

Note that, for disks D whose diameters are comparable to the distance $|z - w|$, one can safely replace the denominator on the right-hand side of (2.1) by a constant. This will suffice for our purposes.

Ingredients

Let $Q = [0, 1]^2 \subset \mathbb{R}^2$ be the unit square, and consider the concentric square $R = [\frac{1}{3}, \frac{2}{3}]^2 \subset Q$. The ingredients in our construction are a sequence of homeomorphisms $g_n\colon Q \to Q$, $n \ge 1$, with the following properties:

(i) The support of each g_n is contained in R;
(ii) For all $n \ge 1$, g_n and g_n^{-1} are regular maps of finite distortion;
(iii) There exist $C > 0$ and a positive integer β such that, for each $n \ge 1$,

$$|Dg_n(z)| \le Cn^{\beta} \quad \text{and} \quad |Dg_n^{-1}(z)| \le Cn^{\beta}$$

for Lebesgue almost every $z \in R$;
(iv) We have $h_{\mathrm{top}}(g_n) \to \infty$ as $n \to \infty$.

For instance, one could take g_n to be a C^1-smooth horseshoe map with n branches supported in the square R, so that $h_{\mathrm{top}}(g_n) = \log n$. Such a horseshoe can be built so as to satisfy (iii) at *every* point.

Construction

Let us write $(0, 1] = \bigcup_{n=1}^{\infty} I_n$, where

$$I_n \;=\; \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right].$$

[2] Here, $2D$ is the disk concentric with D having twice the radius.

Let $Q_n = I_n \times I_n \subset Q$, and let $f_n = A_n \circ g_n \circ A_n^{-1} \colon Q_n \to Q_n$, where $A_n \colon Q \to Q_n$ is the affine map

$$A_n(x, y) = \left(\frac{x+1}{2^n}, \frac{y+1}{2^n} \right) .$$

Note that $|Df_n| = |Dg_n \circ A_n^{-1}|$, so that $|Df_n| \leq Cn^{\beta}$, by (iii) above. Note also that the support of f_n is contained in the rescaled square $R_n = A_n(R)$ so, in particular, $f_n|_{\partial Q_n} \equiv \mathrm{id}$. Next, define $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$f(z) = \begin{cases} f_n(z) & \text{if } z \in Q_n \\ z & \text{if } z \in \mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} Q_n . \end{cases}$$

Then it is clear that f is an orientation-preserving homeomorphism of the plane (see Figure 2.1).

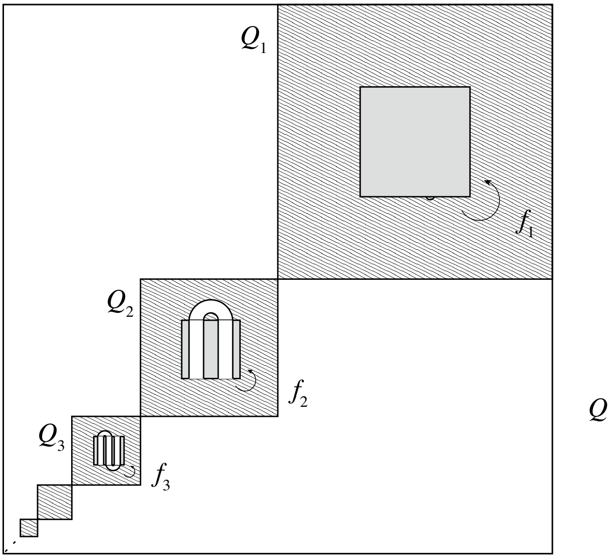

Figure 2.1. Sewing the maps f_n together with the identity yields f.

Theorem 2.2. *The homeomorphism f and its inverse f^{-1} are both regular maps with finite distortion, and have infinite topological entropy. Moreover, there exist constants $C_0 > 0$ and $\delta > 0$ such that, for all $z, w \in \mathbb{R}^2$ with $|z - w| < \delta$ we have*

$$|f(z) - f(w)| \leq C_0 |z - w| \left(\log \frac{1}{|z - w|} \right)^{\beta} , \tag{2.2}$$

and similarly for f^{-1}. In particular, f and f^{-1} are α-Hölder continuous for every $0 < \alpha < 1$. In addition, f and f^{-1} are locally in the Sobolev class $W^{1,p}$ for every $1 \leq p < \infty$.

Proof. We have, rather trivially,

$$h_{\text{top}}(f) = \sup h_{\text{top}}(f_n) = \sup h_{\text{top}}(g_n) = \infty\,.$$

In other words, f has infinite topological entropy, and hence so does f^{-1}. Thus, the real issue is to verify the regularity of f (and f^{-1}). Throughout the proof, we shall denote by $C_1, C_2, \ldots$ positive absolute constants (it is possible to keep track of how they depend on the above data of the construction, but this will not be relevant for our purposes), while still denoting by C the constant appearing in property (iii) above.

First, we check that the map f is locally in $W^{1,p}$, for each $1 \le p < \infty$. If D is any open disk such that $Q \subset D$, then by (iii) above we have

$$\begin{aligned}\int_D |Df|^p &= \int_{D\setminus\bigcup_{n=1}^\infty Q_n} |Df|^p + \int_{\bigcup_{n=1}^\infty Q_n} |Df|^p \\ &\le \text{Area}(D) + C^p \sum_{n=1}^\infty n^{\beta p}\text{Area}(Q_n) \\ &= \text{Area}(D) + C^p \sum_{n=1}^\infty n^{\beta p} 2^{-2n} < \infty\,.\end{aligned}$$

Since f equals the identity outside D, this shows that $f \in W^{1,p}_{\text{loc}}(\mathbb{R}^2)$. The same holds for f^{-1}, and the proof is the same.

Next, we verify the modulus of continuity statement. We do it only for f, the verification for f^{-1} being entirely analogous. Suppose $z, w \in \mathbb{R}^2$ are any two points such that

$$|z - w| < \delta = \min\left\{\frac{1}{16}, e^{-\beta}\right\}.$$

Then there are three cases to consider:

First case. We have $z, w \in \mathbb{R}^2 \setminus \bigcup_{n=1}^\infty R_n$. In this case, $f(z) = z$ and $f(w) = w$, and there is nothing to prove.

Second case. We have $z, w \in \bigcup_{n=1}^\infty R_n$. In this case, we may assume without loss of generality that $z \in R_m$ and $w \in R_n$ with $m \le n$. There are two sub-cases to consider:

(a) $m = n$: in this sub-case, we let D be the closed disk having the segment joining z to w as diameter. Note that, by construction, $2D \subset Q_m$. Applying the inequality (2.1) of Proposition 2.1, we get

$$\begin{aligned}|f(z) - f(w)| &\le C_1 \left(\int_{2D} |Df|^2\right)^{\frac{1}{2}} \\ &\le C_2 \left(m^{2\beta}\text{Area}(2D)\right)^{\frac{1}{2}} = 2C_2\sqrt{\pi}\, m^\beta |z - w|\,.\end{aligned} \tag{2.3}$$

However, since $|z-w| < 2^{-m}$, we have

$$m < \frac{1}{\log 2} \log \frac{1}{|z-w|} . \tag{2.4}$$

Combining (2.3) with (2.4) yields (2.2) in this sub-case;

(b) $m < n$: in this sub-case, the distance between z and w is comparable to the diameter of R_m; in fact,

$$\frac{\sqrt{2}}{2}\frac{1}{2^m} \leq |z-w| \leq \frac{5\sqrt{2}}{3}\frac{1}{2^m} . \tag{2.5}$$

Note that, since we are assuming that $|z-w| < \frac{1}{16}$, it follows from the first inequality in (2.5) that $m > 2$. Let D be the disk with center at the origin and radius $\sqrt{2} \times 2^{-m+1}$, which contains both z and w. Then its double $2D$ contains precisely the squares Q_k with $k \geq m-1$. Thus, we have

$$\begin{aligned}
\int_{2D} |Df|^2 &= \int_{2D \setminus \bigcup_{k=m-1}^{\infty} Q_k} |Df|^2 + \int_{\bigcup_{k=m-1}^{\infty} Q_k} |Df|^2 \\
&\leq \text{Area}(2D) + C^2 \sum_{k=m-1}^{\infty} k^{2\beta} \text{Area}(Q_k) \\
&= \text{Area}(2D) + C^2 \sum_{k=m-1}^{\infty} k^{2\beta} 2^{-2k}.
\end{aligned}$$

Applying the inequality (2.1) of Proposition 2.1, we get

$$|f(z) - f(w)| \leq C_3 \left(\text{Area}(2D) + C^2 \sum_{k=m-1}^{\infty} k^{2\beta} 2^{-2k} \right)^{\frac{1}{2}} . \tag{2.6}$$

Now, on the one hand we have

$$\text{Area}(2D) = \pi \left(\sqrt{2} \cdot 2^{-m+2} \right) < 64\pi |z-w|^2 , \tag{2.7}$$

where we have used (2.5). On the other hand, we can estimate the series in (2.6) as follows. We have (by the integral test)

$$\sum_{k=m-1}^{\infty} k^{2\beta} 2^{-2k} < \int_{m-2}^{\infty} x^{2\beta} e^{-(2\log 2)x} \, dx .$$

Here we use the following general fact[3]: for all $a > 0$, all $\lambda > 1$ and all $\nu \in \mathbb{N}$,

$$\int_a^{\infty} x^{\nu} e^{-\lambda x} \, dx = e^{-\lambda a} \sum_{j=0}^{\nu} \frac{\nu!}{j!} \frac{a^j}{\lambda^{\nu-j+1}} < (\nu+1)! a^{\nu} e^{-\lambda a} .$$

[3] See for instance [8, pages 103–104].

Applying this fact with $a = m - 2$, $\lambda = 2\log 2$ and $\nu = 2\beta$, we deduce using (2.5) that

$$\sum_{k=m-1}^{\infty} k^{2\beta} 2^{-2k} \;<\; C_4 m^{2\beta} 2^{-2m} \;<\; C_5 |z-w|^2 \left(\log \frac{1}{|z-w|}\right)^{2\beta} . \quad (2.8)$$

Combining (2.6) with (2.7) and (2.8) yields (2.2) in this sub-case as well.

Third case. We have $z \in R_m$ for some m, but $w \in \mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} R_n$. In this case, join z to w by a straight line segment and let w' be the unique point at the boundary of R_m belonging to that line segment. We have, of course, $|z - w| = |z - w'| + |w' - w|$. Moreover, $f(w) = w$ and $f(w') = w'$, and since z and w' fall in the second case above, we have

$$\begin{aligned} |f(z) - f(w)| &\le |f(w) - f(w')| + |f(w') - f(z)| \\ &\le |w - w'| + C_6 |z - w'|^{\beta} \log \frac{1}{|z - w'|} . \end{aligned} \quad (2.9)$$

Since $\max\{|w - w'|, |w' - z|\} \le |z - w| < \delta$ and the function $t \mapsto t(\log(1/t))^{\beta}$ is increasing for $0 < t < e^{-\beta}$, we see that (2.9) implies (2.2). This finishes the proof of our theorem. □

For each positive integer n, by performing the same construction as above but just on the union of the squares $Q_n, Q_{n+1}, \ldots$ we also get the following corollary.

Corollary 2.3. *There exists a sequence of homeomorphisms $f_n : Q \to Q$ of the unit square such that, for each n,*

- *f_n has infinite topological entropy;*
- *f_n and f_n^{-1} are regular maps with finite distortion;*
- *For each $\alpha \in (0, 1)$, both f_n and f_n^{-1} are α-Hölder continuous;*
- *For each $1 \le p < \infty$, both f_n and f_n^{-1} are in the Sobolev class $W^{1,p}$.*

Moreover, $\lim_{n\to\infty} f_n^{\pm 1} = \mathrm{id}$*, where convergence is taken*

- *In the C^{α}-topology for each $\alpha \in [0, 1)$;*
- *In the $W^{1,p}$-topology for each $p \in [1, \infty)$.*

Remark 2.4. A similar construction can be performed in the area-preserving category. Namely, if you take a monotone decreasing sequence $1 = r_1 > r_2 > \ldots$ converging to zero, and for each k, let A_k denote the annulus in the plane given in polar coordinates by $\{(r, \theta) : r_k \ge r \ge r_{k+1}\}$. Let A_k^{in} and A_k^{out} denote the inner and outer boundary components. Take a (weak) monotone twist map f_k of the annulus A_k such that $h_{\mathrm{top}}(f_k) = \log k$ and f_k is a rotation in a neighbourhood of ∂A_k, such that $f_k|_{A_k^{\mathrm{in}}} = f_{k+1}|_{A_{k+1}^{\mathrm{out}}}$. Define

$$f(x) = \begin{cases} f_k(x) & x \in A_k \\ 0 & x = 0. \end{cases}$$

Then f is a homeomorphism. For appropriately chosen r_k, f is a bi-Hölder and bi-Sobolev homeomorphism.

3. Hölder mappings

3.1. Preliminaries

We recall some basic definitions and facts concerning Hölder maps. Much of what we state here is classical and proofs are left to the reader.

Hölder mappings between metric spaces

Let Ω and Ω^* be metric spaces. For each $\alpha \in (0, 1)$, let $C^\alpha(\Omega, \Omega^*)$ denote the space of all maps f from Ω to Ω^* satisfying the following *α-Hölder condition*

$$[f]_{\alpha,\Omega} \stackrel{\text{def}}{=} \sup_{x,y\in\Omega; x\neq y} \frac{d_{\Omega^*}(f(x), f(y))}{d_\Omega(x, y)^\alpha} < \infty\ .$$

When the domain of f is clear we will write $[f]_\alpha$ instead of $[f]_{\alpha,\Omega}$. In the case when $\Omega^* = \mathbb{R}^d$, the set $C^\alpha(\Omega, \mathbb{R}^d)$ has a linear structure and $[\cdot]_{\alpha,\Omega}$ defines a semi-norm[4], which we call the *C^α-semi-norm*. Consequently

$$\|f\|_{C^\alpha(\Omega,\mathbb{R}^d)} \stackrel{\text{def}}{=} \|f\|_{C^0(\Omega,\mathbb{R}^d)} + [f]_{\alpha,\Omega}$$

defines a complete norm on $C^\alpha(\Omega, \mathbb{R}^d)$. (Note that, in this case we will often consider the expression $[f - g]_{\alpha,\Omega}$ which obviously has no meaning unless Ω^* is contained in some linear space.)

Let $\mathcal{H}^\alpha(\Omega, \Omega^*)$ denote the space of invertible maps f from Ω to Ω^* for which $f \in C^\alpha(\Omega, \Omega^*)$ and $f^{-1} \in C^\alpha(\Omega^*, \Omega)$. The *bi-$\alpha$-Hölder constant* of f in $\mathcal{H}^\alpha(\Omega, \Omega^*)$ is the positive real number $\max([f]_{\alpha,\Omega}, [f^{-1}]_{\alpha,\Omega^*})$.

Hölder mappings between manifolds

On spaces more general than Euclidean domains, there are several ways to define Hölder continuity. A direct way is to endow the space with a distance function. However, this leads to difficulties in defining a topology on the space of Hölder maps. (Either we could introduce a distance function d on the range and consider $[d(f, g)]_{\alpha,\Omega}$ or, if $\delta_{f,\alpha}(x, y)$ denotes the α-Hölder difference quotient with respect to f, then we could consider $\sup_{x\neq y} |\delta_{f,\alpha}(x, y) - \delta_{g,\alpha}(x, y)|$. Only when the range is contained in a normed linear space and the natural distance function is used do these definitions coincide, with both expressions being equal to $[f - g]_{\alpha,\Omega}$.)

Instead, as we only consider the case when the underlying spaces are manifolds, we proceed with the following construction, which is analogous to the construction of the C^r-Whitney topology [23].

[4] This also induces a pseudo-distance which we will call the *C^α-pseudo-distance*.

Take smooth compact manifolds M and N. We say that $f \in C^0(M,N)$ is α-Hölder continuous if, for any pair of charts (U,φ) on M and (V,ψ) on N, the map $\psi \circ f \circ \varphi^{-1}$ is α-Hölder continuous on the Euclidean domain $\varphi(U \cap f^{-1}(V))$. (Note: in a given pair of charts, since any smooth metric is Lipschitz equivalent to the Euclidean metric, this definition will coincide with the definition above.) Let $C^\alpha(M,N)$ denote the set of α-Hölder continuous maps from M to N. Denote by $\mathcal{H}^\alpha(M,N)$ the subspace of homeomorphisms f such that $f \in C^\alpha(M,N)$ and $f^{-1} \in C^\alpha(N,M)$. When M and N coincide we denote this subspace by $\mathcal{H}^\alpha(M)$.

Spaces of bi-Hölder mappings

We define a topology on $\mathcal{H}^\alpha(M,N)$ as follows. Given $f \in \mathcal{H}^\alpha(M,N)$, take $\epsilon > 0$, charts (U,φ) on M and (V,ψ) on N, such that $f(U) \cap V \neq \emptyset$, and compact sets $K \subset U \cap f^{-1}(V)$, $L \subset f(U) \cap V$, which are the closure of open sets. Denote by $\mathcal{N}_{C^\alpha}(f;(U,\varphi),(V,\psi),K,L,\epsilon)$ the set of maps $g \in \mathcal{H}^\alpha(M,N)$ such that $g(K) \subseteq V$, $g^{-1}(L) \subseteq U$,

$$\|\psi \circ f \circ \varphi^{-1} - \psi \circ g \circ \varphi^{-1}\|_{C^\alpha(\varphi(K),\mathbb{R}^d)} < \epsilon\,,$$

and

$$\|\varphi \circ f^{-1} \circ \psi^{-1} - \varphi \circ g^{-1} \circ \psi^{-1}\|_{C^\alpha(\psi(L),\mathbb{R}^d)} < \epsilon\,.$$

The collection of all sets defined in this way form a subbasis for a topology on $\mathcal{H}^\alpha(M,N)$. We call it the *(weak) α-Hölder-Whitney topology*. As shorthand, we will also refer to it as the *(weak) C^α-Whitney topology*. Observe that the definition is analogous to the (weak) C^r-Whitney topology, the only difference being the choice of norm we use in each chart. As in the C^r-case (see, *e.g.*, [23, Chapter 2]) this topology is Hausdorff, and one can show the following.

Proposition 3.1. *For each $\alpha \in (0,1)$, and each pair of smooth compact manifolds M and N (possibly with boundary), the space $\mathcal{H}^\alpha(M,N)$, endowed with the (weak) C^α-Whitney topology, satisfies the Baire property.*

Remark 3.2. For a proof of this proposition, see [12]. The C^α-Whitney topology is an example of what we call generalized Whitney topologies. Such topologies are always locally metrizable, but may fail to be metrizable. Nevertheless, they always satisfy the Baire property. See [12] for more details.

For Lipschitz maps we can define all the objects above as in the Hölder case. However, for clarity we will use a different notation. Namely, denote by $C^{\mathrm{Lip}}(\Omega,\Omega^*)$ the space of all Lipschitz continuous maps from Ω to Ω^* and denote the Lipschitz constant by $[f]_{\mathrm{Lip},\Omega}$. Abusing notation slightly, we denote by $\mathcal{H}^1(\Omega,\Omega^*)$ the subspace of bi-Lipschitz maps from Ω to Ω^*. The *bi-Lipschitz constant* of the bi-Lipschitz map f in $\mathcal{H}^1(\Omega,\Omega^*)$ is the positive real number $\max([f]_{\mathrm{Lip},\Omega},[f^{-1}]_{\mathrm{Lip},\Omega^*})$.

For manifolds M and N we may also define the *(weak) Lipschitz-Whitney topology* on $\mathcal{H}^1(M,N)$, the space of bi-Lipschitz homeomorphisms from M to N, as in the Hölder case. For $0 \le \alpha < \beta \le 1$, let $\mathcal{H}^\beta_\alpha(M,N)$ denote the closures of $\mathcal{H}^\beta(M,N)$ with respect to the C^α-Whitney topology.

Remark 3.3. As previously mentioned, the C^α-Whitney topology does not require the existence of a distance function on the manifold M. However, we fix now, once and for all, a distance function d_M on M. This is merely to simplify notation in the construction of open sets, etc. In particular, our results do not depend on this metric.

Basic properties of Hölder mappings

In the remainder of this subsection we collect the following straightforward, though useful, results.

Lemma 3.4 (Hölder Arzelà-Ascoli principle). *For $\alpha \in (0,1)$, and $\beta \in (\alpha, 1)$ or $\beta = \mathrm{Lip}$, the space $C^\beta(\Omega, \mathbb{R}^d)$ embeds compactly into $C^\alpha(\Omega, \mathbb{R}^d)$.*

Proposition 3.5 (Hölder rescaling principle). *Let $0 \le \alpha < \beta \le 1$. Let Ω, Ω_0 and Ω_1 be bounded subsets of $\mathbb{R}^d$. Let $f \colon \Omega \to \Omega$ be β-Hölder continuous. Let $\phi_0 \colon \Omega \to \Omega_0$ and $\phi_1 \colon \Omega \to \Omega_1$ be bi-Lipschitz continuous bijections. Let $g = \phi_1 \circ f \circ \phi_0^{-1} \colon \Omega_0 \to \Omega_1$. Then*

$$[g]_\alpha \le [\phi_1]_{\mathrm{Lip}}[f]_\beta[\phi_0^{-1}]_{\mathrm{Lip}}^\beta \mathrm{diam}(\Omega_0)^{\beta-\alpha} .$$

Observe that the following Gluing Principles allow us to show that Hölder maps constructed by gluing with charts are Hölder in the more usual sense, when the manifold is endowed with a smooth metric.

Proposition 3.6 (First Hölder gluing principle). *For $\alpha \in (0,1)$ the following holds. Let $\Omega \subset \mathbb{R}^d$ be a connected bounded open domain. Let $\Omega_1, \Omega_2 \subset \Omega$ be disjoint subdomains such that $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}$. Let $f_1 \in C^\alpha\left(\overline{\Omega}_1, \mathbb{R}^d\right)$ and $f_2 \in C^\alpha\left(\overline{\Omega}_2, \mathbb{R}^d\right)$ have the property that they extend to a continuous function f on Ω. Then f is α-Hölder continuous. In fact,*

$$[f]_\alpha \le C \max\{[f_1]_\alpha, [f_2]_\alpha\} ,$$

where $C > 0$ is a constant depending upon α, Ω_1 and Ω_2 only.

We will say that a collection of pairwise disjoint bounded open subsets $\Omega_1, \Omega_2, \ldots$ of a metric space (Ω, d) are *κ-well-positioned* if

$$\frac{\max_m \mathrm{diam}(\Omega_m)}{\min_{i<j} \mathrm{dist}_H(\Omega_i, \Omega_j)} \le \kappa . \tag{3.1}$$

Proposition 3.7 (Second Hölder gluing principle). *For $\alpha \in (0,1)$ the following holds. Let (Ω, d) and (Ω^*, d^*) be connected metric spaces. Take pairwise disjoint bounded open sets $\Omega_1, \Omega_2, \ldots, \Omega_n \subset \Omega$ which are κ-well positioned for some positive real number κ, and define $\Omega_0 = \Omega \setminus \bigcup_{1 \le m \le n} \Omega_m$. Let $f \in C^0(\Omega, \Omega^*)$ have restrictions $f|_{\Omega_0}$ and $f|_{\overline{\Omega}_m}$, $m = 1, 2, \ldots, n$, which are α-Hölder continuous. Then f is α-Hölder continuous on Ω and*

$$[f]_{\alpha,\Omega} \le K \max_{0 \le m \le n} [f]_{\alpha, \overline{\Omega}_m} ,$$

where K is a constant depending only upon α and κ.

A consequence of the Hölder Rescaling Principle (Proposition 3.5) is the following.

Lemma 3.8. *Let $0 \leq \alpha < \beta \leq 1$. Let (Ω, d) and (Ω^*, d^*) be compact metric spaces. For any $f \in \mathcal{H}_\alpha^\beta(\Omega, \Omega^*)$, $\phi \in \mathcal{H}^1(\Omega)$, and $\psi \in \mathcal{H}^1(\Omega^*)$, the map $\psi \circ f \circ \phi$ lies in $\mathcal{H}_\alpha^\beta(\Omega, \Omega^*)$.*

The Second Hölder gluing principle combines with Lemma 3.8 above to give the following.

Corollary 3.9. *Let $0 \leq \alpha < \beta \leq 1$. Let (Ω, d) and (Ω^*, d^*) be compact metric spaces and let $\Omega_1, \Omega_2, \ldots$ be pairwise disjoint open subsets of Ω. Take $f \in \mathcal{H}_\alpha^\beta(\Omega, \Omega^*)$ and, for $k = 1, 2, \ldots$, take homeomorphisms $\phi_k \in \mathcal{H}^1(\Omega)$, supported in Ω_k. Define*

$$g = \begin{cases} f \circ \phi_k & \text{in } \Omega_k \\ f & \text{in } \Omega \setminus \bigcup_k \Omega_k. \end{cases}$$

Then g lies in $\mathcal{H}_\alpha^\beta(\Omega, \Omega^*)$.

Remark 3.10. In Corollary 3.9 if, instead of pre-composition by bi-Lipschitz mappings with compact pairwise disjoint supports, one considers post-composition, then the equivalent statement is also valid.

3.2. The Hölder closing lemma

In this section we will consider spaces of homeomorphisms on compact manifolds of dimension greater than one. We prove an analogue of Pugh's C^1-Closing Lemma [32,33] in a subspace of bi-Hölder maps.

Recall that, given a continuous self-map f of a topological space X, a point x in X is *non-wandering* if for all neighbourhoods U of x there exists some positive integer n for which $f^n(U) \cap U \neq \emptyset$.

Theorem 3.11 (Hölder closing lemma). *Let M be a smooth compact manifold. For $0 \leq \alpha < \beta \leq 1$, the following holds: Take $f \in \mathcal{H}_\alpha^\beta(M)$ and let y be a non-wandering point of f. For each neighbourhood W of y in M and each neighbourhood $\mathcal{N}$ of f in $\mathcal{H}_\alpha^\beta(M)$ there exists g in $\mathcal{N}$ and a point x in W such that x is a periodic point of the map g.*

Remark 3.12. It suffices to show that such a map g exists in any finite intersection of sub-basic sets of the form $\mathcal{N}_{C^\alpha}(f; (U, \varphi), (V, \psi), K, L, \epsilon)$, since such neighbourhoods form a local basis about f, *i.e.*, any neighbourhood of f will contain such a finite intersection.

To prove Theorem 3.11 we need the following preparatory lemma.

Lemma 3.13. *For each non-wandering point y and each sufficiently small, positive real number η the following holds: there exists a point x in $B(y, \eta)$ and a positive integer k such that $f^k(x)$ also lies in $B(y, \eta)$ and, for all $j = 1, 2, \ldots, k-1$,*

$$f^j(x) \notin B\left(x, \frac{3}{4}\rho\right) \cup B\left(f^k(x), \frac{3}{4}\rho\right),$$

where $\rho = d_M(x, f^k(x))$.

Remark 3.14. As was pointed out to us by Charles Pugh, this is, in essence, the *Fundamental Lemma* given in his paper [32]. However, we include this version here for completeness.

Proof of Lemma 3.13. As y is a non-wandering point, there exists a point z in $B(y, \frac{\eta}{10})$ such that $f^{m_1}(z)$ also lies in $B(y, \frac{\eta}{10})$ for some positive integer m_1. Let $z_m = f^m(z)$ for each integer m and denote the orbit segment $\{z_0, z_1, \dots, z_{m_1}\}$ by O. Also let $m_0 = 0$.

Let $\rho_1 = d_M(z_{m_1}, z_{m_0})$. If

$$\left(B\left(z_{m_0}, \frac{3}{4}\rho_1\right) \cup B\left(z_{m_1}, \frac{3}{4}\rho_1\right)\right) \cap O = \{z_{m_0}, z_{m_1}\}$$

holds, then we are done. Otherwise there exists a point z_{m_2} in the orbit segment O, with m_2 different from m_0 and m_1, such that $d_M(z_{m_2}, z_{m_1}) < \frac{3}{4}d_M(z_{m_1}, z_{m_0})$, say. Let $\rho_2 = d_M(z_{m_2}, z_{m_1})$. If

$$\left(B\left(z_{m_1}, \frac{3}{4}\rho_2\right) \cup B\left(z_{m_2}, \frac{3}{4}\rho_2\right)\right) \cap O = \{z_{m_1}, z_{m_2}\}$$

holds, then we are done. Otherwise there exists a point z_{m_3} in the orbit segment O, with m_3 different from m_1 and m_2, such that $d_M(z_{m_3}, z_{m_2}) < \frac{3}{4}d_M(z_{m_2}, z_{m_1})$, say, etc.

Continuing in this way, we move from one pair of points, z_{m_n} and $z_{m_{n-1}}$, to the next, $z_{m_{n+1}}$ and z_{m_n}. Since there are only finitely many points in the orbit segment O, and since the distance between pairs decreases at least geometrically (which implies that $z_{m_a} \neq z_{m_b}$ for $a \neq b$), it follows that this process must terminate. Hence there are points z_{m_N} and $z_{m_{N-1}}$ in the orbit segment O such that

$$\left(B\left(z_{m_N}, \frac{3}{4}\rho_N\right) \cup B\left(z_{m_{N-1}}, \frac{3}{4}\rho_N\right)\right) \cap O = \{z_{m_N}, z_{m_{N-1}}\},$$

where $\rho_N = d_M(z_{m_N}, z_{m_{N-1}})$. Moreover, as distances between subsequent pairs of points decreases at least geometrically, the distance between the initial point $z_0 = z_{m_0}$ and the terminal point z_{m_N} satisfies the following upper bound

$$d_M(z_0, z_{m_N}) \leq \sum_{n=0}^{N} d_M(z_{m_{n+1}}, z_{m_n}) \leq \sum_{n=0}^{N} \left(\frac{3}{4}\right)^n d_M(z_{m_0}, z_{m_1}) = \frac{4}{5}\eta\,.$$

Consequently the points z_{m_N} and $z_{m_{N-1}}$ also lie in $B(y, \eta)$. Consider the case when $m_N < m_{N-1}$. Then setting $x = z_{m_N}$ and $k = m_{N-1} - m_N$, so that $f^k(x) = z_{m_{N-1}}$, we find that the point x and integer k satisfy the conclusion of the lemma. The case when $m_{N-1} < m_N$ is similar. Hence the lemma is shown. □

We now proceed with the proof of the Hölder Closing Lemma (Theorem 3.11).

Proof of the Hölder Closing Lemma. We will prove the theorem in the case $\beta = 1$, *i.e.*, for maps in the C^α-closure of the space of bi-Lipschitz homeomorphisms. The general case follows analogously as the only property being used here is that the map f satisfies the *little Hölder condition*, *i.e.*, for each $x \in M$, $[f]_{\alpha, B(x,r)} = o(r)$.

Setup: Following Remark 3.12, it suffices to construct the perturbation g so that it lies in the intersection $\mathcal{N}$ of a finite collection of sub-basic sets of the form

$$\mathcal{N}_{C^\alpha}(f; (U_n, \varphi_n), (V_n, \psi_n), K_n, L_n, \epsilon_n) \tag{3.2}$$

as defined in Section 3.1. By adding to the collection of sub-basic sets, if necessary, we may assume that y is contained in $U_0 \cap f^{-1}(V_0)$. Take compact neighbourhoods U_n of K_n in $U_n \cap f^{-1}(V_n)$ and V_n of L_n in $f(U_n) \cap V_n$. We will also take a compact neighbourhood W_0 in U_0 which contains y in its interior.

To simplify notation, for each index n define the map

$$f_n = \psi_n \circ f \circ \varphi_n^{-1} \colon \varphi_n(U_n \cap f^{-1}(V_n)) \longrightarrow \psi_n(f(U_n) \cap V_n)\ . \tag{3.3}$$

When considering a perturbation g of f we will also use the notation

$$g_n = \psi_n \circ g \circ \varphi_n^{-1} \colon \varphi_n(U_n \cap g^{-1}(V_n)) \longrightarrow \psi_n(g(U_n) \cap V_n)\ . \tag{3.4}$$

Fix a positive real number ϵ. This will denote the order of the size of the perturbation. Take a positive real number δ. This will denote the size of the support of the local perturbation. Take δ sufficiently small so that

(a) $B_M(y, \delta)$ is contained in $W \cap \mathsf{W}_0$;
(b) U_n contains a 2δ-neighbourhood of the compact set K_n, and V_n contains a 2δ-neighbourhood of compact set L_n;
(c) $\max\left\{[f_n]_{\alpha, \varphi_n(B(y,\delta))}, [f_n^{-1}]_{\alpha, f_n \circ \varphi_n(B(y,\delta))}\right\} \le \epsilon$.

The neighbourhood $W \cap \mathsf{W}_0$ will contain the support of our perturbation. However, we also need to control the size of the perturbation in charts other than (U_0, φ_0). (This explains why we consider $W \cap \mathsf{W}_0$ and not just the open set W.) Therefore we will also assume that, for any index n,

$$\max\left\{[\varphi_n \circ \varphi_0^{-1}]_{\mathrm{Lip}, \varphi_0(\mathsf{W}_0 \cap \mathsf{U}_n)},\ [\varphi_0 \circ \varphi_n^{-1}]_{\mathrm{Lip}, \varphi_n(\mathsf{W}_0 \cap \mathsf{U}_n)}\right\} < c_1\ . \tag{3.5}$$

Construction of the perturbation. Given a point x^0 in M, for each integer k let $x^k = f^k(x^0)$. For each index n, let $x_n^k = \varphi_n(x^k)$, whenever it is defined. As y is a non-wandering point of f, a consequence of Lemma 3.13, is the following.

Claim 3.15. There exists a positive real number c with the following property: for each sufficiently small positive real number δ there exists a point x^0 in M and a positive integer k_0 such that

(1) $x^0, x^{k_0} \in B(y, \delta)$;

(2) Setting $r_0 = |x_0^0 - x_0^{k_0}|$, for each integer k, where $0 < k < k_0$, either x_0^k is not defined or x_0^k is defined and $x_0^k \notin E(x_0^0, x_0^{k_0}; cr_0)$;
(3) $E(x_0^0, x_0^{k_0}; cr_0) \subset \varphi_0(B(y, \delta))$,

(Claim 3.15(3) follows by applying the Lemma 3.13 to a slightly smaller disk.) Define

$$E = E\big(x_0^0, x_0^{k_0}, cr_0\big) , \qquad E' = E\big(x_0^0, x_0^{k_0}; \tfrac{c}{2} r_0\big)$$

and let $E_M = \varphi_0^{-1}(E)$ and $E'_M = \varphi_0^{-1}(E')$. Applying Lemma A.3 to the neighbourhoods E' and E, there exists a diffeomorphism ϕ supported on E such that

$$\phi(x_0^{k_0}) = x_0^0 . \tag{3.6}$$

Moreover, there exists a positive real number c_1, independent of ϵ, such that

$$[\phi]_{\mathrm{Lip}} \le c_1 . \tag{3.7}$$

Define the self-map g on M by

$$g = \begin{cases} f \circ \varphi_0^{-1} \circ \phi \circ \varphi_0 & \text{in } E_M \\ f & \text{elsewhere.} \end{cases} \tag{3.8}$$

Since ϕ_0 is supported in E, it is clear that g is a homeomorphism. In fact, Corollary 3.9 implies that the map g lies in $\mathcal{H}_\alpha^\beta(M)$. By equality (3.6), and since $x^k \notin E_M$ for $0 < k < k_0$, we also know that

$$g^{k_0}(x^{k_0}) = f^{k_0} \circ \varphi_0^{-1} \circ \phi \circ \varphi_0(x^{k_0}) = x^{k_0} .$$

Thus g possesses a periodic point in the neighbourhood W. Below it will be important to observe that, for each index n, we also have the expression

$$g_n = \begin{cases} f_n \circ \phi_n & \text{in } \varphi_n(E_M) \\ f_n & \text{elsewhere.} \end{cases} \tag{3.9}$$

where

$$\phi_n = \varphi_n \circ \varphi_0^{-1} \circ \phi \circ \varphi_0 \circ \varphi_n^{-1} . \tag{3.10}$$

Size of the perturbation. It remains to estimate the C^α-pseudo-distance between f and g corresponding to each of the sub-basic sets. Fix an index n. First, we must estimate $[f_n - g_n]_{\alpha,\varphi_n(K_n \cap E_M)}$. If K_n and E_M are disjoint, there is nothing to show. Otherwise, by (b) above, E_M is contained in U_n. By the triangle inequality

$$[f_n - g_n]_{\alpha,\varphi_n(K_n\cap E_M)} \le [f_n - g_n]_{\alpha,\varphi_n(E_M)} \le [f_n]_{\alpha,\varphi_n(E_M)} + [g_n]_{\alpha,\varphi_n(E_M)} . \tag{3.11}$$

By the expression (3.9) and the observation that $\phi_n \circ \varphi_n(E_M) = \varphi_n(E_M)$, the Hölder Rescaling Principle (Proposition 3.5) gives

$$[g_n]_{\alpha,\varphi_n(E_M)} \leq [f_n]_{\alpha,\varphi_n(E_M)}[\phi_n]_{\mathrm{Lip},\varphi_n(E_M)} . \tag{3.12}$$

Next, applying the Hölder Rescaling Principle (Proposition 3.5) to the expression (3.10), after observing that $\phi(E) = E$, gives

$$[\phi_n]_{\mathrm{Lip},\varphi_n(E_M)} \leq [\varphi_n \circ \varphi_0^{-1}]_{\mathrm{Lip},\varphi_0(E_M)}[\phi]_{\mathrm{Lip},E}[\varphi_0 \circ \varphi_n^{-1}]_{\mathrm{Lip},\varphi_n(E_M)} .$$

By (3.7) and (3.5), which we apply as E_M is contained in $\mathsf{W}_0 \cap \mathsf{U}_n$, this implies that $[\phi_n]_{\mathrm{Lip},\varphi_n(E_M)}$ is bounded from above independently of ϵ. By (c), together with (3.11) and (3.12), this implies that there exists a positive real number c_3, independent of ϵ, such that

$$[f_n - g_n]_{\mathrm{Lip},\varphi_n(K_n \cap E_M)} \leq c_3\epsilon .$$

Applying the First Hölder gluing principle (Proposition 3.6), there exists a positive real number c_4, also independent of ϵ, for which

$$[f_n - g_n]_{\mathrm{Lip},\varphi_n(K_n)} \leq c_4\epsilon .$$

Since the C^0-distance between f_n and g_n can be made arbitrarily small by shrinking the support of ϕ, the size of the perturbation ϵ may be chosen so that

$$\|f_n - g_n\|_{C^\alpha(\varphi_n(K_n),\mathbb{R}^d)} < \epsilon_n .$$

The same argument applied to the inverse mappings shows that, after shrinking ϵ is necessary, we also have

$$\|f_n - g_n\|_{C^\alpha(\psi_n(L_n),\mathbb{R}^d)} < \epsilon_n .$$

Thus, taking the minimum of all suitable ϵ over all indices n, of which there are finitely many, the resulting map g will lie in the common intersection of all the sub-basic sets above. Thus the theorem is shown. ☐

Remark 3.16. The reader may wonder why the Closing Lemma is much simpler in the Hölder category than in the C^1 category. While in both cases perturbations may be made by pre- or post-composing by diffeomorphisms supported on a small neighbourhood, by shrinking the neighbourhood and conjugating the perturbation by a dilation, this leaves the C^1-size of the perturbation unchanged, whereas, by the Hölder Rescaling Principle, the α-Hölder size of the perturbation can be made arbitrarily small.

The proof of the Hölder Closing Lemma (Theorem 3.11) above also yields the following corollary.

Corollary 3.17. *Let M be a smooth compact manifold. For $0 \leq \alpha < \beta \leq 1$, the following holds: Take $f \in \mathcal{H}_\alpha^\beta(M)$ and let x be a recurrent point of f. For each neighbourhood $\mathcal{N}$ of f in $\mathcal{H}_\alpha^\beta(M)$ there exists g in $\mathcal{N}$ and a positive integer k such that $f^k(x)$ is a periodic point of the map g.*

3.3. Genericity of infinite topological entropy for Hölder mappings

In this section we prove that infinite topological entropy is a generic property in the Hölder context. Theorem A will follow from Theorem 3.18 below. Before we can state this we need to introduce the following terminology. Call $B^{d-1} \times B^1$ the *standard solid cylinder* in $\mathbb{R}^d$. We call images under affine transformations of the standard solid cylinder *rigid solid cylinders* and homeomorphic images of the standard solid cylinder *topological solid cylinders*. Given a rigid solid cylinder C denote the axial length and the coaxial radius of C respectively by $\mathrm{len}(C)$ and $\mathrm{rad}(C)$. Given distinct points a and b in $\mathbb{R}^d$ and $r > 0$, denote by $C(a, b; r)$ the rigid solid cylinder in $\mathbb{R}^d$ whose axis is the line segment $[a, b]$ and whose co-axial radius is r.

Let C be a topological solid cylinder in $\mathbb{R}^d$ with disjoint marked boundary balls C^+ and C^-. We say that an embedding ϕ, from some domain containing C into $\mathbb{R}^d$, *maps C across the standard solid cylinder* $B^{d-1} \times B^1$ if the following properties are satisfied (see Figure 3.1):

(1) $\phi(C)$ intersects $B^{d-1} \times B^1$;
(2) $\phi(C)$ does not intersect $\partial\left(B^{d-1} \times B^1\right) \setminus \left(B^{d-1} \times \partial B^1\right)$;
(3) $\overline{\phi(C^-)}$ and $\overline{\phi(C^+)}$ do not intersect $\overline{B^{d-1} \times B^1}$;
(4) The connected component of $\phi(C) \setminus \left(B^{d-1} \times B^1\right)$ whose boundary contains $\phi(C^\pm)$ has closure intersecting $B^{d-1} \times \{\pm 1\}$ but not $B^{d-1} \times \{\mp 1\}$.

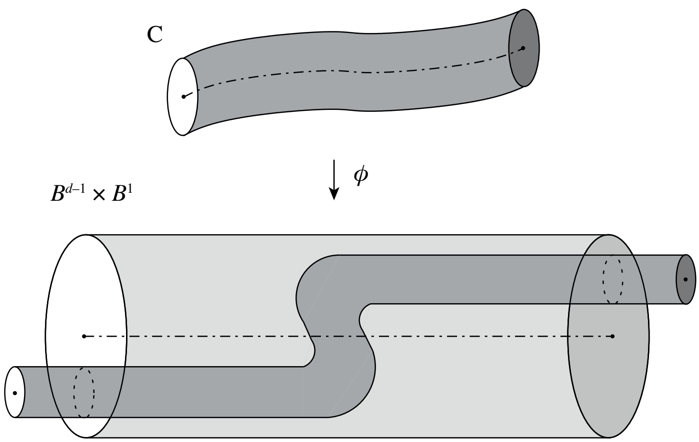

Figure 3.1. The cylinder C maps across the standard solid cylinder $B^{d-1} \times B^1$ via the embedding ϕ.

Given a topological solid cylinder C' in $\mathbb{R}^d$, which is the image of the standard solid cylinder under the homeomorphism ψ, we say that *ϕ maps C across C'* if $\psi^{-1} \circ \phi$ maps C across the standard solid cylinder.

Given a positive integer N, we say that an embedding f maps a (topological) solid cylinder C_0 across a (topological) solid cylinder C_1 *like an N-branched* *horseshoe* if there exist pairwise disjoint subcylinders $C_{0,1}, C_{0,2}, \ldots, C_{0,N}$ of C_0 such that f maps $C_{0,k}$ across C_1 for each $k = 1, 2, \ldots, N$. See Figure 3.2. For results concerning solid cylinders used in this section see Appendix B. Theorem A, stated in Section 1.2, follows directly from the following result.

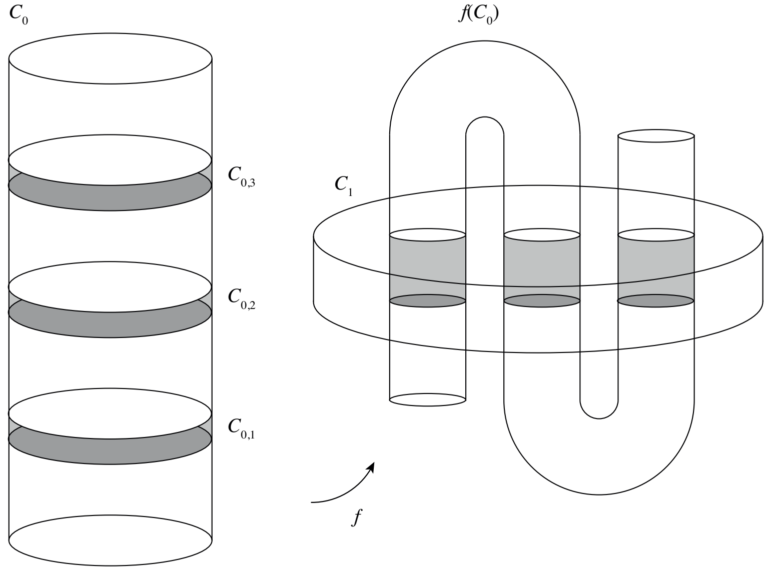

Figure 3.2. The embedding f maps the cylinder C_0 across the cylinder C_1 like a 3-branched horseshoe.

Theorem 3.18. *Let $0 \leq \alpha < 1$. Assume that $f \in \mathcal{H}^1_\alpha(M)$. For each neighbourhood $\mathcal{N}$ of f in $\mathcal{H}^1_\alpha(M)$ and each positive integer N there exists $g \in \mathcal{H}^1_\alpha(M)$ such that:*

(i) *$g \in \mathcal{N}$;*
(ii) *there exists a positive integer k_0, a topological solid cylinder S in M and solid sub-cylinders $S_1, S_2, \ldots, S_{N^{k_0}}$ such that g^{k_0} maps S_j across S for $j = 1, 2, \ldots, N^{k_0}$.*

The second property implies that $h_{\mathrm{top}}(g) \geq \log N$ and that this property is satisfied in an open neighbourhood of g.

Proof. Before starting the proof let us describe the idea. Take a forward-recurrent orbit for f. Take a segment of this orbit, of length k_0 say, whose start- and end-points are sufficiently close. In pairwise disjoint neighbourhoods of each of the points in this orbit segment take a solid cylinder. Perturb f in each of these neighbourhoods so that the solid cylinder maps over the next solid cylinder like an N-branched horseshoe. Finally, 'close-up' the orbit of the horseshoe by mapping the solid cylinder at the end of the the orbit segment across the solid cylinder prescribed at the start of the orbit segment. Observe that if h_0 maps the solid cylinder C_0 across C_1 and if h_1 maps C_1 across C_2 then $h_1 \circ h_0$ maps C_0 across C_1. Thus property (ii) will be satisfied. The discussion below will therefore focus on showing that (i) is satisfied.

Setup. Since $f \in \mathcal{H}^1_\alpha(M)$ we may assume, by making an arbitrarily small perturbation if necessary, that f is bi-Lipschitz. As in the proof of the Hölder Closing Lemma (Theorem 3.11) it suffices to consider the case when the neighbourhood $\mathcal{N}$ is a finite intersection of sub-basic sets of the form

$$\mathcal{N}_{C^\alpha}(f; (U_n, \varphi_n), (V_n, \psi_n), K_n, L_n, \epsilon_n)$$

as defined in Section 3.1. By adding to the collection of sub-basic sets if necessary, we may assume that the collections of neighbourhoods $\{U_n\}$ and $\{V_n\}$ both form open covers of M. For each index n, fix a compact neighbourhood U_n of K_n in $U_n \cap f^{-1}(V_n)$, and a compact neighbourhood V_n of L_n in $f(U_n) \cap V_n$. Let

$$f_n = \psi_n \circ f \circ \varphi_n^{-1} \colon \varphi_n(U_n \cap f^{-1}(V_n)) \longrightarrow \psi_n(f(U_n) \cap V_n) . \tag{3.13}$$

Below we will construct a perturbation g of f. When considering the C^α-pseudo-distance between f and g corresponding to each sub-basic set we will use the notation

$$g_n = \psi_n \circ g \circ \varphi_n^{-1} \colon \varphi_n(U_n \cap g^{-1}(V_n)) \longrightarrow \psi_n(g(U_n) \cap V_n) . \tag{3.14}$$

The map g will be constructed from a finite sequence of local perturbations of f. These perturbations will be in charts. Rather than introduce another set of charts we will use the collection $\{(U_n, \varphi_n)\}$. This is merely to simplify notation, and note that any other set of charts covering M could be used just as well. However, we will need to estimate these perturbations in each pair of charts corresponding to a sub-basic set. To facilitate this, for each index n, take a compact neighbourhood W_n contained in U_n, with the property that $\{\mathsf{W}_n\}$ covers M. The perturbations will be supported in these sets. Since there are only finitely many charts under consideration, all of which are smooth, and all the sets W_n, V_n and U_n are compact, there exists a positive real number c_1 with the property that for any chart (U_m, φ_m) and for each index n,

$$\max\left\{\left[\varphi_m \circ \psi_n^{-1}\right]_{\mathrm{Lip}, \psi_n(\mathsf{W}_m \cap f \mathsf{U}_n)}, \left[\psi_n \circ \varphi_m^{-1}\right]_{\mathrm{Lip}, \varphi_m(\mathsf{W}_m \cap f \mathsf{U}_n)}\right\} \leq c_1 \tag{3.15}$$

and

$$\max\left\{\left[\varphi_m \circ \varphi_n^{-1}\right]_{\mathrm{Lip}, \varphi_n(\mathsf{W}_m \cap \mathsf{U}_n)}, \left[\varphi_n \circ \varphi_m^{-1}\right]_{\mathrm{Lip}, \varphi_m(\mathsf{W}_m \cap \mathsf{U}_n)}\right\} \leq c_1 , \tag{3.16}$$

and similarly for the sets V_n (with f replaced appropriately by f^{-1}).

Fix a positive real number ϵ. This will be the order of the size of the perturbation. Let δ be a positive real number. This will denote the size of the support of the perturbation. Take δ sufficiently small so that:

(a) δ is less than the Lebesgue number of the common refinement of the finite open covers $\{U_n\}$ and $\{V_n\}$. Thus any ball of radius δ or less lies in some set of the form $U_m \cap V_n$;
(b) U_n contains a 2δ-neighbourhood of K_n, and V_n contains a 2δ-neighbourhood of L_n. Importantly, this implies that, given x in M, either $B_M(x,\delta) \cap K_n = \emptyset$ or $B_M(x,\delta) \subset \mathsf{U}_n$, and similarly for L_n and V_n;
(c) For each index n, whenever x lies in a δ-neighbourhood of K_n,

$$\max\left\{[f_n]_{\alpha,\overline{\varphi_n(B_M(x,\delta))}},\ \left[f_n^{-1}\right]_{\alpha,\overline{f_n\circ\varphi_n(B_M(x,\delta))}}\right\} < \epsilon$$

and similarly when x lies in δ-neighbourhood of L_n. (This is possible since, as f and f^{-1} are bi-Lipschitz, they are both little α-Hölder continuous. As all charts are smooth, it follows that for each index n, the maps f_n and f_n^{-1} are also both little α-Hölder continuous.)

Support of the perturbation. We will also need the following notation. Given a point x^0 in M and an integer k, let $x^k = f^k(x^0)$. For any index n let $x_n^k = \varphi_n(x^k)$, whenever this is defined. A slight variation of Claim 1 in the proof of the Hölder Closing Lemma gives us the following.

Claim 3.19. There exist positive real numbers c and κ with the following property: For each sufficiently small positive real number δ, there exists a point x^0 in M, a positive integer k_0, and a chart (U_0, φ_0) such that:

(1) x^0 and x^{k_0} lie in U_0;
(2) If $r_0 = |x_0^0 - x_0^{k_0}|$ then

$$E(x_0^0, x_0^{k_0}; c\cdot r_0) \subset \varphi_0(B_M(x^0,\delta)) \subset \varphi_0(\mathsf{W}_0);$$

(3) $x_0^k \notin E(x_0^0, x_0^{k_0}; c\cdot r_0)$ for $0 < k < k_0$.

Also, there exists a positive real number $r_1 < cr_0$ such that for each integer k satisfying $0 < k < k_0$, there is a chart, denoted by (U_k, φ_k), for which:

(4) $B(x_k^k, r_1) \subset \varphi_k(B_M(x^k,\delta)) \subset \varphi_k(\mathsf{W}_k)$;
(5) The following sets are pairwise disjoint

$$\varphi_0^{-1}(E(x_0^0, x_0^{k_0}; c\cdot r_0)), \varphi_1^{-1}(B(x^1, r_1)), \ldots, \varphi_{k_0-1}^{-1}(B(x^{k_0-1}, r_1))$$

and, in fact, in any chart (U_n, φ_n) these sets are κ-well positioned, *i.e.*, satisfy inequality (3.1).

To simplify notation, set

$$E = E(x_0^0, x_0^{k_0}; c\cdot r_0) \qquad \text{and} \qquad E_M = \varphi_0^{-1}(E)$$

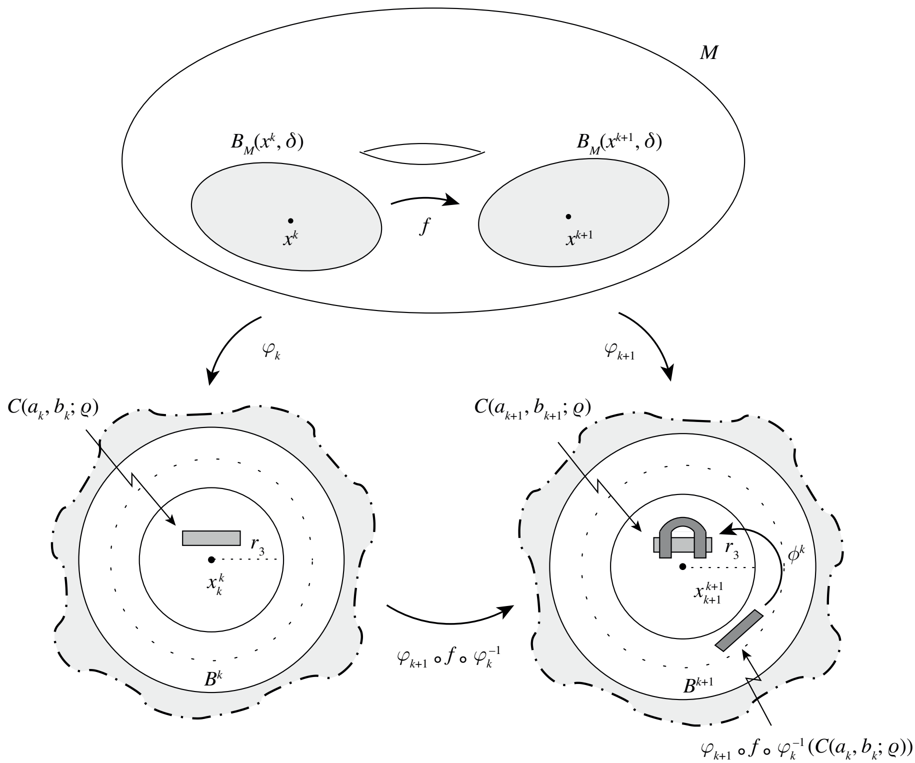

Figure 3.3. The map g_k is constructed as the composition of $\phi^k \circ \varphi_{k+1} \circ f \circ \varphi_k^{-1}$ and maps the cylinder $C(a_k, b_k; \varrho)$ across $C(a_{k+1}, b_{k+1}; \varrho)$ like an N-branched horseshoe.

and, for $0 \le k \le k_0$,

$$B^k = B(x_k^k, r_1) \qquad \text{and} \qquad B_M^k = \varphi_k^{-1}(B^k)\ .$$

By shrinking r_0 if necessary we may thus assume that each of the sets E_M, $f(E_M)$, B_M^k and $f^{-1}(B_M^k)$ are contained in a ball of radius δ or less.

Construction of the perturbation. For the choice of r_1 given above we now take positive real numbers r_2 and r_3 as in the statement of Corollary B.4. Take a positive real number ϱ and, for each $0 \le k \le k_0$, points a_k and b_k in $\mathbb{R}^d$, so that the rigid solid cylinder $C(a_k, b_k; \varrho)$ in $\mathbb{R}^d$ satisfies the following properties:

- (d) $C(a_k, b_k; \varrho)$ is contained in $B(x_k^k, r_3)$;
- (e) The $C(a_k, b_k; \varrho)$ are isometric to one another;
- (f) For each k, $C(a_k, b_k; \varrho)$ and $C(a_{k+1}, b_{k+1}; \varrho)$ satisfy the properties (i)–(iii) of Corollary B.4, where addition in the lower index is taken modulo k_0.

(Since the rigid solid cylinders are isometric, property (ii) of Corollary B.4 is automatically satisfied.) Since f is bi-Lipschitz we may now apply Corollary B.4.

Thus, for $0 \le k < k_0$, there exists a C^1-smooth diffeomorphism ϕ^k supported in B^{k+1} such that

$$\phi^k \circ \varphi_{k+1} \circ f \circ \varphi_k^{-1} : \varphi_k(U_k \cap f^{-1}(U_{k+1})) \longrightarrow \varphi_{k+1}(f(U_k) \cap U_{k+1})$$

maps the solid cylinder $C(a_k, b_k; \varrho)$ across the solid cylinder $C(a_{k+1}, b_{k+1}; \varrho)$ as an N-branched horseshoe. (Here, addition in the lower index is taken modulo k_0.) Moreover, since the ratio of length to radius of each solid cylinder was chosen independently of δ, it follows that there exists a positive real number c_2, independent of δ, such that, for $0 \le k < k_0$,

$$\max\left\{\left[\phi^k\right]_{\mathrm{Lip}, B^{k+1}}, \left[(\phi^k)^{-1}\right]_{\mathrm{Lip}, B^{k+1}}\right\} \le c_2 N . \tag{3.17}$$

Applying Corollary A.8, there also exists a C^1-smooth diffeomorphism ϕ, supported in E, such that ϕ maps the solid cylinder $C(a_{k_0}, b_{k_0}; \varrho)$ across the solid cylinder $C(a_0, b_0; \varrho)$. Moreover, since the radius $c \cdot r_0$ and the length r_0 of E are comparable, with comparability constant independent of ϵ, there exists a positive real number c_3, also independent of ϵ, such that

$$\max\left\{[\phi]_{\mathrm{Lip}, E} , \left[\phi^{-1}\right]_{\mathrm{Lip}, E}\right\} \le c_3 . \tag{3.18}$$

We are now in a position to define the perturbation. Namely, define $g : M \to M$ by

$$g = \begin{cases} \varphi_1^{-1} \circ \phi^0 \circ \varphi_1 \circ f \circ \varphi_0^{-1} \circ \phi \circ \varphi_0 & \text{in } E_M \\ \varphi_{k+1}^{-1} \circ \phi^k \circ \varphi_{k+1} \circ f & \text{in } f^{-1}(B_M^{k+1}),\ 0 < k < k_0 \\ f & \text{elsewhere.} \end{cases} \tag{3.19}$$

Observe that g is a homeomorphism. By Corollary 3.9, the map g lies in $\mathcal{H}_\alpha^1(M)$. Thus, when g is given in charts as per (3.14) we therefore have, for each index n, the expression

$$g_n = \begin{cases} \phi_n^0 \circ f_n \circ \phi_n & \text{in } \varphi_n(E_M) \\ \phi_n^k \circ f_n & \text{in } \varphi_n(f^{-1} B_M^{k+1}),\ 0 < k < k_0 \\ f_n & \text{elsewhere,} \end{cases} \tag{3.20}$$

where

$$\phi_n^k = \psi_n \circ \varphi_{k+1}^{-1} \circ \phi^k \circ \varphi_{k+1} \circ \psi_n^{-1} \quad \text{and} \quad \phi_n = \varphi_n \circ \varphi_0^{-1} \circ \phi \circ \varphi_0 \circ \varphi_n^{-1} . \tag{3.21}$$

Size of the perturbation. Let us estimate the size of the C^α-pseudo-distances between f and g corresponding to each of the sub-basic sets. Fix an index n. We will bound from above

$$[f_n - g_n]_{\alpha, \varphi_n(K_n \cap E_M)} \quad \text{and} \quad [f_n - g_n]_{\alpha, \varphi_n(K_n \cap f^{-1} B_M^k)} .$$

Consider the first quantity. Since E_M lies in a ball of radius δ or less, (b) above implies that either $K_n \cap E_M$ is empty or E_M is contained in U_n. In the first case there is nothing to show. Otherwise, by the triangle inequality

$$[f_n - g_n]_{\alpha,\varphi_n(K_n \cap E_M)} \leq [f_n - g_n]_{\alpha,\varphi_n(E_M)} \leq [f_n]_{\alpha,\varphi_n(E_M)} + [g_n]_{\alpha,\varphi_n(E_M)} \, . \quad (3.22)$$

As the mapping ϕ is supported in the neighbourhood E, we have the equality $\phi_n \circ \varphi_n(E_M) = \varphi_n(E_M)$. From the definition (3.13) it follows that $f_n \circ \varphi_n = \psi_n \circ f$, so the Hölder Rescaling Principle (Proposition 3.5) applied to the expression (3.20) gives

$$[g_n]_{\alpha,\varphi_n(E_M)} \leq \left[\phi_n^0\right]_{\mathrm{Lip},\psi_n \circ f(E_M)} [f_n]_{\alpha,\varphi_n(E_M)} [\phi_n]^{\alpha}_{\mathrm{Lip},\varphi_n(E_M)} \, . \quad (3.23)$$

Consider the first and last factor on the right-hand side. Applying the Hölder Rescaling Principle (Proposition 3.5) again to the expression on the right-hand side in (3.21) and recalling that $E = \varphi_0(E_M)$, we find that

$$[\phi_n]_{\mathrm{Lip},\varphi_n(E_M)} \leq \left[\varphi_n \circ \varphi_0^{-1}\right]_{\mathrm{Lip},\varphi_0(E_M)} [\phi]_{\mathrm{Lip},E} \left[\varphi_0 \circ \varphi_n^{-1}\right]_{\mathrm{Lip},\varphi_n(E_M)} \, .$$

To this expression we apply inequality (3.18), and, since $E_M \subset \mathsf{U}_n \cap \mathsf{W}_0$, we may also apply inequality (3.16). Consequently $[\phi_n]_{\mathrm{Lip},\varphi_n(E_M)}$ is bounded from above by a constant independent of ϵ. Next, using the expression on the left-hand side in (3.21), the Hölder Rescaling Principle (Proposition 3.5) implies that

$$\left[\phi_n^0\right]_{\mathrm{Lip},\psi_n(B_M^1)} \leq \left[\psi_n \circ \varphi_1^{-1}\right]_{\mathrm{Lip},\varphi_1(B_M^1)} \left[\phi^0\right]_{\mathrm{Lip},B^1} \left[\varphi_1 \circ \psi_n^{-1}\right]_{\mathrm{Lip},\psi_n(B_M^1)} \, .$$

To this expression we apply inequality (3.17), and since $B_M^1 \subset f(\mathsf{U}_n) \cap \mathsf{W}_1$ we may apply inequality (3.15). Thus by the First Hölder gluing principle (Proposition 3.6) we find that $\left[\phi_n^0\right]_{\mathrm{Lip},\psi_n \circ f(E_M)}$ is bounded from above, also by a constant independent of ϵ. By the condition (c) above, this therefore implies, together with (3.22) and (3.23), that there exists a positive real number c_5, independent of ϵ, such that

$$[f_n - g_n]_{\alpha,\varphi_n(K_n \cap E_M)} \leq c_5 \epsilon \, .$$

The same argument shows that, after increasing c_5 by a factor independent of ϵ if necessary, that for $0 \leq k < k_0$,

$$[f_n - g_n]_{\alpha,\varphi_n(K_n \cap f^{-1} B_M^k)} \leq c_5 \epsilon \, .$$

Since $f_n = g_n$ on $\varphi_n(K_n) \setminus \varphi_n(E_M \cup \bigcup_k f^{-1}(B_M^k))$ the Second Hölder gluing principle (Proposition 3.7) together with Claim 2(5) implies that there exists a positive real number c_6, independent of ϵ, such that

$$[f_n - g_n]_{\alpha,\varphi_n(K_n)} \leq c_6 \epsilon \, .$$

Also notice that the C^0-distance between f and g can be made arbitrarily small provided that ϵ, and thus δ, is sufficiently small. Combining this observation with the preceding inequality we therefore find that, for ϵ sufficiently small,

$$\| f_n - g_n \|_{C^\alpha(\varphi_n(K_n),\mathbb{R}^d)} < \epsilon_n \ .$$

The same argument applied to the inverse mappings shows that, after shrinking ϵ if necessary, we also have the inequality

$$\| f_n^{-1} - g_n^{-1} \|_{C^\alpha(\psi_n(L_n),\mathbb{R}^d)} < \epsilon_n \ .$$

Thus, taking the minimum of all such δ and ϵ over each index n, the map g must lie in the common intersection of all the sub-basic sets given above. Hence the theorem is shown. □

Analogously to the homeomorphism case [44] we also get the following.

Corollary 3.20. *Let M be a compact manifold of dimension at least two. Let $0 \leq \alpha < 1$. A generic homeomorphism in $\mathcal{H}^1_\alpha(M)$ is not conjugate to any diffeomorphism (or any bi-Lipschitz homeomorphism).*

Recall that horseshoes possess (unique) measures of maximal entropy. Observe that in the proof of Theorem 3.18, the worst that can happen is that the recurrent point being used already lies in a horseshoe. However, as the perturbation being used is arbitrarily small, if the horseshoe of the original map has N branches, we may assume that the perturbed map has a horseshoe with at least N branches. Thus, considering all possible sums of these measures over all possible horseshoes, we get the following corollary.

Corollary 3.21. *Let M be a compact manifold of dimension at least two. Let $0 \leq \alpha < 1$. A generic homeomorphism in $\mathcal{H}^1_\alpha(M)$ has uncountably many measures of maximal entropy.*

Next recall the following. Let X be a compact topological space. Let $\mathcal{M}(X)$ denote the set of Borel probability measures on X. Given $f \in C^0(X, X)$, let $\mathcal{M}(X, f)$ denote the set of f-invariant Borel probability measures. For any $\phi \in C^0(X, \mathbb{R})$, let $P(f, \phi)$ denote the *pressure* of f wth respect to ϕ. Then the Variational Principle [43, Section 9] states that

$$P(f, \phi) = \sup_{\mu \in \mathcal{M}(X,f)} \left(h_\mu(f) + \int \phi \, d\mu \right) \ .$$

Recall that $\mu \in \mathcal{M}(X, f)$ is an *equilibrium state* for (f, ϕ) if

$$P(f, \phi) = h_\mu(f) + \int \phi \, d\mu \ .$$

Lemma 3.22. *If $h_{\mathrm{top}}(f) = +\infty$ then the set of equilibrium states of (f, ϕ) is independent of $\phi \in C^0(X, \mathbb{R})$.*

Proof. By [43, Section 9.2], $h_{\mathrm{top}}(f) = +\infty$ implies that $P(f, \phi) = +\infty$, for all $\phi \in C^0(X, \mathbb{R})$. But $P(f, \phi) = +\infty$ implies that any equilibrium state μ must satisfy either $h_\mu(f) = +\infty$, or $\int \phi \, d\mu = +\infty$. However, μ is a probability measure, so any continuous function ϕ satisfies $|\int \phi \, d\mu| \leq |\phi|_X |\int d\mu| = |\phi|_X < \infty$. Consequently, any equilibrium state μ must satisfy $h_\mu(f) = +\infty$. Conversely, any $\mu \in \mathcal{M}(X, f)$ with $h_\mu(f) = +\infty$ is an equilibrium state for any $\phi \in C^0(X, \mathbb{R})$. The result follows. □

Combining this with Theorem A we therefore get the following corollary.

Corollary 3.23. *Let M be a compact manifold of dimension at least two. Let $0 \leq \alpha < 1$. For a generic homeomorphism f in $\mathcal{H}^1_\alpha(M)$, the set of equilibrium states of (f, ϕ) is independent of $\phi \in C^0(M, \mathbb{R})$. In fact, generically the set of equilibrium states, for any $\phi \in C^0(M, \mathbb{R})$, coincides with the set of measures of maximal entropy.*

4. Sobolev mappings

4.1. Preliminaries

Let us recall some basic definitions and facts about Sobolev functions and maps. For details on the material here we strongly recommend [20,30,31] and [45]. Here and throughout, all open domains in Euclidean spaces will be assumed to have piecewise-smooth boundaries.

Sobolev functions

Let $\Omega \subseteq \mathbb{R}^d$ be open, and let $k \in \mathbb{N}$ and $1 \leq p < \infty$. Recall that a measurable function $u : \Omega \to \mathbb{R}$ is in the Sobolev class $W^{k,p}(\Omega)$ if u has distributional partial derivatives of all orders up to k and, for each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$ with $|\alpha| = \sum_{i=1}^d \alpha_i \leq k$, the corresponding distributional partial derivative $D^\alpha u$ belongs to $L^p(\Omega)$. The space $W^{k,p}(\Omega)$ is a Banach space under the norm

$$\|u\|_{k,p} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_p \, ,$$

where $\| \cdot \|_p$ denotes the standard L^p-norm in Ω. It is known that every Sobolev function u is absolutely continuous on lines (ACL), *i.e.*, its restriction to Lebesgue almost every straight line (parallel to some coordinate axis) is absolutely continuous. In fact, the measurable function u is of class $W^{1,p}$ if and only if u is L^p and ACL with distributional derivatives on almost every straight line (again parallel to some coordinate axis) lying in L^p. (See, *e.g.*, [31, Section 1.1.3] or [45, Theorem 2.1.4].) It is also known that u is differentiable Lebesgue almost everywhere in

Ω provided that $p > d$. (This was proved for $d = 2$ by Cesari [11] and for arbitrary d by Calderón [10].)

Sobolev maps

Let us consider a measurable map $f\colon \Omega \to \mathbb{R}^d$. We say that f is a *Sobolev map* in the class $W^{k,p}$ if, writing $f = (f_1, f_2, \ldots, f_d)$, each component $f_i \in W^{k,p}(\Omega)$. Note that such a map has a *formal* Jacobian matrix $Df(x) = (\partial_{x_j} f_i(x))_{1\le i,j\le d}$ defined at Lebesgue almost every point $x \in \Omega$.

The space of Sobolev maps in the class $W^{k,p}$, which we denote by $W^{k,p}(\Omega,\mathbb{R}^d)$, can be made into a Banach space in several equivalent ways. One natural way is to define, for $f \in W^{k,p}(\Omega, \mathbb{R}^d)$, its Sobolev norm by $\|f\|_{W^{k,p}(\Omega,\mathbb{R}^d)} = \sum_{i=1}^{d} \|f_i\|_{k,p}$, where f_i, $i = 1, \ldots, d$, are the components of f. With this norm $W^{k,p}(\Omega, \mathbb{R}^d)$ is a Banach space.

Continuous Sobolev maps

We are not interested in *all* Sobolev maps, only in those that are *continuous* up to the boundary. Let us write

$$\mathbb{W}^{k,p}\left(\Omega, \mathbb{R}^d\right) = W^{k,p}\left(\Omega, \mathbb{R}^d\right) \cap C^0\left(\overline{\Omega}, \mathbb{R}^d\right) .$$

We need a topology on this space. Rather than giving a general definition covering all cases, we restrict ourselves to the cases when $k = 1$ and $p \geq 1$ is arbitrary; these are the only cases that will be relevant in the present paper. We define a norm in $\mathbb{W}^{1,p}\left(\Omega, \mathbb{R}^d\right)$ as follows. First, given a $d \times d$ matrix $A = (a_{ij})$, we define its *norm* to be $|A| = \sum_{i,j=1}^{d} |a_{ij}|$. Given $f \in \mathbb{W}^{1,p}\left(\Omega, \mathbb{R}^d\right)$, let

$$\|f\|_{\mathbb{W}^{1,p}(\Omega,\mathbb{R}^d)} \;=\; \|f\|_{C^0(\Omega)} + \left(\int_\Omega |Df(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} .$$

This defines a norm, and with this norm $\mathbb{W}^{1,p}\left(\Omega, \mathbb{R}^d\right)$ is a Banach space.

Sobolev homeomorphisms

Let $f\colon \Omega \to f(\Omega) \subseteq \mathbb{R}^d$ be an orientation-preserving homeomorphism, continuous up to the boundary of Ω, and suppose that $f \in \mathbb{W}^{1,p}\left(\Omega, \mathbb{R}^d\right)$. Differentiability properties for homeomorphisms such as f, or more generally for open mappings, are better than for general mappings. More precisely, f is differentiable almost everywhere in Ω provided that $p > d - 1$. (As remarked above, such maps are ACL and the distributional derivatives on almost every line is locally L^p-integrable. By a theorem of Gehring and Lehto [17], ($d = 2$), and its extension by Väisälä [42], ($d > 2$), such maps are differentiable almost everywhere. In fact, for $d = 2$, Gehring and Lehto showed the result also holds when $p = 1$. However, for $d > 2$, Väisälä gave an example with $p = d - 1$ which is nowhere differentiable. This used an explicit construction of a nowhere differentiable function originally due to Serrin [37].) We denote by $J_f(x) = \det Df(x)$ the *Jacobian determinant* of f at x in Ω.

The *chain rule* for Sobolev maps does not always hold. For instance, for $d > 2$ there exist homeomorphisms $f\colon (0,1)^d \to (0,1)^d$ such that f and f^{-1} are of class $\mathbb{W}^{1,d-1}$ but both f and f^{-1} have zero Jacobian matrix at Lebesgue almost every point (see [14]). However, in the present paper, we will need the chain rule for the composition of two Sobolev homeomorphisms only in the case when one of them is a *diffeomorphism*. In this case, the following result is available.

Lemma 4.1 (Chain rule). *Let $U, V \subseteq \mathbb{R}^d$ be open domains. Let $f \in \mathbb{W}^{1,p}\left(U, \mathbb{R}^d\right)$ be a homeomorphism onto its image, and let $\phi\colon V \to \mathbb{R}^d$ be a C^1-diffeomorphism onto its image. Assume that either (a) $d > 2$ and $p > d-1$; or (b) $d = 2$ and $p \geq 1$. Then the following statements are true.*

(i) *If $\phi(V) \subseteq U$, then $f \circ \phi \in \mathbb{W}^{1,p}\left(V, \mathbb{R}^d\right)$, the composition $f \circ \phi$ is differentiable almost everywhere and*

$$D(f \circ \phi)(x) = Df(\phi(x))D\phi(x) \quad \textit{for Lebesgue a.e. } x \in V\ ;$$

(ii) *If $f(U) \subseteq V$, then $\phi \circ f \in \mathbb{W}^{1,p}\left(U, \mathbb{R}^d\right)$, the composition $\phi \circ f$ is differentiable almost everywhere and*

$$D(\phi \circ f)(x) = D\phi(f(x))D\phi(x) \quad \textit{for Lebesgue a.e. } x \in U\ . \tag{4.1}$$

Given a C^1-diffeomorphism ϕ it is known that both pre- and post-composition operators $f \mapsto f \circ \phi$ and $f \mapsto \phi \circ f$ map $\mathbb{W}^{1,p}$ to $\mathbb{W}^{1,p}$ (in the appropriate domains). See, for instance, [1]. For the proof of part (i), one combines the chain rule for Sobolev *functions* as stated, say, in [45, Theorem 2.2.2, p. 52] with the fact that $D(f \circ \phi)$ exists Lebesgue almost everywhere. For the proof of part (ii), note that the set of points x where *both* sides of (4.1) are defined has full measure; then one may write the first-order Taylor expressions for both f and $\phi \circ f$ at x in the direction v, substitute them into the expression $\phi \circ f(x + tv)$, and compare the resulting expressions (after reminding oneself that ϕ is differentiable *everywhere*).

Spaces of bi-Sobolev homeomorphisms

Let $\Omega, \Omega^* \subset \mathbb{R}^d$ be bounded open sets with piecewise-smooth boundary and let $1 \leq p, p^* < \infty$. Denote by $\mathcal{S}^{p,p^*}(\Omega, \Omega^*)$ the space of orientation-preserving homeomorphisms $f\colon \Omega \to \Omega^*$ such that $f \in \mathbb{W}^{1,p}\left(\Omega, \mathbb{R}^d\right)$, $f^{-1} \in \mathbb{W}^{1,p^*}\left(\Omega^*, \mathbb{R}^d\right)$, and f and f^{-1} extend continuously to the boundaries of Ω and Ω^* respectively. Provided that Ω and Ω^* are chosen so that $\mathcal{S}^{p,p^*}(\Omega, \Omega^*)$ is non-empty, $\mathcal{S}^{p,p^*}(\Omega, \Omega^*)$ is a complete metric space when endowed with the distance function

$$\rho(f, g) = \|f - g\|_{\mathbb{W}^{1,p}(\Omega,\mathbb{R}^d)} + \|f^{-1} - g^{-1}\|_{\mathbb{W}^{1,p^*}(\Omega^*,\mathbb{R}^d)}$$

for all $f, g \in \mathcal{S}^{p,p^*}(\Omega, \Omega^*)$. In particular, $\mathcal{S}^{p,p^*}(\Omega, \Omega^*)$ is a Baire space.

As in the Hölder case, there are several ways to define Sobolev classes on spaces more general than Euclidean domains. This can be done when the space is a manifold with a smooth Riemannian metric (or more generally a smooth connection), and it can also be done for embedded manifolds in $\mathbb{R}^n$ (see, *e.g.*, [38]). Instead, we will construct and use a topology analogous to that constructed in the Hölder case in Section 3.1.

First consider bi-Sobolev homeomorphisms between manifolds M and N. Let $1 \leq p, p^* < \infty$. Denote by $\mathbb{W}^{1,p}(M, N)$ the space of maps f from M to N such that, for any pair of charts (U, φ) on M and (V, ψ) on N, the map $\psi \circ f \circ \varphi$ is $\mathbb{W}^{1,p}$ on $\psi(U \cap f^{-1}(V))$. Let $\mathcal{S}^{p,p^*}(M, N)$ denote the space of homeomorphisms f from M to N such that $f \in \mathbb{W}^{1,p}(M, N)$ and $f^{-1} \in \mathbb{W}^{1,p^*}(N, M)$. In the case when M and N coincide we denote this space by $\mathcal{S}^{p,p^*}(M)$.

We define the *(weak)* (p, p^*)*-Sobolev-Whitney topology* on $\mathcal{S}^{p,p^*}(M, N)$ analogously to the (weak) C^α-Whitney topology constructed in Section 3.1. Namely we define sets $\mathcal{N}_{\mathbb{W}^{1,p},\mathbb{W}^{1,p^*}}(f; (U, \varphi), (V, \psi), K, L, \epsilon)$ in exactly the same way as we defined the sub-basic sets $\mathcal{N}_{C^\alpha}(f; (U, \varphi), (V, \psi), K, L, \epsilon)$, except that we replace the C^α-distance between maps and their inverses with the $\mathbb{W}^{1,p}$-distance between maps and $\mathbb{W}^{1,p^*}$-distance between their inverses. Observe that the compact sets K and L used in this construction are required to be the closure of open sets whose boundaries are piecewise-smooth. The collection of sets of the form

$$\mathcal{N}_{\mathbb{W}^{1,p},\mathbb{W}^{1,p^*}}(f; (U, \varphi), (V, \psi), K, L, \epsilon)$$

then forms a subbasis for a topology which we call the *(weak)* (p, p^*)*-Sobolev-Whitney topology*.

As in the Hölder case, the (p, p^*)-Sobolev-Whitney topology is Hausdorff and satisfies the following.

Proposition 4.2. *Let M and N be smooth compact manifolds of dimension d. For $d = 2$ and $1 \leq p, p^* < \infty$; or $d > 2$ and $d - 1 < p, p^* < \infty$, the space $\mathcal{S}^{p,p^*}(M, N)$, endowed with the weak (p, p^*)-Sobolev-Whitney topology, satisfies the Baire property.*

Remark 4.3. For a proof of this proposition, once again see [12]. The Sobolev-Whitney topology is another example of what we call generalized Whitney topologies. As we observed before (*cf.* Remark 3.2), such topologies are always locally metrizable, but may fail to be metrizable. They always satisfy the Baire property, however, as proved in [12].

Further comments

The following facts, concerning certain special classes of Sobolev homeomorphisms, are worth mentioning even though they will not be used in the present paper. The homeomorphism f is said to be a map of *finite distortion* if the quotient $K_f(x) = |Df(x)|^d / J_f(x)$ is finite almost everywhere in Ω. When $K = \|K_f\|_\infty < \infty$, we say that f is a K*-quasiconformal homeomorphism*. Thus a

K-quasiconformal homeomorphism satisfies the inequality $|Df(x)|^d \leq KJ_f(x)$ almost everywhere. An inequality in the opposite direction is possible for general homeomorphisms in $\mathbb{W}^{1,p}(\Omega, \mathbb{R}^d)$ for sufficiently large p, as shown by the following easy lemma.

Lemma 4.4. *If $f \in \mathbb{W}^{1,d}(\Omega, \mathbb{R}^d)$ is a homeomorphism, then the determinant Jacobian J_f belongs to $L^1(\Omega)$.*

Proof. Writing $f_{ij} = \partial_{x_j} f_i$ for the components of the matrix $Df(x)$, we have by definition of the determinant

$$J_f(x) \;=\; \det Df(x) \;=\; \sum_{\sigma \in S_d} (-1)^{\mathrm{sign}(\sigma)} f_{1\sigma(1)}(x) f_{2\sigma(1)}(x) \cdots f_{d\sigma(d)}(x)\ .$$

Taking absolute values on both sides and taking into account that $|f_{ij}(x)| \leq |Df(x)|$, we deduce that

$$|J_f(x)| \;\leq\; d!\,|Df(x)|^d\ . \tag{4.2}$$

Integrating both sides, we deduce that $\|J_f\|_{L^1(\Omega)} \;\leq\; d!\,\|Df\|^d_{L^d(\Omega)} < \infty$, and hence the result. □

We say that a homeomorphism f satisfies *Lusin's N-property* if f maps Lebesgue null-sets onto Lebesgue null-sets. Suppose we know that $Df(x)$ exists Lebesgue almost everywhere, that J_f is integrable, and that

$$\mu(f(E)) \leq \int_E J_f \, d\mu \tag{4.3}$$

for each measurable set E in the domain of f. Then, clearly, f has Lusin's N-property. It has been proved by Reshetnyak in [35] that every homeomorphism $f \in \mathbb{W}^{1,d}(\Omega, \mathbb{R}^d)$ has Lusin's N-property. Despite appearances, this non-trivial result does not follow directly from Lemma 4.4, since we don't know a priori that (4.3) holds.

4.2. The Sobolev closing lemma

As in Section 3.2, we consider spaces of homeomorphisms on smooth compact manifolds of dimension greater than one. Here we prove a version of Pugh's C^1-Closing Lemma for bi-Sobolev mappings.

Theorem 4.5 (Sobolev closing lemma). *Let M be a smooth compact manifold of dimension d. For $d = 2$ and $1 \leq p, p^* < \infty$; or $d > 2$ and $d - 1 < p, p^* < \infty$, the following holds: Take $f \in \mathcal{S}^{p,p^*}(M)$ and let y be a non-wandering point of f. For each neighbourhood W of y in M and each neighbourhood $\mathcal{N}$ of f in $\mathcal{S}^{p,p^*}(M)$ there exists g in $\mathcal{N}$ and a point x in W such that x is a periodic point of the map g.*

Proof. Our approach will be the same as in the proof of the Hölder Closing Lemma (Theorem 3.11). The first significant difference is that, given a finite collection of sub-basic sets for the (p, p^*)-Sobolev-Whitney topology,

$$\mathcal{N}_{\mathbb{W}^{1,p},\mathbb{W}^{1,p^*}}(f; (U_n, \varphi_n), (V_n, \psi_n), K_n, L_n, \varepsilon(n)),$$

rather than constructing a single perturbation and then showing it lies in each sub-basic set, we will construct a sequence of perturbations converging to our original map in the C^0-topology, such that the sequence eventually lands inside each of our (finitely many) sub-basic sets.

Setup. We adopt the notation of the proof of Theorem 3.11. In particular, f_n, U_n, V_n, W_0, etc. are as before. Given a sequence of perturbations g_m of f, we denote by $g_{m,n}$ the map g_m in the pair of charts (U_n, φ_n) and (V_n, ψ_n).

As mentioned above, we take a decreasing sequence ϵ_m of positive real numbers tending to zero, denoting the order of the size of the mth perturbation (instead of just ϵ as in the Hölder case). Similarly, we take a decreasing sequence δ_m of positive real numbers converging to zero, denoting the size of the support of the local perturbation (instead of just δ). We assume that all δ_m are chosen so that

(a) $B_M(y, \delta_m)$ is contained in $W \cap \mathsf{W}_0$;
(b) For each n, U_n contains a $2\delta_m$-neighbourhood of the compact set K_n, and V_n contains a $2\delta_m$-neighbourhood of the compact set L_n;
(c) For any n, given an arbitrary ball B in U_n of radius δ_m or less, the image $f_n \circ \varphi_n(B) = \psi_n \circ f(B)$ has diameter ϵ_m or less;
(d) For any n, given an arbitrary ball B in V_n of radius δ_m or less, the image $f_n^{-1} \circ \psi_n(B) = \varphi_n \circ f^{-1}(B)$ has diameter ϵ_m or less.

Observe that (c) and (d) are possible by compactness of U_n and V_n respectively.

Construction of the perturbation. For each m, the construction of the perturbation g_m is identical to the Hölder case. Namely, by Claim 3.15 in the proof of the Hölder Closing Lemma (Theorem 3.11), for each m there is a point x_m^0 and integer k_m such that x_m^0 and $x_m^{k_m} = f^{k_m}(x_m^0)$ lie in $B_M(y, \delta_m)$ and no other points in the orbit segment $x_m^0, x_m^1, \ldots, x_m^{k_m}$ lie in $B_M(y, \delta_m)$. We set E_m, E'_m, $E_{m,M}$, and $E'_{m,M}$ as before and define ϕ_m via Lemma A.3. Then we define

$$g_m = \begin{cases} f \circ \varphi_0^{-1} \circ \phi_m \circ \varphi_0 & \text{in } E_{m,M} \\ f & \text{elsewhere.} \end{cases}$$

As in the Hölder case, it will be important to note that, for each n,

$$g_{m,n} = \begin{cases} f_n \circ \phi_{m,n} & \text{in } \varphi_n(E_{m,M}) \\ f_n & \text{elsewhere.} \end{cases}$$

where $\phi_{m,n}$ denotes ϕ_m expressed in the chart (U_n, φ_n), *i.e.*,

$$\phi_{m,n} = \varphi_n \circ \varphi_0^{-1} \circ \phi_m \circ \varphi_0 \circ \varphi_n^{-1} .$$

It is clear that g_m is a homeomorphism. Since composition of a Sobolev map with a smooth map is again Sobolev, it also follows that g_m lies in $\mathcal{S}^{p,p^*}(M)$. By the same argument as in the Hölder case, $g_m^{k_m}(x_m^{k_m}) = x_m^{k_m}$. Thus it just remains to show that g_m will lie in $\mathcal{N}$ for m sufficiently large.

Size of the perturbation. For each n, it suffices to estimate the

(i) $\mathbb{W}^{1,p}$-pseudo-distance between f_n and $g_{m,n}$ on $\Omega_n = \varphi_n(K_n)$;
(ii) $\mathbb{W}^{1,p^*}$-pseudo-distance between f_n^{-1} and $g_{m,n}^{-1}$ on $\Omega_n^* = \psi_n(L_n)$.

Below we will construct a subsequence $m_1, m_2, \dots$ of the natural numbers such that

$$\lim_{j\to\infty} \|f_n - g_{m_j,n}\|_{\mathbb{W}^{1,p^*}(\Omega_n,\mathbb{R}^d)} = 0 = \lim_{j\to\infty} \|f_n^{-1} - g_{m_j,n}^{-1}\|_{\mathbb{W}^{1,p^*}(\Omega_n^*,\mathbb{R}^d)} .$$

Applying this inductively for each n, taking a subsequence at each step, will then give the result.

Consider (i). Observe that f and g_m only differ on $E_{m,M}$, and hence that the sets $f(E_{m,M})$ and $g_m(E_{m,M})$ agree. Since $E_{m,M}$ is contained in $B_M(y,\delta_m)$, *i.e.*, a ball of radius δ_m, by (b) above, either $E_{m,M}$ is contained in U_n, or $E_{m,M}$ is disjoint from K_n. In the second case there is nothing to prove. Thus we focus on the first case. Recall that

$$\|f_n - g_{m,n}\|_{\mathbb{W}^{1,p}(\Omega_n,\mathbb{R}^d)} = \|f_n - g_{m,n}\|_{C^0(\Omega_n)} + \left(\int_{\Omega_n} |Df_n - Dg_{m,n}|^p \, d\mu\right)^{\frac{1}{p}} .$$

As f and g_m differ only on $E_{m,N}$ it follows that f_n and $g_{m,n}$ differ only on $\varphi_n(E_{m,M})$. Since $E_{m,M}$ is contained in a ball of radius δ_m, it follows from (c) above that $f_m(\varphi_n(E_{m,M}))$, and hence $g_{m,n}(\varphi_n(E_{m,M}))$, are contained in some ball of radius ϵ_m. Consequently

$$\|f_n - g_{m,n}\|_{C^0(\Omega_n)} \le \epsilon_m .$$

It therefore suffices to show that there is some subsequence J of the natural numbers such that

$$\liminf_{J\ni m\to\infty} \int_{\Omega_n} |Df_n - Dg_{m,n}|^p \, d\mu = 0 . \tag{4.4}$$

In fact, it will be slightly easier to show this on the slightly larger set $\varphi_n(\mathsf{U}_n)$. Namely we will show that for some subsequence J of natural numbers

$$\liminf_{J\ni m\to\infty} \int_{\varphi_n(\mathsf{U}_n)} |Df_n - Dg_{m,n}|^p \, d\mu = 0 . \tag{4.5}$$

Obviously, (4.4) follows directly from (4.5). The chain rule (Lemma 4.1 (i)) gives

$$Dg_{m,n}(x) = Df_n(\phi_{m,n}(x)) D\phi_{m,n}(x)$$

for Lebesgue almost every $x \in \varphi_n(\mathsf{U}_n)$. Let $u_{ij}^{(n)}$, $v_{ij}^{(m,n)}$ and $w_{ij}^{(m,n)}$, $1 \le i, j \le d$, denote the entries of the matrices Df_n, $D\phi_{m,n}$ and $Dg_{m,n}$ so that, for Lebesgue almost every $x \in \varphi_n(\mathsf{U}_n)$

$$w_{ij}^{(m,n)}(x) = \sum_{1 \le k \le d} u_{ik}^{(n)}(\phi_{m,n}(x)) v_{kj}^{(m,n)}(x) .$$

Thus, to show that the integral in (4.5) can be made arbitrarily small we must show that for $1 \le i, j \le d$,

$$\lim_{J \ni m \to \infty} \left\| w_{ij}^{(m,n)} - u_{ij}^{(n)} \right\|_{L^p} = 0 . \tag{4.6}$$

Since the diameter of the support of $\phi_{m,n}$ tends to zero and $\|D\phi_{m,n}\|_{C^0}$ is uniformly bounded over m and n, we know that $\phi_{m,n} \to \mathrm{id}$ in $\varphi_n(\mathsf{U}_n)$ in measure, and $D\phi_{m,n} \to \mathrm{id}_{\mathbb{R}^d}$ in measure, both as $m \to \infty$. Thus $v_{kj}^{(m,n)} \to \delta_{kj}$ in measure as $m \to \infty$ (where δ_{kj} denotes the Kronecker delta). Hence $w_{ij}^{(m,n)} \to u_{ij}^{(n)}$ in measure as $m \to \infty$. By a well-known result in measure theory (see for instance [18]) this implies that there exists a subsequence J_1 of the natural numbers such that for $1 \le i, j \le d$, we have $w_{ij}^{(m,n)}(x) \to u_{ij}^{(n)}(x)$ as $J_1 \ni m \to \infty$, for Lebesgue almost every $x \in \varphi_n(\mathsf{U}_n)$. Now we use the following fact from measure theory (see [36, p. 76]), valid for arbitrary measure spaces with a positive measure:

Fact. Suppose $\sigma \in L^r$, $\sigma_m \in L^r$ *where* $1 < r < \infty$. *If* $\sigma_m(x) \to \sigma(x)$ *almost everywhere and* $\|\sigma_m\|_{L^r} \to \|\sigma\|_{L^r}$ *as* $m \to \infty$, *then* $\lim_{m\to\infty} \|\sigma_m - \sigma\|_{L^r} = 0$.

For each i and j, $1 \le i, j \le d$, we apply the above fact in the case $r = p$, $\sigma = u_{ij}^{(n)}$ and $\sigma_m = w_{ij}^{(m,n)}$, for $m \in J_1$. (As f_n and $g_{m,n}$ lie in $\mathbb{W}^{1,p}$ it follows that $w_{ij}^{(m,n)}$ and $u_{ij}^{(n)}$ lie in L^p for all i and j, $1 \le i, j \le d$.) We already know that $w_{ij}^{(m,n)}(x) \to u_{ij}^{(n)}(x)$ for Lebesgue almost every x along J_1. Hence we only need to check that $\|w_{ij}^{(m,n)}\|_{L^p} \to \|u_{ij}^{(n)}\|_{L^p}$ as $J_1 \ni m \to \infty$. It suffices to show that there exists a subsequence $J_2 \subseteq J_1$ for which

$$\lim_{J_2 \ni m \to \infty} \int_{\varphi_n(\mathsf{U}_n)} \left| w_{ij}^{(m,n)} \right|^p d\mu(x) = \int_{\varphi_n(\mathsf{U}_n)} \left| u_{ij}^{(n)}(x) \right|^p d\mu(x) . \tag{4.7}$$

Observe that $E_{m,M}$ as is contained in U_n, $\phi_{m,n}(\varphi_n(\mathsf{U}_n)) = \varphi_n(\mathsf{U}_n)$. Thus, applying the change of variables $y = \phi_{m,n}(x)$ we can write

$$\begin{aligned} &\int_{\varphi_n(\mathsf{U}_n)} \left| w_{ij}^{(m,n)}(x) \right|^p d\mu(x) \\ &= \int_{\varphi_n(\mathsf{U}_n)} \Big| \sum_{1 \le k \le d} u_{ik}^{(n)}(\phi_{m,n}(x)) v_{kj}^{(m,n)}(x) \Big|^p d\mu(x) \\ &= \int_{\varphi_n(\mathsf{U}_n)} \Big| \sum_{1 \le k \le d} u_{ik}^{(n)}(y) v_{kj}^{(m,n)}(\phi_{m,n}^{-1}(y)) \Big|^p J_{\phi_{m,n}^{-1}}(y)\, d\mu(y) , \end{aligned} \tag{4.8}$$

where $J_{\phi_{m,n}^{-1}}(y) = \det D\phi_{m,n}^{-1}(y)$ denotes the Jacobian of $\phi_{m,n}^{-1}$ at y. Note, using the change of variables formula here is legitimate as $\phi_{m,n}$ is a diffeomorphism. As before, we have $\phi_{m,n}^{-1} \to$ id in measure and $J_{\phi_{m,n}^{-1}} \to 1$ in measure as m tends to infinity, so, passing to a subsequence $J_2 \subseteq J_1$ if necessary, we can once again assume convergence at Lebesgue almost every point of $\varphi_n(\mathsf{U}_n)$. Thus we now know that $v_{kj}^{(m,n)}(\phi_{m,n}^{-1}(y)) \to \delta_{kj}$ and that $J_{\phi_{m,n}^{-1}}(y) \to 1$ for Lebesgue almost every y in $\varphi_n(\mathsf{U}_n)$, as $J_2 \ni m \to \infty$. Hence the integrand in (4.8) converges to $|u_{ij}^{(n)}(y)|^p$ for Lebesgue almost every y in $\varphi_n(\mathsf{U}_n)$. Since $\phi_{m,n}$ leaves $\varphi_n(\mathsf{U}_n)$ invariant, and since $\|D\phi_{m,n}\|_{C^0} \leq K$ for some constant K independent of m, we have that

$$\left\| v_{kj}^{(m,n)} \circ \phi_{m,n}^{-1} \right\|_\infty \leq \left\| v_{kj}^{(m,n)} \right\|_\infty \leq \left\| D\phi_{m,n} \right\|_{C^0} \leq K \,. \tag{4.9}$$

For example, by inequality (4.2) above, we also have

$$J_{\phi_{m,n}^{-1}}(y) = \det D\phi_{m,n}^{-1}(y) \leq d!\, \|D\phi_{m,n}^{-1}(y)\|_{C^0}^d \leq d!\, K^d \,. \tag{4.10}$$

Combining inequalities (4.9) and (4.10), we deduce from Lebesgue's Dominated Convergence Theorem that

$$\lim_{J_2 \ni m \to \infty} \int_{\varphi_n(\mathsf{U}_n)} \Big| \sum_{1 \leq k \leq d} u_{ik}^{(n)}(y) v_{kj}^{(m,n)} \left(\phi_{m,n}^{-1}(y) \right) \Big|^p J_{\phi_{m,n}^{-1}}(y)\, d\mu(y)$$
$$= \int_{\varphi_n(\mathsf{U}_n)} \left| u_{ij}(y) \right|^p \, d\mu(y) \,.$$

This proves inequality (4.7), which in turn show – given the fact stated above – that (4.6) holds for $J = J_2$. Hence (4.4) is satisfied and this concludes the proof of part (i).

Now consider (ii). Recall that

$$\left\| f_n^{-1} - g_{m,n}^{-1} \right\|_{\mathbb{W}^{1,p^*}(\Omega_n^*, \mathbb{R}^d)} = \left\| f_n^{-1} - g_{m,n}^{-1} \right\|_{C^0(\Omega_n^*)} + \left(\int_{\Omega_n^*} \left| Df_n^{-1} - Dg_{m,n}^{-1} \right|^{p^*} \, d\mu \right)^{\frac{1}{p^*}} \,.$$

By the same argument as that given in part (i), the hypotheses (b) and (d) imply that

$$\left\| f_n^{-1} - g_{m,n}^{-1} \right\|_{C^0(\Omega_n^*)} \leq \epsilon_m \,.$$

Hence it suffices to show that, for some subsequence $J' \subseteq J$,

$$\liminf_{J' \ni m \to \infty} \int_{\Omega_n^*} \left| Df_n^{-1} - Dg_{m,n}^{-1} \right|^{p^*} \, d\mu \;=\; 0 \,. \tag{4.11}$$

Since Ω_n^* is contained in $\psi_n(\mathsf{V}_n)$, (4.11) this will follow if we can show that

$$\liminf_{J'\ni m\to\infty} \int_{\psi_n(\mathsf{V}_n)} |Df_n^{-1} - Dg_{m,n}^{-1}|^{p^*}\, d\mu \;=\; 0\,. \tag{4.12}$$

Let $\sigma_{m,n} = \left|Df_n^{-1} - Dg_{m,n}^{-1}\right|^{p^*}$. Once more by the chain rule (Lemma 4.1(ii)), for Lebesgue almost every $x \in \psi_n(\mathsf{V}_n)$ we have

$$Dg_{m,n}^{-1}(x) \;=\; D\phi_{m,n}^{-1}\left(f_n^{-1}(x)\right) Df_n^{-1}(x)\,.$$

Moreover, $\sigma_{m,n} \in L^1\left(\psi_n(\mathsf{V}_n)\right)$ since

$$\begin{aligned}\left|Df_n^{-1} - D\phi_{m,n}^{-1}\circ f_n^{-1} Df_n^{-1}\right|^{p^*} &\le \left|\mathrm{id}_{\mathbb{R}^d} - D\phi_{m,n}^{-1}\circ f_n^{-1}\right|^{p^*}\left|Df_n^{-1}\right|^{p^*}\\ &\le (1+K)^{p^*}\left|Df_n^{-1}\right|^{p^*}\,,\end{aligned}$$

where we have used that $\left\| D\phi_{m,n}^{-1}\right\|_{C^0} \le K$. We claim that the sequence $(\sigma_{m,n})_{m\in\mathbb{N}}$ converges in measure to the zero function. This happens because

$$\begin{aligned}\mu\left(\left\{x : \left|\sigma_{m,n}(x)\right| > 0\right\}\right) &= \mu\left(\left\{x : \left|\mathrm{id}_{\mathrm{R}^d} - D\phi_{m,n}^{-1}(f_n^{-1}(x))\right| > 0\right\}\right)\\ &\le \mu\left(f_n(\varphi_n(E_{m,M}))\right)\,.\end{aligned}$$

But $\mu\left(f_n(\varphi_n(E_{m,M}))\right) \to 0$ as $m \to \infty$, since $\operatorname{diam}\left(E_{m,M}\right) \to 0$ and f_n is uniformly continuous. Once more, this implies that there exists a subsequence J' of J such that $\sigma_{m,n}(x) \to 0$ for Lebesgue almost every x in $\psi_n\left(\mathsf{V}_n\right)$. By Lebesgue's Dominated Convergence Theorem it follows that

$$\lim_{J'\ni m\to\infty} \int_{\psi_n(\mathsf{V}_n)} \sigma_{m,n}\, d\mu \;=\; 0\,,$$

which shows that (4.12) and hence (4.11) holds. This concludes part (ii), and hence the proof is complete. □

4.3. Genericity of infinite topological entropy for Sobolev mappings

In this section we prove that infinite topological entropy is a generic property in the Sobolev context.

First, we give an argument specific to dimension two. The novelty in this approach is that it recovers the "naïve" argument where a periodic point is first 'blown-up' to a periodic disk, and then a horseshoe with an appropriate number of branches is 'glued-in' to this disk. (See the first comment in Section 5 for more details.) It is based on a generalised version of the Radó-Kneser-Choquet Theorem (see [4,26]). However, there is no known generalisation of this result to higher dimensions. In fact, there are explicit counterexamples to the classical Radó-Kneser-Choquet Theorem, see [15, Section 3.7].

Secondly, we present an argument analogous to that used in the Hölder case above, and is applicable in all dimensions greater than one. As much of the argument is the same as in the Hölder case, we only give a sketch, drawing attention to where modifications are necessary.

4.3.1. *First argument*

Our goal in this section is to show that infinite topological entropy is a generic property for homeomorphisms of compact surfaces in certain Sobolev classes. More precisely, we will prove the following result.

Theorem 4.6. *Let $1 < p < \infty$. Let M be a compact oriented surface. The set of orientation-preserving Sobolev homeomorphisms in $\mathcal{S}^{p,1}(M)$ with infinite topological entropy contains a residual subset of $\mathcal{S}^{p,1}(M)$.*

This theorem will be deduced from a corresponding result for maps in the plane, which we proceed to state.

Take $1 < p < \infty$, which we assume to be fixed throughout this section. Let Ω and Ω^* be bounded open sets in the plane. We conform with the notation introduced in Section 4.1. In particular, we denote by ρ the Sobolev distance in $\mathcal{S}^{p,1}(\Omega, \Omega^*)$. Namely

$$\rho(f, g) = \|f - g\|_{\mathbb{W}^{1,p}(\Omega,\mathbb{R}^2)} + \|f^{-1} - g^{-1}\|_{\mathbb{W}^{1,1}(\Omega^*,\mathbb{R}^2)} \, . \tag{4.13}$$

We assume that $\Omega \cap \Omega^* \neq \emptyset$, and for each k, $0 \leq k \leq \infty$, we denote by $\Omega_k = \Omega_k(f)$ the subset of points $x \in \Omega$ such that $f^j(x)$ is defined for all $0 \leq j < k$. Recall that a point $x \in \Omega_\infty$ is (forward) recurrent if it belongs to its own ω-limit set. The set of recurrent points is called the *recurrent set*. Let us denote by $\mathcal{S}^{p,1}_\infty(\Omega, \Omega^*)$ the *closure* of the set of all those Sobolev homeomorphisms $f \in \mathcal{S}^{p,1}(\Omega, \Omega^*)$ with non-trivial recurrent set.

For each $n \in \mathbb{N}$, let us denote by $\mathcal{G}_n$ the set of all $g \in \mathcal{S}^{p,1}(\Omega, \Omega^*)$ for which there exist $k \in \mathbb{N}$ and a topological disk $D \Subset \Omega_k(g)$ such that $g^k|_D \colon D \to g^k(D)$ is an n^k-branched horseshoe map. Note that $\mathcal{G}_n \subset \mathcal{S}^{p,1}_\infty(\Omega, \Omega^*)$, for all $n \in \mathbb{N}$. Note also that if $g \in \mathcal{G}_n$ then $h_{\text{top}}(g^k) \geq \log(n^k) = k \log n$, and therefore we have

$$g \in \mathcal{G}_n \implies h_{\text{top}}(g) \geq \log n \, .$$

With this notation we can now state our theorem as follows.

Theorem 4.7. *Let $1 < p < \infty$. For each $n \in \mathbb{N}$ the set $\mathcal{G}_n$ is dense in $\mathcal{S}^{p,1}_\infty(\Omega, \Omega^*)$. Consequently, the set of homeomorphisms with infinite entropy in $\mathcal{S}^{p,1}_\infty(\Omega, \Omega^*)$ contains a residual subset of $\mathcal{S}^{p,1}_\infty(\Omega, \Omega^*)$.*

Remark 4.8. As $\mathcal{S}^{p,1}_\infty(\Omega, \Omega^*)$ is closed in $\mathcal{S}^{p,1}(\Omega, \Omega^*)$ and $\mathcal{S}^{p,1}(\Omega, \Omega^*)$ has the Baire property, it follows that $\mathcal{S}^{p,1}_\infty(\Omega, \Omega^*)$ also has the Baire property.

It will be straightforward to deduce Theorem 4.6 from Theorem 4.7. As for the latter, note that the second assertion in the statement is an immediate consequence of the first. Namely, since the ρ-distance in $\mathcal{S}^{p,1}(\Omega, \Omega^*)$ is greater than the C^0-distance, and since topological horseshoe maps are stable under small C^0 perturbations, it follows that each $\mathcal{G}_n$ is open in $\mathcal{S}^{p,1}_\infty(\Omega, \Omega^*)$. Therefore the proof of Theorem 4.7 will be complete once we show that each $\mathcal{G}_n$ is *dense* in $\mathcal{S}^{p,1}_\infty(\Omega, \Omega^*)$.

The geometric idea behind the proof of such a density result is very simple, and can informally be described as follows. Starting with an arbitrary $f \in \mathcal{S}^{p,1}_\infty(\Omega, \Omega^*)$, the first step is to apply the Sobolev Closing Lemma to get $g_1 \in \mathcal{S}^{p,1}_\infty(\Omega, \Omega^*)$ close to f which has a periodic orbit. The second step is to then perform a surgery on g_1 in order to get a new $g_2 \in \mathcal{S}^{p,1}_\infty(\Omega, \Omega^*)$ close to g_1 which still has the same periodic orbit as g_1 but which is now a smooth diffeomorphism in a neighbourhood of that periodic orbit. The third step is to use a bump function argument to replace g_2 by yet another homeomorphism $g_3 \in \mathcal{S}^{p,1}_\infty(\Omega, \Omega^*)$ close to g_2, still with the same periodic orbit as g_2, having now a periodic cycle of *disks* around that periodic orbit on which g_3 moves points about by rigid translations – in particular, if k is the period, there is a disk D around a point of the periodic cycle such that $g_3^k|_D$ is the identity. The fourth and final step is to perform another (smooth) surgery to replace g_3 by a new $g_4 \in \mathcal{S}^{p,1}_\infty(\Omega, \Omega^*)$ very close to g_3 having the same periodic disk D as g_3, but now with the property that $g_4^k|_D$ is a horseshoe map with compact support in D and appropriately high entropy. The only difficult step is the second. The Sobolev surgery used in this step requires us to introduce the notion of a *p-harmonic map* as it uses a generalised version of a non-trivial theorem due to Radó, Kneser and Choquet (Theorem 4.10 below), and is inspired by [26].

p-Harmonic maps

As is customary, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Let $1 < p < \infty$ and let $\Omega \subset \mathbb{C}$ be a bounded open set. A function $u\colon \Omega \to \mathbb{R}$ is said to be *p-harmonic* if $u \in W^{1,p}(\Omega)$ and

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0$$

in the sense of distributions. Here and throughout, ∇ denotes the gradient operator, and obviously 'div' denotes the divergence operator. Note that p-harmonic for $p = 2$ simply means harmonic in the usual sense. It is a fact (from the theory of elliptic partial differential equations) that p-harmonic functions are minimizers for the so-called p-energy functional

$$\mathcal{E}_p(u) = \int_\Omega |\nabla u|^p \, d\mu ,$$

where as before μ denotes Lebesgue measure. The norm of the gradient is the standard Euclidean norm, namely $|\nabla u| = \sqrt{u_x^2 + u_y^2}$.

Definition 4.9. A homeomorphism $f = u + iv\colon \Omega \to \Omega^* \subset \mathbb{C}$ is said to be *coordinate-wise p-harmonic*, or simply *p-harmonic*, if its components $u, v\colon \Omega \to \mathbb{R}$ are both p-harmonic.

By analogy with the case of real functions, given a map $f = u + iv \in W^{1,p}(\Omega, \mathbb{C})$, we define its p-energy as the sum of the p-energies of its real and imaginary parts, *i.e.*,

$$\mathcal{E}_p(f) = \int_\Omega \left(|\nabla u|^p + |\nabla v|^p\right) d\mu ,$$

Just as in the case of real functions, p-harmonic homeomorphisms are minimizers of the p-energy.

It is easily seen that the p-energy of $f \in W^{1,p}(\Omega, \mathbb{C})$ controls the L^p-norm of $|Df|$ and vice-versa. Indeed, since in the present context we have $|Df| = |u_x| + |u_y| + |v_x| + |v_y|$, we have the double inequality

$$\mathcal{E}_p(f) \leq \int_\Omega |Df|^p \, d\mu \leq c_p \mathcal{E}_p(f) , \tag{4.14}$$

where $c_p > 1$ is a constant depending only on p [5].

The only non-trivial fact we will use about p-harmonic homeomorphisms is the following generalization due to Alessandrini and Sigalotti [4] of a theorem due to Radó, Kneser and Choquet. The formulation below is adapted from [26].

Theorem 4.10. *Let D, D^* be two Jordan domains in the plane. Assume that both Jordan curves ∂D, ∂D^* are positively oriented, and that D^* is* convex. *Given $1 < p < \infty$ and a homeomorphism $h\colon \partial D \to \partial D^*$ which preserves orientation, there exists an orientation-preserving homeomorphism $\phi\colon \overline{D} \to \overline{D}^*$ such that $\phi|_{\partial D} \equiv h$ and ϕ is p-harmonic. Moreover, $\phi|_D$ is a C^∞-diffeomorphism onto D^*, and in particular its Jacobian is everywhere positive in D.*

When we have a diffeomorphism ϕ in $W^{1,p}$, the p-energy of ϕ always bounds the 1-energy of ϕ^{-1}. This is the content of the following simple lemma, which will be used in combination with Theorem 4.10.

Lemma 4.11. *Let $\phi\colon D \to D^*$ be C^1-diffeomorphism between two bounded domains in the plane, and suppose $\phi \in W^{1,p}(D, \mathbb{C})$ for some $1 < p < \infty$. Then $\phi^{-1} \in W^{1,1}(D^*, \mathbb{C})$, and in fact*

$$\mathcal{E}_1(\phi^{-1}) \leq 4\,(\text{Area}(D))^{1-\frac{1}{p}}\, \mathcal{E}_p(\phi)^{\frac{1}{p}} . \tag{4.15}$$

[5] In fact, one can take $c_p = 2^{\frac{3p}{2}}$.

Proof. There is no loss of generality in assuming that ϕ preserves orientation. Let us write $\phi = u + iv$ and $\phi^{-1} = U + iV$. We need to bound

$$\begin{aligned} \mathcal{E}_1(\phi^{-1}) &= \int_{D^*} |D\phi^{-1}|\, d\mu \\ &= \int_{D^*} |U_x|\, d\mu + \int_{D^*} |U_y|\, d\mu + \int_{D^*} |V_x|\, d\mu + \int_{D^*} |V_y|\, d\mu \end{aligned} \tag{4.16}$$

in terms of the p-energy of ϕ. We proceed to bound each of the four integrals in the right-hand side of (4.16). By the chain rule we have $D\phi \circ \phi^{-1} \cdot D\phi^{-1} = \mathrm{id}$, and therefore

$$D\phi^{-1} = \begin{bmatrix} U_x & U_y \\ V_x & V_y \end{bmatrix} = \frac{1}{J_\phi \circ \phi^{-1}} \begin{bmatrix} v_y \circ \phi^{-1} & -u_y \circ \phi^{-1} \\ -v_x \circ \phi^{-1} & u_x \circ \phi^{-1} \end{bmatrix} = \left(D\phi \circ \phi^{-1} \right)^{-1} ,$$

where $J_\phi = \det(D\phi) = u_x v_y - u_y v_x > 0$ is the Jacobian of ϕ. Note also that $J_\phi \circ \phi^{-1} = (J_{\phi^{-1}})^{-1}$. Comparing entries in the matrices above, we get

$$\begin{aligned} U_x &= v_y \circ \phi^{-1} \cdot J_{\phi^{-1}} \quad , \quad U_y = -u_y \circ \phi^{-1} \cdot J_{\phi^{-1}} \quad , \\ V_x &= -v_x \circ \phi^{-1} \cdot J_{\phi^{-1}} \quad , \quad V_y = u_x \circ \phi^{-1} \cdot J_{\phi^{-1}} \quad . \end{aligned}$$

From the first of the four inequalities above, we deduce by a simple application of the change of variables formula that

$$\int_{D^*} |U_x|\, d\mu = \int_{D^*} |v_y| \circ \phi^{-1} \cdot J_{\phi^{-1}}\, d\mu = \int_D |v_y|\, d\mu .$$

But Hölder's inequality tells us that

$$\int_D |v_y|\, d\mu \le \left(\int_D d\mu \right)^{1-\frac{1}{p}} \left(\int_D |v_y|^p\, d\mu \right)^{\frac{1}{p}} .$$

This obviously implies that

$$\int_{D^*} |U_x|\, d\mu \le (\mathrm{Area}(D))^{1-\frac{1}{p}} \left(\int_D |v_y|^p\, d\mu \right)^{\frac{1}{p}} \le (\mathrm{Area}(D))^{1-\frac{1}{p}}\, \mathcal{E}_p(\phi)^{\frac{1}{p}} .$$

The same estimate holds for the remaining three integrals in the right-hand side of (4.16). Adding up all these estimates yields (4.15), as desired. □

Replacement trick

The following proposition shows that we can always replace a Sobolev homeomorphism in $\mathcal{S}^{p,1}(\Omega, \Omega^*)$ by another which is very close to it and is in fact smooth in the neighbourhood of a point specified in advance. In the proof we will implicitly use, in addition to the auxiliary results of the previous section, the following elementary remark.

Remark 4.12. If $f\colon \Omega \to \Omega^*$ is a homeomorphism and $a^* \in \Omega^*$ and $r_0 > 0$ satisfy $\overline{D}(a^*, r_0) \subset \Omega^*$, then for all but countably many $r \in [0, r_0]$ the Jordan curve $f^{-1}(\partial D(a^*, r))$ has zero Lebesgue measure.

Proposition 4.13. *Let $f\colon \Omega \to \Omega^*$ be a Sobolev homeomorphism in $\mathcal{S}^{p,1}(\Omega, \Omega^*)$, and let $a \in \Omega$ and $a^* \in \Omega^*$ satisfy $f(a) = a^*$. Then for each $\epsilon > 0$ there exists a topological disk $\mathcal{O}^*$, compactly contained in Ω^*, and $g \in \mathcal{S}^{p,1}(\Omega, \Omega^*)$ having the following properties:*

(i) *Both $\mathcal{O}^*$ and its pre-image $\mathcal{O} = f^{-1}(\mathcal{O}^*)$ have diameter less than ϵ;*
(ii) *The map g agrees with f on $\Omega \setminus \mathcal{O}$;*
(iii) *The restriction $g|_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}^*$ is a C^∞ diffeomorphism;*
(iv) *The map g is ϵ-close to f, i.e., $\rho(f, g) < \epsilon$;*
(v) *We have $a \in \mathcal{O}$, $a^* \in \mathcal{O}^*$, and the equality $g(a) = a^*$ holds true.*

Proof. First we prove that for each $\epsilon > 0$ there exists a topological disk $\mathcal{O}^*$ and $g \in \mathcal{S}^{p,1}(\Omega, \Omega^*)$ such that the properties (i)–(iv) hold. We take care of property (v) only at the end of the proof. We proceed by steps in the following way. Let us choose $0 < \epsilon_0 < \epsilon$. (How small ϵ_0 needs to be will be determined in the course of the argument).

(1) By uniform continuity of f^{-1}, there exists $0 < \delta < \epsilon_0/2$ such that $D(a^*, \delta) \subset \Omega^*$ and $\operatorname{diam}\big(f^{-1}(D(a^*, \delta))\big) < \epsilon_0$;
(2) Let $N \in \mathbb{N}$ satisfy $N > \epsilon_0^{-1} \max\big\{c_p \mathcal{E}_p(f), c_1 \mathcal{E}_1(f^{-1})\big\}$. Choose N pairwise disjoint balls (disks) $\Delta_1, \Delta_2, \ldots, \Delta_N \subset D(a^*, \delta)$ and note that, by the inequalities (4.14),

$$\frac{1}{N} \sum_{j=1}^{N} \int_{\Delta_j} |Df^{-1}|\, d\mu \;\le\; \frac{1}{N} \int_{D(a^*,\delta)} |Df^{-1}|\, d\mu \;<\; \epsilon_0 \,.$$

Therefore at least one of the disks $\Delta_1, \Delta_2, \ldots, \Delta_N$, call it Δ, satisfies

$$\int_{\Delta} |Df^{-1}|\, d\mu \;<\; \epsilon_0 \,.$$

Now choose pairwise disjoint balls $B_1, B_2, \ldots, B_N \subset \Delta$ for which we have $\mu(f^{-1}(\partial B_j)) = 0$, for all $1 \le j \le N$. This is possible by Remark 4.12. See Figure 4.1. Then

$$\frac{1}{N} \sum_{j=1}^{N} \int_{f^{-1}(B_j)} |Df|^p\, d\mu \;\le\; \frac{1}{N} \int_{f^{-1}(\Delta)} |Df|^p\, d\mu \;<\; \epsilon_0 \,,$$

and from this it follows that there exists $k \in \{1, 2, \ldots, N\}$ such that

$$\int_{f^{-1}(B_k)} |Df|^p\, d\mu \;<\; \epsilon_0 \,. \tag{4.17}$$

Let us define $\mathcal{O}^* = B_k$ and $\mathcal{O} = f^{-1}(B_k)$. Then the above considerations imply that

$$\mathcal{E}_p(f|_{\mathcal{O}}) < \epsilon_0 \quad \text{and} \quad \mathcal{E}_1(f^{-1}|_{\mathcal{O}^*}) < \epsilon_0 \; ; \tag{4.18}$$

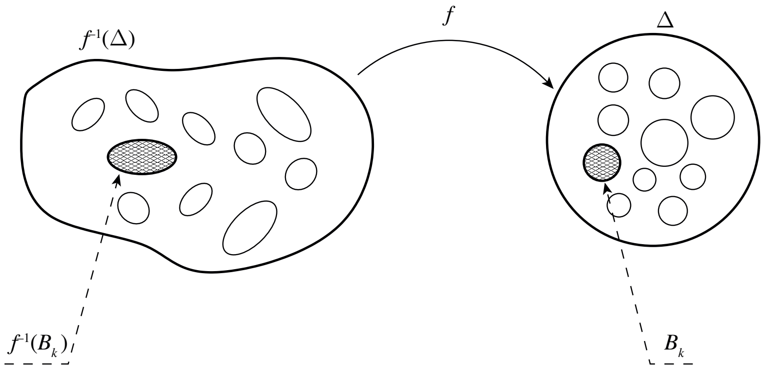

Figure 4.1. A topological disk carrying small Sobolev norm for f.

(3) Let $\phi\colon \overline{\mathcal{O}} \to \overline{\mathcal{O}^*}$ be the p-harmonic homeomorphism with $\phi|_{\partial\mathcal{O}} \equiv f|_{\partial\mathcal{O}}$ whose existence is guaranteed by Theorem 4.10. From (4.18) above and since ϕ minimises p-energy, we know that $\mathcal{E}_p(\phi) \leq \mathcal{E}_p(f|_{\mathcal{O}}) < \epsilon_0$;

(4) Define $g\colon \Omega \to \Omega^*$ by setting

$$g(z) = \begin{cases} f(z) & z \in \Omega \setminus \mathcal{O} \\ \phi(z) & z \in \mathcal{O}. \end{cases}$$

Then g is a homeomorphism. Since $\mu(\partial\mathcal{O}) = \mu(f^{-1}(\partial B_k)) = 0$, we see that $g \in W^{1,p}(\Omega, \mathbb{C})$. Similarly, since $\mu(\partial\mathcal{O}^*) = \mu(\partial B_k) = 0$, we also have $g^{-1} \in W^{1,1}(\Omega^*, \mathbb{C})$. Hence $g \in \mathcal{S}^{p,1}(\Omega, \Omega^*)$. Moreover, $g|_{\mathcal{O}} \equiv \phi\colon \mathcal{O} \to \mathcal{O}^*$ is a C^∞-diffeomorphism;

(5) Let us now estimate the distance $\rho(f, g)$. Since the support of $f - g$ lies in $\mathcal{O} = f^{-1}(\mathcal{O}^*) = g^{-1}(\mathcal{O}^*)$, we have

$$\|f - g\|_{C^0(\Omega)} \leq \operatorname{diam}(\mathcal{O}^*) \; = \; 2\delta \; < \; \epsilon_0 \, . \tag{4.19}$$

Likewise, since the support of $f^{-1} - g^{-1}$ lies in $\mathcal{O}^*$, we have

$$\|f^{-1} - g^{-1}\|_{C^0(\Omega^*)} \leq \operatorname{diam}(\mathcal{O}) < \epsilon_0 \, . \tag{4.20}$$

Moreover, we have

$$\begin{aligned} \|Df - Dg\|_{L^p(\Omega)} &= \|D(f|_{\mathcal{O}}) - D\phi\|_{L^p(\mathcal{O})} \\ &\leq \|D(f|_{\mathcal{O}})\|_{L^p(\mathcal{O})} + \|D\phi\|_{L^p(\mathcal{O})} \\ &\leq c_p^{\frac{1}{p}} \left[\mathcal{E}_p(f|_{\mathcal{O}})^{\frac{1}{p}} + \mathcal{E}_p(\phi)^{\frac{1}{p}} \right] . \end{aligned}$$

Using step (3) above we deduce that

$$\|Df - Dg\|_{L^p(\Omega)} \leq 2(c_p\epsilon_0)^{\frac{1}{p}} . \tag{4.21}$$

Finally, applying Lemma 4.11 to ϕ with $D^* = \mathcal{O}^*$ and $D = \mathcal{O}$, and taking into account that the area of $\mathcal{O}$ is less than $\pi\epsilon_0^2 < 1$ (if ϵ_0 is small enough), we get

$$\begin{aligned} \|Df^{-1} - Dg^{-1}\|_{L^1(\Omega^*)} &= \|D(f^{-1}|_{\mathcal{O}^*}) - D\phi^{-1}\|_{L^1(\mathcal{O}^*)} \\ &\leq c_1\left[\mathcal{E}_1(f^{-1}|_{\mathcal{O}^*}) + \mathcal{E}_1(\phi^{-1})\right] \\ &\leq c_1\left[\mathcal{E}_1(f^{-1}|_{\mathcal{O}^*}) + 4\mathcal{E}_p(\phi)^{\frac{1}{p}}\right] . \end{aligned}$$

Again using step (3) and the second inequality in (4.18), we deduce that

$$\|Df^{-1} - Dg^{-1}\|_{L^1(\Omega^*)} \leq 5c_1\epsilon_0^{\frac{1}{p}} . \tag{4.22}$$

Putting together (4.19), (4.20), (4.21), and (4.22), it follows that $\rho(f, g) < \epsilon$, provided ϵ_0 is chosen so small that $2\epsilon_0 + 2(c_p\epsilon_0)^{\frac{1}{p}} + 5c_1\epsilon_0^{\frac{1}{p}} < \epsilon$.

The proposition is almost proved. The only problem is that the map g we constructed above does not necessarily satisfy property (v). We fix this problem as follows. The argument we have given so far proves that for each $n \in \mathbb{N}$ there exist:

(a) A homeomorphism $f_n \in \mathcal{S}^{p,1}(\Omega, \Omega^*)$ which is ϵ_n-close to f in the Sobolev metric, where $\epsilon_n = 2^{-n}$, say;
(b) Two topological disks $\mathcal{O}_n \subset D(a, \epsilon_n) \subset \Omega$ and $\mathcal{O}_n^* \subset D(a^*, \epsilon_n) \subset \Omega^*$ with $f_n(\mathcal{O}_n) = \mathcal{O}_n^*$ such that $f_n|_{\mathcal{O}_n}$ is a C^∞-diffeomorphism and such that $f_n|_{\Omega\setminus\mathcal{O}_n} \equiv f|_{\Omega\setminus\mathcal{O}_n}$.

For each n, choose a point $a_n \in \mathcal{O}_n$ and let $a_n^* = f_n(a_n) \in \mathcal{O}_n^*$. Using Lemma A.3, we find a smooth diffeomorphism $\varphi_n \colon \Omega \to \Omega$ with support in the disk $D(a, 2\epsilon_n)$ such that $\varphi_n(a) = a_n$, and with the property that the C^1-norms of φ_n and φ_n^{-1} are bounded by a constant independent of n. In the same way we find a smooth diffeomorphism $\psi_n \colon \Omega^* \to \Omega^*$ with support in the disk $D(a^*, 2\epsilon_n)$ such that $\psi_n(a^*) = a_n^*$, also with the property that the C^1-norms of ψ_n and ψ_n^{-1} are bounded independently of n. Now let $g_n \colon \Omega \to \Omega^*$ be the homeomorphism $g_n = \psi_n^{-1} \circ f_n \circ \varphi_n \in \mathcal{S}^{p,1}(\Omega, \Omega^*)$. We clearly have $g_n \to f$ and $g_n^{-1} \to f^{-1}$ uniformly in Ω and Ω^*, respectively. By an argument analogous to the one used in the proof of the Sobolev Closing Lemma (Theorem 4.5) we know that there exists a subsequence $n_k \to \infty$ such that $Dg_{n_k} \to Df$ and $Dg_{n_k}^{-1} \to Df^{-1}$ *in measure*. Passing to a further subsequence if necessary, we may assume that both Dg_{n_k} and $Dg_{n_k}^{-1}$ converge pointwise Lebesgue almost everywhere to Df and Df^{-1}, respectively. Then, just as in the proof of Theorem 4.5, a simple application of Lebesgue's Dominated Convergence Theorem shows that $|Dg_{n_k} - Df| \to 0$ in $L^p(\Omega)$, and $|Dg_{n_k}^{-1} - Df^{-1}| \to 0$ in $L^1(\Omega^*)$. This shows that $\rho(g_{n_k}, f) \to 0$ as $k \to \infty$. But then any $g = g_{n_k}$ for sufficiently large k satisfies all five properties in the statement. This completes the proof. □

Blow-up

The next proposition shows that, for smooth diffeomorphisms, in the neighbourhood of a periodic orbit, the map may be replaced by a translation.

Proposition 4.14. *Let $\Omega, \Omega^* \subseteq \mathbb{R}^d$ be open domains with Ω path-connected. Let $f \in C^2(\Omega, \Omega^*)$ be an orientation-preserving embedding with periodic point x_0 of minimal period k. There exists $C > 0$ with the following property. For each $r_0 > 0$ sufficiently small there exist:*

(i) *An embedding $g \in C^2(\Omega, \Omega^*)$;*
(ii) *Concentric disks $D_{1,j} \subset D_{0,j}$ about $f^j(x_0)$ in Ω, of radius r_0 or less, for each $j = 0, 1, \ldots, k-1$;*

such that

(a) *$g|_{D_{1,j}}$ is a translation from $D_{1,j}$ to $D_{1,j+1}$ (addition taken modulo k), for each $j = 0, 1 \ldots, k-1$;*
(b) *$g|_{\Omega \setminus \bigcup_{j=0}^{k-1} D_{0,j}} = f$;*
(c) *$d_{\mathrm{Lip}}(f, g)$, $d_{\mathrm{Lip}}(f^{-1}, g^{-1}) < C$.*

Proof. The proposition will follow if we can show it in the simplified case when $x_0 = 0$ is a fixed point. Namely, it suffices to show when the origin is fixed that there exists a C^2-embedding g and concentric disks $D_1 \subset D_0$ about $x_0 = 0$ in Ω such that $g|D_1 = \mathrm{id}$, $g|(\Omega \setminus D_0) = f$, and $d_{\mathrm{Lip}}(f, g) < C$. The general case then follows by applying appropriate translations, choosing isometric disks $D_{0,j}$ about $f^j(x_0)$ which are pairwise disjoint, and applying the special case inductively. The first part of the construction is standard, and may be found in, *e.g.*, Hirsch [23]. However, the estimate afterwards, although straightforward, could not be found in the literature, so we include it for completeness.

First, take a disk $D_0 = D(0, r_0)$ about the origin, contained in Ω. Construct a C^2-smooth isotopy

$$A\colon [0,1] \times D_0 \to \Omega^*, \qquad A_0 = Df(0),\ A_1 = f,$$

i.e., a smooth isotopy between $f|_{D'}$ and $Df(0)|_{D'}$. This may be done via Alexanders' trick. Next, we take a C^2-smooth isotopy

$$M\colon [0,1] \to \mathrm{GL}(2, \mathbb{R}), \qquad M_0 = Df(0)^{-1},\ M_1 = \mathrm{id},$$

i.e., a smooth isotopy between id and $Df(0)$. Such an isotopy exists as $GL_+(2,\mathbb{R})$ is connected and f is orientation-preserving. Define

$$F\colon [0,1]\times D_0 \to \Omega^*, \qquad F(t,x) = M_t \cdot A(t,x)\ .$$

Then $F_0 = \mathrm{id}$ and $F_1 = f$. Thus F is a C^2-smooth isotopy between the identity and f. Let X_t denote the time-dependent vector field on (a subset of) Ω induced by F_t. Let $X = \partial_t \times X_t$ denote the corresponding vector field on $\bigcup_{t\in[0,1]}(\{t\}\times F_t(D_0))$ induced by the fat isotopy $\bar{F}(t,x) = (t, F_t(x))$.

We construct a new isotopy G as follows. Since ∂D_0 is compact and 0 is fixed by the isotopy there exists a positive $r < r_0$ such that $|F_t(x)| > r$ for all $x \in \partial D_0$ and $t \in [0,1]$. Take a bump function $\beta \in C^\infty([0,r],\mathbb{R})$ such that $\beta|_{[0,r/3]} \equiv 0$ and $\beta|_{[2r/3,r]} \equiv 1$. Define the vector field on $\bigcup_{t\in[0,1]}(\{t\}\times F_t(D_0))$ given by

$$Y = \begin{cases} \partial_t \times \beta X_t & \text{in } [0,1]\times D(0,r) \\ X & \text{otherwise.} \end{cases}$$

Since $\beta \equiv 1$ in a neighbourhood of r, the vector field Y is smooth. Let Y_t denote the corresponding time-dependent vector field, *i.e.*, $Y = \beta X_t$. Let $G\colon [0,1]\times D_0 \to \Omega^*$ denote the corresponding C^2-smooth isotopy. Denote by g the time-one map.

Set $D_1 = D(0,r_1)$ where $r_1 = r/3$. Since $Y_t|_{D_1} \equiv 0$ for all t, by construction we have that $g|_{D_1} \equiv \mathrm{id}$. Also, as g agrees with f on a collared neighbourhood of ∂D_0 in D_0 it extends smoothly to a map, which we also denote by g, on the whole of Ω.

Since $X = Y$ outside of $[0,1]\times D_0$ and $|F_t(x)| > r$ for all $x \in \partial D_0$, it follows that $F_t(x) = G_t(x)$ for all x in a neighbourhood of ∂D_0 in D_0 and all $t \in [0,1]$. Hence, in this neighbourhood of ∂D_0 in D_0, the time-one maps agree, *i.e.*, $f = g$.

It remains to estimate $d_{\mathrm{Lip}}(f,g)$ and $d_{\mathrm{Lip}}(f^{-1},g^{-1})$. Observe that there exists a positive K, depending upon f only, such that for all $t \in [0,1]$,

$$\max_{x\in D_0} \|\partial_s\partial_x\,(F_t - G_t)\,(x)\|\,,\ \max_{x\in F_s(D_0)} \left\|\partial_s\partial_x\left(F_t^{-1} - G_t^{-1}\right)(x)\right\| \le K\ .$$

This may be seen, for example, by observing that, for each $t \in [0,1]$ and $x \in D_0$,

$$\partial_t F_t(x) = X_t\,(F_t(x)) \quad \text{and} \quad \partial_t G_t(x) = Y_t\,(G_t(x))\ ,$$

so changing the order of differentiations and applying the chain rule together with the explicit expression for Y in terms of X gives the bound. (Observe that the estimate for the inverses requires changing the sign of the time parameter.)

Fix distinct points x_0 and x_1. Define, for all $t \in [0,1]$,

$$\varphi(t) = |(F_t(x_0) - G_t(x_0)) - (F_t(x_1) - G_t(x_1))|\ .$$

Let $z\colon [0, |x_0 - x_1|] \to \Omega$ be an arclength parametrisation of a smooth curve in Ω between $z(0) = x_0$ and $z(|x_0 - x_1|) = x_1$. Since $F_0 = G_0$ we find that

$$\begin{aligned}
\varphi(t) &= \left| \int_0^t \partial_s \left[(F_s(x_0) - G_s(x_0)) - (F_s(x_1) - G_s(x_1)) \right] ds \right| \\
&= \left| \int_0^t \int_0^{|x_0 - x_1|} \partial_s \partial_x \left(F_s - G_s \right) (z(u)) \dot{z}(u) \, du \, ds \right| \\
&\leq \int_0^t \int_0^{|x_0 - x_1|} \left| \partial_s \partial_x \left(F_s - G_s \right) (z(u)) \dot{z}(u) \right| du \, ds \\
&\leq \int_0^t \int_0^{|x_0 - x_1|} \left\| \partial_s \partial_x \left(F_s - G_s \right) (z(u)) \right\| du \, ds \\
&\leq t |x_0 - x_1| \max_s \max_z \left\| \partial_s \partial_x \left(F_s - G_s \right) (z) \right\| .
\end{aligned}$$

Hence, for each $t \in [0, 1]$, $[F_t - G_t]_{\mathrm{Lip}} \leq tK$. Therefore, setting $t = 1$ gives

$$[f - g]_{\mathrm{Lip}} \leq K \ .$$

Since $d_{C^0}(f, g)$ can be made arbitrarily small by making r_0, the radius of D_0, sufficiently small, the uniform bound on $d_{\mathrm{Lip}}(f, g)$ follows. A similar argument also gives the bound for $d_{\mathrm{Lip}}(f^{-1}, g^{-1})$. □

Remark 4.15. The above statement holds more generally in the C^r-category, $r \geq 2$.

Proof of Theorem 4.7. As mentioned in the paragraph following the statement of Theorem 4.7, it suffices to show that, for each positive integer n, the set $\mathcal{G}_n$ is dense in the space $\mathcal{S}_\infty^{p,1}(\Omega, \Omega^*)$. Thus, given a positive real number ϵ and a mapping f in $\mathcal{S}_\infty^{p,1}(\Omega, \Omega^*)$, we wish to show that there exists $g \in \mathcal{S}_\infty^{p,1}(\Omega, \Omega^*)$ such that $\rho(f, g) < \epsilon$ and, for some positive integer k, g^k possesses a horseshoe with n^k branches.

Since $f \in \mathcal{S}_\infty^{p,1}(\Omega, \Omega^*)$, there exists a point $y \in \Omega$ which is a forward recurrent point for f. In particular, it is non-wandering. By the Sobolev Closing Lemma (Theorem 4.5), there exists $g_1 \in \mathcal{S}_\infty^{p,1}(\Omega, \Omega^*)$, a point $x \in \Omega$ and a positive integer k such that $\rho(f, g_1) < \epsilon/4$ and $g_1^k(x) = x$. Assume that k is the minimal period of x, and let $x_j = g_1^j(x)$ for $j = 0, 1, \ldots, k-1$.

Applying the Replacement Trick (Proposition 4.13) inductively around each x_j we find that there exists a map $g_2 \in \mathcal{S}^{p,1}(\Omega, \Omega^*)$, and topological disks $\mathcal{O}_j^*$ about x_{j+1} (where addition is taken mod k) such that

(1) $\mathcal{O}_j^*$, and its preimage $\mathcal{O}_j = g_1^{-1}(\mathcal{O}_j^*)$, have diameter less that $\epsilon/4$;
(2) The collection of sets $\mathcal{O}_j$, $j = 0, 1, \ldots, k-1$, are pairwise disjoint;
(3) g_2 agrees with g_1 outside of $\bigcup_{j=0}^{k-1} \mathcal{O}_j$;

(4) $g_2 \colon (\mathcal{O}_j, x_j) \to (\mathcal{O}_j^*, x_{j+1})$ is smooth for $j = 0, 1, \ldots, k-1$;
(5) $\rho(g_1, g_2) < \epsilon/4$.

Applying Proposition 4.14 to the restriction $g_2|_{\bigcup_{j=0}^{k-1} \mathcal{O}_j}$, we find that there exists a positive real number C such that for any positive r sufficiently small, there exists a positive $r' < r$ and a C^2-smooth embedding $g_3 \colon \bigcup_{j=0}^{k-1} \mathcal{O}_j \to \bigcup_{j=0}^{k-1} \mathcal{O}_j^*$ such that

(6) $D(x_j, r) \subset \mathcal{O}_j$ for all j;
(7) $g_3|_{D(x_j, r')}$ is a translation, for all j;
(8) g_3 agrees with g_2 outside of $\bigcup_{j=0}^{k-1} D(x_j, r)$;
(9) $[g_2 - g_3]_{\mathrm{Lip}}, [g_2^{-1} - g_3^{-1}]_{\mathrm{Lip}} < C$.

Observe that we may extend g_3 to Ω by setting it equal to g_2 outside $\bigcup_{j=0}^{k-1} \mathcal{O}_j$. We wish to estimate the ρ-distance between g_2 and g_3. First, by shrinking r if necessary we may assume that $d_{C^0}(g_2, g_3), d_{C^0}(g_2^{-1}, g_3^{-1}) < \epsilon/8$. Since g_2 and g_3 agree outside $\bigcup_{j=0}^{k-1} D(x_j, r)$ and since there exists K such that, for any smooth map G on a compact domain Ω, $|DG(z)| \leq K[G]_{\mathrm{Lip}}$ for any $z \in \Omega$, we find

$$\begin{aligned}
\int_\Omega |Dg_2 - Dg_3|^p \, d\mu &= \int_{\bigcup_{j=0}^{k-1} D(x_j, r)} |Dg_2 - Dg_3|^p \, d\mu \\
&\leq K^p [g_2 - g_3]_{\mathrm{Lip}}^p \sum_{j=0}^{k-1} \mu\left(D(x_j, r)\right) \\
&= K^p C^p \pi k r^2 .
\end{aligned}$$

Hence, by shrinking r again we may assume that $[g_2 - g_3]_{W^{1,p},\Omega} < \epsilon/8$. Adopting the same argument for the inverse, we may therefore assume that r has been chosen sufficiently small so that $\rho(g_2, g_3) < \epsilon/4$.

Now define g as follows. Let h denote a standard n-branched horseshoe of the unit disk $D(0, 1)$, fixing a neighbourhood of the boundary. We may assume that, for some constant c independent of n, $[h]_{\mathrm{Lip}}, [h^{-1}]_{\mathrm{Lip}} \leq cn$. Choose $r'' < r'$ and let $a_j \colon D(0, 1) \to D(x_j, r')$ be given by $a_j(z) = r''z + x_j$. Define

$$g(z) = \begin{cases} g_3 \circ a_j \circ h \circ a_j^{-1}(z) & z \in D(x_j, r''), \text{ some } j \\ g_3 & \text{otherwise.} \end{cases}$$

Since Lipschitz constants are invariant under affine rescaling, we find that

$$[g_3 - g]_{\mathrm{Lip}}, [g_3^{-1} - g^{-1}]_{\mathrm{Lip}} \leq 1 + cn .$$

By the same argument as before, since g_3 and g agree outside $\bigcup_{j=0}^{k-1} D(x_j, r'')$, for the constant K defined as above, we find

$$\begin{aligned}\int_\Omega |Dg_3 - Dg|^p \, d\mu &= \int_{\bigcup_{j=0}^{k-1} D(x_j, r'')} |Dg_3 - Dg|^p \, d\mu \\ &\le K[g_3 - g]_{\mathrm{Lip}}^p \sum_{j=0}^{k-1} \mu(D(x_j, r'')) \\ &\le K(1 + cn)^p \pi k (r'')^2 \,.\end{aligned}$$

A similar estimate holds for the inverses. Therefore, choosing r'' sufficiently small, we may assume that $\rho(g_3, g) < \epsilon/4$. Thus

$$\rho(f, g) \le \rho(f, g_1) + \rho(g_1, g_2) + \rho(g_2, g_3) + \rho(g_3, g) < \epsilon \,.$$

Finally, observe that since $g|_{D(x_j, r'')}$ is a translation from x_j to x_{j+1}, it follows that $g^k|_{D(x_0, r'')}$ is topologically conjugate to h^k. Thus g lies in $\mathcal{G}_n$ and $\rho(f, g) < \epsilon$. Hence $\mathcal{G}_n$ is dense, and the theorem follows. □

Proof of Theorem 4.6. The proof is totally analogous to the proof of Theorem 4.7 above. Namely, let $\mathcal{G}_n$ denote the subset of $g \in \mathcal{S}^{p,1}(M)$ for which some iterate g^k possesses an n^k-branched horseshoe. By the argument preceding the statement of Theorem 4.7, $\mathcal{G}_n$ is open in $\mathcal{S}^{p,1}(M)$. Thus, to prove Theorem 4.6 it suffices to show that $\mathcal{G}_n$ is dense.

Take $f \in \mathcal{S}^{p,1}(M)$ and a neighbourhood $\mathcal{N}$ of f in $\mathcal{S}^{p,1}(M)$. As M is compact, the non-wandering set of f is non-empty. Take a non-wandering point y in M and apply the Sobolev Closing Lemma. Then there exists $g_1 \in \mathcal{N}$ with periodic point x of some mimimal period k. For $j = 0, 1, \dots, k-1$, take charts (U_j, φ_j) about $x_j = g_1^j(x)$ with pairwise disjoint domains and ranges. Define

$$\Omega = \bigcup_{j=0}^{k-1} \varphi_j(U_j \cap g_1^{-1}(U_{j+1})) \quad \text{and} \quad \Omega^* = \bigcup_{j=0}^{k-1} \varphi_{j+1}(g_1(U_j) \cap U_{j+1}) \,,$$

where, as usual, addition is taken modulo k. Consider the map $G_1 \colon \Omega \to \Omega^*$ which, for $j = 1, 2, \dots, k-1$, is defined on $\varphi_j(U_j \cap g_1^{-1}(U_{j+1}))$ by

$$G_1 = \varphi_{j+1} \circ g_1 \circ \varphi_j^{-1} \,.$$

Then this defines a map in $\mathcal{S}^{p,1}(\Omega, \Omega^*)$. In fact it lies in $\mathcal{S}_\infty^{p,1}(\Omega, \Omega^*)$ as it possesses a periodic orbit. Applying Theorem 4.7, we find that for each positive ϵ there exists $G_2 \in \mathcal{S}^{p,1}(\Omega, \Omega^*)$ with $\rho(G_1, G_2) < \epsilon$, such that in a neighbourhood of $\varphi_0(x_0)$, G_2 possesses an n^k-branched horseshoe, and outside of this neighbourhood G_2 coincides with G_1. Consequently, G_2 induces a Sobolev homeomorphism g_2 in

$\mathcal{S}^{p,1}(M)$, also with the property that g_2^k possesses an n^k-branched horseshoe. Hence g_2 lies in $\mathcal{G}_n$. Moreover, from the definition of the Sobolev-Whitney topology on $\mathcal{S}^{p,1}(M)$, for ϵ sufficiently small, g_2 can be chosen to lie in any neighbourhood of g_1. Thus we may assume g_2 lies in $\mathcal{N}$. Consequently $\mathcal{G}_n$ is dense in $\mathcal{S}^{p,1}(M)$ and the theorem follows. □

4.3.2. *Second argument*

The argument in the Hölder case can be adapted to give a proof of Theorem B. As in the Hölder case, this follows directly from the following result.

Theorem 4.16. *Let M be a compact manifold of dimension d. Assume either*

(a) $d = 2$ *and* $1 \leq p, p^* < \infty$*;*
(b) $d > 2$ *and* $d - 1 < p, p^* < \infty$.

Let $f \in \mathcal{S}^{p,p^*}(M)$. For each neighbourhood $\mathcal{N}$ of f in $\mathcal{S}^{p,p^*}(M)$ and each positive integer N there exists $g \in \mathcal{S}^{p,p^*}(M)$ such that

(i) $g \in \mathcal{N}$*;*
(ii) *There exists a positive integer k_0, a topological solid cylinder S in M and solid sub-cylinders $S_1, S_2, \ldots, S_{N^{k_0}}$ such that g^{k_0} maps S_j across S for $j = 1, 2, \ldots, N^{k_0}$.*

The second property implies that $h_{\mathrm{top}}(g) \geq \log N$ and that this property is satisfied in an open neighbourhood of g.

Proof. The strategy of proof is the same as for the Hölder case (Theorem 3.18). Specifically, the notation, setup and construction of the perturbation will be the same as in that case. We will only remark on the necessary changes, such as the choice of sizes of neighbourhoods, etc., and will go through the estimates for the size of the perturbation in more detail.

Setup. We may assume, by making a small perturbation if necessary, that f is bi-Lipschitz. Take a finite collection of sub-basic sets

$$\mathcal{N}_{\mathbb{W}^{1,p},\mathbb{W}^{1,p^*}}(f; (U_n, \varphi_n), (V_n, \psi_n), K_n, L_n, \varepsilon_n). \tag{4.23}$$

We will use the notation f_n, U_n, V_n, W_n, c_1, etc., as before. Thus c_1 satisfies inequalities (3.15) and (3.16) for f and U_n, as well as their counterparts for f^{-1} and V_n. Given a perturbation g of f and an index n, let g_n denote the map g expressed in the pair of charts (U_n, φ_n) and (V_n, ψ_n).

Fix a positive real number ϵ. This will denote the order of the size of the perturbation. Let δ be a positive real number. This will denote the size of the support of the perturbation. Take δ sufficiently small so that properties (a) and (b) from the proof of the Hölder case are satisfied, together with the following property (c′) For each index n, whenever x lies in a δ-neighbourhood of K_n,

$$\int_{\varphi_n(B(x,\delta))} |Df_n|^p \, d\mu \le \epsilon$$

and similarly, whenever x lies in a δ-neighbourhood of L_n,

$$\int_{\psi_n(B(x,\delta))} |Df_n^{-1}|^p \, d\mu \le \epsilon \, .$$

(Observe that this is possible as the δ-neighbourhoods of K_n and L_n are contained in the compact neighbourhoods U_n and V_n respectively.)

Support of the perturbation. Observe that Claim 3.19 from the proof of Theorem 3.18 also holds for Sobolev mappings. However, rather than Claim 3.19(5) we will require the following, which holds by taking r_1 sufficiently small

(5′) The following sets are pairwise disjoint

$$\varphi_0^{-1}\left(E(x_0^0, x_0^{k_0}; c \cdot r_0)\right), \varphi_1^{-1}(B(x^1, r_1)), \ldots, \varphi_{k_0-1}^{-1}(B(x^{k_0-1}, r_1)) \, .$$

In fact, if $\mathcal{B}$ denotes the collection of all such sets then, for any n,

$$\sum_{B \in \mathcal{B}: f^{-1}B \subset \mathsf{U}_n} \int_{\varphi_n(f^{-1}B)} |Df_n|^p \, d\mu < \epsilon$$

and

$$\sum_{B \in \mathcal{B}: B \subset \mathsf{V}_n} \int_{\psi_n(B)} |Df_n|^p \, d\mu < \epsilon \, .$$

Construction of the perturbation. Define g as in the Hölder case, for the choice of of neighbourhoods (or more specifically, the choice of r_1) as described above.

Then g is a homeomorphism. Pre- and post-composing by smooth mappings preserves the space of bi-Sobolev mappings. Thus the map g also lies in $\mathcal{S}^{p,p^*}(M)$. As in the Hölder case, when g is expressed in the pair of charts (U_n, φ_n) and (V_n, ψ_n), for each index n, we also have

$$g_n = \begin{cases} \phi_n^0 \circ f_n \circ \phi_n & \text{in } \varphi_n(E_M) \\ \phi_n^k \circ f_n & \text{in } \varphi_n(f^{-1}(B_M^{k+1})), \text{ for } 0 < k < k_0 \\ f_n & \text{elsewhere,} \end{cases} \tag{4.24}$$

where

$$\phi_n^k = \psi_n \circ \varphi_{k+1}^{-1} \circ \phi^k \circ \varphi_{k+1} \circ \psi_n^{-1} \quad \text{and} \quad \phi_n = \varphi_n \circ \varphi_0^{-1} \circ \phi \circ \varphi_0 \circ \varphi_n^{-1} \, .$$

Size of the perturbation. We will show that if ϵ is taken sufficiently small, then g lies in each sub-basic set (4.23). Fix an index n.

The argument as in the Hölder case, shows that

$$\|f_n - g_n\|_{C^0(\varphi_n(K_n))} = O(\epsilon) = \|f_n^{-1} - g_n^{-1}\|_{C^0(\psi_n(L_n))} . \tag{4.25}$$

Thus we only need to consider the $W^{1,p}$-semi-norm of $f_n - g_n$ together with the W^{1,p^*}-semi-norm of $f_n^{-1} - g_n^{-1}$. Consider the $W^{1,p}$-semi-norm. Observe that each preimage $f^{-1}(B_M^{k+1})$ lies in a ball of radius δ, and thus, by property (b) above, each $f^{-1}(B_M^{k+1})$ either lies inside U_n or is disjoint from K_n. Similarly for E_M. Let $\mathcal{B}_n$ denote the collection of all sets E_M and $f^{-1}(B_M^{k+1})$, $0 < k < k_0$, that are contained in U_n. Consequently, by (4.24), on U_n the map g_n is expressible as either $g_n = \Phi_n \circ f_n \circ \phi_n$ or $g_n = \Phi_n \circ f_n$, depending upon whether E_M lies in U_n or not. Here Φ_n is the composition of all ϕ_n^k such that $f^{-1}(B_M^{k+1})$ is contained in U_n. (Observe that, as the supports of the ϕ^k are pairwise disjoint, the maps ϕ^k commute. Thus the order in which the ϕ_n^k are composed does not matter.) Then

$$\begin{aligned}[] [f_n - g_n]_{W^{1,p},\varphi_n(K_n)} &\le [f_n - g_n]_{W^{1,p},\varphi_n(\mathsf{U}_n)} \\ &= \left(\sum_{B\in\mathcal{B}_n} \int_{\varphi_n(B)} |Df_n - Dg_n|^p \, d\mu\right)^{\frac{1}{p}} \\ &\le \sum_{B\in\mathcal{B}_n} \left(\int_{\varphi_n(B)} |Df_n - Dg_n|^p \, d\mu\right)^{\frac{1}{p}} . \end{aligned} \tag{4.26}$$

First consider the case when $B \in \mathcal{B}_n$ is of the form $f^{-1}(B_M^{k+1})$. Since ϕ_n^k is smooth and of compact support, it follows that $|\mathrm{id} - D\phi_n^k|$ is bounded, not just essentially bounded. By Hölder's inequality

$$\begin{aligned} &\left(\int_{\varphi_n(f^{-1}(B_M^{k+1}))} |Df_n - Dg_n|^p \, d\mu\right)^{\frac{1}{p}} \\ &\le \left(\int_{\varphi_n(f^{-1}(B_M^{k+1}))} |\mathrm{id} - D\phi_n^k \circ f_n|^p |Df_n|^p \, d\mu\right)^{\frac{1}{p}} \\ &\le \|\mathrm{id} - D\phi_n^k\|_{L^\infty} \left(\int_{\varphi_n(f^{-1}(B_M^{k+1}))} |Df_n|^p \, d\mu\right)^{\frac{1}{p}} . \end{aligned} \tag{4.27}$$

Before considering the case when B in $\mathcal{B}_n$ equals E_M, as in (4.26), observe that the mapping $h_n = \phi_n^0 \circ f_n$ is also Sobolev. Thus, by the chain rule (Lemma 4.1 (ii)) we have

$$\left(\int_{\varphi_n(E_M)} |Dh_n|^p \, d\mu\right)^{\frac{1}{p}} = \left(\int_{\varphi_n(E_M)} |D\phi_n^0 \circ f_n|^p |Df_n|^p \, d\mu\right)^{\frac{1}{p}}$$
$$\leq \|D\phi_n^0\|_{L^\infty} \left(\int_{\varphi_n(E_M)} |Df_n|^p \, d\mu\right)^{\frac{1}{p}} .$$

Now return to the case $B = E_M$. Observe that

$$\left(\int_{\varphi_n(E_M)} |Df_n - Dg_n|^p \, d\mu\right)^{\frac{1}{p}} \leq A_1 + A_2 + A_3 ,$$

where

$$A_1 = \left(\int_{\varphi_n(E_M)} |Df_n - Dh_n|^p \, d\mu\right)^{\frac{1}{p}} ,$$
$$A_2 = \left(\int_{\varphi_n(E_M)} |Dh_n - Dh_n \circ \phi_n|^p \, d\mu\right)^{\frac{1}{p}} ,$$
$$A_3 = \left(\int_{\varphi_n(E_M)} |Dh_n \circ \phi_n - Dg_n|^p \, d\mu\right)^{\frac{1}{p}} .$$

The same argument as for inequality (4.27) gives

$$A_1 = \left(\int_{\varphi_n(E_M)} |Df_n - Dh_n|^p \, d\mu\right)^{\frac{1}{p}}$$
$$\leq \|\mathrm{id} - D\phi_n^0 \circ f_n\|_{L^\infty} \left(\int_{\varphi_n(E_M)} |Df_n|^p \, d\mu\right)^{\frac{1}{p}} ,$$

while the triangle inequality, together with the change of variables formula and Hölder's inequality, implies that

$$A_2 \leq \left(\int_{\varphi_n(E_M)} |Dh_n|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_{\phi_n^{-1} \circ \varphi_n(E_M)} |Dh_n|^p J_{\phi_n^{-1}} \, d\mu\right)^{\frac{1}{p}}$$
$$\leq \left(1 + \|J_{\phi_n^{-1}}\|_{L^\infty}^{\frac{1}{p}}\right) \|D\phi_n^0\|_{L^\infty} \left(\int_{\varphi_n(E_M)} |Df_n|^p \, d\mu\right)^{\frac{1}{p}} .$$

The chain rule (Lemma 4.1 (i)) together with Hölder's inequality also implies that

$$A_3 = \left(\int_{\varphi_n(E_M)} |Dh_n \circ \phi_n - Dh_n \circ \phi_n D\phi_n|^p \, d\mu\right)^{\frac{1}{p}}$$
$$\leq \|\mathrm{id} - D\phi_n\|_{L^\infty} \|D\phi_n^0\|_{L^\infty} \left(\int_{\varphi_n(E_M)} |Df_n|^p \, d\mu\right)^{\frac{1}{p}} .$$

By (3.15) and (3.16), together with Lemma A.3 and Corollary B.4, we know that $\|D\phi_n^k\|_{L^\infty}$, $\|D\phi_n\|_{L^\infty}$, and $\|J_{\phi_n^{-1}}\|_{L^\infty}$ (each taken on the set $\varphi_n(\mathsf{U}_n)$) are bounded from above by some constant independent of ϵ. Therefore, by the above considerations together with property (c') in the case $B = E_M$ and property (5') in the case $B = f^{-1}(B_M^{k+1})$, we find that

$$[f_n - g_n]_{W^{1,p},\varphi_n(K_n)} = O\left(\sum_{B\in\mathcal{B}_n}\left(\int_B |Df_n|^p\, d\mu\right)^{\frac{1}{p}}\right) = O\left(\epsilon^{\frac{1}{p}}\right) .$$

As the above argument deals with both pre- and post-composition by smooth maps, the same argument gives $[f_n^{-1} - g_n^{-1}]_{W^{1,p^*},\psi_n(L_n)} = O(\epsilon^{\frac{1}{p^*}})$ and hence, as ϵ was arbitrary, the result follows. □

We also get the analogue of Yano's corollary on conjugacy classes, and of Corollary 3.20 above, in the Sobolev setting.

Corollary 4.17. *Let M be a compact manifold of dimension d. Assume either*

(a) *$d = 2$ and $1 \leq p, p^* < \infty$, or*
(b) *$d > 2$ and $d - 1 < p, p^* < \infty$.*

Then a generic homeomorphism in $\mathcal{S}^{p,p^*}(M)$ is not topologically conjugate to any diffeomorphism (or any bi-Lipschitz homeomorphism).

The arguments preceding Corollary 3.21 also give us the following.

Corollary 4.18. *Let M be a compact manifold of dimension d. Assume either*

(a) *$d = 2$ and $1 \leq p, p^* < \infty$;*
(b) *$d > 2$ and $d - 1 < p, p^* < \infty$.*

A generic homeomorphism in $\mathcal{S}^{p,p^*}(M)$ has uncountably many measures of maximal entropy.

As in the Hölder case, by the argument following Corollary 3.21 we also get the following.

Corollary 4.19. *Let M be a compact manifold of dimension d. Assume either*

(a) *$d = 2$ and $1 \leq p, p^* < \infty$;*
(b) *$d > 2$ and $d - 1 < p, p^* < \infty$.*

For a generic homeomorphism f in $\mathcal{S}^{p,p^*}(M)$ the set of equilibrium states of (f, ϕ), is independent of $\phi \in C^0(X, \mathbb{R})$. In fact, generically the set of equilibrium states, for any $\phi \in C^0(X, \mathbb{R})$ coincides with the set of measures of maximal entropy.

5. Concluding remarks and open problems

The following is a list of comments and open problems suggested by this work.

(1) To prove genericity of infinite entropy in a given smoothness category CAT there is also a "naïve" perturbation argument whereby we perturb a given initial map to some map with entropy $\log n$ or greater. The strategy is as follows. First, the initial map will possess a non-wandering point so, by the Closing Lemma, this map can be perturbed to a map with a periodic orbit of some period k. Next, we perturb this new map by 'blowing-up' this periodic point to a periodic ball, also of period k, on which the kth iterate is the identity. Finally, a horseshoe with n^k branches is 'glued-in' to this periodic ball. Then the kth iterate of this final map will have a horseshoe with entropy $\log n^k$, and thus the original map will have entropy at least $\log n$. This was the approach in Section 4.3.1. For details in the continuous case, see also [19, Proof of Lemma 3.1]. However, to 'blow-up' we require the Annulus Conjecture to hold in CAT.

CAT**-Annulus Conjecture.** *In any dimension d, for any* CAT*-embedding $\varphi\colon B^d \to \mathbb{R}^d$ with the property that $\varphi(B^d) \Subset B^d$, the sets $B^d \setminus \varphi(B^d)$ and* $[0, 1) \times S^{d-1}$ *are* CAT*-homeomorphic.*

In the continuous category, this was shown by Quinn, following work by Kirby and others [16]. Sullivan [39] proved that the Annulus Conjecture holds for the quasiconformal and bi-Lipschitz categories. (See also [40,41].) Therefore we ask:

Question. Does the Annulus Conjecture hold for bi-Hölder or bi-Sobolev mappings?

If the answer is yes, for either bi-Hölder or bi-Sobolev, then we can recover our results on genericity of infinite entropy more easily via the argument in [19, Proof of Lemma 3.1] and the Closing Lemmas proved in this paper.

(2) Via Morrey's inequality or otherwise, one can show that, on a bounded open set Ω in $\mathbb{R}^d$ with appropriate (*i.e.*, Lipschitz) boundary, $W^{1,p}(\Omega)$ embeds continuously into $C^\alpha(\Omega)$ where $\alpha = 1 - \frac{d}{p}$.

- Does a similar embedding result hold for the spaces of bi-Sobolev and bi-Hölder homeomorphisms, with the Sobolev-Whitney and Hölder-Whitney topologies, introduced in this paper?
- If such an embedding exists, is its image dense? If so, most of the results in Section 3 of this paper would follow from those in Section 4.

(3) Can the results in this paper be generalised to arbitrary metric measure spaces where the measure satisfies a doubling condition? Note that Hölder maps are easily defined for these space, while a definition of Sobolev maps has been given by Hajłasz (see, *e.g.*, [21]).

(4) Does the Sobolev Closing Lemma hold for $1 \leq p, p^* < d$?

(5) Can either the Sobolev or Hölder Closing Lemma be used to give, via an approximation argument, a new proof of Pugh's C^1-Closing Lemma?

Appendices

A. Basic moves

We describe the basic moves from which all perturbations in Sections 3 and 4 are constructed. The geometric properties of these perturbations also allow us to estimate how far they are from the identity with respect to the Lipschitz and Hölder distances.

First we introduce the following terminology, to be used in this and the following Appendix. We call the midpoint of the axis of C the *centre point* of C and we call the centres of the boundary balls C^- and C^+ of C the *endpoints* of C. We say that a topological solid cylinder $C \subset \mathbb{R}^d$ has *separated boundary balls* if the following property is satisfied: Denote the boundary balls of C by C^- and C^+ and denote the corresponding endpoints by c^- and c^+ respectively. Let $\Pi \subset \mathbb{R}^d$ denote the perpendicular bisector of the line segment $[c^-, c^+]$. Then the boundary balls C^+ and C^- lie in different connected components of $\mathbb{R}^d \setminus \Pi$.

Lemma A.1 (Lipschitz gluing principle). *Let (Ω, d_Ω) and $(\Omega', d_{\Omega'})$ be metric spaces, where Ω is geodesically convex. Let $\Omega_1, \Omega_2, \ldots$ be geodesically convex, pairwise disjoint subdomains of Ω and let $\Omega_0 = \Omega \setminus \bigcup_k \Omega_k$. Let $f \in C^0(\Omega, \Omega')$ be a function which is Lipschitz on $\Omega_0, \Omega_1, \Omega_2, \ldots$. Then f is Lipschitz and*

$$[f]_{\mathrm{Lip}} \leq \max_{k=0,1,2\ldots} [f]_{\mathrm{Lip}, \overline{\Omega}_k} .$$

Lemma A.2 (Lipschitz bounds for bump functions). *Given $p, q \in \mathbb{R}^d$, and positive real numbers $r_1 < r_2$, there exists $b \in C^1(\mathbb{R}^d, \mathbb{R})$ satisfying $b|_{E(p,q;r_1)} \equiv 1$, $b|_{\mathbb{R}^d \setminus E(p,q;r_2)} \equiv 0$ and $[b]_{\mathrm{Lip}} \leq K/|r_2 - r_1|$, where K is independent of p, q, r_1 or r_2.*

Lemma A.3. *Take any $p, q \in \mathbb{R}^d$ and $0 < r_1 < r_2$. Then there exists $\phi \in \mathrm{Diff}^1_+(\mathbb{R}^d)$, such that $\phi(p) = q$, $\mathrm{supp}(\phi) \subset \overline{E(p, q; r_2)}$ and*

$$\max \left\{ \left[\phi - \mathrm{id}\right]_{\mathrm{Lip}}, \left[\phi^{-1} - \mathrm{id}\right]_{\mathrm{Lip}} \right\} < K ,$$

where K depends upon $|p - q|/|r_2 - r_1|$ only.

Proof. Let $b \in C^1(\mathbb{R}^d, \mathbb{R})$ denote the bump function given in Lemma A.2 for the domains $E(p, q; r_1) \subset E(p, q; r_2)$. Observe that $[b]_{\mathrm{Lip}} \leq K/|r_2 - r_1|$. Consider the vector field

$$X(x) = b(x)(q - p).$$

There exist positive real numbers L and M such that, for all $x, y \in \mathbb{R}^d$,

$$|X(x)| \leq L, \qquad |X(x) - X(y)| \leq M|x - y| \, . \tag{A.1}$$

(In fact, $L = |q - p|$ and $M = [b]_{\text{Lip}}|q - p| \leq K|q - p|/|r_2 - r_1|$.) Let Φ_t denote the corresponding flow. For x in $E(p, q; r_1)$ and $|t| \leq L \cdot \text{dist}(x, \partial E(p, q; r_1))$ we have that $\Phi_t(x) = x + t(q - p)$. Similarly, for x in $\mathbb{R}^d \setminus E(p, q; r_2)$ we have that $\Phi_t(x) = x$. Let ϕ denote the time-one map Φ_1. Then ϕ is a C^1-smooth diffeomorphism with support in $\overline{E(p, q; r_2)}$ such that $\phi(p) = q$. It remains to bound the Lipschitz constant of ϕ. Fix distinct points x and y in $\mathbb{R}^d$.

Define $f(t) = |(\Phi_t(x) - x) - (\Phi_t(y) - y)|$. Then inequality (A.1) implies

$$\begin{aligned} f(t) &= |(\Phi_t(x) - x) - (\Phi_t(y) - y)| \\ &\leq \int_0^t |X(\Phi_s(x)) - X(\Phi_s(y))| \, ds \\ &\leq \int_0^t M \, |\Phi_s(x) - \Phi_s(y)| \, ds \\ &\leq M \int_0^t |x - y| + |(\Phi_s(x) - x) - (\Phi_s(y) - y)| \, ds \\ &= Mt|x - y| + \int_0^t Mf(s) \, ds \, . \end{aligned}$$

Applying Grönwall's inequality and setting $t = 1$, we get the following inequality

$$[\phi - \text{id}]_{\text{Lip}} \leq K = Me^M \, .$$

By symmetry, the same argument show that $\left[\phi^{-1} - \text{id}\right]_{\text{Lip}} \leq K$. Thus the lemma is shown. □

Applying the construction in Lemma A.3 to a solid cylinder instead of a single point gives the following.

Lemma A.4 (Basic move 1 – translation to the origin). *Let C be a rigid solid cylinder in $\mathbb{R}^d$. Let c denote the centre of C and let r be the least positive real number such that C is contained in the elongated neighbourhood $E(0, c; r)$. Then there exists a diffeomorphism of $\mathbb{R}^d$ with support in $E(0, c; r + |c|)$ such that $\phi|_{E(0,c;r)}$ is translation from c to 0. In particular, $\phi(C)$ is a rigid cylinder with centre 0. Moreover, $\left[\phi\right]_{\text{Lip}}, \left[\phi^{-1}\right]_{\text{Lip}} < K$, where K is a uniform constant.*

The construction in Lemma A.3 can also be performed with any linear flow, not just a translational flow. The case of rotational flows and hyperbolic flows gives the following two Lemmas.

Lemma A.5 (Basic move 2 – rotation about the origin). *Let C and C' be isometric rigid solid cylinders centred at the origin in $\mathbb{R}^d$, both contained in $B(0, r)$. Then there exists a diffeomorphism ϕ of $\mathbb{R}^d$ with support in $B(0, 2r)$ which maps C isometrically onto C'. Moreover, $[\phi]_{\mathrm{Lip}}, [\phi^{-1}]_{\mathrm{Lip}} < K$, where K is a uniform constant (depending only upon the dimension d).*

We note that the uniformity on the constant comes from the compactness of the group of rotations $SO(d)$.

Lemma A.6 (Basic move 3 – cylinder affinity). *Let C' and C'' be rigid solid cylinders in $\mathbb{R}^d$, concentric and with the same axis. Let C denote the smallest rigid solid cylinder, concentric and with the same axis as C' and C'', containing C' and C''. Let E denote the elongated neighbourhood with the same endpoints as C and radius* $\mathrm{rad}(C)$. *Let E' denote the elongated neighbourhood with the same endpoints as C and radius* $\mathrm{rad}(C) + \mathrm{diam}(C)$. *Then there exists a diffeomorphism, supported in E', mapping C' across C''. Moreover $[\phi]_{\mathrm{Lip}}, [\phi^{-1}]_{\mathrm{Lip}} < K$ where K depends only upon* $\max(\|A\|, \|A^{-1}\|)$, *where A is the hyperbolic linear map sending C' to C''.*

Corollary A.7. *There exists $K > 0$ such that the following property is satisfied: Let C and C' be isometric rigid solid cylinders, both contained in $B(p, r) \subset \mathbb{R}^d$. Then there exists a diffeomorphism ϕ of $\mathbb{R}^d$ such that*

(1) *The support of ϕ is contained in $B(p, 10r)$;*
(2) *ϕ maps C isometrically onto C';*
(3) $[\phi]_{\mathrm{Lip}} \leq K$.

Proof. Denote the centres of C and C' by c and c' respectively. An isometry from C to C' can be decomposed as $\tau' \circ \alpha \circ \tau^{-1}$, where τ and τ' are translations by c and c', respectively, and α is a linear isometry. Consequently we may construct diffeomorphisms ϕ_τ, ϕ_α and $\phi_{\tau'}$, associated with τ, α and τ' respectively, by performing the Basic Moves 1 and 2. The resulting diffeomorphism $\phi = \phi_{\tau'}^{-1} \circ \phi_\alpha \circ \phi_\tau$ satisfies the stated properties. □

Combining Corollary A.7 and Basic Move 1 (Lemma A.4) also give the following.

Corollary A.8. *Let $p, q \in \mathbb{R}^d$ and $r > 0$. Let C_p and C_q be isometric solid cylinders contained in $B(p, r)$ and $B(q, r)$ respectively. Then there exists a diffeomorphism ϕ of $\mathbb{R}^d$ and a positive real number K, depending upon $|p - q|/r$ only, such that*

(1) *The support of ϕ is contained in $E(p, q; 10r)$;*
(2) *ϕ maps C_p isometrically onto C_q;*
(3) $[\phi]_{\mathrm{Lip}} \leq K$.

B. Mapping solid cylinders across solid cylinders

In this section we construct certain perturbations that map solid cylinders. For notation and terminology concerning solid cylinders we refer to Section 3.3.

Lemma B.1. *Let Ω_0 and Ω_1 be open domains in $\mathbb{R}^d$. Given $f \in \mathcal{H}^1(\Omega_0, \Omega_1)$ with bi-Lipschitz constant κ there exist positive real numbers K_0 and K_1, depending upon κ only, with the following property: Let C_0 be a rigid solid cylinder in Ω_0 satisfying*

$$\mathrm{rad}(C_0) < K_0 \cdot \mathrm{len}(C_0) . \tag{B.1}$$

Then there exists a rigid solid cylinder C_1 in $\mathbb{R}^d$ such that f maps C_0 across C_1 and

$$\mathrm{rad}(C_1) \leq K_1 \cdot \mathrm{len}(C_1) .$$

Proof. Choose $K_0 = 1/3\kappa^2$. Let C_0 be a rigid solid cylinder in Ω_0 of axial length ℓ_0 and coaxial radius $\varrho_0 \leq K_0\ell_0$. Denote the endpoints of C_0 by c_0^- and c_0^+. We will show that it suffices to take C_1 whose axis is concentric with and parallel to the line segment $[f(c_0^-), f(c_0^+)]$. Below, the length and radius will be determined.

First we consider the length. Denote the boundary ball of C_0 containing $c_0^\pm$ by $C^\pm$. Take any point x in the boundary disk C^-. Then $\varrho_0 \leq K_0\ell_0$ and $K_0 = 1/3\kappa^2$ implies that $\kappa|c_0^- - x| \leq \frac{1}{3}\kappa^{-1}|c_0^- - c_0^+|$. Hence applying the κ-bi-Lipschitz property of f twice gives

$$|f(c_0^-) - f(x)| \leq \kappa|c_0^- - x| \leq \tfrac{1}{3}|f(c_0^-) - f(c_0^+)| .$$

Let $\ell = |f(c_0^-) - f(c_0^+)|$. Then the above implies that $f(C^-)$ is contained in the closure of $B\left(f(c_0^-), \frac{1}{3}\ell\right)$. The same argument also shows that $f(C^+)$ is contained in the closure of $B\left(f(c_0^+), \frac{1}{3}\ell\right)$. Consequently the topological solid cylinder $f(C_0)$ has separated boundary balls.

Next we consider the radius. Choose a point x in C_0 such that the distance ϱ between $f(x)$ and the line segment $[f(c_0^-), f(c_0^+)]$ is maximal. Take the orthogonal projection y of $f(x)$ to the line between $f(c_0^-)$ and $f(c_0^+)$. Let $\delta_1 = |f(c_0^-) - f(x)|$. Then

$$\delta_1 \leq \kappa|c_0^- - x| \leq \kappa(\ell_0^2 + \varrho_0^2)^{1/2} \leq \kappa(1 + K_0^2)^{1/2}\ell_0 .$$

Hence

$$\varrho \leq \delta_1 \leq \kappa(1 + K_0^2)^{1/2}\ell_0 \leq \kappa(1 + K_0^2)^{1/2}\kappa\ell = K\ell .$$

Let C_1 denote the rigid solid cylinder in $\mathbb{R}^d$ with axis

$$[c_1^-, c_1^+] = \left[f(c_0^-) + \tfrac{1}{3}(f(c_0^+) - f(c_0^-)), f(c_0^+) - \tfrac{1}{3}(f(c_0^+) - f(c_0^-))\right]$$

and coaxial radius $\varrho_1 = 2\varrho$. Observe that C_1 has length $\ell_1 = \frac{1}{3}\ell$. Then f maps C_0 across C_1. Moreover $\varrho_1 \leq 2K\ell = 6K\ell_1$. Setting $K_1 = 6K$ finishes the proof. □

Lemma B.2. *Let Ω_0 and Ω_1 be open domains in $\mathbb{R}^d$. Given $f \in \mathcal{H}^1(\Omega_0, \Omega_1)$ there exist positive real numbers K_0, K_1 and γ, depending upon the the bi-Lipschitz constant κ of f only, such that the following property holds: Take $p \in \Omega_0$. Choose r_1 sufficiently small to ensure that*

$$B(p, r_1) \subset \Omega_0\ , \qquad B(f(p), r_1) \subset \Omega_1\ .$$

Let $r_2 = r_1/20(1 + K_1)$ and $r_3 = r_2/\kappa$. Take rigid solid cylinders C_0 and C_1, contained in $B(p, r_3)$ and $B(f(p), r_3)$, such that:

(i) $\mathrm{rad}(C_0) \le K_0 \mathrm{len}(C_0)$;
(ii) $\frac{1}{2}\mathrm{rad}(C_0) \le \mathrm{rad}(C_1) \le 2\mathrm{rad}(C_0)$;
(iii) $r_3 \le \mathrm{len}(C_0), \mathrm{len}(C_1) \le 2r_3$,

Then there exists $\phi \in \mathrm{Diff}^1(\Omega_1)$, supported in $B(f(p), r_1)$, such that

(a) $\max\left\{[\phi]_{\mathrm{Lip}}, [\phi^{-1}]_{\mathrm{Lip}}\right\} \le \gamma$;
(b) $\phi \circ f$ *maps* C_0 *across* C_1.

Proof. Let K_0 and K_1 denote the positive real numbers, depending upon κ only, from Lemma B.1. For $i = 0, 1$, assume that $C_i = C(a_i, b_i; \varrho_i)$ for some a_i, b_i and ϱ_i. Denote by C' the rigid solid cylinder in $\mathbb{R}^d$ from Lemma B.1. Then we may assume that C' has the following properties:

(1) f maps C_0 across C';
(2) The axis of C' is contained in the line segment $[f(c_0^-), f(c_0^+)]$;
(3) C' is symmetric about the perpendicular bisector Π of $f(c_0^-)$ and $f(c_0^+)$;
(4) $\mathrm{len}(C') = \ell' = \frac{1}{3}|f(c_0^-) - f(c_0^+)|$ and $\mathrm{rad}(C') = \rho' = K_1\ell'$.

Moreover C' is contained in $B(f(p), r_1/10)$. Denote by C'' the rigid solid cylinder which is concentric with C', symmetric about bisecting plane Π, and isometric with C_1. Observe that C'' is also contained in $B(f(p), r_1/10)$. Denote the axial length of C'' and the coaxial radius of C'' respectively by ρ'' and ℓ''.

By Basic Move 3 (Lemma A.6), there exists a diffeomorphism σ, supported in some elongated neighbourhood E of $C' \cup C''$ contained in $B(f(p), r_1)$, which maps C' onto C'' and a positive real number γ_1, independent of ℓ', ℓ'', ϱ' and ϱ'', such that

$$[\sigma]_{\mathrm{Lip}} \le c_1 \max\left(\ell''/\ell', \varrho''/\varrho'\right)\ , \qquad [\sigma^{-1}]_{\mathrm{Lip}} \le \gamma_1 \max\left(\ell'/\ell'', \varrho'/\varrho''\right)\ . \quad \text{(B.2)}$$

Observe that, as f is bi-Lipschitz, ℓ_0 and ϱ_0 are comparable, by a constant depending only upon κ, to ℓ' and ϱ' respectively. By hypotheses (i)–(iii), ℓ_0 and ϱ_0 are comparable to $\ell_1 = \ell''$ and $\varrho_1 = \varrho''$. Therefore there exists a positive real number γ_2, depending upon κ only, such that

$$\max\left\{[\sigma]_{\mathrm{Lip}}, [\sigma^{-1}]_{\mathrm{Lip}}\right\} \le \gamma_2\ . \quad \text{(B.3)}$$

Next, since C'' and C_1 are isometric, Corollary A.7 implies that there exists a diffeomorphism τ, supported in $B(f(p), r_1)$, and a positive real number γ_3 such that

$$\max\left\{[\tau]_{\mathrm{Lip}}, [\tau^{-1}]_{\mathrm{Lip}}\right\} \leq \gamma_3 , \tag{B.4}$$

which maps C'' isometrically onto C_1. It follows that the diffeomorphism $\phi = \tau \circ \sigma$ is supported in $B(f(p), r_1)$ and maps $f(C_0)$ across C_1. Therefore $\phi \circ f$ maps C_0 across C_1. Moreover, inequality (B.3) and (B.4) imply

$$\max\left\{[\phi]_{\mathrm{Lip}}, [\phi^{-1}]_{\mathrm{Lip}}\right\} \leq \gamma_2\gamma_3 ,$$

which completes the proof of the lemma. □

Proposition B.3. *Let $C(a, b; \varrho) \subset \mathbb{R}^d$ be a rigid solid cylinder. There exists a positive real number γ, depending upon $|a - b|$ and ϱ only, such that the following property is satisfied: For each positive integer N, there exists an orientation-preserving C^1-diffeomorphism ϕ of $\mathbb{R}^d$, supported in the elongated neighbourhood $E(a, b; 3\varrho)$, and there exist solid sub-cylinders $C_1, C_2, \ldots, C_N$ of $C(a, b; \varrho)$ such that ϕ maps C_j across $C(a, b; \varrho)$ for each $j = 1, 2 \ldots, N$. Moreover $\max\left\{[\phi]_{\mathrm{Lip}}, [\phi^{-1}]_{\mathrm{Lip}}\right\} \leq \gamma N$.*

Proof. The case when $a = (0, \ldots, 0, 1)$, $b = (0, \ldots, 0, -1)$ and $\varrho = 1$ is the classical construction of a horseshoe with N branches. The general case follows by conjugating via a map ψ from an elongated neighbourhood of the standard solid cylinder to $C(a, b; \varrho)$ and applying the Hölder Rescaling Principle (Proposition 3.5). □

Combining the above Proposition B.3 with Lemma B.2 gives the following:

Corollary B.4. *Let Ω_0 and Ω_1 be open domains in $\mathbb{R}^d$. Let N be a positive integer. Given $f \in \mathcal{H}^1(\Omega_0, \Omega_1)$ there exist positive real numbers K_0, K_1 and γ, depending upon the the bi-Lipschitz constant κ of f only, such that the following property holds: Take $p \in \Omega_0$. Choose r_1 sufficiently small to ensure that*

$$B(p, r_1) \subset \Omega_0 , \qquad B(f(p), r_1) \subset \Omega_1 .$$

Let $r_2 = r_1/20(1 + K_1)$ and $r_3 = r_2/\kappa$. Take rigid solid cylinders C_0 and C_1, contained in $B(p, r_3)$ and $B(f(p), r_3)$, such that:

(i) $\mathrm{rad}(C_0) \leq K_0 \mathrm{len}(C_0)$;
(ii) $\frac{1}{2}\mathrm{rad}(C_0) \leq \mathrm{rad}(C_1) \leq 2\mathrm{rad}(C_0)$;
(iii) $r_3 \leq \mathrm{len}(C_0), \mathrm{len}(C_1) \leq 2r_3$.

Then there exists $\phi \in \mathrm{Diff}^1(\Omega_1)$, supported in $B(f(p), r_1)$, and there exist pairwise disjoint subcylinders $C_{0,1}, \ldots, C_{0,N}$, such that:

(a) $\max\left\{[\phi]_{\mathrm{Lip}}, [\phi^{-1}]_{\mathrm{Lip}}\right\} \leq \gamma$;
(b) *$\phi \circ f$ maps the subcylinder $C_{0,k}$ across C_1 for each $k = 1, \ldots, N$.*

Remark B.5. Observe that we can use the same radii, r_2 and r_3 expressed in terms of r_1, in Corollary B.4 as in Lemma B.2. This follows, since given any cylinder $C(a, b; \rho)$ contained in the ball $B(p, r_3)$, the corresponding elongated neighbourhood $E(a, b; 3\rho)$, as used in Proposition B.3, will be contained in the ball $B(p, r_1)$.

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