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Nonlinear elliptic equations with measure valued absorption potentials

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Abstract. We study the semilinear elliptic equation $-\Delta u + g(u)\sigma = \mu$ with Dirichlet boundary conditions in a smooth bounded domain where σ is a nonnegative Radon measure, μ a Radon measure and g is an absorbing nonlinearity. We show that the problem is well posed if we assume that σ belongs to some Morrey class. Under this condition we give a general existence result for any bounded measure provided g satisfies a subcritical integral assumption. We study also the supercritical case when $g(r) = |r|^{q-1}r$, with $q > 1$ and μ satisfies an absolute continuity condition expressed in terms of some capacities involving σ.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2 boundary, σ a nonnegative Radon measure in Ω and $g : \mathbb{R} \to \mathbb{R}$ a continuous function satisfying, for some $r_0 \geq 0$,

$$rg(r) \geq 0 \qquad \text{for all } r \in (-\infty, -r_0] \cup [r_0, \infty). \tag{1.1}$$

In this article we consider the following problem

$$\begin{aligned} -\Delta u + g(u)\sigma &= \mu && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega, \end{aligned} \tag{1.2}$$

where μ is a Radon measure defined in Ω. By a solution we mean a function $u \in L^1(\Omega)$ such that $\rho g(u) \in L^1_\sigma(\Omega)$, where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ and $L^1_\sigma(\Omega)$ is the Lebesgue space of functions integrable with respect to σ, satisfying

$$-\int_\Omega u\Delta\zeta dx + \int_\Omega g(u)\zeta d\sigma = \int_\Omega \zeta d\mu, \tag{1.3}$$

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for all $\zeta \in W_0^{1,\infty}(\Omega)$ such that $\Delta\zeta \in L^\infty(\Omega)$. In the sequel, such a solution is called a *very weak solution*. A measure μ such that the problem admits a solution is called a *good measure*. We emphasize on the particular cases where $g(r) = |r|^{q-1} r$ with $q > 0$, or $g(r) = e^{\alpha r} - 1$ with $\alpha > 0$ and $N = 2$.

When σ is a measure with constant positive density with respect to the Lebesgue measure in $\mathbb{R}^N$, this problem has been initiated by Brezis and Benilan [4, 5] who gave a general existence result for any bounded measure μ under an integrability condition of g at infinity; their proof is based upon an a priori estimate of approximate solutions u_n in Lorentz spaces $L^{q,\infty}(\Omega)$, yielding the uniform integrability of $g(u_n)$ and hence the pre-compactness in $L^1(\Omega)$. If $g(r) = |r|^{q-1} r$, integrability condition is fufilled if and only if $0 < q < \frac{N}{N-2}$ (any $q > 0$ if $N = 2$). In the 2-dim case the integrability condition have been replaced by the exponential order of growth of g in [27]. When $g(u) = |u|^{q-1} u$ with $q \geq \frac{N}{N-2}$ not any bounded measure is eligible for solving (1.2). In fact Baras and Pierre [3] proved that when $N > 2$ and $q > 1$, a bounded Radon measure μ is eligible if and only if it vanishes on Borel sets with $c_{2,q'}$-capacity zero, where $q' = \frac{q}{q-1}$ is the conjugate exponent of q. Contrary to the previous subcritical case, the method for proving the necessity of this condition is based upon a duality-convexity argument, while the sufficiency uses the fact that any positive Radon measure absolutely continuous with respect to the $c_{2,q'}$-capacity can be approximated from below by a non-decreasing sequence of positive measures in $W^{-2,q}(\Omega)$ (see [13]). Furthermore they also gave a necessary and sufficient condition for a compact subset $K \subset \Omega$ to be removable for equation

$$-\Delta u + |u|^{q-1} u = 0 \qquad \text{in } \Omega \setminus K, \tag{1.4}$$

namely that $c_{2,q'}(K) = 0$.

The aim of this paper is to extend the previous constructions of Benilan-Brezis, Baras-Pierre and Vazquez to the case where σ is a general measure. In order to be able to deal with the convergence of approximate solutions we assume that σ belongs to the Morrey class $\mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ for some $\theta \in [0, N]$ which means

$$|B_r(x)|_\sigma := \int_{B_r(x)} d\sigma \leq cr^\theta \qquad \text{for all } (x, r) \in \Omega \times (0, \infty), \tag{1.5}$$

for some $c > 0$. Note that we extend σ by 0 in $\mathbb{R}^N \setminus \Omega$ and slightly abuse notation putting $\frac{N}{N-\theta} = \infty$ when $\theta = N$.

Our first result is the following:

Theorem A. *Assume $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ for some $\theta \in (N-2, N]$ and that g satisfies* (1.1). *Then, for any $\mu \in L^1_\rho(\Omega)$, there exists a very weak solution u of problem* (1.3). *If we assume moreover that g is nondecreasing and if u' is a very weak solution of* (1.3) *with right-hand side $\mu' \in L^1_\rho(\Omega)$, then the following estimates hold*

$$-\int_\Omega |u - u'| \, \Delta\zeta \, dx + \int_\Omega |g(u) - g(u')| \, \zeta \, d\sigma \leq \int_\Omega |\mu - \mu'| \, dx, \tag{1.6}$$

and

$$-\int_\Omega (u-u')_+\Delta\zeta dx + \int_\Omega (g(u)-g(u'))_+\zeta d\sigma \leq \int_\Omega (\mu-\mu')_+ dx \tag{1.7}$$

for all $\zeta \in W_0^{1,\infty}(\Omega)$ *such that* $\Delta\zeta \in L^\infty(\Omega)$ *and* $\zeta \geq 0$.

Note that (1.6) implies the uniqueness of the solution of (1.3), that we denote by u_μ, and (1.7) the monotonicity of the mapping $\mu \mapsto u_\mu$.

The next result extends Benilan-Brezis unconditional existence result for measures.

Theorem B. *Let* $N > 2$ *and* $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ *with* $N \geq \theta > N - \frac{N}{N-1}$. *Assume that* g *satisfies* (1.1) *and* $|g(r)| \leq \tilde{g}(|r|)$ *for all* $|r| \geq r_0$ *where* $\tilde{g}$ *is a continuous nondecreasing function on* $[r_0,\infty)$ *verifying*

$$\int_{r_0}^\infty \tilde{g}(t)t^{-1-\frac{\theta}{N-2}}dt < \infty. \tag{1.8}$$

Then, for any bounded Radon measure μ, *there exists a very weak solution* u *of problem* (1.3) *which moreover belongs to* $L^1_\sigma(\Omega)$. *Moreover, if we assume that* g *is nondecreasing then the solution is unique*.

Note that we recover Benilan-Brezis result when σ is the Lebesgue measure (so that $\theta = N$). Note also that when $g(r) = |r|^{q-1}r$, the integrability condition (1.8) is fullfilled if and only if $0 < q < \frac{\theta}{N-2}$.

In the 2-dimensional case the condition on θ is $2 \geq \theta > 0$ but (1.8) has to be modified. If $f : \mathbb{R} \mapsto \mathbb{R}_+$ is nondecreasing we define its exponential order of growth at ∞ (see [27]) by

$$a_\infty(f) = \inf\left\{\alpha \geq 0 : \int_0^\infty f(s)e^{-\alpha s}ds < \infty\right\}. \tag{1.9}$$

Similarly, if $h : \mathbb{R} \mapsto \mathbb{R}_-$ is nondecreasing its exponential order of growth at $-\infty$ is

$$a_{-\infty}(h) = \sup\left\{\alpha \leq 0 : \int_{-\infty}^0 h(s)e^{\alpha s}ds > -\infty\right\}. \tag{1.10}$$

If $g : \mathbb{R} \mapsto \mathbb{R}$ satisfies (1.1) but is not necessarily nondecreasing, we define the monotone nondecreasing hull g^* of g by

$$g^*(r) = \begin{cases} \sup\{g(s) : s \leq r\} & \text{for all } r \geq r_0 \\ 0 & \text{for all } r \in (-r_0, r_0) \\ \inf\{g(s) : s \geq r\} & \text{for all } r \leq -r_0. \end{cases} \tag{1.11}$$

We set

$$a_\infty(g) = a_\infty(g^*_+) \quad \text{and} \quad a_{-\infty}(g) = a_{-\infty}(g^*_-). \tag{1.12}$$

Theorem C. *Let* $\sigma \in \mathcal{M}^+_{\frac{2}{2-\theta}}(\Omega)$ *with* $2 \geq \theta > 0$ *and* $g : \mathbb{R} \mapsto \mathbb{R}$ *satisfies* (1.1).

(I) *If* $a_\infty(g) = 0 = a_{-\infty}(g)$, *then for any* $\mu \in \mathfrak{M}_b(\Omega)$, *problem* (1.3) *admits a very weak solution.*

(II) *If* $0 < a_\infty(g) < \infty$ *and* $-\infty < a_{-\infty}(g) < 0$ *there exists* $\delta > 0$ *such that if* $\mu \in \mathfrak{M}_b(\Omega)$ *satisfies* $\|\mu\|_{\mathfrak{M}_b} \leq \delta$ *problem* (1.3) *admits a very weak solution.*

In the *supercritical case*, that is when (1.8) is not satisfied, all the measures are not eligible for solving (1.3). Following [16], [28, Theorem 4.2] we can give a sufficient existence condition involving the Green function of the Laplacian. Let $G(.,.)$ be the Green kernel defined in $\Omega \times \Omega$ and $\mathbb{G}[.]$ the corresponding potential operator acting on bounded measures ν namely $\mathbb{G}[\nu](x) = \int_\Omega G(x,y)\,d\nu(y)$. We have the following result:

Theorem D. *Let* $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ *with* $N \geq \theta > N - \frac{N}{N-1}$ *and assume that* g *is nondecreasing and vanishes at* 0.

(I) *If* $\mu \in \mathfrak{M}_b(\Omega)$ *satisfies*

$$\rho g(\mathbb{G}[|\mu|]) \in L^1_\sigma(\Omega), \tag{1.13}$$

then problem (1.3) *admits a unique very weak solution.*

(II) *Let* $\mu = \mu_r + \mu_s$ *where* μ_r *is absolutely continuous with respect to the Lebesgue measure and* μ_s *is singular. Assume that* g *satisfies the* Δ_2 *condition, namely that*

$$\left|g(r+r')\right| \leq a\left(|g(r)| + \left|g(r')\right|\right) + b \qquad \text{for all } r, r' \in \mathbb{R}, \tag{1.14}$$

for some $a > 1$ *and* $b \geq 0$. *Then the previous assertion holds if* (1.13) *is replaced by*

$$\rho g(\mathbb{G}[|\mu_s|]) \in L^1_\sigma(\Omega). \tag{1.15}$$

Notice that (1.13) holds if either (i) σ and μ have disjoint support, or (ii) $\mu \in \mathcal{M}_p(\Omega)$ for some $p > \frac{N}{2}$. Indeed if (i) holds then $\mathbb{G}[|\mu|]$ is bounded pointwise on the support of σ, and if (ii) holds then by Lemma 2.2 $\mathbb{G}[|\mu|]$ is bounded pointwise in Ω. Obviously the same comment holds in the setting of II.

In order to make more explicit conditions (1.13), (1.15), we introduce the following growth assumption on g:

$$|g(r)| \leq c(1 + |r|^q) \qquad \text{for all } r \in \mathbb{R}, \tag{1.16}$$

for some $q > 1$. Notice that $\tilde{g}(r) = 1 + r^q$ satisfies (1.8) if and only if $q < \frac{\theta}{N-2}$. When σ is the Lebesgue measure and $g(r) = |r|^{q-1}r$, Baras and Pierre [3] gave a necessary and sufficient condition for the existence of a solution to (1.2) involving certain capacities associated to the Bessel potential spaces $H^{s,p}(\mathbb{R}^N)$ where $s \in \mathbb{R}$ and $p \in [1, \infty]$. Let us recall that

$$H^{s,p}(\mathbb{R}^N) = \left\{ f : f = \mathbf{G}_s * h, h \in L^p(\mathbb{R}^N) \right\}, \tag{1.17}$$

where $\mathbf{G}_s$ is the Bessel kernel of order s. By extension $\mathbf{G}_0 = \delta_0$, hence $H^{0,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$. When s is a positive integer, it is proved by Calderón [2, Theorem 1.2.3] that $H^{s,p}(\mathbb{R}^N)$ is the standard Sobolev space $W^{s,p}(\mathbb{R}^N)$. If $s > 0$, we denote by $c_{s,p}$ the associated capacity, called the Bessel capacity. It is defined for any compact set $K \subset \mathbb{R}^N$ by

$$c_{s,p}(K) = \inf \{ \|\phi\|^p_{H^{s,p}} : \phi \in \mathcal{S}(\mathbb{R}^N), \ \phi \geq 1 \text{ on } K \}. \tag{1.18}$$

The definition of $c_{s,p}$ is then extended first to open sets and then to arbitrary sets. We refer to [2] for general properties of Bessel spaces and their associated capacities $c_{s,p}$. We say that a measure $\mu \in \mathfrak{M}_b(\Omega)$ is absolutely continuous with respect to the $c_{s,p}$-capacity if for any Borel subset $E \subset \mathbb{R}^N$,

$$c_{s,p}(E) = 0 \Longrightarrow |\mu|(E) = 0.$$

Baras and Pierre's result states that equation (1.2), with σ standing for the Lebesgue measure and $g(r) = |r|^{q-1}r$, has a solution if and only if μ is absolutely continuous with respect to the $c_{2,q'}$-capacity. The next result generalizes the "if" part to the case where σ belongs to some Morrey space.

Theorem E. *Let $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > N - \frac{N}{N-1}$ and assume that g is nondecreasing and satisfies* (1.1) *and* (1.16). *Let $p > 1$ and $s \geq 0$ such that $N > sp > N - \theta$ and $\frac{\theta p}{N-sp} \geq q$. If $\mu \in \mathfrak{M}_b(\Omega)$ is absolutely continuous with respect to the $c_{2-s,p'}$-capacity, then* (1.2) *admits a unique very weak solution.*

As a particular case, we take $p = q$ and obtain that if μ is absolutely continuous with respect to the $c_{2-\frac{N-\theta}{q},q'}$-capacity, then (1.3) admits a unique solution. We thus recover Baras-Pierre's sufficient condition [3] when $\theta = N$.

We give an explicit condition on the measure μ in terms of Morrey spaces implying that it satisfies the conditions of Theorem E.

Proposition 1.1. *Under the assumptions on σ and g of Theorem* E, *if $\mu \in \mathcal{M}_{\frac{N}{N-\theta^*}}(\Omega)$ for some $\theta^* > \frac{(N-2)q-\theta}{q-1}$, then* (1.3) *admits a unique very weak solution.*

Notice that the condition on μ given in Proposition 1.1 is weaker than the one given after Theorem D.

When $g(r) = |r|^{q-1}r$ with $q > 1$, one can find a necessary conditions for the existence of a solution of (1.3) in the supercritical case under additional regularity assumptions on σ. By [2, Definition 2.3.3, Proposition 2.3.5], the following expression

$$c_q^\sigma(E) = \inf\left\{\int_\Omega |v|^{q'}\,d\sigma : v \in L_\sigma^{q'}(\Omega),\ v \geq 0,\ \mathbb{G}[v\sigma] \geq 1 \text{ on } E\right\}, \tag{1.19}$$

where E is any subset of Ω defines an outer capacity. The measure is called θ-regular if

$$\frac{1}{c}r^\theta \leq \int_{B_r(x)} d\sigma \leq cr^\theta \qquad \text{for all } (x,r) \in \Omega \times (0,1].$$

The next result gives a necessary condition for a measure to be a good measure.

Theorem F. *Let $q > 1$ and $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ be θ-regular with $N \geq \theta > N-2$. If $\mu \in \mathfrak{M}_b^+(\Omega)$ is such that problem (1.3) with $g(r) = |r|^{q-1}r$ admits a very weak solution, then μ vanishes on any Borel set E such that $c_q^\sigma(E) = 0$.*

Furthermore the c_q^σ- capacity admits the following representation in terms of Besov capacities. If $\Gamma \subset \Omega$ is the support of σ, we denote by $B_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(\Omega)$ the closed subspace of distributions $\zeta \in B_{q',\infty}^{2-\frac{N-\theta}{q}}(\Omega)$ such that the support of the distribution $\Delta\zeta$ is a subset of Γ. Then

$$c_q^\sigma(K) \sim c_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(K) := \inf\left\{\|\zeta\|^{q'}_{B_{q',\infty}^{2-\frac{N-\theta}{q}}} : \zeta \in B_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(\Omega),\ \zeta \geq \chi_K\right\}, \tag{1.20}$$

for all compact subset $K \subset \Omega$.

Finally a complete characterization of removable sets can be obtained under a much stronger assumption on σ, namely that $d\sigma = wdx$ with $\omega := w^{-\frac{1}{q-1}} \in L^1_{\text{loc}}(\Omega)$. If $K \subset \Omega$ is compact, we set

$$c_q^\omega(K) = \inf\left\{\int_\Omega |\Delta\zeta|^{q'}\,\omega dx : \zeta \in C_0^\infty(\Omega), 0 \leq \zeta \leq 1, \zeta = 1 \text{ in a neighborhood of } K\right\}. \tag{1.21}$$

This defines a capacity on Borel sets of Ω.

Theorem G. *Assume $q > 1$ and there exists a nonnegative Borel function w in Ω in the Muckenhoupt class $A_q(\Omega)$ such that $d\sigma = wdx$. If $K \subset \Omega$ is compact, a function $u \in L^1_{\text{loc}}(\Omega \setminus K)$ such that $|u|^q w \in L^1_{\text{loc}}(\Omega \setminus K)$ which satisfies*

$$-\Delta u + w\,|u|^{q-1}\,u = 0, \tag{1.22}$$

in the sense of distributions in $\Omega \setminus K$ *can be extended as a solution of the same equation in the whole* Ω *if and only if* $c_{q,w}(K) = 0$.

The assumption $w \in A_q(\Omega)$ can be weakened and replaced by $\omega = w^{\frac{1}{1-q}}$ is q'-admissible in the sense of [15, Chapter 1], a condition which implies in particular the validity of the Gagliardo-Nirenberg and the Poincaré inequalities.

2. Preliminaries

In the whole paper c denotes a generic positive constant whose value can change from one ocurrence to another even within a single string of estimates. Sometimes, in order to avoid ambiguity, we are led to introduce other notation for constants, for example c'.

We denote by $\mathfrak{M}_b(\Omega)$ the space of outer regular bounded Borel measures on Ω equipped with the total variation norm, and by $\mathfrak{M}_b^+(\Omega)$ its positive cone. Since Ω is bounded we can identify bounded Radon measures in Ω with measures μ in $\overline{\Omega}$ such that $|\mu|(\partial\Omega) = 0$. All the measures are extended by 0 in $\mathbb{R}^N \setminus \Omega$.

Let $G(.,.)$ be the Green kernel defined in $\Omega \times \Omega$ and $\mathbb{G}[.]$ the corresponding potential operator acting on bounded measures ν namely $\mathbb{G}[\nu](x) = \int_\Omega G(x,y)\,d\nu(y)$. We denote $L^{p,\infty}(\Omega)$ the usual weak L^p space. The next result is classical and valid in a much more general setting (see, *e.g.*, [6,11]).

Lemma 2.1. *Let* $\mu \in \mathfrak{M}_b(\Omega)$ *and* $v = \mathbb{G}[\mu]$ *be the (very weak) solution of*

$$\begin{aligned} -\Delta v &= \mu && \text{in } \Omega \\ v &= 0 && \text{in } \partial\Omega. \end{aligned} \tag{2.1}$$

I- *If* $N \geq 2$, *then* $v \in L^{\frac{N}{N-2},\infty}(\Omega)$, $\nabla v \in L^{\frac{N}{N-1},\infty}(\Omega)$ *and*

$$\|v\|_{L^{\frac{N}{N-2},\infty}} + \|\nabla v\|_{L^{\frac{N}{N-1},\infty}} \leq c\,\|\mu\|_{\mathfrak{M}_b}. \tag{2.2}$$

II- *If* $N = 2$, *then* $v \in BMO(\Omega)$, $\nabla v \in L^{2,\infty}(\Omega)$ *and*

$$\|v\|_{BMO} + \|\nabla v\|_{L^{2,\infty}} \leq c\,\|\mu\|_{\mathfrak{M}_b}. \tag{2.3}$$

This result can be refined when more information is available on the degree of concentration of μ. This leads to the definition of Morrey spaces of measures.

2.1. Morrey spaces of measures

If $1 \leq p \leq \infty$ we define the Morrey space $\mathcal{M}_p(\Omega)$ as the set of bounded outer regular Borel measures μ defined in Ω and extended by 0 in Ω^c, satisfying

$$|B_r(x)|_\mu := \int_{B_r(x)} d\,|\mu| \leq c r^{N(1-\frac{1}{p})} \qquad \text{for all } (x,r) \in \Omega \times \mathbb{R}_+, \tag{2.4}$$

for some $c > 0$. In particular $\mu \in \mathcal{M}_{\frac{N}{N-\theta}}(\Omega)$, $\theta \in [0, N]$, if

$$\int_{B_r(x)} d\,|\mu| \leq cr^\theta \qquad \text{for all } (x, r) \in \Omega \times \mathbb{R}_+.$$

We refer to [19] for a detailed study of $\mathcal{M}_p(\Omega)$ and full proofs of the various results we will recall now. Endowed with the norm

$$\|\mu\|_{\mathcal{M}_p} = \sup_{(x,r)\in\Omega\times\mathbb{R}_+} r^{N(\frac{1}{p}-1)} \, |B_r(x)|_\mu \,, \tag{2.5}$$

$\mathcal{M}_p(\Omega)$ is a Banach space and $\mathcal{M}_p^+(\Omega) = \mathcal{M}_p(\Omega) \cap \mathfrak{M}_b^+(\Omega)$ is its positive cone. We also set $M_p(\Omega) = \mathcal{M}_p(\Omega) \cap L^1_{\text{loc}}(\Omega)$; it is a closed subspace of $\mathcal{M}_p(\Omega)$ and, if $1 < p < \infty$, the following imbedding holds

$$L^p(\Omega) \hookrightarrow L^{p,\infty}(\Omega) \hookrightarrow M_p(\Omega). \tag{2.6}$$

Note that since Ω is bounded and any measure in Ω is extended to $\mathbb{R}^N$ by 0, it is easily seen that if $1 \leq q \leq p \leq \infty$ we have a continuous embedding $\mathcal{M}_p(\Omega) \hookrightarrow \mathcal{M}_q(\Omega)$ with

$$\|v\|_{\mathcal{M}_q} \leq (\text{diam}(\Omega))^{\frac{N}{q}-\frac{N}{p}} \, \|v\|_{\mathcal{M}_p} \qquad \text{for all } v \in \mathcal{M}_p(\Omega). \tag{2.7}$$

Indeed for any $x \in \Omega$ the ball centered at x with radius $\text{diam}(\Omega)$ contains Ω so that it is enough to consider $r \leq \text{diam}(\Omega)$. We have

$$r^{-N(1-1/q)} \, |B_r(x)|_\mu \leq r^{-N(1-1/q)} \, \|\mu\|_{\mathcal{M}_p} \, r^{N(1-1/p)} \leq (\text{diam}(\Omega))^{\frac{N}{q}-\frac{N}{p}} \, \|\mu\|_{\mathcal{M}_p} \,.$$

The following imbedding inequalities holds.

Lemma 2.2. *Let $\mu \in \mathcal{M}_p(\Omega)$ and v be the solution of* (2.1).
I- *If $1 < p < \frac{N}{2}$, then $v \in M_q(\Omega)$ with $\frac{1}{q} = \frac{1}{p} - \frac{2}{N}$ and there holds*

$$\|v\|_{\mathcal{M}_q} \leq c \, \|\mu\|_{\mathcal{M}_p} \,. \tag{2.8}$$

II- *If $p > \frac{N}{2}$, then v is bounded pointwise and*

$$\begin{aligned}
&\text{(i)} \quad v(x) \leq c\, \|\mu\|_{\mathcal{M}_p} \qquad \textit{for all } x \in \Omega, \\
&\text{(ii)} \quad \sup_{x\neq y} \frac{|v(x) - v(y)|}{|x-y|^\alpha} \leq c\, \|\mu\|_{\mathcal{M}_p} \;\; \textit{with} \;\; \alpha = 2 - \frac{N}{p} \quad \textit{if} \;\; N > p > \frac{N}{2}, \\
&\text{(iii)} \quad \sup_{x\neq y} \frac{|v(x) - v(y)|}{|x-y|^\alpha} \leq c\, \|\mu\|_{\mathcal{M}_p} \;\; \textit{with} \;\; \alpha \in (0,1) \quad \textit{if} \;\; N = p, \\
&\text{(iv)} \quad \sup_x |\nabla v(x)| \leq c\, \|\mu\|_{\mathcal{M}_p} \qquad \textit{if} \;\; N < p.
\end{aligned} \tag{2.9}$$

Remark. The previous regularity results are proved in [19, Proposition 3.1, 3.5] when $v = I_\alpha * \mu$ where I_α is the Riesz potential. However it is easily seen that the proof in [19] can be adapted to our setting. In particular for (2.8) we need that $G(x, y) \leq c|x-y|^{2-N}$, for (i) we use (2.7).

Remark. If we assume that $\mu \in \mathfrak{M}_\rho(\Omega) \cap \mathcal{M}_{p,\mathrm{loc}}(\Omega)$, the previous estimates acquire a local aspect and remain valid provided the supremum in the norms on the left-hand sides are taken on compact subsets of Ω.

2.2. Trace embeddings

Some applications of Morrey spaces to imbedding theorems (also called trace inequalities) can be found in Adams-Hedberg's book [2]. For the sake of completeness, we quote here the main result therein we will use in the sequel. If $0 < \alpha < N$ we recall that I_α (respectively G_α) is the Riesz potential (respectively the Bessel potential) of order α in $\mathbb{R}^N$. The next result is [2, Theorem 7.2.2, 7.3.2] (recall that the $c_{I_\alpha,p}$-Riesz capacity of a ball $B_r(x)$ is proportional to $r^{N-\alpha p}$ - see [2, Proposition 5.1.2].)

Proposition 2.3. *Let σ be a nonnegative Radon measure in $\mathbb{R}^N$, $N > \alpha p$ and $1 < p < q < \frac{Np}{N-\alpha p}$.*

(I)- *The following assertions are equivalent:*

$$\|I_\alpha * f\|_{L^q_\sigma(\mathbb{R}^N)} \leq c_1 \|f\|_{L^p(\mathbb{R}^N)} \qquad \text{for all } f \in L^p(\mathbb{R}^N), \tag{2.10}$$

for some $c_1 = c_1(N, \alpha, p, q) > 0$, and

$$\sigma \in \mathcal{M}_r(\mathbb{R}^N) \quad \text{with } \frac{1}{r} = q\left(\frac{1}{q} - \frac{1}{p} + \frac{\alpha}{N}\right). \tag{2.11}$$

(II)- *The mapping $f \mapsto G_\alpha * f$ is continuous from $L^p(\mathbb{R}^N)$ to $L^q_\sigma(\mathbb{R}^N)$ if and only if*

$$\sigma(K)^{\frac{1}{q}} \leq c_2 \left(c_{\alpha,p}(K)\right)^{\frac{1}{p}} \qquad \text{for all } K \subset \mathbb{R}^N, \tag{2.12}$$

where $c_{\alpha,p}$ denotes the Bessel capacity of order α defined in (1.18). *In fact this holds if and only if*

$$\sigma(B_r(x)) \leq c_3 \left(c_{\alpha,p}(B_r(x))\right)^{q/p} \qquad \text{for all } x \in \mathbb{R}^N,\ 0 < r \leq 1. \tag{2.13}$$

(III)- *A necessary and sufficient condition in order the mapping $f \mapsto G_\alpha * f$ be compact from $L^p(\mathbb{R}^N)$ to $L^q_\sigma(\mathbb{R}^N)$ is*

$$\begin{aligned} &\text{(i)} \qquad \lim_{\delta\to 0} \sup_{x\in\mathbb{R}^N,\, r\leq\delta} \frac{\sigma(B_r(x))}{\left(c_{\alpha,p}(B_r(x))\right)^{\frac{q}{p}}} = 0 \\ &\text{(ii)} \qquad \lim_{|x|\to\infty} \sup_{r\leq 1} \frac{\sigma(B_r(x))}{\left(c_{\alpha,p}(B_r(x))\right)^{\frac{q}{p}}} = 0. \end{aligned} \tag{2.14}$$

If $\mathbb{R}^N$ is replaced by a smooth bounded set Ω, we extend any bounded Radon measure in Ω by zero in Ω^c. In view of [2, 5.6.1] the $c_{I_\alpha,p}$-Riesz capacity and $c_{\alpha,p}$-Bessel capacity of balls $B_r(x)$ with $x \in \Omega$ and $r \leq 1$ are then equivalent. It follows that $c_{\alpha,p}(B_r(x)) \simeq r^{N-\alpha p}$. Then, it follows from II and III above, the definition of $H^{\alpha,p}(\mathbb{R}^N)$ and the existence of an extension operator $H^{\alpha,p}(\Omega) \hookrightarrow H^{\alpha,p}(\mathbb{R}^N)$ that the following holds:

Proposition 2.4. *Under the assumptions of Proposition* 2.3*, the embedding* $H^{\alpha,p}(\Omega) \hookrightarrow L^q_\sigma(\Omega)$ *is:*
(I)- *Continuous if and only if* $(\sigma(K))^{\frac{1}{q}} \leq c_2 \left(c_{\alpha,p}(K)\right)^{\frac{1}{p}}$ *for all* $K \subset \mathbb{R}^N$*,* i.e.*, if and only if* $\sigma \in \mathcal{M}^+_r(\mathbb{R}^N)$ *with* $\frac{1}{r} = q\left(\frac{1}{q} - \frac{1}{p} + \frac{\alpha}{N}\right)$*;*
(II)- *Compact if and only if*

$$\lim_{r \to 0} \sup_{x \in \Omega} \frac{\sigma(B_r(x))}{r^{\frac{(N-\alpha p)q}{p}}} = 0. \tag{2.15}$$

As an immediate corollary, we have:

Proposition 2.5. *Let* $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$*,* i.e.*,* $\sigma(B_r(x)) \leq cr^\theta$*,* $N > \alpha p$ *and* $1 < p < q < \frac{Np}{N-\alpha p}$*. Then the embedding*

$$H^{\alpha,p}(\Omega) \hookrightarrow L^q_\sigma(\Omega), \tag{2.16}$$

is continuous if and only if $\sigma(K) \leq c_1 \left(c_{\alpha,p}(K)\right)^{\frac{q}{p}}$ *for all* $K \subset \mathbb{R}^N$ *which holds iff* $q \leq \frac{\theta p}{N-\alpha p}$*. And the embedding* (2.16) *is compact iff* $q < \frac{\theta p}{N-\alpha p}$*.*

Other trace inequalities can be found in [21]. In the case $N = \alpha p$ the following estimate holds, see, *e.g.* [1], [20, Corollary 8.6.2], [31].

Proposition 2.6. *Let* σ *be a nonnegative Radon measure in* $\mathbb{R}^N$ *with compact support and* $N = \alpha p$*,* $p > 1$*. Then there exists a constant* $b = b(N, \alpha, p) > 0$ *such that*

$$\sup_{\|f\|_{L^p} \leq 1} \int_{\mathbb{R}^N} \exp\left(b\,|G_\alpha * f|^{p'}\right) d\sigma < \infty \tag{2.17}$$

if and only if $\sigma \in \mathcal{M}^+_\tau(\mathbb{R}^N)$ *for some* $\tau \in (1, \infty)$*.*

When $p = 1$ the next result is proved in [20, Section 1.4.3]

Proposition 2.7. *Let* σ *be a nonnegative bounded Radon measure in* $\mathbb{R}^N$*,* α *be an integer such that* $1 \leq \alpha \leq N$ *and* $q \geq 1$*. Then the following estimate holds*

$$\|f\|_{L^q_\sigma} \leq c_2 \sum_{|\beta| = \alpha} \|D^\alpha f\|_1 \qquad \text{for all } f \in C_0^\infty(\mathbb{R}^N), \tag{2.18}$$

for some $c_2 = c_2(N, p, q, \alpha) > 0$ *if and only if* $\sigma \in \mathcal{M}^+_{\frac{N}{N-q(N-\alpha)}}(\mathbb{R}^N)$*.*

3. The subcritical case

3.1. The variational construction

We prove in this section that if $\mu \in W^{-1,2}(\Omega)$ then, under some assumptions on g and σ, equation (1.2) has a variational solution.

We assume that $g \in C(\mathbb{R})$ satisfies (1.1), and set $G(r) := \int_0^r g(s)ds$. We will find a solution to (1.2) minimizing the functional

$$J(v) := \frac{1}{2}\int_\Omega |\nabla v|^2\, dx + \int_\Omega G(v)\, d\sigma - \langle \mu, v\rangle, \tag{3.1}$$

over the set

$$X_G(\Omega) := \{v \in W_0^{1,2}(\Omega) : G(v) \in L^1_\sigma(\Omega)\}. \tag{3.2}$$

The next proposition is a variant of a result in [8].

Proposition 3.1. *Assume $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > \frac{N}{2} - 1$. If $\mu \in W^{-1,2}(\Omega)$ there exists $u \in X_G(\Omega)$ which minimizes J in $X_G(\Omega)$. Furthermore u is a weak solution of* (1.2) *in the sense that*

$$\int_\Omega \nabla u.\nabla \zeta\, dx + \int_\Omega g(u)\zeta\, d\sigma = \langle \mu, \zeta\rangle \quad \text{for all } \zeta \in C_0^\infty(\Omega). \tag{3.3}$$

If g is nondecreasing this solution is unique and denoted by u_μ, and the mapping $\mu \mapsto u_\mu$ is nonnecreasing.

Proof. Step 1*: Existence of a minimizer*. If $N > 2$ we apply (2.16) with $\alpha = 1$ and $p = 2$, recalling that by Fourier transform $H^{1,2}(\Omega) = W^{1,2}(\Omega)$ (it is a special case of Calderón's theorem), to obtain that

$$W_0^{1,2}(\Omega) \hookrightarrow L_\sigma^{\frac{2\theta}{N-2}}(\Omega). \tag{3.4}$$

If $N = 2$ with $p = 2$ we take any $\alpha < 1$ and obtain

$$\|f\|_{L_\sigma^{\frac{\theta}{1-\alpha}}} \leq c_1 \|f\|_{W^{\alpha,2}} \leq c_1' \|f\|_{W^{1,2}}\,. \tag{3.5}$$

According to Proposition 2.5 the imbedding of $W_0^{1,2}(\Omega)$ into $L^p_\sigma(\Omega)$ is compact for any $p \in [1, \frac{2\theta}{N-2})$ if $N > 2$ and $1 \leq p < \infty$ if $N = 2$.

Let us first assume that g is bounded. Then $|G(v)| \leq m\,|v|$. Since g is continuous, $G(v) \in L^1_\sigma(\Omega)$ for any $v \in W_0^{1,2}(\Omega)$ and the functional J is well defined and is of class C^1 in $W_0^{1,2}(\Omega)$. Furthermore

$$\lim_{\|v\|_{W^{1,2}} \to \infty} J(v) = +\infty. \tag{3.6}$$

Let $\{u_n\}$ be a minimizing sequence. By (3.6), $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ and thus relatively compact in $L^1_\sigma(\Omega)$ and in $L^2(\Omega)$. Hence there exist $u \in L^2(\Omega)$ and $v \in L^1_\sigma(\Omega)$ such that, up to a subsequence, $u_n \to v$ in $L^1_\sigma(\Omega)$, and $u_n \to u$ strongly in $L^2(\Omega)$ and weakly in $W_0^{1,2}(\Omega)$. We can also assume that $u_n \to u$ $c_{1,2}$-quasi almost everywhere in the sense that there exists $E \subset \Omega$ with $c_{1,2}(E) = 0$ such that $u_n(x) \to u(x)$ for any $x \in \Omega \backslash E$. According to Proposition 2.5, σ is absolutely continuous with respect to the $c_{1,2}$-capacity. It follows that $\sigma(E) = 0$ so that $u_n \to u$ σ-almost everywhere and thus $u = v$ σ-almost everywhere. Thus we have that $u_n \to u$ in $L^2(\Omega)$, in $L^1_\sigma(\Omega)$, σ-almost everywhere and weakly in $W_0^{1,2}(\Omega)$. Then we have that $\langle \mu, u_n\rangle \to \langle \mu, u\rangle$. By the dominated convergence theorem we have also that $G(u_n) \to G(u)$ in $L^1_\sigma(\Omega)$. Therefore

$$J(u) \leq \liminf_{n\to\infty} J(u_n), \tag{3.7}$$

which implies that u is a minimizer of J in $W_0^{1,2}(\Omega)$.

If g is unbounded, we write $g = g_1 + g_2$ where $g_1 = g\chi_{(-r_0,r_0)}$, $g_2 = g\chi_{(-\infty-r_0]\cup[r_0,\infty)}$, where r_0 is defined in (1.1). Hence $G(r) = G_1(r) + G_2(r)$ where $|G_1(r)| \leq m\,|r|$ and $G_2(r)$ is nonnegative. Using again (2.14) we obtain that (3.6) holds. A minimizing sequence $\{u_n\}$ inherits the same property as above, hence $u_n \to u$ σ-almost everywhere in Ω and in $L^1_\sigma(\Omega)$, this implies that $G_1(u_n) \to G_1(u)$ in $L^1_\sigma(\Omega)$ and $G_2(u)$ is σ-measurable. By Fatou's lemma

$$\int G_2(u)d\sigma \leq \liminf_{n\to\infty} \int G_2(u_n)d\sigma,$$

which implies that (3.7) holds. Notice that, among the consequences, X_G is closed subset of $W_0^{1,2}(\Omega)$. Hence u is a minimizer of J in $X_G(\Omega)$.
Uniqueness holds if g is nondecreasing since it implies that J is stricly convex and actually X_G is a closed convex set.

Step 2: The minimizer is a weak solution. For $k > r_0$ we define g_k by

$$g_k(r) = \begin{cases} g(r) & \text{if } |r| \leq k \\ g(k) & \text{if } r > k \\ g(-k) & \text{if } r < -k. \end{cases}$$

Then g_k is continuous and bounded and the minimizer $u_k \in W_0^{1,2}(\Omega)$ of

$$J_k(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx + \int_\Omega G_k(v)\,d\sigma - \langle \mu, v\rangle \quad \text{where } G_k(r) = \int_0^s g_k(s)ds,$$

is a weak solution (*i.e.*, in the sense given by (3.3)) of

$$\begin{aligned} -\Delta u + g_k(u)\sigma &= \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.8}$$

The following energy estimate holds

$$\int_\Omega |\nabla u_k|^2 dx + \int_\Omega u_k g_k(u_k) d\sigma = \langle \mu, u_k \rangle \leq \|\mu\|_{W^{-1,2}} \|u_k\|_{W^{1,2}}, \tag{3.9}$$

and it implies

$$\int_\Omega |\nabla u_k|^2 dx + \int_\Omega |u_k g_k(u_k)| \, d\sigma \leq \|\mu\|^2_{W^{-1,2}} + m\sigma(\Omega) = M, \tag{3.10}$$

for some $m = m(r_0) > 0$. Up to a subsequence, $\{u_k\}_k$ converges to some u as $k \to \infty$, weakly in $W_0^{1,2}(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in Ω. By Proposition 2.4 the imbedding of $W^{1,2}(\Omega)$ in $L^q_\sigma(\Omega)$ is compact for any $q < \frac{2\theta}{N-2}$. Hence the subsequence can be taken such that $u_k \to u$, σ-almost everywhere as $k \to \infty$, and consequently $g_k(u_k) \to g(u)$ σ-almost everywhere. Let $E \subset \Omega$ be a Borel set, then for any $\lambda > r_0$,

$$\begin{aligned} M &\geq \int_E |g_k(u_k)u_k| \, d\sigma \\ &= \int_{E\cap\{|u_k|>\lambda\}} |g_k(u_k)u_k| \, d\sigma + \int_{E\cap\{|u_k|\leq\lambda\}} |g_k(u_k)u_k| \, d\sigma \\ &\geq \lambda \int_{E\cap\{|u_k|>\lambda\}} |g_k(u_k)| \, d\sigma + \int_{E\cap\{|u_k|\leq\lambda\}} |g_k(u_k)u_k| \, d\sigma. \end{aligned}$$

Therefore

$$\begin{aligned} \int_E |g_k(u_k)| \, d\sigma &= \int_{E\cap\{|u_k|>\lambda\}} |g_k(u_k)| \, d\sigma + \int_{E\cap\{|u_k|\leq\lambda\}} |g_k(u_k)| \, d\sigma \\ &\leq \frac{M}{\lambda} + \max\{|g(r)| : |r| \leq \lambda\}\sigma(E). \end{aligned}$$

For $\epsilon > 0$ we first choose λ such that $\frac{M}{\lambda} \leq \frac{\epsilon}{2}$ and then $\sigma(E) \leq \frac{\epsilon}{1+2\max\{|g(r)|\leq\lambda\}}$. This implies the uniform integrability of $\{g_k(u_k)\}_k$ in $L^1_\sigma(\Omega)$. Hence $g_k(u_k) \to g(u)$ in $L^1_\sigma(\Omega)$ by Vitali's convergence theorem. Since u_k is a weak solution of (3.8), there holds for any $\zeta \in C_0^\infty(\Omega)$,

$$\int_\Omega \nabla u_k . \nabla \zeta dx + \int_\Omega g_k(u_k)\zeta d\sigma = \langle \mu, \zeta \rangle. \tag{3.11}$$

Letting $k \to \infty$ we obtain, using the above convergence results,

$$-\int_\Omega \nabla u . \nabla \zeta dx + \int_\Omega g(u)\zeta d\sigma = \langle \mu, \zeta \rangle. \tag{3.12}$$

Hence u is a weak solution. If g is monotone, uniqueness is also a consequence of the weak formulation. Furthermore if μ, μ' belong to $W^{-1,2}(\Omega)$ are such that $\mu - \mu'$ is a nonnegative measure, then $\langle \mu' - \mu, (u'_\mu - u_\mu)_+ \rangle \leq 0$. Taking $(u'_\mu - u_\mu)_+$ for test function in the weak formulation yields $(u'_\mu - u_\mu)_+ = 0$. □

3.2. The L^1 case

In the sequel we set

$$\mathbb{X}(\Omega) = \{\zeta \in C^1(\overline{\Omega}), \zeta = 0 \text{ on } \partial\Omega \text{ and } \Delta\zeta \in L^\infty(\Omega)\}, \tag{3.13}$$

and $\mathbb{X}_+(\Omega) = \mathbb{X}(\Omega) \cap \{\zeta \in C^1(\overline{\Omega}) : \zeta \geq 0 \text{ in } \overline{\Omega}\}$. We recall (see, *e.g.*, [29]) that if $f \in L^1_\rho(\Omega)$ and $u \in L^1(\Omega)$ is a very weak solution of

$$-\Delta u = f \qquad \text{in } \Omega, \tag{3.14}$$

there holds

$$-\int_\Omega |u|\, \Delta\zeta dx \leq \int_\Omega f \,\mathrm{sign}(u)\zeta dx \qquad \text{for all } \zeta \in \mathbb{X}_+(\Omega), \tag{3.15}$$

and

$$-\int_\Omega u^+ \Delta\zeta dx \leq \int_\Omega f \,\mathrm{sign}_+(u)\zeta dx \qquad \text{for all } \zeta \in \mathbb{X}_+(\Omega). \tag{3.16}$$

Proposition 3.2. *Assume $N \geq 2$, $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > N-2$ and $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous nondecreasing function vanishing at 0. If $\mu \in L^1_\rho(\Omega)$ there exists a unique $u := u_\mu \in L^1(\Omega)$ very weak solution of* (1.2). *Furthermore, if $u_\mu, u_{\mu'} \in L^1(\Omega)$ are the very weak solutions of* (1.2) *with right-hand sides $\mu, \mu' \in L^1_\rho(\Omega)$, then*

$$\begin{aligned} &-\int_\Omega \left|u_\mu - u_{\mu'}\right| \Delta\zeta dx + \int_\Omega \left|g(u_\mu) - g(u_{\mu'})\right| \zeta d\sigma \\ &\leq \int_\Omega (\mu - \mu')\mathrm{sign}(u_\mu - u_{\mu'})\zeta dx, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} &-\int_\Omega (u_\mu - u_{\mu'})_+ \Delta\zeta dx + \int_\Omega (g(u_\mu) - g(u_{\mu'}))_+ \zeta d\sigma \\ &\leq \int_\Omega (\mu - \mu')\mathrm{sign}_+(u_\mu - u_{\mu'})\zeta dx \end{aligned} \tag{3.18}$$

for any $\zeta \in \mathbb{X}_+(\Omega)$. In particular the mapping $\mu \to u_\mu$ is nondecreasing.

The following result will be used several time in the sequel. Its proof is standard but we present it for the sake of completeness.

Lemma 3.3. *Assume $N > q \geq 1$ and $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}$ with $N \geq \theta > N - q$. Then σ vanishes on any Borel set with $c_{1,q}$-capacity zero.*

Proof. It suffices to prove the result when E is compact. We define the Λ_θ Hausdorff measure of a set E by

$$\begin{aligned}\Lambda_\theta(E) &= \lim_{\kappa\to 0}\Lambda_\theta^\kappa(E)\\ &:= \lim_{\kappa\to 0}\inf\left\{\sum_{j=1}^\infty r_j^\theta : 0<r_j\le\kappa\le\infty,\ E\subset\bigcup_{j=1}^\infty B_{r_j}(a_j)\right\}.\end{aligned} \tag{3.19}$$

Note that $\Lambda_\theta^\infty(E)$ is the Hausdorff content of E and it is smaller than $(\mathrm{diam}\,(E))^\theta$. For any covering of E by balls $B_{r_j}(a_j)$, $j\ge 1$, we have

$$\sigma(E)\le\sum_{j=1}^\infty\sigma(B_{r_j}(a_j))\le\|\sigma\|_{\frac{N}{N-\theta}}\sum_{j=1}^\infty r_j^\theta.$$

It follows that

$$\sigma(E)\le\|\sigma\|_{\frac{N}{N-\theta}}\Lambda_\theta(E).$$

Next, if $c_{1,q}(E)=0$ then $\Lambda_\theta(E)=0$ according to [2, Theorem 5.1.13], and thus $\sigma(E)=0$ by the previous inequality. □

We introduce the flow coordinates near $\partial\Omega$ defined by

$$\Pi(x)=(\rho(x),\tau(x))\in[0,\epsilon_0]\times\partial\Omega\quad\text{where }\ \tau(x)=\mathrm{proj}_{\partial\Omega}(x).$$

It is well-known that for ϵ_0 small enough, Π is a C^1-diffeomorphism from $\Omega_{\epsilon_0}:=\{x\in\overline{\Omega}:\rho(x)\le\epsilon_0\}$ to $[0,\epsilon_0]\times\partial\Omega$. With this diffeomorphism we can assimilate the surface measure dS_ϵ on $\Sigma_\epsilon=\{x\in\Omega:\rho(x)=\epsilon\}$ with the surface measure dS on $\Sigma_0=\partial\Omega$ by setting

$$\int_{\Sigma_\epsilon}v(x)dS_\epsilon(x)=\int_{\Sigma_0}v(\epsilon,\tau)dS(\tau).$$

Lemma 3.4. *Assume $N\ge 2$ and $\mu\in\mathfrak{M}(\Omega)$ satisfies*

$$\int_\Omega\rho d\,|\mu|<\infty. \tag{3.20}$$

Then $u=\mathbb{G}[\mu]$ satisfies

$$\lim_{\epsilon\to 0}\int_{\Sigma_0}|u|(\epsilon,\tau)dS(\tau)=0. \tag{3.21}$$

Proof. If $u=\mathbb{G}[\mu]$, it is the unique weak solution of $-\Delta u=\mu$ in Ω, $u=0$ on $\partial\Omega$. Hence $u=u_1-u_2$ where $u_1=\mathbb{G}[\mu^+]$ and $u_2=\mathbb{G}[\mu^-]$. Since μ_+ and μ_- satisfy the integrability condition (3.20) both u_1 and u_2 have a zero measure boundary trace (M-boundary trace in the sense of [18, Section 1.3]). Hence, taking for test function the function $\zeta = 1$,

$$\lim_{\epsilon\to 0}\int_{\Sigma_0} u_j(\epsilon,\tau)dS(\tau) = 0, \tag{3.22}$$

which implies (3.20). □

This result allows us to obtain the uniqueness of the solution even if the right-hand side is a measure.

Lemma 3.5. *Assume $N \geq 2$, $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > N-2$ and $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous nondecreasing function. If $\mu \in \mathfrak{M}(\Omega)$ there exists at most one very weak solution of* (1.2).

Proof. By Lemma 3.3 with $\alpha = 1$, $p = 2$, σ is absolutely continuous with respect to the $c_{1,2}$ capacity (it is diffuse in the terminology of [9]), and if $h \in L^1_\sigma(\Omega)$ the measure $h_+\sigma$, which is the increasing limit of $\inf\{n, h_+\}\sigma$ is also diffuse. Similarly $h_-\sigma$ is diffuse and so is $h\sigma$. Next we assume that u and u' are two very weak solutions of (1.2) and set $w = u - u'$. Hence

$$-\Delta w + (g(u) - g(u'))\sigma = 0.$$

Since $\rho(g(u) - g(u')) \in L^1_\sigma(\Omega)$, it follows from Lemma 3.4 that

$$\lim_{\epsilon\to 0}\int_{\Sigma_\epsilon} |w|(\epsilon,\tau)dS(\tau) = 0.$$

We use the Kato inequality for measures as in [10, Theorem 1.1]: Since $w \in L^1(\Omega)$, Δw^+ is a diffuse measure and

$$\Delta w^+ \geq \chi_{\{w\geq 0\}}\Delta w = \chi_{\{w\geq 0\}}(g(u) - g(u'))\sigma \geq 0 \ \text{ in } \Omega.$$

Since w^+ has an M-boundary trace by Lemma 3.4, we can apply [18, Lemma 1.5.8] with $\mu = -\chi_{\{w\geq 0\}}(g(u) - g(u'))\sigma$ which is a measure in $\mathfrak{M}_\rho(\Omega) := \{\nu \in \mathfrak{M}(\Omega) : \rho\nu \in \mathfrak{M}_b(\Omega)\}$. Then there exists $\tau \in \mathfrak{M}^+_\rho(\Omega)$ such that

$$-\Delta w^+ = \mu - \tau.$$

Equivalently

$$-\Delta w^+ + \chi_{\{w\geq 0\}}(g(u) - g(u'))\sigma = -\tau.$$

Since the M-boundary trace of w^+ is zero, it follows that $w^+ = -\mathbb{G}[\chi_{\{w\geq 0\}}(g(u) - g(u'))\sigma + \tau]$. Hence $w^+ = 0$ and $u \leq u'$. Similarly $u' \leq u$. □

The following variant will be useful in the sequel.

Lemma 3.6. *Assume $N \geq 2, \sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > N-2$ and $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous nondecreasing function. If $u, u' \in L^1(\Omega)$ are such that $\rho g(u)$ and $\rho g(u')$ belong to $L^1_\sigma(\Omega)$ and satisfy*

$$-\int_\Omega (u-u')\Delta\zeta dx + \int_\Omega (g(u)-g(u'))\zeta d\sigma = \int_\Omega \zeta d\nu \quad \text{for all } \zeta \in \mathbb{X}_+(\Omega) \quad (3.23)$$

for some $\nu \in \mathfrak{M}_+(\Omega)$ diffuse with respect to the $c_{1,2}$-capacity, then $u \geq u'$ $c_{1,2}$-quasi everywhere in Ω.

Proof. We use Kato's inequality, Lemma 3.4 and [18, Lemma 1.5.8] in the same way as in the proof of Lemma 3.5 since the measures $(g(u) - g(u'))d\sigma$ and ν are diffuse, $\Delta(u' - u)$ is diffuse, hence

$$\Delta(u' - u)_+ \geq \chi_{\{u' \geq u\}} \Delta(u' - u) = (g(u') - g(u))\chi_{\{u' \geq u\}} + \chi_{\{u' \geq u\}}\nu \geq 0.$$

Since $u' - u \in W^{1,q}_0(\Omega)$ for any $1 < q < \frac{N}{N-1}$, we conclude that $(u' - u)_+ = 0$ almost everywhere and $c_{1,2}$-quasi everywhere by [2, Theorem 6.1.4]. □

The next result and the corollary which follows are the key-stone for the proof of Proposition 3.2.

Lemma 3.7. *Let $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > N - 2$, $h \in L^\infty_\sigma(\Omega)$, $f \in L^s(\Omega)$ with $s > \frac{N}{2}$ and $w \in L^1(\Omega)$ be the very weak solution of*

$$\begin{aligned} -\Delta w + h\sigma &= f \qquad \text{in } \Omega \\ w &= 0 \qquad \text{in } \partial\Omega. \end{aligned} \quad (3.24)$$

Then w is continuous in $\overline{\Omega}$ and for any nondecreasing bounded function $\gamma \in C^2(\mathbb{R})$ vanishing at 0, there holds

$$-\int_\Omega j(w)\Delta\zeta dx + \int_\Omega \gamma(w)h\zeta d\sigma \leq \int_\Omega \gamma(w)\zeta f dx \qquad \text{for all } \zeta \in \mathbb{X}_+(\Omega), \quad (3.25)$$

where $j(r) = \displaystyle\int_0^r \gamma(s)ds$.

Proof. The solution is unique and expressed by $w = \mathbb{G}[f - h\sigma]$. Since $\frac{N}{N-\theta} > \frac{N}{2}$, $w \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$ by Lemma 2.2. Hence $\gamma(w)$ is continuous and therefore measurable. We extend σ by zero in Ω^c and denote $\sigma_n = \sigma * \eta_n$ where $\{\eta_n\}$ is a sequence of mollifiers. Then $\sigma_n \to \sigma$ in the narrow topology of Ω. For $n \in \mathbb{N}^*$, let w_n be the solution of

$$\begin{aligned} -\Delta w_n + h\sigma_n &= T_n(f) \qquad \text{in } \Omega \\ w_n &= 0 \qquad \text{in } \partial\Omega, \end{aligned} \quad (3.26)$$

where $T_n(f) = \min\{|f|, n\}\mathrm{sgn}(f)$. Then $w_n \in W^{2,s}(\Omega) \cap W_0^{1,\infty}(\Omega)$ for all $1 < s < \infty$. By Green's formula

$$-\int_\Omega j(w_n)\Delta\zeta dx + \int_\Omega \gamma(w_n)h\zeta d\sigma \leq \int_\Omega \gamma(w_n)\zeta f dx \quad \text{for all } \zeta \in \mathbb{X}_+(\Omega). \quad (3.27)$$

Since $w_n \to w$ uniformly in $\overline{\Omega}$, (3.25) follows. □

Corollary 3.8. *Under the assumptions of Lemma* 3.7*, there holds*

$$-\int_\Omega |w|\,\Delta\zeta dx + \int_\Omega \mathrm{sign}_0(w)h\zeta d\sigma \leq \int_\Omega \mathrm{sign}_0(w)\zeta f dx, \quad (3.28)$$

and

$$-\int_\Omega w_+\Delta\zeta dx + \int_\Omega \mathrm{sign}_+(w)\zeta h d\sigma \leq \int_\Omega \mathrm{sign}_+(w)\zeta f dx, \quad (3.29)$$

for any $\zeta \in \mathbb{X}_+(\Omega)$. *Moreover there exists a constant* $C > 0$ *depending only on* Ω *such that*

$$\int_\Omega \mathrm{sign}_0(w)h d\sigma \leq C\int_\Omega |f|dx. \quad (3.30)$$

Proof. To prove (3.28) we consider a sequence $\{\gamma_k\}$ of odd nondecreasing functions such that

$$\gamma_k(r) = \begin{cases} 1 & \text{if } r \geq 2k^{-1} \\ 0 & \text{if } -k^{-1} \leq r \leq k^{-1} \\ -1 & \text{if } r \leq -2k^{-1} \end{cases}$$

and such that $\{r\gamma_k(r)\}$ is nondecreasing for any r. Using γ_k in place of γ in (3.25) we obtain

$$-\int_\Omega j_k(w)\Delta\zeta dx + \int_\Omega \gamma_k(w)\zeta h d\sigma \leq \int_\Omega \gamma_k(w)\zeta f dx \quad \text{for all } \zeta \in \mathbb{X}_+(\Omega), \quad (3.31)$$

where $j_k(r) = \int_0^r \gamma_k(s)ds$. Since $\gamma_k(w) \uparrow w$ on $\Omega_+ := \{x \in \Omega : w(x) > 0\}$, there holds by the monotone convergence theorem,

$$\int_{\Omega_+} \gamma_k(w)\zeta\,|h|\,d\sigma \uparrow \int_{\Omega_+} w\zeta\,|h|\,d\sigma \qquad \text{as } k \to \infty.$$

Since

$$\left|\int_{\Omega_+} (w - \gamma_k(w))\zeta h d\sigma\right| \leq \int_{\Omega_+} |(w - \gamma_k(w))\zeta h|\,d\sigma = \int_{\Omega_+} (w - \gamma_k(w))\zeta\,|h| d\sigma,$$

we obtain

$$\int_{\Omega_+} \gamma_k(w)h\zeta d\sigma \to \int_{\Omega_+} wh\zeta d\sigma \qquad \text{as } k \to \infty.$$

Similarly, $\gamma_k(w) \downarrow w$ on $\Omega_- := \{x \in \Omega : w(x) < 0\}$ so that

$$\int_{\Omega_-} \gamma_k(w) h\zeta d\sigma \to \int_{\Omega_-} wh\zeta d\sigma \qquad \text{as } k \to \infty.$$

Combining these two results yields

$$\int_\Omega \gamma_k(w)\zeta h d\sigma \to \int_{\Omega_+} w\zeta h d\sigma - \int_{\Omega_-} w\zeta h d\sigma = \int_\Omega \operatorname{sign}_0(w)\zeta h d\sigma.$$

Using the dominated convergence theorem there holds

$$\int_\Omega \gamma_k(w)\Delta\zeta dx \to \int_\Omega \operatorname{sign}_0(w)\Delta\zeta dx,$$

and

$$\int_\Omega \gamma_k(w)\zeta f dx \to \int_\Omega \operatorname{sign}_0(w)\zeta f dx.$$

This implies (3.28). The proof of (3.17) is similar.

Eventually we prove (3.30). Let η_1 be the solution of

$$\begin{aligned} -\Delta\eta_1 &= 1 && \text{in } \Omega \\ \eta_1 &= 0 && \text{in } \partial\Omega. \end{aligned} \tag{3.32}$$

Then $\eta_1 = \mathbb{G}[1] \in \mathbb{X}_+(\Omega)$ and there exists $c, c' > 0$ depending only on Ω such that $c\rho \le \eta_1 \le c'\rho$. Given $\alpha \in (0, 1]$, let $j_\epsilon(r) = (r + \epsilon)^\alpha - \epsilon^\alpha$, $r \ge 0$, and $\zeta = j_\epsilon(\eta_1)$. Note that $\zeta \in C^2(\overline{\Omega}), 0 \le \zeta \le \eta^\alpha, \zeta = 0$ on $\partial\Omega$, $j'_\epsilon > 0$, $j''_\epsilon < 0$, so that $-\Delta\zeta = j'_\epsilon(\eta_1) - j''_\epsilon(\eta_1)|\nabla\eta_1|^2 \ge 0$. We deduce from (3.28) that

$$\int_\Omega \operatorname{sign}_0(w)(\eta + \epsilon)^\alpha h d\sigma \le \int_\Omega \operatorname{sign}_0(w)\eta^\alpha |f| dx + \epsilon^\alpha \int_\Omega \operatorname{sign}_0(w) h d\sigma.$$

We obtain

$$\int_\Omega \operatorname{sign}_0(w)\rho^\alpha h d\sigma \le C \int_\Omega \rho^\alpha |f| dx + \epsilon^\alpha |\tilde{\sigma}(\Omega)|.$$

Letting $\epsilon \to 0$ and then $\alpha \to 0$ we infer the result by dominated convergence. □

We are now in a position to prove Proposition 3.2.

Proof of Proposition 3.2. We divide the proof into several steps.

Step 1*: We assume that* $\mu \in L^\infty(\Omega)$. Let $\{\eta_n\}$ be a sequence of molifiers and $\sigma_n = \sigma * \eta_n$. If $\mu \in L^\infty(\Omega)$, the solution $u_n = u_{n,\mu}$ of

$$\begin{aligned} -\Delta u_n + g(u_n)\sigma_n &= \mu && \text{in } \Omega \\ u_n &= 0 && \text{in } \partial\Omega, \end{aligned} \tag{3.33}$$

is continuous in $\overline{\Omega}$. Since

$$-\mathbb{G}[\mu^-] \le -u_n^- \le 0 \le u_n^+ \le \mathbb{G}[\mu^+] \tag{3.34}$$

by the maximum principle, the sequence $\{u_n\}$ is uniformly bounded. Recalling that g is nondecreasing we have that the sequence $\{g(u_n)\}$ is also uniformly bounded in Ω, hence $g(u_n)\sigma_n$ is bounded in $\mathcal{M}_{\frac{N}{N-\theta}}(\Omega)$ independently of n, and from (2.9) it follows that u_n is bounded in $C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1]$ independently of n. Up to some subsequence, $\{u_n\}$, and thus also $\{g(u_n)\}$, are then uniformly convergent in $\overline{\Omega}$ with limit $u = u_\mu$ and $g(u) = g(u_\mu)$. Because $\sigma * \eta_n$ converges to σ in the narrow topology, u_μ is a very weak solution of (1.2). Notice that being continuous, $g(u)$ is measurable for the measure σ. By Lemma 3.5, u_μ is the unique solution of (1.2), hence the whole sequence $\{u_{\mu_n}\}$ converges to u_μ. Applying Corollary 3.8 with $w = u, \tilde{\sigma} = \sigma$ and $\zeta = \eta_1$ yields

$$\int_\Omega |u|\, dx + \int_\Omega |g(u)|\, \eta_1 d\sigma \le \int_\Omega |\mu|\, \eta_1 dx, \tag{3.35}$$

and (3.29) with $\zeta = \eta_1$ gives

$$\int_\Omega (u - u')_+ dx + \int_\Omega (g(u) - g(u'))_+ \eta_1 d\sigma \le \int_\Omega \eta_1 \operatorname{sign}_+(u - u')(\mu - \mu')_+ dx, \tag{3.36}$$

which implies the monotonicity of the mapping $\mu \mapsto u_\mu$.

Step 2: We assume that $\mu \in L^1(\Omega)$ is bounded from below. Set $\ell = \operatorname{ess\,inf} \mu$. For $k > 0$ set $\mu_k = \min\{k, \mu\}$ and $u_k := u_{\mu_k} \in L^\infty(\Omega)$. The sequence $\{\mu_k\}$ is nondecreasing, hence according to Step 1, the sequence $\{u_k\}$ is a nondecreasing sequence of continuous functions in $\overline{\Omega}$ bounded from below by $\ell\eta_1$, where η_1 is defined in (3.32). Its pointwise limit, denoted by u, is thus lower semicontinuous. Moreover $g(u_k) \to g(u)$ pointwise, hence $g(u)$ is lower semicontinuous and thus σ-measurable. Relation (3.35) applied to μ_k and u_k gives

$$\int_\Omega |u_k|\, dx + \int_\Omega |g(u_k)|\, \eta_1 d\sigma \le \int_\Omega |\mu_k|\, \eta_1 dx.$$

Passing to the limit using Fatou's lemma in the left-hand side and the dominated convergence theorem in the right-hand side yields

$$\int_\Omega |u|\, dx + \int_\Omega |g(u)|\, \eta_1 d\sigma \le \int_\Omega |\mu|\, \eta_1 dx. \tag{3.37}$$

We deduce that $u \in L^1(\Omega)$ and $\rho g(u) \in L^1_\sigma(\Omega)$. We have indeed a more precise result. Since g vanishes at 0 $g(u_k) = g(u_k^+) + g(-u_k^-)$. Hence $\rho g(u_k^+) \to \rho g(u^+)$ in $L^1_\sigma(\Omega)$ by the monotone convergence theorem. Furthermore $g(-u_1^-) \le g(-u_k^-) \le 0$, which implies that $\rho g(-u_k^-) \to \rho g(-u^-)$ in $L^1_\sigma(\Omega)$ by the dominated convergence theorem which finally implies that $\rho g(u_k) \to \rho g(u)$ in $L^1_\sigma(\Omega)$. Using

$\zeta \in \mathbb{X}_+(\Omega)$ as a test function in the very weak formulation of the equation satisfied by u_k gives

$$-\int_\Omega u_k \Delta\zeta dx + \int_\Omega g(u_k)\zeta d\sigma = \int_\Omega \zeta\mu_k dx.$$

Since $u_k \to u$ almost everywhere and $-l\eta_1 \le u_k \le u$ with $u \in L^1(\Omega)$, we can pass to the limit in the first term to obtain $\int_\Omega u_k\Delta\zeta dx \to \int_\Omega u\Delta\zeta dx$. Because $|\mu_k| \le |\mu| \in L^1(\Omega)$ and $\mu_k \to \mu$ almost everywhere, we can also pass to the limit in the last term: $\int_\Omega \zeta\mu_k dx \to \int_\Omega \zeta\mu dx$. It remains to pass to the limit in the nonlinearity. Because $u_k \uparrow u$ and g is nondecreasing, we have $g(u_k) \uparrow g(u)$. Thus by the monotone convergence theorem,

$$-\int_\Omega u\Delta\zeta dx + \int_\Omega g(u)\zeta d\sigma = \int_\Omega \zeta\mu dx,$$

and u is very weak solution of (1.2).

Step 3*: We assume that* $\mu \in L^1(\Omega)$. For $\ell \in \mathbb{R}$, we set $\mu^\ell = \sup\{\mu, \ell\}$ and denote by u^ℓ the solution of (1.2) with right-hand side μ^ℓ. Note that the sequence $\{\mu^\ell\}_\ell$ is increasing, bounded from above by μ^+ so that $u^\ell \le u_{\mu^+}$, where u_{μ^+} is the solution of (1.2) with right-hand side μ^+ which exists according to the previous step, and the sequence $\{u^\ell\}_\ell$ is monotone nondecreasing with ℓ with pointwise limit u when $\ell \to -\infty$. Hence $u \le u^\ell \le u_{\mu^+}$ for any $\ell \le 0$. The sequence $\{g(u^\ell)\}_\ell$ is monotone nondecreasing with limit $g(u)$ when $\ell \to -\infty$, and there holds $g(u) \le g(u^\ell) \le g(u_{\mu^+})$ for any $\ell \le 0$. Since $g(u^\ell)$ is lower semicontinuous and σ-measurable, $g(u)$ shares the same properties.

Applying (3.37) to $\mu = \mu^\ell$ and $u = u^\ell$ gives

$$\int_\Omega \left|u^\ell\right| dx + \int_\Omega \left|g(u^\ell)\right| \eta_1 d\sigma \le \int_\Omega \left|\mu^\ell\right| \eta_1 dx.$$

Passing to the limit in the left-hand side using Fatou's lemma we obtain

$$\int_\Omega |u|\, dx + \int_\Omega |g(u)|\, \eta_1 d\sigma \le \int_\Omega |\mu|\, \eta_1 dx.$$

We deduce that $u \in L^1(\Omega)$ and $\rho g(u) \in L^1_\sigma(\Omega)$. We conclude as in Step 2 that u is solution of (1.2).

Step 4*: Proof of* (3.17) *and* (3.18).

For $\ell < 0 < k$ we set $\mu_k^\ell = \sup\{\ell, \inf\{k, \mu\}\}$ and $(\mu')_k^\ell = \sup\{\ell, \inf\{k, \mu'\}\}$, and denote by u_k^ℓ and $(u')_k^\ell$ the solution of (1.2) with right-hand side μ_k^ℓ and $(\mu')_k^\ell$. Then, by Corollary 3.8, for any $\zeta \in \mathbb{X}(\Omega)$ there holds

$$-\int_\Omega \left|u_k^\ell - (u')_k^\ell\right| \Delta\zeta dx + \int_\Omega \left|g(u_k^\ell) - g((u')_k^\ell)\right| \zeta d\sigma$$
$$\le \int_\Omega \operatorname{sign}_0(u_k^\ell - (u')_k^\ell)(\mu_k^\ell - (\mu')_k^\ell)\zeta dx.$$

Using the previous convergence theorem when $k \to \infty$ and then $\ell \to -\infty$, we derive (3.17). The proof of (3.18) is similar. □

Remark. If it is not assumed that g is nondecreasing, the above proof by monotonicity does not work. However the existence will follow from Theorem B if it is assumed that the extra assumptions in this theorem are satisfied: $\theta > N - q$ for some $q \in (1, \frac{N}{N-1})$ and the growth assumptions of Theorem B.

3.3. Diffuse case

We recall that a measure μ is said to be diffuse with respect to the $c_{s,p}$-capacity defined in (1.18) if $|\mu|$ vanishes on all sets with zero $c_{s,p}$-capacity. An important result due to Feyel and de la Pradelle [13] is the following:

Proposition 3.9. *Let $\alpha > 0$ and $1 < p < \infty$. If $\lambda \in \mathfrak{M}_b^+(\Omega)$ does not charge sets with zero $c_{\alpha,p}$-capacity, there exists an increasing sequence $\{\lambda_n\} \subset H^{-\alpha,p'}(\Omega) \cap \mathfrak{M}_b^+(\Omega)$, λ_n with compact support in Ω, which converges to λ.*

Proposition 3.10. *Assume $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}$ with $N \geq \theta > N - 2$, and that $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous nondecreasing function vanishing at 0. Then for any $\mu \in \mathfrak{M}_b^+(\Omega)$ diffuse with respect to the $c_{1,2}$-capacity there exists a unique very weak solution u to (1.2).*

Proof. According to Proposition 3.9, there exists an increasing sequence of nonnegative measures $\{\mu_n\}$ belonging to $W^{-1,2}(\Omega)$ and converging to μ and by Proposition 3.1, $\{u_{\mu_n}\}$ is a nondecreasing sequence of weak solutions of (1.2) with $\mu = \mu_n$. We claim that $u_{\mu_n} \uparrow u_\mu$ which is a very weak solution of (1.2). There holds,

$$\int_\Omega u_{\mu_n} dx + \int_\Omega g(u_{\mu_n})\eta_1 d\sigma = \int_\Omega \eta_1 d\mu_n \leq \int_\Omega \eta_1 d\mu,$$

where η_1 is defined in (3.32). Since $u_{\mu_n} \geq 0$, $u_{\mu_n} \uparrow u$ and $g(u_{\mu_n}) \uparrow g(u)$. Since u_{μ_n} is σ-measurable by Proposition 3.1, u is also σ-measurable. Hence $g(u)$ shares this measurability property since g is continuous. Hence, by the monotone convergence theorem

$$\int_\Omega u dx + \int_\Omega g(u)\eta_1 d\sigma = \int_\Omega \eta_1 d\mu. \tag{3.38}$$

Furthermore $u_{\mu_n} \to u$ in $L^1(\Omega)$. Indeed it suffices to show that $\{u_{\mu_n}\}$ is uniformly equiintegrable which follows from $0 \leq \int_\omega u_{\mu_n} dx \leq \int_\omega u dx$ and the fact that $u \in L^1(\Omega)$. We show in the same way that $\rho g(u_{\mu_n}) \to \rho g(u)$ in $L^1_\sigma(\Omega)$. This implies that $u = u_\mu$ is the very weak solution of (1.2). □

3.4. Subcritical nonlinearities: proof of Theorem B

Lemma 3.11. *Assume* $N > 2$ *and* $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ *with* $N \geq \theta > N-2$. *If* $\mu \in \mathfrak{M}_b(\Omega)$ *and* $\lambda \geq 0$, *we set* $E_\lambda[\mu] := \{x \in \Omega : \mathbb{G}[|\mu|](x) > \lambda\}$. *Then*

$$e^\sigma_\lambda(\mu) := \int_{E_\lambda[\mu]} d\sigma \leq c\,\|\mu\|^{\frac{\theta}{N-2}}_{\mathfrak{M}_b}\lambda^{-\frac{\theta}{N-2}} \qquad \textit{for all } \lambda > 0. \tag{3.39}$$

Proof. It suffices to prove the result if $\mu \geq 0$. Indeed since $\mathbb{G}[|\mu|] = \mathbb{G}[\mu^+] + \mathbb{G}[\mu^-]$, we have $E_\lambda[\mu] \subset E_{\lambda/2}[\mu^+] \cup E_{\lambda/2}[\mu^-]$ and thus $e^\sigma_\lambda(\mu) \leq e^\sigma_{\lambda/2}(\mu^+) + e^\sigma_{\lambda/2}(\mu^+)$. If the result holds for nonnegative measure, in particular for $\mu^\pm$, then

$$\begin{aligned}\lambda^{\frac{\theta}{N-2}}e^\sigma_\lambda(\mu) &\leq c(\mu^+(\Omega)^{\frac{\theta}{N-2}} + \mu^-(\Omega)^{\frac{\theta}{N-2}}) \leq c(\mu^+(\Omega)+\mu^-(\Omega))^{\frac{\theta}{N-2}} \\ &= c\,\|\mu\|^{\frac{\theta}{N-2}}_{\mathfrak{M}_b}.\end{aligned}$$

Thus, we assume from now on that μ is nonnegative.

If $\mu = \delta_a$ for some $a \in \Omega$, then $\mathbb{G}[\delta_a](x) \leq c_N\,|x-a|^{2-N}$ so that $E_\lambda[\delta_a] \subset B_{(\frac{c_N}{\lambda})^{\frac{1}{N-2}}}(a)$. Since $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ it follows that

$$e^\sigma_\lambda(\delta_a) \leq c\lambda^{-\frac{\theta}{N-2}}. \tag{3.40}$$

Let $E \subset \Omega$ be a Borel set. For any given $t > 0$ there holds

$$\int_E \mathbb{G}[\delta_a]d\sigma = \int_{E\cap E_t[\delta_a]} \mathbb{G}[\delta_a]d\sigma + \int_{E\cap E^c_t[\delta_a]} \mathbb{G}[\delta_a]d\sigma.$$

Clearly $\int_{E\cap E^c_t[\delta_a]} \mathbb{G}[\delta_a]d\sigma \leq t\sigma(E)$ and

$$\int_{E\cap E_t[\delta_a]} \mathbb{G}[\delta_a]d\sigma \leq \int_{E_t[\delta_a]} \mathbb{G}[\delta_a]d\sigma \leq -\int_t^\infty s\,de^\sigma_s(\delta_a) \leq c\frac{\theta t^{1-\frac{\theta}{N-2}}}{\theta+2-N},$$

where the last inequality follows by integration by parts and the help of (3.40). Then

$$\int_E \mathbb{G}[\delta_a]d\sigma \leq t\sigma(E) + c\frac{\theta t^{1-\frac{\theta}{N-2}}}{\theta+2-N}.$$

Minimizing the right-hand side with respect to t, we infer

$$\int_E \mathbb{G}[\delta_a]d\sigma \leq c\sigma(E)^{1-\frac{N-2}{\theta}}. \tag{3.41}$$

We first suppose that $\mu = \sum_{j=1}^{\infty} \alpha_j \delta_{a_j}$ for some $\alpha_j > 0$ and $a_j \in \Omega$. In particular $\sum_{j=1}^{\infty} \alpha_j = \|\mu\|_{\mathfrak{M}_b}$. Using Fubini's theorem and (3.41) we see that for any Borel set $E \subset \Omega$,

$$\int_E \mathbb{G}[\mu](x) d\sigma(x) = \sum_{j=1}^{\infty} \alpha_j \int_E \mathbb{G}[\delta_{a_j}(x)] d\sigma(x) \le c\sigma(E)^{1-\frac{N-2}{\theta}} \|\mu\|_{\mathfrak{M}_b}. \quad (3.42)$$

Taking in particular $E = E_\lambda[\mu]$ we obtain

$$\lambda e_\lambda^\sigma(\mu) \le \int_{E_\lambda[\mu]} \mathbb{G}[\mu](x) d\sigma(x) \le c(e_\lambda^\sigma(\mu))^{1-\frac{N-2}{\theta}} \|\mu\|_{\mathfrak{M}_b},$$

which implies the claim. Notice that the constant c in the right-hand side depends only on N and $\|\sigma\|_{\mathcal{M}_{\frac{N}{N-\theta}}}$.

For a general nonnegative measure $\mu \in \mathfrak{M}_b(\Omega)$, we consider a sequence of nonnegative measures $\{\mu_n\} \subset \mathfrak{M}_b(\Omega)$ where each μ_n is a sum of Dirac masses as before and such that $\mu_n \to \mu$ weakly as $n \to \infty$. Then we have

$$e_\lambda^\sigma(\mu_n) := \int_{E_\lambda[\mu_n]} d\sigma \le c\|\mu_n\|_{\mathfrak{M}_b}^{\frac{\theta}{N-2}} \lambda^{-\frac{\theta}{N-2}},$$

with $\|\mu\|_{\mathfrak{M}_b} \le \liminf_{n\to\infty} \|\mu_n\|_{\mathfrak{M}_b}$. We thus need to prove that

$$\liminf \int_{E_\lambda[\mu_n]} d\sigma \ge \int_{E_\lambda[\mu]} d\sigma. \quad (3.43)$$

We first observe that for any $t > 0$ and $x \in \Omega$ the set $\{y \in \Omega : \mathbb{G}(x, y) > t\}$ is open (with $\mathbb{G}(x, x) = +\infty$). It follows from [7][Theorem 2.1] that $\liminf_{n\to\infty} \mu_n(\{\mathbb{G}(x, \cdot) > t\}) \ge \mu(\{\mathbb{G}(x, \cdot) > t\})$. We can take the lim inf using Fatou's lemma in

$$\int_\Omega \mathbb{G}(x, y)\, d\mu_n(y) = \int_0^{+\infty} \mu_n(\{\mathbb{G}(x, \cdot) > t\})\, dt,$$

to derive

$$\liminf_{n\to\infty} \mathbb{G}[\mu_n](x) \ge \int_0^{+\infty} \mu(\{G(x, \cdot) > t\})\, dt = \int_\Omega G(x, y)\, d\mu(y) = \mathbb{G}[\mu](x).$$

We infer that for any $x \in \Omega$ such that $\chi_{E_\lambda(\mu)}(x) = 1$ we have $\liminf_{n\to\infty} \mathbb{G}[\mu_n](x) > \lambda$, hence $\mathbb{G}[\mu_n](x) > \lambda$ for n large enough. Thus $\chi_{E_\lambda(\mu_n)}(x) = 1$ eventually, and then

$$\liminf_{n\to\infty} \chi_{E_\lambda[\mu_n]}(x) \ge \chi_{E_\lambda[\mu]}(x) \qquad \text{for all } x \in \Omega.$$

The claim (3.43) follows by Fatou's lemma. □

We are now in a position to prove Theorem B.

Proof of Theorem B. We note that if g is nondecreasing, uniqueness follows from estimate Lemma 3.5. Let $\{\eta_n\}$ be a sequence of mollifiers, $\mu_n = \mu * \eta_n$ and $u_n \in W_0^{1,2}(\Omega)$ a minimizing weak solution of

$$\begin{aligned} -\Delta u_n + g(u_n)\sigma = \mu_n \qquad & \text{in } \Omega \\ u_n = 0 \qquad & \text{in } \partial\Omega, \end{aligned} \tag{3.44}$$

given by Proposition 3.1. We write $g(r) = g_1(r) + g_2(r)$ with $g_1 = g\chi_{(-r_0,r_0)}$, $g_2 = g\chi_{(-\infty-r_0]\cup[r_0,\infty)}$, and set $m = \sup\{g(r) : -r_0 \le r \le r_0\} \ge 0$ and $m' = \inf\{g(r) : -r_0 \le r \le r_0\} \le 0$. Then

$$-\mathbb{G}[\mu_n^-] - m\mathbb{G}[\sigma] \le u_n \le \mathbb{G}[\mu_n^+] - m'\mathbb{G}[\sigma].$$

Since $\sigma \in \mathcal{M}_p^+(\Omega)$ for some $p > N/2, \mathbb{G}[\sigma] \in C^{0,\alpha}(\overline{\Omega})$ by Lemma 2.2. Moreover $\mathbb{G}[|\mu_n|] \in C(\overline{\Omega})$ since $|\mu_n| \in C(\overline{\Omega})$. It follows that

$$|u_n| \le \mathbb{G}[|\mu_n|] + M \le c_n, \tag{3.45}$$

where $M, c_n \ge 0$.

Since $u_n \in W_0^{1,2}(\Omega)$, its precise representative (that we identify with u_n) is defined $c_{1,2}$-quasi-everywhere, is $c_{1,2}$-continuous and

$$u_n(x) = \lim_{r\to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} u_n(y)\, dy$$

for any $y \in \Omega \setminus E_n$ with $c_{1,2}(E_n) = 0$ (see [2]). It follows that $|u_n| \le c_n$ in $E := \cup E_n$. Note that $c_{1,2}(E) = 0$ so that $\sigma(E) = 0$ by Lemma 3.3. Hence $|u_n| \le c_n$ σ-almost everywhere, $g(u_n) \in L^\infty_\sigma(\Omega)$, and therefore $g(u_n)\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$. We can then apply Corollary 3.8 to obtain, for any $\zeta \in \mathbb{X}_+(\Omega)$, that

$$-\int_\Omega |u|_n \,\Delta\zeta\, dx + \int_\Omega \operatorname{sign}_0(u_n) g(u_n)\zeta\, d\sigma \le \int_\Omega \operatorname{sign}_0(u_n)\zeta\,\mu_n dx,$$

which implies

$$-\int_\Omega |u|_n \,\Delta\zeta\, dx + \int_\Omega |g_2(u_n)|\zeta\, d\sigma \le \int_\Omega \operatorname{sign}_0(u_n)\zeta\,\mu_n dx + c\int_\Omega \zeta\, d\sigma. \tag{3.46}$$

We take $\zeta = \eta_1$ and obtain

$$\begin{aligned} \int_\Omega |u_n|\, dx + \int_\Omega |g_2(u_n)|\,\eta_1 d\sigma &\le \int_\Omega |\mu_n|\,\eta_1 dx + c \\ &\le \int_\Omega \eta_1 d\,|\mu| + c = c', \end{aligned} \tag{3.47}$$

so that $\{u_n\}$ is bounded in $L^1(\Omega)$. We also have from Corollary 3.8 that

$$\int_\Omega \operatorname{sign}_0(u_n)g(u_n)d\sigma \le C\int_\Omega |\mu_n|\rho dx$$

and so

$$\int_\Omega |g_2(u_n)|d\sigma \le C\int_\Omega |\mu_n|dx + \int_\Omega |g_1(u_n)|d\sigma \le C, \tag{3.48}$$

with C independent of n. We deduce that the sequence of measures $\{g(u_n)\}$ is bounded.

By the standard regularity estimates, the sequence $\{u_n\}$ is bounded in $W^{1,q}(\Omega)$, $q < \frac{N}{N-1}$. Then there exists $u \in W^{1,q}(\Omega)$, $q < \frac{N}{N-1}$, such that, up to a subsequence, $u_n \to u$ in $L^1(\Omega)$ and also pointwise in $\Omega \setminus E$ where $c_{1,q}(E) = 0$. We fix $q \in \left(1, \frac{N}{N-1}\right)$ such that $\theta > N - q$. In view of Lemma 3.3, $\sigma(E) = 0$ so that $g(u_n) \to g(u)$ σ-almost everywhere. Applying Fatou's lemma in (3.48) gives that $g(u) \in L^1_\sigma(\Omega)$.

In order to prove the uniform integrability of $\{g(u_n)\}$ for the measure σ we can assume that $|g_2| \le \tilde g$ with a function satisfying (1.8) still denoted by $\tilde g$ and let $E \subset \Omega$ be a Borel set. Then

$$\begin{aligned}\int_E |g_2(u_n)|\,d\sigma &\le \int_{E\cap\{|u_n|\le t\}} |g_2(u_n)|\,d\sigma + \int_{E\cap\{|u_n|>t\}} |g_2(u_n)|\,d\sigma \\ &\le \tilde g(t)\int_E d\sigma + \int_{\{|u_n|>t\}} \tilde g(|u_n|)d\sigma.\end{aligned}$$

Then we estimate the second integral in the right-hand side: for $\lambda > M$ we set

$$S_n(\lambda) = \{x \in \Omega : |u_n(x)| > \lambda\} \quad \text{and} \quad b_n^\sigma(\lambda) = \int_{S_n(\lambda)} d\sigma.$$

In view of (3.45) we have $|u_n| \le \mathbb{G}(|\mu_n|) + M$ so that $S_n(\lambda) \subset E_{\lambda-M}[\mu_n]$. Hence $b_n^\sigma(\lambda) \le e^\sigma_{\lambda-M}(|\mu_n|)$. This implies

$$\begin{aligned}\int_{\{|u_n|>t\}} \tilde g(|u_n|)d\sigma &= -\int_t^\infty \tilde g(\lambda)db_n^\sigma(\lambda) \\ &\le \int_t^\infty b_n^\sigma(\lambda)d\tilde g(\lambda) \\ &\le \int_t^\infty e^\sigma_{\lambda-M}(|\mu_n|)d\tilde g(\lambda).\end{aligned}$$

Using (3.39) we obtain

$$\begin{aligned}\int_{\{|u_n|>t\}} \tilde g(|u_n|)d\sigma &\le c\,\|\mu\|_{\mathfrak{M}_b}^{\frac{\theta}{N-2}} \int_t^\infty (\lambda - M)^{-\frac{\theta}{N-2}} d\tilde g(\lambda) \\ &\le \frac{c\theta}{N-2}\int_t^\infty (\lambda - M)^{-\frac{\theta}{N-2}-1}\tilde g(\lambda)d\lambda.\end{aligned}$$

In view of assumption (1.8), given $\epsilon > 0$ we fix $t > M$ such that

$$\frac{c\theta}{N-2}\int_t^\infty (\lambda - M)^{-\frac{\theta}{N-2}-1}\tilde{g}(\lambda)d\lambda \leq \frac{\varepsilon}{2}.$$

Then, setting $\delta = \frac{\epsilon}{2\tilde{g}(t)}$, we deduce

$$\int_E d\sigma \leq \delta \Longrightarrow \int_E |g_2(u_n)|\, d\sigma \leq \varepsilon.$$

Since g_1 is bounded, this implies that $\{g(u_n)\}$ is uniformly integrable in $L^1_\sigma(\Omega)$. Since we already know that $g(u_n) \to g(u)$ σ-almost everywhere, it follows by Vitali's convergence theorem that $g(u_n) \to g(u)$ in $L^1_\sigma(\Omega)$. Taking $\zeta \in \mathbb{X}(\Omega)$ and letting $n \to \infty$ in the equality

$$-\int_\Omega u_n \Delta\zeta dx + \int_\Omega g(u_n)\zeta d\sigma = \int_\Omega \zeta d\mu_n$$

yields the result. □

4. The 2-D case

In this section Ω is a bounded C^2 planar domain. The next result is the 2-D version of Lemma 3.11.

Lemma 4.1. *Assume $N = 2$ and $\sigma \in \mathcal{M}^+_{\frac{2}{2-\theta}}(\Omega)$ with $2 \geq \theta > 0$. If $\mu \in \mathfrak{M}_b(\Omega)$ and $\lambda \geq 0$, we set $E_\lambda[\mu] := \{x \in \Omega : \mathbb{G}[|\mu|](x) > \lambda\}$. Then*

$$e^\sigma_\lambda(\mu) := \int_{E_\lambda[\mu]} d\sigma \leq |\Omega|_\sigma\, e^{1-\frac{\lambda}{\gamma\|\mu\|_{\mathfrak{M}_b}}} \qquad \text{for all } \lambda > 0, \tag{4.1}$$

for some $\gamma = \gamma(\theta, \operatorname{diam}(\Omega)) > 0$

Proof. If $\mu = \delta_a$ for some $a \in \Omega$, one has $0 \leq \mathbb{G}[\delta_a](x) \leq \frac{1}{2\pi}\ln\left(\frac{d_\Omega}{|x-a|}\right)$ where $d_\Omega = \operatorname{diam}(\Omega)$. Hence

$$E_\lambda[\delta_a] \subset B_{d_\Omega e^{-2\pi\lambda}} \Longrightarrow e^\sigma_\lambda(\delta_a) = \int_{E_\lambda[\delta_a]} d\sigma \leq c d^\theta_\Omega e^{-2\theta\pi\lambda}.$$

Let $E \subset \Omega$ be a Borel set, $\int_E d\sigma = |E|_\sigma$ and $t > 0$, then, as in Lemma 3.11,

$$\begin{aligned}\int_E \mathbb{G}[\delta_a]d\sigma &\leq t\int_E d\sigma - \int_t^\infty s de^\sigma_s(\delta_a)\\ &\leq t\,|E|_\sigma + c d^\theta_\Omega\left(t + \frac{1}{2\pi\theta}\right)e^{-2\theta\pi t}.\end{aligned}$$

If we choose $e^{-2\theta\pi t} = \frac{|E|_\sigma}{|\Omega|_\sigma}$ we infer

$$\int_E \mathbb{G}[\delta_a]d\sigma \leq \gamma\, |E|_\sigma \left(\ln\left(\frac{|\Omega|_\sigma}{|E|_\sigma}\right) + 1\right). \tag{4.2}$$

For proving (3.39) we can assume that $\mu \geq 0$. Then there exists $\alpha_j > 0$ and $a_j \in \Omega$ such that

$$\mu = \sum_{j=1}^{\infty} \alpha_j \delta_{a_j} \Longrightarrow \sum_{j=1}^{\infty} \alpha_j = \|\mu\|_{\mathfrak{M}_b}.$$

Hence, for any Borel set $E \subset \Omega$,

$$\begin{aligned}\int_E \mathbb{G}[\mu](x)d\sigma(x) &= \sum_{j=1}^{\infty} \alpha_j \int_E \mathbb{G}[\delta_{a_j}(x)]d\sigma(x) \\ &\leq \gamma\, |E|_\sigma \left(\ln\left(\frac{|\Omega|_\sigma}{|E|_\sigma}\right) + 1\right) \|\mu\|_{\mathfrak{M}_b}.\end{aligned} \tag{4.3}$$

If $E = E_\lambda[\mu]$ we infer

$$\lambda e^\sigma_\lambda(\mu) \leq \gamma e^\sigma_\lambda(\mu) \left(\ln\left(\frac{|\Omega|_\sigma}{e^\sigma_\lambda(\mu)}\right) + 1\right) \|\mu\|_{\mathfrak{M}_b},$$

which implies the claim. □

Theorem 4.2. *Assume $N = 2$, $\sigma \in \mathcal{M}^+_{\frac{2}{2-\theta}}(\Omega)$ with $2 \geq \theta > 0$ and $g : \mathbb{R} \mapsto \mathbb{R}$ a continuous function satisfying* (1.1). *If $a_\infty(g) = a_{-\infty}(g) = 0$, for any $\mu \in \mathfrak{M}_b(\Omega)$ problem* (1.2) *admits a very weak solution.*

Proof. Let g^* be the monotone nondecreasing hull of g defined by (1.11). If $m = \sup\{g(r) : -r_0 \leq r \leq r_0\}$ and $m' = \inf\{g(r) : -r_0 \leq r \leq r_0\}$ then $g \leq g^* + m$ on $\mathbb{R}_+$ and $g^* + m' \leq g$ on $\mathbb{R}_-$. If $\{\eta_n\}$ is a sequence of mollifiers and $\mu = \mu^+ - \mu^-$, we set $\mu_n^+ = \mu^+ * \eta_n$, $\mu_n^- = \mu_- * \eta_n$, $\mu_n = \mu_n^+ - \mu_n^-$ and denote by u_n the very weak solution of

$$\begin{aligned} -\Delta u_n + g(u_n)\sigma &= \mu_n && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.4}$$

Since $\|\mu_n\|_{L^1} \leq \|\mu\|_{\mathfrak{M}_b}$, there holds by Proposition 3.2,

$$\|u_n\|_{L^1} + \|\rho g(u_n)\|_{L^1_\sigma} \leq c\, \|\mu\|_{\mathfrak{M}_b} + M, \tag{4.5}$$

and by Lemma 2.1,

$$\|u_n\|_{BMO} + \|\nabla u_n\|_{L^{2,\infty}} \leq c\left(\|\mu\|_{\mathfrak{M}_b} + \|\rho g(u_n)\|_{L^1_\sigma}\right) \leq c'\, \|\mu\|_{\mathfrak{M}_b}. \tag{4.6}$$

Again, there exists a set E with $c_{1,q}(E) = 0$ for any $q \leq 2 - \theta$ such that $u_n(x) \to u(x)$ for all $x \in \Omega \setminus E$, hence $u_n(x) \to u(x)$ and $g(u_n(x)) \to g(u(x))$ $d\sigma$-almost everywhere in Ω. This implies that $g(u)$ is σ-measurable. In order to conclude we have to prove that $g(u_n) \to g(u)$ in $L^1_\sigma(\Omega)$. Estimate (4.1) is valid, hence, for any $t > 0$,

$$\tau_n(t) = \int_{\{|u_n(x)|>t\}} d\sigma \le e^\sigma_{t-M}[\mu_n^+] + e^\sigma_{t-M'}[\mu_n^-] \le ce^{-\frac{t}{\gamma\|\mu\|_{\mathfrak{M}}}},$$

by Lemma 4.1. Since

$$|g(u_n)| \le \left(g_+^*(u_n) - g_-^*(u_n)\right) + m - m',$$

we have that

$$\begin{aligned}
&\int_E |g(u_n)|\, d\sigma \\
&\le \int_E g_+^*(u_n)\, d\sigma - \int_E g_-^*(u_n)\, d\sigma + (m-m')\,|E|_\sigma \\
&\le -\int_t^\infty g_+^*(s) d\,|\{u_n > s\}|_\sigma + \int_{-\infty}^{-t} g_-^*(s) d\,|\{u_n < s\}|_\sigma + \left(m - m'\right)|E|_\sigma \\
&\le -\int_t^\infty \left(g_+^*(s) - g_-^*(-s)\right) d\tau_n(s) + \left(g_+^*(t) - g_-^*(-t) + m - m'\right)|E|_\sigma .
\end{aligned}$$

By integration by parts,

$$\begin{aligned}
&-\int_t^\infty \left(g_+^*(s) - g_-^*(-s)\right) d\tau_n(s) \\
&= \left(g_+^*(t) - g_-^*(-t)\right)\tau_n(t) + \int_t^\infty \tau_n(s) d\left(g_+^*(s) - g_-^*(-s)\right) \\
&\le \left(g_+^*(t) - g_-^*(-t)\right)\left(\tau_n(t) - ce^{-\frac{t}{\gamma\|\mu\|_{\mathfrak{M}_b}}}\right) \qquad (4.7)\\
&\quad + \frac{c}{\gamma\,\|\mu\|_{\mathfrak{M}_b}} \int_t^\infty e^{-\frac{s}{\gamma\|\mu\|_{\mathfrak{M}_b}}} \left(g_+^*(s) - g_-^*(-s)\right) ds \\
&\le \frac{c}{\gamma\,\|\mu\|_{\mathfrak{M}_b}} \int_t^\infty e^{-\frac{s}{\gamma\|\mu\|_{\mathfrak{M}_b}}} \left(g_+^*(s) - g_-^*(-s)\right) ds.
\end{aligned}$$

By assumption the integral on the right-hand side is convergent. We end the proof as in Theorem B, first by fixing t large enough and then $|E|_\sigma$ small enough, and we derive the uniform integrability of $\{g(u_n)\}$. □

A similar result holds when g has nonzero order of growth at infinty.

Theorem 4.3. *Assume* $N = 2$, $\sigma \in \mathcal{M}^+_{\frac{2}{2-\theta}}(\Omega)$ *with* $2 \ge \theta > 0$ *and* $g : \mathbb{R} \mapsto \mathbb{R}$ *a continuous function satisfying* (1.1). *If* $0 < a_\infty(g) < \infty$ *and* $-\infty < a_{-\infty}(g) < 0$, *there exists* $\delta > 0$ *such that for any* $\mu \in \mathfrak{M}_b(\Omega)$ *satisfying* $\|\mu\|_{\mathfrak{M}_b} \le \delta$ *problem* (1.2) *admits a very weak solution.*

Proof. The proof is a straightforward adaptation of the previous one. The choice of δ is such that

$$\|\mu\|_{\mathfrak{M}_b} \leq \delta < \frac{1}{\gamma} \sup\left\{\frac{1}{a_\infty(g)}, -\frac{1}{a_{-\infty}(g)}\right\} \tag{4.8}$$

and the conclusion follows from (4.7). □

5. The supercritical case

5.1. Proof of Theorem D

Proof of assertion I. For $k > 0$ set $g_k(r) = \max\{g(-k), \min\{g(k), g(r)\}\}$ and denote by u_k the very weak solution of

$$\begin{aligned} -\Delta u + g_k(u)\sigma &= \mu \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{5.1}$$

which exists by Theorem B. It follows from the proof of Theorem B (see (3.48) with $g = g_2$ and $g_1 = 0$) that

$$\int_\Omega |g_k(u_k)| d\sigma \leq C, \tag{5.2}$$

where the constant C depends only on Ω and $|\mu|(\Omega)$. Thus the sequence of measures $\{g_k(u_k)\sigma\}$ is bounded. This implies that $\{u_k\}$ is bounded in $W^{1,q}(\Omega)$, $q < \frac{N}{N-1}$, and thus that, up to a subsequence, it converges in $L^1(\Omega)$ to some $u \in W^{1,q}(\Omega)$, $q < \frac{N}{N-1}$. We can also assume that the convergence holds pointwise except on a set E with zero $c_{1,q}$-capacity, which in turn is σ-negligible by Lemma 3.3 if we fix $q \in \left(1, \frac{N}{N-1}\right)$ such that $\theta > N - q$. We also have that u is finite but on a set with zero $c_{1,q}$-capacity hence σ-negligible, therefore

$$g_k(u_k) \to g(u) \qquad \sigma\text{-almost everywhere.}$$

Applying Fatou's lemma in (5.2) yields $g(u) \in L^1_\sigma(\Omega)$.

By the maximum principle

$$-\mathbb{G}[|\mu|] \leq u_k \leq \mathbb{G}[|\mu|], \tag{5.3}$$

hence

$$g\left(-\mathbb{G}[|\mu|]\right) \leq g_k(u_k) \leq g\left(\mathbb{G}[|\mu|]\right), \tag{5.4}$$

since g is nondecreasing.

Because of assumption (1.13) and in view of (5.4), we infer from Lebesgue dominated convergence that $\rho g_k(u_k) \to \rho g(u)$ in $L^1_\sigma(\Omega)$. Thus we can pass to the limit in weak formulation of (5.1) with any $\zeta \in \mathbb{X}(\Omega)$.

Proof of assertion II. We first notice that if g is nondecreasing, vanishes at 0 and satisfies (1.14), then the function g_k defined above also satisfies (1.14) with the same constants a and b. We assume first that $\mu = \mu_r + \mu_s$ is nonnegative and we set $\mu_r^n = \mu_r * \eta_n$ where $\{\eta_n\}$ is a sequence of mollifiers. Let u_k^n be the solution of (5.1) with right-hand side $\mu_r^n + \mu_s$ and v_k^n the one of (5.1) with right-hand side μ_r^n (in both cases existence and uniqueness follows from Theorem B). Then $0 \le u_k^n \le v_k^n + \mathbb{G}[\mu_s]$, $v_k^n \ge 0$ and $\mathbb{G}[\mu_s] \ge 0$. Since g is non-decreasing, we deduce with (1.14) that

$$0 \le g_k(u_k^n) \le g_k\left(v_k^n + \mathbb{G}[\mu_s]\right) \le a\left(g_k(v_k^n) + g_k(\mathbb{G}[\mu_s])\right) + b. \tag{5.5}$$

Since

$$\left\|v_k^n\right\|_{L^1} + \left\|\rho g_k(v_k^n)\right\|_{L^1_\sigma} \le c\left\|\mu_r^n\right\|_{\mathfrak{M}_b} \le c\left\|\mu\right\|_{\mathfrak{M}_b}, \tag{5.6}$$

up to subsequences, the sequences $\{v_k^n\}$ and $\{u_k^n\}$ converge in $L^1(\Omega)$ to some $v^n \in L^1(\Omega)$ and u^n such that $\nabla v^n, \nabla u^n \in L^q(\Omega)$ for any $q < \frac{N}{N-1}$ when $k \to \infty$. As in I, $\{g_k(v_k^n)\}$ and $\{g_k(u_k^n)\}$ converge in $L^1_\sigma(\Omega)$ to $\{g(v^n)\}$ and $\{g(u^n)\}$ respectively. Furthermore v^n and u^n satisfy

$$\begin{aligned} -\Delta v^n + g(v^n)\sigma &= \mu_r^n && \text{in } \Omega \\ v^n &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} -\Delta u^n + g(u^n)\sigma &= \mu_s + \mu_r^n && \text{in } \Omega \\ u^n &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{5.8}$$

respectively and $0 \le u^n \le v^n + \mathbb{G}[\mu_s]$. As in the proof of Proposition 3.2, $v^n \to v$ in $L^1(\Omega)$ and $\rho g(v^n) \to \rho g(v)$in $L^1_\sigma(\Omega)$ as $n \to \infty$, and v is a very weak solution of

$$\begin{aligned} -\Delta v + g(v)\sigma &= \mu_r && \text{in } \Omega \\ v &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{5.9}$$

As above $\{u^n\}$ converge in $L^1(\Omega)$ to some $u \in L^1(\Omega)$ (always up to some subsequence), there holds $u \le v + \mathbb{G}[\mu_s]$ and $g(u^n) \to g(u)$ σ-almost everywhere in Ω since the uniform bound on $\|\nabla u_n\|_{L^{\frac{N}{N-1},\infty}}$ holds. Furthermore

$$\begin{aligned} &0 \le g(u^n) \le a\left(g(v^n) + g(\mathbb{G}[\mu_s])\right) + b \Longrightarrow \\ &0 \le g(u) \le a\left(g(v) + g(\mathbb{G}[\mu_s])\right) + b, \end{aligned} \tag{5.10}$$

and since $g(v^n) \to g(v)$ in $L^1_\sigma(\Omega)$, the sequence $\{g(u^n)\}$ is uniformly integrable in $L^1_\sigma(\Omega)$. Again this implies that $g(u^n) \to g(u)$ in $L^1_\sigma(\Omega)$ and u is a very weak solution of (1.2). If μ is a signed measure, we construct successively the solutions $u_k^n, \overline{u}_k^n$ and $\underline{u}_k^n$ of (5.1) with right-hand side $\mu_r^n + \mu_s$, $\left|\mu_r^n\right| + |\mu_s|$ and $-\left|\mu_r^n\right| - |\mu_s|$ respectively, and the solutions $\overline{v}_k^n$ and $\underline{v}_k^n$ of (5.1) with right-hand side $\left|\mu_r^n\right|$ and $-\left|\mu_r^n\right|$ respectively. Then $\underline{v}_k^n - \mathbb{G}[\mu_s] \le u_k^n \le \overline{v}_k^n + \mathbb{G}[\mu_s]$ which implies by (1.15)

$$a\left(g_k(\underline{v}_k^n) + g_k(-\mathbb{G}[\mu_s])\right) + b \le g_k(u_k^n) \le a\left(g_k(\overline{v}_k^n) + g_k(\mathbb{G}[\mu_s])\right) + b. \tag{5.11}$$

Using the same estimates as above we conclude that $\lim_{n\to\infty}\lim_{k\to\infty} u_k^n = u$ exists in $L^1(\Omega)$, that $\lim_{n\to\infty}\lim_{k\to\infty} g_k(u_k^n) = g(u)$ holds σ almost everywhere in Ω and in $L^1_\sigma(\Omega)$, which ends the proof. □

5.2. Reduced measures

We adapt here some of the results in [9] which turn out to be useful tools in our framework.

Lemma 5.1. *Let $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > N - \frac{N}{N-1}$ and g be nondecreasing satisfying* (1.1). *Assume $\{\mu_n\} \subset \mathfrak{M}^+_b(\Omega)$ is an increasing sequence of good measures for problem* (1.2) *converging to $\mu \in \mathfrak{M}^+_b(\Omega)$. Then μ is a good measure.*

Proof. Let u_{μ_n} be the solutions of (1.2) with right-hand side μ_n then for any $n, k \in \mathbb{N}, k \geq n$, we have since $u_0 \in C^\alpha(\overline{\Omega})$,

$$-m \leq u_0 \leq u_{\mu_n} \leq u_{\mu_k}$$

for some $m \geq 0$ and then

$$g(-m) \leq g(u_0) \leq g(u_{\mu_n}) \leq g(u_{\mu_k}).$$

We use $\zeta := (\eta_1 + \epsilon)^\alpha - \epsilon^\alpha$ as a test-function in the very weak formulation of the equation satisfied by $u_{\mu_n} - u_0$ as in the proof of (3.30); then, recalling that $-\Delta\zeta \geq 0$, we obtain that

$$\int_\Omega (g(u_{\mu_n}) - g(u_0))((\eta_1+\epsilon)^\alpha - \epsilon^\alpha)d\sigma \leq \int_\Omega (\eta_1+\epsilon)^\alpha d\mu_n \leq C\mu_n(\Omega) \leq C\mu(\Omega),$$

where C is independent of n. Letting successively $\epsilon \to 0$ and $\alpha \to 0$ we obtain

$$0 \leq \int_\Omega (g(u_{\mu_n}) - g(u_0))d\sigma \leq C.$$

Hence $\{u_{\mu_n}\}$ is bounded in $W_0^{1,q}(\Omega)$ for any $q < \frac{N}{N-1}$. Thus there exists $u \in W_0^{1,q}(\Omega)$, $q < \frac{N}{N-1}$, such that $u_{\mu_n} \uparrow u$ in $L^1(\Omega)$ and pointwise but for a set E with zero $c_{1,q}$-capacity. Since $\theta > N - \frac{N}{N-1}$ we can find some $q < \frac{N}{N-1}$ such that $\theta > N - q$. It then follows from Lemma 3.3 that $\sigma(E) = 0$.Thus $g(u_{\mu_n}) \uparrow g(u)$ σ-almost everywhere. Fatou's lemma yields $\int_\Omega (g(u) - g(u_0))d\sigma \leq C$, thus $g(u) \in L^1_\sigma(\Omega)$. By the dominated convergence theorem, $g(u_{\mu_n}) \to g(u)$ in L^1_σ. We can then pass to the limit in the equation satisfied by u_{μ_n} to obtain that $u = u_\mu$. □

Proposition 5.2. *Assume σ and g satisfy the assumptions of Lemma* 5.1. *Consider the set*

$$Z = \left\{ x \in \Omega : \int_\Omega \mathbb{G}(x,y)^q \rho(y) d\sigma(y) = \infty \right\}.$$

If $\mu \in \mathfrak{M}_b^+(\Omega)$ is such that $\mu(Z) = 0$ then μ is good.

Proof. We adapt to our case the proof of [30, Theorem 3.10]. Consider the sets

$$C_n = \{x \in \Omega : \int_\Omega \mathbb{G}(x,y)^q \rho(y) d\sigma(y) \leq n\}, \qquad n = 1, 2, \dots.$$

Since the function $x \to \int_\Omega \mathbb{G}(x,y)^q \rho(y) d\sigma(y)$ is lsc (by Fatou's lemma) the sets C_n are closed. Moreover $C_n \subset C_{n+1}$ and $\bigcup_n C_n = \Omega \setminus Z$. Define $\mu_n := 1_{C_n}\mu$, *i.e.*, μ_n is the measure μ restricted to C_n. Then each μ_n satisfies (1.13). Indeed

$$\begin{aligned}
\int_\Omega \mathbb{G}[|\mu_n|]^q \rho d\sigma &\leq \mu_n(\Omega)^{q-1} \int_\Omega \int_\Omega \mathbb{G}(x,y)^{q-1} d\mu_n(x) d\sigma(y) \\
&\leq \mu(\Omega)^{q-1} \int_{C_n} \Big(\int_\Omega \mathbb{G}(x,y)^{q-1} d\sigma(y) \Big) d\mu(x) \\
&\leq n\mu(\Omega)^q.
\end{aligned}$$

It follows from Theorem D that μ_n is good. Since $0 \leq \mu_n \uparrow \mu$ we deduce from Lemma 5.1 that μ is good. □

Lemma 5.3. *Assume σ and g satisfy the assumptions of Lemma* 5.1.

I. *If $\mu \in \mathfrak{M}_b^+(\Omega)$ is a good measure, any $\nu \in \mathfrak{M}_b^+(\Omega)$ such that $\nu \leq \mu$ is a good measure.*

II. *Let $\mu, \mu' \in \mathfrak{M}_b^+(\Omega)$. If μ and $-\mu'$ are good measures, any $\nu \in \mathfrak{M}_b(\Omega)$ such that $-\mu' \leq \nu \leq \mu$ is a good measure.*

Proof. Step 1. Assume $\mu \in \mathfrak{M}_b^+(\Omega)$ is a good measure. For $k > 0$ define g_k by $g_k(r) = \max\{g(-k), \min\{g(k), g(r)\}\}$, and denote by $u_{k,\mu}$ the solution of (5.1), which exists by Theorem B, and by u_μ the solutions of (1.2). Then $-m \leq u_0 \leq \min\{u_\mu, u_{k,\mu}\}$. If $k > m$, then $g_k(u_{k,\mu}) = \min\{g(k), g(u_{k,\mu})\} \leq g(u_{k,\mu})$. Hence

$$-\Delta(u_\mu - u_{k,\mu}) + \big(g_k(u_\mu) - g_k(u_{k,\mu})\big)\sigma \leq 0.$$

Then $u_\mu \leq u_{k,\mu}$ by Lemma 3.6. Similarly $u_{k',\mu} \leq u_{k,\mu}$ for $k' \geq k > m$. Using η_1 as test-function we obtain

$$\int_\Omega (u_{k,\mu} - u_\mu) dx + \int_\Omega (g_k(u_{k,\mu}) - g_k(u_\mu))\eta_1 d\sigma = \int_\Omega (g(u_\mu) - g_k(u_\mu))\eta_1 d\sigma. \quad (5.12)$$

Since $g_k(r) \to g(r)$ for any $r \in \mathbb{R}$ and $|g_k(u_\mu)| \leq |g(u_\mu)|$ with $\rho|g(u_\mu)| \in L^1_\sigma(\Omega)$, the right-hand side converges to 0 as $k \to \infty$ and the second term on the left-hand side is nonnegative. Hence $u_{k,\mu} \to u_\mu$ in $L^1(\Omega)$ as $k \to \infty$, thus $\rho(g_k(u_{k,\mu}) - g_k(u_\mu)) \to 0$ in $L^1_\sigma(\Omega)$ which in turn yields $\rho g_k(u_{k,\mu}) \to \rho g(u_\mu)$ in $L^1_\sigma(\Omega)$.

Step 2: proof of I. Denote by $u_{k,\nu}$ the solution of

$$\begin{aligned} -\Delta u + g_k(u) &= \nu \qquad \text{in } \Omega \\ u &= 0 \qquad \text{in } \partial\Omega. \end{aligned} \tag{5.13}$$

Then $-m \le u_{k,\nu} \le u_{k,\mu}$, $u_{k',\mu} \le u_{k,\mu}$ for $k' \ge k > m$ by Lemma 3.6 and $g_k(u_{k,\nu}) \le g_k(u_{k,\mu})$. Furthermore $\{u_{k,\nu}\}$ is bounded in $W^{1,q}_0(\Omega)$ for $1 < q < \frac{N}{N-1}$ and thus relatively compact in $L^1(\Omega)$. Therefore there exists $u \in W^{1,q}_0(\Omega)$ such that $u_{k,\nu} \downarrow u$ in $L^1(\Omega)$ and also pointwise up to a set with zero $c_{1,q}$-capacity which is therefore a σ-negligible set. By Step 1, the set $\{\rho g_k(u_{k,\nu})\}$ is uniformly integrable in $L^1_\sigma(\Omega)$, this implies that $u = u_\nu$.

Step 3: Proof of II. Because $-\mu' \le \nu \le \mu$ there holds $u_{k,-\mu'} \le u_{k,\nu} \le u_{k,\mu}$ and $g_k(u_{k,-\mu'}) \le g_k(u_{k,\nu}) \le g_k(u_{k,\mu})$. Since the sets $\{u_{k,-\mu'}\}$, $\{u_{k,\nu}\}$ and $\{u_{k,\mu}\}$ are relatively compact in $L^1(\Omega)$ and bounded in $W^{1,q}_0(\Omega)$ for $1 < q < \frac{N}{N-1}$ and the sets $\{g_k(u_{k,-\mu'})\}$ and $\{g_k(u_{k,\mu})\}$ are uniformly integrable in $L^1_\sigma(\Omega)$, then, up to a subsequence, $u_{k,\nu} \to u$ in $L^1(\Omega)$ and σ-almost everywhere as $k \to \infty$. This implies that $g(u) \in L^1_\sigma(\Omega)$ and $\rho g_k(u_{k,\nu}) \to \rho g(u)$ in $L^1_\sigma(\Omega)$. Hence $u = u_\nu$. □

The proof of the next result, based upon Zorn's lemma, is a variant of the one of [9, Theorem 4.1] which uses the inverse maximum principle [9, Corollary 4.8].

Lemma 5.4. *Assume σ and g satisfy the assumptions of Lemma* 5.1. *If $\mu \in \mathfrak{M}^+_b(\Omega)$ there exists a largest good measure smaller than μ, and it is nonnegative.*

Proof. Let $\mathcal{Z}_\mu$ be the subset of all bounded nonnegative good measures smaller than μ. Notice first that $\mathcal{Z}_\mu$ is non-empty since it contains the regular part μ_r of μ with respect to the N-dimensional Hausdorff measure. We now show that $\mathcal{Z}_\mu$ is inductive. Let $\mathcal{C}_I := \{\mu_i\}_{i\in I}$ be a totally ordered subset of $\mathcal{Z}_\mu$. For $\zeta \in C_0(\overline{\Omega})$, $\zeta \ge 0$, the set of nonnegative real numbers

$$\mathcal{C}_I(\zeta) := \left\{ \int_\Omega \zeta \, d\mu_i \right\}$$

is bounded from above by $\displaystyle\int_\Omega \zeta \, d\mu$. Note that can we extend μ as a positive linear form on $C_0(\overline{\Omega})$ since it is a Radon measure and $\mu(\partial\Omega) = 0$. Hence $\mathcal{C}_I(\zeta)$ admits an upper bound $L(\zeta)$ and there exists a sequence $\{i_k\} \subset I$ such that

$$\int_\Omega \zeta \, d\mu_{i_k} \uparrow L(\zeta) \le \int_\Omega \zeta \, d\mu \qquad \text{as } k \to \infty.$$

By the Stone-Weiertrass theorem there exists a dense subset $\{\zeta_n\}$ of the set of nonnegative elements in $C_0(\overline{\Omega})$. By Cantor diagonal process there exists a subsequence $\{i_{n_k}\} \subset I$ such that

$$\int_\Omega \zeta_n d\mu_{i_{n_k}} \uparrow L(\zeta_n) \leq \int_\Omega \zeta_n d\mu \qquad \text{as } k \to \infty.$$

Clearly the map $\zeta_n \mapsto L(\zeta_n)$ is additive, positively homogeneous of order one and satisfies

$$L(\zeta) \leq \int_\Omega \zeta d\mu \qquad \text{for all } \zeta \in C_0(\overline{\Omega}),\ \zeta \geq 0.$$

Hence L extends as a positive linear functional on $C_0(\overline{\Omega})$, dominated by μ denoted by $\mu_{\mathcal{C}_I}$. Since μ is a Radon measure in Ω, $\mu_{\mathcal{C}_I}(\partial\Omega) = 0$, hence it is a Radon measure. Furthermore it is a good measure by Lemma 5.1. It follows that $\mu_{\mathcal{C}_I} \in \mathcal{Z}_\mu$. Moreover since $L(\zeta)$ is an upper bound of $\mathcal{C}_I(\zeta)$ for any nonnegative $\zeta \in C_0(\overline{\Omega})$, we have $\mu_{\mathcal{C}_I} \geq \mu_i$ for any $i \in I$. Hence the set $\mathcal{Z}_\mu$ is inductive.

As a consequence of Zorn's lemma, $\mathcal{Z}_\mu$ admits at least one maximal element that we denote μ^*. If ν is any nonnegative good measure smaller than μ it belongs to $\mathcal{Z}_\mu$ and hence it cannot dominate μ^*. It remains to prove that $\nu \leq \mu^*$. Set $\lambda = \sup\{\nu, \mu^*\}$ and let λ^* be a maximal element of $\mathcal{Z}_\lambda$. Since ν and μ^* are good measures, we have $\nu^* = \nu$ and $(\mu^*)^* = \mu^*$. It follows that $\lambda^* \geq \nu^* = \nu$ and $\lambda^* \geq (\mu^*)^* = \mu^*$ so that $\lambda^* \geq \sup\{\nu, \mu^*\} = \lambda$. This implies that $\lambda^* = \lambda \geq \mu^*$. On the other hand, since $\nu, \mu^* \leq \mu$, we have $\lambda \leq \mu$ and thus $\lambda^* \leq \mu$. By definition of a maximal element it implies that $\lambda^* = \lambda = \mu^*$, and finally $\mu^* = \sup\{\nu, \mu^*\}$. We infer $\nu \leq \mu^*$ and then μ^* is the maximum of $\mathcal{Z}_\mu$. □

Corollary 5.5. *Assume σ and g satisfy the assumptions of Lemma* 5.1. *If $\mu, \nu \in \mathfrak{M}_b^+(\Omega)$ are good measures, then* $\sup\{\mu, \nu\}$ *is a good measure.*

Proof. Set $\lambda = \sup\{\mu, \nu\}$. Then

$$\lambda \geq \lambda^* = (\sup\{\mu, \nu\})^* \geq \sup\{\mu^*, \nu^*\} = \sup\{\mu, \nu\} = \lambda. \tag{5.14}$$

This implies $\lambda = \lambda^*$, hence λ is a good measure. □

5.3. The capacitary framework

We start with the following regularity estimate for the Poisson problem:

Lemma 5.6. *For any $s \geq 0$ and $1 < p < \infty$, the mapping $\mu \mapsto \mathbb{G}[\mu]$ is continuous from $\mathfrak{M}_b(\Omega) \cap H^{s-2,p}(\Omega)$ to $H^{s,p}(\Omega)$.*

Proof. It is classical that the mapping $G_D : \lambda \mapsto u = G_D(\lambda)$ solution of $-\Delta u = \lambda$ in Ω and $u = 0$ on $\partial\Omega$ is continuous from $H^{s-2,p}(\Omega)$ to $H^{s,p}(\Omega)$ for $1 < p < \infty$ and $s > \frac{1}{p}$ (see, *e.g.* [14, Example 3.15 p. 314]). Thus we are left with the case

$0 \le s \le \frac{1}{p}$. If $\lambda \in \mathfrak{M}_b(\Omega)$, then $G_D(\lambda) = \mathbb{G}[\lambda]$ is a very weak solution, hence, since $\mathbb{X}(\Omega) \subset C^1_c(\overline{\Omega}) \cap \left(\bigcap_{1<r<\infty} H^{2,r}(\Omega) \right)$,

$$-\int_\Omega G_D(\lambda)\Delta\zeta dx = \int_\Omega \zeta d\lambda \le \|\zeta\|_{H^{2-s,p'}} \|\lambda\|_{H^{s-2,p}} \quad \text{for all } \zeta \in \mathbb{X}(\Omega).$$

In particular, if $\zeta = \mathbb{G}[v]$, then $\|\zeta\|_{H^{2-s,p'}} \le c\,\|v\|_{H^{-s,p'}}$ since $-s > -2 + 1/p'$, and

$$\int_\Omega G_D(\lambda)v dx \le c\,\|v\|_{H^{-s,p'}} \|\lambda\|_{H^{s-2,p}} \quad \text{for all } v \in \Delta(\mathbb{X}(\Omega)).$$

In particular this inequality holds if $v \in C_c(\overline{\Omega})$ which is dense in $H^{-s,p'}(\Omega)$. Finally this inequality means that the mapping $v \mapsto \int_\Omega G_D(\lambda)v dx$ is a continuous linear form over $H^{-s,p'}(\Omega)$, it thus belongs to $H^{s,p}(\Omega)$. □

Proposition 5.7. *Let σ and g satisfy the assumptions in Theorem* E. *If $\mu \in \mathfrak{M}_b(\Omega)$ is such that $|\mu| \in H^{s-2,p}(\Omega)$ for some $p > 1$ and $s > 0$ such that $N - \theta < sp < N$ and $\frac{\theta p}{N-sp} \ge q$, then* (1.3) *admits a unique very weak solution.*

Proof. By Lemma 5.6, if $|\mu| \in H^{s-2,p}(\Omega)$ then $\mathbb{G}[|\mu|] \in H^{s,p}(\Omega)$. By Proposition 2.4

$$\|\mathbb{G}[|\mu|]\|_{L^q_\sigma} \le c\,\|\mathbb{G}[|\mu|]\|_{H^{s,p}}$$

if and only if $\sigma \in \mathcal{M}^+_r(\Omega)$ with $\frac{1}{r} = q\left(\frac{1}{q} - \frac{1}{p} + \frac{s}{N}\right) = \frac{N-\theta'}{N}$. Then $q = \frac{\theta' p}{N-sp}$. Hence, if $\frac{\theta p}{N-sp} \ge q$ we get $\theta \ge \theta'$ and then $\mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega) \subset \mathcal{M}^+_{\frac{N}{N-\theta'}}(\Omega)$ by [2.7). We conclude by Theorem D. □

Remark. This result covers the case $q = p$, in which any bounded measure such that $|\mu| \in H^{\frac{N-\theta}{q}-2,q}(\mathbb{R}^N)$ is eligible for solving problem (1.2).

Proof of Theorem E. If μ is absolutely continuous with respect to the $c_{2-s,p'}$-capacity, so are μ^+ and $-\mu^-$. By [13] there exists an increasing sequence of positive bounded Radon measures $\mu_j \in H^{s-2,p}(\Omega)$ converging to μ^+. By Proposition 5.7 μ_j is a good measure, hence by Lemma 5.1 μ^+ is a good measure. In the same way $-\mu^-$ is a good measure. Since $-\mu_- \le \mu \le \mu_+$, it follows from Lemma 5.3-II that μ is a good measure. □

Proof of Proposition 1.1. Notice first that if $\mu \in \mathcal{M}_{\frac{N}{N-\theta^*}}(\Omega)$ with $\theta^* > N - sp$, then for any compact $K \subset \Omega$,

$$|\mu|(K) \le c' \left(c_{s,p}(K)\right)^{\frac{1}{p}}. \tag{5.15}$$

In particular μ is absolutely continuous w.r.t $c_{s,p}$-capacity. Indeed under the assumption on θ^* we have $H^{s,p}(\Omega) \hookrightarrow L^1_{|\mu|}(\Omega)$. It follows that for any $v \in H^{s,p}(\Omega)$, $v \geq 1$ on K, we have

$$|\mu|(K) \leq \int_K v d|\mu| \leq \|v\|_{L^1_{|\mu|}} \leq C\|v\|_{H^{s,p}}.$$

We deduce (5.15) taking the infimum over v. To apply Theorem E we need μ to be $c_{2-\frac{N-\theta}{q},q'}$-diffuse. It thus suffices to take $\theta^* > N - sp$ with $s = 2 - \frac{N-\theta}{q}$ and $p = q'$. We obtain exactly the condition on θ^* stated in Proposition 1.1. □

5.4. The case $g(u) = |u|^{q-1} u$

In the sequel we consider the following equation

$$\begin{aligned} -\Delta u + |u|^{q-1} u\sigma &= \mu \qquad &&\text{in } \Omega \\ u &= 0 &&\text{in } \partial\Omega, \end{aligned} \tag{5.16}$$

where $q > 1$. A measure for which there exists a solution, necessarily unique by Lemma 3.5, is called *q-good*. Assume that $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}$ with $N \geq \theta > N - \frac{N}{N-1}$. Then the critical exponent q from the point of view of (1.8) in Theorem B is

$$q_\theta := \frac{\theta}{N-2}, \tag{5.17}$$

which is larger than 1 if $N > 2$.

Let $q > 1$ and $\sigma \in \mathfrak{M}^+_b(\Omega)$. Recall that the Green function G of the Dirichlet Laplacian in Ω is defined on $\overline{\Omega} \times \overline{\Omega}$ with values in $[0, +\infty]$ with $G(x, x) = +\infty$, $x \in \Omega$, and $G(x, y) = 0$ if $x \in \partial\Omega$ or $y \in \partial\Omega$. We extend G to $\mathbb{R}^N \times \overline{\Omega}$ by setting $G(x, y) = 0$ if $(x, y) \in \overline{\Omega}^c \times \overline{\Omega}$. Hence $x \mapsto G(x, y)$ is lower semicontinuous in $\mathbb{R}^N$ and $y \mapsto G(x, y)$ is lower semicontinuous in Ω, and thus is σ-measurable. Following [2, Section 2.3] we then consider the following set function with values in $[0, +\infty]$,

$$c^\sigma_q(E) = \inf\left\{\int_\Omega |v|^{q'} d\sigma : v \in L^{q'}_\sigma(\Omega),\ \mathbb{G}[v\sigma](x) \geq 1 \text{ for all } x \in E\right\}, \tag{5.18}$$

for any $E \subset \Omega$. According to the general theory developped in [2, Section 2.3] c^σ_q is a regular capacity in the sense of Choquet. Using the lower semicontinuity of $y \mapsto \mathbb{G}[v\sigma](y)$ (see [2, Prop 2.3.2]) it is easy to verify that for any compact set $K \subset \Omega$, there holds

$$c^\sigma_q(K) = \inf\left\{\int_\Omega |v|^{q'} d\sigma : v \in L^\infty_\sigma(\Omega),\ \mathbb{G}[v\sigma](x) \geq 1 \text{ for all } x \in K\right\}. \tag{5.19}$$

The dual formulation of the capacity is the following (see [2, Theorem 2.5.1]),

$$\left(c_q^\sigma(K)\right)^{\frac{1}{q'}} = \sup\left\{\lambda(K) : \lambda \in \mathfrak{M}_b^+(K),\ \|\mathbb{G}[\lambda]\|_{L^q_\sigma} \leq 1\right\} \tag{5.20}$$

for $K \subset \Omega$, K compact. Existence of extremal measures satisfying equality in (5.20) is proved in [2, Theorem 2.5.3].

Remark. Note that the $\geq$ inequality in (5.20) follows directly from the following one

$$\nu(K) \leq \left(c_q^\sigma(K)\right)^{\frac{1}{q'}} \|\mathbb{G}\nu\|_{L^q_\sigma}, \tag{5.21}$$

which holds for any $\nu \in \mathfrak{M}_b^+(\Omega)$ such that $\mathbb{G}[\nu] \in L^q_\sigma$ and any $K \subset \Omega$ compact.

We now give some sufficient conditions for a bounded measure to be absolutely continuous with respect to the capacity c_q^σ. First in view of (5.21) and the dual expression of the capacity it is clear that there holds:

Lemma 5.8. *If $\nu \in \mathfrak{M}_b(\Omega)$ is such that $\mathbb{G}[|\nu|] \in L^q_\sigma(\Omega)$, then ν is absolutely continuous with respect to the capacity c_q^σ. This holds in particular if $\nu \in \mathfrak{M}_b(\Omega)$ is such that $|\nu| \in H^{s-2,p}(\Omega)$ for some $p > 1$ and $s > 0$ verifying $N - \theta < sp < N$ and $\frac{\theta p}{N-sp} \geq q$.*

As a direct consequence we have:

Lemma 5.9. *If $\nu \in \mathfrak{M}_b(\Omega)$ is $c_{2-s,p'}$-diffuse where s and p are as in Lemma 5.8, then ν is absolutely continuous with respect to the capacity c_q^σ.*

Proof. If $\nu \geq 0$ there exists a sequence of nonnegative measures $\{\nu_n\} \subset H^{s-2,p}(\Omega)$ such that $\nu_n \uparrow \nu$. If K is a compact such that $c_q^\sigma(K) = 0$ then $\nu_n(K) = 0$ by Lemma 5.8 and thus $\nu(K) = 0$. When ν is a signed measure, we apply the above to its positive and negative part $\nu^\pm$. □

The following particular case will be useful:

Lemma 5.10. *If $\nu \in \mathcal{M}_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > N-2$, then ν is absolutely continuous with respect to the capacity c_q^σ.*

Proof. We have $|\nu| \in \mathcal{M}_p(\Omega)$ for some $p > \frac{N}{2}$. We then obtain from (2.9) that $\mathbb{G}[|\nu|]$ is bounded so that $\mathbb{G}[|\nu|] \in L^q_\sigma(\Omega)$. The conclusion follows from the previous lemma. □

Remark. It is noticeable that if the support of a nonnegative measure μ does not intersect the support of σ, then μ is always q-good. This is due to the fact that $\mathbb{G}[\mu]$ is bounded on the support of σ, hence $\mathbb{G}[\mu] \in L^q_\sigma(\Omega)$ for any $q < \infty$ and the result follows from Theorem D. Hence, a more accurate necessary condition must involve a notion of density of σ on its support, a property which has been developed by Triebel [26] in connection with fractal measures.

We recall that the θ-dimensional Hausdorff measure H^θ, $0 \le \theta \le N$, is defined on subsets E of $\mathbb{R}^N$ by

$$H^\theta(E) = \lim_{\delta \to 0} \left(\inf \left\{ \sum_{j=1}^{\infty} (\operatorname{diam} U_j)^\theta : E \subset \bigcup_{j=1}^{\infty} U_j, \operatorname{diam} U_j \le \delta \right\} \right). \tag{5.22}$$

Definition 5.11. A nonnegative Radon measure σ on $\overline{\Omega}$ with support Γ is θ-regular with $0 \le \theta \le N$ if there exists $c > 0$ such that

$$\frac{1}{c} r^\theta \le |B_r(x)|_\sigma \le c r^\theta \qquad \text{for all } x \in \Gamma\,, \text{ for all } r > 0. \tag{5.23}$$

The support Γ of σ is called a θ-set.

By [26, Theorem 3.4] σ is equivalent in $\overline{\Omega}$ to the restriction $H^\theta \lfloor_\Gamma$ of H^θ to Γ in the sense that there exists $c' > 0$ such that

$$\frac{1}{c'} H^\theta(E \cap \Gamma) \le \sigma(E) \le c' H^\theta(E \cap \Gamma) \qquad \text{for all } E \subset \overline{\Omega}\,,\ E \text{ Borel}. \tag{5.24}$$

The description of $L^p_\sigma(\Gamma)$ necessitates to introduce the scale of Besov spaces and their *trace* on Γ. For $0 < s < 1$, $1 \le p, q \le \infty$, we denote by $B^s_{p,q}(\Omega)$ the space obtained by the real interpolation method by

$$B^s_{p,q}(\Omega) = \left[W^{1,p}(\Omega), L^p(\Omega) \right]_{s,q}. \tag{5.25}$$

Details can be found in [23]. Its norm is equivalent to

$$\|\phi\|_{B^s_{p,q}} = \|v\|_{L^p} + \left(\int_0^\infty \frac{\left(\omega_p(t; v)\right)^q}{t^{sq}} \frac{dt}{t} \right)^{\frac{1}{q}}, \tag{5.26}$$

if $q < \infty$ and

$$\|\phi\|_{B^s_{p,\infty}} = \|v\|_{L^p} + \sup_{t>0} \frac{\omega_p(t; v)}{t^s}, \tag{5.27}$$

where

$$\omega_p(t; \phi) = \sup_{|h|<t} \|v(. + h) - v(.)\|_{L^p}.$$

For $k \in \mathbb{N}_*$, $B^{k+s}_{p,q}(\Omega) = \{v \in W^{k,p}(\Omega) : D^\alpha v \in B^s_{p,q}(\Omega)\,,$ for all $\alpha \in \mathbb{N}^N,\ |\alpha| = k\}$ with norm

$$\|v\|_{B^{k+s}_{p,q}} = \|v\|_{W^{k-1,p}} + \sum_{|\alpha|=k} \left\| D^\alpha v \right\|_{B^s_{p,q}}.$$

If $\Gamma \subset \mathbb{R}^N$ is a closed set with zero Lebesgue measure, we consider the set

$$B^{s,\Gamma}_{p,q}(\mathbb{R}^N) = \left\{ v \in B^s_{p,q}(\mathbb{R}^N) : \langle v, \phi \rangle = 0 \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^N) \text{ s.t. } \phi\lfloor_\Gamma = 0 \right\}, \tag{5.28}$$

endowed with the $B^s_{p,q}(\mathbb{R}^N)$ norm, where $\langle v,\phi\rangle$ is the pairing between $\mathcal{S}'(\mathbb{R}^N)$ and $\mathcal{S}(\mathbb{R}^N)$. If $v\in L^p_\sigma(\Omega)$ and σ has support $\Gamma\subset\overline{\Omega}$, the linear map

$$\phi\mapsto T^\sigma_v(\phi)=\int_\Gamma\phi v d\sigma \tag{5.29}$$

defined on $\mathcal{S}(\mathbb{R}^N)$ is a tempered distribution in $\mathbb{R}^N$. The following results are proved in [26, Theorem 18.2, 18.6].

Proposition 5.12. *Assume σ is θ-regular, $0<\theta<N$, with support $\Gamma\subset\mathbb{R}^N$, and let $v\in L^q_\sigma(\Omega)$ with $1<p\leq+\infty$. There holds*

$$\left|T^\sigma_v(\phi)\right|\leq c\,\|v\|_{L^p_\sigma}\,\|\phi\|_{B^{\frac{N-\theta}{p'}}_{p',1}}\qquad \text{for all } \phi\in\mathcal{S}(\mathbb{R}^N). \tag{5.30}$$

It follows that $T^\sigma_v\in B^{-\frac{N-\theta}{p'},\Gamma}_{p,\infty}$ with $\|T^\sigma_v\|_{B^{-\frac{N-\theta}{p'}}_{p,\infty}}\leq c\,\|v\|_{L^p_\sigma}$.
Morover the map $v\in L^p_\sigma(\Gamma)\to T^\sigma_v\in B^{-\frac{N-\theta}{p'},\Gamma}_{p,\infty}$ is a linear isomorphism. We can thus denote $L^p_\sigma(\Gamma)\sim\left(B^{\frac{N-\theta}{p'},\Gamma}_{p',1}\right)'=B^{-\frac{N-\theta}{p'},\Gamma}_{p,\infty}$.

Proposition 5.13. *Assume σ is θ-regular, $0<\theta<N$ with support $\Gamma\subset\mathbb{R}^N$. Then for any $1<p\leq\infty$ the restriction operation from $\mathcal{S}(\mathbb{R}^N)$ to $C(\Gamma)$, $\phi\mapsto\phi\lfloor_\Gamma$ can be extended as a continuous linear operator from $B^{\frac{N-\theta}{p}}_{p,1}(\mathbb{R}^N)$ to $L^p_\sigma(\Gamma)$ that we denote Tr_Γ. Furthermore this operator is onto.*

Definition 5.14. If $\sigma\in\mathfrak{M}^+_b(\Omega)$ is θ-regular, $N\geq\theta>N-2$ with support $\Gamma\subset\Omega$ and $m,q>1$, we set

$$c^{2-\frac{N-\theta}{q},\Gamma}_{q',\infty}K)=\inf\left\{\|\zeta\|^{q'}_{B^{2-\frac{N-\theta}{q}}_{q',\infty}}:\zeta\in B^{2-\frac{N-\theta}{q},\Gamma}_{q',\infty}(\Omega)\text{ s.t. }\zeta\geq\chi_K\right\}, \tag{5.31}$$

where

$$B^{2-\frac{N-\theta}{q},\Gamma}_{q',\infty}(\Omega)=\left\{\zeta\in B^{2-\frac{N-\theta}{q}}_{q',\infty}(\Omega)\text{ s.t. }\Delta\zeta\in B^{-\frac{N-\theta}{q},\Gamma}_{q',\infty}(\Omega)\right\}. \tag{5.32}$$

Notice that $B^{2-\frac{N-\theta}{q},\Gamma}_{q',\infty}(\Omega)$ is a closed subspace of $B^{2-\frac{N-\theta}{q}}_{q',\infty}(\Omega)$.

Proposition 5.15. *Assume $\sigma\in\mathfrak{M}^+_b(\Omega)$ is θ-regular, $N\geq\theta>N-2$ with support $\Gamma\subset\Omega$ and $q>1$. Then there exists a positive constant $M>0$ such that*

$$\frac{1}{M}c^\sigma_q(K)\leq c^{2-\frac{N-\theta}{q},\Gamma}_{q',\infty}(K)\leq Mc^\sigma_q(K), \tag{5.33}$$

for all compact set $K\subset\Omega$.

Proof. By standard elliptic equations and interpolation theory (see [23,24]), for any $\psi \in B_{q',\infty}^{-\frac{N-\theta}{q},\Gamma}(\Omega)$, $\mathbb{G}[\psi\sigma] \in B_{q',\infty}^{2-\frac{N-\theta}{q}}(\Omega)$ and there holds

$$\frac{1}{c}\,\|\mathbb{G}[\psi\sigma]\|_{B_{q',\infty}^{2-\frac{N-\theta}{q}}} \leq \|\psi\|_{B_{q',\infty}^{-\frac{N-\theta}{q},\Gamma}} \leq c\,\|\mathbb{G}[\psi\sigma]\|_{B_{q',\infty}^{2-\frac{N-\theta}{q}}}\,. \tag{5.34}$$

By Proposition 5.12 we can replace $\|\psi\|_{B_{q',\infty}^{-\frac{N-\theta}{q},\Gamma}}$ by $\|\psi\|_{L_\sigma^{q'}}$ in the above inequality, up to a change of constants c. Let $\{v_k\} \subset L_\sigma^\infty(\Omega)$ be such that $v_k \geq 0$, $\zeta_k := \mathbb{G}[v_k\sigma] \geq 0$ on K and $\|v_k\|_{L_\sigma^{q'}} \downarrow \left(c_q^\sigma(K)\right)^{\frac{1}{q'}}$. Since (5.32) is equivalent to

$$\frac{1}{c}\,\|\zeta_k]\|_{B_{q',\infty}^{2-\frac{N-\theta}{q}}} \leq \|v_k\|_{L_\sigma^{q'}} \leq c\,\|\zeta_k]\|_{B_{q',\infty}^{2-\frac{N-\theta}{q}}}\,,$$

we derive $c_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(K) \geq \frac{1}{c^{q'}}c_q^\sigma(K)$. Similarly $c_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(K) \leq c^{q'}c_q^\sigma(K)$. $\square$

Proof of Theorem F. By Lemma 5.10 the measure u^q vanishes on Borel sets with zero c_q^σ-capacity. Since $u \in L_\sigma^q(\Omega)$ the mapping

$$\phi \mapsto \int_\Gamma u\phi d\sigma = \langle u, \phi\rangle$$

is a tempered distribution that we denote by T_u^σ, hence

$$|\langle\Delta u, \phi\rangle| = |\langle u, \Delta\phi\rangle| = \left|\int_\Omega u\Delta\phi d\sigma\right| \leq \|u\|_{L_\sigma^q}\,\|\Delta\phi\|_{L_\sigma^{q'}}\,.$$

Using Proposition 5.12

$$\|\Delta\phi\|_{L_\sigma^{q'}} \leq c\,\|\Delta\phi\|_{B_{q',\infty}^{-\frac{N-\theta}{q},\Gamma}} \leq c'\,\|\phi\|_{B_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}}\,.$$

Therefore the nonnegative measure T_u^σ is a continuous linear form on $B_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(\Omega)$. Therefore it vanishes on Borel sets with zero $c_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}$-capacity, which actually coincide with Borel sets with zero zero c_q^σ-capacity. $\square$

5.5. Removable singularities

It is easy to prove that for any compact set $K \subset \Omega$, there exists $\mu_K \in \mathfrak{M}_b^+(K)$ such that $\int_\Omega (\mathbb{G}[\mu_K])^q d\sigma = 1$ and $c_q^\sigma(K) = \mu_K(K)$ (see [2, Theorem 2.5.3]). Since μ_K is an admissible measure, it follows from Theorem D that (1.3) is solvable with $\mu = \mu_K$, hence K is not removable. Although it could be conjectured that a compact set with zero c_q^σ-capacity is removable we can prove this assertion only for sigma-moderate solutions.

Definition 5.16. Let $q > 1, \sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ where $N \geq \theta > N-2$ and $K \subset \Omega$ a compact set. A nonnegative function $u \in L^1_{\mathrm{loc}}(\overline{\Omega} \setminus K) \cap L^q_{\sigma,\mathrm{loc}}(\overline{\Omega} \setminus K)$ is a sigma-moderate solution of

$$\begin{aligned} -\Delta u + |u|^{q-1} u\sigma &= 0 \qquad \text{in } \Omega \setminus K \\ u &= 0 \qquad \text{in } \partial\Omega, \end{aligned} \tag{5.35}$$

if there exists an increasing sequence $\{\mu_n\} \subset \mathfrak{M}^+_b(K)$ of q-good measures such that $u_{\mu_n} \to u$ in $L^1_{\mathrm{loc}}(\overline{\Omega} \setminus K) \cap L^q_{\sigma,\,\mathrm{loc}}(\overline{\Omega} \setminus K)$.

Theorem 5.17. *Under the assumptions on q, σ and K of Definition* 5.16, *if $c^\sigma_q(K) = 0$ then the only sigma-moderate solution of* (5.35) *is the trivial one.*

Proof. Since $c^\sigma_q(K) = 0$ the set of nonnegative q-good measures with support in K is reduced to the zero function by Theorem F. This implies the claim. □

Remark. We conjecture that for any compact set $K \subset \Omega$, any nonnegative local solution of (5.12) is sigma-moderate. This would imply that a necessary and sufficient condition for a local nonnegative solution of (5.12) to be a solution in Ω is $c^\sigma_q(K) = 0$. However this type of result is usually difficult to prove, see [12,17,22] in the framework of semilinear equations with measure boundary data.

In order to find necessary and sufficient conditions for the removability of a compact set $K \subset \Omega$, we assume that σ is a positive measure in Ω absolutely continuous with respect to the Lebesgue measure, with a nonnegative density w. For proving our results we will assume that the function $\omega = w^{-\frac{1}{q-1}}$ is q'-admissible in the sense of [15, Chapter 1]. One sufficient condition is that w belongs to the Muckenhoupt class A_q, that is

$$\sup_B \left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B w^{-\frac{1}{q-1}} dx\right)^{\frac{1}{p-1}} = m_{w,q} < \infty \tag{5.36}$$

for all ball $B \subset \mathbb{R}^N$.

If $K \subset \Omega$ is compact, we set

$$c^\omega_q(K) = \inf\left\{\int_\Omega |\Delta\zeta|^{q'}\, \omega dx : \zeta \in C^\infty_0(\Omega),\ \zeta \geq 1 \text{ in a neighborhood of } K\right\}. \tag{5.37}$$

This defines a capacity on Borel subsets of Ω. Since ω is q'-admissible, it satisfies Poincaré inequality, hence a set with zero c^ω_q-capacity is ω-negligible. Furthermore, following the proof of [2, Theorem 3.3.3], c^ω_q is equivalent to $\dot{c}^\omega_q$ defined by

$$\begin{aligned} &\dot{c}^\omega_q(K) \\ &= \inf\left\{\|\zeta\|^{q'}_{W^{2,q'}_\omega} : \zeta \in C^\infty_0(\Omega),\ 0 \leq \zeta \leq 1,\ \zeta \geq 1 \text{ in a neighborhood of } K\right\}. \end{aligned} \tag{5.38}$$

The dual definition is (see [2, Theorem 2.5.1])

$$\left(c_q^{\omega}(K)\right)^{\frac{1}{q'}} = \sup\left\{\lambda(K) : \lambda \in \mathfrak{M}_b^+(K),\ \|\mathbb{G}[\lambda]\|_{L^q_\omega} \le 1\right\}. \tag{5.39}$$

Proof of Theorem G. *Step* 1*: The condition is sufficient*. We assume first that $L^q_{w,\mathrm{loc}}(\Omega\setminus K)\cap u \in L^1(\Omega\setminus K)$ is a nonnegative subsolution of (1.22) in the sense of distributions in $\Omega\setminus K$ where $K\subset\Omega$ is a compact subset with c_q^{ω}-capacity zero. There exists a sequence of functions $\{\zeta_k\}\subset C_0^\infty(\Omega)$ with value in [0, 1], value 1 in a neighborhood of K and such that $\|\Delta\zeta_k\|_{L^{q'}_\omega}\to 0$ when $k\to\infty$. Let $\rho\in C_0^\infty(\Omega)$, $0\le\rho\le 1$, such that $\rho=1$ in a neighborhood of K containing the support of the ζ_k. Using $\phi_k := (1-\zeta_k)^\alpha\rho^\alpha$, with $\alpha>1$, in the very weak formulation of equation (1.22) we obtain,

$$\begin{aligned}
&\int_\Omega u^q\phi_k w dx\\
&\le \int_\Omega u\Delta\phi_k dx\\
&\le \alpha\int_\Omega u(1-\zeta_k)^\alpha\rho^{\alpha-1}\Delta\rho dx - 2\alpha\int_\Omega u(1-\zeta_k)^{\alpha-1}\nabla\zeta_k.\nabla\rho^\alpha dx\\
&\quad -\alpha\int_\Omega u(1-\zeta_k)^{\alpha-1}\rho^\alpha\Delta\zeta_k dx + \alpha(\alpha-1)\int_\Omega u(1-\zeta_k)^{\alpha-2}\rho^\alpha|\nabla\zeta_k|^2 dx\\
&\quad +\alpha(\alpha-1)\int_\Omega u(1-\zeta_k)^\alpha\rho^{\alpha-2}|\nabla\rho|^2 dx.
\end{aligned} \tag{5.40}$$

Notice that the second integral in the right-hand side vanishes since $\nabla\zeta_k.\nabla\rho^\alpha = 0$ by the assumption on their support. If we choose $\alpha = 2q'$, we can bound the remaining integrals as follows:

$$\begin{aligned}
\left|\int_\Omega u(1-\zeta_k)^{2q'-1}\rho^{2q'}\Delta\zeta_k dx\right| &\le \left(\int_\Omega u^q\phi_k w dx\right)^{\frac1q}\left(\int_\Omega |\Delta\zeta_k|^{q'}(1-\zeta_k)^{q'}\rho^{2q'}\omega dx\right)^{\frac{1}{q'}}\\
&\le \left(\int_\Omega u^q\phi_k w dx\right)^{\frac1q}\left(\int_\Omega |\Delta\zeta_k|^{q'}\omega dx\right)^{\frac{1}{q'}},\\
\left|\int_\Omega u(1-\zeta_k)^{2q'}\rho^{2q'-1}\Delta\rho dx\right| &\le \left(\int_\Omega u^q\phi_k w dx\right)^{\frac1q}\left(\int_\Omega |\Delta\rho|^{q'}(1-\zeta_k)^{2q'}\rho^{q'}\omega dx\right)^{\frac{1}{q'}}\\
&\le \left(\int_\Omega u^q\phi_k w dx\right)^{\frac1q}\left(\int_\Omega |\Delta\rho|^{q'}\omega dx\right)^{\frac{1}{q'}},\\
\left|\int_\Omega u(1-\zeta_k)^{2q'-2}\rho^{2q'}|\nabla\zeta_k|^2 dx\right| &\le \left(\int_\Omega u^q\phi_k w dx\right)^{\frac1q}\left(\int_\Omega |\nabla\zeta_k|^{2q'}\rho^{2q'}\omega dx\right)^{\frac{1}{q'}}\\
&\le \left(\int_\Omega u^q\phi_k w dx\right)^{\frac1q}\left(\int_\Omega |\nabla\zeta_k|^{2q'}\omega dx\right)^{\frac{1}{q'}},
\end{aligned}$$

and finally

$$\left|\int_\Omega u(1-\zeta_k)^{2q'}\rho^{2q'-2}|\nabla\rho|^2\,dx\right| \le \left(\int_\Omega u^q\phi_k w dx\right)^{\frac{1}{q}}\left(\int_\Omega |\nabla\rho|^{2q'}(1-\zeta_k)^{2q'}\omega dx\right)^{\frac{1}{q'}}$$
$$\le \left(\int_\Omega u^q\phi_k w dx\right)^{\frac{1}{q}}\left(\int_\Omega |\nabla\rho|^{2q'}\omega dx\right)^{\frac{1}{q'}}.$$

Since the Gagliardo-Nirenberg inequality holds with the q'-admissible weight ω, we have for some $\tau \in (0,1)$ and some $c = c(q,N) > 0$,

$$\begin{aligned}\left(\int_\Omega |\nabla\zeta_k|^{2q'}\omega dx\right)^{\frac{1}{2q'}} &\le c\left(\int_\Omega |\Delta\zeta_k|^{q'}\omega dx\right)^{\frac{\tau}{q'}}\|\zeta_k\|_{L^\infty}^{1-\tau}\\ &\le c'\left(\int_\Omega |\Delta\zeta_k|^{q'}\omega dx\right)^{\frac{\tau}{q'}}.\end{aligned} \tag{5.41}$$

Therefore, if we set

$$X_k = \left(\int_\Omega u^q\phi_k w dx\right)^{\frac{1}{q}} \quad\text{and}\quad Z_k = \left(\int_\Omega |\Delta\zeta_k|^{q'}\omega dx\right)^{\frac{1}{q'}},$$

we obtain the inequation

$$X_k^q \le c_1X_kZ_k + c_2X_k + c_3X_kZ_k^\tau, \tag{5.42}$$

for some positive constants c_1, c_2, c_3 depending on q, N and ρ. By definition of ζ_k we have $Z_k \to 0$. We thus deduce that $X_k^q \le cX_k$ with $q > 1$ and then that the sequence $\{X_k\}$ is bounded. Since $\zeta_k \to 0$ almost everywhere, we have $\phi_k \to \rho^{2q'}$ almost everywhere. It then follows by Fatou's lemma that

$$\int_\Omega u^q\rho^{2q'}w dx \le c. \tag{5.43}$$

We deduce that $u \in L^q_{w,\mathrm{loc}}(\Omega)$. Since $\omega^{-\frac{q'}{q}} \in L^1_{\mathrm{loc}}(\Omega)$, we obtain that $L^1_{\mathrm{loc}}(\Omega)$ by Hölder's inequality. If $u \in L^q_{w,\mathrm{loc}}(\Omega\setminus K) \cap u \in L^1(\Omega\setminus K)$ is a distributional solution of (1.22) in $\Omega\setminus K$, then $|u|$ is a nonnegative subsolution with the same integrability constraints and we derive $u \in L^q_{w,\mathrm{loc}}(\Omega)\cap L^1_{\mathrm{loc}}(\Omega)$.

If $\phi \in C_0^\infty(\Omega)$, we take $\phi(1-\zeta_k)^{2q'}$ for test function of equation (1.22) in $\mathcal{D}'(\Omega\setminus K)$,

$$-\int_\Omega u\Delta(\phi(1-\zeta_k)^{2q'})\,dx + \int_\Omega |u|^{q-1}u\phi(1-\zeta_k)^{2q'}w\,dx = 0.$$

Since $u \in L^q_{w,\text{loc}}(\Omega)$, ϕ has compact support, and $\zeta_k \to 0$ almost everywhere, we can pass to the limit as $k \to +\infty$ in the second integral using Lebesgue convergence theorem and obtain

$$\int_\Omega |u|^{q-1}u\phi(1-\zeta_k)^{2q'}w\,dx \to \int_\Omega |u|^{q-1}u\phi w\,dx.$$

Moreover we can pass to the limit in the first integral expanding the laplacian. Using that $u \in L^1_{\text{loc}}(\Omega)$ and that $\Delta\zeta_k \to 0$ in $L^{q'}_\omega$, it is easy to prove from the previous computation that

$$\int_\Omega u(1-\zeta_k)^{q'}\Delta\phi dx \to \int_\Omega u\Delta\phi dx \quad \text{as } k \to \infty,$$

and

$$\lim_{k\to\infty}\int_\Omega u(1-\zeta_k)^{2q'-1}\nabla\zeta_k.\nabla\phi dx = 0 = \lim_{k\to\infty}\int_\Omega u(1-\zeta_k)^{2q'-1}\phi\Delta\zeta_k dx.$$

Hence

$$-\int_\Omega u\Delta\phi dx + \int_\Omega u^q\phi w dx = 0. \tag{5.44}$$

Step 2: The condition is necessary. Let K be a compact set with positive c^ω_q-capacity. According to [2, Theorem 2.5.3] there exists an extremal $\mu_k \in \mathfrak{M}^+_b(K)$ in the dual formulation (5.39) of the capacity. According to Theorem D, problem (5.16) with $\mu = \mu_K$ admits a positive solution which is therefore a positive solution of (5.35). □

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