

A Fejér theorem for boundary quotients arising from algebraic dynamical systems

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Abstract. A Fejér-type theorem is proved within the framework of C^* -algebras associated with certain irreversible algebraic dynamical systems. This makes it possible to strengthen a result on the structure of the relative commutant of a family of generating isometries in a boundary quotient.

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1. Introduction

The uniform convergence of the Fourier series $S_n(f)$ of a given continuous function $f \in C(\mathbb{T})$ is notoriously a delicate matter. Indeed, although strong sufficient conditions are easy to find, for instance that f is continuously differentiable, optimal conditions are not as easy to spot. No wonder, fairly pathological examples of continuous functions can be exhibited whose Fourier series fails to converge uniformly. As a matter of fact, far more pathological examples can be given for which the Fourier series behaves so badly as to diverge at some point. This was already known as long ago as 1876 to du Bois-Reymond, who is generally credited with having produced the first example of this sort. Even so, it is still possible to have uniform convergence provided that one considers the Cesàro mean of the sequence $\{S_n(f) : n \in \mathbb{N}\}$ instead. In other words, the sequence $\frac{1}{N} \sum_{n=0}^{N-1} S_n(f)$ does converge uniformly to f , which is nothing but the content of a classical theorem proved by Fejér in 1904. As well as being an interesting result in its own right, Fejér's theorem has proved to be a good source of inspiration for modern research too. On the one hand, one may exploit the nexus of ideas and techniques involved in the proof of the classical result to treat fine properties, such as weak amenability, of

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locally compact groups, as is done in, *e.g.*, [17]. On the other hand, one may also endeavour to generalize the content of the classical theorem to cover a wider variety of situations, when $C(\mathbb{T})$ is replaced by a non-commutative C^* -algebra. The first generalizations considered in this direction [18] dealt with C^* -algebras obtained as crossed products (also see, *e.g.*, [12] for a simplified treatment where crossed products are only considered with respect to the action of \mathbb{Z} , and [16, Proposition B1] for C^* -algebras acted upon by an n -dimensional torus). Greater generality was then achieved in [9,10]. In recent times, crossed products by locally compact groups with the approximation property have been addressed in [11] in a von Neumann algebra setting also. In this note, though, the focus is on the so-called boundary quotient C^* -algebras, which of late have been given a good deal of attention in a series of papers [5–8]. If we are to take a step further into the analysis of these C^* -algebras, suitable approximation properties appear to be the needed tool to make finer computations. Therefore, what the note aims to do is bridge this gap by providing a theorem à la Fejér, which we believe may and indeed does come in useful at least to treat a natural subclass of boundary quotient C^* -algebras.

2. Preliminaries and notation

In this section we rather quickly recall the definitions and the properties of the C^* -algebras that we will be treating in the present work. These are obtained out of so-called algebraic dynamical systems, for which much wider information can be found in [5] and some of the references therein. The algebraic dynamical systems we are actually interested in are of a rather special kind. Namely, they are a triple (G, P, θ) , where:

- (a) G is a countable discrete group;
- (b) P is a discrete (countable) Abelian cancellative unital semigroup such that the intersection of two principal ideals is still a (possibly empty) principal ideal; and
- (c) θ is an action of P upon G through injective homomorphisms that *preserve the order*, that is $pP \cap qP = rP$ implies $\theta_p(G) \cap \theta_q(G) = \theta_r(G)$.

Furthermore, P can be made into a directed set by introducing the order relation $p \geq q \Leftrightarrow p \in qP$. We denote by $\mathcal{Q}(G \rtimes_{\theta} P)$ the associated boundary quotient C^* -algebra, which is by definition the universal C^* -algebra generated by a unitary representation u of the group G and a representation s of the semigroup P by isometries satisfying the relations

- (I) $s_p u_g = u_{\theta_p(g)} s_p$;
- (II) $s_p^* u_g s_q = \begin{cases} u_{g_1} s_{p'} s_{q'}^* u_{g_2} & \text{if } g = \theta_p(g_1) \theta_q(g_2) \text{ and } pP \cap qP = pp'P, pp' = qq' \\ 0 & \text{otherwise;} \end{cases}$
- (III) $\sum_{\bar{g} \in G/\theta_p(G)} e_{g,p} = 1$ if $N_p < \infty$;

where $e_{g,p} \doteq u_g s_p s_p^* u_g^*$ and $N_p \doteq [G : \theta_p(G)]$.

As a cancellative Abelian semigroup, P embeds into its Grothendieck group $H \doteq P^{-1}P$, which is obviously countable and discrete. As observed in [5, Section 2] its Pontryagin dual T , which is a compact metrizable group, acts on $\mathcal{Q}(G \rtimes_{\theta} P)$ by a gauge action γ defined by $\gamma_{\chi}(u_g s_p) \doteq \chi(p) u_g s_p$. The corresponding fixed-point algebra is denoted by \mathcal{F} and coincides with $\overline{\text{span}}\{u_g s_p s_p^* u_h^* \mid g, h \in G, p \in P\}$.

Denote by $P(\perp) \doteq \{(p, q) \in P \times P \mid pP \cap qP = pqP\}$ the collection of all relatively prime pairs in P . We define an equivalence relation \sim on $P(\perp) \times P(\perp)$ by saying that two pairs $(p, q), (\tilde{p}, \tilde{q}) \in P(\perp)$ are equivalent whenever they satisfy $p^{-1}q = \tilde{p}^{-1}\tilde{q}$, or equivalently if there is an $x \in P^*$ such that $\tilde{p} = xp$ and $\tilde{q} = xq$. For each $(p, q) \in P(\perp)$, we define a contractive linear map $F_{(p,q)}: \mathcal{Q}(G \rtimes_{\theta} P) \rightarrow \mathcal{F}$ by $a \mapsto \int_T \gamma_{\chi}(s_p a s_q^*) d\chi$, where the integration is with respect to the normalized Haar measure on the compact Abelian group T . Given an element a in $\mathcal{Q}(G \rtimes_{\theta} P)$, the terms $F_{(p,q)}(a)$ should be regarded as its Fourier coefficients and, within this framework, one can show [5, Section 4] that if the element actually sits in $\text{span}\{u_g s_{p'} s_{q'}^* u_h \mid g, h \in G, p', q' \in P\}$, which is dense in $\mathcal{Q}(G \rtimes_{\theta} P)$, there is a uniquely determined finite set $A(a) \subset P(\perp)/\sim$ with the property that

$$a = \sum_{[(p,q)] \in A(a)} s_p^* F_{(p,q)}(a) s_q, \tag{2.1}$$

In the next section we will show that for an arbitrary element of $\mathcal{Q}(G \rtimes_{\theta} P)$ a similar result holds.

3. Main result

Since P is an Abelian semigroup, its Grothendieck group $H = P^{-1}P$ is also an Abelian group. As such, it is in particular amenable. Now there are many a way in which amenability can be expressed. One that is particularly suited to our context is exploiting the existence of the so-called Følner sequences. A Følner sequence for a discrete group H is a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets $F_n \subset H$ such that for every $h \in H$ $\lim_n \frac{|hF_n \Delta F_n|}{|F_n|} = 0$, where Δ denotes the symmetric difference between two sets, i.e., $A \Delta B \doteq (A \cup B) \setminus (A \cap B)$. As can be proved, it is always possible to produce out of a given Følner sequence a new Følner sequence which is also increasing, that is $F_n \subset F_{n+1}$ and exhausting, namely $\cup_n F_n = H$. Henceforth our Følner sequences will always be assumed both increasing and exhausting.

Associated with every Følner sequence, there is a sequence of real-valued functions defined as $\varphi_n(h) \doteq \frac{|hF_n \cap F_n|}{|F_n|}$, $h \in H$. Their most relevant properties are summarized in a couple of easy yet useful lemmas.

Lemma 3.1. *The functions $\varphi_n : H \rightarrow \mathbb{R}$ are finitely supported, and $\{\text{supp } \varphi_n : n \in \mathbb{N}\}$ is an increasing and exhausting family of subsets of H .*

Proof. By definition, the support of φ_n is the set $\{h \in H : hF_n \cap F_n \neq \emptyset\}$, which is easily seen to coincide with $F_n F_n^{-1}$. Therefore, we have $\text{supp } \varphi_n \subset \text{supp } \varphi_{n+1}$ and $\cup_n \text{supp } \varphi_n = H$. \square

Lemma 3.2. *The functions φ_n are all positive definite. Moreover, for every $h \in H$ $\lim_n \varphi_n(h) = 1$.*

Proof. The statement about positive definiteness is clear, as φ_n is nothing but $\frac{\langle \lambda_h \chi_{F_n} \cdot \chi_{F_n} \rangle}{\langle \chi_{F_n} \cdot \chi_{F_n} \rangle}$, where $\lambda : H \rightarrow \mathcal{U}(\ell_2(H))$ is the left regular representation of H . Finally, the limit property comes from the Følner property, i.e., $\lim_n \frac{|hF_n \Delta F_n|}{|F_n|} = 0$ for every $h \in H$. Indeed, we certainly have $\limsup_n \varphi_n(h) \leq 1$. On the other hand, we also have

$$\liminf_n \varphi_n(h) = \liminf_n \frac{|hF_n \cap F_n|}{|F_n|} \geq \liminf_n \frac{|hF_n \cup F_n|}{|F_n|} - \lim_n \frac{|hF_n \Delta F_n|}{|F_n|} \geq 1$$

This concludes the proof. \square

As for every integer n the support of the function φ_n is finite, the sum

$$S_n(x) \doteq \sum_{h=[(p,q)] \in H} \varphi_n(h) s_p^* F_{(p,q)}(x) s_q \quad \text{for every } x \in \mathcal{Q}(G \rtimes_{\theta} P)$$

is well defined. We now want to prove the above sums have good approximation properties. To this aim, the above S_n 's are better understood as linear operators acting on the C^* -algebra $\mathcal{Q}(G \rtimes_{\theta} P)$, merely thought of as a Banach space with respect to its norm.

Lemma 3.3. *The linear operators S_n are uniformly bounded. More precisely, we have $\|S_n\| = 1$ for every $n \in \mathbb{N}$.*

Proof. Inserting the expression of the Fourier coefficients $F_{(p,q)}(x)$ into the definition of the sums S_n , the following representation formula is easily arrived at

$$S_n(x) = \int_T K_n(\chi) \gamma_{\chi}(x) d\chi,$$

where the integral kernel K_n is explicitly given by $K_n(\chi) = \sum_{h \in H} \varphi_n(h) \chi(h)$, and γ_{χ} is the gauge automorphism of $\mathcal{Q}(G \rtimes_{\theta} P)$ corresponding to the character $\chi \in T = \widehat{H}$. Among other things, the representation formula above also has the merit of showing the sums S_n do not depend on the representatives (p, q) of $h = [(p, q)]$. The thesis will be fully proved as soon as we show $\|K_n\|_{L^1(K)} = 1$ for every $n \in \mathbb{N}$. This is done in two steps. We first prove that $K_n(\chi) \geq 0$ for every $\chi \in T$. This is in turn achieved by an application of Bochner's theorem that

a function is positive if and only if its Fourier transform is positive definite. In our case, this Fourier transform is easily computed. Indeed,

$$\widehat{K}_n(k) = \int_T K_n(\chi)\overline{\chi(k)}d\chi = \sum_{h \in H} \varphi_n(h) \int_T \chi(h)\overline{\chi(k)}d\chi = \varphi_n(k)$$

Therefore, the functions K_n are positive thanks to Lemma 3.2. This allows us to compute their L^1 -norms exactly. Indeed, we have

$$\|K_n\|_{L^1(T)} = \int_T K_n(\chi)d\chi = \sum_{h \in H} \varphi_n(h) \int_T \chi(h)d\chi = \varphi_n(1_G) = 1$$

thanks to the fact that $\int_T \chi(h)d\chi = 0$ unless $h = 1_G$, in which case the integral is 1. □

We are now in a position to prove the main result of this paper.

Theorem 3.4. *Let $\mathcal{Q}(G \rtimes_{\theta} P)$ be the boundary quotient C^* -algebra associated to an algebraic dynamical system (G, P, θ) , as at the beginning of Section 2. For every $x \in \mathcal{Q}(G \rtimes_{\theta} P)$ the sums $S_n(x)$ converge to x in norm when n goes to infinity.*

Proof. Phrased differently, all we have to prove is $\{S_n : n \in \mathbb{N}\}$ converges strongly to the identity operator on $\mathcal{Q}(G \rtimes_{\theta} P)$. As the the sequence is bounded by Lemma 3.3, it is enough to check the statement on a convenient dense subspace of our C^* -algebra. In our previous work [5] it was shown that for every x in the norm-dense subalgebra

$$\mathcal{A} \doteq \text{span}\{u_g s_{p'} s_{q'}^* u_h \mid g, h \in G, p', q' \in P\} \subset \mathcal{Q}(G \rtimes_{\theta} P)$$

there is a finite set $F_x \subset H$ such that $x = \sum_{h=[(p,q)] \in F_x} s_p^* F_{(p,q)}(x) s_q$ which is moreover uniquely determined under the requirement that $F_x \subset P(\perp)/\sim \subset H$. It is obvious that for such a x the sequence $S_n(x) = \sum_{h=[(p,q)] \in H} \varphi_n(h) s_p^* F_{(p,q)}(x) s_q$ must converge to x , since we will eventually have $F_x \subset \text{supp } \varphi_n$ and $\varphi_n(h) \rightarrow 1$ by Lemmas 3.1 and 3.2 respectively. We can also be more explicit and display the usual underlying $\frac{\epsilon}{3}$ -argument in full detail. So given any $x \in \mathcal{Q}(G \rtimes_{\theta} P)$, pick an element $x_{\epsilon} \in \mathcal{A}$ such that $\|x - x_{\epsilon}\| < \frac{\epsilon}{3}$. Then we have

$$\|x - S_n(x)\| \leq \|x - x_{\epsilon}\| + \|x_{\epsilon} - S_n(x_{\epsilon})\| + \|S_n(x_{\epsilon} - x)\| < \frac{2\epsilon}{3} + \|x_{\epsilon} - S_n(x_{\epsilon})\|$$

Since $S_n(x_{\epsilon})$ converges to x_{ϵ} , there exists $N_{\epsilon} \in \mathbb{N}$ such that $\|S_n(x_{\epsilon}) - x_{\epsilon}\| < \frac{\epsilon}{3}$ for every $n \geq N_{\epsilon}$. This finally says that $\|S_n(x) - x\| < \epsilon$ for every $n \geq N_{\epsilon}$. □

Example 3.5. The 2-adic ring C^* -algebra \mathcal{Q}_2 is by definition the universal C^* -algebra generated by a unitary U and a (proper) isometry S_2 such that $S_2 U = U^2 S_2$ and $S_2 S_2^* + U S_2 S_2^* U^* = 1$ (for more information see, e.g., [1–4, 13]). The Cuntz

algebra \mathcal{O}_2 , as is known, is the universal C^* -algebra generated by two isometries S_1, S_2 such that $S_1 S_1^* + S_2 S_2^* = 1$. It is rather obvious that \mathcal{O}_2 embeds into \mathcal{Q}_2 . Indeed, there exists an injective $*$ -homomorphism that sends S_1 to $U S_2$ and S_2 to S_2 . As shown in [1], \mathcal{Q}_2 can also be realized as the boundary quotient C^* -algebra arising from the algebraic dynamical system in which P is the unital semigroup generated by 2 in \mathbb{N}^\times acting on Z by multiplication. Clearly, the Grothendieck group H of P is isomorphic with \mathbb{Z} . If we choose the Følner sequence $\{F_n : n \in \mathbb{N}\}$ given by $F_n \doteq [-n, n]$, then the functions $\varphi_n : \mathbb{Z} \rightarrow \mathbb{R}$ are seen at once to be $\varphi_n(i) = 1 - |i|/(2n + 1)$, $-n \leq i \leq n$ and $\varphi_n(i) = 0$ otherwise. In this setting Theorem 3.4 gives the following formula, which holds in the norm topology

$$\begin{aligned} x &= \lim_n F_{(1,1)}(x) + \sum_{i=1}^n \left(1 - \frac{|i|}{2n + 1}\right) (S_2^*)^i F_{(2,1)}(x) \\ &\quad + \left(1 - \frac{|i|}{2n + 1}\right) F_{(1,2^i)}(x) S_2^i \\ &= \lim_n \int_{\mathbb{T}} \tilde{\alpha}_z(x) dz + \sum_{i=1}^n \left(1 - \frac{|i|}{2n + 1}\right) \int_{\mathbb{T}} (\tilde{\alpha}_z(x) z^i + \tilde{\alpha}_z(x) z^{-i}) dz \end{aligned}$$

where dz is the normalized Haar measure on the one-dimensional torus. Of course, in the previous formula the isometry S_2 may be replaced by S_1 and thus the limit may be expressed in terms of the maps $F_i : \mathcal{Q}_2 \rightarrow \mathcal{Q}_2^{\mathbb{T}}$ and $F_{-i} : \mathcal{Q}_2 \rightarrow \mathcal{Q}_2^{\mathbb{T}}$, with $i \geq 0$, defined in [1, Section 3.3]. We also note that when $x \in \mathcal{O}_2$, we recover the well-known result for the Cuntz algebra \mathcal{O}_2 (see, e.g. [15]). Finally, a similar formula could also be written for the p -adic ring C^* -algebra \mathcal{Q}_p for all $2 \leq p < \infty$. In [14] Nekrashevych found a natural embedding of the Thompson groups into $\mathcal{U}(\mathcal{O}_2)$, which in turn embeds into $\mathcal{U}(\mathcal{Q}_2)$. The Thompson group F is here represented as the subgroup generated by the elements $x_0 \doteq S_2^2 S_2^* + S_2 S_1 (S_1 S_2)^* + S_1 (S_1^*)^2$ and $x_1 \doteq S_2 S_2^* + S_1 S_2^2 (S_1 S_2)^* + S_1 S_2 S_1 (S_1^* S_2)^* + S_1^2 (S_1^*)^3$. We find it interesting to rewrite these elements in terms of the Fourier coefficients defined above. After some easy computations one gets

$$x_0 = F_{(1,2)}(x_0) S_2 + F_{(1,1)}(x_0) + S_2^* F_{(2,1)}(x_0)$$

where $F_{(1,2)}(x_0) = S_2^2 (S_2^*)^*$, $F_{(1,1)}(x_0) = S_2 S_1 (S_1 S_2)^*$, $F_{(2,1)}(x_0) = S_2 S_1 (S_1^*)^*$, and

$$x_1 = F_{(1,2)}(x_1) S_2 + F_{(1,1)}(x_1) + S_2^* F_{(2,1)}(x_1)$$

where $F_{(1,2)}(x_1) = S_1 S_2^2 (S_2 S_1 S_2)^*$, $F_{(1,1)}(x_1) = S_2 S_2^* + S_1 S_2 S_1 (S_1^* S_2)^*$, $F_{(2,1)}(x_1) = S_2 S_1^2 (S_2^*)^*$.

Example 3.6. As already observed in [5], the well-known C^* -algebra $\mathcal{Q}_{\mathbb{N}}$ may be recovered as the boundary quotient $\mathcal{Q}(\mathbb{Z} \rtimes \mathbb{N}^\times)$ where $P = \mathbb{N}^\times$ acts on \mathbb{Z} by multiplication. In this case the Grothendieck group H is given by the positive rational

integers $\mathbb{Q}_+^* \cong \bigoplus_{i=1}^\infty \mathbb{Z}$ and its dual is $T \cong \prod_{i=1}^\infty \mathbb{T}$. A typical Følner sequence for H is given by

$$F_n \doteq \{(x_i) \in H \mid -n \leq x_j \leq n \forall 1 \leq j \leq n, x_i = 0 \forall i > n\}.$$

We observe that if $h = (h_1, \dots, h_k, 0, \dots), n > \max\{k, |h_1|, \dots, |h_k|\}$, then

$$\varphi_n(h) = \frac{\prod_{i=1}^k (2n + 1 - |h_i|)}{(2n + 1)^k}.$$

In this particular example, Theorem 3.4 gives the following slightly less usual formula, which holds for any $x \in \mathcal{Q}_{\mathbb{N}}$

$$x = \lim_n \sum_{h \in H} \sum_{(f_i) \in F_n} \varphi_n(h) \int_T \left(\prod_{i=1}^\infty z_i^{f_i} \right) \alpha_{(z_k)}(x) d\mu$$

where $d\mu$ is the normalized Haar measure on T , the sequence (z_i) is an element in T and $\alpha_{(z_i)}$ is the gauge automorphism mapping s_p to $z_p s_p, p \in \mathbb{N}^\times$. Of course the infinite product is well defined because f_i is eventually zero.

4. An application: the relative commutant of a family of generating isometries

In [5] the relative commutant $C^*({s_p : p \in P})' \cap \mathcal{Q}(G \rtimes_\theta P)$ has been proved to be as small as possible, to wit it is nothing but the C^* -subalgebra generated by the set of unitaries $\{s_p : p \in P^*\}$, under the hypothesis that P^* is finite. Here $P^* \subset P$ is the group of all invertible elements of P . As an application of the Fejér-type theorem obtained above, we now intend to show that the finiteness assumption is actually unnecessary.

Theorem 4.1. *If $\bigcap_{p \in P} \theta_p(G) = \{1_G\}$, then the relative commutant $C^*({s_p : p \in P})' \cap \mathcal{Q}(G \rtimes_\theta P)$ coincides with $C^*(P^*)$. In particular, the relative commutant reduces to the multiples of the identity if $P^* = \{1_P\}$.*

Proof. We only have to deal with the inclusion $C^*({s_p : p \in P})' \cap \mathcal{Q}(G \rtimes_\theta P) \subset C^*(P^*)$, for the reverse inclusion is trivially checked. If $w \in \mathcal{Q}(G \rtimes_\theta P)$ is a unitary that commutes with s_p for every $p \in P$, then $F_{(p,q)}(w) = 0$ unless $[(p,q) \in P^*$, in which case the Fourier coefficient is just a scalar, as we had already proved in [5]. For such a w the sequence $\{S_n(w) : n \in \mathbb{N}\}$ is therefore all contained in $C^*(P^*)$, and thus w is also an element of $C^*(P^*)$ since it is the limit of the sequence $\frac{1}{N} \sum_{i=0}^{N-1} S_n(w)$. □

Remark 4.2. It might be worth stressing that the above application really seems to be out of the reach of the Fejér-type theorems we already knew of before writing the present note.

We end with a result concerning the structure of $C^*(P^*)$. As one would expect, $C^*(P^*)$ can be proved to be isomorphic with the group C^* -algebra $C_{\text{red}}^*(P^*)$ in a number of relevant cases. To this aim, we recall that for any discrete group Γ , the reduced C^* -algebra of Γ , here denoted by $C_{\text{red}}^*(\Gamma)$, is the C^* -subalgebra of

$\mathcal{B}(\ell_2(\Gamma))$ generated by the left regular representation of Γ . Phrased differently, $C_{\text{red}}^*(\Gamma)$ is the concrete C^* -algebra generated by the set of unitaries $\{\lambda_\gamma : \gamma \in \Gamma\}$ acting on $\ell_2(\Gamma)$ as $\lambda_\gamma \delta_k = \delta_{\gamma k}$, for any $k \in \Gamma$, where $\{\delta_k : k \in \Gamma\}$ is the canonical basis of $\ell_2(\Gamma)$. We denote by $C_\pi^*(\Gamma)$ the full C^* -algebra of Γ . This is the enveloping C^* -algebra of the convolution algebra $L^1(\Gamma)$, namely the completion of the $*$ -Banach algebra $L^1(\Gamma)$ under the maximal C^* -norm. The map given by

$$\ell_1(\Gamma) \ni \sum_{\gamma \in \Gamma} \alpha_\gamma \delta_\gamma \rightarrow \sum_{\gamma \in \Gamma} \alpha_\gamma \lambda_\gamma \in C^*(\Gamma)_{\text{red}}$$

extends to a surjective $*$ -homomorphism $\pi : C_\pi^*(\Gamma) \rightarrow C_{\text{red}}^*(\Gamma)$ by the very definition of the maximal C^* -norm. Furthermore, the map is well known to be also injective if and only if Γ is amenable, which is certainly true when Γ is Abelian.

The cases where our claim on $C^*(P^*)$ holds are singled out by choosing a suitable class of algebraic dynamical systems. More precisely, we need the following condition to be fulfilled: there exists $g_0 \in G$ such that $\theta_p(g_0) = g_0$ with $p \in P^*$ implies $p = 1_P$. Because g_0 must not depend on $p \in P^*$, the condition is actually stronger than merely requiring that $P^* \ni p \rightarrow \theta_p \in \text{Hom}(G)$ is injective. However, it is satisfied in many of the examples we discussed in [5].

Proposition 4.3. *The C^* -algebra $C^*(P^*)$ is isomorphic with $C_{\text{red}}^*(P^*)$ if P is Abelian and (G, P, θ) satisfies the condition above.*

Proof. The map $P^* \ni p \rightarrow s_p \in \mathcal{U}(\ell_2(G))$ is a unitary representation of the discrete group P^* , therefore it can be lifted to a surjective $*$ -homomorphism $\rho : C_\pi^*(P^*) \rightarrow C^*(P^*)$. Being Abelian, P^* is amenable as well, hence the above homomorphism can be seen as an epimorphism from $C_{\text{red}}^*(P^*)$ to $C^*(P^*)$, which we still denote by ρ . The conclusion is immediately got to if ρ is shown to be injective. To this aim, we consider the isometry $W : \ell_2(P^*) \rightarrow \ell_2(G)$ that acts on the canonical basis of $\ell_2(P^*)$ as $W\delta_p = \delta_{\theta_p(g_0)}$ for every $p \in P^*$, where $g_0 \in G$ is such that $\theta_p(g_0) = g_0$ implies $p = 1_P$, which says $\theta_p(g_0) = \theta_q(g_0)$ is possible only when $p = q$. Because the intertwining relation $W\lambda_p = s_p W$ is easily checked for every $p \in P^*$ (indeed, $W\lambda_p \delta_q = W\delta_{pq} = \delta_{\theta_{pq}(g_0)} = \delta_{\theta_p(\theta_q(g_0))} = s_p W\delta_q$), we see that $\sum_{p \in F} \alpha_p \lambda_p = W^*(\sum_{p \in F} \alpha_p s_p)W$, for every finite set $F \subset P^*$, where the coefficients α_p are complex numbers. In particular, we find

$$\left\| \sum_{p \in F} \alpha_p \lambda_p \right\| = \left\| W^* \left(\sum_{p \in F} \alpha_p s_p \right) W \right\| \leq \left\| \sum_{p \in F} \alpha_p s_p \right\| = \left\| \rho \left(\sum_{p \in F} \alpha_p s_p \right) \right\|$$

Therefore, the inequality $\|x\| \leq \|\rho(x)\|$ holds for every $x \in C_{\text{red}}^*(P^*)$ by density. \square

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