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Differential p-forms and q-vector fields with constant coefficients

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Abstract. Differential p-forms and q-vector fields with constant coefficients are studied. Differential p-forms of degrees $p = 1, 2, n-1, n$ with constant coefficients on a smooth n-dimensional manifold M are characterized. In the contravariant case, the obstruction for a q-vector field V_q to have constant coefficients is proved to be the Schouten-Nijenhuis bracket of V_q with itself. The q-vector fields with constant coefficients of degrees $q = 1, 2, n-1, n$ are also characterized. The notions of differential p-forms and q-vector fields with conformal constant coefficients are introduced. For arbitrary degrees p and q, such differential p-forms and q-vector fields are seen to be the solutions to two second-order partial differential systems on $J^2(M, \mathbb{R}^n)$, which are reducible to two first-order partial differential systems by adding variables. Computational aspects in solving these systems are discussed and examples and applications are also given.

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1. Definitions and examples

The notion of "having constant coefficients" plays an important role in both Geometry and Analysis; for example, when does a differential operator have constant coefficients? [3], or when an exterior differential system [1, Chapter XVIII, Section 9] is defined by forms with constant coefficients? etc. In Geometry this notion is usually related to concepts such as locally symmetric spaces, flat semi-Riemannian manifolds, etc. Below we tackle this notion in the case of p-differential forms and q-vector fields. First of all we introduce the formal definitions and show some examples.

1.1. The covariant case

Definition 1.1. Let M be a smooth manifold of dimension n. A differential form $\omega_p \in \Omega^p(M)$ is said to have *constant coefficients* on a neighbourhood of the point

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$x \in M$ if there exists a system of coordinates $(u^1, \dots, u^n)$ centred at x, such that all the functions $\omega_p\left(\frac{\partial}{\partial u^{i_1}}, \dots, \frac{\partial}{\partial u^{i_p}}\right)$, $1 \le i_1 < \dots < i_p \le n$, are constant.

Notation 1.2. For every $p \in \mathbb{N}^+$, let $\mathcal{I}_p^n$ denote the set of all multi-indices $I = (i_1, \dots, i_p) \in \mathbb{N}^p$ such that $1 \le i_1 < \dots < i_p \le n$. If $(u^1, \dots, u^n)$ is a system of C^∞ functions on M, we write $du^I = du^{i_1} \wedge \dots \wedge du^{i_p}$, for any $I \in \mathcal{I}_p^n$.

Proposition 1.3. *Every form ω_p with constant coefficients is closed. In this case, the differential system $D = \{X \in T_xM : i_X(\omega_p)_x = 0, x \in M\}$ has a locally constant rank and it is involutive.*

Proof. If ω_p has constant coefficients, by writing ω_p on a coordinate system as in Definition 1.1, we have $\omega_p = \sum_{I\in\mathcal{I}_p^n} \lambda_I du^I$, $\lambda_I \in \mathbb{R}$, and by taking its exterior differential, we have $d\omega_p = \sum_{I\in\mathcal{I}_p^n} d\lambda_I \wedge du^I = 0$.

The tangent vectors $X \in T_xM$ in D are in one-to-one correspondence with the solutions

$$\mathcal{X} = \{(X^1, \dots, X^n) \in \mathbb{R}^n\}, \quad X^i = du^i(X), \quad 1 \le i \le n,$$

to the following system of $\binom{n}{p-1}$ linear equations with constant coefficients:

$$0 = \sum\nolimits_{I\in\mathcal{I}_p^n} (-1)^{h-1} X^h \lambda_I \left(du^{i_1}\right)_x \wedge \dots \wedge \widehat{\left(du^{i_h}\right)_x} \wedge \dots \wedge \left(du^{i_p}\right)_x .$$

As the vector space $\mathcal{X}$ is independent of the point x, it follows that D has a basis of vector fields with constant coefficients $V^1, \dots, V^k$ on the domain of the system $(u^1, \dots, u^n)$. Hence $[V^i, V^j] = 0$, $\forall i, j = 1, \dots, k$, and accordingly, D is involutive. □

Remark 1.4. In degrees $1, 2, n-1$ and n the property of having constant coefficients is generic in the space of closed forms, as proved in the next four examples.

Example 1.5. An 1-form ω_1 that does not vanish at x has constant coefficients if and only if ω_1 is closed. If ω_1 has constant coefficients, then it is closed as follows from Proposition 1.3; hence ω_1 is locally exact: $\omega_1 = df$. We can suppose $f(x) = 0$ and $\frac{\partial f}{\partial x^1}(x) \ne 0$, permuting indices if necessary. In this case the functions $u^1 = f$, $u^h = x^h$, $2 \le h \le n$, are a coordinate system centred at the point x, as the Jacobian of the system $(u^i)_{i=1}^n$ with respect to $(x^j)_{j=1}^n$ at x is $\frac{\partial(u^1,\dots,u^n)}{\partial(x^1,\dots,x^n)}(x) = \frac{\partial f}{\partial x^1}(x)$.

Example 1.6. A form of maximum degree $\omega_n \in \Omega^n(M)$ has constant coefficients on a neighbourhood of $x \in M$ if either ω_n vanishes on a neighbourhood of x or ω_n does not vanish at x. If $\omega_n = f dx^1 \wedge \dots \wedge dx^n$ and $f(x) \ne 0$, then by defining $u^1 = \int_0^{x^1} f\left(t, x^2, \dots, x^n\right) dt$, $u^i = x^i$, $2 \le i \le n$, we have $\omega_n = du^1 \wedge \dots \wedge du^n$.

Example 1.7. A 2-form ω_2 of constant class (*cf.* [6, Appendix 4, Section 3.5]) on a neighbourhood of $x \in M$ has constant coefficients if and only it is closed, as follows from Darboux's theorem (*cf.* [2, VI, Section 4.5]).

Example 1.8. An $(n-1)$-form $\omega_{n-1} \in \Omega^{n-1}(M)$ that does not vanish at $x \in M$ has constant coefficients if and only it is closed. In fact, if ω_n is a volume form on an open neighbourhood U of $x \in M$, then there exists a vector field $X \in \mathfrak{X}(U)$ such that $i_X \omega_n = \omega_{n-1}$, and $X_x \neq 0$; hence, there exist coordinates (x^i) centred at x reducing X to a canonical form, *i.e.*, $X = \frac{\partial}{\partial x^1}$. On these coordinates: $\omega_n = \rho dx^1 \wedge \ldots \wedge dx^n$ and $\omega_{n-1} = \rho dx^2 \wedge \cdots \wedge dx^n$. By differentiating we thus have $0 = d\omega_{n-1} = \frac{\partial \rho}{\partial x^1} dx^1 \wedge \cdots \wedge dx^n$. Accordingly, ρ depends only on $x^2, \ldots, x^n$, and we can conclude by virtue of Example 1.6.

Remark 1.9. The cases not included in the four examples above are $n \geq 5$, $3 \leq p \leq n-2$.

Example 1.10. A differential p-form ω_p on M of constant class equal to p has constant coefficients, as every $x \in M$ admits a coordinate neighbourhood $(U; x^1, \ldots, x^n)$ such that $\omega_p|_U = dx^1 \wedge \cdots \wedge dx^p$; see [2, VI, 3.4].

1.2. The contravariant case

Definition 1.11. Let M be a smooth manifold of dimension n. A q-vector field $V_q \in \wedge^q \mathfrak{X}(M)$ is said to have *constant coefficients* on a neighbourhood of the point $x \in M$ if there exists a system of coordinates $(u^1, \ldots, u^n)$ centred at x, such that all the functions $V_q\left(du^{i_1}, \ldots, du^{i_q}\right)$, $1 \leq i_1 < \ldots < i_q \leq n$, are constant.

Notation 1.12. We set $\partial u^I = \frac{\partial}{\partial u^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial u^{i_q}}$ for all $I \in \mathcal{I}^n_q$.

Example 1.13. A 1-vector field $X \in \mathfrak{X}(M)$ that does not vanish at x has constant coefficients on a neighbourhood of this point, as there exists a system of coordinates $(U; x^1, \ldots, x^n)$ centred at x, such that: $X|_U = \frac{\partial}{\partial x^1}$.

Proposition 1.14. *If a q-vector field V_q on M has constant coefficients, then the Schouten-Nijenhuis bracket V_q with itself vanishes,* i.e., $[V_q, V_q] = 0$. *In this case, the Pfaffian system $P = \{w \in T^*_x M : i_w(V_q)_x = 0, x \in M\}$ has a locally constant rank and for every $x \in M$ there exists an open neighbourhood U_x such that $\Gamma(U_x, P)$ admits a basis of closed 1-forms; hence, P is integrable.*

Proof. If $V_q = \sum_{I \in \mathcal{I}^n_q} \lambda_I \partial u^I$, $\lambda_I \in \mathbb{R}$, in a coordinate system $(U; u^1, \ldots, u^n)$, the first part of the statement follows by virtue of the following formula (*cf.* [8]):

$$\begin{aligned} [A, B] = {} & \textstyle\sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_q \wedge Y_1 \wedge \\ & \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_r, \\ A = {} & X_1 \wedge \cdots \wedge X_q, \\ B = {} & Y_1 \wedge \cdots \wedge Y_r, \\ \forall X_i, Y_j \in \mathfrak{X}(M), \; & 1 \leq i \leq q, 1 \leq j \leq r. \end{aligned}$$

The 1-covectors $w \in T^*_x M$ in P are in one-to-one correspondence with the solutions

$$\mathcal{W} = \{(w^1, \ldots, w^n) \in \mathbb{R}^n\}, \quad w^i = w\left(\frac{\partial}{\partial u^i}\right)_x, \quad 1 \leq i \leq n,$$

to the following system of $\binom{n}{q-1}$ linear equations with constant coefficients:

$$0=\sum\nolimits_{I\in\mathcal{I}_q^n}(-1)^{h-1}w^h\lambda_I\left(\frac{\partial}{\partial u^{i_1}}\right)_x\wedge\cdots\wedge\widehat{\left(\frac{\partial}{\partial u^{i_h}}\right)_x}\wedge\cdots\wedge\left(\frac{\partial}{\partial u^{i_q}}\right)_x.$$

As the vector space $\mathcal{W}$ is independent of the point x, it follows that P has a basis of 1-forms $\zeta^1,\dots,\zeta^k$ with constant coefficients and consequently $d\zeta^j=0$, $1\le j\le k$. Hence

$$d\left(\sum_{j=1}^k f_j\zeta^j\right)=\sum_{j=1}^k df_j\wedge\zeta^j,\quad \forall f_j\in C^\infty(M),\ 1\le j\le k.$$

This proves that P is integrable. □

Corollary 1.15. *If $n=\dim M$ is even, then a 2-vector field V_2 of rank n has constant coefficients if and only if $[V_2,V_2]=0$. If n is odd, then a 2-vector field V_2 of rank $n-1$ has constant coefficients if and only if the Pfaffian system $P=\{w\in T_x^*M : i_wV_2=0\}$ is integrable and the restriction of $[V_2,V_2]$ to each integral submanifold of P vanishes.*

Proof. If n is odd, then $\operatorname{rank} P=1$, as $\operatorname{rank} V_2=n-1$. Hence, there exists a local section of P with constant coefficients, $\omega_1=\lambda_i du^i\in\Gamma(U,P)$, and any other section of P is of the form $f\omega_1$.

If n is even, the mapping $V_2^\natural\colon T^*M\to TM$, $V_2^\natural(w)=i_wV_2$, $\forall w\in T^*M$, is an isomorphism, as $\operatorname{rank} V_2=n$, and a 2-form can be defined by the formula $\omega_2(X,Y)=V_2((V_2^\natural)^{-1}X,(V_2^\natural)^{-1}Y)$, $\forall X,Y\in TM$. According to [7] (also see [6, III.8.12–2]), ω_2 is closed if and only if $[V_2,V_2]=0$. In this case, by virtue of Darboux's theorem, there exist coordinates (p_i,q^i), $1\le i\le\frac{n}{2}$, such that $\omega_2=dp_i\wedge dq^i$; hence $V_2=\frac{\partial}{\partial p_i}\wedge\frac{\partial}{\partial q^i}$ has constant coefficients.

If n is odd and P is integrable, then the distribution $\ker P\subset\mathfrak{X}(M)$ is involutive, since if ω_1 is an arbitrary local section of P, then $d\omega_1=\omega_1\wedge\xi_1$ for some 1-form ξ_1. Therefore, for all $X,Y\in\Gamma(U,\ker P)$ we have

$$d\omega_1(X,Y)=(\omega_1\wedge\xi_1)(X,Y)=\omega_1(X)\xi_1(Y)-\omega_1(Y)\xi_1(X)=0,$$

since $\omega_1(X)=\omega_1(Y)=0$. Furthermore

$$\begin{aligned}0=d\omega_1(X,Y)&=X(\omega_1(Y))-Y(\omega_1(X))-\omega_1([X,Y])\\&=-\omega_1([X,Y]).\end{aligned}$$

Accordingly $[X,Y]\in\Gamma(U,\ker P)$. From Frobenius theorem, there exist coordinates $x^1,\dots,x^n$ such that $\ker P=\left\langle\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^{n-1}}\right\rangle$, so that in this system we have $P=\langle dx^n\rangle$; hence $V_2=\sum_{1\le i<j\le n-1}F_{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^i}$, and one concludes by applying the proof of the even-dimensional case to the restriction of V_2 to each integral submanifold $x^n=\lambda$ of P. □

Proposition 1.16. *If V_{n-1} is a $(n-1)$-vector field such that $(V_{n-1})_{x_0} \neq 0$ at a point $x_0 \in M$, then V_{n-1} has constant coefficients on an open neigbourhood U of x_0 if and only if the following two conditions hold:*

(1) *The Pfaffian system P associated to V_{n-1} according to Proposition* 1.14 *satisfies the properties therein stated;*
(2) *Let η be the $\wedge^n TM$-valued differential* 1*-form defined by*

$$\eta(X) = X \wedge V_{n-1}, \quad \forall X \in \mathfrak{X}(M).$$

There exists a derivation law

$$\nabla\colon \mathfrak{X}(M) \times \wedge^n \mathfrak{X}(M) \to \wedge^n \mathfrak{X}(M),\ (X, W_n) \mapsto \nabla_X W_n,$$

*with vanishing curvature (*i.e., $R_\nabla = 0$*) such that $d^\nabla \eta = 0$.*

Proof. If V_{n-1} has constant coefficients, *i.e.*, $V_{n-1} = c_i \frac{\partial}{\partial x^1} \wedge \cdots \wedge \widehat{\frac{\partial}{\partial x^i}} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$, then $\eta = (-1)^{i-1} c_i dx^i \otimes \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$ and it suffices to consider the derivation law given by

$$\nabla_X \left(f \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n} \right) = X(f) \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n},\ \forall f \in C^\infty(M),\ \forall X \in \mathfrak{X}(M).$$

Conversely, assume that items (1) and (2) hold.

Let U be the neigbourhood of x_0 defined by $\{x \in M : (V_{n-1})_x \neq 0\}$. Shrinking U if necessary we can assume that U is a coordinate domain. If V_n^0 is a basis for $\Gamma(U, \wedge^n TM)$, then there exists a unique ω_1 in $\Omega^1(U)$ such that $i_{\omega_1} V_n^0 = V_{n-1}$; hence $i_{\omega_1} V_{n-1} = 0$, and thus rank $P_{|U} \geq 1$. According to the assumption in item (i) we conclude that there exists a function f in $C^\infty(U)$ such that $(df)_x \neq 0$, $\forall x \in U$, and $i_{df} V_{n-1} = 0$. Again by shrinking U, we can assume that there exists a coordinate system $(U; x^1, \dots, x^n)$ such that $f = x^n$. We have $V_{n-1} = F \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^{n-1}}$, $\eta = \alpha \otimes V_n$, with $F \in C^\infty(U)$, $\alpha = (-1)^{n-1} F dx^n$ and $V_n = \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$. The connection form ω of ∇ is defined by the formula $\nabla_X V_n = \omega(X) V_n$, and the curvature is then given by $R_\nabla = d\omega \otimes V_n$. As $R_\nabla = 0$, we have $\omega = d\rho$, and substituting $\exp(-\rho) V_n$ for V_n, it follows $\nabla_X (V_n) = 0$. Therefore $d^\nabla \eta = d\alpha \otimes V_n = (-1)^{n-1} dF \wedge dx^n \otimes V_n = 0$, and, consequently, $F = F(x^n)$. As $F(x_0) \neq 0$, the following formulas $y^1 = \frac{x^1}{F}$, $y^i = x^i$, $2 \leq i \leq n$, determine a change of coordinates and we have $V_{n-1} = \frac{\partial}{\partial y^1} \wedge \cdots \wedge \frac{\partial}{\partial y^{n-1}}$. □

Remark 1.17. If ∇ is a derivation law as in Proposition 1.16 and D_∇ is the differential operator generating Schouten-Nijenhuis bracket introduced in [5, Section 2], then $(D_\nabla)^2 = 0$, as follows from the formula [5, (2.2)].

Remark 1.18. For $q \geq \frac{1}{2}(n+2)$ the condition $[V_q, V_q] = 0$ holds automatically, as $\deg[V_q, V_q] \geq 2q - 1 \geq n + 1$.

Example 1.19. An n-vector field V_n has constant coefficients on a neighbourhood of $x \in M$ if and only if, either V_n vanishes on a neighbourhood of x, or V_n does not vanish at x. In fact, in the second case, there exists a unique n-form ω_n such that $\omega_n(V_n) = 1$, and we conclude by virtue of Example 1.6.

1.3. Conformal constant coefficients

Each volume form ω_n induces an isomorphism

$$\begin{aligned}
&\iota_{pq}\colon \wedge^q T(M) \to \wedge^p T^*(M), \quad n = p+q,\\
&\iota_{pq}\left(X_1 \wedge \cdots \wedge X_q\right) = i_{X_1\wedge\cdots\wedge X_q}\omega_n = i_{X_1}\left(\ldots i_{X_q}(\omega_n)\right),\\
&\forall X_1,\ldots,X_q \in T_xM.
\end{aligned}$$

If $\omega_n = \rho dx^1 \wedge \cdots \wedge dx^n$, then for every $I = (i_1,\ldots,i_q) \in \mathcal{I}_q^n$, we have

$$\begin{aligned}
\iota_{pq}\left(\frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_q}}\right) &= i_{\frac{\partial}{\partial x^{i_1}}}\left(\ldots i_{\frac{\partial}{\partial x^{i_q}}}(\omega_n)\right)\\
&= \rho i_{\frac{\partial}{\partial x^{i_1}}}\left(\ldots (-1)^{i_q-1} dx^1 \wedge \cdots \wedge \widehat{dx^{i_q}} \wedge \cdots \wedge dx^n\right)\\
&= (-1)^{|I|-q}\rho dx^1 \wedge \cdots \wedge \widehat{dx^{i_1}} \wedge \cdots \wedge \widehat{dx^{i_q}} \wedge \cdots \wedge dx^n.
\end{aligned}$$

Hence

$$\begin{aligned}
&\iota_{pq}\left(\sum_{I\in\mathcal{I}_q^n} F_I \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_q}}\right)\\
&= \rho \sum_{I\in\mathcal{I}_q^n} (-1)^{|I|-q} F_I dx^1 \wedge \cdots \wedge \widehat{dx^{i_1}} \wedge \cdots \wedge \widehat{dx^{i_q}} \wedge \cdots \wedge dx^n.
\end{aligned}$$

Dually, an isomorphism can also be defined

$$\begin{aligned}
&\iota^*_{qp}\colon \wedge^p T^*(M) \to \wedge^q T(M), \quad n = p+q,\\
&\iota^*_{qp}\left(w_1 \wedge \cdots \wedge w_p\right) = i_{w_1\wedge\cdots\wedge w_p} V_n = i_{w_1}\left(\ldots i_{w_p}(V_n)\right),\\
&\forall w_1,\ldots,w_p \in T^*_x M,
\end{aligned}$$

V_n being an n-vector field that does not vanish at any point.

Definition 1.20. A q-vector field V_q (respectively a differential p-form ω_p) is said to have *conformal constant coefficients* if there exists a q-vector field V_q' (respectively a differential p-form ω_p') with constant coefficients and function $f \in C^\infty(M)$ such that $V_q = fV_q'$ (respectively $\omega_p = f\omega_p'$).

Remark 1.21. From the previous formulas it follows that if V_q has conformal constant coefficients, then the p-form $\iota_{pq}(V_q)$ has also conformal constant coefficients. In fact, if V_q' has constant coefficients on $(u^1, \dots, u^n)$, then $\omega_n = \rho du^1 \wedge \dots \wedge du^n$; hence $\iota_{pq}(V_q) = f\rho i_{V_q'}\left(du^1 \wedge \dots \wedge du^n\right)$, which is a p-form with conformal constant coefficients.

Remark 1.22. If V_q and ω_n have constant coefficients in the same coordinate system, then $i_{V_q}\omega_n$ has also constant coefficients in that system. If $n = pk$, ω_p has constant coefficients and $\omega_n = \omega_p \wedge \overset{(k}{\dots} \wedge \omega_p$ is a volume form, then the q-vector field V_q, $q = p(k-1)$, defined by $i_{V_q}\omega_n = \omega_p$, has also constant coefficients.

Example 1.23. An i-form ω_i on M with $i = 1$ or $i = 2$ of constant class and such that $(\omega_i)_x \neq 0$ at point $x \in M$ has conformal constant coefficients if and only if $\omega_i \wedge d\omega_i = 0$. In fact, if $\omega_i = \rho\omega_i'$, with $\rho \in C^\infty(M)$ and ω_i' has constant coefficients, then $d\omega_i = d\rho \wedge \omega_i'$, and hence $\omega_i \wedge d\omega_i = 0$. Conversely, if the previous equation holds, then the class of ω_i is 2 and by virtue of Darboux's theorem (*cf.* [2, VI, Section 4.1]) there exists a coordinate system $(x^i)_{i=1}^n$ such that either $\omega_1 = (1 + x^1)dx^2$ or $\omega_2 = \rho dx^1 \wedge dx^2$.

Example 1.24. If V_{n-1} is a $(n-1)$-vector field on M such that $(V_{n-1})_{x_0} \neq 0$ on a point $x_0 \in M$ and the Pfaffian system P associated to V_{n-1} according to Proposition 1.14 satisfies the properties therein stated, then V_{n-1} has conformal constant coefficients. This is proved by proceeding as in the proof of Proposition 1.16.

The results of this section prove that for very high or very low degrees p or q, differential p-forms and q-vector fields with constant coefficients admit simple geometric characterizations obtained by using classical theorems.

For intermediate degrees the situation is different and one must resort to consider partial differential systems of geometric nature that allow to characterize differential forms and vector fields with constant coefficients. This is the purpose of the next two sections.

2. The associated linear connection

Given a local basis $\mathbf{f} = (X_1, \dots, X_n)$ of the bundle of linear frames $F(M)$, or equivalently, $(X_1, \dots, X_n)$ is a local basis of the module $\mathfrak{X}(M)$, let $\nabla^{\mathbf{f}}$ denote the only linear connection on M parallelizing $\mathbf{f}$, *i.e.*, $\nabla^{\mathbf{f}}_{X_i} X_j = 0, \forall i, j = 1, \dots, n$. If $\mathbf{u} = (u^1, \dots, u^n)$ is a coordinate system on M, we write $\nabla^{\mathbf{u}} = \nabla^{\mathbf{f}}$, with $X_i = \frac{\partial}{\partial u^i}$, $1 \leq i \leq n$. The curvature of $\nabla^{\mathbf{f}}$ vanishes (*i.e.*, $\nabla^{\mathbf{f}}$ is flat) and its torsion vanishes if and only if there exist coordinates such that $\nabla^{\mathbf{f}} = \nabla^{\mathbf{u}}$.

Proposition 2.1. *A p-form ω_p (respectively a q-vector field V_q) has constant coefficients on the coordinate system $\mathbf{u} = (u^1, \dots, u^n)$ if and only if: $\nabla^{\mathbf{u}}\omega_p = 0$ (respectively $\nabla^{\mathbf{u}} V_q = 0$).*

In other words, ω_p (respectively V_q) has constant coefficients if and only if there exists a symmetric linear connection ∇ on M such that $R^\nabla = 0$, $\nabla\omega_p = 0$ (respectively $\nabla V_q = 0$).

Proof. The condition $\nabla^{\mathbf{u}}\omega_p = 0$ means $\nabla^{\mathbf{u}}_X\omega_p = 0$, $\forall X \in \mathfrak{X}(M)$. We have $\nabla^{\mathbf{u}}(du^J) = 0$, as

$$\left(\nabla^{\mathbf{u}}_{\frac{\partial}{\partial u^i}}(du^J)\right)\frac{\partial}{\partial u^k} = \frac{\partial}{\partial u^i}\left(du^J\left(\frac{\partial}{\partial u^k}\right)\right) - du^J\left(\nabla^{\mathbf{u}}_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^k}\right) = 0.$$

Consequently, if $\omega_p = \sum_{I\in\mathcal{I}^n_p} F_I du^I$ on the system $\mathbf{u} = (u^1, \dots, u^n)$, then the equation $\nabla^{\mathbf{u}}_X\omega_p = 0$ is equivalent to saying $X(F_I) = 0$ for all $X \in \mathfrak{X}(M)$ and all multi-indices $I \in \mathcal{I}^n_p$; in other words, all the coefficients of ω_p are constant.

The proof for the contravariant case is analogous. □

Lemma 2.2. *If $\nabla^{\mathbf{u}}_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) = \Gamma^h_{ij}\frac{\partial}{\partial x^h}$, then $\Gamma^b_{ij} = v^b_h\frac{\partial^2 u^h}{\partial x^i\partial x^j}$, where (v^i_j) denotes the inverse matrix of the Jacobian matrix $\left(\frac{\partial u^i}{\partial x^j}\right)$, $b, h, i, j = 1, \dots, n$.*

Proof. From $\frac{\partial}{\partial x^j} = \frac{\partial u^h}{\partial x^j}\frac{\partial}{\partial u^h}$, it follows:

$$\nabla^{\mathbf{u}}_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial^2 u^h}{\partial x^i\partial x^j}\frac{\partial}{\partial u^h} + \frac{\partial u^h}{\partial x^j}\nabla^{\mathbf{u}}_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial u^h}\right).$$

As the second term in the right-hand side vanishes, taking the formulas $\frac{\partial u^a}{\partial x^b}v^b_c = \delta^a_c$, $v^b_c\frac{\partial}{\partial x^b} = \frac{\partial u^a}{\partial x^b}v^b_c\frac{\partial}{\partial u^a} = \delta^a_c\frac{\partial}{\partial u^a} = \frac{\partial}{\partial u^c}$ into account, we have

$$\nabla^{\mathbf{u}}_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial^2 u^h}{\partial x^i\partial x^j}\frac{\partial}{\partial u^h} = v^b_h\frac{\partial^2 u^h}{\partial x^i\partial x^j}\frac{\partial}{\partial x^b},$$

thus concluding. □

3. The associated partial differential system

In Proposition 2.1 we have proved that a p-form or a q-vector field have constant coefficients if and only if there exists a linear connection of vanishing curvature parallelizing them. Below, this condition is proved to be equivalent to the existence of solution to a partial differential system of equations, which we describe explicitly.

Theorem 3.1. *Let $n = \dim M$ and let $O^2_M \subset J^2(M, \mathbb{R}^n)$ be the open subbundle of 2-jets $j^2_x(u^1, \dots, u^n)$ such that $(du^1 \wedge \cdots \wedge du^n)_x \neq 0$.*

Given a differential p-form and q-vector field,

$$\omega_p = \sum_{I \in \mathcal{I}_p^n} F_I dx^I, \quad V_q = \sum_{I \in \mathcal{I}_q^n} \bar{F}_I \partial x^I,$$

on M, let $\Phi_{j,J}$ (respectively $\bar{\Phi}_{j,J}$) be the functions defined by the following formulas:

$$\begin{aligned} \sum_{J \in \mathcal{I}_p^n} \Phi_{j,J} dx^J &= \sum_{I \in \mathcal{I}_p^n} \sum_{h=1}^{p} \sum_{i=1}^{n} \Gamma_{ji}^{i_h} F_I dx^{i_1} \wedge \cdots \wedge \underset{(i_h}{\underline{dx^i}} \wedge \cdots \wedge dx^{i_p}, \\ \textstyle\sum_{J \in \mathcal{I}_q^n} \bar{\Phi}_{j,J} \partial x^J &= \sum_{I \in \mathcal{I}_q^n} \sum_{h=1}^{q} \sum_{i=1}^{n} \Gamma_{ji}^{i_h} \bar{F}_I \partial x^{i_1} \wedge \cdots \wedge \underset{(i_h}{\underline{\partial x^i}} \wedge \cdots \wedge \partial x^{i_q}, \\ 1 \le j \le n, & \end{aligned} \tag{3.1}$$

where Γ_{bc}^a are the functions introduced in Lemma 2.2 *and the notation $\underset{(i_h}{\underline{dx^i}}$ and $\underset{(i_h}{\underline{\partial x^i}}$ mean that the* 1*-form dx^i and the vector field ∂x^i must be inserted in the i_h-th place of the exterior product.*

The form ω_p (respectively the vector field V_q) has constant coefficients on a neighbourhood of $x \in M$ if and only if the following second-order partial differential system defined on $J^2(M, \mathbb{R}^n)$:

$$\left.\begin{aligned} &\text{(i)}\ \tfrac{\partial F_J}{\partial x^j} = \Phi_{j,J}, \quad \forall J \in \mathcal{I}_p^n \\ &\text{(ii)}\ \frac{\partial \bar{F}_J}{\partial x^j} = -\bar{\Phi}_{j,J}, \ \forall J \in \mathcal{I}_q^n \end{aligned}\right] \ 1 \le j \le n, \tag{3.2}$$

admits a local solution $(u^1, \ldots, u^n)$ defined on a neighbourhood of x in O_M^2.

Proof. According to Proposition 2.1 we must impose $\nabla^{\mathbf{u}} \omega_p = 0$, *i.e.*,

$$\nabla^{\mathbf{u}}_{\frac{\partial}{\partial x^j}} \omega_p = \sum_{I \in \mathcal{I}_p^n} \left(\frac{\partial F_I}{\partial x^j} dx^I + \sum_{h=1}^{p} F_I dx^{i_1} \wedge \cdots \wedge \nabla^{\mathbf{u}}_{\frac{\partial}{\partial x^j}} \left(dx^{i_h}\right) \wedge \cdots \wedge dx^{i_p} \right),$$

where $I = (i_1, \ldots, i_p)$. Moreover, we have

$$\nabla^{\mathbf{u}}_{\frac{\partial}{\partial x^j}} \left(dx^{i_h}\right) \left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^j} \left(dx^{i_h} \left(\frac{\partial}{\partial x^i} \right) \right) - dx^{i_h} \left(\nabla^{\mathbf{u}}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} \right) = -\Gamma_{ji}^{i_h}.$$

Hence $\nabla^{\mathbf{u}}_{\frac{\partial}{\partial x^j}} \left(dx^{i_h}\right) = -\Gamma_{ji}^{i_h} dx^i$, and thus

$$\begin{aligned} 0 &= \nabla^{\mathbf{u}}_{\frac{\partial}{\partial x^j}} \omega_p \\ &= \textstyle\sum_{I \in \mathcal{I}_p^n} \left(\frac{\partial F_I}{\partial x^j} dx^I - \sum_{h=1}^{p} \Gamma_{ji}^{i_h} F_I dx^{i_1} \wedge \cdots \wedge \underset{(i_h}{\underline{dx^i}} \wedge \cdots \wedge dx^{i_p} \right), \end{aligned}$$

which allows one to conclude.

The proof of the contravariant case is analogous, replacing the formulas $\nabla^{\mathbf{u}}_{\frac{\partial}{\partial x^j}}\left(dx^h\right) = -\Gamma^h_{ji}dx^i$ by $\nabla^{\mathbf{u}}_{\frac{\partial}{\partial x^j}}\left(\frac{\partial}{\partial x^h}\right) = \Gamma^i_{jh}\frac{\partial}{\partial x^i}$. □

The right-hand side of (3.1) can be computed in order to obtain the function $\Phi_{j,J}$ explicitely, by simply applying the rules of calculation in the exterior algebra.

Example 3.2. As stated in Remark 1.9, the first case not included in the examples in Section 1.1 is obtained for the values $p = 3, n = 5$. In this case, we have

$$\begin{aligned}\omega_3 = \; & F_{123}dx^1 \wedge dx^2 \wedge dx^3 + F_{124}dx^1 \wedge dx^2 \wedge dx^4 + F_{125}dx^1 \wedge dx^2 \wedge dx^5 \\ & +F_{134}dx^1 \wedge dx^3 \wedge dx^4 + F_{135}dx^1 \wedge dx^3 \wedge dx^5 + F_{145}dx^1 \wedge dx^4 \wedge dx^5 \\ & +F_{234}dx^2 \wedge dx^3 \wedge dx^4 + F_{235}dx^2 \wedge dx^3 \wedge dx^5 + F_{245}dx^2 \wedge dx^4 \wedge dx^5 \\ & +F_{345}dx^3 \wedge dx^4 \wedge dx^5.\end{aligned}$$

For simplicity's sake we write $F_{hij} = 0$ if there is a repeated index in (h, i, j); otherwise $F_{hij} = \varepsilon F_{abc}$, with $a < b < c$ and ε is the sign of the permutation $a \mapsto h, b \mapsto i, c \mapsto j$. Expanding on the first formula of item (1) in Theorem 3.1, we have

$$\begin{aligned}& \Phi_{j,123}dx^1 \wedge dx^2 \wedge dx^3 + \Phi_{j,124}dx^1 \wedge dx^2 \wedge dx^4 + \Phi_{j,125}dx^1 \wedge dx^2 \wedge dx^5 \\ & + \Phi_{j,134}dx^1 \wedge dx^3 \wedge dx^4 + \Phi_{j,135}dx^1 \wedge dx^3 \wedge dx^5 + \Phi_{j,145}dx^1 \wedge dx^4 \wedge dx^5 \\ & + \Phi_{j,234}dx^2 \wedge dx^3 \wedge dx^4 + \Phi_{j,235}dx^2 \wedge dx^3 \wedge dx^5 + \Phi_{j,245}dx^2 \wedge dx^4 \wedge dx^5 \\ & + \Phi_{j,345}dx^3 \wedge dx^4 \wedge dx^5 \\ = & \sum_{I\in\mathcal{I}^5_3}\sum_{i=1}^{5}\Gamma^{i_1}_{ji}F_{i_1i_2i_3}dx^i \wedge dx^{i_2} \wedge dx^{i_3} + \sum_{I\in\mathcal{I}^5_3}\sum_{i=1}^{5}\Gamma^{i_2}_{ji}F_{i_1i_2i_3}dx^{i_1} \wedge dx^i \wedge dx^{i_3} \\ & + \sum_{I\in\mathcal{I}^5_3}\sum_{i=1}^{5}\Gamma^{i_3}_{ji}F_{i_1i_2i_3}dx^{i_1} \wedge dx^{i_2} \wedge dx^i.\end{aligned}$$

Hence

$$\begin{aligned}\Phi_{j,123} = & \Gamma^h_{j1}\left(F_{h23} - F_{2h3} + F_{23h}\right) + \Gamma^h_{j2}\left(F_{1h3} - F_{h13} - F_{13h}\right) \\ & + \Gamma^h_{j3}\left(F_{h12} - F_{1h2} + F_{12h}\right), \\ \Phi_{j,124} = & \Gamma^h_{j1}\left(F_{h24} - F_{h24} + F_{24h}\right) + \Gamma^h_{j2}\left(F_{1h4} - F_{h14} - F_{14h}\right) \\ & + \Gamma^h_{j4}\left(F_{h12} - F_{1h2} + F_{12h}\right), \\ \Phi_{j,125} = & \Gamma^h_{j1}\left(F_{h25} - F_{h25} + F_{25h}\right) + \Gamma^h_{j2}\left(F_{1h5} - F_{h15} - F_{15h}\right) \\ & + \Gamma^h_{j5}\left(F_{h12} - F_{1h2} + F_{12h}\right), \\ \vdots \quad & \quad \vdots\end{aligned}$$

and thus analogously for the remaining components.

Moreover, the cases studied in Section 1.2 (the contravariant case) are: $q = 1$ (Example 1.13), $q = 2$ and V_2 being of maximum rank (Corollary 1.15), $q = n-1$ (Proposition 1.16), and $q = n$ (Example 1.19) with arbitrary n. Therefore, the first case not included in the examples in Section 1.1 is also obtained for the values $q = 3, n = 5$. Consequently, replacing Γ^h_{ij} by $-\Gamma^i_{jh}$ in the previous formulas, the corresponding values for $\bar{\Phi}_{j,abc}$, $1 \le a < b < c \le 5$, are also obtained.

Corollary 3.3. *Let* $\Psi_{j,J}$, $J \in \mathcal{I}^n_p$ *(respectively* $\bar{\Psi}_{j,J}$, $J \in \mathcal{I}^n_q$*) be the function obtained by replacing* $\Gamma^a_{cd} = v^a_b \frac{\partial^2 u^b}{\partial x^c \partial x^d}$ *by* $\Gamma^a_{cd} = -\frac{\partial v^a_b}{\partial x^d}\left(V^{-1}\right)^b_c$ *in the formulas above, where* $V = (v^i_j)$ *is the inverse matrix of the Jacobian matrix* $\left(\frac{\partial u^i}{\partial x^j}\right)$. *The systems* (3.2)-(i) *and* (3.2)-(ii) *of PDS of second order of Theorem* 3.1*, defined in* O^2_M*, are respectively equivalent to the following systems of first order in the variables* v^i_j*,* $i, j = 1, \dots, n$*, both defined on* $J^1(M, GL(n, \mathbb{R}))$*:*

$$\begin{aligned} \frac{\partial F_J}{\partial x^j} &= \Psi_{j,J}, \qquad \forall J \in \mathcal{I}^n_p, 1 \le j \le n, \\ v^j_m \frac{\partial v^k_l}{\partial x^j} &= \frac{\partial v^k_m}{\partial x^i} v^i_l \qquad 1 \le k \le n, 1 \le l < m \le n, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \frac{\partial \bar{F}_J}{\partial x^j} &= \bar{\Psi}_{j,J}, \qquad \forall J \in \mathcal{I}^n_q, 1 \le j \le n, \\ v^j_m \frac{\partial v^k_l}{\partial x^j} &= \frac{\partial v^k_m}{\partial x^i} v^i_l 1 \le k \le n, 1 \le l < m \le n. \end{aligned} \tag{3.4}$$

Proof. By taking derivatives on $v^a_b \frac{\partial u^b}{\partial x^c} = \delta^a_c$ with respect to x^d it follows: $\frac{\partial v^a_b}{\partial x^d}\frac{\partial u^b}{\partial x^c} + v^a_b \frac{\partial^2 u^b}{\partial x^c \partial x^d} = 0$. According to Lemma 2.2: $\Gamma^a_{cd} = v^a_b \frac{\partial^2 u^b}{\partial x^c \partial x^d}$; hence

$$\Gamma^a_{cd} = -\frac{\partial v^a_b}{\partial x^d}\frac{\partial u^b}{\partial x^c} = -\frac{\partial v^a_b}{\partial x^d}\left(V^{-1}\right)^b_c.$$

The system (3.2) of Theorem 3.1 is thus of first order with respect to the variables v^i_j, and hence, it is defined on $J^1(M, GL(n, \mathbb{R}))$; but this is not enough to solve it by taking the functions v^i_j as unknowns, since it is not ensured that the inverse matrix V^{-1} is a Jacobian matrix. For this, the following equations must also be verified: $\frac{\partial (V^{-1})^h_i}{\partial x^j} = \frac{\partial (V^{-1})^h_j}{\partial x^i}$, $h, i, j = 1, \dots, n$, or equivalently,

$$(V^{-1})^h_a \frac{\partial v^a_b}{\partial x^j}(V^{-1})^b_i = (V^{-1})^h_a \frac{\partial v^a_b}{\partial x^i}(V^{-1})^b_j, \quad h, i, j = 1, \dots, n, \tag{3.5}$$

as follows by taking derivatives in the equality $v_b^a (V^{-1})_c^b = \delta_c^a$ with respect to x^d and solving it for $\frac{\partial (V^{-1})_c^b}{\partial x^d}$. Finally, by multiplying the two sides of (3.5) by v_h^k and by summing up over h, it follows: $\frac{\partial v_b^k}{\partial x^j}(V^{-1})_i^b = \frac{\partial v_b^k}{\partial x^i}(V^{-1})_j^b$, and by multiplying both sides by v_m^i and summing up over i, we thus obtain

$$\frac{\partial v_l^k}{\partial x^j} = \frac{\partial v_b^k}{\partial x^i}(V^{-1})_j^b v_l^i.$$

Furthermore, by multiplying both sides by v_m^j and summing up over j, we conclude. □

Corollary 3.4. *The integrability conditions of the system* (3.2)-(i) *(respectively* (3.2)-(ii)*) with respect to the unknowns F_J (respectively $\bar{F}_J$) hold identically.*

Accordingly, if M is of class C^ω, given a point $x_0 \in M$, scalars λ_J (respectively $\bar{\lambda}_J$), $1 \leq j \leq n$, and a coordinate system $(u^1, \dots, u^n)$ of class C^ω on M, there exists a unique differential p-form ω_p (respectively a q-vector field V_q) on M with constant coefficients in the system $(u^1, \dots, u^n)$ such that $F_J(x_0) = \lambda_J$ (respectively $\bar{F}_J(x_0) = \bar{\lambda}_J$).

Proof. From the equations (3.2)-(i) it follows:

$$\frac{\partial^2 F_J}{\partial x^j \partial x^l} = \frac{\partial \Phi_{j,J}}{\partial x^l}, \qquad \frac{\partial^2 F_J}{\partial x^l \partial x^j} = \frac{\partial \Phi_{l,J}}{\partial x^j}.$$

Hence, integrability conditions of the system (3.2)-(i) with respect to the functions F_J are $0 = \frac{\partial \Phi_{j,J}}{\partial x^l} - \frac{\partial \Phi_{l,J}}{\partial x^j}$.

As $\nabla^{\mathbf{u}}$ is flat, we have $\frac{\partial \Gamma_{bd}^a}{\partial x^c} - \frac{\partial \Gamma_{bc}^a}{\partial x^d} = \Gamma_{bc}^e \Gamma_{de}^a - \Gamma_{bd}^e \Gamma_{ce}^a$, and as a calculation shows, the integrability conditions written above hold identically.

The proof in the contravariant case is analogous. □

Remark 3.5. Certainly a p-form ω_p or a q-vector field V_q have constant coefficients in an open subset U if there exists a coordinate system $(u^i)_{i=1}^n$ defined on U such that $L_{\frac{\partial}{\partial u^i}} \omega_p = 0$ or $L_{\frac{\partial}{\partial u^i}} V_q = 0$, respectively, for $1 \leq i \leq n$. By directly imposing these conditions, systems of equations equivalent to (3.2) are obtained. However, we prefer the torsion-free flat linear connection method because it reveals the geometry underlying the problem considered and is closer to the classical formulations given in Geometry; for example, a pseudo-Riemannian metric has constant coefficients if and only if the curvature of its associated Levi-Civita connection vanishes.

4. Computational aspects

Remark 4.1. For all $n \geq 7$, $3 \leq p \leq n-3$, the system (3.3) has $n\binom{n}{p} + n\binom{n}{2}$ equations in the $n^2 + n^3$ variables v^a_b, $\frac{\partial v^a_b}{\partial x^d}$, and we have $n\binom{n}{p} + n\binom{n}{2} \geq n^2 + n^3$. In fact, only for $3 \leq p \leq 4$, $n = 7$ the inequality above turns into an equality, *i.e.*, $7\binom{7}{p} + 7\binom{7}{2} = 7^2 + 7^3$. Accordingly, from dimension 8 there are more equations than unknowns and the system (3.3) is generically compatible and determined, so that the connection $\nabla^{\mathbf{u}}$, if exists, is unique.

Remark 4.2. The systems (i) and (ii) in (3.2) can be viewed as two linear systems in the n^3 unknowns Γ^a_{bc} each with $n\binom{n}{p}$ equations.

Moreover, as a computation shows, we have

$$n\binom{n}{p} < n^3, \text{ if } 2 \leq n \leq 7 \text{ and } 2 \leq p \leq n-2,$$

$$n\binom{n}{p} < n^3, \text{ if } n = 8 \text{ and } p = 2, 3, 5, 6,$$

$$n\binom{n}{p} \geq n^3, \text{ if } n = 8 \text{ and } p = 4,$$

$$n\binom{n}{p} < n^3, \text{ if } n \geq 9 \text{ and } n-2 \leq p \leq n,$$

$$n\binom{n}{p} \geq n^3, \text{ if } n \geq 9 \text{ and } 3 \leq p \leq n-3.$$

Remark 4.3. Let $\mathcal{M}(x)$ be the coefficient matrix of (3.2)-(i) at a point $x \in M$, and let $\mathcal{M}'(x)$ be its augmented matrix.

We have $\operatorname{rank} \mathcal{M}(x) \leq \operatorname{rank} \mathcal{M}'(x) \leq n^3$, and from Rouché-Capelli theorem it follows:

- If $\operatorname{rank} \mathcal{M}(x) < \operatorname{rank} \mathcal{M}'(x)$, then $\operatorname{rank} \mathcal{M}(x') < \operatorname{rank} \mathcal{M}'(x')$ for every x' in a neighbourhood of x, and the p-form $\omega_p = \sum_{I \in \mathcal{I}^n_p} F_I dx^I$ does not have constant coefficients on a neighbourhood of x
  A similar result also holds for a q-vector field;
- If $\operatorname{rank} \mathcal{M}(x') = \operatorname{rank} \mathcal{M}'(x')$ for every x' in a neighbourhood of x, then the system (3.2)-(i) admits at least a solution Γ^a_{bc}, $a, b, c = 1, \ldots, n$; but we need to impose that the curvature and torsion fields of the linear connection ∇ given by the formula $\nabla_{\frac{\partial}{\partial x^b}} \frac{\partial}{\partial x^c} = \Gamma^a_{bc} \frac{\partial}{\partial x^a}$ should vanish, or equivalently, such that $\nabla = \nabla^{\mathbf{u}}$ for some coordinate system $\mathbf{u} = (u^1, \ldots, u^n)$.
  In other words, on the solution to (3.2)-(i) we must impose the following condi-

tions:

$$
\begin{aligned}
&(\mathrm{C}) \quad \frac{\partial \Gamma^a_{bd}}{\partial x^c} - \frac{\partial \Gamma^a_{bc}}{\partial x^d} = \Gamma^e_{bc}\Gamma^a_{de} - \Gamma^e_{bd}\Gamma^a_{ce},\\
&(\mathrm{T}) \qquad\qquad \Gamma^a_{cb} = \Gamma^a_{bc},\\
&\qquad\qquad a, b, c, d, e = 1, \dots, n.
\end{aligned}
$$

Remark 4.4. If $n\binom{n}{p} \geq n^3$, $3 \leq p \leq n-3$, as in Remark 4.2 and furthermore we have $\operatorname{rank} \mathcal{M}(x') = \operatorname{rank} \mathcal{M}'(x') = n^3$, then the solution to the system (3.2)-(i) can be written in the form $\Gamma^a_{bc} = Q^a_{bc}$, $a, b, c = 1, \dots, n$, where the functions Q^a_{bc} are rational fractions of the functions F_J, $\frac{\partial F_J}{\partial x^j}$, $J \in \mathcal{I}^n_p$, $1 \leq j \leq n$. The contravariant case is completely similar. Hence, in this case, there exist two second-order differential operators

$$
D \colon \mathcal{O} \subset \wedge^p \Omega(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)^k, \quad \bar{D} \colon \bar{\mathcal{O}} \subset \wedge^q \mathfrak{X}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)^{\bar{k}},
$$

for some integers k and $\bar{k}$, where $\mathcal{O}$ and $\bar{\mathcal{O}}$ are dense open subsets, such that a differential p-form $\omega_p \in \mathcal{O}$ and a q-vector field $V_q \in \bar{\mathcal{O}}$ have constant coefficients if and only if $D(\omega_p) = 0$ and $\bar{D}(V_q) = 0$, respectively.

Although the operators D and $\bar{D}$ can be computed feasibly, since their computation is reduced to Linear Algebra operations, their expressions become increasingly longer when the value of n increases.

If $\operatorname{rank} \mathcal{M}(x') = \operatorname{rank} \mathcal{M}'(x')$ does not reach its maximum value, the order of the operators D and $\bar{D}$ may be higher than 2, if they exist.

Remark 4.5. The case of a $(n-1)$-vector field

$$
V_{n-1} = \sum\nolimits_{i=1}^n F_i \frac{\partial}{\partial x^1} \wedge \cdots \wedge \widehat{\frac{\partial}{\partial x^i}} \wedge \cdots \wedge \frac{\partial}{\partial x^n}
$$

is exceptional because from Remark 4.2 we know that the system (ii) in (3.2) is a linear system in the n^3 unknowns Γ^a_{bc} with $n\binom{n}{n-1} = n^2$ equations. In fact, taking the symmetry $\Gamma^a_{bc} = \Gamma^a_{cb}$ into account, it follows that the number of unknowns in this case is $\frac{1}{2}n^2(n+1) > n^2$. Hence the system (ii)-(3.2) is underdetermined and the associated torsion-free flat linear connection is not unique.

With the previous notation, the system (ii)-(3.2) can be written as follows:

$$
\begin{aligned}
\frac{\partial F_l}{\partial x^j} = &\left(\delta_{l,1} - 1\right) \sum_{h=1}^{l-1} \Gamma^h_{jh} F_l + \left(\delta_{l,n} - 1\right) \sum_{h=l+1}^{n} \Gamma^h_{jh} F_l \\
&+ \left(1 - \delta_{i,1}\right)\left(1 - \delta_{l,n}\right) \sum_{i=l+1}^{n} (-1)^{i-l} \Gamma^i_{jl} F_i \\
&+ \left(1 - \delta_{i,n}\right)\left(1 - \delta_{l,1}\right) \sum_{i=1}^{l-1} (-1)^{l-i} \Gamma^i_{jl} F_i, \qquad j, l = 1, \dots, n.
\end{aligned} \tag{4.1}
$$

As a computation shows, on the dense open subset defined by $F_1 \cdots F_n \neq 0$ the rank of the $n^2 \times \frac{1}{2}n^2(n+1)$ coefficient matrix of (4.1) is n^2, *i.e.*, the rank is maximum. Hence n^2 of the unknowns Γ^a_{bc}, $a, b, c = 1, \dots, n$, $b \leq c$, can be written in terms of the $\frac{1}{2}n^2(n+1) - n^2 = n\binom{n}{2}$ remaining unknowns.

We can also compute the first prolongation $\mathcal{P}^1$ of (4.1).

Furthemore, we must impose the vanishing of the curvature, *i.e.*, the equations (C) in Remark 4.3.

Therefore, by joining the systems (4.1), its first prolongation and the independent components of the curvature, a system is obtained, which has $n^2 + \frac{1}{2}n^2(n+1) + \frac{1}{3}n^2(n^2-1) = \frac{1}{6}n^2(3n+2n^2+7)$ equations in the $\frac{1}{2}n^2(n+1) + \frac{1}{2}n^3(n+1) = \frac{1}{2}n^2(n+1)^2$ unknowns Γ^h_{ij}, $\frac{\partial \Gamma^h_{ij}}{\partial x^k}$, $1 \leq i \leq j \leq n$, $h, k = 1, \dots, n$. We cannot write down the formulas that solve this system because they are very involved; for example, for $n = 3$, 9 unknown Gammas Γ^h_{ij}, $1 \leq h \leq 3$, $1 \leq i \leq j \leq 3$, can be written as functions of the 9 remaining Gammas, F_j, and $\frac{\partial F_j}{\partial x^l}$, $j, l = 1, \dots, 3$ in the system (4.1), whereas in the system $\mathcal{P}^1 \cup \{R^i_{jkl} = 0 : 1 \leq k \leq j \leq n, 1 \leq k < l \leq n\}$, 36 unknowns $\frac{\partial \Gamma^h_{ij}}{\partial x^k}$, $h, k = 1, 2, 3$, $1 \leq i \leq j \leq 3$, can be written as functions of the 15 remaining derivatives of Gamma's, F_j, $\frac{\partial F_l}{\partial x^l}$, and $\frac{\partial^2 F_l}{\partial x^k \partial x^l}$, $j, k, l = 1, 2, 3$, $k \leq l$, but in addition there appear 3 constraints that must be fulfilled identically $C_1 = 0$, $C_2 = 0$, $C_3 = 0$. They can be written as functions of the coefficient $C = -\frac{1}{2}dx^1 \wedge dx^2 \wedge dx^3([V_2, V_2])$, namely

$$C_1 = F_2 \frac{\partial}{\partial x^3}\left(\frac{C}{F_2}\right),$$

$$C_2 = F_2 \frac{\partial}{\partial x^2}\left(\frac{C}{F_2}\right),$$

$$\begin{aligned} C_3 = &-\frac{F_1^2 F_2}{F_3^2}\left(F_1\Gamma^1_{33} - F_2\Gamma^2_{33} + F_3\Gamma^3_{33}\right)C + \left(F_3\Gamma^3_{12} + F_2\Gamma^3_{13}\right)C \\ &- F_1 C_2 + F_2^2 \frac{\partial}{\partial x^1}\left(\frac{C}{F_2}\right) - \frac{F_1}{F_3^2}\left(F_2\frac{\partial F_3}{\partial x^3} - F_3\frac{\partial F_3}{\partial x^2} - F_3\frac{\partial F_2}{\partial x^3}\right)C, \end{aligned}$$

and we can conclude that the constraints vanish by simply applying Corollary 1.15.

5. Some applications

1. Two 2-forms ω_2, ω_2' on a manifold M have constant coefficients if there exist linear connections ∇, ∇' with vanishing torsion and curvature tensors such that $\nabla\omega_2 = 0$, $\nabla'\omega_2' = 0$. Furthermore, to say that ω_2, ω_2' have constant coefficients "simultaneously" means that $\nabla = \nabla'$. As ∇ parallelizes ω_2, we have $X(\omega_2(Y, Z)) = \omega_2(\nabla_X Y, Z) + \omega_2(Y, \nabla_X Z)$. If ω_2 is of maximum rank, *i.e.*,

rank $\omega_2 = 2n = \dim M$, then we can define an endomorphism $J\colon TM \to TM$ by setting $\omega_2(X, J(Y)) = \omega_2'(X, Y), \forall X, Y \in T_x M$, and as a calculation shows, we have $\left(\nabla_X \omega_2'\right)(Y, Z) = \omega_2\left(Y, (\nabla_X J)Z\right), \forall X, Y, Z \in \mathfrak{X}(M)$.
Hence ∇ also parallelizes ω_2' if and only if it parallelizes J, *i.e.*, $\nabla J = 0$. The previous result can be applied to the study of complex valued 2-forms $\omega_2 + i\omega_2' \in \Omega^2(M) \otimes_{\mathbb{R}} \mathbb{C}$ with constant coefficients.

2. An almost Hermitian manifold (M, g, J) is Kähler if and only if its fundamental 2-form Φ is closed. The manifold M is Kähler if and only if $d\Phi = 0$. As Φ is defined by $\Phi(X, Y) = g(X, JY)$ it follows that Φ is of maximum rank, and we can conclude from Example 1.7 above. Moreover, if there exists an analytic coordinate system $z = (z^j)_{j=1}^n$, $n = \dim M$, such that $\Phi = \sum_{\alpha=1}^n dx^\alpha \wedge dy^\alpha = -\frac{1}{2i}\sum_{\alpha=1}^n dz^\alpha \wedge d\bar{z}^\alpha$, with $x^\alpha = \operatorname{Re} z^\alpha$, $y^\alpha = \operatorname{Im} z^\alpha$, $1 \le \alpha \le n$, then g is flat, as $\Phi = i\sum_{\alpha,\beta=1}^n g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta = -\frac{1}{2i}\sum_{\alpha=1}^n dz^\alpha \wedge d\bar{z}^\alpha$. Hence the coefficients $g_{\alpha\bar{\beta}}$ are constant.
3. Let ω_{p+q} be a form of degree $p + q$ and type (p, q) on a complex manifold M. If $z = (z^j)_{j=1}^n$, $n = \dim_{\mathbb{C}} M$, is an analytic coordinate system on an open domain $U \subset M$ such that
$$\omega_{p+q} = \sum_{i_1 < \ldots < i_p, i_1 < \ldots < i_p} \lambda_{i_1,\ldots,i_p,j_1,\ldots,j_q}(z, \bar{z}) dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{i_1} \wedge \ldots \wedge d\bar{z}^{i_p},$$
with $\lambda_{i_1,\ldots,i_p,j_1,\ldots,j_q} \in \mathbb{C}$, then the real and imaginary parts of ω_{p+q} have constant coefficients, as writing $x^j = \operatorname{Re} z$, $y^j = \operatorname{Im} z^j$, $1 \le j \le n$, we deduce $dz^1 \wedge \ldots \wedge dz^r = \sum_{|I|+|J|=r, I\cap J=\emptyset} \varepsilon_{IJ} (dx)^I \wedge (dy)^J$, where I and J are multi-indices $I = (i_1, \ldots, i_k) \in \mathbb{N}^k$, $0 \le k \le r$, $J = (j_1, \ldots, j_l) \in \mathbb{N}^l$, $k + l = r$, and $\varepsilon_{IJ} \in \{\pm 1, \pm i\}$. Hence $\operatorname{Re} \omega_{p+q}$ and $\operatorname{Im} \omega_{p+q}$ are constant coefficients when written in the real coordinate system $(x^j, y^j)_{j=1}^n$. The converse however is not true as shows the case of the fundamental 2-form Φ (which is the type $(1, 1)$) of a Kähler manifold with non-flat Riemannian metric as proved in the previous item.
4. If $(\omega^1, \ldots, \omega^n)$ is basis of Maurer-Cartan forms on a connected Lie group G, then G is Abelian if and only if each of these forms ω^i, $1 \le i \le n$, have constant coefficients. In fact, let $(X_1, \ldots, X_n)$ be the dual basis to $(\omega^1, \ldots, \omega^n)$ of left-invariant vector fields. If the forms ω^i, $1 \le i \le n$, have constant coefficients, then they are closed, hence $0 = (d\omega^i)(X_j, X_k) = -\omega^i([X_j, X_k]), \forall i, j, k = 1, \ldots, n$. Therefore $[X_j, X_k] = 0$, thus proving that the Lie algebra of G is Abelian and consequently so is G, because it is connected. Conversely, if G is Abelian, then the torsion and curvature tensors of the linear connection $\nabla^{\mathbf{f}}$ parallelizing the linear frame $\mathbf{f} = (X_1, \ldots, X_n)$ vanish; hence $\nabla^{\mathbf{f}} = \nabla^{\mathbf{u}}$ for some coordinate system $\mathbf{u} = (u^i)_{i=1}^n$, and we have $\nabla^{\mathbf{f}} \omega^i$, $1 \le i \le n$.
5. Let $V = \mathbb{R}^n$, let $t \in T_p^q(V) = (\otimes^p V^*) \otimes (\otimes^q V)$ be a given tensor and let $G \subseteq GL(n, \mathbb{R})$ be the isotropy subgroup of t; namely, $G = \{A \in GL(n, \mathbb{R}) : A \cdot t = A\}$, where the dot denotes the natural action of $GL(n, \mathbb{R})$ on $T_p^q(V)$. If M is an n-dimensional C^∞ manifold and $\pi\colon P \to M$ is a G-structure,

then a tensor field τ of type (p, q) on M can be defined as follows: A linear frame $u \in \pi^{-1}(x)$ determines an isomorphism $u\colon V \to T_x M$, which induces a $GL(n, \mathbb{R})$-equivariant isomorphism $u_q^p : T_p^q V \to T_p^q(T_x M)$, and we set $\tau(x) = u_q^p(x)$. This formula does not depend on u because of the definition of G. Then P is integrable if and only if each point of M has a coordinate neigbourhood $(U; x^1, \dots, x^n)$ with respect to which the components of τ are constant functions on U; *e.g.*, see [4].

6. There is a close relationship between the group of symmetries of a p-form and the set of torsion-free flat linear connections that parallelize it. For every $\omega_p \in \Omega^p(M)$, we set $\mathrm{Sym}(\omega_p) = \{\phi \in \mathrm{Diff}\, M : \phi^*\omega_p = \omega_p\}$ and let $P(\omega_p)$ be the set of coordinate systems whose associated linear connection parallelizes ω_p; namely $P(\omega_p) = \{\mathbf{u} = (u^1, \dots, u^n) : \nabla^{\mathbf{u}}\omega_p = 0\}$.

   To say that ω_p has constant coefficients is equivalent to saying that $P(\omega_p)$ is non-empty. When making a linear change of coordinates $u^i = \alpha^i_j x^j$, the matrix $A = (\alpha^i_j)^n_{i,j=1}$ in $GL(n, \mathbb{R})$, a form with constant coefficients ω_p. Consequently, $GL(n, \mathbb{R})$ acts on the left on $P(\omega_p)$ by composition, namely $(A, \mathbf{x}) \mapsto A \cdot \mathbf{x} = A \circ (x^1, \dots, x^n), \forall A \in GL(n, \mathbb{R}), \forall \mathbf{x} \in P(\omega_p)$.

   Once a point $x_0 \in M$ and a coordinate system $\mathbf{x} = (x^1, \dots, x^n)$ have been fixed, every coordinate system $\mathbf{u} = (u^1, \dots, u^n)$ defined on an open neighbourhood U of x_0 can be written as follows: $\mathbf{u} = A \circ \mathbf{u}'$, where the matrix A is as above, and $\alpha^i_j = \frac{\partial u^i}{\partial x^j}(x_0)$, $\frac{\partial u'^i}{\partial x^j}(x_0) = \delta^i_j, i, j = 1, \dots, n$. Therefore, the quotient set $P(\omega_p)/GL(n, \mathbb{R})$ can be identified to the subset $P^0(\omega_p) \subset P(\omega_p)$ of the coordinate systems whose Jacobian matrix at x_0 with respect to $\mathbf{x}$ is the identity map.

   In summary: Once a point $x_0 \in M$ and a coordinate system $\mathbf{x}$ have been fixed, there exists a one-to-one correspondence between the group $\mathrm{Sym}(\omega_p)$ and the set $P^0(\omega_p)$.

   The treatment for a q-vector field is entirely analogous.

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