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Riesz transform on manifolds and Poincaré inequalities

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Abstract. We study the validity of the L^p inequality for the Riesz transform when $p > 2$ and of its reverse inequality when $1 < p < 2$ on complete Riemannian manifolds under the doubling property and some Poincaré inequalities.

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Introduction

Let M be a non-compact complete Riemannian manifold. Denote by μ the Riemannian measure, and by ∇ the Riemannian gradient. Denote by $|.|$ the length in the tangent space, and by $\|.\|_p$ the norm in $L^p(M, \mu)$, $1 \leq p \leq \infty$. One defines Δ, the Laplace-Beltrami operator, as a self-adjoint positive operator on $L^2(M, \mu)$ by the formal integration by parts

$$(\Delta f, f) = \||\nabla f|\|_2^2$$

for all $f \in \mathcal{C}_0^\infty(M)$, and its positive self-adjoint square root $\Delta^{1/2}$ by

$$(\Delta f, f) = \|\Delta^{1/2} f\|_2^2.$$

As a consequence,

$$\| |\nabla f| \|_2^2 = \|\Delta^{1/2} f\|_2^2. \qquad (E_2)$$

To identify the spaces defined by (completion with respect to) the seminorms $\| |\nabla f| \|_p$ and $\left\| \Delta^{1/2} f \right\|_p$ on $\mathcal{C}_0^\infty(M)$ for some $p \in (1, \infty)$, it is enough to prove that there exist $0 < c_p \leq C_p < \infty$ such that for all $f \in \mathcal{C}_0^\infty(M)$

$$c_p \left\| \Delta^{1/2} f \right\|_p \leq \| |\nabla f| \|_p \leq C_p \left\| \Delta^{1/2} f \right\|_p . \qquad (E_p)$$

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This equivalence splits into two inequalities of different nature. The right-hand inequality may be reformulated by saying that the Riesz transform $\nabla\Delta^{-1/2}$ is bounded from $L^p(M,\mu)$ to the space of L^p vector fields,[1] in other words

$$\left\| \, |\nabla\Delta^{-1/2} f| \, \right\|_p \leq C_p \|f\|_p \,. \tag{R_p}$$

The left-hand inequality is what we call the reverse inequality

$$\left\| \, \Delta^{1/2} f \, \right\|_p \leq C_p \, \| \, |\nabla f| \, \|_p \,. \tag{RR_p}$$

It is well-known (see [5], Section 4, or [10], Section 2.1) that (R_p) implies $(RR_{p'})$ where p' is the conjugate exponent of p but the converse is not clear (in fact, it is false, see below). We mention a partial converse which we shall use and prove in the sequel.

Lemma 0.1. *The conjunction of* $(RR_{p'})$ *and* (Π_p) *implies* (R_p).

Here, (Π_p) is the inequality describing the boundedness on L^pT^*M of the orthogonal projector $d\Delta^{-1}\delta$ of 1-forms onto exact forms. Namely, for all $\omega \in \mathcal{C}_0^\infty(T^*M)$,

$$\left\| \, |d\Delta^{-1}\delta\omega| \, \right\|_p \leq C_p \, \| \, \omega \, \|_p \,, \tag{Π_p}$$

where d is the exterior derivative and δ is its formal adjoint.

The question is to find which geometrical properties on M insure each of these inequalities, and in the end (E_p) for a range of p's.

We first recall the result of [9] which deals with (R_p) for $1 < p < 2$. Denote by $B(x,r)$ the open ball of radius $r > 0$ and center $x \in M$, and by $V(x,r)$ its measure $\mu(B(x,r))$. One says that M satisfies the doubling property if there exists $C > 0$ such that, for all $x \in M$ and $r > 0$,

$$V(x,2r) \leq C\,V(x,r). \tag{D}$$

By an observation in [23], the non-compactness of M together with (D) implies that $\mu(M) = \infty$. We were not aware of this remark in [2]. Let $p_t(x,y)$, $t > 0$, $x, y \in M$ be the heat kernel of M, that is the kernel of the heat semigroup $e^{-t\Delta}$.

Theorem 0.2 ([9]). *Let M be a complete non-compact Riemannian manifold satisfying* (D). *Assume that for all $x \in M$, $t > 0$ and some constant $C > 0$,*

$$p_t(x,x) \leq \frac{C}{V(x,\sqrt{t})}. \tag{DUE}$$

Then (R_p) *holds for* $1 < p < 2$, *hence* (RR_p) *for* $2 < p < \infty$.

[1] In the case where M has finite measure, one should replace $L^p(M)$ by the subspace $L^p_0(M)$ of functions with zero mean. However, we shall work in a situation where M has infinite measure. See below.

It is also shown in [9] that the Riesz transform is unbounded on L^p for every $p > 2$ on the manifold consisting of two copies of the Euclidean plane glued smoothly along their unit circles, although this manifold satisfies (D) and (DUE).

A stronger assumption is therefore required to obtain (R_p) when $p > 2$.

It is natural to assume in addition the Poincaré inequalities, although it is known that they are not sufficient for (R_p) to hold for all $p > 2$ ([22, 11]), nor necessary for (R_p) to hold for some $p > 2$ ([7]). One says that M satisfies the (scaled) Poincaré inequalities (P_2) if there exists $C > 0$ such that, for every ball $B = B(x, r)$, $x \in M$, $r > 0$, and every f with $f, \nabla f$ locally in L^2,

$$\int_B |f - f_B|^2 \, d\mu \leq Cr^2 \int_B |\nabla f|^2 \, d\mu, \tag{P_2}$$

where f_E denotes the mean of f on the set E.

Even under (D) and (P_2) alone, it is not clear that (R_p) holds for some $p > 2$ because of the following result proved in [2] which tells us that the gradient of the semigroup should have some boundedness properties (it is also shown there that these properties are equivalent to some L^p estimates of the gradient of the heat kernel).

Theorem 0.3. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_2). Let $p_0 \in (2, \infty]$. The following assertions are equivalent:*

1. *For all $p \in (2, p_0)$, there exists C_p such that for all $t > 0$*

$$\| |\nabla e^{-t\Delta}| \|_{p \to p} \leq \frac{C_p}{\sqrt{t}}.$$

2. *(R_p) holds for $p \in (2, p_0)$.*

Our main result states that, in the situation of Theorem 0.3, there always exists a $p_0 = 2 + \varepsilon > 2$ such that condition 2 is satisfied.

Theorem 0.4. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_2). Then there exists $\varepsilon > 0$ such that (R_p) holds for $2 < p < 2 + \varepsilon$.*

Our proof does not rely on Theorem 0.3, and in fact we shall add a list of assertions equivalent to condition 2, one of them being easier to check. But in view of Theorem 0.3, this also says that there is an automatic improvement of L^p estimates for the gradient of the semigroup, which is reminiscent (and, as we shall see, equivalent) to the self-improvement "à la Meyers" of Sobolev $W^{1,p}$ estimates for weak solutions of elliptic equations (see [24]).

It is well-known (see [25, 26]) that the conjunction of (D) and (P_2) is equivalent to the full Li-Yau type estimate

$$\frac{c}{V(y, \sqrt{t})} \exp\left(-C\frac{d^2(x, y)}{t}\right) \leq p_t(x, y) \leq \frac{C}{V(y, \sqrt{t})} \exp\left(-c\frac{d^2(x, y)}{t}\right), \tag{LY}$$

for all $x, y \in M$, $t > 0$ and some constants $C, c > 0$. Hence, (D) and (P_2) imply (D) and (DUE). Therefore combining Theorems 0.2 and 0.4, we obtain

Corollary 0.5. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_2). Then there exists $p_0 \in (2, \infty)$ such that (E_p) holds when $p_0' < p < p_0$.*

A crucial step towards Theorem 0.4 consists in giving a sufficient condition for the reverse inequality (RR_p) for $1 < p < 2$ in terms of the L^p version of (P_2). Let $1 \le p < \infty$. One says that M satisfies (P_p) if there exists $C > 0$ such that, for every ball $B = B(x, r)$ and every f with $f, \nabla f$ locally p-integrable,

$$\int_B |f - f_B|^p \, d\mu \le Cr^p \int_B |\nabla f|^p \, d\mu. \tag{P_p}$$

It is known that (P_p) implies (P_q) when $p < q$ (see for instance [18]). Thus the set of p's such that (P_p) holds is, if it is not empty, an interval unbounded on the right. A recent deep result asserts in a general context of metric measured spaces that this interval is open in $[1, +\infty[$. In our case, it states as follows.

Lemma 0.6 ([21]). *Let M be a complete non-compact Riemannian manifold satisfying (D). Assume $p > 1$. Then (P_p) self-improves to $(P_{p-\varepsilon})$ for some $\varepsilon > 0$.*

We shall prove

Theorem 0.7. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_q) for some $q \in [1, 2]$. Then (RR_p) holds for $q < p < 2$. If $q = 1$, there is a weak-type $(1, 1)$ estimate.*

Define $q_0 = \inf\{p \in [1, 2]; (P_p) \text{ holds}\}$. Note that if (P_p) holds for some $p \in (1, 2]$, then $q_0 < p$ according to Lemma 0.6. As a consequence of Theorem 0.7 and Lemma 0.6, if $q_0 < 2$, that is to say if (P_2) holds, (RR_p) holds for $p \in (q_0, 2]$.

As a corollary of Theorems 0.2, 0.4 and 0.7 we obtain for instance

Corollary 0.8. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_1). Then (E_p) holds when $1 < p < 2 + \varepsilon$ for some $\varepsilon > 0$.*

One may observe that our proofs do not use completeness in itself, but rather stochastic completeness, that is the property

$$\int_M p_t(x, y) \, d\mu(y) = 1, \tag{0.1}$$

for all $x \in M$ and $t > 0$, which does hold for complete manifolds satisfying (D) (see [15]), but also for instance for conical manifolds with closed basis (see [22]).

Note that the class of manifolds satisfying (D) and (P_1) (therefore also (P_2)) contains all complete manifolds that are quasi-isometric to a manifold with non-negative Ricci curvature (see [26]).

It is proved in [10] that for any $q \in (1, 2)$, there exists a complete Riemannian manifold with (D) such that (RR_p) fails for all $1 < p < q$.[2] The point is that there are manifolds satisfying an L^2 Sobolev inequality at infinity associated with a certain dimension, but, for p close to 1, only a L^p Sobolev inequality associated with a much lower dimension, and, for $p = 1$, a trivial isoperimetric inequality, whereas (RR_p) would impose a tighter connection between L^2 and L^p Sobolev inequalities. In other words, (RR_p) imposes that the heat kernel dimension and the isoperimetric dimension cannot differ too much.

It has been proved by Li Hong-Quan in [22] that, on conical manifolds with closed basis, (R_p) holds if and only if $1 < p < p_0$, where the threshold $p_0 > 2$ depends on the λ_1 of the basis. Now, all these manifolds satisfy (P_2) (see [11]) and one can see that they even satisfy (P_1) by using the methods in [17]. In particular, there is no hope that the assumptions of Corollary 0.8 suffice for (R_p) to hold for all $p > 2$.

In view of Corollary 0.8, this also shows that, as we mentioned above, (RR_p) does not imply $(R_{p'})$, even in the class of manifolds with doubling, in the range $1 < p < 2$.

Let us summarize the situation for (stochastically) complete Riemannian manifolds, satisfying (D), going from weakest to strongest hypotheses.

1. It is known that (R_p) may be false for $2 < p$ and that (RR_p) may be false for $1 < p < 2$. What can be said about the other cases, that is (R_p) for $1 < p < 2$ and (RR_p) for $p > 2$?
2. Assume (DUE). Then (R_p) holds for $1 < p \le 2$, (RR_p) for $p \ge 2$ and (R_p) may be false for all $p > 2$. What can be said about (RR_p) for $1 < p < 2$?
3. Assume (P_2). Then (R_p) holds for $1 < p < p_0$ with some $p_0 > 2$, (RR_p) for $q_0 < p < \infty$ with some $1 \le q_0 < 2$. Can one give estimates on p_0 and q_0?
4. Assume (P_1). Then (R_p) holds for $1 < p < p_0$ with some $p_0 > 2$, (RR_p) for $1 < p < \infty$. Can one give estimates on p_0?

The proof of Theorem 0.7 in Section 1 uses methods of the first author in [1] adapted to the present situation and in particular a Calderón-Zygmund lemma for Sobolev functions, which allows us to do a Marcinkiewicz type interpolation.

As said before, we do not rely on Theorem 0.3 to prove Theorem 0.4. Instead, we use ideas of Shen in [27] developed for elliptic operators on Euclidean space and extend them to the class of manifolds we consider. This yields a new characterization of the L^p boundedness of Riesz transforms for $p > 2$ (with a restriction

[2] We remark that the positive result in [10] concerning (RR_p), namely Theorem 6.1, has a gap, since it depends on another result in the same paper, Proposition 5.4, which has a mistake in the argument. The mistake is located in the last line of p. 1744 where it is said that the (usual) Calderón-Zygmund decomposition preserves exact forms. This is exactly the obstacle that we get around in Section 1 with a modified Calderón-Zygmund decomposition and it is not clear that the same ideas can be employed under the assumption taken in [10].

that p should be close to 2) in terms of local, scale invariant estimates on harmonic functions (Theorem 2.1) which are more tractable in practice.

In passing, we show that this is also equivalent to the L^p boundedness of $d\Delta^{-1}\delta$. Actually the main tool in [27] is a theorem (Theorem 3.1) for boundedness of operators with no kernels which is essentially similar to Theorem 2.1 in [2].

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1. Reverse inequalities (RR_p) for $1 < p < 2$

In this section, we prove Theorem 0.7. Let $1 \le q < 2$. We assume that (D) and (P_q) hold and prove (RR_p) for $q < p < 2$.

We first establish a Calderón-Zygmund lemma for Sobolev functions. Next, we apply this lemma to establish the preliminary weak-type estimate

$$\left\| \Delta^{1/2} f \right\|_{q,\infty} \le C_q \, \| \, |\nabla f| \, \|_q \, , \ \forall f \in \mathcal{C}_0^\infty(M). \tag{1.1}$$

Finally, we proceed via an interpolation argument.

1.1. A Calderón-Zygmund lemma for Sobolev functions

We present here in the Riemannian context a result first proved by one of us [1] in the Euclidean setting with Lebesgue measure (see also the extension to weighted Lebesgue measure in [3]).

Proposition 1.1. *Let M be a complete non-compact Riemannian manifold satisfying (D).*[3] *Let $1 \le q < \infty$ and assume that (P_q) holds. Let $f \in \mathcal{C}_0^\infty(M)$*[4] *be such that $\| \, |\nabla f| \, \|_q < \infty$. Let $\alpha > 0$. Then one can find a collection of balls B_i, C^1 functions b_i and a (almost everywhere) Lipschitz function g such that the following properties hold:*

$$f = g + \sum b_i, \tag{1.2}$$
$$|\nabla g(x)| \le C\alpha, \quad \text{for } \mu - a.e. \ x \in M, \tag{1.3}$$
$$\operatorname{supp} b_i \subset B_i \text{ and } \int_{B_i} |\nabla b_i|^q \, d\mu \le C\alpha^q \mu(B_i), \tag{1.4}$$
$$\sum_i \mu(B_i) \le C\alpha^{-q} \int |\nabla f|^q \, d\mu, \tag{1.5}$$
$$\sum_i \mathbf{1}_{B_i} \le N, \tag{1.6}$$

where C and N only depend on q and on the constant in (D).

[3] Recall that this implies $\mu(M) = \infty$.

[4] Of course, f can be taken more general than this.

Proof. Let $f \in C_0^\infty(M)$ and $\alpha > 0$. Consider $\Omega = \{x \in M; \mathcal{M}(|\nabla f|^q)(x) > \alpha^q\}$, where $\mathcal{M}$ is the uncentered maximal operator over balls of M. If Ω is empty, then set $g = f$, $b_i = 0$; (1.3) is satisfied thanks to Lebesgue differentiation theorem. Otherwise, the maximal theorem gives us

$$\mu(\Omega) \le C\alpha^{-q} \int |\nabla f|^q \, d\mu. \tag{1.7}$$

Let F be the complement of Ω. Again by the Lebesgue differentiation theorem, $|\nabla f| \le \alpha$ μ-almost everywhere on F. Since Ω is open, let $(\underline{B}_i)$ be a Whitney decomposition of Ω. That is, Ω is the union of the $\underline{B}_i$'s, and there are constants $C_2 > C_1 > 1$ depending only on the metric such that the balls $B_i = C_1\underline{B}_i$ are contained in Ω and have the bounded overlap property, but each ball $\overline{B}_i = C_2\underline{B}_i$ intersects F (see [8]). As usual, CB is the ball co-centered with B with radius $Cr(B)$. Condition (1.6) is nothing but the bounded overlap property and (1.5) follows from (1.6) and (1.7). Furthermore, $\overline{B_i} \cap F \neq \emptyset$ and the doubling property imply

$$\int_{B_i} |\nabla f|^q \, d\mu \le \int_{\overline{B_i}} |\nabla f|^q \, d\mu \le \alpha^q \mu(\overline{B_i}) \le C\alpha^q \mu(B_i).$$

Let us now define the functions b_i. Let $(\mathcal{X}_i)$ be a partition of unity of Ω subordinated to the covering $(\underline{B}_i)$ so that for each i, $\mathcal{X}_i$ is a C^1 function supported in B_i with $\|\nabla \mathcal{X}_i\|_\infty \le \frac{C}{r_i}$, $r_i = r(B_i)$. Set

$$b_i = (f - f_{B_i})\mathcal{X}_i.$$

It is clear that b_i is supported in B_i. Let us estimate $\int_{B_i} |\nabla b_i|^q \, d\mu$. Since

$$\nabla \big((f - f_{B_i})\mathcal{X}_i \big) = \mathcal{X}_i \nabla f + (f - f_{B_i})\nabla \mathcal{X}_i,$$

we have by (P_q) and the above estimate on ∇f that

$$\int_{B_i} |\nabla \big((f - f_{B_i})\mathcal{X}_i \big)|^q \, d\mu \le C\alpha^q \mu(B_i).$$

Thus (1.4) is proved.

Set $g = f - \sum_i b_i$. Then g is defined μ-almost everywhere since the sum is locally finite on Ω and vanishes on F, and g is also defined in the sense of distributions on M (not just on Ω which is trivial: in fact the argument shows that g is a locally integrable function on M). For the latter claim, if $\varphi \in C_0^\infty(M)$, we observe that for x in the support of b_i, we have $d(x, F) \ge r_i$, so that

$$\int \sum_i |b_i||\varphi| \, d\mu \le \left(\int \sum_i \frac{|b_i|}{r_i} \, d\mu \right) \sup_{x \in M} (d(x, F)|\varphi(x)|).$$

By the Hölder inequality and (P_q),

$$\int \frac{|b_i|}{r_i}\,d\mu \le (\mu(B_i))^{1/q'} \left(\int_{B_i} |\nabla f|^q\,d\mu\right)^{1/q} \le C\alpha\mu(B_i).$$

Hence

$$\int \sum_i |b_i||\varphi|\,d\mu \le C\alpha\,\mu(\Omega)\ \sup_{x\in M}(d(x,F)|\varphi(x)|),$$

which proves the claim.

It remains to prove (1.3). Note that $\sum_i \mathcal{X}_i(x) = 1$ and $\sum_i \nabla\mathcal{X}_i(x) = 0$ for $x \in \Omega$. It follows that

$$\begin{aligned} \nabla g &= \nabla f - \sum_i \nabla b_i \\ &= \nabla f - (\sum_i \mathcal{X}_i)\nabla f - \sum_i (f - f_{B_i})\nabla\mathcal{X}_i \\ &= \mathbf{1}_F(\nabla f) + \sum_i f_{B_i}\ \nabla\mathcal{X}_i. \end{aligned}$$

Note that by the definition of F, $|\mathbf{1}_F(\nabla f)| \le \alpha$. We claim that a similar estimate holds for $h = \sum_i f_{B_i}\ \nabla\mathcal{X}_i$, that is $|h(x)| \le C\alpha$ for all $x \in M$ for some constant C independent of x. Note that this sum vanishes on F and is locally finite on Ω. Fix now $x \in \Omega$. Let B_j be some Whitney ball containing x and let I_x be the set of indices i such that $x \in B_i$. We know that $\sharp I_x \le N$. Also for $i \in I_x$ we have that $C^{-1}r_i \le r_j \le Cr_i$ where the constant C depends only on doubling (see [28, Chapter I, 3] for the Euclidean case). We also have $|f_{B_i} - f_{B_j}| \le Cr_j\alpha$. Indeed, one has $B_i \subset AB_j$ with $A = 2C + 1$, so that by (P_q) one obtains

$$\begin{aligned} |f_{B_i} - f_{AB_j}| &\le \frac{1}{\mu(B_i)}\int_{B_i} |f - f_{AB_j}| \\ &\le \frac{C}{\mu(B_j)}\int_{AB_j} |f - f_{AB_j}| \\ &\le CAr_j((|\nabla f|^q)_{AB_j})^{1/q} \\ &\le CAr_j\alpha \end{aligned}$$

and similarly for $|f_{AB_j} - f_{B_j}|$. Hence,

$$|h(x)| = \left|\sum_{i\in I_x}(f_{B_i} - f_{B_j})\nabla\mathcal{X}_i(x)\right| \le C\sum_{i\in I_x}|f_{B_i} - f_{B_j}|r_i^{-1} \le CN\alpha.$$

This proves (1.3), and finishes the proof of Proposition 1.1. □

Remarks. 1) It follows from the construction that $\sum \nabla b_i \in L^q$ with norm bounded by $C \| |\nabla f| \|_q$, hence $\| |\nabla g| \|_q \le (C+1) \| |\nabla f| \|_q$.

2) g is equal almost everywhere to a Lipschitz function on M and $|g(x) - g(y)| \le C\alpha d(x, y)$ almost everywhere. The point is that the Lipschitz constant is controlled by α. This can be shown by similar arguments as for obtaining (1.2). Alternatively, once (1.2) is proved, one can show that g satisfies (P_q) on arbitrary balls by using the definition of g as $f - \sum b_i$ since f and each b_i do. At this point, we invoke Theorem 3.2 in [18] and the L^∞ bound on $|\nabla g|$ to conclude.

3) Observe that $g = \mathbf{1}_F f + \sum f_{B_i} \mathcal{X}_i$ so that, in particular, f is equal almost everywhere to a Lipschitz function on F. Hence, g is some sort of Whitney extension of the restriction of f to F where averages of f on B_i (since f was already defined on the complement of F) replace evaluation at some point inside F at distance Cr_i to B_i.

1.2. A weak-type estimate

Assume (P_q) for some $q \in [1, 2)$. Let $f \in C_0^\infty(M)$. We wish to establish the estimate

$$\mu\left(\left\{x \in M; |\Delta^{1/2} f(x)| > \alpha\right\}\right) \le \frac{C}{\alpha^q} \int_M |\nabla f|^q \, d\mu, \tag{1.8}$$

for all $\alpha > 0$. We use the following resolution of $\Delta^{1/2}$:

$$\Delta^{1/2} f = c \int_0^\infty \Delta e^{-t\Delta} f \, \frac{dt}{\sqrt{t}}$$

where $c = \pi^{-1/2}$ is forgotten from now on. It suffices to obtain the result for the truncated integrals $\int_\varepsilon^R \ldots$ with bounds independent of ε, R, and then to let $\varepsilon \downarrow 0$ and $R \uparrow \infty$. For the truncated integrals, all the calculations are justified. We henceforth assume that $\Delta^{1/2}$ is replaced by one of the truncations above but we keep writing $\Delta^{1/2}$ and the limits of the integral as $0, \infty$ to keep the notation simple.

Apply the Calderón-Zygmund decomposition of Proposition 1.1 to f at height α with exponent q and write $f = g + \sum_i b_i$.

Since g and b_i are no longer $C_0^\infty(M)$, we have to give a meaning to $\Delta^{1/2} g$ and $\Delta^{1/2} b_i$. As $\Delta^{1/2}$ is replaced by approximations, it suffices to define $\Delta e^{-t\Delta} g$ and $\Delta e^{-t\Delta} b_i$ for $t > 0$. Since (D) and (P_q) imply (D) and (P_2), we have the Gaussian upper bounds for the kernel of $e^{-t\Delta}$ and by analyticity for the kernel of $t\Delta e^{-t\Delta}$. As b_i has support in a ball and is integrable (see the proof of Proposition 1.1), $\Delta e^{-t\Delta} b_i(x)$ is defined by the convergent integral $\int_M \partial_t p_t(x, y) b_i(y) \, d\mu(y)$.

As for g, we know it equals almost everywhere a Lipschitz function with Lipschitz constant bounded by $C\alpha$ (see Remarks 1 and 2 at the end of Section 1.1). We fix any point z where $g(z)$ exists and we have that $\int_M \partial_t p_t(x, y) g(y) \, d\mu(y)$ is a smooth function bounded by $C\alpha t^{-1}(d(x, z) + t^{1/2})$ (we use the fact that $\int_M \partial_t p_t(x, y) \, d\mu(y) = 0$). We take this as our definition of $\Delta e^{-t\Delta} g(x)$.

Next, we prove

$$\mu\left\{x \in M; |\Delta^{1/2}g(x)| > \frac{\alpha}{3}\right\} \leq \frac{C}{\alpha^q}\int_M |\nabla f|^q \, d\mu.$$

Since

$$\mu\left\{x \in M; |\Delta^{1/2}g(x)| > \frac{\alpha}{3}\right\} \leq \frac{9}{\alpha^2}\int_M |\Delta^{1/2}g|^2 \, d\mu,$$

it remains to justify

$$\int_M |\Delta^{1/2}g|^2 \, d\mu \leq \int_M |\nabla g|^2 \, d\mu. \tag{1.9}$$

Indeed, once this is done, we conclude by using $\int_M |\nabla g|^2 d\mu \leq C\alpha^{2-q} \int_M |\nabla f|^q \, d\mu$ which follows from $\| \, |\nabla g| \, \|_q \leq C \, \| \, |\nabla f| \, \|_q$ and (1.3) since $q < 2$.

Note that (1.9) (since we have replaced $\Delta^{1/2}$ by truncations) would be valid if g were in $C_0^\infty(M)$. For $\varphi \in C_0^\infty(M)$, we have by the Fubini's theorem

$$\begin{aligned}\int_M \Delta e^{-t\Delta}g(x)\varphi(x)\, d\mu(x) &= \int_M g(y)\Delta e^{-t\Delta}\varphi(y)\, d\mu(y) \\ &= \lim_{r\to+\infty}\int_M \eta_r(y)g(y)\Delta e^{-t\Delta}\varphi(y)\, d\mu(y).\end{aligned}$$

Here η_r is a smooth function which is bounded by 1 on M, equal to 1 on a ball B_r of radius r, 0 outside the ball $2B_r$, and with $\| \, |\nabla \eta_r| \, \|_\infty \leq C/r$. By the Stokes theorem, the last integral is equal to

$$\int_M \eta_r \nabla g \cdot \nabla e^{-t\Delta}\varphi \, d\mu + \int_M g\nabla\eta_r \cdot \nabla e^{-t\Delta}\varphi \, d\mu.$$

Under our assumptions, we have the weighted L^2 estimate from [16] (see also [9]): for some $\gamma > 0$ and all $y \in M$, $t > 0$,

$$\int_M |\nabla_x \, p_t(x,y)|^2 e^{\gamma \frac{d^2(x,y)}{t}} \, d\mu(x) \leq \frac{C}{t\, V(y,\sqrt{t})} \tag{1.10}$$

where ∇_x means that the gradient is taken with respect to the x variable. Given the fact that ∇g is square integrable and g is Lipschitz, it is not difficult to pass to the limit as $r \to \infty$ and to conclude that

$$\int_M \Delta e^{-t\Delta} g \, \varphi \, d\mu = \int_M \nabla g \cdot \nabla e^{-t\Delta}\varphi \, d\mu.$$

Thus, we obtain (again, $\Delta^{1/2}$ is replaced by truncated integrals)

$$\langle \Delta^{1/2}g, \varphi\rangle = \langle \nabla g, \nabla\Delta^{-1/2}\varphi\rangle,$$

so that a duality argument from the equality (E_2) (or rather its approximation) yields (1.9).

To compute $\Delta^{1/2}b_i$, let $r_i = 2^k$ if $2^k \le r(B_i) < 2^{k+1}$ and set $T_i = \int_0^{r_i^2} \Delta e^{-t\Delta} \frac{dt}{\sqrt{t}}$ and $U_i = \int_{r_i^2}^{\infty} \Delta e^{-t\Delta} \frac{dt}{\sqrt{t}}$. It is enough to estimate $A = \mu\{x \in M; |\sum_i T_i b_i(x)| > \alpha/3\}$ and $B = \mu\{x \in M; |\sum_i U_i b_i(x)| > \alpha/3\}$.

First

$$A \le \mu(\cup_i 4B_i) + \mu\left(\left\{x \in M \setminus \cup_i 4B_i; \left|\sum_i T_i b_i(x)\right| > \frac{\alpha}{3}\right\}\right),$$

and by (1.5) and (D), $\mu(\cup_i 4B_i) \le \frac{C}{\alpha^q} \int_M |\nabla f|^q \, d\mu$.

For the other term, we have

$$\mu\left(\left\{x \in M \setminus \cup_i 4B_i; \left|\sum_i T_i b_i(x)\right| > \frac{\alpha}{3}\right\}\right) \le \frac{C}{\alpha^2} \int_M \left|\sum_i h_i\right|^2 d\mu$$

with $h_i = \mathbf{1}_{(4B_i)^c} |T_i b_i|$. To estimate the L^2 norm, we follow ideas in [6, 19] and dualize against $u \in L^2(M, \mu)$ with $\|u\|_2 = 1$ and write

$$\int_M |u| \sum_i h_i \, d\mu = \sum_i \sum_{j=2}^{\infty} A_{ij}$$

where

$$A_{ij} = \int_{C_j(B_i)} |T_i b_i||u| \, d\mu$$

with $C_j(B_i) = 2^{j+1}B_i \setminus 2^j B_i$. By the Minkowski integral inequality

$$\|T_i b_i\|_{L^2(C_j(B_i))} \le \int_0^{r_i^2} \|\Delta e^{-t\Delta} b_i\|_{L^2(C_j(B_i))} \frac{dt}{\sqrt{t}}$$

and by the Gaussian upper bounds for the kernel of $\Delta e^{-t\Delta}$ (see above),

$$|\Delta e^{-t\Delta} b_i(x)| \le \int_M \frac{C}{t V(y, \sqrt{t})} e^{-\frac{cd^2(x,y)}{t}} |b_i(y)| \, d\mu(y).$$

Now, y is in the support of b_i, that is B_i, and $x \in C_j(B_i)$, hence one may replace $d(x, y)$ by $2^j r_i$ in the Gaussian term since $r_i \sim r(B_i)$. Also, if y_i denotes the center of B_i, write

$$\frac{V(y_i, \sqrt{t})}{V(y, \sqrt{t})} = \frac{V(y_i, \sqrt{t})}{V(y_i, r_i)} \frac{V(y_i, r_i)}{V(y, r_i)} \frac{V(y, r_i)}{V(y, \sqrt{t})}.$$

By (D) and $\frac{V(z,r)}{V(z,s)} \le c(\frac{r}{s})^\beta$ for $r > s$, as $t \le r_i^2$, we have

$$\frac{V(y_i, \sqrt{t})}{V(y, \sqrt{t})} \le c\left(\frac{r_i}{\sqrt{t}}\right)^\beta .$$

Using this estimate, $\int_{B_i} |b_i|\, d\mu \le C\mu(B_i) r_i \alpha$ and $\mu(B_i) \sim V(y_i, r_i)$, we obtain

$$\begin{aligned} |\Delta e^{-t\Delta} b_i(x)| &\le \frac{C}{tV(y_i, \sqrt{t})}\left(\frac{r_i}{\sqrt{t}}\right)^\beta e^{-\frac{c4^j r_i^2}{t}} \int_{B_i} |b_i|\, d\mu \\ &\le \frac{Cr_i}{t}\left(\frac{r_i}{\sqrt{t}}\right)^{2\beta} e^{-\frac{c4^j r_i^2}{t}} \alpha. \end{aligned}$$

Thus,

$$\|\Delta e^{-t\Delta} b_i\|_{L^2(C_j(B_i))} \le \frac{Cr_i}{t}\left(\frac{r_i}{\sqrt{t}}\right)^{2\beta} e^{-\frac{c4^j r_i^2}{t}} (\mu(2^{j+1}B_i))^{1/2}\alpha.$$

Plugging this estimate inside the integral, we obtain

$$\|T_i b_i\|_{L^2(C_j(B_i))} \le Ce^{-c4^j}(\mu(2^{j+1}B_i))^{1/2}\alpha$$

for some $C, c > 0$.

Now remark that for any $y \in B_i$ and any $j \ge 2$,

$$\left(\int_{C_j(B_i)} |u|^2\, d\mu\right)^{1/2} \le \left(\int_{2^{j+1}B_i} |u|^2\, d\mu\right)^{1/2} \le \mu(2^{j+1}B_i)^{1/2}\big(\mathcal{M}(|u|^2)(y)\big)^{1/2}.$$

Applying Hölder inequality and doubling, one obtains

$$A_{ij} \le C\alpha 2^{j\beta} e^{-c4^j} \mu(B_i)\big(\mathcal{M}(|u|^2)(y)\big)^{1/2}.$$

Averaging over $y \in B_i$ yields

$$A_{ij} \le C\alpha 2^{j\beta} e^{-c4^j} \int_{B_i} \big(\mathcal{M}(|u|^2)\big)^{1/2}\, d\mu.$$

Summing over $j \ge 2$ and i, we have

$$\int_M |u| \sum_i h_i\, d\mu \le C\alpha \int_M \sum_i \mathbf{1}_{B_i} \big(\mathcal{M}(|u|^2)\big)^{1/2}\, d\mu.$$

Using finite overlap (1.6) of the balls B_i and Kolmogorov's inequality, one obtains

$$\int_M |u| \sum_i h_i\, d\mu \le C' N\alpha\mu\big(\cup_i B_i\big)^{1/2} \||u|^2\|_1^{1/2}.$$

Hence, by (1.6) and (1.5),

$$\mu\left\{x \in M \setminus \cup_i 4B_i ;\left|\sum_i T_i b_i(x)\right| > \frac{\alpha}{3}\right\} \leq C\mu\big(\cup_i\, B_i\big) \leq \frac{C}{\alpha^q}\int_M |\nabla f|^q\, d\mu.$$

It remains to handle the term B. Define

$$\beta_k = \sum_{i, r_i = 2^k} \frac{b_i}{r_i}$$

for $k \in \mathbb{Z}$. With this definition, it is easy to see that

$$\sum_i U_i b_i = \sum_{k\in\mathbb{Z}} \int_{4^k}^{\infty} \left(\frac{2^k}{\sqrt{t}}\right) t\Delta e^{-t\Delta} \beta_k \frac{dt}{t} = \int_0^{\infty} t\Delta e^{-t\Delta} f_t \frac{dt}{t}$$

where

$$f_t = \sum_{k\,;4^k\leq t} \left(\frac{2^k}{\sqrt{t}}\right)\beta_k.$$

By using duality from the well-known Littlewood-Paley estimate

$$\left\|\left(\int_0^{\infty} |t\Delta e^{-t\Delta} f|^2 \frac{dt}{t}\right)^{1/2}\right\|_{q'} \leq C\|f\|_{q'}$$

(see [29]), we find that

$$\left\|\sum_i U_i b_i\right\|_q \leq C\left\|\left(\int_0^{\infty} |f_t|^2 \frac{dt}{t}\right)^{1/2}\right\|_q.$$

Now, by the Cauchy-Schwarz inequality,

$$|f_t|^2 \leq 2\sum_{k\,;4^k\leq t}\left(\frac{2^k}{\sqrt{t}}\right)|\beta_k|^2$$

and it is easy to obtain

$$\left\|\left(\int_0^{\infty} |f_t|^2 \frac{dt}{t}\right)^{1/2}\right\|_q \leq C\left\|\left(\sum_{k\in\mathbb{Z}} |\beta_k|^2\right)^{1/2}\right\|_q.$$

Using the bounded overlap property (1.6), one has that

$$\left\|\left(\sum_{k\in\mathbb{Z}} |\beta_k|^2\right)^{1/2}\right\|_q^q \leq C\int_M \sum_i \frac{|b_i|^q}{r_i^q}\, d\mu$$

and by a similar argument to the one in the proof of Proposition 1.1,

$$\int_M \sum_i \frac{|b_i|^q}{r_i^q} \, d\mu \le C\alpha^q \sum_i \mu(B_i).$$

Hence, by (1.5)

$$\mu\left\{x \in M; \left|\sum_i U_i b_i(x)\right| > \frac{\alpha}{3}\right\} \le C \sum_i \mu(B_i) \le \frac{C}{\alpha^q} \int_M |\nabla f|^q \, d\mu.$$

This concludes the proof of (1.8).

1.3. An interpolation argument

It is not known whether the spaces defined by the seminorms $\| \, |\nabla f| \, \|_q$ interpolate by the real method. So it is not immediate to obtain (RR_p) for $q < p < 2$ directly from (E_2) and (1.1). We next prove this fact by adapting the Marcinkiewicz theorem argument which bears again on our Calderón-Zygmund decomposition.

Fix $q < p < 2$ and $f \in \mathcal{C}_0^\infty(M)$. We want to show that

$$\left\| \Delta^{1/2} f \right\|_p \le C_p \, \| \, |\nabla f| \, \|_p \, .$$

Choose $0 < \delta < 1$ so that $q < p\delta$. For $\alpha > 0$, we can apply the Calderón-Zygmund decomposition of Proposition 1.1 with exponent $p\delta$ and threshold α. We may do this since $\| \, |\nabla f| \, \|_{p\delta} < \infty$ and $(P_{p\delta})$ holds. Of course we do not want to use $\| \, |\nabla f| \, \|_{p\delta}$ in a quantitative way. We obtain that $f = g_\alpha + b_\alpha$ with $b_\alpha = \sum_i b_i$.

Write

$$\begin{aligned}
\left\| \Delta^{1/2} f \right\|_p^p &= p2^p \int_0^\infty \alpha^{p-1} \mu\{x \in M; |\Delta^{1/2} f(x)| > 2\alpha\} \, d\alpha \\
&\le p2^p \int_0^\infty \alpha^{p-1} \mu\{x \in M; |\Delta^{1/2} g_\alpha(x)| > \alpha\} \, d\alpha \\
&\qquad + p2^p \int_0^\infty \alpha^{p-1} \mu\{x \in M; |\Delta^{1/2} b_\alpha(x)| > \alpha\} \, d\alpha \\
&\le I + II
\end{aligned}$$

with

$$I = Cp2^p \int_0^\infty \alpha^{p-1} \frac{\| \, |\nabla g_\alpha| \, \|_2^2}{\alpha^2} \, d\alpha$$

and

$$II = Cp2^p \int_0^\infty \alpha^{p-1} \frac{\| \, |\nabla b_\alpha| \, \|_q^q}{\alpha^q} \, d\alpha,$$

where we used (E_2) and assumption (1.1). To estimate these integrals, we need to come back to the construction of ∇g_α and ∇b_α. Write F_α as the complement of $\Omega_\alpha = \{\mathcal{M}(|\nabla f|^{p\delta}) > \alpha^{p\delta}\}$. Then recall that $\nabla g_\alpha = \mathbf{1}_{F_\alpha}(\nabla f) + \mathbf{1}_{\Omega_\alpha} h$ where $|h| \le C\alpha$ and $|\nabla f| \le \alpha$ on F_α. Thus I splits into $I_1 + I_2$ according to this decomposition. The treatment of I_1 is done using the definition of F_α, Fubini's theorem and $p < 2$ as follows:

$$\begin{aligned} I_1 &= \frac{Cp2^p}{2-p} \int_M |\nabla f|^2 \left(\mathcal{M}(|\nabla f|^{p\delta})\right)^{\frac{p-2}{p\delta}} \, d\mu \\ &\le \frac{Cp2^p}{2-p} \int_M |\nabla f|^p \, d\mu, \end{aligned}$$

where we used $|\nabla f|^2 = |\nabla f|^p \, |\nabla f|^{2-p} \le |\nabla f|^p \left(\mathcal{M}(|\nabla f|^{p\delta})\right)^{\frac{2-p}{p\delta}}$ almost everywhere. For I_2, we only use the bound of h to obtain

$$\begin{aligned} I_2 &\le Cp2^p \int_0^\infty \alpha^{p-1} \mu(\Omega_\alpha) \, d\alpha \\ &= C2^p \int_M \left(\mathcal{M}(|\nabla f|^{p\delta})\right)^{\frac{1}{\delta}} \, d\mu \\ &\le C \int_M |\nabla f|^p \, d\mu \end{aligned}$$

using the strong type $(\frac{1}{\delta}, \frac{1}{\delta})$ of the maximal operator.

Next, we turn to the term II. We have $\nabla b_\alpha = \mathbf{1}_{\Omega_\alpha}(\nabla f) - \mathbf{1}_{\Omega_\alpha} h$, so that $II \le 2^q(II_1 + II_2)$. For II_1, we have by using Hölder's inequality and the strong type $(\frac{1}{\delta}, \frac{1}{\delta})$ of the maximal operator

$$\begin{aligned} II_1 &= \frac{Cp2^p}{p-q} \int_M |\nabla f|^q \left(\mathcal{M}(|\nabla f|^{p\delta})\right)^{\frac{p-q}{p\delta}} \, d\mu \\ &\le \frac{Cp2^p}{p-q} \left(\int_M |\nabla f|^p \, d\mu\right)^{q/p} \left(\int_M \left(\mathcal{M}(|\nabla f|^{p\delta})\right)^{(\frac{p-q}{p\delta})(\frac{p}{q})'} \, d\mu\right)^{1/(\frac{p}{q})'} \\ &\le C \int_M |\nabla f|^p \, d\mu. \end{aligned}$$

The treatment of the term II_2 is similar to the one of I_2.

2. (R_p) for $p > 2$

In this section, we prove Theorem 0.4 as a consequence of the next two results.

Theorem 2.1. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_2). Then there exists $p_0 \in (2, \infty]$ such that for any $q \in (2, p_0)$ the following assertions are equivalent.*

1. (R_p) *holds for* $2 < p < q$,
2. (Π_p) *holds for* $2 < p < q$,
3. *For any* $p \in (2, q)$*, there exists a constant* $C > 0$ *such that for any ball* B *and any harmonic function* u *in 3B, one has the reverse Hölder inequality*

$$\left(\frac{1}{\mu(B)}\int_B |\nabla u|^p \, d\mu\right)^{\frac{1}{p}} \leq C\left(\frac{1}{\mu(2B)}\int_{2B} |\nabla u|^2 \, d\mu\right)^{\frac{1}{2}}. \qquad (RH_p)$$

Proposition 2.2. *Let* M *be a complete non-compact Riemannian manifold satisfying* (D) *and* (P_2)*. Then there is* $p_1 \in (2, \infty]$ *such that* (RH_p) *holds for* $2 < p < p_1$.

The value of p_1 in Proposition 2.2 is not known. The same is true for p_0 in Theorem 2.1. However, if we assume (P_q) for $q \in (1, 2)$ then the argument shows that $p_0 > q'$ and for $q = 1$, $p_0 = \infty$.

We shall first prove Proposition 2.2. Of course, harmonic functions are smooth, but the point of (RH_p) is that the estimate is scale invariant. Then we shall prove Theorem 2.1, in establishing successively that 3. $\Longrightarrow$ 2. $\Longrightarrow$ 1. $\Longrightarrow$ 3. This will prove (R_p) for $2 < p < \inf(p_0, p_1)$.

2.1. Reverse Hölder inequality for the gradient of harmonic functions

Assume (D) and (P_2). First we have a Caccioppoli inequality: Let u be a harmonic function on $3B$ where B is some fixed ball. Let B' be a ball such that $3B' \subset 3B$. Then, we have

$$\left(\frac{1}{\mu(B')}\int_{B'} |\nabla u|^2 \, d\mu\right)^{\frac{1}{2}} \leq \frac{C}{r(B')}\left(\frac{1}{\mu(2B')}\int_{2B'} |u - u_{2B'}|^2 \, d\mu\right)^{\frac{1}{2}}. \qquad (2.1)$$

The proof of this fact is entirely similar to the one in the Euclidean setting under (D) and (P_2). We skip details and refer, *e.g.*, to Giaquinta's book [14].

Next, we use Lemma 0.6 which tells us that $(P_{2-\varepsilon})$ holds for some $\varepsilon > 0$. According to [12], Corollary 3.2, we have the $L^{2-\varepsilon} - L^2$ Poincaré inequality

$$\left(\frac{1}{\mu(2B')}\int_{2B'} |u - u_{2B'}|^2 \, d\mu\right)^{\frac{1}{2}} \leq Cr(B')\left(\frac{1}{\mu(2B')}\int_{2B'} |\nabla u|^{2-\varepsilon} \, d\mu\right)^{\frac{1}{2-\varepsilon}} \qquad (2.2)$$

provided for any ball B and subball B'

$$\frac{r(B')}{r(B)} \lesssim \left(\frac{\mu(B')}{\mu(B)}\right)^{\frac{1}{2-\varepsilon}-\frac{1}{2}}. \qquad (2.3)$$

Admit (2.3) and combine (2.2) with (2.1) to obtain a reverse Hölder inequality,

$$\left(\frac{1}{\mu(B')}\int_{B'} |\nabla u|^2 \, d\mu\right)^{\frac{1}{2}} \leq C\left(\frac{1}{\mu(2B')}\int_{2B'} |\nabla u|^{2-\varepsilon} \, d\mu\right)^{\frac{1}{2-\varepsilon}}.$$

Applying Gehring's self-improvement of reverse Hölder inequality [13] (see also [20], [14]), which holds since we work in a doubling space, we conclude that there is $\delta > 0$ and a constant C such that

$$\left(\frac{1}{\mu(B)}\int_B |\nabla u|^{2+\delta}\,d\mu\right)^{\frac{1}{2+\delta}} \le C\left(\frac{1}{\mu(2B)}\int_{2B} |\nabla u|^2\,d\mu\right)^{\frac{1}{2}}.$$

It remains to verify (2.3). Write $B = B(x,r)$ and $B' = B(y,s)$ with $s < r$. Then observe that (D) and $d(x,y) < r$ imply that $V(x,r) \sim V(y,r)$. Hence, we may assume that $x = y$ and (2.3) becomes

$$\frac{s}{r} \lesssim \left(\frac{V(x,s)}{V(x,r)}\right)^a$$

with $a = \frac{1}{2-\varepsilon} - \frac{1}{2} > 0$. The doubling property (D) implies that for some $\beta > 0$,

$$\frac{V(x,r)}{V(x,s)} \lesssim \left(\frac{r}{s}\right)^\beta,$$

hence it suffices to have $\beta a \le 1$. Choosing ε smaller if necessary, we obtain (2.3). Finally, (RH_p) holds for $2 < p < 2+\delta$.

2.2. From reverse Hölder to Hodge projection

The main tool is the adaptation to spaces of homogeneous type of a result by the Shen in [27] essentially similar to Theorem 2.1 in [2]. For the sake of completeness we include its proof in Section 3. Let $\mathcal{M}$ denote the Hardy-Littlewood maximal function.

Theorem 2.3. *Let (E,d,μ) be a measured metric space satisfying the doubling property (D). Let T be a bounded sublinear operator from $L^2(E,\mu)$ to $L^2(E,\mu)$. Assume that for $q \in (2,\infty]$, $1 < \alpha < \beta$ and $C > 0$, we have*

$$\left(\frac{1}{\mu(B)}\int_B |Tf|^q\,d\mu\right)^{1/q} \le C\left(\frac{1}{\mu(\alpha B)}\int_{\alpha B} |Tf|^2\,d\mu\right)^{1/2} \tag{2.4}$$

for all balls B in E and $f \in L^2(E,\mu)$ supported on $E \setminus \beta B$. Then, T is bounded from $L^p(E,\mu)$ to $L^p(E,\mu)$ for $2 < p < q$. More precisely, there exists a constant C' such that for any $f \in L^p \cap L^2(E,\mu)$, we have $Tf \in L^p(E,\mu)$ and

$$\|Tf\|_p \le C'\|f\|_p.$$

In this statement, the functions f can be vector-bundle-valued and $|f|$ is then the norm of f while Tf is real valued.

We now prove 3. $\Longrightarrow$ 2. in Theorem 2.1. We assume the reverse Hölder condition. Let T be the sublinear bounded operator from L^2T^*M into $L^2(M,\mu)$ such that $T\omega = |d\Delta^{-1}\delta\omega|$ when $\omega \in L^2T^*M$. Let $2 < p < \tilde{p} < q$ where q is the exponent in condition 3. Let B be a ball in M and $\omega \in L^2T^*M \cap L^pT^*M$ be supported on $M \setminus 4B$. Let u be a distribution defined by $\|\,|du|\,\|_2 < +\infty$ and $\Delta u = \delta\omega$, so that $|du| = T\omega$. Given the support of ω, it follows that u is harmonic in $3B$. The reverse Hölder condition yields (2.4) with q replaced by $\tilde{p}$, hence, according to Theorem 2.3,

$$\|T\omega\|_p \leq C\|\omega\|_p.$$

A density argument concludes the proof.

2.3. From Hodge projection to Riesz transform

We begin with the proof of Lemma 0.1. To do this, we look at the form version of the Riesz transform, $d\Delta^{-1/2}$, where d is the exterior derivative. We assume that, for $f \in \mathcal{C}_0^\infty(M)$,

$$\|\Delta^{1/2} f\|_{p'} \leq C_{p'} \,\|\,|df|\,\|_{p'}$$

and, for $\omega \in \mathcal{C}_0^\infty(T^*M)$,

$$\left\| \, |d\Delta^{-1}\delta\omega| \, \right\|_p \leq C_p \, \|\, \omega \,\|_p \, . \tag{Π_p}$$

Since $d\Delta^{-1}\delta$ is self-adjoint, the last inequality holds with p replaced by p'.

Let $\omega \in \mathcal{C}_0^\infty(T^*M)$. Then using successively $(RR_{p'})$ and $(\Pi_{p'})$,

$$\left\| \, \Delta^{-1/2}\delta\omega \, \right\|_{p'} = \left\| \, \Delta^{1/2}\Delta^{-1}\delta\omega \, \right\|_{p'} \leq C \left\| \, |d\Delta^{-1}\delta\omega| \, \right\|_{p'} \leq C \, \|\, \omega \,\|_{p'} \, .$$

Hence, by duality, $d\Delta^{-1/2}$ is bounded on L^p.

The proof that 2. $\Longrightarrow$ 1. in Theorem 2.1 is now easy. By combining Theorem 0.7 with Lemma 0.6, we have (RR_p) for $2 - \varepsilon < p < 2$. Let $p_0 = (2-\varepsilon)'$ and $2 < q < p_0$. If we assume (Π_p) for $2 < p < q$, then Lemma 0.1 gives us (R_p) for $2 < p < q$.

2.4. From Riesz transform to reverse Hölder inequalities

We show here the necessity of the reverse Hölder inequalities (RH_p). We assume that the Riesz transform is bounded on L^p for $2 < p < q$. Fix such a p.

Let B be a ball, r its radius and let u be harmonic function in $3B$. Let φ a C^1 function, supported in $2B$ with $\varphi = 1$ on $\frac{3}{2}B$, $\|\varphi\|_\infty \leq 1$ and $\|\nabla\varphi\|_\infty \leq C/r$. We assume that $\int_{2B} u = 0$, so that it follows from (P_2) that

$$r^{-2}\int_{2B} |u|^2 d\mu + \int_{2B} |\nabla(u\varphi)|^2 \, d\mu \leq C \int_{2B} |\nabla u|^2 \, d\mu.$$

To estimate $\int_B |\nabla u|^p d\mu$, it suffices to estimate $\int_B |\nabla(u\varphi)|^p \, d\mu$. Using an idea in [4], p. 35, we can write

$$u\varphi = e^{-r^2\Delta}(u\varphi) + u\varphi - e^{-r^2\Delta}(u\varphi) = e^{-r^2\Delta}(u\varphi) - \int_0^{r^2} e^{-s\Delta}\Delta(u\varphi)\, ds,$$

hence

$$\nabla(u\varphi) = \nabla e^{-r^2\Delta}(u\varphi) - \int_0^{r^2} \nabla e^{-s\Delta}\Delta(u\varphi)\, ds.$$

Let $p < \rho < q$. Since the Riesz transform is bounded on L^ρ, by the easy part of the necessary and sufficient condition in Theorem 0.3, we have that $\sqrt{t}\nabla e^{-t\Delta}$ is bounded on L^ρ uniformly with respect to t. It essentially follows from Lemma 3.2 in [2] that

$$\left(\frac{1}{\mu(B)}\int_B |\nabla e^{-s\Delta} f|^p \, d\mu\right)^{1/p} \leq \frac{Ce^{-\frac{\alpha 4^j r^2}{s}}}{\sqrt{s}} \left(\frac{1}{\mu(c_2 2^j B)}\int_{C_j(B)} |f|^2 \, d\mu\right)^{1/2} \tag{2.5}$$

for some constants C and α depending only on (D), (P_2), p and ρ whenever f is supported in $C_j(B)$ and $s \lesssim r^2(B)$. Here $C_1(B)$ is a fixed multiple of B, and for $j \geq 2$, $C_j(B)$ is a ring based on B: there are constants c_1, c_2 such that for all $j \geq 1$, if $x \in C_j(B)$ then $c_1 2^j r \leq d(x, B) \leq c_2 2^j r$.

It suffices to apply this inequality to $f = u\varphi$ which is supported in $2B$ to treat the L^p average of $\nabla e^{-r^2\Delta}(u\varphi)$ on B.

In the other term, a computation yields

$$\Delta(u\varphi) = -du \cdot d\varphi - \delta(ud\varphi).$$

We replace $\Delta(u\varphi)$ by its expression and observe that the support condition of $d\varphi$ allows us to use the previous estimates (2.5) for $\nabla e^{-s\Delta}(du \cdot d\varphi)$ when $j \geq 2$. Then, by the Minkowski inequality,

$$\left(\frac{1}{\mu(B)}\int_B \left|\int_0^{r^2} \nabla e^{-s\Delta}(du \cdot d\varphi)\, ds\right|^p d\mu\right)^{\frac{1}{p}} \leq C\left(\frac{1}{\mu(2B)}\int_{2B} |\nabla u|^2 \, d\mu\right)^{\frac{1}{2}}.$$

For the remaining term, it suffices to prove

$$\left(\frac{1}{\mu(B)}\int_B |\nabla e^{-s\Delta}\delta f|^p \, d\mu\right)^{1/p} \leq \frac{Ce^{-\frac{cr^2}{s}}}{s} \left(\frac{1}{\mu(2B)}\int_{2B\setminus\frac{3}{2}B} |f|^2 \, d\mu\right)^{1/2} \tag{2.6}$$

whenever f is supported in $2B \setminus \frac{3}{2}B$ and $s \leq r^2$ since this yields

$$\left(\frac{1}{\mu(B)}\int_B \left|\int_0^{r^2} \nabla e^{-s\Delta}\delta(ud\varphi)\, ds\right|^p d\mu\right)^{\frac{1}{p}} \leq \frac{C}{r}\left(\frac{1}{\mu(2B)}\int_{2B} |u|^2 \, d\mu\right)^{\frac{1}{2}},$$

which concludes the proof of (RH_p).
To see (2.6), the strategy is as follows. We use that $\nabla e^{-t\Delta}\delta = (\nabla e^{-t/2\Delta})(e^{-t/2\Delta}\delta)$. For the second operator we have the Gaffney type estimate

$$\|\sqrt{t}\, e^{-t\Delta}\delta\omega\|_{L^2(F)} \leq Ce^{-\frac{\alpha d(E,F)^2}{t}}\|\omega\|_{L^2(E)}.$$

whenever ω is a 1-form supported on E and E, F are closed subsets of M and $t > 0$. This estimate is for example proved in [2] for the dual operator $de^{-t\Delta}$. Make use of it with $E = 2B \setminus \frac{3}{2}B$ and successively $F = \frac{5}{4}B, 4B \setminus \frac{5}{4}B$, and $2^{j+1}B \setminus 2^jB$ for $j \geq 2$ and combine them with (2.5) to conclude. Similar calculations are shown in [2] and we skip further details.

3. Proof of Theorem 2.3

We split the argument in several steps. The following lemma is a localization result and is applied in the proof of a good lambda inequality which is the key step. The latter yields L^p inequalities, which applied to our particular hypotheses concludes the proof.

Lemma 3.1. *There is K_0 depending only on the doubling constant of E such that the following holds. Given $f \in L^1_{\text{loc}}(E,\mu)$, a ball B and $\lambda > 0$ such that there exists $\bar{x} \in B$ for which $\mathcal{M}f(\bar{x}) \leq \lambda$, then for any $K \geq K_0$,*

$$\{\chi_B\mathcal{M}f > K\lambda\} \subset \{\mathcal{M}(f\chi_{3B}) > \frac{K}{K_0}\lambda\}.$$

Proof. Recall that $\mathcal{M}$ is comparable to the centered maximal function $\mathcal{M}_c$: there is K_0 depending only on the doubling constant such that $\mathcal{M} \leq K_0\mathcal{M}_c$.

Let $x \in B$ with $\mathcal{M}f(x) > K\lambda$. Then $\mathcal{M}_cf(x) > \frac{K}{K_0}\lambda$. Hence, there is a ball $B(x,r)$ centered at x with radius r such that

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f\,d\mu > \frac{K}{K_0}\lambda.$$

If $\frac{K}{K_0} \geq 1$, $\bar{x} \notin B(x,r)$ since $\mathcal{M}f(\bar{x}) \leq \lambda$. The conditions $x \in B$, $\bar{x} \in B$ and $\bar{x} \notin B(x,r)$ imply $B(x,r) \subset 3B$. Hence,

$$\frac{K}{K_0}\lambda < \frac{1}{\mu(B(x,r))}\int_{B(x,r)} (f\chi_{3B})\,d\mu \leq \mathcal{M}(f\chi_{3B})(x).$$

This proves the lemma. □

We continue with a two parameters family of good lambda inequalities.

Proposition 3.2. *Fix $1 < q \le \infty$ and $a > 1$. Let $F, G \in L^1_{\mathrm{loc}}(E, \mu)$, non-negative. We say that $(F, G) \in \mathcal{E}_{q,a}$ if one can find for every ball B non-negative measurable functions G_B, H_B defined on B with*

$$F \le G_B + H_B \quad a.e. \text{ on } B$$

such that

$$\left(\frac{1}{\mu(B)}\int_B (H_B)^q \, d\mu\right)^{1/q} \le a \inf_{x\in B} \mathcal{M}F(x) + \inf_{x\in B} G(x),$$

$$\frac{1}{\mu(B)}\int_B G_B \, d\mu \le \inf_{x\in B} G(x).$$

There exist $C = C(q, (D), a)$ and $K_0' = K_0'(a, (D))$ such that for $(F, G) \in \mathcal{E}_{q,a}$, for all $\lambda > 0$, for all $K > K_0'$ and $\gamma \le 1$,

$$\mu\{\mathcal{M}F > K\lambda, G \le \gamma\lambda\} \le C\left(\frac{1}{K^q} + \frac{\gamma}{K}\right)\mu\{\mathcal{M}F > \lambda\}$$

provided $\{\mathcal{M}F > \lambda\}$ is a proper subset of E.

If $q = \infty$, we understand the average in L^q as an essential supremum. In this case, we set $\frac{1}{K^q} = 0$.

Proof. Let $E_\lambda = \{\mathcal{M}F > \lambda\}$. This is an open proper subset of E. The Whitney decomposition for E_λ yields a family of boundedly overlapping balls B_i such that $E_\lambda = \cup_i B_i$. There exists $c > 1$ such that, for all i, cB_i contains at least one point $\overline{x_i}$ outside E_λ, that is

$$\mathcal{M}F(\overline{x_i}) \le \lambda.$$

Let $B_\lambda = \{\mathcal{M}F > K\lambda, G \le \gamma\lambda\}$. If $K \ge 1$ then $B_\lambda \subset E_\lambda$, hence

$$\mu(B_\lambda) \le \sum_i \mu(B_\lambda \cap B_i) \le \sum_i \mu(B_\lambda \cap cB_i).$$

Fix i. If $B_\lambda \cap cB_i = \emptyset$, we have nothing to do. If not, there is a point $\overline{y_i} \in cB_i$ such that

$$G(\overline{y_i}) \le \gamma\lambda.$$

By the localization lemma applied to F on cB_i, if $K \ge K_0$, then

$$\mu(B_\lambda \cap cB_i) \le \mu(\{\mathcal{M}F > K\lambda\} \cap cB_i) \le \mu\left\{\mathcal{M}(F\chi_{3cB_i}) > \frac{K}{K_0}\lambda\right\}.$$

Now use $F \le G_i + H_i$ on $3cB_i$ with $G_i = G_{3cB_i}$ and $H_i = H_{3cB_i}$ to deduce

$$\mu\left\{\mathcal{M}(F\chi_{3cB_i}) > \frac{K}{K_0}\lambda\right\} \le \mu\left\{\mathcal{M}(G_i\chi_{3cB_i}) > \frac{K}{2K_0}\lambda\right\} + \mu\left\{\mathcal{M}(H_i\chi_{3cB_i}) > \frac{K}{2K_0}\lambda\right\}.$$

Now by using the weak-type (1, 1) and (q, q) of the maximal operator with respective constants c_1 and c_q, we have

$$\mu\{\mathcal{M}(G_i\chi_{3B_i}) > \frac{K}{2K_0}\lambda\} \le \frac{2K_0c_1}{K\lambda}\int_{3cB_i} G_i\,d\mu \le \frac{2K_0c_1}{K\lambda}\mu(3cB_i)G(\overline{y_i})$$
$$\le \frac{2K_0c_1\gamma}{K}\mu(3cB_i),$$

and, if $q < \infty$,

$$\mu\{\mathcal{M}(H_i\chi_{3cB_i}) > \frac{K}{2K_0}\lambda\} \le \left(\frac{2K_0c_q}{K\lambda}\right)^q \int_{3cB_i} H_i^q\,d\mu$$
$$\le \left(\frac{2K_0c_q}{K\lambda}\right)^q \mu(3cB_i)(a\mathcal{M}F(\overline{x_i}) + G(\overline{y_i}))^q$$
$$\le \left(\frac{2K_0c_q(a+1)}{K}\right)^q \mu(3cB_i).$$

Hence, summing over i yields

$$\mu(B_\lambda) \le C\left(\frac{1}{K^q} + \frac{\gamma}{K}\right)\sum_i \mu(3cB_i) \le C'\left(\frac{1}{K^q} + \frac{\gamma}{K}\right)\mu(E_\lambda)$$

by applying the doubling property together with the bounded overlap. If $q = \infty$, then

$$\|\mathcal{M}(H_i\chi_{3cB_i})\|_\infty \le \|H_i\chi_{3cB_i}\|_\infty \le a\mathcal{M}F(\overline{x_i}) + G(\overline{y_i}) \le (a+1)\lambda,$$

so that, choosing $K \ge 2K_0(a+1)$ leads us to $\{\mathcal{M}(H_i\chi_{3B_i}) > \frac{K}{2K_0}\lambda\} = \emptyset$. The rest of the proof is unchanged. This proves the proposition. □

Corollary 3.3. *Assume that* $(F, G) \in \mathcal{E}_{q,a}$. *Let* $1 < \rho < q$ *and assume that* $\|G\|_\rho < \infty$ *and* $\|F\|_1 < \infty$. *Then, we have*

$$\|\mathcal{M}F\|_\rho \le C\left(\|G\|_\rho + \mu(E)^{\frac{1}{\rho}-1}\|F\|_1\right),^5$$

where the constant C *depends on* (D), ρ, q, a.

Proof. We begin with the case $\mu(E) = \infty$. Define $\Phi(t) = \rho\int_0^t \lambda^{\rho-1}\mu\{\mathcal{M}F > \lambda\}\,d\lambda$ for $t \ge 0$. Since $\|F\|_1 < \infty$, the maximal theorem implies that $\lambda\mu\{\mathcal{M}F >$

[5] In the case $\mu(E) = \infty$ which is the situation of interest here, the last term vanishes but we still need some a priori knowledge such as $F \in L^1$ to conclude.

$\lambda\}$ is bounded on $\mathbb{R}^+$. As $1 < \rho$, Φ is a well-defined positive and non-decreasing function on $\mathbb{R}^+$ into $\mathbb{R}^+$.

By the maximal theorem and $\|F\|_1 < \infty$, $\{\mathcal{M}F > \lambda\}$ is a proper subset in E, hence the good lambda inequality is valid and integration leads us to

$$\Phi(Kt) \leq CK^\rho\left(\frac{1}{K^q} + \frac{\gamma}{K}\right)\Phi(t) + \left(\frac{K}{\gamma}\right)^\rho \|G\|_\rho^\rho.$$

Since $\rho < q$, one can choose K large enough and γ small enough so that

$$CK^\rho\left(\frac{1}{K^q} + \frac{\gamma}{K}\right) \leq \frac{1}{2}.$$

hence, for this choice, for all $t \geq 0$

$$\Phi(Kt) \leq \frac{1}{2}\Phi(t) + \left(\frac{K}{\gamma}\right)^\rho \|G\|_\rho^\rho.$$

An easy iteration proves that Φ is bounded and this proves the corollary in this case as $\Phi(\infty)$ is $\|\mathcal{M}F\|_\rho^\rho$.

In the case where $\mu(E) < \infty$, we have $\lambda\mu\{\mathcal{M}F > \lambda\} \leq C\|F\|_1$, hence for $\lambda > a$ with $a = \frac{C}{\mu(E)}\|F\|_1$, the good lambda inequality applies. If we define Φ as before, the previous argument gives us a control of $\Phi(\infty) - \Phi(a)$ by $C\|G\|_\rho^\rho$ and it remains to controlling $\Phi(a)$. But $\Phi(a) \leq a^\rho\mu(E)$ and the conclusion follows. □

Now, we may prove Theorem 2.3. We let $f \in L^p \cap L^2(E, \mu)$ and $F = |Tf|^2$. We let $G_B = 2|T(\chi_{\beta B}f)|^2$ and $H_B = 2|T((1-\chi_{\beta B})f)|^2$. On the one hand, for C depending only on (D) and the norm $\|T\|$ of T on L^2,

$$\frac{1}{\mu(B)}\int_B G_B\,d\mu \leq \frac{2\|T\|^2}{\mu(B)}\int_{\beta B}|f|^2 \leq C\inf_{x\in B}\mathcal{M}(|f|^2)(x).$$

On the other hand, since $(1-\chi_{\beta B})f$ is supported away from βB, the assumption (2.4) yields

$$\left(\frac{1}{\mu(B)}\int_B (H_B)^{q/2}\,d\mu\right)^{2/q} \leq \frac{C}{\mu(\alpha B)}\int_{\alpha B} H_B\,d\mu,$$

and we have

$$\int_{\alpha B} H_B\,d\mu \leq 4\int_{\alpha B} F\,d\mu + 2\int_{\alpha B} G_B\,d\mu,$$

hence for some $a > 0$,

$$\left(\frac{1}{\mu(B)}\int_B (H_B)^{q/2}\,d\mu\right)^{2/q} \leq a\inf_{x\in B}\mathcal{M}F(x) + C\inf_{x\in B}\mathcal{M}(|f|^2)(x).$$

Thus we conclude with $G = C\mathcal{M}(|f|^2)$ that if $2 < p < q$, since $Tf \in L^2$ hence $F \in L^1$, then

$$\|F\|_{p/2} \le C \left(\|G\|_{p/2} + \mu(E)^{\frac{2}{p}-1} \|F\|_1 \right).$$

Observe then that $\|G\|_{p/2} \sim \|f\|_p^2$ and by the L^2 boundedness of T and Hölder inequality,

$$\mu(E)^{\frac{2}{p}-1} \|F\|_1 \le C\mu(E)^{\frac{2}{p}-1} \|f\|_2^2 \le C\|f\|_p^2.$$

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