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Analytic adjoint ideal sheaves associated to plurisubharmonic functions. II

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Abstract. In this article we will introduce a coherent analytic adjoint ideal sheaf associated to plurisubharmonic function along a divisor. As an application, we will characterize the equivalence of canonical and rational singularities for complex hypersurfaces in the analytic setting.

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1. Introduction

The adjoint ideal sheaves, as a variant of the multiplier ideal sheaves, turned out to be a powerful tool in algebraic geometry in recent years; one can refer to [12–14] for a general exposition of the algebro-geometric side of the theory. In [9], H. Guenancia generalized the notion of the adjoint ideal sheaf to the analytic setting via the Ohsawa-Takegoshi-Manivel extension theorem, and then proved the coherence for the locally Hölder continuous plurisubharmonic weights (see also [10]). However, the analytic adjoint ideal sheaves defined in [9, 10] are not coherent in general and one can find an explicit example given by Guan and the author in [6].

In order to define an analytic adjoint ideal sheaves along a divisor (maybe singular) and establish the so-called adjunction exact sequence as in the algebraic setting, it is important for us to construct the multiplier ideal sheaves on singular hypersurfaces. In the present paper, we will firstly construct multiplier ideals for the singular hypersurface case (see Section 3), by which we then obtain a version of the definition of (coherent) analytic adjoint ideal sheaves and establish the associated adjunction exact sequence as follows.

Let X be a complex manifold and $H \subset X$ be a reduced complex hypersuface (not necessarily smooth). Let $\varphi \in \mathrm{Psh}(X)$ be a plurisubharmonic function on X such that $\varphi|_H \not\equiv -\infty$ on every irreducible component of H. Then, we have:

Theorem 1.1 (Theorem 3.5). *There exists an ideal sheaf*

$$Adj_H(\varphi) \subset \mathcal{O}_X,$$

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called the analytic adjoint ideal sheaf *associated to φ along H, sitting in an exact sequence*

$$0 \longrightarrow \mathscr{I}(\varphi) \otimes \mathcal{O}_X(-H) \xrightarrow{\iota} Adj_H(\varphi) \xrightarrow{\rho} i_*\mathscr{I}(\varphi|_H) \longrightarrow 0, \qquad (\star)$$

where $i : H \hookrightarrow X$, ι and ρ are the natural inclusion and restriction maps respectively.

Remark 1.2. (1) Similar to the proof of Proposition 2.11 in [9], it follows that $Adj_H(\varphi)$ coincides with the algebraic adjoint ideal sheaf along a divisor whenever φ has analytic singularities.

(2) If φ is a Hölder plurisubharmonic function and H is smooth, our definition of $Adj_H(\varphi)$ is the same as that given by Guenancia (see Theorem 2.16 in [9]). Similarly, we always have $Adj_H(\varphi) \subset \mathscr{I}(\varphi)$, and $Adj_H(\varphi)_x = \mathscr{I}(\varphi)_x$ if $x \notin H$ as well.

Remark 1.3. In fact, Theorem 1.1 also holds if $H \subset X$ is locally a complete intersection. One can refer to Appendix A for more details.

Whereas the algebraic adjoint ideals encode much information on the singularities of the divisor, as an application, we can establish the following characterization of rationality of hypersuface singularities.

Theorem 1.4. *Let $\varphi \in \mathrm{Psh}(X)$ be a plurisubharmonic function such that the slope $\nu(\varphi|_H; x) = 0$ for every $x \in H$. Then, the analytic adjoint ideal sheaf $Adj_H(\varphi) = \mathcal{O}_X$ if and only if $\mathscr{I}(\varphi|_H) = \mathcal{O}_H$ if and only if H is normal and has only rational singularities if and only if (H, x) is canonical for any $x \in H$.*

Remark 1.5. (1) Note that our multiplier (respectively adjoint) ideals measure both the singularities of φ and H together. The above result also implies that if a hypersurface has at most rational singularities, the plurisubharmonic weights with zero slope do not add the singularities in the sense of multiplier or adjoint ideals.

(2) If X is a smooth complex algebraic variety and φ is trivial, Theorem 1.4 is nothing but [12, Proposition 9.3.48 (ii)], which turned out to be very useful to study the singularities of theta divisors on principally polarized Abelian varieties.

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2. Some useful results

We start by recalling some basic facts. Throughout this paper, all complex spaces are assumed to be reduced and paracompact; we refer to [5] for main reference on the theory of complex spaces.

For the sake of convenience, we state the following special case of the Ohsawa-Takegoshi-Manivel extension theorem (see [4,7]):

Theorem 2.1 (Ohsawa-Takegoshi-Manivel). *Let $X \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, E be a Hermitian holomorphic vector bundle of rank r on X and s be a global holomorphic section of E. Assume that s is generically transverse to the zero section and $|s| \leq e^{-1}$ on X, and set $Y = \{x \in X | s(x) = 0, \Lambda^r ds(x) \neq 0\}$.*

Then, there is a constant $C > 0$ (depending only on E) such that for every plurisubharmonic function φ on X and every holomorphic function f on Y with $\int_Y |f|^2 |\Lambda^r(ds)|^{-2} e^{-2\varphi} dV_Y < +\infty$, there exists a holomorphic extension $\tilde{f}$ of f to X such that

$$\int_X \frac{|\tilde{f}|^2}{|s|^{2r} \log^2 |s|} e^{-2\varphi} dV_X \leq C \cdot \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} e^{-2\varphi} dV_Y.$$

In order to prove Theorem 1.4, we need the following characterization on the slope of plurisubharmonic functions in [2, Corollary 9.3]:

Lemma 2.2. *Let X be a normal complex space and $\varphi \in \mathrm{Psh}(X)$ $(\not\equiv -\infty)$. If the slope $\nu(\varphi; x) = 0$ for some point $x \in X$, then, for any resolution of singularities $\pi : \tilde{X} \to X$ of X, $\nu(\varphi \circ \pi; \tilde{x}) = 0$ for each $\tilde{x} \in \pi^{-1}(x)$.*

Here, a plurisubharmonic function $\varphi \in \mathrm{Psh}(X)$ means that φ is an upper semi-continuous function on X with values in $[-\infty, \infty)$, which extends to a plurisubharmonic function in some local embedding $X \to \mathbb{C}^N$. The slope of φ at x is defined by

$$\nu(\varphi; x) := \sup \left\{ \gamma \geq 0 \middle| \, \varphi \leq \gamma \log \sum_k |f_k| + O(1) \right\} \in [0, +\infty),$$

where (f_k) are local generators of the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$.

Remark 2.3. If X is a complex manifold, the slope $\nu(\varphi; x)$ is exactly the usual Lelong number of φ at x and $\nu(\varphi; x) < 1$ implies the integrability of $e^{-2\varphi}$ near x by a result of Skoda (see [4, Lemma 5.6]).

3. Multiplier and adjoint ideal sheaves

In this section we will give the definitions of analytic multiplier and adjoint ideal sheaves, and then discuss some properties of them.

Definition 3.1 (Multiplier ideal sheaves on singular complex hypersurfaces). Let X be a complex manifold of dimension n and $H \subset X$ a reduced complex hypersuface (not necessarily smooth). Let $\varphi \in \mathrm{Psh}(X)$ be a plurisubharmonic function such that $\varphi|_H \not\equiv -\infty$ on every irreducible component of H.

The *multiplier ideal sheaf* $\mathscr{I}(\varphi|_H) \subset \mathcal{O}_H$ on H is defined to be the ideal sheaf of germs of holomorphic functions $f \in \mathcal{O}_{H,x}$ such that $\frac{|f|^2 e^{-2\varphi}}{|dh|^2}$ is locally integrable at x on H with respect to the Lebesgue measure, where h is the minimal defining function of H near x.

Remark 3.2. Indeed, we can only assume that φ is well-defined on H, not necessary on the whole X.

Proposition 3.3. *With the same notation as above, we have the following:*

(1) (Coherence). $\mathscr{I}(\varphi|_H) \subset \mathcal{O}_H$ *is a coherent ideal sheaf;*
(2) (Restriction formula). $\mathscr{I}(\varphi|_H) \subset \mathscr{I}(\varphi)|_H$*;*
(3) (Bimeromorphic transformation rule). *Let* $\pi : \tilde{H} \to H$ *be a proper modification with* $\tilde{H}$ *non-singular,* i.e., *a generically 1:1 holomorphic mapping. Then,*
$$\pi_*(\mathcal{O}(K_{\tilde{H}}) \otimes \mathscr{I}(\varphi|_H \circ \pi)) = \mathcal{O}(K_H) \otimes \mathscr{I}(\varphi|_H),$$
where $\mathcal{O}(K_H) := \tilde{i}_*\mathcal{O}(K_{H_{\mathrm{reg}}})$ *and* $\tilde{i} : H_{\mathrm{reg}} \to H$ *is the natural inclusion;*
(4) (Strong openness). $\mathscr{I}(\varphi|_H) = \mathscr{I}_+(\varphi|_H) := \bigcup_{\varepsilon>0} \mathscr{I}((1+\varepsilon)\varphi|_H)$.

Proof. (1) As the statement is local, without loss of generality, we may assume that $X \subset \mathbb{C}^n$ $(n \geq 2)$ is a bounded Stein domain and h is the minimal defining function of H on X. Let $\mathscr{I}$ be the ideal sheaf generated by the global sections of $\mathscr{I}(\varphi|_H)$. It follows from the strong Noetherian property of coherent analytic sheaves that $\mathscr{I} \subset \mathcal{O}_H$ is a coherent ideal sheaf. Since $\mathcal{O}_{H,x}$ is a Noetherian local ring with maximal ideal $\mathfrak{m}_{H,x}$ for any $x \in H$, by the Krull's intersection theorem, it suffices to prove
$$\mathscr{I}_x + \mathscr{I}(\varphi|_H)_x \cap \mathfrak{m}^k_{H,x} = \mathscr{I}(\varphi|_H)_x$$
for every $k \in \mathbb{N}$.

Let $f \in \mathscr{I}(\varphi|_H)_x$ be defined on a neighborhood $V \subset\subset H$ of x, and χ a cut-off function with support in V such that $\chi = 1$ near x. Let $v := \bar\partial(\chi \cdot f \wedge \sigma)$, where $\sigma = dz_1 \wedge \cdots \wedge dz_{n-1} \big/ \frac{\partial h}{\partial z_n}$ is a nowhere vanishing holomorphic $(n-1)$-form on H_{reg}. Then, v is a $\bar\partial$-closed $(n-1, 1)$-form on H_{reg}, and for some constant $C > 0$,
$$\int_H |v|^2 e^{-2\varphi} |z-x|^{-2(n+k)} \leq C \cdot \int_V (\sqrt{-1})^{(n-1)^2} |f|^2 e^{-2\varphi} \sigma \wedge \bar\sigma < +\infty.$$

Since [3, Theorem 8.5] amounts to solvability and L^2-estimates for $(n-1, q)$-forms on the regular part of an analytic variety with codimension one in pseudoconvex domain X (see [3, Appendix B]), by solving the equation $\bar\partial u = v$ with the strictly plurisubharmonic function weight $\hat\varphi(z) = \varphi(z) + (n+k)\log|z-x| + |z|^2$, we obtain a solution u such that
$$\int_H (\sqrt{-1})^{(n-1)^2} e^{-2\varphi} |z-x|^{-2(n+k)} u \wedge \bar u < +\infty.$$

On the other hand, there is a smooth function g on H_{reg} such that $u = g \cdot \sigma$ because σ is nowhere vanishing on H_{reg}. Then, we infer from $\bar{\partial} u = v$ that $F := \chi \cdot f - g$ is holomorphic on H_{reg} with

$$\int_{H_{\text{reg}}} (\sqrt{-1})^{(n-1)^2} |F|^2 e^{-2\varphi} \sigma \wedge \bar{\sigma} < +\infty,$$

and there is some Stein neighborhood $\Omega \subset\subset X$ of x with $g \in \mathcal{O}_H(\Omega \cap H_{\text{reg}})$. Thus, by a standard application of Ohsawa-Takegoshi-Manivel extension theorem, there exists a holomorphic function $\tilde{F} \in \mathcal{O}_X(X)$ with $\tilde{F}|_{H_{\text{reg}}} = F$, and $\tilde{g} \in \mathcal{O}_X(\Omega)$ such that $\tilde{g}|_{\Omega \cap H_{\text{reg}}} = g|_{\Omega \cap H_{\text{reg}}}$ and $\tilde{g}_x \in \mathfrak{m}_{X,x}^{k+1}$, which implies that F and g have natural extension to H and $H_{\text{reg}} \cup (\Omega \cap H)$ respectively, still denoted by F and g, with $F = \tilde{F}|_H$ and $g|_{\Omega \cap H} = \tilde{g}|_{\Omega \cap H}$. Therefore, $F \in \Gamma(H, \mathscr{I}(\varphi|_H))$ and

$$f_x - F_x = (\tilde{g}|_{\Omega \cap H})_x \in \mathscr{I}(\varphi|_H)_x \cap \mathfrak{m}_{H,x}^{k+1},$$

which concludes the proof.

(2) It is a direct consequence of Ohsawa-Takegoshi-Manivel extension theorem, just by extending locally a germ of holomorphic function on H near $x \in H$ to a holomorphic function on a Stein neighborhood of x in X.

(3) Let $S \subsetneq H$ be an analytic set such that $\pi : \tilde{H} \backslash \pi^{-1}(S) \to H \backslash S$ is a biholomorphism. Since $\frac{|f|^2 e^{-2\varphi}}{|dh|^2} dV_H$ is equal to $(\sqrt{-1})^{(n-1)^2} e^{-2\varphi} (f\sigma) \wedge \overline{(f\sigma)}$ up to a constant multiple, then, for an open set $U \subset H$, $\mathcal{O}(K_H) \otimes \mathscr{I}(\varphi)(U)$ consists of all holomorphic $(n-1)$-forms α on U_{reg} with $(\sqrt{-1})^{(n-1)^2} e^{-2\varphi} \alpha \wedge \bar{\alpha} \in L^1_{\text{loc}}(U)$. Since the holomorphic forms in $L^2_{\text{loc}}(U_{\text{reg}})$ which are only defined on $U_{\text{reg}} \backslash S$ have unique holomorphic extension to the whole U_{reg}, due to the change of variable formula

$$\int_{U \backslash S} (\sqrt{-1})^{(n-1)^2} e^{-2\varphi} \alpha \wedge \bar{\alpha} = \int_{\pi^{-1}(U \backslash S)} (\sqrt{-1})^{(n-1)^2} e^{-2\varphi \circ \pi} \pi^* \alpha \wedge \overline{\pi^* \alpha},$$

it follows that $\alpha \in \Gamma(U, \mathcal{O}(K_H) \otimes \mathscr{I}(\varphi))$ if and only if $\pi^* \alpha \in \Gamma(\pi^{-1}(U), \mathcal{O}(K_{\tilde{H}}) \otimes \mathscr{I}(\varphi \circ \pi))$, *i.e.*,

$$\pi_*(\mathcal{O}(K_{\tilde{H}}) \otimes \mathscr{I}(\varphi \circ \pi)) = \mathcal{O}(K_H) \otimes \mathscr{I}(\varphi).$$

(4) The desired result immediately follows from (3) and the strong openness property of multiplier ideal sheaves established by Guan and Zhou in [8]. □

Remark 3.4. If H is a normal complex hypersurface and φ has analytic singularities, it follows from the bimeromorphic transformation rule that our definition coincides with the algebraic counterpart given by Lazarsfeld (see [12, Definition 9.3.55]). One can refer to [4, Remark 5.9] for the smooth H case.

Specifically, let $\varphi = \frac{c}{2}\log(|f_1|^2+\cdots+|f_j|^2)+O(1)$ with $\mathscr{I} := (f_1, \ldots, f_j)\cdot \mathcal{O}_H$ on H, $\pi : \tilde{H} \to H$ a log resolution of H such that $\pi^*\mathscr{I} = \mathcal{O}_{\tilde{H}}(-\Delta)$ with $\Delta = \sum b_k\Delta_k$ a simple normal crossing divisor on $\tilde{H}$ and $K_{\tilde{H}} = \pi^*K_H + \sum a_k\Delta_k$. Then, we have

$$\mathscr{I}(\varphi) = \pi_*\mathcal{O}_{\tilde{H}}\left(\sum(a_k - \lfloor cb_k\rfloor)\Delta_k\right).$$

Theorem 3.5 (Theorem 1.1). *With hypotheses as in Definition* 3.1*, there exists an ideal sheaf*

$$Adj_H(\varphi) \subset \mathcal{O}_X,$$

called the analytic adjoint ideal sheaf *associated to φ along H, sitting in an exact sequence*

$$0 \longrightarrow \mathscr{I}(\varphi)\otimes\mathcal{O}_X(-H) \stackrel{\iota}{\longrightarrow} Adj_H(\varphi) \stackrel{\rho}{\longrightarrow} i_*\mathscr{I}(\varphi|_H) \longrightarrow 0, \qquad (\star)$$

where $i : H \hookrightarrow X$, ι and ρ are the natural inclusion and restriction maps respectively.

Proof. Without loss of generality, we may assume that X is still a bounded Stein domain in $\mathbb{C}^n$ $(n \geq 2)$, h is the minimal defining function of H and $\mathscr{J} \subset \mathcal{O}_X$ is an ideal sheaf such that $\mathscr{J}|_H = \mathscr{I}(\varphi|_H)$, which implies that $\mathscr{J} + \mathscr{I}_H$ is independent of the choice of $\mathscr{J}$.

Let

$$Adj_H(\varphi) := \bigcup_{\varepsilon>0} Adj_H^0((1+\varepsilon)\varphi) \cap (\mathscr{J} + \mathscr{I}_H),$$

where $\mathscr{I}_H$ is the ideal sheaf of the hypersurface H and $Adj_H^0(\varphi) \subset \mathcal{O}_X$ is an ideal sheaf of germs of holomorphic functions $f \in \mathcal{O}_{X,x}$ such that

$$\frac{|f|^2e^{-2\varphi}}{|h|^2\log^2|h|}$$

is locally integrable with respect to the Lebesgue measure near x on X. By the definition, $Adj_H(\varphi)$ is independent of the choices of $\mathscr{J}$ and h. Then, the restriction map ρ is well-defined and the surjectivity of ρ follows from the strong openness (4) in Proposition 3.3 and a local version of Ohsawa-Takegoshi-Manivel extension theorem, with the weight $(1+\varepsilon)\varphi$. In addition, the inclusion map ι is also well-defined by the strong openness property of multiplier ideal sheaves.

Now we only need to prove $\mathscr{K}er\ \rho = \mathscr{I}m\ \iota$ for exactness of the sequence $(\star)$. It is clear that $\mathscr{I}m\ \iota \subset \mathscr{K}er\ \rho$. Without loss of generality, we assume that $f \in Adj_H(\varphi)_x \neq \mathcal{O}_{X,x}$ and $\rho(f) \in \mathscr{I}_{H,x}$ with $x \in H$, then there exists a neighborhood U of x and $g \in \mathcal{O}_X(U)$ such that $f = g\cdot h$, $\varphi(z) < 0$ on U and for some $0 < \varepsilon_0 < 1$,

$$\int_U \frac{|g|^2e^{-2(1+\varepsilon_0)\varphi}}{\log^2|h|}d\lambda_n < +\infty.$$

On the other hand, for any $\delta > 0$, we have

$$\begin{aligned}
&\int_{U\cap\{\varphi\le\delta\log|h|\}} |g|^2 e^{-2(1+\varepsilon_0/2)\varphi} d\lambda_n \\
&= \int_{U\cap\{\varphi\le\delta\log|h|\}} (e^{\varepsilon_0\varphi}\log^2|h|) \frac{|g|^2 e^{-2(1+\varepsilon_0)\varphi}}{\log^2|h|} d\lambda_n \\
&\le \int_{U\cap\{\varphi\le\delta\log|h|\}} (|h|^{\varepsilon_0\delta}\log^2|h|) \frac{|g|^2 e^{-2(1+\varepsilon_0)\varphi}}{\log^2|h|} d\lambda_n \\
&\le \sup_{z\in U}(|h(z)|^{\varepsilon_0\delta/2}\log|h(z)|)^2 \cdot \int_{U\cap\{\varphi\le\delta\log|h|\}} \frac{|g|^2 e^{-2(1+\varepsilon_0)\varphi}}{\log^2|h|} d\lambda_n < +\infty
\end{aligned}$$

and

$$\int_{U\cap\{\varphi>\delta\log|h|\}} |g|^2 e^{-2(1+\varepsilon_0/2)\varphi} d\lambda_n \le \sup_{z\in U}|g(z)|^2 \cdot \int_U |h|^{-2(1+\varepsilon_0/2)\delta} d\lambda_n.$$

Therefore, for small enough $\delta > 0$, we obtain that

$$\begin{aligned}
&\int_U |g|^2 e^{-2(1+\varepsilon_0/2)\varphi} d\lambda_n \\
&= \int_{U\cap\{\varphi\le\delta\log|h|\}} |g|^2 e^{-2(1+\varepsilon_0/2)\varphi} d\lambda_n + \int_{U\cap\{\varphi>\delta\log|h|\}} |g|^2 e^{-2(1+\varepsilon_0/2)\varphi} d\lambda_n < +\infty,
\end{aligned}$$

by shrinking U if necessary. Thus, it follows that $g \in \mathscr{I}(\varphi)_x$, which implies $\mathscr{K}er\ r \subset \mathscr{I}m\ \iota$, and the proof is thereby concluded. □

Remark 3.6. The exact sequence ($\star$) yields the coherence of $Adj_H(\varphi)$, and

$$Adj_H(\varphi) \cdot \mathcal{O}_H = \mathscr{I}(\varphi|_H)$$

after twisting through by $\mathcal{O}_H$. Moreover, we can infer from the strong openness property of multiplier ideal sheaves that

$$Adj_H(\varphi) \subset \mathscr{I}(\varphi + \delta\log|h|)$$

for any $0 \le \delta < 1$.

4. Singularities of complex hypersurfaces

As an application, in the present section we will characterize the singularities of hypersurfaces via our multiplier and adjoint ideals. Firstly, let us recall some concepts on the singularities of complex spaces, which are defined exactly as in the algebraic context; we refer to [11].

Definition 4.1. Let X be a complex space, $x \in X$ a point and $n = \dim_x X$. Let $\pi : \tilde{X} \to X$ be a resolution of singularities of X. x is called a *weakly rational singularity* of X if $(R^{n-1}\pi_* \mathcal{O}_{\tilde{X}})_x = 0$.

Additionally, if X is normal, x is called a *rational singularity* of X if $(R^q \pi_* \mathcal{O}_{\tilde{X}})_x = 0$ for all $q > 0$.

Remark 4.2 (see [1]). (1) The above definition is independent of the resolution π and every smooth point of X is a weakly rational singularity.

(2) If x is a weakly rational singularity, then x is rational if and only if $\mathcal{O}_{X,x}$ is a Cohen-Macaulay local ring.

(3) If X is a normal complex space of dimension n, then x is a weakly rational singularity if and only if every holomorphic n-form defined on the regular points of a neighborhood of x is locally square integrable around x with respect to the Lebesgue measure of X.

Definition 4.3. Let X be a normal complex space and $x \in X$ a point. (X, x) is called an *(analytically) canonical singularity* if there exists an open neighborhood U of x such that:

(1) the canonical class K_X is $\mathbb{Q}$-Cartier on U;

(2) if $\pi : \tilde{U} \to U$ is any resolution of singularities of U with exceptional divisors $E_1, \dots, E_m$, and we write

$$K_{\tilde{U}} = \pi^* K_U + \sum_{k=1}^{m} a_k E_k,$$

then $a_k \geq 0$ holds for every k.

Proof of Theorem 1.4. (1st "if and only if") By Remark 3.6, it follows that $Adj_H(\varphi) = \mathcal{O}_X$ if and only if $\mathscr{I}(\varphi|_H) = \mathcal{O}_H$.
(2nd "if and only if") If $\mathscr{I}(\varphi|_H)_x = \mathcal{O}_{H,x}$, $\forall x \in H$, then there exists an arbitrarily small Stein neighborhood $\Omega \subset X$ of x such that

$$\int_{\Omega \cap H_{\mathrm{reg}}} (\sqrt{-1})^{(n-1)^2} e^{-2\varphi} \sigma \wedge \bar{\sigma} < +\infty,$$

where $\sigma = dz_1 \wedge \cdots \wedge dz_{n-1} \big/ \frac{\partial h}{\partial z_n}$ is a nowhere vanishing holomorphic $(n-1)$-form on H_{reg}. Thus, for any bounded holomorphic function f on $\Omega \cap H_{\mathrm{reg}}$, we have

$$\int_{\Omega \cap H_{\mathrm{reg}}} \frac{|f|^2 e^{-2\varphi}}{|dh|^2} dV_H \leq C \cdot \int_{\Omega \cap H_{\mathrm{reg}}} (\sqrt{-1})^{(n-1)^2} e^{-2\varphi} \sigma \wedge \bar{\sigma} < +\infty.$$

As a consequence of the Ohsawa-Takegoshi-Manivel extension theorem, we have a holomorphic function $\tilde{f}$ on Ω such that $\tilde{f}|_{H_{\mathrm{reg}}} = f$, which implies that f has a holomorphic extension $\tilde{f}|_H$ to $\Omega \cap H$, *i.e.*, x is a normal point of H. Moreover, the local integrability of $(\sqrt{-1})^{(n-1)^2} \sigma \wedge \bar{\sigma}$ implies that every holomorphic $(n-1)$-form defined on the regular points near $x \in H$ is locally square integrable, *i.e.*, H has weakly rational singularities. Then, we infer from the Cohen-Macaulay property of complex hypersuface that H has only rational singularities (see Remark 4.2).

Conversely, if H has only rational singularities, then H is normal and any holomorphic $(n-1)$-form defined on H_{reg} is locally square integrable on H. Thus, it follows from Lemma 2.2 and Remark 2.3 that

$$\int_{\Omega\cap H_{\text{reg}}} (\sqrt{-1})^{(n-1)^2} e^{-2\varphi}\sigma \wedge \bar{\sigma} < +\infty$$

(shrinking Ω if necessary), *i.e.*, $\mathscr{I}(\varphi|_H) = \mathcal{O}_H$.

(3rd "if and only if") Let $\pi : \tilde{V} \to V$ be a desingularization of $V \subset H$ near x such that

$$\pi^{-1}(V_{\text{sing}}) = \sum a_k \Delta_k$$

is the exceptional divisor. Write $K_{\tilde{V}} = \pi^* K_V + \sum a_k \Delta_k$. Then, by (3) of Proposition 3.3, we infer from Lemma 2.2 and Remark 2.3 that

$$\mathscr{I}(\varphi|_V) = \pi_* \left(\mathcal{O}_{\tilde{V}}\left(\sum a_k \Delta_k\right) \otimes \mathscr{I}(\varphi|_V \circ \pi)\right) = \pi_* \mathcal{O}_{\tilde{V}}\left(\sum a_k \Delta_k\right).$$

Since H has only rational singularities, it follows that $\mathscr{I}(\varphi|_H) = \mathcal{O}_H$, which implies $a_k \geq 0$, *i.e.*, (H, x) is canonical.

On the contrary, if (H, x) is a canonical singularity, then there exists a desingularization $\pi : \tilde{V} \to V$ of V near x such that $K_{\tilde{V}} = \pi^* K_V + \sum a_k \Delta_k,\ a_k \geq 0$, and

$$\mathscr{I}(\varphi|_V) = \pi_* \left(\mathcal{O}_{\tilde{V}}\left(\sum a_k \Delta_k\right) \otimes \mathscr{I}(\varphi|_V \circ \pi)\right) = \pi_* \mathcal{O}_{\tilde{V}}\left(\sum a_k \Delta_k\right)$$

as above. Since V_{sing} has codimension at least two, then it follows from $\pi_* \mathscr{M}_{\tilde{V}} = \mathscr{M}_V$ that $\mathscr{I}(\varphi|_V) = \mathcal{O}_V$. Therefore, $x \in H$ is a rational singularity. $\square$

Remark 4.4. With the same assumption on φ as above, we can also get the following short exact sequence by a similar discussion as in [12, Proposition 9.3.48]:

$$0 \longrightarrow \mathcal{O}_X(K_X) \longrightarrow \mathcal{O}_X(K_X + H) \otimes Adj_H(\varphi) \longrightarrow \pi_* \mathcal{O}_{\tilde{H}}(K_{\tilde{H}}) \longrightarrow 0,$$

where $\pi : \tilde{X} \to X$ is an embedded resolution of (X, H) such that the proper transform $\tilde{H} \subset \tilde{X}$ of H is non-singular.

A. Appendix

In [14], Takagi introduced the notion of (algebraic) adjoint ideal sheaves along closed subvarieties of higher codimension and studied its local properties by characteristic p methods. Thanks to the Ohsawa-Takegoshi-Manivel extension theorem, we will be able to generalize the analytic adjoint ideal sheaves to a higher codimension case as follows.

Let X be an $(n+r)$-dimensional complex manifold and $Y \subset X$ a closed complex subspace of codimension r, which is locally a complete intersection with $s = (s_1, \dots, s_r)$ a system of generators of $\mathscr{I}_Y$ near $x \in X$. Let $\varphi \in \mathrm{Psh}(X)$ such that $\varphi|_Y \not\equiv -\infty$ on every irreducible component of Y.

Definition A.1 (Multiplier ideal sheaves on locally complete intersection). The *multiplier ideal sheaf* $\mathscr{I}(\varphi|_Y) \subset \mathcal{O}_Y$ on Y is defined to be the ideal sheaf of germs of holomorphic functions $f \in \mathcal{O}_{Y,x}$ such that $\frac{|f|^2}{|\Lambda^r(ds)|^2} e^{-2\varphi}$ is locally integrable at x on Y with respect to the Lebesgue measure.

Note that $\sigma := dz_1 \wedge \cdots \wedge dz_n \Big/ \frac{\partial(s_1,\dots,s_r)}{\partial(z_{n+1},\dots,z_{n+r})}$ is a nowhere vanishing holomorphic n-form on Y_{reg} near x. By a similar discussion of Proposition 3.3, we have the following

Proposition A.2. *Proposition* 3.3 *still holds if we replace the hypersurface H by a locally complete intersection subvariety $Y \subset X$.*

Furthermore, similar to Theorem 1.1, we can obtain:

Theorem A.3. *There exists an ideal sheaf*

$$Adj_Y(\varphi) \subset \mathcal{O}_X,$$

called the analytic adjoint ideal sheaf *associated to φ along Y, sitting in an exact sequence*

$$0 \longrightarrow \mathscr{I}(\varphi + r\log|\mathscr{I}_Y|) \stackrel{\iota}{\longrightarrow} Adj_Y(\varphi) \stackrel{\rho}{\longrightarrow} i_*\mathscr{I}(\varphi|_Y) \longrightarrow 0 \qquad (\star\star)$$

where $i : Y \hookrightarrow X$, ι and ρ are the natural inclusion and restriction maps respectively, and $\log|\mathscr{I}_Y| := \log|s| = \frac{1}{2}\log(|s_1|^2 + \cdots + |s_r|^2)$ *near every point $x \in X$.*

Proof. We may still assume that X is still a bounded Stein domain in $\mathbb{C}^n$ $(n \geq 2)$, $s = (s_1, \dots, s_r)$ is the system of generators of $\mathscr{I}_Y$ and $\mathscr{J} \subset \mathcal{O}_X$ is an ideal sheaf such that $\mathscr{J}|_Y = \mathscr{I}(\varphi|_Y)$, which implies that $\mathscr{J} + \mathscr{I}_Y$ is independent of the choice of $\mathscr{J}$.

Let

$$Adj_Y(\varphi) := \bigcup_{\varepsilon > 0} Adj_Y^0((1+\varepsilon)\varphi) \cap (\mathscr{J} + \mathscr{I}_Y),$$

where $\mathscr{I}_Y$ is the ideal sheaf of the locally complete intersection Y and $Adj_Y^0(\varphi) \subset \mathcal{O}_X$ is an ideal sheaf of germs of holomorphic functions $f \in \mathcal{O}_{X,x}$ such that

$$\frac{|f|^2 e^{-2\varphi}}{|s|^{2r}\log^2|s|}$$

is locally integrable with respect to the Lebesgue measure near x on X. Besides, $Adj_Y(\varphi)$ is independent of the choices of $\mathscr{J}$ and s. By a similar discussion to the proof of Theorem 1.1, it follows that the mapping ι and ρ are well-defined, and the sequence ($\star\star$) is exact. □

Remark A.4. By the same argument, it follows that $Adj_Y(\varphi) \subset \mathcal{O}_X$ is a coherent sheaf of ideals and

$$Adj_Y(\varphi) \cdot \mathcal{O}_Y = \mathscr{I}(\varphi|_Y).$$

Moreover, we can also infer that

$$Adj_H(\varphi) \subset \mathscr{I}(\varphi + (r-\delta)\log|s|)$$

for any $0 < \delta \leq r$.

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