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The vortex-wave system with gyroscopic effects

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Abstract. In this paper, we study the well-posedness for a coupled PDE/ODE system describing the interaction of several massive point vortices moving within a vorticity background in a 2D ideal incompressible fluid. The points are driven by the velocity induced by the background vorticity, by the other vortices, and by a Kutta-Joukowski-type lift force creating an additional gyroscopic effect. This system reduces to the so-called vortex-wave system, introduced by Marchioro and Pulvirenti [13, 14], when the point vortices are massless.

On the one hand, we establish existence of a weak solution before the first collision. We show moreover that the background vorticity is transported by the flow associated to the total velocity field. On the other hand, we establish uniqueness in the case where the vorticity is initially constant in a neighborhood of the point vortices. When all the densities of the point vortices have the same sign, no collision occurs in finite time and our results are then global in time. Our proofs strongly rely on the definition of a suitable energy functional.

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1. Introduction

The purpose of this article is to investigate the well-posedness of the following PDE/ODE system:

$$\begin{cases} \partial_t \omega + \operatorname{div}(v\omega) = 0 \\ v = u + \displaystyle\sum_{k=1}^{N} \frac{\gamma_k}{2\pi} \frac{(x-h_k)^\perp}{|x-h_k|^2} \quad u = K * \omega \quad K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} \\ m_k \ddot{h}_k = \gamma_k \Big(\dot{h}_k - u(t,h_k) - \displaystyle\sum_{j\neq k} \frac{\gamma_j}{2\pi} \frac{(h_k-h_j)^\perp}{|h_k-h_j|^2} \Big)^\perp \text{ for } k = 1,\dots,N, \end{cases} \tag{1.1}$$

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where

$$\omega : [0, T] \times \mathbb{R}^2 \to \mathbb{R}, \quad h_k : [0, T] \to \mathbb{R}^2 \text{ for } k = 1, \dots, N,$$

and where

$$(m_k, \gamma_k) \in \mathbb{R}_*^+ \times \mathbb{R} \text{ for } k = 1, \dots, N.$$

We supplement (1.1) with the initial conditions

$$\begin{aligned} &\omega(0, \cdot) = \omega_0 \in L^\infty(\mathbb{R}^2) \text{ compactly supported in some } B(0, R_0), \\ &(h_k, \dot{h}_k)(0) = (h_{k,0}, \ell_{k,0}) \text{ for } k = 1, \dots, N, \text{ with } h_{k,0} \text{ distincts.} \end{aligned} \tag{1.2}$$

System (1.1) for $N = 1$ was derived by Glass, Lacave and Sueur [7] as an asymptotical system for the dynamics of a body immersed in a 2D perfect incompressible fluid, when the size of the body vanishes whereas the mass is assumed to be constant. The position of the body at time t is represented by the position $h(t)$, the fluid is described by its divergence-free velocity $u(t, x)$ and vorticity $\omega(t, x) = \operatorname{curl} u(t, x)$. Under suitable decay assumptions, the divergence free condition enables to recover the velocity explicitly in terms of the vorticity by the Biot-Savart law [14]: $u = x^\perp/(2\pi|x|^2) * \omega$. The quantities m and γ are reminiscent of the mass of the body and of the circulation of the velocity around the body, respectively. The second order differential equation verified by h means that the body is accelerated by a force that is orthogonal to the difference between the body speed and the fluid velocity at this point. This gyroscopic force is similar to the well-known Kutta-Joukowski-type lift force revealed in the case of a single body in an irrotational unbounded flow, see for instance [10, 14, 20]. Therefore, a byproduct of [7] is the existence of a global weak solution of (1.1) when $N = 1$.

In the case $N > 1$, it is not known whether the previous convergence result holds. The main goal of this paper is to establish the existence and the uniqueness (under an additional assumption on the initial data, see below) of solutions for any $N \geqslant 1$. In particular, we will prove that the trajectories of the points h_k never collide if all the circulations γ_k have the same sign. Such a result is important for example to justify the 2D spray inviscid model established by Moussa and Sueur [16], which was derived as a mean-field limit $N \to \infty$ of (1.1). We refer to that article for a comparison of the recent spray models introduced in the literature.

Before giving the precise statements of our theorems, we mention that (1.1) reduces to the so-called vortex-wave system when setting $m_k = 0$:

$$\begin{cases} \partial_t \omega + \operatorname{div}(v\omega) = 0 \\ v = u + \displaystyle\sum_{k=1}^N \frac{\gamma_k}{2\pi} \frac{(x - h_k)^\perp}{|x - h_k|^2} \quad u = K * \omega \quad K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} \\ \dot{h}_k = u(t, h_k) + \displaystyle\sum_{j \neq k} \frac{\gamma_j}{2\pi} \frac{(h_k - h_j)^\perp}{|h_k - h_j|^2} \text{ for } k = 1, \dots, N. \end{cases} \tag{1.3}$$

And indeed, for $N = 1$, Glass, Lacave and Sueur showed in [8] that the asymptotical dynamics of a small solid with vanishing mass evolving in a 2D incompressible fluid is governed by the vortex-wave system. The vortex-wave system was previously derived by Marchioro and Pulvirenti [13, 14] to describe the interaction of a background vorticity ω interacting with one or several point vortices h_k with circulations γ_k. Very recently, Nguyen and Nguyen have also justified the vortex-wave system as the inviscid limit of the Navier-Stokes equations [17]. For System (1.3), existence of a weak solution (according to Definition 1.1 below) is proved up to the first collision time between the vortex trajectories. Concerning uniqueness, it is open in general, and it holds in the particular case when the vorticity ω is initially constant near the point vortices (namely the condition appearing in Theorem 1.5 below), as suggested in [13,14] and proved in [9,15]. It is also proved in [13] that if all the γ_k have the same sign then no collision occurs in finite time therefore global existence holds.

As for the spray model, these results are the first key to get a time of existence that is independent of N, in order to consider the homogenized limit (or mean-field limit) $N \to \infty$, for instance, used by Schochet [18] to justify the vortex method in $\mathbb{R}^2$. The main goal of this paper is to establish the corresponding existence and uniqueness results for the vortex-wave system with gyroscopic effects (1.1). From now on we will refer to the points h_k in (1.1) as "massive" point vortices.

Main results

The first part of our analysis focuses on the existence issue for (1.1).

Definition 1.1. Let $T > 0$. We say that $(\omega, \{h_k\}_{1 \leqslant k \leqslant N})$ is a *weak solution* of (1.1) on $[0, T]$, with initial data given by (1.2), if:

- $\omega \in L^\infty([0, T], L^1 \cap L^\infty(\mathbb{R}^2)) \cap C([0, T], L^\infty(\mathbb{R}^2) - w^*)$, $h_k \in C^2([0, T])$ for $k = 1, \dots, N$;
- The PDE in (1.1) is satisfied in the sense of distributions, and the ODEs in (1.1) are satisfied in the classical sense.

Theorem 1.2. *Let ω_0 and $(\{h_{k,0}\}, \{\ell_{k,0})\})$ be as in* (1.2). *There exists $T_* > 0$ such that for any $T \in (0, T_*)$, there exists a weak solution $(\omega, \{h_k\})$ to* (1.1) *on* $[0, T]$. *Moreover, if we assume that γ_k have the same sign for all $k = 1, \dots, N$, then $T_* = +\infty$.*

Remark. The maximal time T_* such that Theorem 1.2 holds corresponds to the first collision between some of the massive points, and we will prove that no collision occurs in finite time if all the γ_k have the same sign.

Remark. If the initial vorticity ω_0 was only assumed to be in $L^p_c(\mathbb{R}^2)$ for some $p > 2$, then one could still prove (global if all γ_k have the same sign) existence of a weak solution to (1.1) such that $\omega \in L^\infty(L^p)$. However in this case no uniqueness result is known.

As already said, the same existence result is known to hold for the vortex-wave system (1.3), see [13]. The proof of Theorem 1.2, given in Section 2, follows the same method as in [13], namely passing to the limit in an iterative scheme after establishing uniform estimates on the solution $(\omega_n, \{h_{k,n}\})$. To do so, we introduce a functional $\mathcal{H}_n$ in (2.9). This functional is well-adapted to System (1.1) because it controls both the minimal distance between the vortex trajectories and the velocities; moreover, it can be shown that its time derivative is uniformly bounded. Except for the estimates we perform for this new functional $\mathcal{H}_n$, the proof of Theorem 1.2 is quite straightforward and is not the main point of this paper.

Our next result is that any weak solution as in Theorem 1.2 is actually transported by the regular Lagrangian flow relative to the total velocity field. We refer to the recent papers [1–4] for the subsequent definition of regular Lagrangian flow:

Definition 1.3. Let $T > 0$ and let $v \in L^1_{\mathrm{loc}}([0, T] \times \mathbb{R}^2)$. We say that $X : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is a *regular Lagrangian flow* relative to v if:

- For a.e. $x \in \mathbb{R}^2$, the map $t \mapsto X(t, x)$ is an absolutely continuous solution to the ODE $\frac{d}{dt}X(t, x) = v(t, X(t, x))$ with $X(0, x) = x$, *i.e.*, a continuous function verifying $X(t, x) = x + \int_0^t v(s, X(s, x))\, ds$ for all $t \in [0, T]$;
- There exists a constant $L > 0$ independent of t such that
$$\mathcal{L}^2(X(t, \cdot)^{-1}(A)) \leqslant L\mathcal{L}^2(A), \quad \forall t \in [0, T], \forall A \text{ Borel set of } \mathbb{R}^2,$$
where $\mathcal{L}^2$ is the Lebesgue measure on $\mathbb{R}^2$.

Such a definition is intended to generalize the classical notion of flow associated to smooth vector fields. It was proved by Ambrosio [1] that such flow exists and is unique under BV space regularity for the vector field. In [4,9], a similar result was established for vector fields composed of a smooth part and of a part with a specific localized singularity that is explicit. In the present setting, where the total velocity field in (1.1) contains singularities created by the point vortices, we will rely on those last results to establish the following general result.

Theorem 1.4. *Let $\{h_k\}$ be any given maps belonging to $W^{2,\infty}([0, T]; \mathbb{R}^2)$ without collision:*
$$\min_{k \neq p} \min_{t \in [0,T]} |h_k(t) - h_p(t)| \geqslant \rho > 0.$$
For $\omega_0 \in L^\infty_c(\mathbb{R}^2)$, let ω be a weak solution on $[0, T]$ (in the sense of Definition 1.1) to
$$\begin{cases} \partial_t \omega + \operatorname{div}(v\omega) = 0 \\ v = u + \displaystyle\sum_{k=1}^{N} \frac{\gamma_k}{2\pi} \frac{(x - h_k)^\perp}{|x - h_k|^2} \quad u = K * \omega \quad K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \end{cases} \tag{1.4}$$
such that $\omega(0, \cdot) = \omega_0$. Then, there exists a unique regular Lagrangian flow X relative to the total velocity field v and ω is transported by this flow:
$$\omega(t, \cdot) = X(t, \cdot)_{\#}\omega_0.$$

Moreover, the vorticity $\omega(t,\cdot)$ is compactly supported in $B(0, R_T)$ for all $t \in [0,T]$, where R_T depends on T, on $\|h_k\|_{L^\infty([0,T])}$ and on the initial data.

Furthermore, we have the additional non collision information:

$$\text{for a.e. } x \in \mathbb{R}^2, \quad X(t,x) \neq h_k(t), \quad \forall t \in [0,T], \quad \forall k = 1,\dots,N.$$

Finally, if we assume

$$\omega_0 = \alpha_k \quad \text{on } B(h_k(0), \delta_0) \quad \forall k = 1,\dots,N \tag{1.5}$$

for some $\alpha_k \in \mathbb{R}$ and $\delta_0 > 0$, there exists a positive δ depending only on T, δ_0, $\|\omega_0\|_{L^\infty}$, $\|h_k\|_{W^{2,\infty}([0,T])}$ and R_0, such that

$$\omega(t,\cdot) = \alpha_k \quad \text{on } B(h_k(t), \delta), \quad \forall t \in [0,T].$$

We emphasize that Theorem 1.4 does not rely on the equation verified by the point vortices and thus it holds not only for (1.1) but also for any system (1.4).

We finally turn to the uniqueness issue of (1.2).

Theorem 1.5. *Let ω_0 and $(\{h_{k,0}\}, \{\ell_{k,0})\})$ be as in (1.2). Assume moreover that*

$$\omega_0 = \alpha_k \quad \text{on } B(h_{k,0}, \delta_0), \quad \forall k = 1,\dots,N$$

for some $\alpha_k \in \mathbb{R}$ and $\delta_0 > 0$. Then for any $T > 0$, there exists at most one weak solution $(\omega, \{h_k\})$ to (1.1) on $[0,T]$ with this initial condition.

The proof of Theorem 1.5 is a straightforward adaptation of the uniqueness proof given for the vortex-wave system in [9] when the vorticity is constant for all time in the neighborhood of the point vortices. Hence the main difficulty in order to prove uniqueness under Assumption (1.5) is to prove the last point of Theorem 1.4.

Theorem 1.5 together with Theorem 1.2 thus implies global existence and uniqueness if all the γ_k have the same sign, and existence and uniqueness up to the first collision otherwise.

The plan of this paper is as follows. In the next section, we prove Theorem 1.2 after collecting a few well-known properties. Then in Section 3 we establish Theorem 1.4. Finally, in Section 4 we show how it implies Theorem 1.5 by adapting the arguments of [9,15]. For simplicity we focus for this on the case of one point, but the case of $N \geqslant 1$ points is similar. The last section is devoted to some additional properties satisfied by solutions of System (1.1).

With respect to the above-mentioned previous works, the main novelty for the proofs here is the use of a new local energy functional

$$F_k(t) = \sum_{j=1}^{N} \frac{\gamma_j}{2\pi} \ln|X(t,x) - h_j(t)| + \varphi(t, X(t,x)) + \langle X(t,x), \dot{h}_k^{\perp}(t)\rangle, \tag{1.6}$$

defined as long as $X(t,x) \neq h_j(t)$, where φ is the stream function associated to u (namely $u = \nabla^\perp \varphi$, see (2.2)). It turns out that the two last terms in the definition

(1.6) are uniformly bounded. Hence controlling the distances between the fluid particles and the massive point vortices (thus controlling the behavior of $\omega(t,\cdot)$ near those points) is made possible by proving that $F_k(t)$ is bounded. In the case of one point vortex, the following formal computation on the derivative of $F(t) := F_1(t)$ shows that the most singular terms cancel, which motivates our definition (1.6)[1]:

$$\begin{aligned} F' =& \left\langle \frac{\gamma}{2\pi}\frac{X-h}{|X-h|^2}, u(t,X) + \gamma K(X-h) - \dot{h} \right\rangle \\ &+ \partial_t\varphi(t,X) + \langle \dot{X}, \nabla\varphi(t,X)\rangle + \langle \dot{X}, \dot{h}^\perp\rangle + \langle X, \ddot{h}^\perp\rangle \\ =& \left\langle \frac{\gamma}{2\pi}\frac{X-h}{|X-h|^2}, u(t,X) - \dot{h} \right\rangle + \partial_t\varphi(t,X) \\ &- \langle \dot{X}, u^\perp(t,X)\rangle + \langle \dot{X}, \dot{h}^\perp\rangle + \langle X, \ddot{h}^\perp\rangle. \end{aligned}$$

Since

$$\frac{\gamma}{2\pi}\frac{X-h}{|X-h|^2} = -\dot{X}^\perp + u^\perp(t,X),$$

we observe that the singular terms in the previous expression actually cancel. Finally, we get

$$F'(t) = -\langle u^\perp(t,X), \dot{h}\rangle + \partial_t\varphi(t,X) + \langle X, \ddot{h}^\perp\rangle.$$

Thus it only remains to notice that this expression only involves bounded terms so that $|F'(t)| \leqslant C$ on $[0,T]$, as wanted. The rigorous proof of this bound for several points will be established in Proposition 3.3.

Notations. From now on C will refer to a constant depending only on T, on ρ, on $\|h_k\|_{W^{2,\infty}([0,T])}$, and on the initial data ($R_0, m_k, \gamma_k, h_{k,0}, \ell_{k,0}$ and $\|\omega_0\|_{L^\infty}$), but not on δ_0. It will possibly changing value from one line to another.

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2. Proof of Theorem 1.2

2.1. Some general regularity properties

We start with the following well-known property, see [14, Appendix 2.3] for instance.

[1] We set $X = X(t,x)$ and $h = h(t)$ for more clarity.

Proposition 2.1. *Let $\Omega \in L^\infty([0,T], L^1 \cap L^\infty(\mathbb{R}^2))$. Let $U = K * \Omega$. Then we have*

$$\|U\|_{L^\infty} \leqslant C\|\Omega\|^{1/2}_{L^\infty(L^1)}\|\Omega\|^{1/2}_{L^\infty}.$$

Moreover, U is log-Lipschitz uniformly in time:

$$\|U(\cdot,x) - U(\cdot,y)\|_{L^\infty(0,T)} \leqslant C(\|\Omega\|_{L^\infty(L^1\cap L^\infty)})|x-y|(1+|\ln|x-y||),$$
$$\forall (x,y) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

We also have the Calderón-Zygmund inequality [19, Chapter II, Theorem 3]

Proposition 2.2. *There exists C such that for all $p \geqslant 2$*

$$\|\nabla U(t,\cdot)\|_{L^p} \leqslant Cp\|\Omega(t,\cdot)\|_{L^p} \quad \textit{for all } t \in [0,T].$$

In particular, it follows that any such velocity field satisfies

$$U \in L^\infty([0,T]\times\mathbb{R}^2)) \cap L^\infty([0,T], W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)), \quad \mathrm{div}(U) = 0. \tag{2.1}$$

2.2. Some basic properties for weak solutions of (1.1)-(1.2)

In all this paragraph, $(\omega, \{h_k\})$ denotes a weak solution of (1.1)-(1.2) on $[0,T]$, so that in particular u satisfies Proposition 2.1 and the regularity property (2.1). We assume *moreover* that $\omega(t,\cdot)$ is compactly supported in some $B(0,R)$ for all $t \in [0,T]$.

We introduce the stream function

$$\varphi(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y|\omega(t,y)\,dy, \tag{2.2}$$

so that

$$u(t,x) = \nabla^\perp\varphi(t,x).$$

For the subsequent computations, in order to make the arguments rigorous, we introduce a regularized version of the stream function: for $\varepsilon > 0$ and $\ln_\varepsilon$ a smooth function coinciding with ln on $[\varepsilon, +\infty)$ and satisfying $|\ln'_\varepsilon(r)| \leqslant C/\varepsilon$ for all $r > 0$, we set

$$\varphi_\varepsilon(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln_\varepsilon|x-y|\omega(t,y)\,dy. \tag{2.3}$$

Note that by assumption on the support of $\omega(t,\cdot)$ the following estimate holds for φ_ε:

$$|\varphi_\varepsilon(t,x)| \leqslant C\ln(2+|x|), \tag{2.4}$$

with C also independent of ε.

The following bound will be useful in order to establish a bound on the local energies in Proposition 3.3:

Proposition 2.3. *There exists C_R depending only on R, $\|\omega\|_{L^\infty}$, $\|h_k\|_{L^\infty}$, and the initial data, such that*

$$\|\partial_t \varphi_\varepsilon\|_{L^\infty} \leqslant C_R.$$

Proof. Using the weak formulation for ω in (1.1), we have

$$\begin{aligned}
&\partial_t \varphi_\varepsilon(t,x)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln_\varepsilon'(|x-y|)\frac{y-x}{|y-x|}\cdot\left(u(t,y)+\sum_{k=1}^N \gamma_k K(y-h_k(t))\right)\omega(t,y)\,dy,
\end{aligned}$$

therefore

$$\begin{aligned}
|\partial_t \varphi_\varepsilon(t,x)| \leqslant & C\|u\|_{L^\infty}\int_{\mathbb{R}^2}\frac{|\omega(t,y)|}{|x-y|}\,dy\\
&+ C\sum_{k=1}^N \int_{\mathbb{R}^2}\left|\frac{(x-y)\cdot(y-h_k(t))^\perp}{|x-y|^2|y-h_k(t)|^2}\omega(t,y)\right|\,dy.
\end{aligned}$$

By the estimates (1.39) to (1.43) in [12], there exists a constant C depending only on R, on $\|\omega\|_{L^\infty}$, and on $\|h_k\|_{L^\infty}$, such that

$$|\partial_t \varphi_\varepsilon(t,x)| \leqslant C_R.$$

The conclusion follows. □

In the previous computation, we needed the smoothness of $\ln_\varepsilon$ in order to use the weak formulation for ω. This explains why we have to replace φ in the definition of F_k (1.6) by φ_ε (see (3.5)) when we compute the derivative.

In the following proposition we state that $\nabla\varphi_\varepsilon$ approaches uniformly $\nabla\varphi$.

Proposition 2.4. *We have*

$$\nabla^\perp \varphi_\varepsilon = u + R_\varepsilon,$$

where R_ε satisfies

$$\|R_\varepsilon\|_{L^\infty} \leqslant \varepsilon C\|\omega\|_{L^\infty}.$$

Proof. We have

$$\begin{aligned}
\nabla^\perp\varphi_\varepsilon(t,x) &= u(t,x) + \int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\left(|x-y|\ln_\varepsilon'|x-y|-1\right)\omega(t,y)\,dy\\
&= u(t,x) + R_\varepsilon(t,x),
\end{aligned}$$

where

$$|R_\varepsilon(t,x)| \leqslant C\int_{|y-x|\leqslant\varepsilon}\frac{1}{|x-y|}|\omega(t,y)|\,dy \leqslant C\|\omega(t,\cdot)\|_{L^\infty}\varepsilon.$$ □

2.3. Proof of Theorem 1.2

The proof of Theorem 1.2 is divided into two steps.

Step 1: iterative scheme. Let $\rho \in (0, \min_{k\neq p} |h_{k,0} - h_{p,0}|)$ which will be fixed later. We consider the following iterative scheme: for $n \in \mathbb{N}^*$, given

$$\omega_{n-1} \in L^\infty([0, T_{n-1}], L^1 \cap L^\infty(\mathbb{R}^2)),$$

and given N trajectories $h_{k,n-1}$ in $C^2([0, T_{n-1}])$ such that

$$\min_{t\in[0,T_{n-1}]} \min_{k\neq p} |h_{k,n-1}(t) - h_{p,n-1}(t)| > 0,$$

for some $T_{n-1} > 0$, we set

$$u_{n-1} = K * \omega_{n-1},$$

having in mind to solve the linear PDE

$$\begin{cases} \partial_t \omega_n + \operatorname{div}\left[\left(u_{n-1} + \displaystyle\sum_{k=1}^N \frac{\gamma_k}{2\pi} \frac{(x - h_{k,n-1}(t))^\perp}{|x - h_{k,n-1}(t)|^2}\right)\omega_n\right] = 0 \\ \omega_n(0) = \omega_0, \end{cases} \tag{2.5}$$

and the non linear system of ODEs: for $k = 1, \dots, N$,

$$\begin{cases} \ddot{h}_{k,n}(t) = \dfrac{\gamma_k}{m_k}\left(\dot{h}_{k,n}(t) - u_{n-1}(t, h_{k,n}(t)) - \displaystyle\sum_{j\neq k} \frac{\gamma_j}{2\pi} \frac{(h_{k,n}(t) - h_{j,n}(t))^\perp}{|h_{k,n}(t) - h_{j,n}(t)|^2}\right)^\perp \\ (h_{k,n}(0), \dot{h}_{k,n}(0)) = (h_{k,0}, \ell_{k,0}), \end{cases} \tag{2.6}$$

on $[0, T_n]$, where $T_n \in (0, T_{n-1}]$ will be chosen such that

$$\min_{t\in[0,T_n]} \min_{k\neq p} |h_{k,n}(t) - h_{p,n}(t)| \geqslant \rho. \tag{2.7}$$

For $n = 0$ we take ω_0 and $(h_{k,0}, \ell_{k,0})$ as data (with $T_0 = +\infty$).

Proposition 2.5. *For all $n \in \mathbb{N}$, there exist $T_n \in (0, T_{n-1}]$ and a unique weak solution ω_n to* (2.5) *and $\{h_{k,n}\}$ to* (2.6) *on $[0, T_n]$ such that* (2.7) *is satisfied.*

Moreover,

$$\|\omega_n(t, \cdot)\|_{L^1\cap L^\infty} \leqslant \|\omega_0\|_{L^1\cap L^\infty} \quad \forall t \in [0, T_n]$$

and there exists $\widetilde{T}$ depending only on ρ, $h_{k,0}$, $\ell_{k,0}$, R_0 and $\|\omega_0\|_{L^\infty}$ such that $T_n \geqslant \widetilde{T}$ for all n.

Finally, if all the γ_k have the same sign, then for any $T > 0$, one can choose ρ depending on T (and on $h_{k,0}$, $\ell_{k,0}$ and $\|\omega_0\|_{L^\infty}$, R_0) such that $T_n = T$ for all $n \in \mathbb{N}$.

Proof. Given $(\omega_{n-1}, \{h_{k,n-1}\})$ satisfying the bound of Proposition 2.5, we solve the *linear* transport equation (2.5) with initial data ω_0 and velocity field given by

$$v_{n-1}(t,x) = u_{n-1}(t,x) + \sum_{k=1}^{N} \frac{\gamma_k}{2\pi} \frac{(x - h_{k,n-1}(t))^{\perp}}{|x - h_{k,n-1}(t)|^2}.$$

The existence of such a weak solution $\omega_n \in L^\infty([0, T_{n-1}], L^1 \cap L^\infty(\mathbb{R}^2))$ follows from classical arguments on linear transport equation. For the uniqueness issue, we refer to Lemma A.1 (derived from [15, Chapter 1]), which proves that any field v_{n-1} given as above, with u_{n-1} satisfying the regularity property (2.1) and with the maps $h_{k,n-1}$ Lipschitz continuous and not intersecting on $[0, T_{n-1}]$, has the renormalization property (see [5, Definition 1.5] for the definition of renormalization). By the usual arguments for linear transport equations, see [6], uniqueness therefore holds in $L^\infty([0, T_{n-1}], L^1 \cap L^\infty(\mathbb{R}^2))$ for the linear transport equation associated to v_{n-1}.

Moreover, it follows from Corollary A.2 in the Appendix that the norms $\|\omega_n(t,\cdot)\|_{L^p}$ are constant in time for all p, therefore we get the desired bound for $\|\omega_n\|_{L^1\cap L^\infty}$. Recalling Proposition 2.1, it follows that $\|u_n\|_{L^\infty} \leqslant C$. Furthermore, the weak time continuity for ω_n established in [9, Proposition 4.1] (see also [15]) implies that u_n is uniformly continuous in space-time.

Next, in view of the almost-Lipschitz property and the time regularity for u_{n-1}, Osgood's lemma ensures that there exists a unique solution $\{h_{k,n}\}$ to (2.6) on some maximal open interval $I_n \subset [0, T_{n-1}]$ such that

$$\min_{k\neq p} |h_{k,n}(t) - h_{p,n}(t)| > 0, \quad \forall t \in I_n.$$

We consider then $T_n \leqslant T_{n-1}$ such that $[0, T_n) \subset I_n$ and T_n is the largest time for which (2.7) holds:

$$\min_{k\neq p} |h_{k,n}(t) - h_{p,n}(t)| > \rho, \quad \forall t \in [0, T_n).$$

Taking the scalar product of (2.6) with $\dot{h}_{k,n}(t)$ and using Proposition 2.1 and the lower bound (2.7), we get on $[0, T_n]$:

$$m_k \frac{d|\dot{h}_{k,n}(t)|^2}{dt} \leqslant C|\dot{h}_{k,n}(t)| \leqslant C|\dot{h}_{k,n}(t)|^2 + 1$$

hence we deduce by Gronwall that

$$|\dot{h}_{k,n}(t)| \leqslant C \quad \text{on } [0, \min(T_n, 1)], \tag{2.8}$$

(where we emphasize that C depends on ρ), so

$$T_n \geqslant \widetilde{T} := \min\Big(1, \frac{\min_{k\neq p} |h_{k,0} - h_{p,0}| - \rho}{2C}\Big).$$

It remains to study the case where all γ_k have the same sign (say positive), where we have to derive an inequality like (2.8) which is independent of ρ. We fix $T > 0$ and we assume that $T_{n-1} = T$. We want to show that $T_n = T$. In the sequel of this proof, C depends only on the initial data and T. We seek for a uniform lower bound for the distances $|h_{k,n} - h_{p,n}|$ and for a uniform upper bound for $|\dot{h}_{k,n}|$ on $[0, T_n)$. To this aim, we introduce the quantity

$$\mathcal{H}_n(t) = \sum_{j\neq k} \frac{\gamma_j\gamma_k}{2\pi} \ln|h_{j,n}(t) - h_{k,n}(t)| - \sum_{k=1}^{N} m_k|\dot{h}_{k,n}(t)|^2, \tag{2.9}$$

defined on $[0, T_n]$. As we shall see below, bounding $|\mathcal{H}_n|$ uniformly with respect to n allows to obtain the desired bounds on $|\dot{h}_{k,n}|$ and on $|h_{k,n} - h_{p,n}|$. In order to obtain a suitable estimate on $|\mathcal{H}_n|$, we compute the time derivative:

$$\begin{aligned}\dot{\mathcal{H}}_n &= \sum_{j\neq k} \frac{\gamma_j\gamma_k}{2\pi}(\dot{h}_{k,n} - \dot{h}_{j,n}) \cdot \frac{h_{k,n} - h_{j,n}}{|h_{k,n} - h_{j,n}|^2} - 2\sum_{k=1}^{N} m_k\dot{h}_{k,n} \cdot \ddot{h}_{k,n}\\ &= 2\sum_{k=1}^{N} \dot{h}_{k,n} \cdot \left(\gamma_k \sum_{j\neq k} \frac{\gamma_j}{2\pi} \frac{h_{k,n} - h_{j,n}}{|h_{k,n} - h_{j,n}|^2} - m_k\ddot{h}_{k,n}\right),\end{aligned}$$

where we have exchanged j and k in order to pass from the first line to the second one. Thus by (2.6), it only remains:

$$\dot{\mathcal{H}}_n(t) = 2\sum_{k=1}^{N} \gamma_k\dot{h}_{k,n}(t) \cdot u_{n-1}(t, h_{k,n}(t))^{\perp}.$$

Using the bound $\|u_{n-1}\|_\infty \leqslant C$ we get

$$|\dot{\mathcal{H}}_n(t)| \leqslant C\sum_{k=1}^{N} \gamma_k|\dot{h}_{k,n}(t)|. \tag{2.10}$$

On the other hand, we notice that for all k, for all $t \in [0, T_n]$, using that $\ln|x - y| \leqslant |x| + |y|$ we have

$$\begin{aligned}|\dot{h}_{k,n}(t)| \leqslant& C|\mathcal{H}_n(t)|^{1/2} + C\left(\sum_{j=1}^{N} \gamma_j|h_{j,n}(t)|\right)^{1/2}\\ \leqslant& C|\mathcal{H}_n(t)|^{1/2} + C\left(\sum_{j=1}^{N} \gamma_j|h_{j,n}(0)| + \sum_{j=1}^{N} \gamma_j \int_0^t |\dot{h}_{j,n}(\tau)|\,d\tau\right)^{1/2}\\ \leqslant& C|\mathcal{H}_n(t)|^{1/2} + C\left(\sum_{j=1}^{N} \gamma_j|h_{j,0}| + \max_k \max_{[0,T_n]} |\dot{h}_{k,n}|\right)^{1/2},\end{aligned}$$

hence

$$\begin{aligned}\max_k \max_{[0,T_n]} |\dot{h}_{k,n}| \leqslant & C \max_{[0,T_n]} |\mathcal{H}_n|^{1/2} + C\sqrt{1 + \max_k \max_{[0,T_n]} |\dot{h}_{k,n}|} \\ \leqslant & C \max_{[0,T_n]} |\mathcal{H}_n|^{1/2} + C + \frac{\max_k \max_{[0,T_n]} |\dot{h}_{k,n}|}{2},\end{aligned}$$

where we have used that $C\sqrt{1+a} \leqslant C\sqrt{2} + C^2 + a/2$ for $a > 0$. Therefore

$$\max_k \max_{[0,T_n]} |\dot{h}_{k,n}| \leqslant C \max_{[0,T_n]} |\mathcal{H}_n|^{1/2} + C. \tag{2.11}$$

Inserting (2.11) in (2.10) we also obtain

$$\max_{[0,T_n]} |\dot{\mathcal{H}}_n| \leqslant C \left(1 + \max_{[0,T_n]} |\mathcal{H}_n|^{1/2}\right),$$

therefore we get

$$\max_{[0,T_n]} |\mathcal{H}_n| \leqslant C \quad \text{and} \quad \max_{[0,T_n]} |\dot{\mathcal{H}}_n| \leqslant C.$$

Coming back to (2.11), it follows that

$$\max_k \max_{[0,T_n]} |\dot{h}_{k,n}| \leqslant C, \tag{2.12}$$

so that

$$\max_k \max_{[0,T_n]} |h_{k,n}| \leqslant C.$$

Finally, by the definition of $\mathcal{H}_n(t)$ and by the previous bounds, using again that $\ln|x-y| \leqslant |x| + |y|$ we have for all $j \neq k$:

$$\begin{aligned}\gamma_j\gamma_k \ln|h_{j,n}(t) - h_{k,n}(t)| \geqslant & -2\pi|\mathcal{H}_n(t)| - \sum_{p,\ell=1}^{N} \gamma_p\gamma_\ell \left(|h_{p,n}(t)| + |h_{\ell,n}(t)|\right) \\ \geqslant & -C,\end{aligned}$$

which means that there exists $\rho > 0$ depending only on T and the initial data such that

$$\min_{j\neq k} |h_{j,n}(t) - h_{k,n}(t)| > \rho > 0, \quad \forall t \in [0, \min(T_n, T)).$$

Choosing this ρ from the beginning, we conclude that $T_n = T$, and that the proposition is proved. □

Step 2: Passing to the limit. We only sketch the subsequent arguments. By the previous estimates, extracting if necessary, we find that $\{\omega_n\}_{n\in\mathbb{N}}$ converges to some ω in L^∞ weak-$*$ on $[0, \widetilde{T}] \times \mathbb{R}^2$. Moreover, setting $u = K * \omega$, we infer that

$\{u_n\}_{n\in\mathbb{N}}$ converges to u locally uniformly on $[0, \widetilde{T}] \times \mathbb{R}^2$ (see for instance [7, Sect. 6.1]). On the other hand, the bounds (2.7)-(2.8) (or (2.12)) imply that each sequence $\{\ddot{h}_{k,n}\}_{n\in\mathbb{N}}$ is uniformly bounded on $[0, \widetilde{T}]$. By Ascoli's theorem, extracting again if necessary, we obtain that each $\{(h_{k,n}, \dot{h}_{k,n})\}_{n\in\mathbb{N}}$ converges uniformly to some $(h_k, \dot{h}_k)$ on $[0, \widetilde{T}]$, and passing to the limit in (2.6), we see that the points $\{h_k\}$ satisfy the desired system of ODE in (1.1). Note that in particular that they satisfy

$$\max_k \max_{[0,\widetilde{T}]} |h_k| \leqslant C, \quad \min_{j\neq k} \min_{[0,\widetilde{T}]} |h_j - h_k| \geqslant \rho > 0 \tag{2.13}$$

and

$$\max_k \max_{[0,\widetilde{T}]} |\dot{h}_k| \leqslant C. \tag{2.14}$$

Finally, coming back to (2.5), we can pass to the limit exploiting the previous types of convergence to show that ω is a weak solution of the first PDE in (1.1) on $[0, \widetilde{T}]$.

Iterating this construction we reach existence up to the first time of collisions.

If all the circulations have the same sign, we take $T > 0$, and we can replace $\widetilde{T}$ by T in all the arguments above since for all n we have a solution ω_n and $\{h_{k,n}\}$ on $[0, T_n = T]$. This shows that no collision occurs in finite time.

3. Proof of Theorem 1.4

In all this section, ω denotes any weak solution of (1.4) on $[0, T]$, where $\{h_k\}$ are given trajectories belonging in $W^{2,\infty}([0, T])$. We assume the analog initial condition as (1.2):

$$\omega(0, \cdot) = \omega_0 \in L^\infty(\mathbb{R}^2) \text{ compactly supported in some } B(0, R_0). \tag{3.1}$$

We assume that no collision occurs, *i.e.*,

$$\min_{t\in[0,T]} \min_{k\neq p} |h_k(t) - h_p(t)| \geqslant \rho \tag{3.2}$$

for some $\rho > 0$. The purpose of this section is to show Theorem 1.4. We emphasize that the proof does not use the dynamics of $\{h_k\}$.

3.1. Regular Lagrangian flow

We show here that there exists a unique regular Lagrangian flow as defined in Definition 1.1.

Recall the general following abstract result by Ambrosio [1, Theorems 3.3 and 3.5]. Given a vector field v in $L^1_{\text{loc}}([0, T] \times \mathbb{R}^d)$, if existence and uniqueness for the continuity equation

$$\partial_t \omega + \operatorname{div}(v\omega) = 0, \quad \omega(0, \cdot) = \omega_0 \in L^1 \cap L^\infty$$

hold in $L^\infty([0,T], L^1 \cap L^\infty)$ then the regular Lagrangian flow X for v exists and is unique, and the unique solution is then given by $\omega(t,\cdot) = X(t,\cdot)_\# \omega_0$.

In order to apply this result to the present setting, we introduce the divergence-free field

$$v(t,x) = u(t,x) + \sum_{j=1}^{N} \gamma_j K(x - h_j(t)). \tag{3.3}$$

By Corollary A.2 in the Appendix, the transport equation associated to v admits a unique solution (which is renormalized by Lemma A.1: for any continuous function β growing not too fast at infinity, the function $\beta(\omega)$ is also a solution). Therefore Ambrosio's result yields the existence and uniqueness of the regular Lagrangian flow X associated to v, and we have $\omega(t,\cdot) = X(t,\cdot)_\# \omega_0$. This proves the first claim of Theorem 1.4.

Again by Corollary A.2, the renormalization property ensures that

$$\|\omega(t,\cdot)\|_{L^p} = \|\omega_0\|_{L^p}, \quad 1 \leqslant p \leqslant +\infty. \tag{3.4}$$

We first derive the following property:

Proposition 3.1. *There exists C depending only on T, on $\|h_k\|_{L^\infty([0,T])}$ and on the initial data such that*

$$\sup_{t\in[0,T]} |X(t,x)| \leqslant |x| + C, \quad \textit{for a.e. } x \in \mathbb{R}^2.$$

Proof. Let us introduce the set of tubular neighborhoods

$$\Sigma := \bigcup_{t\in[0,T]} \bigcup_{k=1}^{N} B(h_k(t), 1),$$

which is bounded set, let say included in a ball $B(0, C_1)$. Outside Σ, the velocity v is uniformly bounded by C_2 (see Proposition 2.1).

Therefore, for any $x \in \mathbb{R}^2$ such that $X(\cdot, x)$ is absolutely continuous on $[0,T]$, the map $t \to X(t,x)$ starts from x, has a Lipschitz variation outside Σ and can evolve with a diverging velocity inside Σ, but remaining bounded.

Thus, setting $C = C_1 + C_2 T$ proves the proposition. □

The following corollary gives the second point in Theorem 1.4.

Corollary 3.2. *The vorticity $\omega(t,\cdot)$ is compactly supported for all $t \in [0,T]$, with* $\operatorname{supp}(\omega(t,\cdot)) \subset B(0, R_T)$ *for some R_T depending only on the initial data, on $\|h_k\|_{L^\infty([0,T])}$ and on T.*

Proof. We have $\omega(t,\cdot) = X(t,\cdot)_\# \omega_0$ and ω_0 is compactly supported in $B(0,R_0)$, so it follows from Proposition 3.1 that $\omega(t,\cdot)$ is compactly supported for all $t \in [0,T]$, with $\operatorname{supp}(\omega(t,\cdot)) \subset B(0,R_T)$ for $R_T = R_0 + C$ (with C given in Proposition 3.1) □

3.2. Vorticity trajectories

For the third point in Theorem 1.4, we have to show that for almost every $x \in \mathbb{R}^2$, we have $X(t,x) \neq h_k(t)$ for all $t \in [0,T]$ and for all $k = 1, \dots, N$.

For almost every $x \in \mathbb{R}^2 \setminus \cup_j \{h_{j,0}\}$ such that $X(\cdot, x)$ is an absolutely continuous solution on $[0,T]$ to the ODE with field v defined in (3.3), by time continuity, there exists $T^*(x)$ such that

$$\min_j \min_{t \in [0,T^*(x)]} |X(t,x) - h_j(t)| > 0.$$

We may then consider the local microscopic energies near the points $h_k(t)$ on $[0, T^*(x)]$:

$$F_k(t) = \sum_{j=1}^N \frac{\gamma_j}{2\pi} \ln |X(t,x) - h_j(t)| + \varphi_\varepsilon(t, X(t,x)) + \langle X(t,x), \dot{h}_k^\perp(t)\rangle, \quad (3.5)$$

where we recall that φ_ε denotes the regularization of the stream function, see (2.3).

On the other hand, the result in [9, Proposition 4.1] states the continuity of u on $[0,T] \times \mathbb{R}^2$. Therefore, the field $v(\cdot, X(\cdot, x))$ is continuous on $[0, T^*(x)]$. So we infer that $X(\cdot, x)$ is differentiable on $[0, T^*(x)]$ with $\frac{d}{dt} X(t,x) = v(t, X(t,x))$. This enables to perform the following estimate on the local energies.

Proposition 3.3. *We have for $t \in [0, T^*(x)]$ and for all $k = 1, \dots, N$,*

$$|F_k'(t)| \leqslant C\Big(1 + |x| + \varepsilon |X(t,x) - h_k(t)|^{-1} + \sum_{j \neq k} |X(t,x) - h_j(t)|^{-1}\Big).$$

In the previous statement, C is independent of ε, whereas F_k depends on ε.

Proof. In the subsequent proof we set for clarity:

$$X = X(t,x), \quad u = u(t, X(t,x)), \quad \varphi_\varepsilon = \varphi_\varepsilon(t, X(t,x)), \quad h_k = h_k(t),$$

and we compute on $[0, T^*(x)]$

$$\begin{aligned}
F_k' &= \sum_{j=1}^N \left\langle \frac{\gamma_j}{2\pi} \frac{X - h_j}{|X - h_j|^2}, \sum_{m=1}^N \gamma_m K(X - h_m) + u - \dot{h}_j \right\rangle \\
&\quad + \partial_t \varphi_\varepsilon + \langle \dot{X}, \nabla \varphi_\varepsilon \rangle + \langle \dot{X}, \dot{h}_k^\perp \rangle + \langle X, \ddot{h}_k^\perp \rangle \\
&= \left\langle \sum_{j=1}^N \frac{\gamma_j}{2\pi} \frac{X - h_j}{|X - h_j|^2}, \sum_{m=1}^N \frac{\gamma_m}{2\pi} \frac{(X - h_m)^\perp}{|X - h_m|^2} \right\rangle + \sum_{j=1}^N \left\langle \frac{\gamma_j}{2\pi} \frac{X - h_j}{|X - h_j|^2}, u - \dot{h}_j \right\rangle \\
&\quad + \partial_t \varphi_\varepsilon + \langle \dot{X}, \nabla \varphi_\varepsilon \rangle + \langle \dot{X}, \dot{h}_k^\perp \rangle + \langle X, \ddot{h}_k^\perp \rangle \\
&= \left\langle \frac{\gamma_k}{2\pi} \frac{X - h_k}{|X - h_k|^2}, u - \dot{h}_k \right\rangle + \sum_{j \neq k} \left\langle \frac{\gamma_j}{2\pi} \frac{X - h_j}{|X - h_j|^2}, u - \dot{h}_j \right\rangle \\
&\quad + \partial_t \varphi_\varepsilon + \langle \dot{X}, \nabla \varphi_\varepsilon \rangle + \langle \dot{X}, \dot{h}_k^\perp \rangle + \langle X, \ddot{h}_k^\perp \rangle.
\end{aligned}$$

Next, using again that X satisfies the ODE with field v defined in (3.3), we have

$$\frac{\gamma_k}{2\pi}\frac{X-h_k}{|X-h_k|^2} = -\dot{X}^\perp + u^\perp - \sum_{j\neq k}\frac{\gamma_j}{2\pi}\frac{X-h_j}{|X-h_j|^2}, \tag{3.6}$$

hence we get

$$\begin{aligned}
F_k' =& \langle -\dot{X}^\perp + u^\perp, u-\dot{h}_k\rangle - \sum_{j\neq k}\left\langle \frac{\gamma_j}{2\pi}\frac{X-h_j}{|X-h_j|^2}, u-\dot{h}_k\right\rangle \\
&+ \sum_{j\neq k}\left\langle \frac{\gamma_j}{2\pi}\frac{X-h_j}{|X-h_j|^2}, u-\dot{h}_j\right\rangle + \partial_t\varphi_\varepsilon + \langle \dot{X}, \nabla\varphi_\varepsilon\rangle + \langle \dot{X}, \dot{h}_k^\perp\rangle + \langle X, \ddot{h}_k^\perp\rangle \\
=& \langle \dot{X}^\perp, -u+\nabla^\perp\varphi_\varepsilon\rangle - \langle u^\perp, \dot{h}_k\rangle \\
&+ \sum_{j\neq k}\left\langle \frac{\gamma_j}{2\pi}\frac{X-h_j}{|X-h_j|^2}, \dot{h}_k-\dot{h}_j\right\rangle + \partial_t\varphi_\varepsilon + \langle X, \ddot{h}_k^\perp\rangle.
\end{aligned}$$

Hence, plugging the equality $\nabla^\perp\varphi_\varepsilon = u + R_\varepsilon$, with R_ε defined in Proposition 2.4, we have

$$F_k' = \langle \dot{X}^\perp, R_\varepsilon\rangle - \langle u^\perp, \dot{h}_k\rangle + \sum_{j\neq k}\left\langle \frac{\gamma_j}{2\pi}\frac{X-h_j}{|X-h_j|^2}, \dot{h}_k-\dot{h}_j\right\rangle + \partial_t\varphi_\varepsilon + \langle X, \ddot{h}_k^\perp\rangle.$$

By Proposition 2.4 together with (3.4) and (3.6), we have on the one hand

$$\left|\langle \dot{X}^\perp, R_\varepsilon\rangle\right| \leqslant C\varepsilon\|\omega_0\|_{L^\infty}\left(\sum_{j=1}^N |X-h_j|^{-1} + \|u\|_{L^\infty}\right).$$

On the other hand, as $h_k \in W^{2,\infty}$, we obtain by Proposition 3.1

$$\begin{aligned}
&\Big| -\langle u^\perp, \dot{h}_k\rangle + \sum_{j\neq k}\left\langle \frac{\gamma_j}{2\pi}\frac{X-h_j}{|X-h_j|^2}, \dot{h}_k-\dot{h}_j\right\rangle + \langle X, \ddot{h}_k^\perp\rangle\Big| \\
&\leqslant C\Big(\|u\|_{L^\infty} + \sum_{j\neq k}|X-h_j|^{-1} + |x| + 1\Big).
\end{aligned}$$

Finally, recalling that $\|\partial_t\varphi_\varepsilon\|_{L^\infty} \leqslant C$ by Proposition 2.3 and that $\|u\|_{L^\infty} \leqslant C$ by Proposition 2.1, the conclusion follows. □

Corollary 3.4. *For almost every x in $\mathbb{R}^2$ we can take $T^*(x) = T$, more precisely*

$$\min_j \min_{t\in[0,T]} |X(t,x) - h_j(t)| > 0, \quad \textit{for a.e. } x \in \mathbb{R}^2.$$

Proof. We argue by contradiction, assuming that $T^*(x) = T$ is impossible for some $x \in \mathbb{R}^2 \setminus \{h_{k,0}\}$ where the flow exists, so that there exist $k \in \{1, \dots, N\}$ and $\tilde{T} < T$ such that $\liminf_{t \to \tilde{T}} |X(t,x) - h_k(t)| = 0$ and $\min_j \min_{t \in [0,T^*]} |X(t,x) - h_j(t)| > 0$ for any $T^* < \tilde{T}$. We further set $X(t) = X(t,x)$. Let $t_n \to \tilde{T}$ such that $|X(t_n) - h_k(t_n)| \to 0$ as $n \to +\infty$. We recall that ρ is defined by (3.2). For n sufficiently large we have $|X(t_n) - h_k(t_n)| < \rho/K$, with $K > 3$ large to be determined later on. We take t_n' maximal such that on $[t_n, t_n')$ we have $|X(t) - h_k(t)| < \rho/3$. In particular, by (3.2), for $j \neq k$ we have $|X(t) - h_j(t)| \geqslant 2\rho/3$ on $[t_n, t_n')$. We assume first that $t_n' < \widetilde{T}$: then we have $|X(t_n') - h_k(t_n')| = \rho/3$. We fix $n \in \mathbb{N}$. For all $\varepsilon > 0$, by the definition of F_k, we write

$$\begin{aligned}\frac{\gamma_k}{2\pi} \ln\left(\frac{|X(t_n') - h_k(t_n')|}{|X(t_n) - h_k(t_n)|}\right) = \sum_{j \neq k}^{N} \frac{\gamma_j}{2\pi} \ln\left(\frac{|X(t_n) - h_j(t_n)|}{|X(t_n') - h_j(t_n')|}\right) + \int_{t_n}^{t_n'} F_k'(\tau)\, d\tau \\ + \varphi_\varepsilon(t_n, X(t_n)) - \varphi_\varepsilon(t_n', X(t_n')) \\ + \langle X(t_n), \dot{h}_k^\perp(t_n)\rangle - \langle X(t_n'), \dot{h}_k^\perp(t_n')\rangle,\end{aligned} \tag{3.7}$$

so by (2.4), Proposition 3.3 and the previous estimates we get

$$\frac{|\gamma_k|}{2\pi} \ln\left(\frac{K}{3}\right) \leqslant C\Big(1 + |x| + \varepsilon \int_{t_n}^{t_n'} |X(\tau) - h_k(\tau)|^{-1}\, d\tau\Big). \tag{3.8}$$

Letting $\varepsilon \to 0$ for fixed n, we find

$$\ln\left(\frac{K}{3}\right) \leqslant C,$$

which is a contradiction for K sufficiently large (depending on x and on the initial conditions). So we have $t_n' = \widetilde{T}$, hence

$$|X(t,x) - h_k(t)| < \frac{\rho}{3} \text{ and } |X(t,x) - h_j(t)| \geqslant \frac{2\rho}{3} \text{ on } [t_n, \widetilde{T}). \tag{3.9}$$

We have therefore localized the fluid trajectory $X(t)$ in the neighborhood of one point vortex trajectory $h_k(t)$, namely we have proved that if the trajectory goes too close to h_k, it stays in a neighborhood of radius $\rho/3$. We fix $n_0 \in \mathbb{N}$ sufficiently large so that $|X(t_{n_0}) - h_k(t_{n_0})| < \rho/K$. We come back to (3.7), replacing t_n' by any $t \in [t_n, \widetilde{T})$, and we apply again Proposition 3.3:

$$\ln|X(t) - h_k(t)| \geqslant \ln|X(t_{n_0}) - h_k(t_{n_0})| - C\Big(1 + |x| + \varepsilon \int_{t_{n_0}}^{t} |X(\tau) - h_k(\tau)|^{-1}\, d\tau\Big).$$

Letting $\varepsilon \to 0$ we find

$$|X(t) - h_k(t)| \geqslant |X(t_{n_0}) - h_k(t_{n_0})| e^{-C(1+|x|)} \text{ on } [t_{n_0}, \widetilde{T}),$$

which contradicts the fact that $\liminf_{t \to \widetilde{T}} |X(t,x) - h_k(t)| = 0$.

Hence we conclude that $T^*(x) = T$ is possible. □

Corollary 3.5. *For a.e. $x \in \mathbb{R}^2$, the map $X(\cdot, x)$ is the unique differentiable solution on $[0, T]$ of the ODE*

$$\frac{d}{dt}\gamma(t) = u(t, \gamma(t)) + \sum_{j=1}^{N} \gamma_j K(\gamma(t) - h_j(t)), \quad \gamma(0) = x,$$

such that $\min_j \min_{[0,T]} |\gamma(t) - h_j(t)| > 0$.

Proof. We gather the already mentioned time continuity of u, the log-Lipschitz space regularity for u stated in Proposition 2.1, the no collision property of Corollary 3.4, and the fact that K is Lipschitz away from the origin. Invoking Osgood's Lemma, we can then conclude. □

We finish this paragraph with an additional estimate on the Lagrangian trajectories, which can be derived easily from the proof of Corollary 3.4.

Proposition 3.6. *Let ω be any weak solution of* (1.4) *on* $[0, T]$ *with initial datum* (3.1)*, where $\{h_k\}$ are given trajectories in $W^{2,\infty}([0, T])$ satisfying the no collision property* (3.2)*. There exist $0 < \delta < \min(\rho/3, 1)$, $0 < \delta_1 < 1$ and $0 < \delta_2 < 1$, depending only on T, $\|h_k\|_{W^{2,\infty}([0,T])}$, ρ, R_0 and $\|\omega_0\|_{L^\infty}$, satisfying the following property:*

Let $x \in \operatorname{supp}(\omega_0)$ such that $\min_j \min_{t\in[0,T]} |X(t, x) - h_j(t)| > 0$.

If $|X(t_0, x) - h_k(t_0)| < \delta$ for some $t_0 \in [0, T]$ and $k \in \{1, \dots, N\}$, then

$$\delta_1 |x - h_k(0)| \leqslant |X(t, x) - h_k(t)| < \frac{\rho}{3}, \quad \forall t \in [0, T].$$

If $\min_j |X(t_0, x) - h_j(t_0)| > \delta$ *for some $t_0 \in [0, T]$, then*

$$\delta_2 \leqslant \min_j |X(t, x) - h_j(t)|, \quad \forall t \in [0, T].$$

Proof. We start with the first estimate. We come back to the proof of Corollary 3.4 above, with t_n replaced by t_0, t_n' replaced by T. With $K > 3$ a sufficiently large number to be chosen, we set $\delta = \rho/K$. By (3.8) we obtain, using that $|x| \leqslant R_0$ since x belongs to $\operatorname{supp}(\omega_0)$:

$$\frac{|\gamma_k|}{2\pi} \ln\left(\frac{K}{3}\right) \leqslant C\Big(1 + \varepsilon \int_{t_0}^{T} |X(\tau) - h_k(\tau)|^{-1}\, d\tau\Big).$$

Letting $\varepsilon \to 0$, we find a contradiction if K is sufficiently large (depending only on T, $\|h_k\|_{W^{2,\infty}([0,T])}$, R_0 and $\|\omega_0\|_{L^\infty}$). Hence by the same arguments as those leading to (3.9) we obtain:

$$|X(t, x) - h_k(t)| < \frac{\rho}{3}, \quad |X(t, x) - h_j(t)| \geqslant \frac{2\rho}{3} \text{ for } j \neq k \text{ on } [t_0, T].$$

We can invoke the same arguments to obtain the estimates above on $[0, t_0]$. Therefore, by Proposition 3.3, this yields:

$$\lim_{\varepsilon \to 0} \int_0^T |F_k'(\tau)|\, d\tau \leqslant C,$$

so that, using again (3.7) with t_n replaced by 0 and t_n' by t, we get

$$\ln |X(t,x) - h_k(t)| \geqslant \ln |x - h_k(0)| - C \text{ on } [0, T],$$

for a constant C, so the first part is proved by setting $\delta_1 = e^{-C}$.

We turn now to the second part. Let $\widetilde{K} \geqslant 1$ be a number to be determined later on. Let $(t_1, t_2) \subset (0, T)$ containing t_0 and be maximal such that $\min_j |X(t,x) - h_j(t)| > \delta/\widetilde{K}$ on (t_1, t_2). If $(t_1, t_2) \neq (0, T)$, let $k \in \{1, \dots, N\}$ such that $|X(t_1, x) - h_k(t_1)| = \delta/\widetilde{K}$ (or $|X(t_2, x) - h_k(t_2)| = \delta/\widetilde{K}$). Repeating the first part of the proof of Corollary 3.4 with $t_n = t_0$ and $t_n' = t_1$ (or $t_n' = t_2$), we find $|\ln \widetilde{K}| \leqslant C$ which is a contradiction provided $\widetilde{K}$ is sufficiently large (depending only on T, $\|h_k\|_{W^{2,\infty}([0,T])}$, R_0 and $\|\omega_0\|_{L^\infty}$). So, setting $\delta_2 = \delta/\widetilde{K}$, the conclusion follows. □

3.3. Decomposition of the vorticity and reduction to the case of one point vortex

In all this subsection we assume moreover that ω_0 is constant in a neighborhood of $\{h_k(0)\}$, namely that (1.5) holds. The purpose here is to show the last property of Theorem 1.4: the vorticity remains constant in a neighborhood of each point vortex. To this aim, we will first reduce the problem to the case of one single point vortex. In the next subsection, we will then establish the desired property.

Let δ, δ_1 and δ_2 be the constants introduced in Proposition 3.6. We decompose ω_0 as

$$\omega_0 = \sum_{k=1}^N \omega_{0,k} + \omega_{0,r},$$

where

$$\omega_{0,k} = \omega_0 \mathbb{1}_{B(h_k(0),\delta)}, \quad k = 1, \dots, N$$

and

$$\omega_{0,r} \quad \text{is supported in } \mathbb{R}^2 \setminus \cup_{j=1}^N B(h_j(0), \delta).$$

By uniqueness of the weak solution to the linear transport equation associated to the field

$$v(t,x) = u(t,x) + \sum_{j=1}^N \gamma_j K(x - h_j(t)),$$

(see the Appendix and the beginning of Subsection 3.1), ω may then be decomposed as

$$\omega(t,\cdot) = \sum_{k=1}^{N} X(t,\cdot)_{\#}\omega_{0,k} + X(t,\cdot)_{\#}\omega_{0,r} = \sum_{k=1}^{N} \omega_k(t,\cdot) + \omega_r(t,\cdot).$$

Let $K_\delta = 1/(2\pi)\nabla^\perp \ln_\delta$, where $\ln_\varepsilon$ is defined in Subsection 2.2. So K_δ is a smooth, divergence-free map coinciding with K on $\mathbb{R}^2 \setminus B(0,\delta)$ such that $\|K_\delta\|_{L^\infty} \leqslant C\delta^{-1}$.

Let $k = 1, \dots, N$. By the first part of Proposition 3.6, by definition of δ, we have

$$|X(t,x) - h_k(t)| < \frac{\rho}{3}, \quad \forall t \in [0,T], \quad \text{for a.e. } x \in \operatorname{supp}(\omega_{0,k}). \tag{3.10}$$

Therefore,

$$\min_{j\neq k} |X(t,x) - h_j(t)| \geqslant \frac{2\rho}{3} > \delta, \quad \forall t \in [0,T] \quad \text{for a.e. } x \in \operatorname{supp}(\omega_{0,k}).$$

So by Corollary 3.5, we have

$$X(t,x) = \widetilde{X}_k(t,x), \quad \text{for a.e. } x \in \operatorname{supp}(\omega_{0,k}), \tag{3.11}$$

where $\widetilde{X}_k$ is the unique regular Lagrangian flow associated to the field

$$\widetilde{v}_k(t,x) = u(t,x) + \sum_{j\neq k} \gamma_j K_\delta(x - h_j(t)) + \gamma_k K(x - h_k(t)). \tag{3.12}$$

In particular,

$$\omega_k(t,\cdot) = X(t,\cdot)_{\#}\omega_{0,k} = \widetilde{X}_k(t,\cdot)_{\#}\omega_{0,k}. \tag{3.13}$$

We observe here for later use that the same argument applied to ω^2 (noting that it is also a distributional solution of (1.4) with initial datum ω_0^2) yields

$$\omega_k^2(t,\cdot) = X(t,\cdot)_{\#}\omega_{0,k}^2 = \widetilde{X}_k(t,\cdot)_{\#}\omega_{0,k}^2, \quad k = 1, \dots, N. \tag{3.14}$$

So we are left with the case of a linear transport equation with field $\widetilde{v}_k$ given by the superposition (3.12) of a regular part

$$u_k(t,x) = u(t,x) + \sum_{j\neq k} \gamma_j K_\delta(x - h_j(t)) \tag{3.15}$$

and a singular part generared by only one point vortex:

$$\gamma_k K(x - h_k(t)).$$

The analysis of this case was performed in [4]. It was proved in particular that for all $t \in [0,T]$, the regular Lagrangian flow $\widetilde{X}_k$ associated to $\widetilde{v}_k$ is the limit in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$

of the sequence $\widetilde{X_{k,n}}(t,\cdot)$, where $\widetilde{X_{k,n}}$ is the flow associated to any regularization of $\widetilde{v}_k$:

$$\widetilde{v}_{k,n}(t,x) = u_{k,n}(t,x) + \frac{\gamma_k}{2\pi}\frac{(x-h_k(t))^\perp}{|x-h_k(t)|^2+n^{-2}},$$

with $u_{k,n}$ a smooth and divergence-free approximation of u_k. By Liouville's theorem, $\widetilde{X_{k,n}}(t,\cdot)$ thus preserves Lebesgue's measure. Moreover, Proposition 3.1 also applies to $\widetilde{X_{k,n}}$ (with a constant independent of n). Therefore, passing to the limit, we conclude that $\widetilde{X_k}(t,\cdot)$ preserves Lebesgue's measure:

$$\widetilde{X_k}(t,\cdot)_\# dx = dx, \quad \forall t \in [0,T]. \tag{3.16}$$

We next derive a localization property for $\widetilde{X_k}(t,\cdot)$.

Proposition 3.7. *For all $R > 0$, there exists C_R depending only on R, T, $\|h_k\|_{W^{2,\infty}([0,T])}$, ρ, R_0 and $\|\omega_0\|_{L^\infty}$, but not on δ_0, such that*

$$C_R|x-h_k(0)| \leqslant |\widetilde{X_k}(t,x)-h_k(t)|,$$
$$\forall t \in [0,T], \quad \textit{for all } x \in B(0,R)\setminus\{h_k(0)\}, \quad k=1,\dots,N.$$

Remark 3.8. By (3.11) and by Proposition 3.6, we already know that this holds for a.e. x in $\mathrm{supp}(\omega_{0,k})$.

Proof. As long as $\widetilde{X_k}(t,x) \neq h_k(t)$, we introduce the new energy

$$\begin{aligned}\widetilde{F_k}(t) = {}& \frac{\gamma_k}{2\pi}\ln|\widetilde{X_k}(t,x)-h_k(t)| \\ &+ \varphi_\varepsilon(t,\widetilde{X_k}(t,x)) + \psi_\delta(t,\widetilde{X_k}(t,x)) + \langle \widetilde{X_k}(t,x), \dot{h}_k(t)^\perp\rangle,\end{aligned}$$

where

$$\psi_\delta(t,x) = \sum_{j\neq k}\frac{\gamma_j}{2\pi}\ln_\delta|x-h_j(t)|,$$

so that

$$\nabla^\perp\psi_\delta(t,x) = \sum_{j\neq k}\gamma_j K_\delta(x-h_j(t)).$$

Exactly as in the proof of Proposition 3.3, we compute, recalling the definition (3.15) of u_k,

$$\begin{aligned}\widetilde{F_k}'(t) = {}& -\Big\langle u_k^\perp(t,\widetilde{X_k}(t,x)), \dot{h}_k(t)\Big\rangle + \partial_t\varphi_\varepsilon(t,\widetilde{X_k}(t,x)) + \partial_t\psi_\delta(t,\widetilde{X_k}(t,x)) \\ &+ \Big\langle \dot{\widetilde{X_k}}(t,x)^\perp, R_\varepsilon(t,\widetilde{X_k}(t,x))\Big\rangle + \Big\langle \widetilde{X_k}(t,x), \ddot{h}_k^\perp(t)\Big\rangle.\end{aligned}$$

Using the uniform bounds on u_k, $\partial_t\psi_\delta$, h_j, $\dot{h}_j$ and $\ddot{h}_j$ for $j=1,\dots,N$, and using the previous bounds for $\partial_t\varphi_\varepsilon$ and R_ε we therefore get for all $\varepsilon > 0$

$$|\widetilde{F_k}'(t)| \leqslant C\big(|x| + \varepsilon|\widetilde{X_k}(t,x)-h_k(t)|^{-1}+1\big).$$

We may now conclude exactly as in the proof of the first part of Proposition 3.6: as long as $\widetilde{X}_k(t,x) \neq h_k(t)$, letting ε tend to zero after integrating the inequality above on $[0,t]$, we get

$$\left|\ln\left(\frac{|\widetilde{X}_k(t,x)-h_k(t)|}{|x-h_{k,0}|}\right)\right| \leqslant C(1+|x|),$$

where C depends on δ, T, $\|h_k\|_{W^{2,\infty}([0,T])}$, ρ and on the initial data. So, setting $C_R = e^{-C(1+R)}$, the conclusion follows. □

Proposition 3.9. *We have*

$$|x-h_k(0)| \leqslant C(|\widetilde{X}_k(t,x)-h_k(t)|+1),$$

where C depends only on T, $\|h_k\|_{W^{2,\infty}([0,T])}$, ρ, R_0 and $\|\omega_0\|_{L^\infty}$, but not on δ_0.

Proof.

$$\begin{aligned}
&\frac{d}{dt}|\widetilde{X}_k(t,x)-h_k(t)|^2\\
&= 2\big\langle \widetilde{X}_k(t,x)-h_k(t), u_k(t,\widetilde{X}_k(t,x)) + \gamma_k K(\widetilde{X}_k(t,x)-h_k(t)) - \dot{h}_k(t)\big\rangle\\
&= 2\big\langle \widetilde{X}_k(t,x)-h_k(t), u_k(t,\widetilde{X}_k(t,x)) - \dot{h}_k(t)\big\rangle\\
&\geqslant -2|\widetilde{X}_k(t,x)-h_k(t)|(\|\dot{h}_k\|_{L^\infty}+\|u_k\|_{L^\infty}),
\end{aligned}$$

hence

$$\frac{d}{dt}|\widetilde{X}_k(t,x)-h_k(t)| \geqslant -C,$$

so the conclusion follows. □

3.4. The vorticity remains constant in the neighborhood of the point vortices

We finally establish that the vorticity remains constant in a neighborhood of the point vortices. Let C be the constant of Proposition 3.9. We set

$$R = 2C + |h_k(0)|,$$

and we consider the corresponding constant C_R of Proposition 3.7. We may decrease δ_0 so that

$$\max(\delta_0, C_R\delta_0) < \min\left(\delta, \delta_2, 1, \frac{2\rho}{3}\right),$$

where we recall δ and δ_2 were found in Proposition 3.6.

We fix $t \in [0,T]$. We claim that

$$\omega_j(t,y) = 0, \quad \text{for a.e. } y \in B(h_k(t), C_R\delta_0), \quad \forall j \neq k. \tag{3.17}$$

Indeed, by (3.14), considering the L^1 function $\varphi_k = \mathbb{1}_{B(h_k(t), C_R\delta_0)}$, we find

$$\int_{\mathbb{R}^2} \omega_j^2(t, y)\varphi_k(y)\, dy = \int_{\mathbb{R}^2} \omega_{0,j}^2(x)\varphi_k(X(t, x))\, dx.$$

On the other hand, for $x \in \operatorname{supp}(\omega_{0,j})$, we have by (3.10) $|X(t, x) - h_j(t)| < \rho/3$ and therefore $|X(t, x) - h_k(t)| > 2\rho/3 > C_R\delta_0$. So the right hand side above vanishes, which establishes (3.17).

Using that $\delta_2 > C_R\delta_0$, by the same arguments as above, the second part of Proposition 3.6 yields that

$$\omega_r(t, y) = 0, \quad \text{for a.e. } y \in \bigcup_{k=1}^{N} B(h_k(t), C_R\delta_0). \tag{3.18}$$

Finally, we show that

$$\omega_k(t, y) = \alpha_k, \quad \text{for a.e. } y \in B(h_k(t), C_R\delta_0). \tag{3.19}$$

Indeed, since $\omega_k(t, \cdot) = \widetilde{X}_k(t, \cdot)_\# \omega_{0,k}$, $\omega_k^2(t, \cdot) = \widetilde{X}_k(t, \cdot)_\# \omega_{0,k}^2$, and $\widetilde{X}_k(t, \cdot)_\# dx = dx$ by (3.13), (3.14) and (3.16), we compute

$$\begin{aligned}
&\int_{\mathbb{R}^2} (\omega_k(t, y) - \alpha_k)^2 \varphi_k(y)\, dy \\
&= \int_{\mathbb{R}^2} \omega_k(t, y)^2 \varphi_k(y)\, dy - 2\alpha_k \int_{\mathbb{R}^2} \omega_k(t, y)\varphi_k(y)\, dy + \alpha_k^2 \int_{\mathbb{R}^2} \varphi_k(y)\, dy \\
&= \int_{\mathbb{R}^2} \omega_{0,k}^2(x)\varphi_k(\widetilde{X}_k(t, x))\, dx - 2\alpha_k \int_{\mathbb{R}^2} \omega_{0,k}(x)\varphi_k(\widetilde{X}_k(t, x))\, dx \\
&\quad + \alpha_k^2 \int_{\mathbb{R}^2} \varphi_k(\widetilde{X}_k(t, x))\, dx \\
&= \int_{\mathbb{R}^2} (\omega_{0,k}(x) - \alpha_k)^2 \varphi_k(\widetilde{X}_k(t, x))\, dx \\
&= \int_{\widetilde{X}_k(t,\cdot)^{-1}(B(h_k(t), C_R\delta_0))} (\omega_{0,k}(x) - \alpha_k)^2\, dx.
\end{aligned}$$

Now we observe that since $C_R\delta_0 < 1$, by Proposition 3.9, we get

$$\widetilde{X}_k(t, \cdot)^{-1}\Big(B(h_k(t), C_R\delta_0)\Big) \subset B(h_k(0), 2C) \subset B(0, R).$$

Thus, we are allowed to use Proposition 3.7, and we have for $x \in \widetilde{X}_k(t, \cdot)^{-1}(B(h_k(t), C_R\delta_0))$:

$$C_R|x - h_k(0)| \leqslant |\widetilde{X}_k(t, x) - h_k(t)| \leqslant C_R\delta_0.$$

We get therefore

$$\int_{\mathbb{R}^2} (\omega(t, y) - \alpha_k)^2 \varphi_k(y)\, dy \leqslant \int_{B(h_k(0), \delta_0)} (\omega_0(x) - \alpha_k)^2\, dx = 0,$$

and the conclusion follows.

In view of (3.17), (3.18) and (3.19), we finally conclude that

$$\omega(t,\cdot) = \alpha_k, \quad \text{a.e. on } B(h_k(t), C_R\delta_0). \tag{3.20}$$

4. Proof of Theorem 1.5.

Step 1: uniqueness in the case of one point vortex.
We start with the case $N = 1$. Let (ω, h) and $(\widetilde{\omega}, \widetilde{h})$ two solutions of (1.1) with initial datum (ω_0, h_0, ℓ_0) satisfying the assumption of Theorem 1.5. So Theorem 1.4 holds for both solutions: ω and $\widetilde{\omega}$ remain constant in a neighborhood of the trajectories of h and $\tilde{h}$.

Noting that $u - \widetilde{u} = K * (\omega - \widetilde{\omega})$ with $\int(\omega - \widetilde{\omega}) = 0$ and $\omega, \widetilde{\omega}$ compactly supported, we have $u - \widetilde{u} \in L^2(\mathbb{R}^2)$ (see [11, Proposition 3.3]) and we may consider the quantity

$$D(t) = \|u(t,\cdot) - \widetilde{u}(t,\cdot)\|_{L^2}^2 + |h(t) - \widetilde{h}(t)|^2 + |\dot{h}(t) - \dot{\widetilde{h}}(t)|^2, \quad t \in [0, T].$$

In what follows we establish a Gronwall inequality for $D(t)$.

We remark that the only difference between (1.1) and the vortex-wave system (1.3) is the ODE for the point vortex, since the PDE for the vorticity is the same. Thus we may directly use the estimates derived for (1.3) in [9, Subsection 3.4] for the quantity $\|u(t,\cdot) - \widetilde{u}(t,\cdot)\|_{L^2}^2$. More precisely, by the estimate (3.9) in [9] we have for $t \in [0, T^*)$ and for all $p \geqslant 2$

$$\|u(t,\cdot) - \widetilde{u}(t,\cdot)\|_{L^2}^2 \leqslant C \int_0^t \left(r(\tau) + \sqrt{r(\tau)} f(\sqrt{r(\tau)}) + p\, r(\tau)^{1-1/p} \right) d\tau,$$

where

$$r(t) = \|u(t,\cdot) - \widetilde{u}(t,\cdot)\|_{L^2}^2 + |h(t) - \widetilde{h}(t)|^2,$$

and where

$$f(\tau) = \tau |\ln \tau|.$$

Here, $T^* \in [0, T]$ is the largest time such that $|h(t) - \widetilde{h}(t)| < \min(1, \delta/2)$ on $[0, T^*)$. So using that $r(t) \leqslant D(t)$, and the inequalities $\tau f(\tau) \leqslant f(\tau^2)$, $\tau \leqslant f(\tau)$ for $\tau \leqslant 1$ and $f(\tau) \leqslant p\tau^{1-1/p}$ (for all $p \geqslant 2$), we get for $t \in [0, T^*)$ and for all $p \geqslant 2$

$$\|u(t,\cdot) - \widetilde{u}(t,\cdot)\|_{L^2}^2 \leqslant C\, p \int_0^t D(\tau)^{1-1/p}\, d\tau. \tag{4.1}$$

We emphasize that the property obtained in Theorem 1.4 is crucial in order to obtain the previous estimate, by implying in particular that $u - \widetilde{u}$ is harmonic in the neighborhood of h and $\tilde{h}$.

We turn next to the estimate for the point vortices. We compute

$$\begin{aligned}
&\frac{d}{dt}|h-\widetilde{h}|^2+\frac{d}{dt}|\dot{h}-\dot{\widetilde{h}}|^2\\
=&2\langle h-\widetilde{h},\dot{h}-\dot{\widetilde{h}}\rangle-2\frac{\gamma}{m}\langle\dot{h}-\dot{\widetilde{h}},u(t,h)^{\perp}-\widetilde{u}(t,\widetilde{h})^{\perp}\rangle\\
\leqslant&D(t)+2\frac{\gamma}{m}\sqrt{D(t)}|u(t,h)-u(t,\widetilde{h})|+2\frac{\gamma}{m}\sqrt{D(t)}|(u-\widetilde{u})(t,\widetilde{h})|.
\end{aligned}$$

On the one hand, since u is log-Lipschitz we have $|u(t,h)-u(t,\widetilde{h})|\leqslant Cf(|h-\widetilde{h}|)\leqslant Cf(\sqrt{D(t)})$. On the other hand, exactly as in Step 2 in the proof of [9, Proposition 3.10], we rely on [9, Lemma 3.9]: using the analyticity of $u-\widetilde{u}$ near h and $\widetilde{h}$, that lemma enables to obtain

$$|u(t,h)-\widetilde{u}(t,h)|\leqslant C\|u(t,\cdot)-\widetilde{u}(t,\cdot)\|_{L^2}.$$

Hence we get finally that for all $p\geqslant 2$,

$$\frac{d}{dt}|h-\widetilde{h}|^2+\frac{d}{dt}|\dot{h}-\dot{\widetilde{h}}|^2\leqslant Cf(D(t))\leqslant CpD(t)^{1-1/p},\quad \forall t\in[0,T^*). \tag{4.2}$$

Finally, gathering (4.1) and (4.2), we find

$$D(t)\leqslant C\,p\int_0^t D(\tau)^{1-1/p}\,d\tau,\quad \forall p\geqslant 2.$$

So we conclude by usual arguments (see [11, Chapter 8]) that $D\equiv 0$ on $[0,T^*)$. Thus by definition of T^* we get $T^*=T$ and uniqueness follows on $[0,T]$.

Step 2: Proof of Theorem 1.5 completed
Once the case of one point is settled, the conclusion of Theorem 1.5 follows easily by adapting the proof above to the case of several points, using (3.20), (2.13) and (2.14). We refer also to the proof of uniqueness in [15, Theorem 2.1, Chapter 2] dealing with several points.

5. Some additional properties

We prove in this section some additional properties for System (1.1) in the case where the circulations and the vorticity have positive sign.

Proposition 5.1. *Let ω_0 and $(\{h_{k,0}\},\{\ell_{k,0}\})$ be as in* (1.2) *and let $(\omega,\{h_k\})$ be any corresponding weak solution to* (1.1) *on* $[0,T]$. *The following quantities are conserved:*

- *The energy,*

$$\begin{aligned}\mathcal{H}_0 =& \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y|\omega(t,y)\omega(t,x)\,dx\,dy \\ &+ \frac{1}{\pi}\sum_{k=1}^{N}\gamma_k\int_{\mathbb{R}^2}\ln|x-h_k(t)|\omega(t,x)\,dx \\ &+ \sum_{j\neq k}\frac{\gamma_k\gamma_j}{2\pi}\ln|h_k(t)-h_j(t)| - \sum_{k=1}^{N} m_k|\dot{h}_k(t)|^2;\end{aligned}$$

- *The momentum,*

$$\mathcal{I}_0 = \int_{\mathbb{R}^2}|x|^2\omega(t,x)\,dx + \sum_{k=1}^{N}\gamma_k|h_k(t)|^2 - 2\sum_{k=1}^{N} m_k h_k(t)^{\perp}\cdot\dot{h}_k(t).$$

Proof (sketch). For $\varepsilon < \frac{1}{3}\min_{j\neq k}\min_{t\in[0,T]}|h_j(t)-h_k(t)|$, we replace ln by the smooth function $\ln_\varepsilon$ defined in the first section and we set $\varphi_\varepsilon = \frac{1}{2\pi}\ln_\varepsilon * \omega$ as in (2.3), so that, setting

$$\begin{aligned}\mathcal{H}_\varepsilon =& \int_{\mathbb{R}^2}\varphi_\varepsilon(t,x)\omega(t,x)\,dx + 2\sum_{k=1}^{N}\gamma_k\varphi_\varepsilon(t,h_k(t)) \\ &+ \sum_{j\neq k}\frac{\gamma_j\gamma_k}{2\pi}\ln_\varepsilon|h_j(t)-h_k(t)| - \sum_{k=1}^{N} m_k|\dot{h}_k(t)|^2,\end{aligned}$$

we have $\sup_{[0,T]}|\mathcal{H}_0-\mathcal{H}_\varepsilon| \leqslant C\varepsilon$, with the quantity C depending only on $\|\omega\|_{L^\infty}$, $\|h_k\|_{L^\infty}, m_k, \gamma_k$ etc.

It suffices then to compute the time derivative of $\mathcal{H}_\varepsilon$ using the weak formulation for ω and the ODE for the $h_k's$, which yields $\sup_{[0,T]}|\dot{\mathcal{H}}_\varepsilon| \leqslant C\varepsilon$. Letting ε tend to zero, the conclusion follows.

For $\mathcal{I}_0$ we compute directly the time derivative using the weak formulation for ω and the ODE for the $h_k's$ and we show that it vanishes, which yields the result. □

With these conservations, we can prove that the massive point vortices are confined if ω and $\{\gamma_k\}$ have the same sign.

Corollary 5.2. *Assume moreover that*

$$\omega_0 \geqslant 0,\ \textit{a.e. on } \mathbb{R}^2, \quad \gamma_k > 0,\ k = 1,\dots,N.$$

Let $(\omega,\{h_k\})$ *be any corresponding weak solution to* (1.1) *on* $[0,T]$. *Then there exists* $C > 0$ *and* $d > 0$, *depending only on* $\mathcal{H}_0$, $\mathcal{I}_0$, m_k, γ_k *and* $\|\omega_0\|_{L^\infty}$ *and* R_0, *but not on* T, *such that*

$$\sup_{t\in[0,T]}\left(|\dot{h}_k(t)|^2 + |h_k(t)|^2\right) \leqslant C$$

and

$$\inf_{t\in[0,T]} \min_{j\neq k} |h_j(t) - h_k(t)| \geqslant d.$$

Proof. Since ω is transported by the flow, we have $\omega(t,\cdot) \geqslant 0$ almost everywhere for $t \in [0,T]$. Similarly to the proof of Proposition 2.5, picking $m \neq n$, we have, using that $\ln(|x-y|) \leqslant |x| + |y|$,

$$\begin{aligned}
&\frac{\gamma_m\gamma_n}{2\pi} \ln|h_m(t) - h_n(t)| \\
&\geqslant \mathcal{H}_0 - \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(|x| + |y|\big)\omega(t,y)\omega(t,x)\,dx\,dy \\
&\quad - \frac{1}{\pi}\sum_{k=1}^{N}\gamma_k \int_{\mathbb{R}^2} \big(|x| + |h_k(t)|\big)\omega(t,x)\,dx - \sum_{j\neq k}\frac{\gamma_k\gamma_j}{2\pi}\big(|h_k(t)| + |h_j(t)|\big)
\end{aligned}$$

therefore by the Cauchy-Schwartz inequality:

$$\begin{aligned}
&\frac{\gamma_m\gamma_n}{2\pi} \ln|h_m(t) - h_n(t)| \\
&\geqslant \mathcal{H}_0 - C\left(\int_{\mathbb{R}^2} |x|^2\omega(t,x)\,dx + \sum_{k=1}^{N}\gamma_k|h_k(t)|^2\right)^{1/2},
\end{aligned} \tag{5.1}$$

where C depends only on $\|\omega(t,\cdot)\|_{L^1} = \|\omega_0\|_{L^1}$ and on γ_k.

By the same estimates we also obtain

$$\sum_{k=1}^{N} m_k|\dot{h}_k(t)|^2 \leqslant -\mathcal{H}_0 + C\left(\int_{\mathbb{R}^2} |x|^2\omega(t,x)\,dx + \sum_{k=1}^{N}\gamma_k|h_k(t)|^2\right)^{1/2}. \tag{5.2}$$

On the other hand, by the Cauchy-Schwartz inequality:

$$\begin{aligned}
&\int_{\mathbb{R}^2} |x|^2\omega(t,x)\,dx + \sum_{k=1}^{N}\gamma_k|h_k(t)|^2 \\
&\leqslant \mathcal{I}_0 + C\left(\sum_{k=1}^{N} m_k|\dot{h}_k(t)|^2\right)^{1/2}\left(\sum_{k=1}^{N}\gamma_k|h_k(t)|^2\right)^{1/2} \\
&\leqslant \mathcal{I}_0 + C\left(-\mathcal{H}_0 + C\left(\int_{\mathbb{R}^2} |x|^2\omega(t,x)\,dx + \sum_{k=1}^{N}\gamma_k|h_k(t)|^2\right)^{1/2}\right)^{1/2} \\
&\quad \times \left(\sum_{k=1}^{N}\gamma_k|h_k(t)|^2\right)^{1/2} \\
&\leqslant \mathcal{I}_0 + C\left(1 + \int_{\mathbb{R}^2} |x|^2\omega(t,x)\,dx + \sum_{k=1}^{N}\gamma_k|h_k(t)|^2\right)^{3/4},
\end{aligned}$$

where we have used (5.2). We conclude that

$$\int_{\mathbb{R}^2} |x|^2 \omega(t,x)\,dx + \sum_{k=1}^{N} \gamma_k |h_k(t)|^2 \leqslant C$$

with C depending only $\mathcal{I}_0, \mathcal{H}_0$ and $\|\omega_0\|_{L^1}$. Coming back to (5.1) and (5.2), the conclusion follows. □

Appendix

A. Some results included in [15]

In this Appendix we gather several results from [15, Chapter 1]. Since that reference is in French we provide here the statements in English and refer to [15] for the proofs. Similar results and proofs in the case of one point vortex are also to be found in [9].

Lemma A.1. *Let $\{h_k\}$ be N Lipschitz trajectories on $[0,T]$ without collisions:*

$$\min_{t\in[0,T]} \min_{k\neq p} |h_k(t) - h_p(t)| \geqslant \rho$$

for some $\rho > 0$. Let ω be a weak solution of the PDE

$$\partial_t \omega + \operatorname{div}(v\omega) = 0 \quad \text{on } [0,T], \tag{A.1}$$

where v is the divergence-free velocity field given by

$$v(t,x) = u(t,x) + \sum_{j=1}^{N} \gamma_j K(x - h_j(t)) \tag{A.2}$$

with u a divergence-free vector field satisfying

$$u \in L^\infty([0,T]\times\mathbb{R}^2) \quad \text{and } u(t,\cdot) \text{ is log-Lipschitz uniformly in time.} \tag{A.3}$$

Let $\beta : \mathbb{R} \to \mathbb{R}$ be C^1 such that

$$|\beta'(z)| \leqslant C(1 + |z|^p), \qquad \forall z \in \mathbb{R},$$

for some $p \geqslant 0$. Then for all test function $\psi \in C_c^\infty([0,T]\times\mathbb{R}^2)$, we have

$$\frac{d}{dt}\int_{\mathbb{R}^2} \psi\beta(\omega)\,dx = \int_{\mathbb{R}^2} \beta(\omega)(\partial_t\psi + v\cdot\nabla\psi)\,dx \ \text{ in } L^1([0,T]).$$

This lemma is stated in [15, Chapitre 1, Lemme 1.5] in the case where $(\omega, \{h_k\})$ is a weak solution of the vortex-wave system. However a straightforward adaptation of the proof shows that this holds for the linear transport equation (A.1) with any vector field v given by the decomposition (A.2), where u satisfies the regularity properties (A.3) and where the $\dot{h}_j$ are Lipschitz continuous on $[0, T]$ and do not intersect. We emphasize that their precise dynamics is not used to show the renormalization property.

As a consequence of Lemma A.1 it is observed in [15, Chapter 1, Remarque 1.3] (or in [9, Lemma 3.2] for the case of one point) that

Corollary A.2. *Under the same assumption as in Lemma A.1 for the $\{h_k\}$, let ω be a weak solution of the PDE* (A.1). *Then for all $1 \leqslant p \leqslant +\infty$ we have $\|\omega(t, \cdot)\|_{L^p} = \|\omega(0, \cdot)\|_{L^p}$. In particular, uniqueness of the weak solution holds.*

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