# LINEAR QUANTITATIVE RIGIDITY FOR ALMOST–CMC SURFACES

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ABSTRACT. We prove a quantitative rigidity result for almost constant mean curvature spheres in $\mathbb{R}^3$. Under a sub–two–sphere Willmore bound and a small $L^2$–CMC defect, we show that an almost–CMC surface is close to the round sphere, with linear control of the $W^{2,2}$–distance of the parametrization and the $L^\infty$–norm of the conformal factor. An analogous statement holds under an a priori area bound below that of two spheres. The proof relies on a linearized analysis around the sphere. A previously established qualitative rigidity result provides the initial closeness required to enter the perturbative regime. The estimate further extends to integral 2–varifolds of unit density using known regularity and density results.

## 1. INTRODUCTION

A classical theorem of Alexandrov asserts that an embedded closed hypersurface $\Sigma = \partial\Omega$ in Euclidean space $\mathbb{R}^{n+1}$ with constant mean curvature must be a round sphere. In recent years, a broad literature has developed qualitative or quantitative rigidity results for Alexandrov-type characterizations of spheres under various regularity assumptions and smallness of natural CMC deficits, with conclusions ranging from closeness in weak geometric distances to stronger graphical descriptions under additional non–degeneracy assumptions that preclude bubbling. We refer to [3–5, 8, 11] and the references therein for related results in this direction.

In this paper we study *almost–CMC* surfaces in $\mathbb{R}^3$ and prove a quantitative rigidity estimate with a *linear rate* in the critical $L^2$–defect $\|H-\overline{H}\|_{L^2}$. From an analytic viewpoint a natural way to measure deviation from the CMC condition is through scale–invariant norms of $H-\overline{H}$; in dimension $n$ this leads to the critical quantity $\|H-\overline{H}\|_{L^n}$.

In the two–dimensional setting in $\mathbb{R}^3$, a natural topology for such critical problems is suggested by the conformal parametrization framework of Müller–Šverák [13] and De Lellis–Müller [6, 7]: one measures the deviation from the round sphere through a conformal map and controls it in $W^{2,2}$, together with an $L^\infty$ bound for the conformal factor. Within this framework, De Lellis–Müller quantify nearness to a round sphere by the $L^2$–smallness of the trace–free second fundamental form, whereas we quantify nearness to the CMC condition by the oscillation of the scalar mean curvature $\|H-\overline{H}\|_{L^2}$. Since small oscillation of $H$ alone does not exclude bubbling, and bubbling is not naturally captured by the conformal topology considered here, we impose in addition an a priori two–sphere–threshold $L^2$–bound on $H$, namely $\int_\Sigma |H|^2\, d\mu_g < 32\pi$.

Quantitative results under scale–invariant assumptions in more general settings, including higher dimensions and configurations with multiple spherical components, are naturally formulated in topologies that are robust under bubbling phenomena; see, for instance, the quantitative Alexandrov theorem of Julin–Niinikoski [10].

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As a starting point we use a qualitative stability result from [1], which shows that small $L^2$ CMC defect (together with the two–sphere–threshold curvature bound) forces the surface into a perturbative $W^{2,2}$–conformal neighborhood of the round sphere. The main purpose of the present paper is to upgrade the resulting modulus of continuity $\Psi(\delta)$ to a linear bound.

**Theorem 1.1** (Qualitative rigidity for almost–CMC spheres [1, Theorem 1.2])**.** *For any $\alpha \in (0, \frac{1}{2})$ there exists $\delta_0 = \delta_0(\alpha) > 0$ with the following property.*

*Let $F : \Sigma \to \mathbb{R}^3$ be a smooth immersion of a closed connected surface, let $g := dF \otimes dF$ and $d\mu_g$ be the induced area measure, and let $\vec{N}$ be a globally defined unit normal, with scalar mean curvature $H := -\langle \vec{H}, \vec{N} \rangle$ and average*

$$\overline{H} := \fint_\Sigma H \, d\mu_g := \frac{1}{\mu_g(\Sigma)} \int_\Sigma H \, d\mu_g.$$

*Assume*

$$\int_\Sigma |H - \overline{H}|^2 \, d\mu_g \leqslant \delta \leqslant \delta_0, \qquad \int_\Sigma |H|^2 \, d\mu_g \leqslant 32\pi(1-\alpha). \tag{1.1}$$

*Then $\Sigma$ is homeomorphic to $\mathbb{S}^2$.*

*Moreover, after rescaling so that $\mu_g(\Sigma) = 4\pi$, and after composing $F$ with a suitable translation of $\mathbb{R}^3$, there exists a conformal parametrization $f : \mathbb{S}^2 \to F(\Sigma)$ with*

$$df \otimes df = e^{2u} g_{\mathbb{S}^2},$$

*such that,*

$$\|f - \mathrm{id}_{\mathbb{S}^2}\|_{W^{2,2}(\mathbb{S}^2)} + \|u\|_{L^\infty(\mathbb{S}^2)} \leqslant \Psi(\delta), \tag{1.2}$$

*where $\Psi(\delta) \to 0$ as $\delta \downarrow 0$.*

We emphasize that our curvature assumption

$$\int_\Sigma |H|^2 \, d\mu_g \leqslant 32\pi(1-\alpha) \tag{1.3}$$

is primarily a *single–bubble* condition, in the sense that it rules out the formation of two spherical bubbles, whether disjoint or touching. We will refer to (1.3) as a *two–sphere–threshold* (or *sub–two–spheres*) Willmore bound. In particular, this bound implies that $\Sigma$ is connected. Indeed, by the Willmore inequality, every connected component $\Sigma_j$ satisfies $\int_{\Sigma_j} |H|^2 \, d\mu_g \geqslant 16\pi$. Since (1.3) is strictly below $32\pi$, there can be only one connected component.

Although Theorem 1.1 is stated for smooth immersions, this assumption has further geometric consequences. By the classical result of Li–Yau [12], any closed immersed surface in $\mathbb{R}^3$ with Willmore energy strictly below $8\pi$ is necessarily embedded. In particular, under the curvature assumption (1.3), the immersion $F$ in Theorem 1.1 is an embedding.

Hence, without loss of generality, we may regard $F$ as the standard embedding

$$\iota : \Sigma \hookrightarrow \mathbb{R}^3,$$

and, by a mild abuse of notation, identify $\Sigma$ with its image $\iota(\Sigma) \subset \mathbb{R}^3$. Since $\Sigma$ is a closed embedded surface, there exists a precompact domain $\Omega \subset \mathbb{R}^3$ such that $\Sigma = \partial\Omega$.

In our quantitative analysis it is more convenient to normalize by the enclosed volume $|\Omega| = \frac{4\pi}{3}$ rather than by the area $\mu_g(\Sigma) = 4\pi$. This choice is immaterial, since in the perturbative regime the two normalizations are equivalent up to a fixed scaling.

We now state our main result. As discussed above, it provides a *linear upgrade* of the qualitative rigidity estimate for almost–CMC spheres, yielding quantitative control in the critical $L^2$–CMC defect.

**Theorem 1.2** (Linear quantitative rigidity for almost–CMC spheres)**.** *For any $\alpha \in (0, \frac{1}{2})$ there exist $\delta_0 = \delta_0(\alpha) > 0$ and $C = C(\alpha) < \infty$ with the following property.*

*Let $\iota : \Sigma \to \mathbb{R}^3$ be a smooth immersion of a closed surface satisfying* (1.1)*. After rescaling so that $|\Omega| = \frac{4\pi}{3}$, up to a translation of $\mathbb{R}^3$, there exists a conformal parametrization $f : \mathbb{S}^2 \to \Sigma$ with*

$$df \otimes df = e^{2u} g_{\mathbb{S}^2},$$

*such that*

$$\|h\|_{W^{2,2}(\mathbb{S}^2)} + \|u\|_{L^\infty(\mathbb{S}^2)} \leqslant C \|H - \overline{H}\|_{L^2(\Sigma)}, \tag{1.4}$$

*where $h := f - \mathrm{id}_{\mathbb{S}^2}$.*

**Relation to perimeter–based formulations.** Very recently, Julin–Morini–Oronzio–Spadaro [9] proved a sharp quantitative Alexandrov inequality in $\mathbb{R}^3$: for every $\delta_0 > 0$ there exists $C = C(\delta_0) > 0$ such that any $C^2$–regular set $E \subset \mathbb{R}^3$ with $|E| = |B_1| = \frac{4\pi}{3}$ and

$$P(E) \leqslant 4\pi \sqrt[3]{2} - \delta_0 \tag{1.5}$$

satisfies

$$P(E) - P(B_1) \leqslant C \|H_E - \overline{H}_E\|^2_{L^2(\partial E)}. \tag{1.6}$$

Moreover, if $\partial E$ is smooth and the CMC defect is sufficiently small, then $\partial E$ is diffeomorphic to $\mathbb{S}^2$. The perimeter constraint (1.5) can be viewed as a natural non–degeneracy assumption excluding the two–bubble threshold $4\pi\sqrt[3]{2}$.

For our purposes it is useful to relate (1.5) to the two–sphere–threshold Willmore bound (1.3). This implication can be deduced directly from the quantitative Alexandrov theorem of Julin–Niinikoski [10]. Since the argument is short, we include it in Section 4 for completeness. As a consequence, our main theorem admits an equivalent formulation in which the Willmore assumption is replaced by the area bound (1.5), at the expense of assuming that the surface arises as the boundary of a smooth domain (since without the Willmore bound embeddedness is not available a priori).

**Theorem 1.3** (Linear quantitative stability under an area bound)**.** *For any $\beta > 0$ there exist $\delta_0 = \delta_0(\beta) > 0$ and $C = C(\beta) < \infty$ with the following property.*

*Let $\Omega \subset \mathbb{R}^3$ be a bounded domain whose boundary $\Sigma$ is a smooth embedded surface, and let $\iota : \Sigma \to \mathbb{R}^3$ denote the inclusion map. After rescaling so that $|\Omega| = \frac{4\pi}{3}$, assume that*

$$\mu_g(\Sigma) \leqslant 4\pi \sqrt[3]{2} - \beta, \qquad \|H - \overline{H}\|_{L^2(\Sigma)} \leqslant \delta_0. \tag{1.7}$$

*Then there exists a conformal parametrization $f : \mathbb{S}^2 \to \Sigma$ with*

$$df \otimes df = e^{2u} g_{\mathbb{S}^2},$$

*such that, after composing $f$ with a suitable translation and writing $h := f - \mathrm{id}_{\mathbb{S}^2}$, one has*

$$\|h\|_{W^{2,2}(\mathbb{S}^2)} + \|u\|_{L^\infty(\mathbb{S}^2)} \leqslant C \|H - \overline{H}\|_{L^2(\Sigma)}.$$

With this reformulation, the assumptions of Theorem 1.3 align with those in [9]. In this sense, our result may be viewed as a geometric counterpart of their quantitative Alexandrov inequality.

**A varifold formulation.** Although Theorem 1.2 is stated for smooth immersions, the same linear control admits a convenient varifold formulation. Throughout the varifold statements below, *unit density* means $\Theta^2(\|V\|, x) = 1$ for every $x \in \mathrm{spt}\|V\|$. For the compact varifolds considered below, [2, Theorem B.4] shows that the support is a compact embedded $W^{2,2}$-surface. We orient each connected component by its exterior unit normal $\vec{N}$, defined $\|V\|$-a.e.

**Theorem 1.4** (Varifold stability for almost–CMC spheres)**.** *Fix $\alpha \in (0, \frac{1}{2})$. There exist $\delta_0 = \delta_0(\alpha) > 0$ and $C = C(\alpha) < \infty$ with the following property.*

*Let $V = \underline{v}(\Sigma, 1)$ be an integral 2–varifold in $\mathbb{R}^3$ with unit density and generalized mean curvature $\vec{H} \in L^2(d\|V\|)$. Assume that $\Sigma := \operatorname{spt}\|V\|$ is compact and that*

$$\int_\Sigma |H - \overline{H}|^2 \, d\|V\| \leqslant \delta^2, \qquad \int_\Sigma |H|^2 \, d\|V\| \leqslant 32\pi(1-\alpha), \qquad 0 < \delta \leqslant \delta_0, \tag{1.8}$$

*where $H := -\langle \vec{H}, \vec{N} \rangle$ and*

$$\overline{H} := \fint_\Sigma H \, d\|V\| := \frac{1}{\|V\|(\mathbb{R}^3)} \int_\Sigma H \, d\|V\|.$$

*Then $\Sigma$ is homeomorphic to $\mathbb{S}^2$. Moreover, after a translation of $\mathbb{R}^3$ and rescaling so that the enclosed volume equals $\frac{4\pi}{3}$, there exists a homeomorphism*

$$f : \mathbb{S}^2 \longrightarrow \Sigma$$

*with $f \in W^{2,2}(\mathbb{S}^2; \mathbb{R}^3)$ and a function $u \in L^\infty(\mathbb{S}^2)$ such that*

$$df \otimes df = e^{2u} g_{\mathbb{S}^2} \quad \text{a.e. on } \mathbb{S}^2,$$

*and the quantitative estimate*

$$\|f - \mathrm{id}_{\mathbb{S}^2}\|_{W^{2,2}(\mathbb{S}^2)} + \|u\|_{L^\infty(\mathbb{S}^2)} \leqslant C\delta \tag{1.9}$$

*holds.*

*Derivation from the smooth case.* Under (1.8), one may approximate $\Sigma$ by a sequence of smooth embedded surfaces $\Sigma_k$ in $W^{2,2}$ while preserving unit density and the $L^2$–control of the mean curvature; this is provided by Corollary 5.7 in [14] or Theorem B.4 in [2]. Applying Theorem 1.2 to $\Sigma_k$ yields uniform linear bounds as in (1.9), and a compactness argument then passes the estimate to the varifold limit $V$.

*Remark* 1.5 (A finite–perimeter variant). In Theorem 1.4 the two–sphere–threshold Willmore bound is used as a single–bubble assumption. In view of the area formulation (Theorem 1.3), one can replace this curvature bound by a perimeter bound, provided the surface is known a priori to arise as the boundary of a set of finite perimeter.

More precisely, for every $\beta > 0$ there exists $\delta_0 = \delta_0(\beta) > 0$ with the following property. Let $E \subset \mathbb{R}^3$ be a bounded Caccioppoli set, set $\mu := |D\chi_E|$ and $\Sigma := \operatorname{spt}\mu$, and let $V_E$ be the associated boundary varifold. Assume that $V_E$ has generalized mean-curvature vector $\vec{H} \in L^2(\mu; \mathbb{R}^3)$. Let $\nu_E$ be the measure-theoretic exterior unit normal, and define $H := -\vec{H} \cdot \nu_E$ and

$$\overline{H} := \frac{1}{P(E)} \int_\Sigma H \, d\mu.$$

After rescaling so that $|E| = \frac{4\pi}{3}$, assume that

$$P(E) \leqslant 4\pi\sqrt[3]{2} - \beta, \qquad \int_\Sigma |H - \overline{H}|^2 \, d\mu \leqslant \delta^2, \qquad 0 < \delta \leqslant \delta_0.$$

Then the conclusion of Theorem 1.3 holds for $\Sigma$, with constants depending only on $\beta$.

Indeed, Lemma 4.2 gives a strict sub-two-sphere Willmore bound, so Simon's monotonicity formula yields $\Theta^2(\mu, x) < 2$ for every $x \in \Sigma$. By dimension reduction, a nonplanar tangent cone would give a nonflat one-dimensional stationary cone of density less than 2, hence a triple junction, which cannot occur as the measure-theoretic boundary of a set of locally finite perimeter. Thus every tangent cone is a multiplicity-one plane, and consequently $\Theta^2(\mu, x) = 1$ for every $x \in \Sigma$. After decreasing $\delta_0(\beta)$, if necessary, Theorem 1.4 applies.

**Strategy of the proof.** Theorem 1.1 reduces the quantitative problem to a perturbative regime: after normalization and a suitable choice of rigid motion (and conformal gauge), we may assume that $f$ is a $W^{2,2}$–conformal immersion close to the standard embedding $f_0(x) = x$ on $\mathbb{S}^2$, with a small conformal factor $u$.

The core of the paper consists of a linearized analysis around the round sphere. In Section 2 we derive a precise quadratic expansion of the geometric energy governing the area excess and identify the leading quadratic form in the normal component $z$ of the perturbation $h$. By exploiting the spectral properties of the spherical Laplacian together with the normalization and orthogonality conditions, we obtain a coercive estimate that controls the $W^{1,2}$–norm of the perturbation in terms of the CMC defect.

In Section 3 we estimate the deviation of the average mean curvature $\overline{H}$ from the spherical value 2 and use elliptic regularity to upgrade the control to the full $W^{2,2}$–norm of $h$, as well as to derive an $L^\infty$–bound for the conformal factor $u$. This yields the desired linear estimate (1.4) and completes the proof of Theorem 1.2.

Finally, in Section 4 we give a short argument showing that the two–sphere–threshold Willmore bound condition can equivalently be expressed in terms of an area bound, and we derive the corresponding area formulation of our main theorem (Theorem 1.3).

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## 2. $W^{1,2}$–Estimate

Throughout this section we assume the smallness condition

$$\|f - f_0\|_{W^{2,2}(\mathbb{S}^2)} + \|u\|_{C^0(\mathbb{S}^2)} \ll 1, \tag{2.1}$$

together with the volume normalization

$$|\Omega| = \frac{4\pi}{3}, \tag{2.2}$$

which have been discussed in the Introduction.

We normalize the parametrization of $f$ by minimizing its $L^2$–distance to the standard embedding $f_0(x) = x$ in the translational and Möbius parameters. For $a \in \mathbb{R}^3$ and a Möbius transformation $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$, define

$$\mathscr{E}(a,\varphi) := \int_{\mathbb{S}^2} |f \circ \varphi + a - f_0|^2 \, dx.$$

**Lemma 2.1** (Existence of a minimizer)**.** *Under the smallness assumption* (2.1)*, there exist* $\varphi_0 \in \mathrm{Mob}(\mathbb{S}^2)$ *and* $a_0 \in \mathbb{R}^3$ *such that* $(a_0, \varphi_0)$ *minimizes* $\mathscr{E}(a,\varphi)$ *among all translations* $a \in \mathbb{R}^3$ *and all Möbius transformations* $\varphi \in \mathrm{Mob}(\mathbb{S}^2)$.

*Moreover, there exists* $C > 0$ *such that,*

$$|a_0| \leqslant C\|f - f_0\|_{L^\infty(\mathbb{S}^2)}, \qquad \|\varphi_0 - \mathrm{Id}\|_{L^2(\mathbb{S}^2)} \leqslant C\|f - f_0\|_{L^\infty(\mathbb{S}^2)}. \tag{2.3}$$

*Proof.* Set $\delta_\infty := \|f - f_0\|_{L^\infty(\mathbb{S}^2)}$, which is small by (2.1) and Sobolev embedding.

We first compare $\mathscr{E}$ with the auxiliary quantity

$$G(a,\varphi) := \int_{\mathbb{S}^2} |\varphi + a - f_0|^2 \, dx, \qquad a \in \mathbb{R}^3,\ \varphi \in \mathrm{Mob}(\mathbb{S}^2).$$

Using $f_0(x) = x$ and the identity

$$f\circ\varphi + a - f_0 = (f - f_0)\circ\varphi + (\varphi + a - f_0),$$

we obtain by the triangle inequality

$$\|f\circ\varphi + a - f_0\|_{L^2} \geqslant \|\varphi + a - f_0\|_{L^2} - \|(f - f_0)\circ\varphi\|_{L^2} \geqslant \|\varphi + a - f_0\|_{L^2} - \sqrt{4\pi}\,\delta_\infty,$$

and therefore

$$\mathscr{E}(a,\varphi) = \|f\circ\varphi + a - f_0\|_{L^2}^2 \geqslant \Big(\|\varphi + a - f_0\|_{L^2} - \sqrt{4\pi}\,\delta_\infty\Big)^2.$$

Since $(X-Y)^2 \geqslant \frac{1}{2}X^2 - Y^2$, this implies the rough bound

$$\mathscr{E}(a,\varphi) \geqslant \frac{1}{2}\,G(a,\varphi) - 4\pi\,\delta_\infty^2. \tag{2.4}$$

Next, $\mathscr{E}(a,\varphi)\to\infty$ as $|a|\to\infty$, uniformly in $\varphi$. Indeed, for $\delta_\infty \leqslant 1$ we have the pointwise estimate $|f\circ\varphi - f_0| \leqslant |f\circ\varphi - f_0\circ\varphi| + |f_0\circ\varphi - f_0| \leqslant \delta_\infty + 2 \leqslant 3$. Therefore,

$$\begin{aligned}\mathscr{E}(a,\varphi) &= 4\pi|a|^2 + 2a\cdot\int_{\mathbb{S}^2}(f\circ\varphi - f_0)\,dx + \int_{\mathbb{S}^2}|f\circ\varphi - f_0|^2\,dx\\ &\geqslant 4\pi|a|^2 - 24\pi|a| + \int_{\mathbb{S}^2}|f\circ\varphi - f_0|^2\,dx.\end{aligned}$$

It follows that any minimizing sequence must remain in a bounded region $\{|a|\leqslant A\}$ for some $A<\infty$.

For the Möbius parameter, we use the standard decomposition $\varphi = R\circ\phi_v$ with $R\in SO(3)$ and $v\in B^3 = \{v\in\mathbb{R}^3 : |v|<1\}$, where

$$\phi_v(x) := \frac{(1-|v|^2)x + 2(1+v\cdot x)\,v}{1+|v|^2+2v\cdot x} \in \mathbb{S}^2.$$

If $v = r_i p_i$ with $p_i\in\mathbb{S}^2$ and $r_i\uparrow 1$, then after passing to a subsequence, we can assume $p_i\to p$ and $a_i\to a$, and $\phi_{r_ip_i}(\cdot)\to p\in\mathbb{S}^2$ almost everywhere and hence in $L^2(\mathbb{S}^2)$.

As a result,

$$\int_{\mathbb{S}^2}|\phi_{r_ip_i} + a_i - x|^2\,dx \to \int_{\mathbb{S}^2}|p+a-x|^2 dx = \int_{\mathbb{S}^2}(|p+a|^2+1)dx \geqslant 4\pi.$$

Since $SO(3)$ is compact and rotations preserve the $L^2$–norm, the same conclusion holds for $\varphi = R\circ\phi_v$. In particular, there exists $\eta\in(0,1)$ and $c_*>0$ such that

$$\inf_{a\in\mathbb{R}^3} G(a,\varphi) \geqslant c_* \qquad \text{whenever } \varphi = R\circ\phi_v \text{ with } |v|\geqslant 1-\eta. \tag{2.5}$$

Choosing $\delta_\infty$ so small that $4\pi\delta_\infty^2 \leqslant \frac{1}{4}c_*$, the comparison (2.4) yields

$$\inf_{a\in\mathbb{R}^3}\mathscr{E}(a,\varphi) \geqslant \frac{1}{2}c_* - 4\pi\delta_\infty^2 \geqslant \frac{1}{4}c_* \qquad \text{for all } \varphi \text{ with } |v|\geqslant 1-\eta,$$

whereas

$$\mathscr{E}(0,\mathrm{Id}) = \int_{\mathbb{S}^2}|f-f_0|^2\,dx \leqslant 4\pi\delta_\infty^2 \leqslant \frac{1}{4}c_*.$$

Consequently, a minimizing sequence cannot degenerate to the boundary $|v|\to 1$, and together with the coercivity in $a$ this shows that it suffices to minimize $\mathscr{E}$ over the compact set

$$K := \{a\in\mathbb{R}^3 : |a|\leqslant A\}\times\{R\circ\phi_v : R\in SO(3),\ |v|\leqslant 1-\eta\}.$$

The map $(a,\varphi)\mapsto\mathscr{E}(a,\varphi)$ is continuous on $K$, hence it attains its minimum at some $(a_0,\varphi_0)\in K$. By construction, $(a_0,\varphi_0)$ is also a global minimizer among all translations and all Möbius transformations.

For the estimate, minimality gives

$$\mathscr{E}(a_0,\varphi_0) \leqslant \mathscr{E}(0,\mathrm{Id}) \leqslant 4\pi\delta_\infty^2.$$

Combining this with (2.4) yields

$$G(a_0,\varphi_0) \leqslant C\delta_\infty^2. \tag{2.6}$$

We next bound $a_0$ quantitatively. Since $\varphi_0(x) \in \mathbb{S}^2$ for all $x$, we have

$$|\varphi_0(x)+a_0-x| = |\varphi_0(x)-(x-a_0)| \geqslant \mathrm{dist}(x-a_0,\mathbb{S}^2) = ||x-a_0|-1|.$$

Consequently,

$$G(a_0,\varphi_0) \geqslant \int_{\mathbb{S}^2}\big(||x-a_0|-1|\big)^2\,dx =: J(a_0).$$

Recall from the coercivity in the translation parameter that any minimizer satisfies $|a_0| \leqslant A$ for some fixed $A<\infty$. For such $a$ we estimate

$$||x-a|-1| = \frac{||x-a|^2-1|}{|x-a|+1} = \frac{||a|^2-2a\cdot x|}{|x-a|+1} \geqslant \frac{||a|^2-2a\cdot x|}{|a|+2} \geqslant \frac{||a|^2-2a\cdot x|}{A+2}.$$

Squaring and integrating yields

$$J(a) \geqslant \frac{1}{(A+2)^2}\int_{\mathbb{S}^2}\big(|a|^2-2a\cdot x\big)^2\,dx.$$

Using $\int_{\mathbb{S}^2} x\,dx = 0$ and $\int_{\mathbb{S}^2}(a\cdot x)^2\,dx = \frac{4\pi}{3}|a|^2$, we compute

$$\int_{\mathbb{S}^2}\big(|a|^2-2a\cdot x\big)^2\,dx = 4\pi|a|^4+\frac{16\pi}{3}|a|^2 \geqslant \frac{16\pi}{3}|a|^2,$$

and therefore

$$J(a) \geqslant \frac{16\pi}{3(A+2)^2}|a|^2 \qquad \text{for all } |a| \leqslant A. \tag{2.7}$$

Applying (2.7) to $a_0$ and combining with $J(a_0) \leqslant G(a_0,\varphi_0)$ and (2.6), we conclude

$$|a_0|^2 \leqslant C\,J(a_0) \leqslant C\,G(a_0,\varphi_0) \leqslant C\delta_\infty^2, \qquad \text{hence} \qquad |a_0| \leqslant C\delta_\infty.$$

Finally,

$$\|\varphi_0-\mathrm{Id}\|_{L^2(\mathbb{S}^2)} = \|\varphi_0-f_0\|_{L^2(\mathbb{S}^2)} \leqslant \|\varphi_0+a_0-f_0\|_{L^2(\mathbb{S}^2)}+\|a_0\|_{L^2(\mathbb{S}^2)} \leqslant G(a_0,\varphi_0)^{1/2}+\sqrt{4\pi}\,|a_0| \leqslant C\delta_\infty,$$

which proves (2.3). □

We now replace $f$ by the normalized map

$$\tilde f := f\circ\varphi_0+a_0, \qquad h := \tilde f - f_0.$$

Since $\varphi_0$ lies in the fixed compact subset of $\mathrm{Mob}(\mathbb{S}^2)$ obtained in the proof of Lemma 2.1, finite-dimensional norm equivalence gives $\|\varphi_0-\mathrm{Id}\|_{C^2} \leqslant C\|\varphi_0-\mathrm{Id}\|_{L^2}$. Using

$$\widetilde f - f_0 = (f-f_0)\circ\varphi_0 + (f_0\circ\varphi_0 - f_0) + a_0$$

and Lemma 2.1, we obtain

$$\|\widetilde f - f_0\|_{W^{2,2}(\mathbb{S}^2)} \leqslant C\|f-f_0\|_{W^{2,2}(\mathbb{S}^2)}.$$

Moreover, $e^{2\tilde u}g_{\mathbb{S}^2} = d\tilde f\otimes d\tilde f = e^{2u(\varphi_0(x))}\mathrm{Jac}_{\varphi_0}(x)g_{\mathbb{S}^2}$ implies

$$\|\tilde u\|_{L^\infty} = \|u(\varphi_0(x))+\frac12\ln\mathrm{Jac}_{\varphi_0}(x)\|_{L^\infty} \leqslant \|u\|_{L^\infty}+C\|\varphi_0-\mathrm{Id}\|_{C^1} \leqslant C(\|u\|_{L^\infty}+\|f-f_0\|_{W^{2,2}}).$$

So, $\tilde f$ still satisfies the smallness condition (2.1) and volume renormalization condition (2.2). From now on, we use $f$ again to denote the normalized map $\widetilde f$. Then, by construction, the point

$(a,\varphi) = (0,\mathrm{Id})$ is a *critical point* of $\mathscr{E}$ with respect to variations in $a$ and in $\varphi$ generated by infinitesimal Möbius transformations. This yields the following orthogonality identities.

**Lemma 2.2** (Orthogonality identities)**.** *For the normalized map $f$, the deviation $h = f - f_0$ satisfies*

$$\int_{\mathbb{S}^2} h\,dx = 0, \tag{2.8}$$

$$\int_{\mathbb{S}^2} h\cdot f_*(X)\,dx = 0 \qquad \forall \text{ conformal Killing fields } X. \tag{2.9}$$

*Proof.* Varying $a_t = t\xi$,

$$0 = \frac{d}{dt}\Big|_{t=0}\mathscr{E}(a_t,\mathrm{Id}) = 2\int_{\mathbb{S}^2} h\cdot\xi\,dx,$$

so $\int h = 0$. If $\varphi_t$ is generated by a conformal Killing field $X$, then $\frac{d}{dt}|_{t=0}(f\circ\varphi_t) = f_*(X)$, hence

$$0 = \frac{d}{dt}\Big|_{t=0}\mathscr{E}(0,\varphi_t) = 2\int h\cdot f_*(X)\,dx.$$

□

We next use the pointwise orthogonal decomposition

$$h = v + zn,$$

where $n := f_0$ is the outward unit normal along the standard embedding $f_0 : \mathbb{S}^2 \to \mathbb{R}^3$, $z := h\cdot n$, and $v := h - zn$ is tangent to $\mathbb{S}^2$ along $f_0$.

**Lemma 2.3** (Consequences of the normalization)**.** *For each coordinate function $x^i$ the following hold:*

$$\int_{\mathbb{S}^2} v\cdot\nabla x^i\,dx + \int_{\mathbb{S}^2} z x^i\,dx = 0, \tag{2.10}$$

$$\Big|\int_{\mathbb{S}^2} v\cdot Af_0\,dx\Big| \leqslant C\|h\|^2_{W^{1,2}(\mathbb{S}^2)}, \qquad \forall A\in\mathfrak{so}(3), \tag{2.11}$$

$$\Big|\int_{\mathbb{S}^2} v\cdot\nabla x^i\,dx\Big| \leqslant C\|h\|^2_{W^{1,2}(\mathbb{S}^2)}, \tag{2.12}$$

$$\Big|\int_{\mathbb{S}^2} z x^i\,dx\Big| \leqslant C\|h\|^2_{W^{1,2}(\mathbb{S}^2)}. \tag{2.13}$$

*Proof.* Since $\int_{\mathbb{S}^2} h\,dx = 0$ and $h = v + zn$ with $n = f_0$, we have

$$\int_{\mathbb{S}^2} v\,dx + \int_{\mathbb{S}^2} zn\,dx = 0.$$

Taking the Euclidean inner product with $e_i$ and using $v\cdot e_i = v\cdot\nabla x^i$ and $n\cdot e_i = x^i$ gives

$$\int_{\mathbb{S}^2} v\cdot\nabla x^i\,dx + \int_{\mathbb{S}^2} z x^i\,dx = 0,$$

which is (2.10).

For (2.11), fix $A\in\mathfrak{so}(3)$ and apply (2.9) with the conformal Killing field $X = Af_0$:

$$0 = \int_{\mathbb{S}^2} h\cdot f_*(Af_0)\,dx.$$

Since $f = f_0 + h$ and $df = df_0 + dh$, we have

$$f_*(Af_0) = df(Af_0) = df_0(Af_0) + dh(Af_0) = Af_0 + dh(Af_0),$$

because $df_0$ is the identity on $T\mathbb{S}^2$. Thus

$$0 = \int_{\mathbb{S}^2} h \cdot Af_0 \, dx + \int_{\mathbb{S}^2} h \cdot dh(Af_0) \, dx.$$

Decomposing $h = v + zn$ and using $n \cdot Af_0 = 0$ (since $Af_0$ is tangent), we obtain

$$\int_{\mathbb{S}^2} v \cdot Af_0 \, dx = -\int_{\mathbb{S}^2} h \cdot dh(Af_0) \, dx.$$

The pointwise bound $|dh(Af_0)| \leqslant C|\nabla h|$ yields

$$\Big|\int_{\mathbb{S}^2} v \cdot Af_0 \, dx\Big| \leqslant C \int_{\mathbb{S}^2} |h|\,|\nabla h| \, dx \leqslant C \, \|h\|_{L^2(\mathbb{S}^2)} \, \|\nabla h\|_{L^2(\mathbb{S}^2)} \leqslant C \, \|h\|^2_{W^{1,2}(\mathbb{S}^2)}.$$

For (2.12), we apply (2.9) with $X = \nabla x^i$:

$$0 = \int_{\mathbb{S}^2} h \cdot f_*(\nabla x^i) \, dx = \int_{\mathbb{S}^2} h \cdot \big(\nabla x^i + dh(\nabla x^i)\big) \, dx.$$

Since $h = v + zn$ and $n \cdot \nabla x^i = 0$, we get

$$\int_{\mathbb{S}^2} v \cdot \nabla x^i \, dx = -\int_{\mathbb{S}^2} h \cdot dh(\nabla x^i) \, dx.$$

Using $|\nabla x^i| \leqslant C$ and $|dh(\nabla x^i)| \leqslant C|\nabla h|$,

$$\Big|\int_{\mathbb{S}^2} v \cdot \nabla x^i \, dx\Big| \leqslant C \int_{\mathbb{S}^2} |h|\,|\nabla h| \, dx \leqslant C \, \|h\|_{L^2(\mathbb{S}^2)} \, \|\nabla h\|_{L^2(\mathbb{S}^2)} \leqslant C \, \|h\|^2_{W^{1,2}(\mathbb{S}^2)}.$$

Finally, combining (2.10) and (2.12) gives

$$\Big|\int_{\mathbb{S}^2} z x^i \, dx\Big| = \Big|\int_{\mathbb{S}^2} v \cdot \nabla x^i \, dx\Big| \leqslant C \, \|h\|^2_{W^{1,2}(\mathbb{S}^2)},$$

which is (2.13). □

We now recall the Cauchy–Riemann operator that encodes the trace-free part of the conformality condition. The conformality of $f$ reads

$$(df_0 + dh) \otimes (df_0 + dh) = e^{2u} g_{\mathbb{S}^2},$$

so, using $df_0 \otimes df_0 = g_{\mathbb{S}^2}$, we obtain

$$df_0 \otimes dh + dh \otimes df_0 + dh \otimes dh = (e^{2u} - 1) \, g_{\mathbb{S}^2}. \tag{2.14}$$

Taking the trace-free part of (2.14), we obtain

$$\big(df_0 \otimes dh + dh \otimes df_0 + dh \otimes dh\big)_0 = 0,$$

where by $(T)_0$ we mean the trace free part of a tensor $T$. Let $(e_1, e_2)$ be a local $g_{\mathbb{S}^2}$-orthonormal frame, and write

$$h_i := dh(e_i) \in \mathbb{R}^3, \qquad v_{i,j} := \langle \nabla_{e_j} v, e_i \rangle.$$

Since $h = v + zn$, $df_0 \otimes n dz = f_0 df_0 \otimes dz = 0$ and $dn(e_i) = e_i = df_0(e_i)$ on the unit sphere, the normal component $zn$ contributes only a pure-trace term to $(df_0 \otimes dh)_0$, hence it drops out after taking the trace-free part. In these coordinates the trace-free conformality conditions are the two scalar equations

$$\begin{aligned} v_{1,1} - v_{2,2} &= -\frac{1}{2}\big(|h_1|^2 - |h_2|^2\big), \\ v_{1,2} + v_{2,1} &= -\langle h_1, h_2 \rangle. \end{aligned} \tag{2.15}$$

Equivalently, if we define the Cauchy–Riemann operator on vector fields by

$$Dv := \big(\mathscr{L}_v g_{\mathbb{S}^2}\big)_0 \in \Gamma(S_0^2 T^*\mathbb{S}^2), \qquad (Dv)_{11} = v_{1,1} - v_{2,2}, \ \ (Dv)_{12} = v_{1,2} + v_{2,1},$$

then (2.15) can be written compactly as

$$Dv = Q(dh), \tag{2.16}$$

where $Q(dh)$ is the trace-free quadratic form

$$Q(dh) := -(dh\otimes dh)_0, \quad \text{i.e.} \quad Q_{11} = -\frac{1}{2}\big(|h_1|^2 - |h_2|^2\big), \ \ Q_{12} = -\langle h_1, h_2\rangle.$$

In particular, $Q(dh)$ is quadratic in $dh$ and satisfies the pointwise bound

$$|Q(dh)| \leqslant C|dh|^2 \leqslant C|\nabla h|^2. \tag{2.17}$$

Before applying elliptic theory, we record the complex form of $D$. Assume that the frame $(e_1, e_2)$ above is positively oriented, let $J$ be the induced complex structure, and set $Z := \frac{1}{2}(e_1 - ie_2)$ and $\overline{Z} := \frac{1}{2}(e_1 + ie_2)$. For $q \in S_0^2 T^*\mathbb{S}^2$, let $q^\sharp$ be obtained by raising one index. Since $q_{22} = -q_{11}$, we have

$$q^\sharp = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & -q_{11}\end{pmatrix}, \qquad q^\sharp\overline{Z} = (q_{11} + iq_{12})Z.$$

Consequently, $\Phi(q) := q^\sharp|_{T^{0,1}\mathbb{S}^2}$ defines a real-linear bundle isomorphism

$$\Phi : S_0^2 T^*\mathbb{S}^2 \xrightarrow{\cong_{\mathbb{R}}} \mathrm{Hom}_{\mathbb{C}}\big(T^{0,1}\mathbb{S}^2, T^{1,0}\mathbb{S}^2\big).$$

For $v = v_1 e_1 + v_2 e_2$, write $v^{1,0} := \frac{1}{2}(v - iJv) = (v_1 + iv_2)Z$. At a fixed point, choosing the frame so that $\nabla e_i = 0$, we compute

$$\begin{aligned} 2(\bar\partial v^{1,0})(\overline{Z}) &= \big[v_{1,1} - v_{2,2} + i(v_{1,2} + v_{2,1})\big]Z \\ &= \big[(Dv)_{11} + i(Dv)_{12}\big]Z = \Phi(Dv)(\overline{Z}). \end{aligned}$$

Thus $\Phi(Dv) = 2\bar\partial v^{1,0}$. The holomorphic tangent bundle of the Riemann sphere satisfies $T^{1,0}\mathbb{S}^2 \simeq \mathscr{O}(2)$, where $\mathscr{O}(2)$ is the holomorphic line bundle of degree 2. Hence its space of holomorphic sections has complex dimension 3, while $H^{0,1}(\mathbb{S}^2, T^{1,0}\mathbb{S}^2) = 0$. Therefore $\ker D$ is the six-dimensional real space of conformal Killing fields, and $D$ is surjective. Standard elliptic theory gives the following estimate.

**Lemma 2.4** (Elliptic estimate for the tangential part)**.** *Let $1 < p < \infty$. There exists a constant $C_p < \infty$ such that the following holds: if $v$ is the tangential part of $h = f - f_0$ as above, then*

$$\|v\|_{W^{1,p}(\mathbb{S}^2)} \leqslant C_p\Big(\|Q(dh)\|_{L^p(\mathbb{S}^2)} + \|h\|^2_{W^{1,2}(\mathbb{S}^2)}\Big). \tag{2.18}$$

*In particular, using* (2.17)*,*

$$\|v\|_{W^{1,p}(\mathbb{S}^2)} \leqslant C_p\Big(\|\nabla h\|^2_{L^{2p}(\mathbb{S}^2)} + \|h\|^2_{W^{1,2}(\mathbb{S}^2)}\Big). \tag{2.19}$$

*Proof.* Decompose $v$ into its component orthogonal to $\ker D$ and its projection onto $\ker D$:

$$v = v_0 + v_{\ker}, \qquad v_0 \perp \ker D, \quad v_{\ker} \in \ker D,$$

where the orthogonality is taken with respect to the $L^2(\mathbb{S}^2, g_{\mathbb{S}^2})$ inner product. Using (2.16), there is an elliptic estimate

$$\|v_0\|_{W^{1,p}} \leqslant C_p\|Dv_0\|_{L^p} = C_p\|Dv\|_{L^p} = C_p\|Q(dh)\|_{L^p}.$$

Next, choose a fixed basis $A_1, A_2, A_3$ of $\mathfrak{so}(3)$. Then

$$\mathscr{B} := \{\nabla x^1, \nabla x^2, \nabla x^3, A_1 f_0, A_2 f_0, A_3 f_0\}$$

is a basis of $\ker D$. Since $v - v_{\ker} \perp \ker D$, the $L^2$-pairings of $v_{\ker}$ with the elements of $\mathscr{B}$ agree with those of $v$. Hence (2.11) and (2.12), together with finite-dimensional norm equivalence, give

$$\|v_{\ker}\|_{W^{1,p}(\mathbb{S}^2)} \leqslant C_p \|h\|^2_{W^{1,2}(\mathbb{S}^2)}.$$

Combining the two parts, we obtain

$$\|v\|_{W^{1,p}(\mathbb{S}^2)} \leqslant C_p\Big(\|Q(dh)\|_{L^p(\mathbb{S}^2)} + \|h\|^2_{W^{1,2}(\mathbb{S}^2)}\Big),$$

which is (2.18).

Finally, by (2.17) we have

$$\|Q(dh)\|_{L^p(\mathbb{S}^2)} \leqslant C\,\||\nabla h|^2\|_{L^p} = C\,\|\nabla h\|^2_{L^{2p}(\mathbb{S}^2)},$$

and substituting this into (2.18) yields (2.19). □

We write

$$E[h] := \int_{\mathbb{S}^2} \big(|\nabla(f_0+h)|^2 - |\nabla f_0|^2\big)\,dx$$

for the Dirichlet energy difference.

**Lemma 2.5.** *Under the above assumptions, the following identities hold:*

$$E[h] = 4\int_{\mathbb{S}^2} z\,dx + \int_{\mathbb{S}^2} |\nabla h|^2\,dx, \tag{2.20}$$

$$E[h] = -2\int_{\mathbb{S}^2} \Delta(f_0+h)\cdot h\,dx - \int_{\mathbb{S}^2} |\nabla h|^2\,dx, \tag{2.21}$$

*Proof.* By definition,

$$E[h] = \int_{\mathbb{S}^2} \big(|\nabla(f_0+h)|^2 - |\nabla f_0|^2\big)\,dx = 2\int_{\mathbb{S}^2} \nabla f_0\cdot\nabla h\,dx + \int_{\mathbb{S}^2} |\nabla h|^2\,dx. \tag{2.22}$$

Noting $-\Delta f_0 = 2f_0$, we know

$$2\int_{\mathbb{S}^2} \nabla f_0\cdot\nabla h\,dx = -2\int_{\mathbb{S}^2} \Delta f_0\cdot h dx = 4\int_{\mathbb{S}^2} f_0\cdot h dx = 4\int_{\mathbb{S}^2} z dx.$$

Substituting this into (2.22) yields (2.20). Again by (2.22), we know

$$E[h] = 2\int_{\mathbb{S}^2} \nabla(f_0+h)\cdot\nabla h - \int_{\mathbb{S}^2} |\nabla h|^2 dx = -2\int_{\mathbb{S}^2} \Delta(f_0+h)\cdot h dx - \int_{\mathbb{S}^2} |\nabla h|^2 dx,$$

which is (2.21). □

In what follows, $n_f$ and $n := f_0$ denote the exterior unit normals to $\Sigma = \partial\Omega$ and $\mathbb{S}^2 = \partial B_1$, respectively.

**Lemma 2.6** (Relation between the averages of $z$ and $z^2$)**.** *Under the normalization and smallness assumption $\delta := \|h\|_{W^{2,2}(\mathbb{S}^2)} \ll 1$, the scalar normal component $z$ satisfies*

$$\Big|\int_{\mathbb{S}^2} z\,dx + \int_{\mathbb{S}^2} z^2\,dx\Big| \leqslant C\delta^3. \tag{2.23}$$

*Proof.* By the divergence theorem,

$$\int_\Sigma f\cdot n_f\,d\mu_f = 3|\Omega|, \qquad \int_{\mathbb{S}^2} f_0\cdot n\,d\mu_{f_0} = 3|B_1|,$$

and by our volume normalization $|\Omega| = |B_1|$ these integrals coincide.

In local conformal coordinates we have

$$f \cdot (f_1 \times f_2)\, dy = f \cdot n_f \, d\mu_f, \qquad f_0 \cdot (f_0)_1 \times (f_0)_2 \, dy = f_0 \cdot n \, d\mu_{f_0},$$

so

$$\int_{\mathbb{S}^2} \big(f \cdot f_1 \times f_2 - f_0 \cdot (f_0)_1 \times (f_0)_2\big)\, dx = 0. \tag{2.24}$$

Fix a point and choose an oriented orthonormal frame $(e_1, e_2, n)$ in $\mathbb{R}^3$ with

$$e_1 = (f_0)_1, \quad e_2 = (f_0)_2, \quad n = e_1 \times e_2 = f_0.$$

Write $h = v + zn$ with $v \perp n$, and denote $f_i := \partial_i f$, $h_i := \partial_i h$, etc. Then

$$\begin{aligned} f \cdot f_1 \times f_2 - f_0 \cdot (f_0)_1 \times (f_0)_2 &= (f_0 + h) \cdot (f_0 + h)_1 \times (f_0 + h)_2 - n \cdot e_1 \times e_2 \\ &= h \cdot e_1 \times e_2 + n \cdot h_1 \times e_2 + n \cdot e_1 \times h_2 \\ &\quad + h \cdot h_1 \times e_2 + h \cdot e_1 \times h_2 + n \cdot h_1 \times h_2 + h \cdot h_1 \times h_2. \end{aligned}$$

We analyse the terms one by one in the frame $(e_1, e_2, n)$.

Firstly, since $h = v + zn$ and $e_1 \times e_2 = n$, we have

$$h \cdot e_1 \times e_2 = h \cdot n = z.$$

For the other terms, we use $h = v + zn$ and the identities

$$n \times e_1 = e_2, \qquad e_2 \times n = e_1.$$

For the first order term with derivative of $h$, the above identity implies

$$n \cdot h_1 \times e_2 = h_1 \cdot e_2 \times n = h_1 \cdot e_1 \quad \text{ and } \quad h_2 \cdot n \times e_1 = h_2 \cdot e_2.$$

Noting $\nabla z \cdot n = 0$, this further implies

$$n \cdot h_1 \times e_2 + n \cdot e_1 \times h_2 = \operatorname{div} h = \operatorname{div} v + 2z.$$

For the quadratic and cubic terms, expanding the cross products gives

$$\begin{aligned} h \times h_1 &= (v + zn) \times h_1 = v \times h_1 + zn \times (v_1 + z_1 n + z e_1) = z^2 e_2 + R_1, \\ h_2 \times h &= h_2 \times (v + zn) = h_2 \times v + (v_2 + z_2 n + z e_2) \times zn = z^2 e_1 + R_2, \\ h_1 \times h_2 &= v_1 \times h_2 + (z_1 n + z e_1) \times (v_2 + z_2 n + z e_2) = -z \nabla z + z^2 n + R_3, \end{aligned}$$

where

$$\begin{aligned} R_1 &:= v \times h_1 + zn \times v_1, \\ R_2 &:= h_2 \times v + z v_2 \times n, \\ R_3 &:= v_1 \times h_2 + (z_1 n + z e_1) \times v_2. \end{aligned}$$

These remainders satisfy

$$|R_1| + |R_2| + |R_3| \leqslant C\big(|v||\nabla h| + |\nabla v|(|h| + |\nabla h|)\big).$$

Consequently,

$$\begin{aligned} I &:= h \cdot h_1 \times e_2 = e_2 \cdot (z^2 e_2 + R_1) = z^2 + O(|R_1|), \\ II &:= h \cdot e_1 \times h_2 = e_1 \cdot (z^2 e_1 + R_2) = z^2 + O(|R_2|), \\ III &:= n \cdot h_1 \times h_2 = n \cdot (-z \nabla z + z^2 n + R_3) = z^2 + O(|R_3|), \end{aligned}$$

and the fully cubic term $h \cdot h_1 \times h_2$ is $O(|h|\,|\nabla h|^2)$.

Putting everything together, we obtain the pointwise expansion

$$f \cdot f_1 \times f_2 - f_0 \cdot (f_0)_1 \times (f_0)_2 = 3z + 3z^2 + \operatorname{div} v + R, \tag{2.25}$$

where

$$|R| \leqslant C\big((|v|+|\nabla v|)(|h|+|\nabla h|)+|h|\,|\nabla h|^2\big).$$

Integrating (2.25) over $\mathbb{S}^2$ and using (2.24) and $\int_{\mathbb{S}^2} \operatorname{div} v = 0$, we obtain

$$\int_{\mathbb{S}^2} z\,dx + \int_{\mathbb{S}^2} z^2\,dx = -\frac{1}{3}\int_{\mathbb{S}^2} R\,dx.$$

By Lemma 2.4 (with $p=2$) we have $\|v\|_{W^{1,2}} \leqslant C\delta^2$, while $\|h\|_{L^\infty}$ and $\|\nabla h\|_{L^2}$ are $O(\delta)$ by Sobolev embedding. Thus

$$\int_{\mathbb{S}^2} |R| \leqslant C\int_{\mathbb{S}^2}\big((|v|+|\nabla v|)(|h|+|\nabla h|)+|h|\,|\nabla h|^2\big) \leqslant C\delta^2\delta + C\delta\delta^2 \leqslant C\delta^3.$$

This yields (2.23). □

**Lemma 2.7** (Refined expansion of $|\nabla h|^2$)**.** *On the unit sphere $\mathbb{S}^2$, with $h = v + zn$, $n = f_0$ and $v \perp n$, we have the pointwise identity*

$$|\nabla(zn)|^2 = |\nabla z|^2 + 2z^2. \tag{2.26}$$

*In particular, there exists $C<\infty$ such that*

$$\left|\int_{\mathbb{S}^2} |\nabla h|^2\,dx - \int_{\mathbb{S}^2}\big(|\nabla z|^2+2z^2\big)\,dx\right| \leqslant C\delta^3. \tag{2.27}$$

*Proof.* The pointwise identity (2.26) follows from the decomposition

$$\nabla(zn) = (\nabla z)n + z\nabla n$$

and the relations $\nabla n\cdot n = 0$ together with $|\nabla n|^2 = 2$ on $\mathbb{S}^2$. For the integral estimate (2.27), by the decomposition

$$\nabla h = \nabla v + \nabla(zn)$$

and the relations $\nabla_a v\cdot n = -v\cdot a$, we rewrite

$$|\nabla h|^2 - (|\nabla z|^2+2z^2) = |\nabla v|^2 + 2\nabla v\cdot\nabla(zn) = |\nabla v|^2 + O(|\nabla v|(|h|+|\nabla h|))$$

Using the bounds on $v$ from Lemma 2.4 (with $p=2$) and the smallness of $h$, we obtain $\|v\|_{W^{1,2}} \lesssim \|\nabla h\|_{L^4}^2 + \|h\|_{W^{1,2}}^2 \lesssim \delta^2$, hence all terms involving $v$ contribute at most $C\delta^3$ after integration, which gives (2.27). □

**Lemma 2.8** (Quadratic expansion with parameter)**.** *Under the normalization and smallness assumption $\delta := \|h\|_{W^{2,2}(\mathbb{S}^2)} \ll 1$, for every fixed $c\in\mathbb{R}$, there exists $C = C(c) < \infty$ such that*

$$\begin{aligned} E[h] = &-2\int_{\mathbb{S}^2}\Big[\Delta(f_0+h) + c(f_1\times f_2)\Big]\cdot h\,dx \\ &- \int_{\mathbb{S}^2}|\nabla z|^2\,dx + 2(c-1)\int_{\mathbb{S}^2} z^2\,dx + \widetilde{R}_c[h], \end{aligned} \tag{2.28}$$

*where the error satisfies*

$$|\widetilde{R}_c[h]| \leqslant C(c)\,\delta^3. \tag{2.29}$$

*Proof.* It remains to treat the last term

$$2c\int_{\mathbb{S}^2}(f_1\times f_2)\cdot(f-f_0)\,dx \;=\; 2c\int_{\mathbb{S}^2}(f_1\times f_2)\cdot h\,dx.$$

We estimate it exactly as in Lemma 2.6. Using the volume constraint (2.24), we may rewrite

$$\int_{\mathbb{S}^2}(f_1\times f_2)\cdot h\,dx = \int_{\mathbb{S}^2}(e_1\times e_2 - f_1\times f_2)\cdot f_0\,dx.$$

Expanding $f = f_0 + h$ (hence $f_i = e_i + h_i$) yields

$$e_1 \times e_2 - f_1 \times f_2 = -(e_1 \times h_2 + h_1 \times e_2 + h_1 \times h_2),$$

and therefore

$$2c \int_{\mathbb{S}^2} (f_1 \times f_2) \cdot h\, dx = -2c \int_{\mathbb{S}^2} (e_1 \times h_2 + h_1 \times e_2 + h_1 \times h_2) \cdot f_0\, dx.$$

By the computation in Lemma 2.6, we get

$$(e_1 \times h_2 + h_1 \times e_2 + h_1 \times h_2) \cdot f_0 = \operatorname{div} v + 2z + z^2 + R',$$

where $R' = O(|\nabla v|(|h| + |\nabla h|))$. After integration and applying Lemma 2.6, we get

$$\begin{aligned} 2c \int_{\mathbb{S}^2} (f_1 \times f_2) \cdot h\, dx &= -4c \int_{\mathbb{S}^2} z\, dx - 2c \int_{\mathbb{S}^2} z^2\, dx + O(\delta^3) \\ &= 2c \int_{\mathbb{S}^2} z^2\, dx + O(\delta^3) \end{aligned}$$

Substituting this into

$$\begin{aligned} E[h] &= -2 \int_{\mathbb{S}^2} \Delta(f_0 + h) \cdot h\, dx - \int_{\mathbb{S}^2} |\nabla h|^2\, dx \\ &= -2 \int_{\mathbb{S}^2} \Big[\Delta(f_0 + h) + c f_1 \times f_2\Big] \cdot h\, dx - \int_{\mathbb{S}^2} |\nabla h|^2\, dx + 2c \int_{\mathbb{S}^2} (f_1 \times f_2) \cdot h\, dx, \end{aligned} \tag{2.30}$$

then applying Lemma 2.7 gives (2.28) with remainder estimate (2.29). □

**Proposition 2.9.** *Fix $c < 4$. There exist $\delta_0 = \delta_0(c) > 0$ and $C = C(c) < \infty$ such that the following holds. Assume the normalization and volume constraint above, and let*

$$\delta := \|h\|_{W^{2,2}(\mathbb{S}^2)} \leqslant \delta_0.$$

*Then*

$$E[h] \leqslant C\Big(\|H - c\|^2_{L^2(\Sigma)} + \delta^3\Big), \tag{2.31}$$

*where $H$ denotes the scalar mean curvature of $\Sigma = f(\mathbb{S}^2)$, and the $L^2$-norm is taken with respect to $d\mu_f$. Moreover, the constants $C$ and $\delta_0$ may be chosen uniformly when $c$ ranges in a compact subset of $(-\infty, 4)$.*

*Proof.* For a conformal immersion $f$ we have

$$\Delta f = -H(f_1 \times f_2),$$

so

$$\Delta(f_0 + h) + c(f_1 \times f_2) = (c - H)(f_1 \times f_2).$$

In particular,

$$|\Delta(f_0 + h) + c(f_1 \times f_2)| = \tfrac{1}{2}|H - c|\,|\nabla f|^2, \tag{2.32}$$

and hence

$$\int_{\mathbb{S}^2} |\Delta(f_0 + h) + c(f_1 \times f_2)|^2\, dx = \frac{1}{2} \int_{\Sigma} |H - c|^2 |\nabla f|^2\, d\mu_f. \tag{2.33}$$

By Cauchy–Schwarz and (2.32),

$$\left|\int_{\mathbb{S}^2} \Big[\Delta(f_0 + h) + c(f_1 \times f_2)\Big] \cdot h\, dx\right| \leqslant \Big\||H - c||\nabla f|\Big\|_{L^2(\Sigma)} \|h\|_{L^2(\mathbb{S}^2)}. \tag{2.34}$$

Since $|\nabla f|^2 = 2e^{2u}$ is uniformly comparable to 1, $\||H - c||\nabla f|\|_{L^2(\Sigma)} \leqslant C\|H - c\|_{L^2(\Sigma)}$.

Next, decomposing $h = v + zn$ and using Lemma 2.4 with $p = 2$, we obtain

$$\|h\|_{L^2} \leqslant \|v\|_{L^2} + \|z\|_{L^2} \leqslant C\delta^2 + \|z\|_{L^2}. \tag{2.35}$$

Combining (2.34) and (2.35) yields

$$\left|\int_{\mathbb{S}^2} \Big[\Delta(f_0 + h) + c(f_1 \times f_2)\Big] \cdot h\, dx\right| \leqslant C\,\|H - c\|_{L^2(\Sigma)}\big(\|z\|_{L^2} + \delta^2\big). \tag{2.36}$$

By Young's inequality, for any $\varepsilon > 0$,

$$C\,\|H - c\|_{L^2(\Sigma)}\big(\|z\|_{L^2} + \delta^2\big) \;\leqslant\; C_\varepsilon\,\|H - c\|^2_{L^2(\Sigma)} + \varepsilon\,(\|z\|^2_{L^2} + \delta^4). \tag{2.37}$$

We now study the quadratic form in $z$ appearing in (2.28):

$$-\int_{\mathbb{S}^2} |\nabla z|^2\, dx + 2(c-1)\int_{\mathbb{S}^2} z^2\, dx. \tag{2.38}$$

Decompose $z$ into spherical harmonics. By Lemma 2.3, the $L^2$–projection of $z$ onto the first eigenspace (eigenvalue $\lambda_1 = 2$) is $O(\delta^2)$. Moreover, Lemma 2.6 implies

$$\left|\int_{\mathbb{S}^2} z\, dx\right| \leqslant C\int_{\mathbb{S}^2} z^2\, dx + C\delta^3,$$

so the constant part of $z$ is also of size $O(\delta^2)$. Thus the low-frequency component of $z$ is $O(\delta^2)$, and on the orthogonal complement the Laplacian has spectrum bounded below by $\lambda_2 = 6$, yielding

$$\int_{\mathbb{S}^2} |\nabla z|^2\, dx \geqslant 6\int_{\mathbb{S}^2} z^2\, dx - C\delta^4. \tag{2.39}$$

Plugging (2.39) into (2.38) we get

$$-\int_{\mathbb{S}^2} |\nabla z|^2\, dx + 2(c-1)\int_{\mathbb{S}^2} z^2\, dx \leqslant -(6 - 2(c-1))\int_{\mathbb{S}^2} z^2\, dx + C\delta^4. \tag{2.40}$$

For $c < 4$ we have $6 - 2(c-1) > 0$, so the quadratic form in $z$ is strictly negative modulo an $O(\delta^4)$ error. In particular, for $\varepsilon > 0$ small and $\delta$ sufficiently small, the $\varepsilon\|z\|^2_{L^2}$ term in (2.37) can be absorbed into the negative contribution (2.40), and the remaining error is bounded by $C\delta^3$.

Finally, combining (2.28), (2.36), (2.37), (2.40) and the error bound (2.29), we obtain

$$E[h] \;\leqslant\; C\,\|H - c\|^2_{L^2(\Sigma)} + C\,\delta^3$$

for all $c < 4$ and $\delta$ sufficiently small. This proves (2.31) for the fixed $c < 4$. The preceding estimates also show that $C$ and $\delta_0$ may be chosen uniformly when $c$ ranges in a compact subset of $(-\infty, 4)$. □

**Corollary 2.10** ($W^{1,2}$–control and almost-orthogonality of $z$)**.** *Fix $c < 4$. There exist $\delta_0 = \delta_0(c) > 0$ and $C = C(c) < \infty$ such that the following holds. Assume the normalization and volume constraint above, and let*

$$h := f - f_0, \qquad h = v + zn, \qquad n = f_0, \qquad \delta := \|h\|_{W^{2,2}(\mathbb{S}^2)} \leqslant \delta_0.$$

*Let $H$ be the scalar mean curvature of $\Sigma = f(\mathbb{S}^2)$. Then*

$$\|h\|^2_{W^{1,2}(\mathbb{S}^2)} \leqslant C\Big(\|H - c\|^2_{L^2(\Sigma)} + \delta^3\Big), \tag{2.41}$$

$$\left|\int_{\mathbb{S}^2} z\, dx\right| \leqslant C\Big(\|H - c\|^2_{L^2(\Sigma)} + \delta^3\Big), \tag{2.42}$$

$$\left|\int_{\mathbb{S}^2} z\, x^i\, dx\right| \leqslant C\Big(\|H - c\|^2_{L^2(\Sigma)} + \delta^3\Big), \qquad i = 1, 2, 3, \tag{2.43}$$

*or equivalently*

$$\left|\int_{\mathbb{S}^2} z\,n\,dx\right| \leqslant C\Big(\|H-c\|^2_{L^2(\Sigma)}+\delta^3\Big). \tag{2.44}$$

*Proof.* From Proposition 2.9 we have

$$E[h] \leqslant C\Big(\|H-c\|^2_{L^2(\Sigma)}+\delta^3\Big). \tag{2.45}$$

Combining this with the identity (Lemma 2.5)

$$E[h] = 4\int_{\mathbb{S}^2} z\,dx + \int_{\mathbb{S}^2} |\nabla h|^2\,dx,$$

the refined gradient expansion (Lemma 2.7) and the relation between the average of $z$ and $z^2$ (Lemma 2.6), we obtain the quadratic form estimate

$$Q(z) := \int_{\mathbb{S}^2}\big(|\nabla z|^2 - 2z^2\big) \leqslant C\Big(\|H-c\|^2_{L^2(\Sigma)}+\delta^3\Big). \tag{2.46}$$

Let $P_{\leqslant 1}$ denote the $L^2$-orthogonal projection onto $\operatorname{span}\{1,x^1,x^2,x^3\}$. By Lemmas 2.3 and 2.6, $\|P_{\leqslant 1}z\|_{L^2(\mathbb{S}^2)} \leqslant C\delta^2$. Since the next eigenvalue of $-\Delta_{\mathbb{S}^2}$ is 6,

$$\|z\|^2_{W^{1,2}(\mathbb{S}^2)} \leqslant C\Big(Q(z)+3\|P_{\leqslant 1}z\|^2_{L^2(\mathbb{S}^2)}\Big) \leqslant C\Big(\|H-c\|^2_{L^2(\Sigma)}+\delta^3\Big).$$

For $v$, Lemma 2.4 (with $p=2$) yields

$$\|v\|_{W^{1,2}(\mathbb{S}^2)} \leqslant C\delta^2.$$

Thus

$$\|h\|^2_{W^{1,2}} \leqslant C\big(\|z\|^2_{W^{1,2}}+\|v\|^2_{W^{1,2}}\big) \leqslant C\Big(\|H-c\|^2_{L^2(\Sigma)}+\delta^3\Big), \tag{2.47}$$

which establishes (2.41).

Finally, Lemmas 2.3 and 2.6 imply

$$\left|\int_{\mathbb{S}^2} z\,x^i\,dx\right| \leqslant C\,\|h\|^2_{W^{1,2}(\mathbb{S}^2)} \quad \text{and} \quad \left|\int_{\mathbb{S}^2} z\,dx\right| \leqslant \int_{\mathbb{S}^2} z^2\,dx + C\delta^3,$$

and inserting (2.47) gives (2.43) and (2.42). Since $n=(x^1,x^2,x^3)$, (2.44) follows. As in Proposition 2.9, the constants may be chosen uniformly when $c$ ranges in a compact subset of $(-\infty,4)$. □

We note that, except for Proposition 2.9, the estimates in this section do not rely on an a priori Lipschitz bound for $f$ (or an $L^\infty$–bound for the conformal factor $u$). We denote by $n_f = n+\nu$ the unit normal to $\Sigma$ along $f$, so that $\nu$ measures the normal deviation of $f$ from $f_0$. The following lemma gives a quantitative control of the normal error $\nu$. As expected from standard computations in differential geometry, we show that $\nu$ is well approximated by $-\nabla z$, up to higher–order terms. This estimate will be useful in the sequel; its proof will make essential use of the qualitative $L^\infty$–smallness of $u$.

**Lemma 2.11** (Control of $\nu+\nabla z$ in $L^p$)**.** *Let $\delta := \|h\|_{W^{2,2}(\mathbb{S}^2)}$. Then for every $p\in[1,\infty)$ there exists a constant $C_p<\infty$ such that*

$$\|\nu+\nabla z\|_{L^p(\mathbb{S}^2)} \leqslant C_p\,\delta^2. \tag{2.48}$$

*In particular,*

$$\int_{\mathbb{S}^2} |\nu+\nabla z|^2\,dx \leqslant C\,\delta^4. \tag{2.49}$$

*Proof.* We express the normal variation $\nu$ via the normalized cross product. In local coordinates $(y^1, y^2)$ we have

$$\nu = 2|\nabla(f_0+h)|^{-2}(f_0+h)_1 \times (f_0+h)_2 - (f_0)_1 \times (f_0)_2.$$

Our basic smallness assumption implies that

$$\big||\nabla(f_0+h)|^2 - 2\big| \leqslant 1$$

pointwise. Writing

$$|\nabla(f_0+h)|^2 = 2+\delta_1,$$

where $\delta_1 := 2\nabla f_0 \cdot \nabla h + |\nabla h|^2 = O(|\nabla h| + |\nabla h|^2)$, we have

$$\frac{2}{|\nabla(f_0+h)|^2} = \frac{2}{2+\delta_1} = 1 - \frac{\delta_1}{2} + R_1 = 1 - \nabla f_0 \cdot \nabla h - \frac{|\nabla h|^2}{2} + R_1,$$

where $R_1 = \frac{\delta_1^2}{2(2+\delta_1)}$ and hence

$$|R_1| \leqslant C\delta_1^2 = C\big||\nabla(f_0+h)|^2 - 2\big|^2 \leqslant C(|\nabla h|^2 + |\nabla h|^4).$$

Expanding the cross product then yields

$$\nu = (f_0)_1 \times h_2 + h_1 \times (f_0)_2 - (\nabla f_0 \cdot \nabla h)n + R_2,$$

with $|R_2| \leqslant C(|\nabla h|^2 + |\nabla h|^4)$. Writing $h = v + zn$ and using

$$h_1 = v_1 + z_1 n + z e_1, \qquad h_2 = v_2 + z_2 n + z e_2,$$

we obtain

$$h_1 \times (f_0)_2 = v_1 \times e_2 - z_1 e_1 + zn, \qquad (f_0)_1 \times h_2 = e_1 \times v_2 - z_2 e_2 + zn,$$

thus by $\nabla f_0 \cdot \nabla h = \operatorname{div} h = \operatorname{div} v + 2z$, we get

$$\nu = -\nabla z + 2zn + O(|\nabla v|) - 2zn + R_2 = -\nabla z + R_3, \qquad |R_3| \leqslant C\big(|\nabla v| + |\nabla h|^2 + |\nabla h|^4\big). \tag{2.50}$$

By Sobolev embedding, for every $q < \infty$ there exists $C_q$ with

$$\|\nabla h\|_{L^q(\mathbb{S}^2)} \leqslant C_q \|h\|_{W^{2,2}(\mathbb{S}^2)} \leqslant C_q \delta.$$

From (2.50) and Lemma 2.4, we get, for $p \in [1,\infty)$,

$$\|\nu + \nabla z\|_{L^p} = \|R_3\|_{L^p} \leqslant C\Big(\|\nabla h\|^2_{L^{2p}} + \|h\|^2_{W^{1,2}} + \|\nabla h\|^4_{L^{4p}}\Big) \leqslant C_p\big(\delta^2 + \delta^4\big) \leqslant C_p \delta^2$$

for $\delta$ sufficiently small, which proves (2.48). In particular, taking $p = 2$ gives (2.49). □

## 3. CONTROLLING THE DEVIATION BY THE CMC DEFECT

In this section we prove that the geometric deviation

$$\|h\|_{W^{2,2}(\mathbb{S}^2)} + \|u\|_{L^\infty(\mathbb{S}^2)}$$

is controlled by the $L^2$–oscillation of the mean curvature. As a consequence, we prove Theorem 1.2.

Recall that the scalar mean curvature $H$ is defined with respect to the exterior normal $n_f = n + \nu$, and we denote its average by

$$\overline{H} := \frac{1}{\mathcal{H}^2(\Sigma)} \int_\Sigma H \, d\mu_f.$$

Under our assumptions we know that $|\overline{H} - 2| \ll 1$, thus $\overline{H} \in [1,3]$; hence the constants in Proposition 2.9 and Corollary 2.10 can be chosen uniformly for $c = \overline{H}$.

We first show that the average mean curvature is close to 2, with a quantitative bound in terms of $H - \overline{H}$ and $\delta$.

**Lemma 3.1** (Control of $\overline{H}-2$)**.** *There exists $C<\infty$ such that, for $\delta := \|h\|_{W^{2,2}(\mathbb{S}^2)}$ sufficiently small,*

$$|\overline{H}-2| \leqslant C\Big(\|H-\overline{H}\|^2_{L^2(\Sigma)} + \delta\,\|H-\overline{H}\|_{L^2(\Sigma)} + \delta^3\Big). \tag{3.1}$$

*Proof.* Let $\varphi := (n+\nu)\cdot f$. By the divergence theorem (or Minkowski formula) we have

$$\int_\Sigma H\varphi\, d\mu_f = 2\mathscr{H}^2(\Sigma).$$

We write

$$H\varphi = (H-\overline{H})(\varphi-\overline{\varphi}) + \overline{H}\,\overline{\varphi} + (H-\overline{H})\overline{\varphi} + \overline{H}(\varphi-\overline{\varphi}),$$

so that

$$2\mathscr{H}^2(\Sigma) = \int_\Sigma H\varphi\, d\mu_f = \int_\Sigma (H-\overline{H})(\varphi-\overline{\varphi})\, d\mu_f + \overline{H}\,\overline{\varphi}\,\mathscr{H}^2(\Sigma), \tag{3.2}$$

because the mixed terms integrate to zero.

On the other hand, by definition of $\varphi$ and the divergence theorem in $\mathbb{R}^3$,

$$\int_\Sigma \varphi\, d\mu_f = \int_\Sigma (n+\nu)\cdot f\, d\mu_f = \int_\Omega \operatorname{div} x\, dx = 3|\Omega| = 4\pi,$$

since $|\Omega| = |B_1| = 4\pi/3$. Thus

$$\overline{\varphi} = \frac{1}{\mathscr{H}^2(\Sigma)}\int_\Sigma \varphi\, d\mu_f = \frac{4\pi}{\mathscr{H}^2(\Sigma)}.$$

By the volume normalization and the isoperimetric inequality, $\mathscr{H}^2(\Sigma) \geqslant 4\pi$; hence $E[h] = 2(\mathscr{H}^2(\Sigma) - 4\pi) \geqslant 0$. Proposition 2.9 therefore gives

$$|\mathscr{H}^2(\Sigma) - 4\pi| = \frac{1}{2}E[h] \leqslant C\Big(\|H-\overline{H}\|^2_{L^2(\Sigma)} + \delta^3\Big).$$

We obtain

$$|\overline{\varphi}-1| \leqslant C\Big(\|H-\overline{H}\|^2_{L^2(\Sigma)} + \delta^3\Big). \tag{3.3}$$

We now estimate the fluctuation $\varphi - \overline{\varphi}$. Writing

$$\varphi - 1 = (n+\nu)\cdot f - n\cdot f_0 = (n+\nu)\cdot h + \nu\cdot f_0,$$

and using the bounds on $h$ and $\nu$ ( Lemma 2.11) and the bound of the conformal factor, we have

$$\|\varphi-1\|_{L^2(\Sigma)} \leqslant C\delta.$$

This implies

$$|\overline{\varphi}-1| \leqslant C\delta, \qquad \|\varphi-\overline{\varphi}\|_{L^2(\Sigma)} \leqslant C\delta. \tag{3.4}$$

Rewriting (3.2), we have

$$(2-\overline{H}\,\overline{\varphi})\,\mathscr{H}^2(\Sigma) = \int_\Sigma (H-\overline{H})(\varphi-\overline{\varphi})\, d\mu_f.$$

Hence

$$|2-\overline{H}\,\overline{\varphi}| \leqslant \frac{1}{\mathscr{H}^2(\Sigma)}\|H-\overline{H}\|_{L^2(\Sigma)}\,\|\varphi-\overline{\varphi}\|_{L^2(\Sigma)} \leqslant C\delta\,\|H-\overline{H}\|_{L^2(\Sigma)},$$

using (3.4) and the area bounds.

Finally,

$$\overline{H} - 2 = \overline{H}(1-\overline{\varphi}) + (\overline{H}\,\overline{\varphi} - 2),$$

so

$$|\overline{H}-2| \leqslant C|\overline{\varphi}-1| + C\delta\,\|H-\overline{H}\|_{L^2(\Sigma)} \leqslant C\Big(\|H-\overline{H}\|^2_{L^2(\Sigma)} + \delta\,\|H-\overline{H}\|_{L^2(\Sigma)} + \delta^3\Big),$$

where we used (3.3) in the last step. This is (3.1). $\square$

3.1. **Control of the vectorial defect $\vec H+2x$.** Let $\vec H$ denote the mean-curvature vector of $\Sigma$. With the convention above, $\vec H = -Hn_f = -H(n+\nu)$. We also write $x = f$ for the position vector along $\Sigma$.

**Lemma 3.2** (Control of $\vec H+2x$ in $L^2$)**.** *There exists $C<\infty$ such that, for $\delta := \|h\|_{W^{2,2}(\mathbb{S}^2)}$ sufficiently small,*

$$\|\vec H+2x\|^2_{L^2(\Sigma)} \leqslant C\Big(\|H-\overline{H}\|^2_{L^2(\Sigma)} + \delta\|H-\overline{H}\|_{L^2(\Sigma)} + \delta^3\Big). \tag{3.5}$$

*Proof.* By the Minkowski formula,

$$\int_\Sigma \vec H\cdot x\,d\mu_f = -\int_\Sigma H(x\cdot n_f)\,d\mu_f = -2\mathscr{H}^2(\Sigma).$$

Consequently,

$$\begin{aligned}\int_\Sigma |\vec H+2x|^2\,d\mu_f &= \int_\Sigma |\vec H|^2\,d\mu_f + 4\int_\Sigma \vec H\cdot x\,d\mu_f + 4\int_\Sigma |x|^2\,d\mu_f\\ &= \int_\Sigma \big(|\vec H|^2-4\big)\,d\mu_f + 4\int_\Sigma\big(|x|^2-1\big)\,d\mu_f.\end{aligned}\tag{3.6}$$

We first estimate the curvature term. Since $\int_\Sigma (H-\overline{H})\,d\mu_f = 0$, we have

$$\int_\Sigma \big(|\vec H|^2-4\big)\,d\mu_f = \int_\Sigma (H-\overline{H})^2\,d\mu_f + (\overline{H}^2-4)\mathscr{H}^2(\Sigma).$$

Since $\overline{H}^2-4 = (\overline{H}+2)(\overline{H}-2)$ and $|\overline{H}+2|$ is uniformly bounded for $\delta$ sufficiently small, Lemma 3.1 yields

$$\left|\int_\Sigma \big(|\vec H|^2-4\big)\,d\mu_f\right| \leqslant C\Big(\|H-\overline{H}\|^2_{L^2(\Sigma)} + \delta\|H-\overline{H}\|_{L^2(\Sigma)} + \delta^3\Big). \tag{3.7}$$

We next estimate the position term. Since $x = f = f_0+h$, $|f_0| = 1$, and $h = \nu + zn$, we have

$$|x|^2-1 = |f_0+h|^2-1 = 2f_0\cdot h + |h|^2 = 2z+|h|^2.$$

It follows that

$$\int_\Sigma (|x|^2-1)\,d\mu_f = 2\int_{\mathbb{S}^2} ze^{2u}\,dx + \int_{\mathbb{S}^2}|h|^2e^{2u}\,dx.$$

Using $2(e^{2u}-1) = 2\nabla f_0\cdot\nabla h + |\nabla h|^2$ and Corollary 2.10, we obtain

$$\begin{aligned}\left|\int_{\mathbb{S}^2} ze^{2u}\,dx\right| &\leqslant \left|\int_{\mathbb{S}^2} z(e^{2u}-1)\,dx\right| + \left|\int_{\mathbb{S}^2} z\,dx\right|\\ &\leqslant C\Big(\|H-\overline{H}\|^2_{L^2(\Sigma)}+\delta^3\Big).\end{aligned}\tag{3.8}$$

Similarly, Corollary 2.10 and the uniform boundedness of $e^{2u}$ give

$$\int_{\mathbb{S}^2}|h|^2e^{2u}\,dx \leqslant C\Big(\|H-\overline{H}\|^2_{L^2(\Sigma)}+\delta^3\Big).$$

Therefore,

$$\left|\int_\Sigma (|x|^2-1)\,d\mu_f\right| \leqslant C\Big(\|H-\overline{H}\|^2_{L^2(\Sigma)}+\delta^3\Big). \tag{3.9}$$

Combining (3.6), (3.7), and (3.9) proves (3.5). □

3.2. **Elliptic upgrade to $W^{2,2}$ and control of the conformal factor.** We now use the mean curvature equation to upgrade the $W^{1,2}$–control of $h$ to a $W^{2,2}$–estimate, and at the same time bound the conformal factor $u$. This completes the proof of Theorem 1.2.

Recall that for a conformal immersion $f$ we have

$$\Delta f = \vec{H}\, e^{2u},$$

while for the standard embedding $f_0(x) = x$ we have

$$\Delta f_0 = -2 f_0.$$

Thus, for $h = f - f_0$,

$$\Delta h = \vec{H}\, e^{2u} + 2 f_0. \tag{3.10}$$

**Proposition 3.3** (CMC stability estimate)**.** *There exist $\varepsilon_0 > 0$ and $C < \infty$ such that if $\|h\|_{W^{2,2}(\mathbb{S}^2)} + \|u\|_{C^0(\mathbb{S}^2)} \leqslant \varepsilon_0$, then*

$$\|h\|_{W^{2,2}(\mathbb{S}^2)} + \|u\|_{C^0(\mathbb{S}^2)} \leqslant C\, \|H - \overline{H}\|_{L^2(\Sigma)}. \tag{3.11}$$

*Proof.* We first estimate the right-hand side of (3.10). Write

$$\vec{H}\, e^{2u} + 2 f_0 = (\vec{H} + 2f)\, e^{2u} - 2f\,(e^{2u} - 1) + 2(f_0 - f),$$

where $f$ is the position vector along $\Sigma$ and we identify $\Sigma$ with $\mathbb{S}^2$ via $f$. The last term is simply

$$2(f_0 - f) = -2h,$$

so its $L^2$-norm is controlled by $\|h\|_{L^2}$.

To estimate the factor $e^{2u} - 1$, note the pointwise identity

$$2(e^{2u} - 1) = 2\nabla f_0 \cdot \nabla h + |\nabla h|^2.$$

Thus

$$\|e^{2u} - 1\|_{L^2(\mathbb{S}^2)} \leqslant C\Big(\|\nabla h\|_{L^2(\mathbb{S}^2)} + \|\nabla h\|^2_{L^4(\mathbb{S}^2)}\Big).$$

Since $f = f_0 + h$, $|f_0| = 1$, and $W^{2,2}(\mathbb{S}^2) \hookrightarrow L^\infty(\mathbb{S}^2)$, the smallness assumption gives $\|f\|_{L^\infty(\mathbb{S}^2)} \leqslant 1 + C\|h\|_{W^{2,2}(\mathbb{S}^2)} \leqslant C$. Hence

$$\|2f\,(e^{2u} - 1)\|_{L^2(\mathbb{S}^2)} \leqslant C\,\|e^{2u} - 1\|_{L^2(\mathbb{S}^2)} \leqslant C\Big(\|\nabla h\|_{L^2(\mathbb{S}^2)} + \|\nabla h\|^2_{L^4(\mathbb{S}^2)}\Big).$$

Finally, for the term $(\vec{H}+2f)\, e^{2u}$ we use the $L^2$control of $\vec{H}+2f$ from Lemma 3.2 and the fact that $e^{2u}$ is uniformly bounded when $\delta$ is small (this follows from the smallness of $h$ and the conformality relation). Thus

$$\|(\vec{H} + 2f)\, e^{2u}\|_{L^2(\mathbb{S}^2)} \leqslant C\,\|\vec{H} + 2f\|_{L^2(\Sigma)}.$$

Collecting these estimates we obtain

$$\|\vec{H}\, e^{2u} + 2f_0\|_{L^2(\mathbb{S}^2)} \leqslant C\Big(\|\vec{H} + 2f\|_{L^2(\Sigma)} + \|h\|_{W^{1,2}(\mathbb{S}^2)} + \|\nabla h\|^2_{L^4(\mathbb{S}^2)}\Big).$$

We may combine this with Lemma 3.2 and Corollary 2.10 to conclude that

$$\|\vec{H}\, e^{2u} + 2f_0\|_{L^2(\mathbb{S}^2)} \leqslant C\Big(\|H - \overline{H}\|_{L^2(\Sigma)} + \delta^{3/2} + \sqrt{\|H - \overline{H}\|_{L^2(\Sigma)}\delta}\Big). \tag{3.12}$$

We now apply the standard elliptic estimate

$$\|h\|_{W^{2,2}(\mathbb{S}^2)} \leqslant C\Big(\|\Delta h\|_{L^2(\mathbb{S}^2)} + \|h\|_{L^2(\mathbb{S}^2)}\Big).$$

Using (3.10), (3.12) and the $L^2$–bound for $h$ from Corollary 2.10, we obtain

$$\delta = \|h\|_{W^{2,2}(\mathbb{S}^2)} \leqslant C\Big(\|H - \overline{H}\|_{L^2(\Sigma)} + \delta^{3/2} + \sqrt{\|H - \overline{H}\|_{L^2(\Sigma)}\delta}\Big) \leqslant C_\varepsilon \|H - \overline{H}\|_{L^2(\Sigma)} + (\varepsilon + C\delta^{\frac{1}{2}})\delta.$$

For $\delta$ sufficiently small such that $C\delta^{\frac{1}{2}} \leqslant \frac{1}{4}$, taking $\varepsilon = \frac{1}{4}$, the terms involving $\delta$ on the right-hand side can be absorbed into the left-hand side, yielding

$$\|h\|_{W^{2,2}(\mathbb{S}^2)} \leqslant C\,\|H-\overline{H}\|_{L^2(\Sigma)}. \tag{3.13}$$

We finally estimate the conformal factor $u$ in terms of the geometric deviation $h = f - f_0$. Note that $u$ is already known to be small in $L^\infty$ by the qualitative theory.

We work in local conformal coordinates on $\mathbb{S}^2$. Fix a finite collection of conformal charts

$$\iota_\alpha : D^2 \longrightarrow \mathbb{S}^2,$$

with $D^2 \subset \mathbb{R}^2$ the unit disc, such that $\{\iota_\alpha(D^2)\}$ covers $\mathbb{S}^2$. In each chart we have conformal representations

$$(f_0\circ\iota_\alpha)^* g_{\mathbb{R}^3} = e^{2w_{0,\alpha}} g_{D^2}, \qquad (f\circ\iota_\alpha)^* g_{\mathbb{R}^3} = e^{2w_\alpha} g_{D^2},$$

for some $w_{0,\alpha} \in C^\infty(\overline{D^2})$ and $w_\alpha \in W^{2,2}(D^2)\cap L^\infty(D^2)$. By definition of $u$ we have

$$w_\alpha - w_{0,\alpha} = u\circ\iota_\alpha.$$

Fix $\alpha$ and set $\iota := \iota_\alpha$. For notational convenience, we write

$$f_0 := f_0\circ\iota, \qquad f := f\circ\iota, \qquad w_0 := w_{0,\alpha}, \qquad w := w_\alpha, \qquad u_\iota := u\circ\iota = w - w_0.$$

We denote by $f_i := \partial_i f$ and $(f_0)_i := \partial_i f_0$ the coordinate derivatives on $D^2$, so that

$$|f_i| = e^{w}, \qquad |(f_0)_i| = e^{w_0}.$$

We now introduce the normalized tangent frames

$$e_i^0 := \frac{(f_0)_i}{|(f_0)_i|} = e^{-w_0}(f_0)_i, \qquad e_i := \frac{f_i}{|f_i|} = e^{-w} f_i,$$

so that $\{e_1^0, e_2^0\}$ and $\{e_1, e_2\}$ are orthonormal frames for the pullback metrics of $f_0$ and $f$ on $D^2$, respectively. As is well known, the Gauss curvature can be written in terms of these frames as

$$-\Delta w_0 = *(de_1^0\wedge de_2^0), \qquad -\Delta w = *(de_1\wedge de_2),$$

where $*$ denotes the Hodge star on $D^2$.

Subtracting the two identities yields an equation for the difference $u_\iota = w - w_0$:

$$-\Delta u_\iota = *\big(d(e_1 - e_1^0)\wedge de_2\big) + *\big(de_1^0\wedge d(e_2 - e_2^0)\big) \;=: F_\iota. \tag{3.14}$$

Since $|f_i| = e^w$ and $u$ is small in $L^\infty$, we have a uniform lower bound $|f_i| \geqslant c > 0$ on $D^2$. Differentiating the definition $e_i = f_i/|f_i|$ and using the corresponding formula for $e_i^0$, one checks that

$$\|e_i - e_i^0\|_{L^2(D^2)} + \|d(e_i - e_i^0)\|_{L^2(D^2)} \;\leqslant\; C\,\|h\|_{W^{2,2}(D^2)}, \qquad i = 1,2. \tag{3.15}$$

On the other hand, $de_i$ and $de_i^0$ are uniformly bounded in $L^2$ on $D^2$ (the latter because $f_0$ is fixed and smooth, the former because $f$ is conformal and $h$ is small in $W^{2,2}$). Choose $D' \Subset D := D^2$ such that $\{\iota_\alpha(D')\}$ still covers $\mathbb{S}^2$. Let

$$E : W^{1,2}(D) \longrightarrow W^{1,2}(\mathbb{R}^2)$$

be a bounded extension operator, and set $\widetilde{e}_i := E(e_i)$ and $\widetilde{e}_i^0 := E(e_i^0)$. Then

$$\|\widetilde{e}_i\|_{W^{1,2}(\mathbb{R}^2)} + \|\widetilde{e}_i^0\|_{W^{1,2}(\mathbb{R}^2)} \leqslant C$$

and

$$\|d(\widetilde{e}_i - \widetilde{e}_i^0)\|_{L^2(\mathbb{R}^2)} \leqslant C\|h\|_{W^{2,2}(D)}, \qquad i = 1,2.$$

By Wente's inequality [13, 16, 17], there exists a unique solution $v_\iota$ of

$$-\Delta v_\iota = \widetilde{F}_\iota := *\big(d(\widetilde{e}_1 - \widetilde{e}_1^0)\wedge d\widetilde{e}_2\big) + *\big(d\widetilde{e}_1^0\wedge d(\widetilde{e}_2 - \widetilde{e}_2^0)\big)$$

with $v_\iota(\infty) = 0$, and

$$\|v_\iota\|_{L^\infty(\mathbb{R}^2)} \leqslant C\big(\|d(\widetilde{e}_1 - \widetilde{e}_1^0)\|_{L^2}\|d\widetilde{e}_2\|_{L^2} + \|d\widetilde{e}_1^0\|_{L^2}\|d(\widetilde{e}_2 - \widetilde{e}_2^0)\|_{L^2}\big) \leqslant C\|h\|_{W^{2,2}(D)}.$$

Since $-\Delta u_\iota = F_\iota = \widetilde{F}_\iota|_D$, the function $u_\iota - v_\iota$ is harmonic in $D$. Hence

$$\|u_\iota - v_\iota\|_{L^\infty(D')} \leqslant C\|u_\iota - v_\iota\|_{L^1(D)},$$

and consequently

$$\|u_\iota\|_{L^\infty(D')} \leqslant C\Big(\|u_\iota\|_{L^1(D)} + \|h\|_{W^{2,2}(D)}\Big).$$

Taking the maximum over the finitely many charts gives

$$\|u\|_{L^\infty(\mathbb{S}^2)} \leqslant C\Big(\|h\|_{W^{2,2}(\mathbb{S}^2)} + \|u\|_{L^1(\mathbb{S}^2)}\Big). \tag{3.16}$$

It remains to estimate $\|u\|_{L^1}$. Using the conformality relation $|\nabla f|^2 = 2e^{2u}, |\nabla f_0|^2 = 2$, we have

$$2(e^{2u} - 1) = |\nabla f|^2 - |\nabla f_0|^2 = 2\nabla f_0 \cdot \nabla h + |\nabla h|^2.$$

On the other hand, the inequality

$$|e^{2u} - 1 - 2u| \leqslant Cu^2$$

implies

$$\int_{\mathbb{S}^2} |u|\, dx \leqslant \frac{1}{2}\int_{\mathbb{S}^2} \Big|e^{2u} - 1\Big|\, dx + C\int_{\mathbb{S}^2} u^2\, dx.$$

Using the identity for $2(e^{2u} - 1)$,

$$\int_{\mathbb{S}^2} \Big|e^{2u} - 1\Big|\, dx \leqslant \int_{\mathbb{S}^2} |\nabla f_0 \cdot \nabla h|\, dx + \frac{1}{2}\int_{\mathbb{S}^2} |\nabla h|^2\, dx \leqslant C\|h\|_{W^{1,2}(\mathbb{S}^2)}.$$

Therefore

$$\int_{\mathbb{S}^2} |u|\, dx \leqslant C\|h\|_{W^{1,2}(\mathbb{S}^2)} + C\|u\|^2_{L^2(\mathbb{S}^2)}. \tag{3.17}$$

Inserting (3.17) into (3.16) and using $\|u\|_{L^\infty(\mathbb{S}^2)} \ll 1$, we finally obtain

$$\|u\|_{L^\infty(\mathbb{S}^2)} \leqslant C\|h\|_{W^{2,2}(\mathbb{S}^2)}. \tag{3.18}$$

Substituting this into (3.13), we get

$$\|u\|_{L^\infty(\mathbb{S}^2)} + \|h\|_{W^{2,2}(\mathbb{S}^2)} \leqslant C\|H - \overline{H}\|_{L^2(\Sigma)}.$$

This completes the proof of (3.11). □

## 4. From an area bound to the two–sphere–threshold Willmore bound

In this section we show that, in the small–defect regime, an area bound below the two–bubble threshold forces the two–sphere–threshold Willmore bound (1.3). The key input is the quantitative Alexandrov theorem of Julin–Niinikoski [10, Theorem 1.2].

**Proposition 4.1** (Area bound ⇒ two–sphere–threshold Willmore bound)**.** *Fix $\alpha \in (0, \frac{1}{2})$ and $\beta > 0$. There exists $\delta_0 = \delta_0(\alpha, \beta) > 0$ with the following property.*

*Let $\Omega \subset \mathbb{R}^3$ be a bounded domain whose boundary $\Sigma = \partial\Omega$ is a smooth embedded surface. After rescaling assume that $|\Omega| = \frac{4\pi}{3}$. If*

$$\mu_g(\Sigma) \leqslant 4\pi\sqrt[3]{2} - \beta, \qquad \|H - \overline{H}\|_{L^2(\Sigma)} \leqslant \delta_0, \tag{4.1}$$

*then*

$$\int_\Sigma |H|^2\, d\mu_g \leqslant 32\pi(1-\alpha). \tag{4.2}$$

*Proof.* Argue by contradiction. Then there exist $\delta_i \downarrow 0$ and bounded smooth domains $\Omega_i \subset \mathbb{R}^3$ with smooth embedded boundaries $\Sigma_i = \partial\Omega_i$ such that, after rescaling,

$$|\Omega_i| = \tfrac{4\pi}{3}, \qquad \mu_{g_i}(\Sigma_i) \leqslant 4\pi\sqrt[3]{2} - \beta, \qquad \|H_i - \overline{H}_i\|_{L^2(\Sigma_i)} \leqslant \delta_i, \tag{4.3}$$

but

$$\int_{\Sigma_i} |H_i|^2\, d\mu_{g_i} \;>\; 32\pi(1-\alpha) \qquad \text{for all } i. \tag{4.4}$$

Set $E_i := \Omega_i$ and $\lambda_i := \overline{H}_i$. Since $\mu_{g_i}(\Sigma_i)$ is uniformly bounded and $|E_i| = \frac{4\pi}{3}$, the hypotheses $P(E_i) \leqslant C_0$ and $|E_i| \geqslant 1/C_0$ in [10, Theorem 1.2] hold for a suitable $C_0 = C_0(\beta)$. For $i$ large we also have $\|H_{E_i} - \lambda_i\|_{L^2(\partial E_i)} = \|H_i - \overline{H}_i\|_{L^2(\Sigma_i)} \leqslant \delta_i$, so [10, Theorem 1.2] applies (with $n = 2$) and yields:

- a uniform bound $1/C \leqslant \lambda_i \leqslant C$, hence also $R_i := 2/\lambda_i \in [1/C', C']$;
- an integer $N_i \in \mathbb{N}$ and points $x_{i,1}, \dots, x_{i,N_i}$ with $|x_{i,k} - x_{i,\ell}| \geqslant 2R_i$ for $k \neq \ell$;
- writing $F_i := \bigcup_{k=1}^{N_i} B_{R_i}(x_{i,k})$, one has the perimeter estimate

$$|\mu_{g_i}(\Sigma_i) - 4\pi N_i R_i^2| \;\leqslant\; C\delta_i^{\frac{1}{64}}, \tag{4.5}$$

  and a corresponding Hausdorff–type closeness of $E_i$ to $F_i$ (in the sense stated in [10, Theorem 1.2]).

The Hausdorff–type closeness implies the volumes of $E_i$ and $F_i$ are asymptotically the same, hence

$$|F_i| \;=\; N_i \frac{4\pi}{3} R_i^3 \;=\; |E_i| + o(1) \;=\; \frac{4\pi}{3} + o(1).$$

In particular, after passing to a subsequence we may assume $N_i \equiv N$ is constant and $R_i \to R$, so that $NR^3 = 1$ and therefore $R = N^{-1/3}$. Taking limits in (4.5) gives

$$\lim_{i\to\infty} \mu_{g_i}(\Sigma_i) = 4\pi N R^2 = 4\pi N^{1/3}.$$

Combining with the uniform area bound in (4.3) yields

$$4\pi N^{1/3} \;\leqslant\; 4\pi\sqrt[3]{2} - \beta,$$

hence $N < 2$ and therefore $N = 1$. Consequently $R = 1$ and $\lambda_i = \overline{H}_i = 2/R_i \to 2$.

With $N = 1$, (4.5) becomes

$$|\mu_{g_i}(\Sigma_i) - 4\pi R_i^2| \;\leqslant\; C\delta_i^{\frac{1}{64}}, \qquad R_i = \frac{2}{\overline{H}_i}.$$

Equivalently,

$$\overline{H}_i^2\, \mu_{g_i}(\Sigma_i) \;=\; 16\pi + o(1). \tag{4.6}$$

On the other hand, since $\int_{\Sigma_i} (H_i - \overline{H}_i)\, d\mu_{g_i} = 0$, we have the exact identity

$$\int_{\Sigma_i} |H_i|^2\, d\mu_{g_i} \;=\; \int_{\Sigma_i} |H_i - \overline{H}_i|^2\, d\mu_{g_i} \;+\; \overline{H}_i^2\, \mu_{g_i}(\Sigma_i). \tag{4.7}$$

Using (4.3) and (4.6) in (4.7) we obtain

$$\int_{\Sigma_i} |H_i|^2\, d\mu_{g_i} \;=\; o(1) + 16\pi \;\longrightarrow\; 16\pi,$$

which contradicts (4.4) since $32\pi(1-\alpha) > 16\pi$ for $\alpha \in (0, \frac{1}{2})$. This contradiction proves (4.2). □

Proposition 4.1, together with Theorem 1.2, proves Theorem 1.3. In fact, for this implication it is enough to obtain a strict gap below the two–sphere threshold, with the size of the gap depending only on $\beta$. Proposition 4.1 gives the stronger statement for smooth boundaries, allowing any prescribed $\alpha \in (0, \frac{1}{2})$. We now prove the corresponding finite–perimeter statement for some $\alpha_\beta \in (0, \frac{1}{2})$.

**Lemma 4.2.** *For every $\beta > 0$, there exist $\alpha_\beta \in (0, \frac{1}{2})$ and $\delta_\beta > 0$ with the following property.*

*Let $E \subset \mathbb{R}^3$ be a bounded Caccioppoli set, write $\mu := |D\chi_E|$ and $\Sigma := \operatorname{spt}\mu$, and assume that the associated boundary varifold has generalized mean-curvature vector $\vec{H} \in L^2(\mu;\mathbb{R}^3)$. Let $\nu_E$ be the exterior unit normal and set $H := -\vec{H}\cdot\nu_E$. If*

$$|E| = \frac{4\pi}{3}, \qquad P(E) \leqslant 4\pi\sqrt[3]{2} - \beta, \qquad \|H - \overline{H}\|_{L^2(\mu)} \leqslant \delta_\beta,$$

*where $\overline{H} := P(E)^{-1}\int_\Sigma H\,d\mu$, then*

$$\int_\Sigma |H|^2\,d\mu \leqslant 32\pi(1-\alpha_\beta).$$

*Proof.* Set $\eta := \|H - \overline{H}\|_{L^2(\mu)}$. Let $\{\Sigma_j\}_{j\in J}$ be the connected components of $\Sigma$, and set $\mu_j := \mu \llcorner \Sigma_j$ and $m_j := \mu_j(\mathbb{R}^3)$. Simon's monotonicity formula [15] gives $\int_{\Sigma_j} |H|^2\,d\mu_j \geqslant 16\pi$ for every $j$. Thus $J$ is finite; write $J = \{1,\dots,N\}$. Moreover, $\int_{\Sigma_j} \nu_E\,d\mu_j = 0$ for every $j$.

Choose $a_j \in \Sigma_j$. The diameter estimate in [15, Lemma 1.1] gives

$$\operatorname{diam}\Sigma_j \leqslant C m_j^{1/2}\|H\|_{L^2(\mu_j)}. \tag{4.8}$$

Since the components are separated, following the proof of [10, Proposition 3.3], choose $X_j \in C^1_c(\mathbb{R}^3;\mathbb{R}^3)$ such that $X_j(x) = x - a_j$ near $\Sigma_j$ and $X_j = 0$ near the other components. Since the tangential divergence of $x - a_j$ is 2, the first variation formula and $\vec{H} = -H\nu_E$ give

$$2m_j = -\int_{\Sigma_j} \vec{H}\cdot(x-a_j)\,d\mu_j = \int_{\Sigma_j} H(x-a_j)\cdot\nu_E\,d\mu_j.$$

It follows that

$$2m_j - \overline{H}\int_{\Sigma_j}(x-a_j)\cdot\nu_E\,d\mu_j = \int_{\Sigma_j}(H-\overline{H})(x-a_j)\cdot\nu_E\,d\mu_j.$$

Since $\int_{\Sigma_j}\nu_E\,d\mu_j = 0$, the Gauss–Green formula gives

$$\sum_{j=1}^N \int_{\Sigma_j}(x-a_j)\cdot\nu_E\,d\mu_j = \int_{\partial^* E} x\cdot\nu_E\,d\mu = 3|E|.$$

Summing the componentwise identities, using (4.8), and applying Cauchy–Schwarz, we obtain

$$\begin{aligned}|2P(E) - 3\overline{H}|E|| &\leqslant C\sum_{j=1}^N m_j\|H-\overline{H}\|_{L^2(\mu_j)}\|H\|_{L^2(\mu_j)}\\ &\leqslant CP(E)\,\|H-\overline{H}\|_{L^2(\mu)}\|H\|_{L^2(\mu)}.\end{aligned} \tag{4.9}$$

Since $3|E| = 4\pi$, $4\pi \leqslant P(E) \leqslant 4\pi\sqrt[3]{2} - \beta$, and

$$\|H\|^2_{L^2(\mu)} = \eta^2 + \overline{H}^2 P(E),$$

estimate (4.9), together with $\|H\|_{L^2(\mu)} \leqslant \eta + P(E)^{1/2}|\overline{H}|$, implies, after decreasing $\delta_\beta$ and absorbing the resulting error,

$$\left|\overline{H} - \frac{P(E)}{2\pi}\right| \leqslant C_\beta\eta. \tag{4.10}$$

Consequently,

$$\int_\Sigma |H|^2\, d\mu = \eta^2 + \overline{H}^2 P(E) \leqslant \frac{P(E)^3}{4\pi^2} + C_\beta \eta \leqslant \frac{\left(4\pi\sqrt[3]{2} - \beta\right)^3}{4\pi^2} + C_\beta \eta.$$

The constant term on the right is strictly smaller than $32\pi$. We may therefore choose $\alpha_\beta \in (0, \frac{1}{2})$, and then choose $\delta_\beta > 0$ sufficiently small, so that

$$\int_\Sigma |H|^2\, d\mu \leqslant 32\pi(1 - \alpha_\beta).$$

□

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