THE POLARIZED DEGREE OF IRRATIONALITY OF $K3$ SURFACES

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Abstract. Given a polarized variety (X, L), we construct and study projections of low degree $X \dashrightarrow \mathbb{P}(H^0(L^\vee)) \dashrightarrow \mathbb{P}^n$ using the associated kernel bundles. As an application, we can show that the degree of irrationality of a very general $(1, 6)-$polarized abelian surface, as well as that of a very general $K3$ surface of genus 6 is 3. We also give new upper bounds on the degree of irrationality of $K3$ surfaces of any genus. We study the family of projections of minimal degree of a very general $K3$ surface of genus $4, 5, 6$. As a different application of our construction, we exhibit new generically finite rational maps of low degree from some hyper-Kähler varieties and abelian varieties to projective spaces.

Introduction

Given a quasi-projective variety X over an algebraically closed field, a natural invariant measuring how far X is from being rational is its *degree of irrationality*

$$\mathrm{irr}(X) = \min\Big\{\deg(\varphi) \;\Big|\; \varphi : X \dashrightarrow \mathbb{P}^{\dim(X)} \text{ is a generically finite rational map}\Big\}.$$

The degree of irrationality of a curve is called its *gonality* and is a classical invariant. In contrast, when $\dim(X) \geq 2$ very little is known in general, and the problem has only recently received considerable attention. The invariant $\mathrm{irr}(X)$ was introduced in [20]. This invariant has been studied in a number of cases, including hypersurfaces [3, 5, 9, 26, 30], abelian varieties [2, 7, 10, 17, 27, 28, 31], hyper-Kähler varieties [29], and other classes of examples [1, 11]. For a recent survey the reader may consult [8]. Giving sharp lower bounds is, in general, a difficult problem. In this paper, among other things, we compute the degree of irrationality of a very general $K3$ surface of genus 6 and of a very general $(1, 6)$-polarized abelian surface. The latter was one of the remaining open cases for very general $(1, d)$-polarized abelian surfaces in view of [7, 17]. The only other case in which the degree of irrationality of a very general $(1, d)$-polarized abelian surface was known to be 3 is $d = 2$, as proven in [27].

More generally, we introduce and study a natural invariant of polarized projective varieties closely related to the degree of irrationality. Let (X, L) be a polarized projective variety of dimension $n \geq 2$, where L is an ample line bundle. For any subspace $V \subset H^0(L)$ of dimension $n + 1$ we obtain a rational map

$$\varphi_V : X \dashrightarrow \mathbb{P}^n.$$

We define

$$\mathrm{irr}_L(X) = \inf\Big\{\deg(\varphi_V) \;\Big|\; V \in \mathrm{Gr}(n{+}1, H^0(L)),\ V \text{ generates } L \text{ in codimension 1 and } \varphi_V \text{ is generically finite}\Big\}.$$

Equivalently, given the map $X \dashrightarrow \mathbb{P}H^0(L)^\vee$, we aim to study the projections $\mathbb{P}H^0(L)^\vee \dashrightarrow \mathbb{P}^n$ such that the induced map $X \dashrightarrow \mathbb{P}^n$ is generically finite and has minimal degree.

The curve case. Let C be a general curve of genus g, and let $L \in \mathrm{Pic}^e(C)$ be a general line bundle of degree $e \geq g + 1$. In this case it is elementary to see that

$$\mathrm{irr}_L(C) = \min\Big\{ d \;\Big|\; L \in W^1_d(C) + W^0_{e-d}(C)\Big\},$$

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where $W^r_d(C) \subset \mathrm{Pic}^d(C)$ denotes the Brill–Noether locus of line bundles of degree d carrying at least $r+1$ linearly independent global sections and $W^1_d(C) + W^0_{e-d}(C)$ is the image of the summation map

$$W^1_d(C) \times W^0_{e-d}(C) \longrightarrow \mathrm{Pic}^e(C), \qquad (A, B) \longmapsto A \otimes B.$$

Since L is general and the summation map is of maximal rank, the condition $L \in W^1_d(C) + W^0_{e-d}(C)$ is equivalent to requiring that the locus $W^1_d(C) + W^0_{e-d}(C)$ has dimension g (i.e. dominates $\mathrm{Pic}^e(C)$). Thus $\mathrm{irr}_L(C)$ is the minimal d for which

$$\dim W^1_d(C) + \dim W^0_{e-d}(C) \ \geq \ g.$$

By Brill–Noether theory, for a general curve one has

$$\dim W^1_d(C) = \min\{\rho(g,1,d), g\} = \min\{2d-g-2, g\}, \qquad \dim W^0_{e-d}(C) = \min\{e-d, g\}.$$

Plugging these expressions into the inequality above yields $\mathrm{irr}_L(C) = \max\{2g+2-d, \lfloor \frac{g+3}{2} \rfloor\}$.

For a surface S and a non-trivial indecomposable class $L \in \mathrm{Pic}(S)$ (i.e. L cannot be written as the tensor product of two non-trivial effective line bundles), the V generates L in codimension 1 and the rational map φ_V is generically finite for any $V \in \mathrm{Gr}(3, H^0(L))$. For $K3$ surfaces we obtain the following.

Theorem A. *Let (S, L) be a $K3$ surface of genus $g = 2 + 2n(n+1) + k$. Then*

$$\mathrm{irr}_L(S) \leq 2 + n + \left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{k}{4} \right\rfloor.$$

Moreover, if $g = 3, 4, 6$ and S is very general, then $\mathrm{irr}_L(S) = 3$. If $g = 5$ and S is very general, then $\mathrm{irr}_L(S) = 4$.

In the paper we present the bound for $K3$ surfaces of Picard rank 1, and the corresponding statement for arbitrary $K3$ surfaces follows from general specialization results; see [9, Corollary C]. The bounds we obtain improve those in [26]. In low genus they are optimal up to genus 8 (in [21] it is proven that $\mathrm{irr}_L(S) = 4$ for $g = 7, 8, 9, 11$), and asymptotically the improvement over [26] is by a factor between 3 and 6, depending on the genus. Let us also point out that after the appearance of the first version of this paper the bounds were further improved: in [21] the bounds were sharpened for $9 \leq g \leq 14$ by combining the method of the present paper with derived category tools (Fourier–Mukai transforms and Bridgeland stability conditions), while in [14] our approach was optimized to yield better asymptotic bounds for $k \gg 0$ in Theorem A.

Our technique relies on an elementary yet powerful observation, which we briefly outline below. It can be seen as a natural variation of the Lazarsfeld–Mukai bundles introduced to study linear series on curves on $K3$ surfaces; see [16]. Let X be a variety of dimension n and let $|V| \subset |L|$ be a linear system inducing a rational map $\varphi_V : X \dashrightarrow \mathbb{P}^n$. One may then construct the associated kernel reflexive sheaf, fitting into an exact sequence

$$0 \longrightarrow E^\vee \longrightarrow V \otimes \mathcal{O}_X \longrightarrow L.$$

Via the identification $\mathrm{Gr}(n, V) = \mathbb{P}V^\vee \subset \mathbb{P}(H^0(E))$, the fibers of φ_V can be described as zero loci of sections of E. More precisely, the fiber above $[s] \in \mathbb{P}V^\vee = \mathbb{P}^n$ is $Z(s)$ outside the base locus of the map (see Proposition 1.5). This provides a quick way to compute the degree of φ_V.

The main idea of this paper is that, under suitable hypotheses on E, this construction can be reversed. Namely, if one is given a sufficiently general vector bundle E of rank n together with at least $n+1$ linearly independent global sections generating E in codimension 1, then one obtains a generically finite rational map of degree $\leq c_n(E)$. A simple corollary, proved along these lines in [22], is that if X carries a globally generated vector bundle E with $c_n(E) > 0$, then $\mathrm{irr}(X) \leq c_n(E)$.

Moreover, if E has many global sections, one can impose base points and make the degree drop. The basic example to keep in mind is a general $K3$ surface S of genus 6. In this case there is a stable vector bundle E of rank 2 on S with $h^0(E) = 5$ and $c_2(E) = 4$. Then, for any point $P \in S$, the map defined by

$$V = H^0(E \otimes \mathcal{I}_P)^\vee \subset H^0(L)$$

has degree $c_2(E) - 1 = 3$; see also Theorem 2.5. More generally, Theorem A is a direct application of Proposition 1.5. We now outline some further applications.

For a surface S with $\mathrm{NS}(S) = \mathbb{Z} \cdot L$, this technique can be used to give a scheme structure to the locus of special maps; see Theorem 6.6. We endow the locus

$$W^r_d(S, L) = \left\{ V \in \mathrm{Gr}(r+1, H^0(L)) \;\middle|\; \deg(\varphi_V) \leq d \right\}$$

with a natural structure of algebraic variety. Some explicit computations are as follows.

Theorem B. *Let (S, L) be a $K3$ surface of genus g with $\mathrm{Pic}(S) = \mathbb{Z} \cdot L$. The following holds:*

(1) if $g = 4$, then $W^2_3(S, L) = \mathbb{P}^3$;
(2) if $g = 5$, then $W^2_3(S, L) = \emptyset$ and $W^2_4(S, L)$ has two components, one birational to S and one birational to a $\mathbb{P}^3$-bundle over the moduli space of stable rank 2 vector bundles with $c_2 = 4$;
(3) if $g = 6$, then $W^2_3(S, L) = S$.

A more geometric interpretation of Theorem B is the following. If $S \subset \mathbb{P}^4$ is a very general $K3$ surface of degree 6, then the locus of trisecant lines[1] to S is isomorphic to $\mathbb{P}^3$. If $S \subset \mathbb{P}^5$ is a very general $K3$ surface of degree 8, then there are no 4-secant planes tangent to S. If $S \subset \mathbb{P}^6$ is a very general $K3$ surface of degree 10, then for any $p \in S$ there exists a unique 6-secant codimension-3 linear subspace tangent to S at p. A similar theorem for $K3$ surfaces up to genus 14 is developed in [21, Theorem A and Theorem B], suggesting interesting patterns for the dimension of the loci $W^2_d(S, L)$; see [21, Section 8]. Other kinds of polarized surfaces are also interesting from this perspective: for example, the case of $(S, 3\Theta)$ for a very general principally polarized abelian surface (S, Θ) should be accessible with the methods of the present paper.

We give a few elementary applications to other kinds of varieties. For a surface S we denote by $S^{[n]}$ the Hilbert scheme of zero-dimensional subschemes of length n.

Theorem C. *The following bounds hold:*

(1) if (S, L) is a very general $(1, 6)$-polarized abelian surface, then $\mathrm{irr}(S) = \mathrm{irr}_L(S) = 3$;
(2) if (S, L) is a very general $K3$ surface of genus 6, then $\mathrm{irr}(S^{[2]}) \leq 6$;
(3) if (A, L) is a very general abelian threefold of type $(1, 3, 12)$ or $(1, 6, 6)$, then $\mathrm{irr}_L(A) \leq 8$;
(4) if (S, L) is a very general $K3$ surface of genus 10, then $\mathrm{irr}(S^{[3]}) \leq 20$.

Let us remark that the case of $(1, 6)$-polarized abelian surfaces settles one of the remaining exceptional cases for abelian surfaces. More precisely, the degree of irrationality of a very general $(1, d)$-polarized abelian surface is either 3 or 4 for every positive integer d. It is known to be 4 when $d \neq 1, 2, 3, 6$ (see [7, 17]), and it is known to be 3 when $d = 2$ (see [27]). The first item in Theorem C addresses the case $d = 6$. In a somewhat similar spirit, Mason [18] completed the classification of the degree of irrationality of hyperelliptic surfaces initiated by Yoshihara [32].

For very general abelian threefolds, the best lower bound currently known is 5 (see [10]). In [22], Picard bundles on Jacobians are studied, yielding the upper bound 2^g for the degree of irrationality of genus g Jacobians.

[1] A linear space is k-secant to S if its scheme theoretic intersection with S is of length k.

For hyper-Kähler varieties of dimension $2n$ Voisin shows in [29] that the degree of irrationality is at least $n+1$.

Structure of the paper. In section 1 we explain how to construct and compute the degree of a generically finite rational map starting from the kernel sheaf. In section 2 the technique is applied to study projections of the lowest possible degree for a very general $K3$ surface $S \subset \mathbb{P}^g$ of genus $4, 5, 6$ (see Theorem B). In section 3 we study the case of $(1, 6)$-polarized abelian surfaces. In section 4 we obtain the asymptotic bound of Theorem A. We prove Theorem C in section 5. Finally, in section 6 we prove that the loci $W^r_d(S, L)$ are algebraic varieties (see Theorem 6.6).

Each section presents an independent application of the method introduced in section 1, and the sections are mutually independent. The contents of section 1 and section 3 hold over any algebraically closed field. The contents of the other sections hold over $\mathbb{C}$.

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1. Constructing rational maps via the kernel bundle.

We begin with some notation and definitions.

Set-up 1.1. We denote by

- X a smooth projective variety of dimension n;
- E a reflexive sheaf of rank n on X;
- $V^\vee$ an $(n+1)$-dimensional subspace of $H^0(E)$, i.e. $V^\vee \in \mathrm{Gr}(n+1, H^0(E))$;
- $\mathrm{Bl}(V^\vee) \subset X$ the union of the locus where the sections of $V^\vee$ do not generate E and the locus where E is not locally free.[2]

For a general section $s \in V^\vee$, let $Z_{\mathrm{cycle}}(s)$ denote the zero-cycle associated with the zero-dimensional part of $Z(s)$.

We define A_s to be the maximal effective subcycle of $Z_{\mathrm{cycle}}(s)$ supported on $X \setminus \mathrm{Bl}(V^\vee)$. We then set

$$c_n(V^\vee) := \deg(A_s).$$

[2] We will sometimes write Bl instead of $\mathrm{Bl}(V^\vee)$ when the dependence on $V^\vee$ is clear from the context.

Note that, since $V^\vee$ generates E away from $\mathrm{Bl}(V^\vee)$, a general section $s \in V^\vee$ has a zero-dimensional vanishing locus on $X \setminus \mathrm{Bl}(V^\vee)$, so $c_n(V^\vee)$ is well defined.

If, moreover, a general section $s \in V^\vee$ vanishes along a zero-dimensional subscheme, we define $bl(V^\vee)$ as the degree of the complementary part supported on $\mathrm{Bl}(V^\vee)$. Namely, writing

$$Z_{\mathrm{cycle}}(s) = A_s + A'_s,$$

where A'_s is supported on $\mathrm{Bl}(V^\vee)$, and set

$$bl(V^\vee) := \deg(A'_s).$$

Remark 1.2. In general, the support of A'_s is a proper subset of $\mathrm{Bl}(V^\vee)$.

We will be interested in *pairs* $(E, V^\vee)$ that satisfy some special properties. For the purposes of this paper we will refer to them as *good pairs*.

Definition 1.3. A pair $(E, V^\vee)$, with E a reflexive sheaf of rank n and $V^\vee \in \mathrm{Gr}(n+1, H^0(E))/\mathrm{Aut}(E)$, is a *good pair* if $V^\vee$ generates E in codimension 1 and $H^0(E^\vee) = 0$.

Given a good pair $(E, V^\vee)$, the kernel of the evaluation $\mathrm{ev} : V^\vee \otimes \mathcal{O}_X \to E$ is $L^\vee$, where $L = \det(E)$. Indeed, since the evaluation is surjective in codimension 1, we get $c_1(\mathrm{Im}(ev)) = c_1(E)$ and $\mathrm{Im}(ev)^\vee = E^\vee$ (recall that E is reflexive). Dualizing, we get an exact sequence

$$0 \longrightarrow E^\vee \longrightarrow V \otimes \mathcal{O}_X \longrightarrow L.$$

In particular, to any good pair $(E, V^\vee)$, we can associate the rational map induced by (L, V) (notice that since $h^0(E^\vee) = 0$ we have $V \subset H^0(L)$, therefore, the map is non-degenerate). We will denote the induced rational map by $\varphi_V : X \dashrightarrow \mathbb{P}V^\vee$. It is easy to see that the set-theoretic support of the base locus of the rational map coincides with $\mathrm{Bl}(V^\vee)$. For instance, one can observe that V coincides with the image of $\bigwedge^n V^\vee \to H^0(L)$ induced by the determinant map on the global sections of E. In fact, the locus where V does not generate L coincides with the union of the locus where $V^\vee$ does not generate E and the locus where E is not locally free.

On the other hand, as mentioned in the introduction, if one starts with a non-degenerate rational map $X \dashrightarrow \mathbb{P}^n$ (hence, a line bundle $L \in \mathrm{Pic}(X)$ together with $V \in \mathrm{Gr}(n+1, H^0(L))$ generating L in codimension 1), one may construct the kernel sheaf $E^\vee = \ker(V \otimes \mathcal{O}_X \to L)$ together with $V^\vee \subset H^0(E)$. We have the following.

Lemma 1.4. *Given a non-degenerate rational map $X \dashrightarrow \mathbb{P}^n$ the pair $(E, V^\vee)$ is a good pair.*

Proof. On a smooth variety any rational map is well defined in codimension 1, hence the sequence

$$0 \longrightarrow E^\vee \longrightarrow V \otimes \mathcal{O}_X \longrightarrow L,$$

is exact in codimension 1. Dualizing, we get that $V^\vee$ generates E in codimension 1. Moreover, E is the kernel of a morphism between vector bundles, so it is reflexive. The morphism $V \to H^0(L)$ is injective, this implies $H^0(E^\vee) = 0$. We showed that $(E, V^\vee)$ is a good pair. □

We can now prove the key proposition that will be used throughout the paper: there is a bijective correspondence between rational maps of degree d and good pairs $(E, V^\vee)$ with $c_n(V^\vee) = d$. The invariant $c_n(V^\vee)$ will make the computation of the degree of certain rational maps much easier.

Proposition 1.5. *Let X be a smooth projective variety of dimension n. There is a* one-to-one *correspondence*

$$\{\text{non-degenerate rational maps } X \dashrightarrow \mathbb{P}^n \text{ of degree } d\} \leftrightarrow \{\text{good pairs } (E, V^\vee) \text{ with } c_n(V^\vee) = d\},$$

$$(L, V) \longrightarrow (E, V^\vee).$$

Moreover, the fibers of the rational map $\varphi_V : X \dashrightarrow \mathbb{P}V^\vee$ are computed by $\varphi_V^{-1}[s] \cap (X - \mathrm{Bl}) = Z(s) \cap (X - \mathrm{Bl})$, where s can be seen both as a point of the target projective space (left hand side of the equality) and as a section of the vector bundle E (right hand side of the equality). In particular, if E is a vector bundle and a general element of $V^\vee$ vanishes along a zero-dimensional subscheme, then

$$c_n(V^\vee) = \deg(\varphi_V) = c_n(E) - bl(V^\vee).$$

Proof. The map $(E, V^\vee) \mapsto (L, V)$ is clearly a bijection between good pairs and non-degenerate rational maps $X \dashrightarrow \mathbb{P}^n$. Indeed, given a rational map one can reconstruct the kernel sheaf

$$E^\vee = \ker\big(V \otimes \mathcal{O}_X \to L\big)$$

together with $V^\vee \subset H^0(E)$ (uniquely up to the action of $\mathrm{Aut}(E)$ on $H^0(E)$), and Lemma 1.4 shows that $(E, V^\vee)$ is a good pair.

It now suffices to show the second part of the proposition, as it implies that rational maps of degree d correspond to good pairs $(E, V^\vee)$ with $c_n(V^\vee) = d$. We abuse notation and denote by φ_V the induced morphism

$$\varphi_V : (X \setminus \mathrm{Bl}) \to \mathbb{P}V^\vee.$$

The Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}V^\vee}(-1) \longrightarrow V^\vee \otimes \mathcal{O}_{\mathbb{P}V^\vee} \longrightarrow T_{\mathbb{P}V^\vee}(-1) \longrightarrow 0$$

pulls back to

$$0 \longrightarrow L^\vee_{|X\setminus \mathrm{Bl}} \longrightarrow V^\vee \otimes \mathcal{O}_{X\setminus \mathrm{Bl}} \longrightarrow E_{|X\setminus \mathrm{Bl}} \longrightarrow 0$$

on $X \setminus \mathrm{Bl}$. In particular,

$$E_{|X\setminus \mathrm{Bl}} \cong \varphi_V^* T_{\mathbb{P}V^\vee}(-1),$$

and pullback induces an identification

$$\varphi_V^* : H^0(T_{\mathbb{P}V^\vee}(-1)) \xrightarrow{\sim} V^\vee \subset H^0(E).$$

In particular, a general section of $T_{\mathbb{P}V^\vee}(-1)$ vanishes at the corresponding point $[s] \in \mathbb{P}V^\vee$ (and at no other point, since $c_n(T_{\mathbb{P}V^\vee}(-1)) = 1$). Hence the fiber over $[s]$ of the map $\varphi_V : X \setminus \mathrm{Bl} \to \mathbb{P}V^\vee$ is $Z(\varphi_V^* s)$.

The degree statement follows because $c_n(V^\vee)$ is, by definition, the degree of $Z(s) \cap (X \setminus \mathrm{Bl})$ for $s \in V^\vee$ general.

Finally, the equality $c_n(V^\vee) = c_n(E) - bl(V^\vee)$ follows from the definition of $c_n(V^\vee)$ together with the standard fact that if a section of a rank-n vector bundle E vanishes along a zero-dimensional subscheme, then its length is exactly $c_n(E)$. Indeed, for $s \in V^\vee$ general we can write the associated zero-cycle as

$$Z_{\mathrm{cycle}}(s) = A_s + A'_s,$$

where A_s is supported on $X \setminus \mathrm{Bl}(V^\vee)$ and A'_s is supported on $\mathrm{Bl}(V^\vee)$. By definition, $\deg(A_s) = c_n(V^\vee)$ and $\deg(A'_s) = bl(V^\vee)$, while $\deg(Z_{\mathrm{cycle}}(s)) = c_n(E)$. Therefore

$$c_n(V^\vee) = c_n(E) - bl(V^\vee),$$

as claimed. □

Remark 1.6. Note that in the above correspondence we have $L = \det(E)$.

It is sometimes easy to verify when $(E, V^\vee)$ is a *good pair*.

Corollary 1.7. *Let X be a smooth projective variety* of dimension n *such that* $\mathrm{NS}(X) = \mathbb{Z} \cdot L$*. Then there is a one-to-one correspondence*

$$\Big\{ V \in \mathrm{Gr}(n+1, H^0(L)) \;\Big|\; \deg(\varphi_V) = d \Big\} \longleftrightarrow \Big\{ (E, V^\vee) \;\Big|\; \begin{pmatrix} E \text{ is stable reflexive of rank } n, \\ c_1(E) = L, \quad c_n(V^\vee) = d \end{pmatrix} \Big\}.$$

In particular, for any $V \in \mathrm{Gr}(n+1, H^0(L))$ the associated kernel sheaf is stable; conversely, for any stable reflexive sheaf E of rank n and any $V^\vee \in \mathrm{Gr}(n+1, H^0(E))$, the pair $(E, V^\vee)$ is a good pair.

Proof. Consider $V \in \mathrm{Gr}(n+1, H^0(L))$. Since $\mathrm{NS}(X) = \mathbb{Z} \cdot L$, the evaluation map $V \otimes \mathcal{O}_X \to L$ is surjective in codimension 1 (if the base locus had a divisorial component, then there would be an injective map from a non-trivial effective line bundle to L).

Now suppose, by way of contradiction, that the associated kernel sheaf E is not stable. Then there is a destabilizing exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0,$$

with $c_1(M) \geq c_1(E) = L$ numerically, hence $c_1(N) \leq 0$ numerically. Since E is generically globally generated, any quotient of E is generically globally generated as well. A generically globally generated torsion-free sheaf with $c_1 \leq 0$ must be of the form $N \simeq \mathcal{O}_X^{\oplus k}$. Dualizing, we obtain a non-zero map $\mathcal{O}_X^{\oplus k} \to E^\vee$, contradicting Lemma 1.4.

On the other hand, if we start with a stable reflexive sheaf E, stability implies $h^0(E^\vee) = 0$. Consider any $V^\vee \in \mathrm{Gr}(n+1, H^0(E))$ and suppose, by way of contradiction, that the evaluation map $\mathrm{ev} : V^\vee \otimes \mathcal{O}_X \to E$ is not surjective in codimension 1. If $\mathrm{Im}(\mathrm{ev})$ has generic rank $\leq n-1$, then $c_1(\mathrm{Im}(\mathrm{ev})) > 0$, hence $c_1(\mathrm{Im}(\mathrm{ev})) \geq L$ numerically since $\mathrm{NS}(X) = \mathbb{Z} \cdot L$. This contradicts stability of E. If $\mathrm{Im}(\mathrm{ev})$ has generic rank n, then again $c_1(\mathrm{Im}(\mathrm{ev})) \geq L$, and this implies[3] $c_1(\mathrm{Im}(\mathrm{ev})) = L$ and the map ev is surjective in codimension 1. Indeed, if the cokernel were supported on a divisor, then $\mathrm{Im}(\mathrm{ev})$ and E would not have the same first Chern class. Hence, for any stable reflexive sheaf E of rank n with $c_1(E) = L$ and any $V^\vee \in \mathrm{Gr}(n+1, H^0(E))$, the pair $(E, V^\vee)$ is a good pair. ∎

In the case of surfaces, the picture is simpler. In what follows, by the intersection of a family of effective zero-cycles, we mean the effective zero-cycle of maximal degree that is a subcycle of every zero-cycle in the family.

Corollary 1.8. *Let S be a smooth surface with $\mathrm{NS}(S) = \mathbb{Z} \cdot L$, and let $(E, V^\vee)$ be a good pair with $c_1(E) = L$. Then:*

(i) E is a vector bundle;

(ii) for a general section $s \in V^\vee$, the cycle supported on $\mathrm{Bl}(V^\vee)$ *is rigid, in the following sense: there exists a non-empty Zariski open subset $U \subset V^\vee \setminus \{0\}$ such that A'_s does not depend on $s \in U$. In particular,*

$$bl(V^\vee) = \deg\Big(\bigcap_{s \in U} Z_{\mathrm{cycle}}(s)\Big) = \deg(A'_s);$$

(iii) the degree of φ_V satisfies

$$\deg(\varphi_V) = d = c_2(E) - bl(V^\vee) > 0.$$

[3]We have an induced non-zero map $\det(\mathrm{Im}(\mathrm{ev})) \to \det(E) = L$, hence $c_1(\mathrm{Im}(\mathrm{ev})) \leq c_1(E)$.

Proof. (i) On a smooth surface, any reflexive sheaf is locally free; hence E is a vector bundle.

(ii) The locus $\mathrm{Bl}(V^\vee)$ is zero-dimensional. For $s \in V^\vee$ general, the part of the zero-cycle $Z_{\mathrm{cycle}}(s)$ supported on $\mathrm{Bl}(V^\vee)$ is therefore rigid in the sense above. This implies the stated formula for $bl(V^\vee)$.

(iii) The degree computation follows from Proposition 1.5, together with the fact that any section of E vanishes along a zero-dimensional subscheme: otherwise it would induce an injective map from a non-trivial effective line bundle to L. The inequality $d > 0$ follows from the indecomposability of L. □

We end this section with a couple of remarks.

Remark 1.9. In (ii), the scheme-theoretic structure of the vanishing locus of $s \in V^\vee$ at points of $\mathrm{Bl}(V^\vee)$ may vary with s. Moreover, the degree of the associated cycle supported on $\mathrm{Bl}(V^\vee)$ may jump for special choices of $s \in V^\vee$.

Remark 1.10. In the surface case, the length of the base locus satisfies $\mathrm{length}(\mathrm{Bl}) = L^2 - c_2(E)$. Thus $c_2(E)$ measures how *secant* the linear space

$$\mathbb{P}\big(\ker(H^0(L)^\vee \to V^\vee)\big) \subset \mathbb{P}H^0(L)^\vee$$

is with respect to S (scheme-theoretically). On the other hand, $bl(V^\vee)$ measures the number of points (counted with multiplicity) at which this linear space is tangent to S.

2. Brill–Noether loci for the very general $K3$ surface of genus $4, 5, 6$

In this section (S, L) will denote a polarized smooth surface. We will use the framework provided by the previous section to study the loci

$$W^2_d(S, L) = \Big\{ V \in \mathrm{Gr}(3, H^0(L)) \;\Big|\; \deg(\varphi_V) \leq d \Big\}.$$

In the last section of the paper we will see that, as soon as $\mathrm{NS}(S) = \mathbb{Z} \cdot L$, these loci can always be endowed with a natural scheme structure. Under this hypothesis, points of $W^2_d(S, L)$ correspond via Proposition 1.5 to pairs $(E, V^\vee)$, where E is a stable rank 2 vector bundle by Corollary 1.7.

We begin by studying $K3$ surfaces of genus 4, 5, and 6 with Picard rank one. We recall the following preliminary result, essentially due to Mukai (see [24, Corollary 0.2] or [25, Corollary 2.5]). It can also be viewed as a consequence of [16, Theorem, p. 80], which shows that curves with indecomposable class on a very general $K3$ surface do not carry linear series with negative Brill–Noether number. In particular, the inequality below follows from the condition $\rho(g, r-1, d) \geq 0$. We include a proof, as we could not find a sufficiently elementary reference. For a modern treatment in the more general setting of Bridgeland-stable objects, see [6, Theorem 6.10], which also implies the theorem below.

Theorem 2.1 (Mukai)**.** *Let (S, L) be a $K3$ surface of genus g with $\mathrm{Pic}(S) = \mathbb{Z} \cdot L$. A stable vector bundle F of rank r with $c_1(F) = L$ and $c_2(F) = d$ exists if and only if*

$$d \geq \frac{(r-1)(g+r)}{r}.$$

Moreover, if equality holds, there exists a unique vector bundle with these invariants.

Proof. If such a bundle exists, it must satisfy the inequality by [25, Corollary 2.5]. In the proof of Corollary 2.3 below we only need the *non-existence* direction of the theorem; we may therefore use Corollary 2.3 to establish the remaining implications.

For the existence direction, assume $\rho(g, r-1, d) \geq 0$, equivalently $d \geq \frac{(r-1)(g+r)}{r}$. Choose a smooth curve $C \in |L|_{\mathrm{sm}}$ which is Brill–Noether–Petri general (see [16] together with [15]) and a globally generated line bundle $A \in W^{r-1}_d(C)$. Consider the Lazarsfeld–Mukai bundle $E_{C,A}$, defined by the exact sequence

$$0 \longrightarrow E^\vee_{C,A} \longrightarrow H^0(A) \otimes \mathcal{O}_S \longrightarrow A \longrightarrow 0.$$

From the defining sequence we get $H^0(E^\vee_{C,A}) = 0$ and $E_{C,A}$ generically globally generated, which implies that $E_{C,A}$ is stable: indeed, a destabilizing quotient $E_{C,A} \twoheadrightarrow Q$ would be generically globally generated (by $H^0(A)^\vee \subset H^0(E_{C,A})$) and satisfy $c_1(Q) \leq 0$, hence $Q \simeq \mathcal{O}_S^{\oplus k}$. This would yield $0 \neq H^0(Q^\vee) \subset H^0(E^\vee_{C,A})$, a contradiction. The bundle $E_{C,A}$ is the stable vector bundle with the desired invariants.

Finally, assume $\rho(g, r-1, d) = 0$ and let E, F be stable bundles with the same invariants $c_1^2 = 2g-2$, $c_2 = d$, and $\mathrm{rk} = r$. Then one computes

$$\chi(E \otimes F^\vee) = 2 - 2\rho(g, r-1, d) = 2.$$

In particular, there exist non-zero morphisms $\varphi : E \to F$ and $\psi : F \to E$. Since E is simple, the endomorphism $\varphi \circ \psi$ is either 0 or an isomorphism. Suppose, by way of contradiction, that $\varphi \circ \psi = 0$. Then φ is not of maximal rank. By Corollary 2.3 (applied with $d' = d$), both E and F are globally generated. Since both $\mathrm{coker}(\varphi)$ and $\mathrm{Im}(\varphi)$ are (generically torsion-free) quotients of generically globally generated bundles, they have $c_1 \geq 0$. As $L = \det(F) = \det(E)$ is indecomposable, this forces either $c_1(\mathrm{coker}(\varphi)) = 0$ or $c_1(\mathrm{Im}(\varphi)) = 0$. In either case one gets a contradiction, since $h^2(E) = h^2(F) = 0$ and (again, a generically globally generated sheaf with $c_1 = 0$ has torsion free part isomorphic to $\mathcal{O}_S^{\oplus k}$). Therefore, $\varphi \circ \psi$ is an isomorphism, and $E \simeq F$, proving uniqueness. □

Remark 2.2. One could also obtain a (somewhat more involved) proof of the *non-existence* direction of Theorem 2.1 by reversing the Lazarsfeld–Mukai construction and combining it with the *non-existence* part of Brill–Noether theory for curves on $K3$ surfaces (i.e. starting with a stable bundle with fixed invariants and a suitable space of sections construct a curve in $|L|_{sm}$ together with a special linear series). This is similar, in spirit, to the approach taken in the present paper to study linear systems $V \in \mathrm{Gr}(k, H^0(L))$.

Theorem 2.1 has the following corollary of independent interest, which is likely known to experts.

Corollary 2.3. *Let F be a stable vector bundle on a $K3$ surface (S, L) of Picard rank one, of rank r, with $c_1(F) = L$ and $c_2(F) = d$. Then for any $d' < \frac{r(g+r+1)}{r+1}$. and any $\xi \in S^{[d'-d]}$ one has*

$$h^1(F \otimes \mathcal{I}_\xi) = 0.$$

Proof. Suppose, by way of contradiction, that there exists $\xi \in S^{[d'-d]}$ with $h^1(F \otimes \mathcal{I}_\xi) \geq 1$ with $d' < \frac{r(g+r+1)}{r+1}$. Then there is a non-trivial extension

$$0 \longrightarrow F^\vee \longrightarrow G \longrightarrow \mathcal{I}_\xi \longrightarrow 0,$$

so that $c_2(G) = c_2(F) + \deg(\xi) = d'$ and $c_1(G) = -c_1(F)$.

We claim that $G^\vee$ is stable. Indeed:

- replacing G by its double dual, we may assume that G is locally free (this amounts to replacing ξ by a subscheme $\xi' \subseteq \xi$, and hence only decreases $c_2(G)$, so in particular $c_2(G) \leq d'$ still holds);
- if $K \hookrightarrow G$ is a destabilizing locally free subsheaf, then $c_1(K) \geq 0$. Moreover, the induced map $K \to \mathcal{I}_\xi$ cannot be injective (otherwise $K \simeq \mathcal{O}_S$ would split the extension);
- thus $\ker(K \to \mathcal{I}_\xi)$ is a non-zero locally free subsheaf of $F^\vee$ with $c_1 \geq 0$, contradicting the stability of $F^\vee$.

Therefore $G^\vee$ is stable (of rank $r+1$), and Theorem 2.1 yields the the desired inequality, contradicting $d' < \frac{r(g+r+1)}{r+1}$. □

Proposition 2.4. *Let (S, L) be a $K3$ surface of genus 4 with* $\mathrm{Pic}(S) = \mathbb{Z} \cdot L$. *Then* $W^2_3(S, L) = \mathbb{P}^3$.

Proof. By Theorem 2.1, there is a unique stable rank 2 vector bundle E on S with $c_1(E) = L$ and $c_2(E) = 3$. We claim that, via the correspondence of Corollary 1.7, one has

$$W^2_3(S, L) = \mathrm{Gr}(3, H^0(E)) = \mathbb{P}H^0(E)^\vee \simeq \mathbb{P}^3.$$

On the one hand, we clearly have $\mathrm{Gr}(3, H^0(E)) \subset W^2_3(S, L)$ through the correspondence $(E, V^\vee) \mapsto (L, V)$.

On the other hand, giving a degree-3 projection $S \subset \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ is the same as projecting away from a line $l \subset \mathbb{P}^4$ which is 3-secant to S. The evaluation exact sequence associated with this rational map then takes the form

$$0 \longrightarrow E_l^\vee \longrightarrow V \otimes \mathcal{O}_S \longrightarrow L \otimes \mathcal{I}_{l\cap S} \longrightarrow 0.$$

The kernel E_l is a stable rank 2 vector bundle with

$$c_2(E_l) = L^2 - \deg(l \cap S) = 6 - 3 = 3 \qquad \text{and} \qquad c_1(E_l) = L$$

(see Corollary 1.7). Therefore, by the uniqueness part of Theorem 2.1, $E_l \simeq E$, and hence $V^\vee \in \mathrm{Gr}(3, H^0(E))$, as desired. $\square$

We now move to $K3$ surfaces of genus 6.

Theorem 2.5. *Let (S, L) be a $K3$ surface of genus 6 with* $\mathrm{Pic}(S) = \mathbb{Z} \cdot L$. *Then* $W^2_3(S, L) = S$. *In particular,* $\mathrm{irr}(S) = \mathrm{irr}_L(S) = 3$.

Proof. By Theorem 2.1, on such a surface there is a unique stable rank 2 vector bundle E with invariants $c_1(E) = L$ and $c_2(E) = 4$, and one has $h^0(E) = 5$. Moreover, since E is globally generated by Corollary 2.3, we have $h^0(E \otimes \mathcal{I}_P) = 3$ for every $P \in S$.

By the correspondence of Corollary 1.7, for any $P \in S$ the pair $(E, H^0(E \otimes \mathcal{I}_P)^\vee)$ is a good pair and induces a map of degree

$$d = c_2(E) - bl\big(H^0(E \otimes \mathcal{I}_P)^\vee\big) = 4 - 1 = 3.$$

This gives an inclusion $S \hookrightarrow W^2_3(S, L)$; we view S as the image of this inclusion.

Moreover, it can be shown that any pair $(L, V) \in W^2_3(S, L)$ with $\deg(\varphi_V) \leq 3$ and $V \subset H^0(L)$ gives rise to a kernel bundle satisfying $c_2(E) \leq 4$; see [21, Corollary 2.3]. Since E is stable by Corollary 1.7, Theorem 2.1 implies that $c_2(E) \geq 4$. Hence $c_2(E) = 4$ and E must be the unique stable rank 2 vector bundle with $c_1(E) = L$ and $c_2(E) = 4$.

Finally, in view of Corollary 1.8, since $d = 3$ we have $bl(V^\vee) = c_2(E) - d = 1$, so the sections of $V^\vee$ have a base point. Therefore $V^\vee = H^0(E \otimes \mathcal{I}_P)^\vee$ for some $P \in S$, and thus (L, V) belongs to the image of the inclusion $S \hookrightarrow W^2_3(S, L)$. We conclude that $W^2_3(S, L) = S$.

The fact that $\mathrm{irr}(S) \neq 2$ follows from the fact that a very general $K3$ surface does not admit any non-trivial dominant rational map to $\mathbb{P}^2$ of degree 2 (see, for instance, [13, Corollary 2.12]). $\square$

Remark 2.6. The geometric interpretation of this projection is transparent. We have an exact sequence

$$0 \longrightarrow E^\vee \longrightarrow H^0(E \otimes \mathcal{I}_P)^\vee \otimes \mathcal{O}_S \longrightarrow L \otimes \mathcal{I}_{\mathrm{Bl}} \longrightarrow 0,$$

where we are considering the base locus Bl scheme-theoretically. It is easy to see that $\deg(\mathrm{Bl}) = L^2 - c_2(E) = 6$. Hence we are projecting from a 6-secant linear subspace of codimension 3. Moreover, the map

$$H^0(E \otimes \mathcal{I}_P) \otimes k(P) \to E \otimes k(P)$$

is zero, hence $\mathcal{I}_{\mathrm{Bl}} \otimes k(P)$ is 3-dimensional. This implies that the codimension-3 linear subspace is tangent to S at P.

We now study projections $S \subset \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$ for a very general $K3$ surface of genus 5.

Theorem 2.7. *Let (S, L) be a $K3$ surface of genus 5 such that the Picard group has rank 1. Then there are no projections $S \subset \mathbb{P}(H^0(L)^\vee) \dashrightarrow \mathbb{P}^2$ of degree ≤ 3.*

Proof. Let S be a $K3$ surface of genus 5 and suppose that there exists a rational map $\varphi : S \dashrightarrow \mathbb{P}^2$ of degree 3 induced by a 3-dimensional subspace $V \subset H^0(L)$. By Theorem 2.1 and [21, Corollary 2.3], the associated kernel bundle E satisfies $c_2(E) = 4$. By Corollary 2.3 we have $h^1(E) = 0$, hence by Riemann–Roch $h^0(E) = \chi(E) = 4$. Again Corollary 2.3 implies that $h^1(E \otimes \mathcal{I}_P) = 0$ and $h^0(E \otimes \mathcal{I}_P) = 2$ for every $P \in S$. Hence, for any $V \in \mathrm{Gr}(3, H^0(E))$ the induced map φ_V has degree 4 by Corollary 1.8.

□

Remark 2.8. The content of the previous theorem can be rephrased as follows: a 4-secant plane to $S \subset \mathbb{P}^5$ is never tangent to S (more precisely, its intersection with S is always a local complete intersection subscheme).

We can also describe the Brill–Noether locus $W^2_4(S, L)$ in this case.

Corollary 2.9. *The locus $W^2_4(S, L) \subset \mathrm{Gr}(3, H^0(L))$ consists of two disjoint components, one birational to S and the other birational to a $\mathbb{P}^3$-bundle over $\mathcal{M}$, where $\mathcal{M}$ denotes the moduli space of stable rank 2 vector bundles on S with $c_1 = L$ and $c_2 = 4$.*

Proof. The image of the map

$$\begin{aligned} S &\hookrightarrow W^2_4(S, L) \subset \mathrm{Gr}(3, H^0(L)) \\ P &\longmapsto H^0(L \otimes \mathcal{I}_P^2) \end{aligned} \tag{2.1}$$

is the component birational to S. It parametrizes projections from planes in $\mathbb{P}^5$ that are tangent to S.

For the other component, let $\mathcal{M}$ be the moduli space of stable rank 2 vector bundles on S with $c_1 = L$ and $c_2 = 4$, and let $\mathcal{G} \to \mathcal{M}$ be the relative Grassmannian of hyperplanes,

$$\mathcal{G}_E = \mathrm{Gr}(3, H^0(E)) \simeq \mathbb{P}^3.$$

By the correspondence of Corollary 1.7, we obtain a morphism

$$\begin{aligned} \mathcal{G} &\longrightarrow W^2_4(S, L) \\ (E, V) &\longmapsto V^\vee. \end{aligned} \tag{2.2}$$

This component parametrizes maps whose scheme-theoretic base locus has degree 4.

The two components are disjoint. Indeed, if they intersected we would obtain a tangent plane to S meeting S in at least two points. Projecting from this plane would give a rational map $S \dashrightarrow \mathbb{P}^2$ of degree ≤ 3, contradicting the previous theorem.

Finally, we show that there are no other components by considering the possible degrees of the base locus. If the base locus has degree ≤ 2, then $\deg(\varphi_V) \geq 6$. If the base locus has degree 3, then it must be of the form $\operatorname{Spec} \mathcal{O}_S/\mathcal{I}_P^2$; otherwise the map would have degree ≥ 5. In this case the rational map belongs to the component birational to S.

If the base locus has degree 4, then the associated kernel is a stable rank 2 vector bundle with $c_2 = 4$, hence the map lies in the component coming from (2.2). Finally, the base locus cannot have higher degree, because there are no stable rank 2 vector bundles with $c_2 \leq 3$ by Theorem 2.1. This completes the proof. □

As mentioned in the introduction, the interested reader may find further computations of the loci $W^2_d(S,L)$ for $K3$ surfaces up to genus 14 in [21].

3. The case of the $(1,6)$-polarized abelian surface

In this section we consider a very general $(1,6)$-polarized abelian surface (A,L_A). Every such surface admits a presentation as a quotient

$$(A,L_A)\simeq (B/\langle b\rangle,\, L_A),$$

where (B,L_B) is a $(1,3)$-polarized abelian surface and $b\in B[2]$ is a non-trivial 2-torsion point.[4] Starting from (B,L_B) and $b\in B[2]$, let

$$\pi : B\longrightarrow A := B/\langle b\rangle$$

be the quotient map. The polarization L_A is induced by pushing forward divisors in $|L_B|$; for instance, by the projection formula one checks that $L_A^2=12$. Conversely, given (A,L_A), choosing a non-trivial 2-torsion point in the kernel of the dual polarization recovers such a presentation.

Assumption. We assume that L_B contains no invertible subsheaf of the form $\mathcal{O}_B(nF)$ with $n\geq 2$ for some elliptic curve $F\subset B$, and that divisors in $|L_B|$ do not contain elliptic curves invariant under translation by b. These conditions hold for a general abelian surface.

Our goal is to construct generically finite rational maps $A \dashrightarrow \mathbb{P}^2$ of degree 3. We will use the rank-2 vector bundle

$$E := \pi_* L_B$$

to do so. We begin by recording some basic properties of E. If one assumes $\mathrm{NS}(A)=\mathbb{Z}\cdot L_A$, then E is stable, and the proofs of Lemma 3.1 and Corollary 3.2 can be simplified substantially in view of Corollary 1.7 and Corollary 1.8.

Lemma 3.1. *The push-forward $E=\pi_*L_B$ is a rank 2 vector bundle with invariants $c_1(E)=L_A$ and $c_2(E)=3$. Moreover, E is globally generated in codimension 1, $h^0(E^\vee)=0$, the determinant map*

$$\bigwedge^2 H^0(E)\longrightarrow H^0(L_A)$$

is injective and, in particular, the couple $(E,H^0(E))$ is a good pair.

Proof. We observe that

$$\pi^*E\simeq L_B\oplus t_b^*L_B,$$

so the computation of the Chern classes is immediate. Moreover, under the identification

$$i : H^0(L_B)\xrightarrow{\ \sim\ } H^0(E),$$

for $s\in H^0(L_B)$ we have

$$\pi^* i(s) = (s, t_b^* s),\qquad \text{hence}\qquad Z(\pi^* i(s)) = Z(s)\cap t_b^* Z(s).$$

Injectivity of the determinant map. Since $h^0(E)=3$, every element of $\bigwedge^2 H^0(E)$ is a simple tensor. If $i(s_1)\wedge i(s_2)=0$, then $i(s_1)$ and $i(s_2)$ span an invertible subsheaf $M\subset E$. In particular, $Z(i(s_1))$ moves in a positive-dimensional linear system. Pulling back to B, the subsheaf $\pi^*M\subset\pi^*E\simeq L_B\oplus t_b^*L_B$ yields an invertible subsheaf of L_B with two independent sections, whose vanishing divisor is invariant under translation by b (since it is the pullback of a divisor on $A=B/\langle b\rangle$).

This can only occur if $M^2=0$. Indeed, otherwise M would descend to a line bundle on A of type (a,b) with $2ab\geq 4$, and its pullback cannot be an invertible subsheaf of L_B, which has type $(1,3)$. Therefore

[4] In fact, there are three such presentations, corresponding to the three non-trivial 2-torsion points in the kernel of the dual polarization on $\mathrm{Pic}^0(A)$.

$\pi^*M \simeq \mathcal{O}_B(nF)$ for some elliptic curve $F \subset B$ and some $n \geq 2$ (since $h^0(M) \geq 2$), contradicting our assumptions on L_B. This proves injectivity.

Zero-dimensional vanishing for a general section. We claim that for a general $s \in H^0(L_B)$ the section $i(s) \in H^0(E)$ has vanishing locus of dimension 0. Otherwise, for every $s \in H^0(L_B)$ the scheme $Z(\pi^* i(s)) = Z(s) \cap Z(t_b^* s)$ would contain a divisor D_s, and hence we would obtain an injection $\mathcal{O}_B(D_s) \hookrightarrow L_B$. To simplify notation, we suppress the dependence on s and write $D = D_s$.

Since D is t_b-invariant, there are two possibilities:

- D induces a $(1,2)$ polarization; or
- D is an elliptic curve (it cannot be a union of elliptic curves by our assumptions on L_B). In this case D does not depend on s, since s varies in $\mathbb{P}^2$ and D has no non-trivial linearly equivalent translates.

In the first case, a general section of $H^0(\mathcal{O}_B(D)) \subset H^0(L_B)$ is not t_b-invariant, hence D cannot occur as the divisorial part of $Z(s) \cap Z(t_b^* s)$ for all s. In the second case, if every section vanished along a fixed t_b-invariant elliptic curve D, then D would be a fixed component of $|L_B|$, and $H^0(L_B) = H^0(L_B(-D))$. This forces $h^1(L_B(-D)) > 0$ (since $L_B(-D)^2 < 6$), hence any divisor in $|L_B(-D)|$ is a union of elliptic curves, contradicting our assumptions. This proves the claim.

*A degree-*3 *map and global generation in codimension* 1. Set

$$E_1 := \left(\operatorname{Im}(H^0(E) \otimes \mathcal{O}_A \to E)^\vee\right)^\vee.$$

Via the correspondence of Proposition 1.5, we obtain a non-degenerate rational map

$$\varphi : A \dashrightarrow \mathbb{P}H^0(E) = \mathbb{P}H^0(E_1).$$

Since a general section of $H^0(E)$ has zero-dimensional vanishing locus of length $c_2(E) = 3$ as a section of E, it follows that a general section of $H^0(E_1)$ has zero-dimensional vanishing locus of length ≤ 3. Hence, by the fiber description in Proposition 1.5, the degree of φ is ≤ 3. By the main theorem of [2], the degree cannot be 1 or 2, so either φ has degree 3 or its image is a (non-degenerate) curve $C \subset \mathbb{P}H^0(E_1)$.

We claim that the latter does not occur. Indeed, if $\operatorname{Im}(\varphi) = C$, then pulling back a line $l \subset \mathbb{P}H^0(E_1)$ would produce a divisor $D_l \subset A$ which is numerically a positive multiple $k \cdot D'$, where $k = \deg(C)$ and D' is the pullback of a point of C. If $k = 2$, then C is rational and $h^0(\mathcal{O}_A(D')) \geq 2$, so either $D'^2 \geq 4$ (contradicting $\mathcal{O}_A(2D') \subset L_A$) or $D'^2 = 0$ and D' is a union of (at least) two elliptic curves, contradicting our assumptions after pulling back to B. The argument is analogous for $k \geq 3$ when C is rational.

If C is not rational, then C is an elliptic curve and the fibers of $A \to C$ are unions of k' elliptic curves. Writing $D' = k' \cdot F$, if $k' \geq 2$ then $\pi^*\mathcal{O}_A(D') \subset L_B$ gives an invertible subsheaf of L_B, contradicting our assumptions. If $k' = 1$, then by the fiber description in Proposition 1.5 a section $[i(s)] \in C$ vanishes along an elliptic curve as a section of E_1, hence also as a section of E. Pulling back to B and using $\pi^* Z(i(s)) = Z(s) \cap Z(t_b^* s)$, we conclude that s either vanishes along a t_b-invariant elliptic curve, or vanishes along two elliptic curves exchanged by translation by b, again contradicting our assumptions on L_B. This proves the claim.

Therefore the map associated to $(E_1, H^0(E))$ has degree 3. To conclude the proof, we show that $E_1 = E$, i.e. that E is globally generated in codimension 1. The group $K(L_B) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ acts on $H^0(L_B)$, hence on $H^0(E) = H^0(E_1)$ and therefore on $\bigwedge^2 H^0(E_1)^\vee \subset H^0(\det(E_1))$ (see also [12, Theorem 2.2]). Since E_1 is globally generated in codimension 1, the base locus Bl of $|\bigwedge^2 H^0(E_1)|$ has codimension 2 and is invariant under $K(L_B)$. As $K(L_B)$ acts on A by translations (hence has no fixed points), we deduce that $|\mathrm{Bl}| \geq 9$. Since $\deg(\varphi) \geq \det(E_1)^2 - |\mathrm{Bl}|$, we obtain $\det(E_1)^2 \geq 12$.

Finally, since $E_1 \subset E$, we have an inclusion $\det(E_1) \subset \det(E) = L_A$. On an abelian surface, if $M \subset N$ are effective line bundles with $M^2 \geq N^2 > 0$, then $M = N$. Hence $\det(E_1) = L_A$, and since $E_1 \subset E$ with both locally free of the same rank, we conclude that $E_1 = E$. ☐

Corollary 3.2. *The degree of irrationality of a general* $(1,6)$*-polarized abelian surface is* 3.

Proof. Since $(E, H^0(E))$ is a good pair, the correspondence of Proposition 1.5 yields a rational map $f : A \dashrightarrow \mathbb{P}^2$ of degree ≤ 3. We claim that $\deg(f) > 0$. Indeed, if the image were a (non-degenerate) curve $C \subset \mathbb{P}^2$, then we would obtain

$$L_A \simeq f^*(z)$$

for some $z \in \mathrm{Pic}(C)$ of degree at least 2, contradicting the numerical primitivity of L_A. Finally, the degree cannot be 2 by the main theorem of [2]. Hence $\deg(f) = 3$. ☐

Remark 3.3. In [12] the geometry of the degree 3 rational map $A \dashrightarrow \mathbb{P}^2$ is studied in detail, and the invariants of its Galois closure (a Lagrangian surface in $A \times A$) are computed. A similar program was carried out for $(1,2)$-polarized abelian surfaces and the degree 3 map constructed by Tokunaga and Yoshihara (see [27]) in [4].

With some extra effort it is possible to prove a theorem analogous to Theorem B for a very general $(1,6)$-polarized abelian surface $A \subset \mathbb{P}^5 = \mathbb{P}H^0(L_A)^\vee$. One uses the geometry of stable vector bundles of rank 2 on A with $c_1 = L_A$ and $c_2 = 3$, which turn out to be pushforwards of line bundles on $(1,3)$-polarized abelian surfaces under a degree-2 isogeny (see [23, Theorem 5.8]). This implies several equivariance properties of the degree 3 rational map $A \dashrightarrow \mathbb{P}^2$ (see [12, Section 2]). This yields, for such an embedding $A \subset \mathbb{P}^5$, the existence of exactly four planes in $\mathbb{P}^5$ which are 9-secant to A, together with a description of their intersection with A.

4. The upper bound

We use the construction of the previous sections to produce generically finite rational maps of low degree from very general $K3$ surfaces of genus g to $\mathbb{P}^2$. We establish our bounds for very general $K3$ surfaces, and general specialization results (see [9, Corollary C]) imply the same bounds for arbitrary $K3$ surfaces. Our construction improves the best previously known bounds due to Stapleton by a factor between 3 and 6, depending on the genus; see [26].

The following table collects some of the resulting upper bounds for small values of g:

g	6	8	10	12	14	16	18	20	22	24	26	42	62	$2+2n(n+1)$
$\mathrm{irr}(S) \leq$	3	4	4	5	4	5	5	6	6	7	5	6	7	$2+n$

We first deal with the case $g = 2 + 2n(n+1)$. In view of Theorem 2.1, the minimal possible second Chern class for a rank 2 bundle with primitive c_1 is

$$c_2(E) = 2 + n(n+1).$$

We will prove that

$$\mathrm{irr}(S) \leq 2 + \frac{n(n+1)}{2} - \frac{(n-1)n}{2} = 2 + n \sim \frac{1}{\sqrt{2}}\sqrt{g}.$$

For this series of genera, the resulting estimate is asymptotically a $1/6$th of Stapleton's bound. The theorem is an easy consequence of the construction in the previous section.

Theorem 4.1. *Let* (S, L) *be a* $K3$ *surface of genus* $g = 2 + 2n(n+1)$*, then* $\mathrm{irr}(S) \leq 2 + n$.

Proof. Consider the rank 2 vector bundle E with invariants

$$c_1(E) = L, \quad c_2(E) = 2 + n(n+1), \quad h^0(E) = 3 + n(n+1).$$

Since $\mathrm{colength}(\mathcal{I}_P^n) = \frac{n(n+1)}{2}$ for any $P \in S$ there exists a vector space $V_P^\vee \subset H^0(E \otimes \mathcal{I}_P^n)$ of dimension 3. Any section of $V_P^\vee$ vanishes at P with order n^2 (since it is locally defined by two functions vanishing with order n at P). We deduce that $bl(V_P^\vee) = \deg(\cap_{s \in V_P^\vee} Z_{cycle}(s)) \geq n^2$. By Proposition 1.5 and Corollary 1.8

$$\deg(\varphi_{V_P}) = c_2(V_P)^\vee \leq c_2(E) - n^2 = 2 + n,$$

where φ_{V_P} is the map $S \dashrightarrow \mathbb{P}^2$ induced by $V_P \subset H^0(L)$. □

The general case is an easy consequence.

Corollary 4.2. *Let (S, L) be a $K3$ surface of genus $g = 2 + 2n(n+1) + k < 2 + 2(n+1)(n+2)$ (i.e. $k < 4n + 4$), then*

$$\mathrm{irr}(S) \leq 2 + n + \left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{k}{4} \right\rfloor.$$

Proof. In view of Theorem 2.1 on S there is a stable rank 2 vector bundle E with

$$c_1(E) = L, \quad c_2(E) = 2 + n(n+1) + \left\lceil \frac{k}{2} \right\rceil, \quad h^0(E) = 3 + n(n+1) + \left\lceil \frac{k}{2} \right\rceil.$$

Consider distinct points $P, Q_1, \dots, Q_{\lfloor \frac{k}{4} \rfloor} \in S$ and

$$V^\vee \in \mathrm{Gr}(3, H^0(E \otimes \mathcal{I}_P^n \otimes \mathcal{I}_{Q_1} \otimes \cdots \otimes \mathcal{I}_{Q_{\lfloor \frac{k}{4} \rfloor}}),$$

then, in view of Proposition 1.5 and Corollary 1.8, φ_V is of degree

$$\deg(\varphi_V) = c_2(V^\vee) \leq c_2(E) - n^2 - \left\lfloor \frac{k}{4} \right\rfloor = 2 + n + \left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{k}{4} \right\rfloor.$$

□

5. An application to higher-dimensional varieties

In this section we provide a few applications of Proposition 1.5 to higher-dimensional varieties. We begin with a sufficient criterion ensuring that a good pair $(E, V^\vee)$ induces a *generically finite* rational map. Note that, by Proposition 1.5, the associated map is generically finite if and only if $c_n(V^\vee) > 0$.

Lemma 5.1. *Let $(E, V^\vee)$ be a good pair on an n-dimensional smooth projective variety X. Suppose that for every $k \geq 0$ the locus*

$$\left\{ s \in V^\vee \;\middle|\; Z(s) \text{ has an irreducible component of dimension } \geq k \right\}$$

has dimension $< n - k$. Then $c_n(V^\vee) > 0$, i.e. φ_V induces a generically finite rational map.

In particular, if E is a vector bundle and every section in $V^\vee$ has vanishing locus of dimension 0 on $X \setminus \mathrm{Bl}(V^\vee)$, then $c_n(V^\vee) > 0$.

Proof. By Proposition 1.5, the fibers of $\varphi_V : X \setminus \mathrm{Bl}(V^\vee) \to \mathbb{P}V^\vee$ are precisely the vanishing loci of sections in $V^\vee$ restricted to $X \setminus \mathrm{Bl}(V^\vee)$. Hence the locus

$$\bigcup_{s \in V^\vee} \left(Z(s) \cap (X \setminus \mathrm{Bl}(V^\vee)) \right)$$

covers $X \setminus \mathrm{Bl}(V^\vee)$. The conclusion follows by a dimension count using the hypothesis on the loci of sections with positive-dimensional components. □

Proposition 5.2. *The following estimates hold:*

- *if (S,L) is a very general $K3$ surface of genus 6, then* $\operatorname{irr}(S^{[2]}) \leq 6$*;*
- *if (A,L) is a very general abelian threefold of type $(1,3,12)$ or $(1,6,6)$, then* $\operatorname{irr}_L(A) \leq 8$*;*
- *if (S,L) is a very general $K3$ surface of genus 10, then* $\operatorname{irr}(S^{[3]}) \leq 20$.

Proof.

- *The case of $S^{[2]}$.* Let E be the stable rank 2 vector bundle on S with $c_1(E) = L$, $c_2(E) = 4$, and $h^0(E) = 5$. It canonically induces a rank 4 vector bundle $\mathcal{E}$ on $S^{[2]}$, defined by
$$\mathcal{E} \otimes k(\xi) \;=\; H^0(E \otimes \mathcal{O}_\xi) \qquad \text{for } \xi \in S^{[2]}.$$
Moreover, one has $c_4(\mathcal{E}) = 6$, and there is a natural identification
$$i : H^0(E) \xrightarrow{\sim} H^0(\mathcal{E}).$$
We briefly describe the vanishing loci of sections of $\mathcal{E}$. If $Z(s) = \zeta \in S^{[4]}$, then
$$Z(i(s)) \subset \Big\{ \xi \in S^{[2]} \;\Big|\; \xi \subset \zeta \Big\}.$$
This set is finite if ζ is curvilinear and of cardinality exactly 6 if ζ is reduced. By Lemma 5.3, for general $s \in H^0(E)$ the subscheme $Z(s)$ is reduced (hence curvilinear), and therefore for general $s \in H^0(E)$ the vanishing locus $Z(i(s))$ is zero-dimensional of length 6. It follows that $H^0(\mathcal{E})$ generically generates $\mathcal{E}$ (otherwise a general section would factor through a rank 3 subsheaf and would have vanishing locus of dimension at least 1). In particular, the pair $\big((\operatorname{Im}(\mathrm{ev})^\vee)^\vee,\, H^0(\mathcal{E})\big)$ is a good pair.
If ζ is not curvilinear, then locally $\mathcal{I}_\zeta = (x^2, y^2)$ at its support, and the locus
$$\Big\{ \xi \in S^{[2]} \;\Big|\; \xi \subset \zeta \Big\}$$
is 1-dimensional (it consists of length-2 subschemes supported at the same point). By Lemma 5.3, for every $P \in S$ the vector space $H^0(E \otimes \mathcal{I}_P^2)$ has dimension at most 1, so the locus of sections vanishing along a positive dimensional subscheme is of dimension ≤ 2. Therefore Lemma 5.1 yields $c_4\big(H^0(\mathcal{E})\big) > 0$. On the other hand, since a general section vanishes at 6 points outside the base locus, we have $c_4\big(H^0(\mathcal{E})\big) \leq 6$. Hence, by Proposition 1.5, the good pair $\big((\operatorname{Im}(\mathrm{ev})^\vee)^\vee,\, H^0(\mathcal{E})\big)$ induces a generically finite rational map of degree ≤ 6, proving $\operatorname{irr}(S^{[2]}) \leq 6$.
- *The case of abelian threefolds.* A very general abelian threefold (A, L_A) of type $(1,3,12)$ (resp. $(1,6,6)$) admits a description as $(B, L_B)/\langle b \rangle$, where (B, L_B) is a very general $(1,1,4)$ (resp. $(1,2,2)$) abelian threefold and b is a 3-torsion point. Let $\pi : B \to A$ be the quotient map and set $E := \pi_* L_B$. Then E is a rank 3 vector bundle with
$$c_1(E) = L_A, \qquad c_3(E) = 8,$$
since there is an isomorphism $\pi^* E \simeq L_B \oplus t_b^* L_B \oplus t_{2b}^* L_B$. If $\mathrm{NS}(A) = \mathbb{Z} \cdot L_A$ (as for a very general abelian threefold), then $(E, H^0(E))$ is a good pair in view of Corollary 1.7. Moreover, for a very general A, one checks that every section $s \in H^0(E)$ has vanishing locus of dimension 0. For instance, one can argue by degeneration. Degenerate the abelian threefold B to a product $E \times S$, where E is an elliptic curve and S is a very general $(1,4)$-polarized abelian surface. In this situation, one checks that a section vanishes either along a zero-dimensional subscheme or along a disjoint union of smooth elliptic curves. In particular, such vanishing loci cannot occur as flat limits of vanishing loci containing a curve on a very general (simple) abelian threefold. Therefore Lemma 5.1 and Proposition 1.5 yield a generically finite rational map of degree $\leq c_3(E) = 8$.
- *The case of $S^{[3]}$.* The argument is analogous to the case of $S^{[2]}$, using the corresponding tautological bundle on $S^{[3]}$, and we omit the details.

□

Here is a proof of a few technical results used in the above proof.

Lemma 5.3. *Let (S,L) be a $K3$ surface of genus 6 with $\mathrm{Pic}(S)=\mathbb{Z}\cdot L$, and let E be the stable rank 2 vector bundle with $c_1(E)=L$ and $c_2(E)=4$. Then:*

- *for general $s\in H^0(E)$ the zero locus $Z(s)$ is reduced (hence curvilinear);*
- *for any $P\in S$ one has $h^0(E\otimes\mathcal{I}_P^2)\le 1$.*

Proof. Recall that, in view of Theorem 2.1, such a vector bundle is unique. For the first point, take a general smooth curve $C\in|L|$. By [16], C is Brill–Noether general, hence it carries a base-point-free g^1_4; let $A\in W^1_4(C)$ be the associated line bundle. Consider the Lazarsfeld–Mukai bundle

$$E_{C,A}^\vee := \ker\bigl(H^0(A)\otimes\mathcal{O}_S\to A\bigr).$$

One checks that $h^0(E_{C,A}^\vee)=0$, which implies stability: any destabilizing subsheaf would have $c_1\ge 0$ (by $\mathrm{Pic}(S)=\mathbb{Z}\cdot L$) and would force $h^0(E_{C,A}^\vee)>0$. Moreover $c_1(E_{C,A})=L$ and $c_2(E_{C,A})=4$, hence $E_{C,A}\simeq E$ by uniqueness. Finally, the sections in $H^0(A)^\vee\subset H^0(E)$ vanish along a divisor in $|A|$, and a general divisor in $|A|$ is reduced. This proves the first claim.

For the second statement, suppose, by way of contradiction, that $h^0(E\otimes\mathcal{I}_P^2)=2$ for some $P\in S$. Then two independent sections $s_1,s_2\in H^0(E\otimes\mathcal{I}_P^2)$ vanish with multiplicity at least 4 at P (their zero locus is locally given by two equations vanishing with order 2 at P). Since $c_2(E)=4$ and no section of E vanishes along a divisor (otherwise E would be destabilized by our assumptions on $\mathrm{Pic}(S)$), these sections cannot vanish on $S\setminus P$. On the other hand, the wedge $s_1\wedge s_2$ defines a non-zero section of $\det(E)\simeq L$ vanishing along a curve $C\in|L|$. Pick $Q\in C\setminus\{P\}$. Since s_1 and s_2 are linearly dependent at Q, a suitable linear combination vanishes at Q and with multiplicity ≥ 4 at P, contradicting $c_2(E)=4$. $\square$

Remark 5.4. The cases of abelian varieties and Hilbert schemes of $K3$ surfaces can be generalized to higher dimensions. In the abelian threefold case, the kernel bundle is semi-homogeneous, i.e. a pushforward of a line bundle under an isogeny (as for the $(1,6)$-polarized abelian surface; see Remark 3.3). This implies, in particular, equivariance and symmetry properties for the rational maps we construct. For more details on semi-homogeneous vector bundles on abelian varieties, see the work of Mukai [23].

As the proof of Proposition 5.2 illustrates, it can be nontrivial to use Proposition 1.5 to compute the degree of the rational map associated with a good pair $(E,V^\vee)$. In [22] it is shown that, if E is a globally generated vector bundle on a smooth projective variety X of dimension n, then for general $V^\vee\in\mathrm{Gr}(n+1,H^0(E))$ the rational map associated with $(E,V^\vee)$ via Proposition 1.5 has degree exactly $c_n(E)$.

6. Towards a Brill–Noether theory for surfaces

We fix a smooth projective surface S with $\mathrm{NS}(S)=\mathbb{Z}\cdot L$ and $h^0(L)\ge 3$. The goal of this section is to show that the locus of special projections

$$W^r_d(S,L)=\Bigl\{\,V\in\mathrm{Gr}(r+1,H^0(L))\;\Big|\;\deg(\varphi_V)\le d\Bigr\}$$

is naturally a scheme.

We also remark that Mendes-Lopes, Pardini and Pirola introduced Brill–Noether loci for irregular varieties in [19] as cohomology jump loci in $\mathrm{Pic}^0(S)$. Our definition is of a different nature.

6.1. **The correspondence.** Let $V\subset H^0(L)$ be an $(r+1)$-dimensional subspace inducing a rational map

$$\varphi_V : S \dashrightarrow \mathbb{P}^r=\mathbb{P}(V^\vee),$$

and let $E^\vee := \ker\bigl(V \otimes \mathcal{O}_S \to L\bigr)$ be the associated kernel bundle, so that we have an exact sequence

$$0 \longrightarrow E^\vee \longrightarrow V \otimes \mathcal{O}_S \longrightarrow L.$$

(Recall that on a surface, reflexive implies locally free.)

Definition 6.1. A pair $(E, V^\vee)$, where E is a rank r vector bundle and $V^\vee \in \mathrm{Gr}(r+1, H^0(E))/\mathrm{Aut}(E)$, is a *good r-pair* if $V^\vee$ generates E in codimension 1 and $H^0(E^\vee) = 0$.

We have the following analogue of Corollary 1.7.

Lemma 6.2. *Let S be a smooth surface with $\mathrm{NS}(S) = \mathbb{Z} \cdot L$. Then there is a one-to-one correspondence*

$$\Bigl\{ V \in \mathrm{Gr}(r+1, H^0(L)) \Bigr\} \longleftrightarrow \Bigl\{ (E, V^\vee) \Bigm| E \text{ stable of rank } r,\ c_1(E) = L,\ V^\vee \in \mathrm{Gr}(r+1, H^0(E)) \Bigr\}.$$

Proof. Same proof as Corollary 1.7. ☐

Let $(E, V^\vee)$ be a good r-pair and let $W \in \mathrm{Gr}(r-1, V^\vee)$. Consider the evaluation map

$$\mathrm{ev}_W : W \otimes \mathcal{O}_S \longrightarrow E.$$

We claim that ev_W is injective. If, by contradiction, ev_W is not injective we would have $c_1(\mathrm{Im}(\mathrm{ev}_W)) \geq L \in \mathrm{NS}(S)$, so that $\mathrm{Im}(\mathrm{ev}_W)$ would destabilize the (stable) vector bundle E. Hence ev_W is injective and we obtain an exact sequence

$$0 \longrightarrow W \otimes \mathcal{O}_S \longrightarrow E \longrightarrow L \otimes \mathcal{I}_\xi \longrightarrow 0,$$

for some $\xi \in S^{[c_2(E)]}$.

Definition 6.3. Let $(E, V^\vee)$ be a good r-pair with $c_1(E) = L$. We define:

- the base locus $\mathrm{Bl}(V^\vee) \in \mathrm{Sym}^{L^2 - c_2(E)}(S)$ as the (zero-dimensional) locus where $V^\vee$ does not generate E;
- for $W \in \mathrm{Gr}(r-1, V^\vee)$,
$$Z(W) := \mathrm{Spec}\, \mathcal{O}_S/\mathrm{coker}(\mathrm{ev}_W) \ \in S^{[c_2(E)]},$$
$$Z_{\mathrm{cycle}}(W) := \mathrm{HC}(Z(W)) \ \in \mathrm{Sym}^{c_2(E)}(S),$$
where $\mathrm{HC} : S^{[c_2(E)]} \to \mathrm{Sym}^{c_2(E)}(S)$ is the Hilbert–Chow morphism;
- $c_2(V^\vee)$ as the degree of the part of $Z_{\mathrm{cycle}}(W)$ supported on $S \setminus \mathrm{Bl}(V^\vee)$ for general $W \in \mathrm{Gr}(r-1, V^\vee)$;
- $bl(V^\vee) := \deg\Bigl(\bigcap_{W \in \mathrm{Gr}(r-1, V^\vee)} Z_{\mathrm{cycle}}(W)\Bigr)$.

Remark 6.4. Since $\mathrm{Bl}(V^\vee)$ is zero-dimensional, we have

$$bl(V^\vee) = \deg\bigl(Z_{\mathrm{cycle}}(W) \cap \mathrm{Bl}(V^\vee)\bigr)$$

for general $W \in \mathrm{Gr}(r-1, V^\vee)$. We also remark that $Z(W)$ is well defined for every W because ev_W is injective for every $W \in \mathrm{Gr}(r-1, V^\vee)$.

The above constructions globalize to the relative setting. More precisely, let:

- $\mathcal{M}_{r,c}$ be the moduli space of stable vector bundles of rank r with $c_1 = L$ and $c_2 = c$;
- $\mathcal{G}_{r,c} \to \mathcal{M}_{r,c}$ be the relative Grassmannian whose fiber over $E \in \mathcal{M}_{r,c}$ is $\mathrm{Gr}(r+1, H^0(E))$;[5]
- $\mathcal{W}_{r,c} \to \mathcal{G}_{r,c}$ be the relative Grassmannian whose fiber over $(E, V^\vee)$ is $\mathrm{Gr}(r-1, V^\vee)$;
- $\mathcal{G}_r := \bigcup_c \mathcal{G}_{r,c}$ and $\mathcal{W}_r := \bigcup_c \mathcal{W}_{r,c}$.

[5] The construction can be made locally analytically on $\mathcal{M}_{r,c}$, since universal bundles exist locally analytically. It can also be globalized even when no global universal bundle exists; for the $K3$ case see [25].

By Lemma 6.2, the space $\mathcal{G}_r$ parametrizes good r-pairs and comes equipped with a natural bijection

$$\mathcal{G}_r \longrightarrow \mathrm{Gr}(r+1, H^0(L)).$$

Moreover, Definition 6.3 globalizes to give morphisms

$$Z : \mathcal{W}_{r,c} \to S^{[c]}, \qquad Z_{\mathrm{cycle}} : \mathcal{W}_{r,c} \to \mathrm{Sym}^c(S), \qquad \mathrm{Bl} : \mathcal{G}_{r,c} \to \mathrm{Sym}^{L^2-c}(S).$$

The construction is based on the following observation, which generalizes Proposition 1.5 in the surface case.

Corollary 6.5. *Let $\varphi_V : S \dashrightarrow \mathbb{P}^r = \mathbb{P}(V^\vee)$ be a rational map with kernel bundle E and base locus $\mathrm{Bl}(V^\vee)$. Then for any $W \in \mathrm{Gr}(r-1, V^\vee)$ one has an identification*

$$\varphi_V^{-1}\big(\mathbb{P}(W)\big) \cap (S \setminus \mathrm{Bl}(V^\vee)) = Z(W) \cap (S \setminus \mathrm{Bl}(V^\vee)).$$

In particular,

$$\deg(\varphi_V) = c_2(V^\vee) = c_2(E) - bl(V^\vee).$$

Proof. We abuse notation and denote by φ_V the induced morphism $\varphi_V : S \setminus \mathrm{Bl}(V^\vee) \to \mathbb{P}(V^\vee)$. The Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(V^\vee)}(-1) \longrightarrow V^\vee \otimes \mathcal{O}_{\mathbb{P}(V^\vee)} \longrightarrow T_{\mathbb{P}(V^\vee)}(-1) \longrightarrow 0$$

pulls back to

$$0 \longrightarrow L^\vee_{|_{S\setminus \mathrm{Bl}(V^\vee)}} \longrightarrow V^\vee \otimes \mathcal{O}_{S\setminus \mathrm{Bl}(V^\vee)} \longrightarrow E_{|_{S\setminus \mathrm{Bl}(V^\vee)}} \longrightarrow 0,$$

so that $E_{|_{S\setminus \mathrm{Bl}(V^\vee)}} \simeq \varphi_V^* T_{\mathbb{P}(V^\vee)}(-1)$ and pullback induces an identification,

$$\varphi_V^* : H^0\big(T_{\mathbb{P}(V^\vee)}(-1)\big) \xrightarrow{\sim} V^\vee \subset H^0(E).$$

and, more generally, an inclusion

$$\mathrm{Gr}\big(r-1, H^0(T\mathbb{P}(V^\vee)(-1))\big) \subset \mathrm{Gr}\big(r-1, H^0(E)\big).$$

For a general $W \in \mathrm{Gr}\big(r-1, H^0(T_{\mathbb{P}(V^\vee)}(-1))\big)$, the map

$$W \otimes \mathcal{O}_{\mathbb{P}(V^\vee)} \longrightarrow T_{\mathbb{P}(V^\vee)}(-1)$$

drops rank along the linear subspace $\mathbb{P}(W) \subset \mathbb{P}(V^\vee)$ (and at no other point, since $c_r\big(T_{\mathbb{P}(V^\vee)}(-1)\big) = \mathcal{O}(1)^r$). Hence, for $W \in \mathrm{Gr}(r-1, V^\vee)$, the fiber of

$$\varphi_V : S \setminus \mathrm{Bl} \longrightarrow \mathbb{P}(V^\vee)$$

above $\mathbb{P}(W)$ is $Z(W)$. The degree statement follows because $c_2(V^\vee)$ is precisely the degree of $Z_{\mathrm{cycle}}(W)$ supported outside the base locus Bl (see also Remark 6.4), and because $\deg(\varphi_V)$ equals the degree of the pullback of a general codimension-2 linear subspace $\mathbb{P}(W) \subset \mathbb{P}(V^\vee)$, with $W \in \mathrm{Gr}(r-1, V^\vee)$. $\square$

We are now ready to show that the loci $W^r_d(S, L)$ are quasi-projective algebraic varieties.

Theorem 6.6. *Let (S, L) be a polarized surface with $\mathrm{NS}(S) = \mathbb{Z} \cdot L$. Then the locus*

$$W^r_d(S, L) = \Big\{ V \in \mathrm{Gr}(r+1, H^0(L)) \;\Big|\; \deg(\varphi_V) \leq d \Big\}$$

is a quasi-projective algebraic variety.

Proof. By Lemma 6.2 we have a bijection $\mathcal{G}_r \to \mathrm{Gr}(r+1, H^0(L))$. We construct the loci inside $\mathcal{G}_r = \bigcup_c \mathcal{G}_{r,c}$, which filters $\mathrm{Gr}(r+1, H^0(L))$ according to the second Chern class of the kernel bundle. Let $\pi : \mathcal{W}_{r,c} \to \mathcal{G}_{r,c}$ be the relative Grassmannian. To a point $(E, V^\vee, W) \in \mathcal{W}_{r,c}$ we associate the two cycles $\mathrm{Bl}(V^\vee)$ and $Z_{\mathrm{cycle}}(W)$.

In view of Corollary 6.5, the point $(E, V^\vee)$ gives a rational map of degree $\leq d$ if and only if for every $W \in \mathrm{Gr}(r-1, V^\vee)$ the cycle $Z_{\mathrm{cycle}}(W)$ contains at least $c-d$ points (counted with multiplicity) supported on $\mathrm{Bl}(V^\vee)$. Define

$$\Delta_{c,d} := \left\{ (P,Q) \in \mathrm{Sym}^c(S) \times \mathrm{Sym}^{L^2-c}(S) \;\middle|\; \deg(P \cap \mathrm{Supp}(Q)) \geq c-d \right\}.$$

Then the locus

$$W^r_d(S,L)_c := \left\{ (E, V^\vee) \in \mathcal{G}_{r,c} \;\middle|\; \pi^{-1}(E, V^\vee) \subset (Z_{\mathrm{cycle}}, \mathrm{Bl} \circ \pi)^{-1}(\Delta_{c,d}) \right\}$$

parametrizes good r-pairs with $c_2(E) = c$ and $\deg(\varphi_V) \leq d$. The loci $W^r_d(S,L)_c$ are quasi-projective (by general facts on flag varieties), and therefore the finite union

$$W^r_d(S,L) = \bigcup_c W^r_d(S,L)_c$$

is a quasi-projective algebraic variety. □

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