EXISTENCE OF CURVATURE FLOW WITH FORCING IN A CRITICAL SOBOLEV SPACE

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ABSTRACT. Suppose that a closed 1-rectifiable set $\Gamma_0 \subset \mathbb{R}^2$ of finite 1-dimensional Hausdorff measure and a vector field u in a dimensionally critical Sobolev space are given. It is proved that, starting from Γ_0, there exists a non-trivial flow of curves with the normal velocity given by the sum of the curvature and the given vector field u. The motion law is satisfied in the sense of Brakke and the flow exists through singularities.

1. INTRODUCTION

The present paper studies a general existence problem related to the mean curvature flow with a critical driving force, the meaning of "critical" being explained in the next paragraph. Given an n-dimensional submanifold $\Gamma_0 \subset \mathbb{R}^{n+1}$ and a vector field $u(x,t) : \mathbb{R}^{n+1} \times [0,\infty) \to \mathbb{R}^{n+1}$, consider the existence problem of a family $\Gamma(t) \subset \mathbb{R}^{n+1}$ such that $\Gamma(0) = \Gamma_0$ and $\Gamma(t)$ moves by the motion law of

$$v(x,t) = h(x,t) + u(x,t)^{\perp}. \tag{1.1}$$

Here, $v(x,t)$ is the normal velocity vector of $\Gamma(t)$ at x, $h(x,t)$ is the mean curvature vector of $\Gamma(t)$ at x, and $u(x,t)^{\perp}$ is the projection of $u(x,t)$ to $(\mathrm{Tan}_x\Gamma(t))^{\perp}$. One may view the problem as the mean curvature flow with a given background flow field u as a forcing term, and the short-time existence of unique classical solution $\Gamma(t)$ satisfying (1.1) can be proved if Γ_0 and u are sufficiently smooth. When the given vector u is not smooth, it is interesting to study the minimal requirement for the existence of solution, where the meaning of solution to (1.1) itself can be an essential part of the study. Though the motivation of the problem with a low-regularity assumption on u may be considered as purely theoretical, there is a coupled two-phase flow problem of Navier-Stokes equation proposed as a sharp interface limit of the Navier-Stokes/Allen-Cahn equation (see [6, 12, 13] and the references therein) where the presence of interface $\Gamma(t)$ influences the incompressible flow u itself via the surface tension and $\Gamma(t)$ moves by the motion law (1.1) (when $\sigma_2 = 1$ below). We briefly mention this problem, where $u(x,t)$, $\Pi(x,t)$ (pressure) and $\Gamma(t)$ satisfy in the distributional

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sense

$$\begin{aligned} & u_t + u\cdot\nabla u = \Delta u - \nabla\Pi + \sigma_1 h\mathcal{H}^n\lfloor_{\Gamma(t)} && \text{on } \mathbb{R}^{n+1}\times(0,\infty),\\ & \nabla\cdot u = 0 && \text{on } \mathbb{R}^{n+1}\times(0,\infty),\\ & v = \sigma_2 h + u^{\perp} && \text{on } \Gamma(t) \text{ for } t{>}0. \end{aligned} \tag{1.2}$$

Here σ_1, σ_2 are positive constants and $h\,\mathcal{H}^n\lfloor_{\Gamma(t)}$ is an n-dimensional surface measure on $\Gamma(t)$ multiplied by the mean curvature vector h of $\Gamma(t)$. This system has a natural energy law of

$$\frac{d}{dt}\Big(\int_{\mathbb{R}^{n+1}}\frac{|u|^2}{2}\,dx + \sigma_1\mathcal{H}^n(\Gamma(t))\Big) = -\int_{\mathbb{R}^{n+1}}|\nabla u|^2\,dx - \sigma_1\sigma_2\int_{\Gamma(t)}|h|^2\,d\mathcal{H}^n \tag{1.3}$$

so that the sum of the kinetic energy of the fluid and the n-dimensional surface area of interface $\Gamma(t)$ decreases in time. In the present paper, we emphasize that we take u to be a *given datum*, on the other hand, leaving the application to the coupled problem for the future.

The main result of the present paper is a global-in-time existence of solution for the initial value problem of (1.1) in the case of $n = 1$ under the assumption that the given vector field u satisfies

$$u \in L^\infty_{loc}([0,\infty); L^2(\mathbb{R}^2)) \cap L^2_{loc}([0,\infty); W^{1,2}(\mathbb{R}^2)). \tag{1.4}$$

Here $W^{1,2}(\mathbb{R}^2)$ is the Sobolev space with L^2-integrable weak first derivatives. Moreover, for all $T > 0$, writing

$$c_1(u,T) := \Big(\operatorname*{ess\,sup}_{t\in[0,T]}\int_{\mathbb{R}^2}|u(x,t)|^2\,dx\Big)\Big(\int_0^T\int_{\mathbb{R}^2}|\nabla u(x,t)|^2\,dxdt\Big), \tag{1.5}$$

the solution satisfies

$$\mathcal{H}^1(\Gamma(T)) \le \mathcal{H}^1(\Gamma_0)\exp(c_2\,c_1(u,T)) \tag{1.6}$$

for all $T > 0$ with an absolute constant $c_2 > 0$. Here $\mathcal{H}^1(A)$ for $A \subset \mathbb{R}^2$ is the one-dimensional Hausdorff measure of A. The solution $\Gamma(t)$ is guaranteed to be non-trivial at least for some positive time interval and may go through various occurrences of singularities. The functional space (1.4) is a natural class to solve the above mentioned problem (1.2), but more interestingly, the quantities appearing in (1.5) are invariant under the natural parabolic scaling (when $T = \infty$ and note that u has the same scaling as x/t, see [4]). As the most relevant results, the second-named author and collaborators showed ([11, 20]) such existence results for general $n \ge 1$ if the given vector field u satisfies

$$\int_0^T\Big(\int_{\mathbb{R}^{n+1}}|u(x,t)|^p + |\nabla u(x,t)|^p\,dx\Big)^{q/p}dt < \infty \text{ with } p > \frac{(n+1)q}{2(q-1)},\ q > 2 \tag{1.7}$$

and $p \ge \frac{4}{3}$ in addition if $n = 1$. These conditions on p, q imply that, under the parabolic scaling, the effect of u in small scale becomes negligible and u may be regarded as a perturbation to the mean curvature flow. More precisely, for $\lambda > 0$, let $\tilde{x} = \lambda^{-1}x$, $\tilde{t} = \lambda^{-2}t$ and $\tilde{u}(\tilde{x},\tilde{t}) = \lambda u(x,t)$ (which should behave like $\tilde{x}/\tilde{t}$). Then the change of variables shows

$$\Big(\int\Big(\int|\nabla u(x,t)|^p\,dx\Big)^{\frac{q}{p}}dt\Big)^{\frac{1}{q}} = \Big(\int\Big(\int|\nabla\tilde{u}(\tilde{x},\tilde{t})|^p\,d\tilde{x}\Big)^{\frac{q}{p}}d\tilde{t}\Big)^{\frac{1}{q}}\lambda^{-2+\frac{n+1}{p}+\frac{2}{q}}, \tag{1.8}$$

and the condition on the exponents p, q of (1.7) means $-2+(n+1)/p+2/q<0$. This implies that the effect of $\tilde{u}$ becomes smaller as one looks at a finer scale $\lambda \approx 0$. For this reason, it is fitting to say that the problem (1.1) with (1.7) is subcritical. In contrast, the case of $n+1=p=q=2$ is a critical case which just fails to satisfy (1.7). The solution in the present paper has the feature that the effect of curvature and that of forcing are analytically comparable in strength since the parabolic change of variables, which is invariant under the mean curvature flow, is also invariant for the norms of u considered here. There is no known general existence theorem for the "supercritical" case, where at least one of the inequalities in (1.7) is $<$. We expect that there is no solution of the similar type in the sense that the forcing field is too turbulent for the evolving surface to even retain the finiteness of the surface measure in general.

We next discuss the technical difference between the critical and subcritical cases of the problem. One important point for both cases is that, unlike the mean curvature flow ($u=0$) which simply reduces the surface area $\mathcal{H}^n(\Gamma(t))$ in time, $\mathcal{H}^n(\Gamma(t))$ may blow up immediately with irregular u. The strategies of having a good bound for $\mathcal{H}^n(\Gamma(t))$ can be explained as follows. If u and $\Gamma(t)$ are smooth and (1.1) is satisfied, then by the first variation formula of the surface measure, we have

$$\frac{d}{dt}\mathcal{H}^n(\Gamma(t)) = -\int_{\Gamma(t)} v\cdot h\, d\mathcal{H}^n = -\int_{\Gamma(t)} (|h|^2 + h\cdot u^{\perp})\, d\mathcal{H}^n. \tag{1.9}$$

We may proceed to use $-h\cdot u^{\perp} \le |h|^2/2+|u|^2/2$, so that we would have an apriori estimate for $\mathcal{H}^n(\Gamma(t))$ in term of $\mathcal{H}^n(\Gamma(0))$ if we have a control of the term

$$\int_0^T \int_{\Gamma(t)} |u|^2\, d\mathcal{H}^n dt. \tag{1.10}$$

Note that $|u|^2$ is integrated with respect to the surface measure, so that we need to control the L^2-norm of the *trace* of Sobolev function u on $\Gamma(t)$. For this purpose, we take advantage of the Meyers-Ziemer inequality [15] (see Theorem 3.2), namely, if the upper density ratio

$$\mathfrak{D}(t) = \sup_{B_r(x)\subset\mathbb{R}^{n+1}} \frac{\mathcal{H}^n(\Gamma(t))}{r^n} \tag{1.11}$$

is bounded, then the Meyers-Ziemer inequality gives

$$\int_{\Gamma(t)} |u|^2\, d\mathcal{H}^n \le c(n)\mathfrak{D}(t)\int_{\mathbb{R}^{n+1}} |\nabla|u|^2|\, dx, \tag{1.12}$$

and if the right-hand side is appropriately bounded, we would be able to obtain the apriori bound for $\mathcal{H}^n(\Gamma(t))$. For the subcritical case of (1.7) (as in [20]), it turned out that the local L^∞ bound of $\mathfrak{D}(t)$ in time can be controlled using an analogue of Huisken's monotonicity formula [7]. For the present critical case, a similar strategy does not work since the contribution coming from u cannot be controlled in the computation of Huisken's monotonicity formula. Instead, we utilize a monotonicity formula for one-dimensional general varifold which gives

$$\mathfrak{D}(t) \leq \int_{\Gamma(t)} |h| \, d\mathcal{H}^1 \tag{1.13}$$

in essence (see Proposition 3.1 for the precise statement) and which allows us to close the above estimate. In this sense, the strategy is particular to one-dimension and is different from the subcritical case. In the end, we only obtain

$$\int_0^T \mathfrak{D}(t)^2 \, dt \leq \sup_{t\in[0,T]} \mathcal{H}^1(\Gamma(t)) \int_0^T \int_{\Gamma(t)} |h|^2 \, d\mathcal{H}^1 \tag{1.14}$$

and not necessarily the local L^∞ bound on $\mathfrak{D}(t)$ as in the subcritical case.

We also utilize global-in-time existence results [9, 10, 19] for the mean curvature flow, which can be modified for the flow with a smooth forcing term, and which prove the global-in-time existence of approximate flows. We show a priori estimates on the length of the curve, L^2-norm of the curvature and L^2-norm of the vector field u (as a trace) depending only on $\mathcal{H}^1(\Gamma_0)$ and $c_1(u,T)$. It is proved that the limit of the approximate flow satisfies a set of desired properties similar to the results in [20]. As in [9, 10, 19], the setting of the problem is for that of general multi-phase and the flow typically keeps triple junctions as it evolves.

The organization of the paper is as follows. In the next Section 2, notation and main results are stated. In Section 3, key a priori estimates are presented. Section 4 states the existence results of the flow in the case of smooth forcing term and for general dimensions, and whose proof is relegated to the Appendix. Section 5 analyzes the limiting process and proves the main theorem, and in Section 6, we mention a few additional comments.

2. Notation and main theorem

2.1. **Notation.** Though the main result of the present paper is for $n = 1$, here we write the notation for general dimensions because of Section 4. We define $G_n(\mathbb{R}^{n+1}) := \mathbb{R}^{n+1} \times \mathbf{G}(n+1,n)$ where $\mathbf{G}(n+1,n)$ is the space of n-dimensional subspaces of $\mathbb{R}^{n+1}$. The subspace S is often identified with the $(n+1)\times(n+1)$ matrix representing the orthogonal projection $\mathbb{R}^{n+1} \to S$. A general n-varifold is a Radon measure on $G_n(\mathbb{R}^{n+1})$ (see [2, 17] for more comprehensive presentation). The set of all general n-varifolds is denoted by $\mathbf{V}_n(\mathbb{R}^{n+1})$. For $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$, let $\|V\|$ be the weight measure of V, namely,

$$\|V\|(\phi) := \int_{G_n(\mathbb{R}^{n+1})} \phi(x) \, dV(x,S)$$

for $\phi \in C_c(\mathbb{R}^{n+1})$. A set $M \subset \mathbb{R}^{n+1}$ is said to be *countably n-rectifiable* if there exists a countable number of Lipschitz functions $F_j : \mathbb{R}^n \to \mathbb{R}^{n+1}$ ($j \in \mathbb{N}$) such that $\mathcal{H}^n(M \setminus \cup_{j=1}^\infty F_j(\mathbb{R}^n)) = 0$ (see [17] for the other properties). $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ is called rectifiable if there exist an $\mathcal{H}^n$-measurable countably n-rectifiable set $M \subset \mathbb{R}^{n+1}$ and a locally $\mathcal{H}^n$-integrable function θ defined on M such that

$$V(\phi) = \int_M \phi\,(x, \operatorname{Tan}_x M)\, \theta(x) \, d\mathcal{H}^n(x) \tag{2.1}$$

for $\phi \in C_c\,(G_n(\mathbb{R}^{n+1}))$. Here $\mathrm{Tan}_x\, M$ is the approximate tangent space of M at x which exists $\mathcal{H}^n$ a.e. on M. Any rectifiable n-varifold is uniquely determined by its weight measure $\|V\| = \theta\,\mathcal{H}^n\lfloor_M$ through the formula (2.1). For this reason, we naturally say that a Radon measure μ is rectifiable when one can associate a rectifiable varifold V such that $\|V\| = \mu$. The approximate tangent space $\mathrm{Tan}_x\, M$ is denoted in this case by $\mathrm{Tan}_x\,\mu$ or $\mathrm{Tan}_x\,\|V\|$ without fear of confusion. We let $\mathbf{var}(M, \theta)$ denote the varifold defined as in (2.1). If $\theta \in \mathbb{N}$ for $\mathcal{H}^n$ a.e. on M, we say V is integral. The set of all integral n-varifolds is denoted by $\mathbf{IV}_n(\mathbb{R}^{n+1})$. If $\theta = 1$ for $\mathcal{H}^n$ a.e. on M, we say that V is a unit density n-varifold.

For $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ let δV be the first variation of V, namely,

$$\delta V(g) := \int_{G_n(\mathbb{R}^{n+1})} \nabla g(x) \cdot S\, dV(x, S) \quad \text{for } g \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \tag{2.2}$$

Here $\nabla g \cdot S := \sum_{i,j=1}^{n+1} g^i_{x_j} S_{ij}$ which is the divergence of g on S. If

$$\sup_{g \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1}),\, |g| \le 1} \delta V(g) < \infty, \tag{2.3}$$

we say that the first variation is bounded, and in this case, let $\|\delta V\|$ be the total variation measure. If $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, by the Radon-Nikodym theorem, we have a $\|V\|$-measurable vector field $h(x, V)$ such that

$$\delta V(g) = -\int_{\mathbb{R}^{n+1}} g(x) \cdot h(x, V)\, d\|V\|(x). \tag{2.4}$$

The vector field $h(x, V)$ (often written as $h(V)$) is called the generalized mean curvature of V (or curvature in the case of $n = 1$). For any $V \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ with bounded first variation, Brakke's perpendicularity theorem for generalized mean curvature [3, Chapter 5] says that

$$S^\perp(h(x, V)) = h(x, V) \quad \text{for } V \text{ a.e. } (x, S) \in G_n(\mathbb{R}^{n+1}), \tag{2.5}$$

i.e. $h(x, V)$ is perpendicular to the approximate tangent space for $\|V\|$ a.e. $x \in \mathbb{R}^{n+1}$. For $V \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ and $\|V\|$-integrable function u, we use the notation

$$u^\perp(x) := (\mathrm{Tan}_x\,\|V\|)^\perp(u(x)), \tag{2.6}$$

which is well-defined for $\|V\|$ a.e. $x \in \mathbb{R}^{n+1}$.

For a set of finite perimeter $E \subset \mathbb{R}^{n+1}$, let $\partial^* E$ be the reduced boundary of E. Let $\nabla\chi_E$ be the distributional derivative of the characteristic function of E and let $\|\nabla\chi_E\|$ be the total variation of $\nabla\chi_E$. By the well-known structure theorem for sets of finite perimeter (see [5]), $\mathcal{H}^n\lfloor_{\partial^* E} = \|\nabla\chi_E\|$. We also write $|E|$ for the Lebesgue measure of E.

2.2. **Measure-theoretic formulation of the normal velocity.** In this subsection we review what it means for a one-parameter family of varifolds $\{V_t\}_{t\ge 0}$ to be a solution of (1.1). To motivate the definition, let us assume first that we have a smooth family of n-dimensional surfaces without boundary $\Gamma(t) \subset \mathbb{R}^{n+1}$ moving by the normal velocity $v = v(x, t)$. For any test function $\phi \in C^1_c(\mathbb{R}^{n+1} \times [0, \infty))$, we can prove that

$$\frac{d}{dt}\int_{\Gamma(t)} \phi\, d\mathcal{H}^n(x) = \int_{\Gamma(t)} (\nabla\phi - \phi\, h) \cdot v + \frac{\partial\phi}{\partial t}\, d\mathcal{H}^n \tag{2.7}$$

is satisfied. Conversely, if $\tilde{v} = \tilde{v}(x,t)$ is a smooth normal vector field defined on $\Gamma(t)$ which satisfies

$$\frac{d}{dt}\int_{\Gamma(t)} \phi \, d\mathcal{H}^n(x) \le \int_{\Gamma(t)} (\nabla\phi - \phi\, h)\cdot \tilde{v} + \frac{\partial \phi}{\partial t}\, d\mathcal{H}^n \tag{2.8}$$

for all *non-negative* $\phi \in C^1_c(\mathbb{R}^{n+1}\times[0,\infty))$, then one can prove that $\tilde{v}$ has to be the normal velocity vector, i.e., $\tilde{v} = v$ (see [21, Section 2.1] for the proof). Thus (2.8) can be used to characterize the normal velocity vector, and by considering the integrated version, we have the following

Proposition 2.1. *Suppose $\{\Gamma(t)\}_{t\in[0,T]}$ is a smooth family of n-dimensional surfaces without boundary. Then a normal vector $\tilde{v}$ defined on $\Gamma(t)$ coincides with the normal velocity vector v if and only if the following holds. For any $0 \le t_1 < t_1 \le T$ and for any non-negative $\phi \in C^1_c(\mathbb{R}^{n+1}\times[0,T])$,*

$$\int_{\Gamma(t_2)} \phi(x,t_2)\, d\mathcal{H}^n(x) - \int_{\Gamma(t_1)} \phi(x,t_1)\, d\mathcal{H}^n(x) \le \int_{t_1}^{t_2}\int_{\Gamma(t)} (\nabla\phi - \phi\, h)\cdot\tilde{v} + \frac{\partial\phi}{\partial t}\, d\mathcal{H}^n dt \tag{2.9}$$

is satisfied.

Motivated by this, we may define the normal velocity of a family of varifolds $\{V_t\}_{t\in[0,T]}$. First, as a generalization of the n-dimensional surface, we may require $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ for a.e. $t \in [0,T]$. Since we want to consider $\tilde{v} = h + u^\perp$, we require that V_t has the generalized mean curvature vector $h(x,V_t)$ for a.e. t, which is also L^2-integrable, i.e., $\int_0^T\int_{\mathbb{R}^{n+1}} |h(x,V_t)|^2\, d\|V_t\| dt < \infty$. Under these additional conditions and replacing $\tilde{v}$ in (2.9) by $h+u^\perp$, $\mathcal{H}^n\lfloor_{\Gamma(t)}$ by $\|V_t\|$, respectively, we say that $\{V_t\}_{t\in[0,T]}$ is a solution of (1.1) if (2.9) holds true for any $0 \le t_1 < t_2 \le T$ and for any non-negative $\phi \in C^1_c(\mathbb{R}^{n+1}\times[0,T])$. See (V3) (which shows, in particular, that $V_t \in \mathbf{IV}_1(\mathbb{R}^2)$ for a.e. t), (V4), (V5) and (V7) in the next subsection.

2.3. **Main results.** We first state the assumptions for the main theorem.

Assumption 2.2. We consider the following (A1)-(A4):

(A1) N is an integer with $N \ge 2$.
(A2) $E_{0,1},\ldots,E_{0,N}$ are non-empty open sets in $\mathbb{R}^2$ which are mutually disjoint.
(A3) $\Gamma_0 := \mathbb{R}^2 \setminus \cup_{i=1}^N E_{0,i}$ is countably 1-rectifiable and $\mathcal{H}^1(\Gamma_0) < \infty$.
(A4) u is a given vector field as in (1.4) and $c_1(u,T)$ is as in (1.5).

Here, the simplest case is $N = 2$, which may be considered as a "two-phase" case. In general, N corresponds to the number of "phases" $E_{0,1},\ldots,E_{0,N}$ and they partition the whole $\mathbb{R}^2$. Note that the connectedness of each of $E_{0,1},\ldots,E_{0,N}$ is not assumed. Since Γ_0 cannot have interior due to $\mathcal{H}^1(\Gamma_0) < \infty$, we have $\Gamma_0 = \cup_{i=1}^N \partial E_{0,i}$ and the initial datum Γ_0 for the flow is given as the topological boundary of the partition.

We next state the main existence theorem as Theorem 2.3, and we give an itemized explanation in Section 2.4.

Theorem 2.3. *Assume* (A1)-(A4). *Then there exist a family* $\{V_t\}_{t\geq 0} \subset \mathbf{V}_1(\mathbb{R}^2)$ *and a family of sets of finite perimeter* $\{E_i(t)\}_{t\geq 0}$ *in* $\mathbb{R}^2$ *for each* $i = 1, \ldots, N$ *with the following: Properties for* V_t*:*

(V1) $\|V_0\| = \mathcal{H}^1\lfloor_{\Gamma_0}$.

(V2) *If* $\mathcal{H}^1(\cup_{i=1}^N(\partial E_{0,i} \setminus \partial^* E_{0,i})) = 0$*, then* $\lim_{t\to 0+} \|V_t\| = \|V_0\|$.

(V3) *For a.e.* $t \geq 0$*, for* t *with* $\|V_t\|(\mathbb{R}^2) > 0$*, there exist a finite number (denoted by* $P(t) \in \mathbb{N}$*) of connected embedded* $W^{2,2}$ *curves* $\ell_1(t), \ldots, \ell_{P(t)}(t)$*, each of them being either a closed curve or a curve with two endpoints, such that*

$$V_t = \sum_{i=1}^{P(t)} \mathbf{var}(\ell_i(t), 1). \tag{2.10}$$

Here, it is not ruled out that some $\ell_i(t)$ *and* $\ell_j(t)$ *with* $i \neq j$ *may be identical. Moreover, there is a finite set of points where the endpoints of curves* $\ell_1(t), \ldots, \ell_{P(t)}(t)$ *meet at angles of either 0, 60 or 120 degrees. If* $N = 2$*, each curve is an embedded closed curve which may intersect other curves only tangentially.*

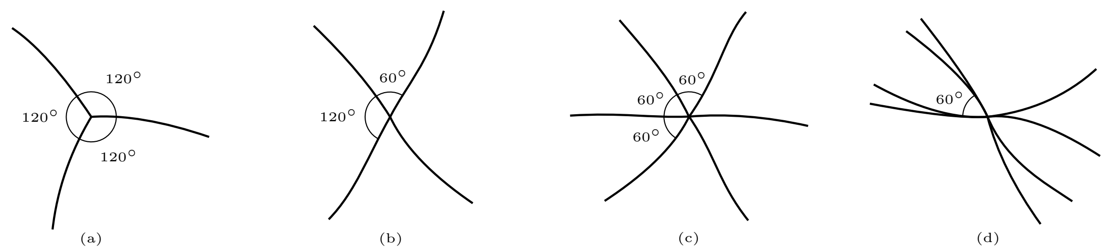

FIGURE 1.

(V4) *For a.e.* $t \geq 0$*,* δV_t *is bounded and absolutely continuous with respect to* $\|V_t\|$ *(thus* $h(x, V_t)$ *exists).*

(V5) *There exists an absolute constant* $c_2 > 0$ *(cf. Theorem 3.2) such that for all* $T > 0$*, and writing* $c_1(u, T)$ *(cf. (1.5)) as* c_1*,*

$$\int_0^T \int_{\mathbb{R}^2} |h(x, V_t)|^2 \, d\|V_t\| dt \leq 4\mathcal{H}^1(\Gamma_0)(1 + c_2 c_1 \exp(c_2 c_1)), \tag{2.11}$$

$$\int_0^T \int_{\mathbb{R}^2} |u|^2 \, d\|V_t\| dt \leq 4\sqrt{c_1 c_2}\mathcal{H}^1(\Gamma_0)\sqrt{1 + c_2 c_1 \exp(c_2 c_1)} \exp(c_2 c_1/2). \tag{2.12}$$

(V6) *For all* $T > 0$*,*

$$\sup_{t\in[0,T]} \|V_t\|(\mathbb{R}^2) \leq \mathcal{H}^1(\Gamma_0) \exp\left(c_2 \, c_1(u, T)\right). \tag{2.13}$$

(V7) *For all* $0 \le t_1 < t_2 < \infty$ *and* $\phi \in C^1_c(\mathbb{R}^2 \times [0,\infty))$ *with* $\phi \ge 0$*, it holds*

$$\begin{aligned} &\|V_{t_2}\|(\phi(\cdot,t_2)) - \|V_{t_1}\|(\phi(\cdot,t_1)) \le \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \frac{\partial \phi}{\partial t}(x,t)\, d\|V_t\|(x)dt \\ &+ \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \Big\{\nabla\phi(x,t) - \phi(x,t)h(x,V_t)\Big\}\cdot\Big\{h(x,V_t)+u^{\perp}(x,t)\Big\}\, d\|V_t\|(x)dt. \end{aligned} \tag{2.14}$$

Properties of $E_i(t)$*:*

(E1) $E_i(0) = E_{0,i}$ *for* $i = 1,\dots,N$.

(E2) $E_1(t),\dots,E_N(t)$ *are mutually disjoint sets of finite perimeter for all* $t \ge 0$.

(E3) *For all* $t \ge 0$, $\|V_t\| \ge \|\nabla\chi_{E_i(t)}\|$ *for each* $i = 1,\dots,N$ *and* $2\|V_t\| \ge \sum_{i=1}^N \|\nabla\chi_{E_i(t)}\|$.

(E4) $S(i) := \{(x,t) : x \in E_i(t), t \ge 0\}$ *is a set of finite perimeter in* $\mathbb{R}^2 \times [0,T)$ *for all* $T > 0$ *and* $i = 1,\dots,N$.

(E5) *There exists a constant* $c_3 = c_3(c_1(u,T),\mathcal{H}^1(\Gamma_0))$ *such that, for each* $i = 1,\dots,N$ *and* $0 \le t_1 < t_2 \le T$,

$$|E_i(t_2)\triangle E_i(t_1)| \le c_3(t_2-t_1)^{\frac{1}{2}}. \tag{2.15}$$

(E6) *There exists a Radon measure* β *on* $[0,\infty)$ *with* $\beta([0,T]) \le c_4 = c_4(c_1(u,T),\mathcal{H}^1(\Gamma_0))$ *such that, for any* $0 \le t_1 < t_2 \le T$, $\phi \in C_c(\mathbb{R}^2\times[0,\infty))$ *and* $i = 1,\dots,N$*, we have*

$$\begin{aligned} &\Big|\int_{E_i(t_2)} \phi(x,t_2)\,dx - \int_{E_i(t_1)} \phi(x,t_1)\,dx - \int_{t_1}^{t_2}\int_{E_i(t)} \frac{\partial\phi}{\partial t}\,dxdt\Big| \\ &\qquad \le (\beta([t_1,t_2]))^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\phi^2\,d\|V_t\|dt\right)^{1/2}. \end{aligned} \tag{2.16}$$

(E7) *Let* $\theta_t(x)$ *be the density of* $\|V_t\|$*. Then for a.e.* $t \ge 0$ *and* $\mathcal{H}^1$ *a.e.* $x \in \cup_{i=1}^{P(t)} \ell_i(t)$ *(cf.* (2.10)*)*

(1) *if* $N \ge 3$*, then,* $\theta_t(x) = 1$ *implies* $x \in \cup_{i=1}^N \partial^* E_i(t)$.

(2) *If* $N = 2$,

$$\theta_t(x) = \begin{cases} \text{odd integer for } x \in \partial^*E_1(t) (= \partial^*E_2(t)), \\ \text{even integer for } x \in \cup_{i=1}^{P(t)}\ell_i(t)\setminus\partial^*E_1(t). \end{cases} \tag{2.17}$$

(E8) *If* $\{V_t\}_{0\le t\le T}$ *is of unit density for a.e.* $t \in [0,T]$*, then* $V_t = \mathbf{var}(\cup_{i=1}^N \partial^*E_i(t),1)$ *for a.e.* $t \in [0,T]$.

2.4. **Discussion on the main results.** We give comments on the stated results.

(V1) This specifies that the initial condition is satisfied.

(V2) In general, $\partial^*E_{0,i} \subset \partial E_{0,i}$, that is, the measure-theoretic boundary is contained in the topological boundary, but they are not equal when there is a piece of topological boundary "within the set", to put it heuristically. Such a set typically vanishes instantaneously under our construction, so that we cannot expect the stated continuity in general. On the other hand, if the difference has null measure as assumed in (V2), then $\|V_t\|$ is continuous at $t = 0$.

(V3) This states that, for a.e. $t \geq 0$ and for t with $\|V_t\|(\mathbb{R}^2) > 0$, the flow looks like a network of $W^{2,2}$ curves with triple junctions with 120-degree angles or the degenerate triple junctions (such as "infinitesimal hexagons" as in Figure 1(c)). This regularity is a special feature of the construction of [9, 10]. For the case of $N = 2$ ("two-phase"), there is no junction and each curve is an embedded and closed loop, and they may only intersect tangentially. We know the local description of the support of $\|V_t\|$, thus the number of curves is finite, which we denote as $P(t)$. On the other hand, we do not know any further property of $P(t)$.

(V4) This property in particular implies that the sum of the unit conormals of the curves at the junction (counting multiplicities) must be 0.

(V5,6) These give control of the relevant integrals in terms of the initial length and norm of u in an explicit manner.

(V7) The inequality (2.14) is a characterization of the normal velocity being equal to $h + u^{\perp}$ as described in Section 2.2 in a generalized sense of Brakke. Unlike the subcritical case of [20] where the regularity theorem of [8] is applicable, no regularity theorem of a similar kind is known for this flow due to the criticality.

We next discuss the results on $E_i(t)$.

(E1) Each $E_i(t)$ starts with the given initial open set $E_{0,i}$. As already stated, $E_{0,i}$ may not be connected in general.

(E2) At each time t, $E_1(t), \ldots, E_N(t)$ give a partition of $\mathbb{R}^2$, and note that some of them may be empty sets. Unlike the subcritical case, where $E_i(t)$ is open (see (E2')), we can only conclude that $E_i(t)$ is a set of finite perimeter.

(E3) The perimeter measure of $E_i(t)$ bounds $\|V_t\|$ from below and this property is useful to guarantee the non-triviality of $\|V_t\|$ in the sense that, as long as there are at least two elements of $E_1(t), \ldots, E_N(t)$ with positive Lebesgue measure, we have $\|V_t\| \neq 0$.

(E4) Each $E_i(t)$ may be obtained as the time-slice of $S(i) \subset \mathbb{R}^2 \times [0, \infty)$, which is itself a set of finite perimeter in $\mathbb{R}^2 \times [0, T]$ for all T.

(E5) The Lebesgue measure of $E_i(t)$ is Hölder continuous in t. In particular, it cannot suddenly vanish discontinuously.

(E6) More detailed information is contained in (2.16) on the continuity of each $E_i(t)$.

(E7) This is an intuitively expected result in that, if the density is one, then, there should be only one curve with unit density in the neighborhood, so that there should be exactly two $E_i(t)$'s whose boundaries coincide with the curve. More precise information is available if $N = 2$. If the multiplicity is $\theta \geq 2$, then, there are θ curves counting multiplicities. Crossing each curve transversely, one enters from $E_1(t)$ to $E_2(t)$ or vice-versa. Thus, depending on the parity, the point is either in the reduced boundary or inside the domain. The idea of proof is somewhat similar in spirit.

(E8) If we have a unit density flow, then the measure $\|V_t\|$ is precisely the one coming from the reduced boundary of $E_i(t)$. In other words, the higher multiplicity occurs with positive space-time measure only when the discrepancy $\|V_t\| > \mathbf{var}(\cup_{i=1}^{N} \partial^* E_i(t), 1)$ happens with positive measure of time.

We give more comments in Section 4 comparing the subcritical case and the present critical case in particular.

3. Key a priori estimates

In this section, we establish the length estimate (1.6) for an evolving curve with the velocity given by (1.4) in the sense expressed in (2.14). We also obtain the L^2 control of u. The first proposition is for a fixed time, so we drop the reference to t in V_t. The result is certainly not new and we include the proof from [16] for the reader's convenience.

Proposition 3.1. *Let $V \in \mathbf{V}_1(\mathbb{R}^2)$ with $\|V\|(\mathbb{R}^2) < \infty$ and $\|\delta V\|(\mathbb{R}^2) < \infty$. Then the following quantity*

$$\mathfrak{D}(\|V\|) := \sup_{B_r(x)\subset\mathbb{R}^2} \frac{\|V\|(B_r(x))}{r} \tag{3.1}$$

satisfies the estimate

$$\mathfrak{D}(\|V\|) \leq \|\delta V\|(\mathbb{R}^2). \tag{3.2}$$

If in addition that $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, then we have

$$\mathfrak{D}(\|V\|) \leq (\|V\|(\mathbb{R}^2))^{\frac{1}{2}} \Big(\int_{\mathbb{R}^2} |h(V)|^2 \, d\|V\| \Big)^{\frac{1}{2}}. \tag{3.3}$$

Proof. For $0 < \sigma < \rho$ and $x_0 \in \mathbb{R}^2$, we consider the vector field

$$X(x) = \left(\frac{1}{|x - x_0|_\sigma} - \frac{1}{\rho} \right)_+ (x - x_0),$$

where $(\cdot)_+$ denotes the nonnegative part and $|\cdot|_\sigma = \max\{|\cdot|, \sigma\}$. For $S \in \mathbf{G}(2,1)$, we can compute as

$$S \cdot \nabla X(x) = \begin{cases} \frac{1}{\sigma} - \frac{1}{\rho} & \text{on } B_\sigma(x_0), \\ \frac{|S^\perp(x-x_0)|^2}{|x-x_0|^3} - \frac{1}{\rho} & \text{on } B_\rho(x_0)\setminus \bar{B}_\sigma(x_0), \end{cases}$$

where $S^\perp$ represents the orthogonal projection to the orthogonal complement of S. Integrating $S \cdot \nabla X$ with respect to V,

$$\delta V(X) = \frac{\|V\|(B_\sigma(x_0))}{\sigma} - \frac{\|V\|(B_\rho(x_0))}{\rho} + \int_{G_1(B_\rho(x_0)\setminus B_\sigma(x_0))} \frac{|S^\perp(x - x_0)|^2}{|x - x_0|^3} \, dV(x, S)$$

while using $|X| \leq 1$, we have

$$\frac{\|V\|(B_\sigma(x_0))}{\sigma} - \frac{\|V\|(B_\rho(x_0))}{\rho} \leq \|\delta V\|(\mathbb{R}^2).$$

By letting $\rho \to \infty$, we obtain (3.2) and using (2.4), we have (3.3). □

We next recall the Meyers-Ziemer inequality [15] that bridges the integral with respect to a Radon measure and the Lebesgue measure of compactly supported functions.

Theorem 3.2. *There exists a constant $c_2 > 0$ depending only on n with the following property. Suppose that μ is a Radon measure on $\mathbb{R}^{n+1}$ with*

$$\mathfrak{D}(\mu) := \sup_{B_r(x)\subset\mathbb{R}^{n+1}} \frac{\mu(B_r(x))}{r^n} < \infty. \tag{3.4}$$

Then for any $\phi \in C^1_c(\mathbb{R}^{n+1})$, we have

$$\int_{\mathbb{R}^{n+1}} \phi(x)\, d\mu(x) \le \sqrt{c_2}\,\mathfrak{D}(\mu) \int_{\mathbb{R}^{n+1}} |\nabla\phi(x)|\, dx. \tag{3.5}$$

We use Theorem 3.2 and Proposition 3.1 to obtain the following a priori estimate for flows satisfying (1.1) with a regular u.

Proposition 3.3. *Consider a family of varifolds $\{V_t\}_{t\ge 0} \subset \mathbf{V}_1(\mathbb{R}^2)$ with the following properties.*

(a) *For a.e. $t \ge 0$, $V_t \in \mathbf{IV}_1(\mathbb{R}^2)$, δV_t is bounded and absolutely continuous with respect to $\|V_t\|$, and $\int_0^T\int_{\mathbb{R}^2} |h(x,V_t)|^2\, d\|V_t\| dt < \infty$ for all $T > 0$.*
(b) *For all $T > 0$, $\sup_{t\in[0,T]} \|V_t\|(\mathbb{R}^2) < \infty$.*
(c) *For some $u \in C^1_c(\mathbb{R}^2\times[0,\infty);\mathbb{R}^2)$, the property (V7) in Theorem 2.3 is satisfied.*

Set $c_1 = c_1(u,T)$ as in (1.5). *Then we have*

$$\sup_{t\in[0,T]} \|V_t\|(\mathbb{R}^2) \le \|V_0\|(\mathbb{R}^2)\exp(c_2\, c_1), \tag{3.6}$$

$$\int_0^T \int_{\mathbb{R}^2} |h(V_t)|^2\, d\|V_t\| dt \le 4\|V_0\|(\mathbb{R}^2)(1 + c_2 c_1 \exp(c_2\, c_1)), \tag{3.7}$$

$$\int_0^T \int_{\mathbb{R}^2} |u|^2\, d\|V_t\| dt \le 4\sqrt{c_1 c_2}\|V_0\|(\mathbb{R}^2)\sqrt{1 + c_2 c_1\exp(c_2 c_1)}\exp(c_2 c_1/2). \tag{3.8}$$

Proof. We consider a sequence $\{\phi_\ell\}_{\ell\in\mathbb{N}} \subset C^1_c(\mathbb{R}^2;[0,1])$ which equals 1 in B_ℓ and vanishes outside $B_{2\ell}$ with $|\nabla\phi_\ell| \le 1/\ell$. Using (2.14) and the dominated convergence theorem, we may take the limit $\ell\to\infty$ in (2.14) with $\phi = \phi_\ell$ and obtain

$$\begin{aligned}\|V_t\|(\mathbb{R}^2)\Big|_{t=t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathbb{R}^2} |h(V_t)|^2\, d\|V_t\| dt &\le \int_{t_1}^{t_2}\int_{\mathbb{R}^2} -h(V_t)\cdot u^\perp\, d\|V_t\| dt \\ &\le \frac12\int_{t_1}^{t_2}\int_{\mathbb{R}^2} |h(V_t)|^2\, d\|V_t\| dt + \frac12 \int_{t_1}^{t_2}\int_{\mathbb{R}^2} |u|^2\, d\|V_t\| dt.\end{aligned} \tag{3.9}$$

To estimate the last integral, we argue for a.e. $t\in[t_1,t_2]$ that

$$\begin{aligned}\int_{\mathbb{R}^2} |u|^2\, d\|V_t\| &\overset{(3.5)}{\le} \sqrt{c_2}\,\mathfrak{D}(\|V_t\|)\int_{\mathbb{R}^2} |\nabla |u|^2|\, dx \\ &\overset{(3.3)}{\le} 2\sqrt{c_2}(\|V_t\|(\mathbb{R}^2))^{\frac12}\Big(\int_{\mathbb{R}^2} |h(V_t)|^2\, d\|V_t\|\Big)^{\frac12}\Big(\int_{\mathbb{R}^2} |\nabla u|^2\, dx\Big)^{\frac12}\Big(\int_{\mathbb{R}^2} |u|^2\, dx\Big)^{\frac12} \\ &\le \frac12\int_{\mathbb{R}^2} |h(V_t)|^2\, d\|V_t\| + 2c_2\|V_t\|(\mathbb{R}^2)\int_{\mathbb{R}^2} |\nabla u|^2\, dx \int_{\mathbb{R}^2} |u|^2\, dx.\end{aligned} \tag{3.10}$$

The above two inequalities together imply

$$\begin{aligned}&\|V_t\|(\mathbb{R}^2)\Big|_{t=t_1}^{t_2} + \frac{1}{4}\int_{t_1}^{t_2}\int_{\mathbb{R}^2} |h(V_t)|^2\, d\|V_t\| dt \\ &\le c_2 \sup_{t\in[t_1,t_2]} \|u(\cdot,t)\|_{L^2}^2 \int_{t_1}^{t_2} \Big(\|V_t\|(\mathbb{R}^2)\int_{\mathbb{R}^2} |\nabla u|^2\, dx\Big) dt.\end{aligned} \tag{3.11}$$

This combined with Gronwall's inequality leads to (3.6). Using (3.6) in (3.11), we obtain (3.7), and using (3.6) and (3.7) in (3.10), one obtains (3.8). □

4. Existence result for flow with regular forcing

Given a vector field with a compact support $u \in C_c^1(\mathbb{R}^{n+1}\times[0,\infty);\mathbb{R}^{n+1})$ and a suitable initial datum, by modifying the proof of [9, 10, 19], we may establish the existence of a flow of multi-phase boundary moving with $v = h + u^\perp$. Since the modifications are somewhat of technical nature, we relegate the proof to Appendix A. We state the result for general dimensions since no change is needed. The stated results are not exhaustive and the reader is referred to [19] for additional properties, such as [19, Theorem 2.10,2.12,2.13], which hold equally for the current situation.

Assumption 4.1. Consider the following (A1)–(A3), (A4′), $n \in \mathbb{N}$:

- (A1) N is an integer with $N \ge 2$.
- (A2) $E_{0,1},\ldots,E_{0,N}$ are non-empty open sets in $\mathbb{R}^{n+1}$ which are mutually disjoint.
- (A3) $\Gamma_0 := \mathbb{R}^{n+1}\setminus \cup_{i=1}^N E_{0,i}$ is countably n-rectifiable and $\mathcal{H}^n(\Gamma_0) < \infty$.
- (A4') u is a vector field in $C_c^1(\mathbb{R}^{n+1}\times[0,\infty);\mathbb{R}^{n+1})$.

Theorem 4.2. *Assume* $(\mathrm{A1})-(\mathrm{A3}),(\mathrm{A4}')$. *Then there exist a family* $\{V_t\}_{t\ge 0}\subset \mathbf{V}_n(\mathbb{R}^{n+1})$ *and a family of open sets* $\{E_i(t)\}_{t\ge0}$ *for each* $i = 1,\ldots,N$ *with the following properties:*
Properties for V_t*:*

- (V1') $\|V_0\| = \mathcal{H}^n\lfloor_{\Gamma_0}$.
- (V2') *If* $\mathcal{H}^n(\cup_{i=1}^N(\partial E_{0,i}\setminus\partial^* E_{0,i})) = 0$*, then* $\lim_{t\to 0+}\|V_t\| = \|V_0\|$.
- (V3') *For a.e.* $t\ge 0$*,* $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$*, and if* $n=1$*,* (V3) *of Theorem 2.3 holds.*
- (V4') *For a.e.* $t \ge 0$*,* δV_t *is bounded and absolutely continuous with respect to* $\|V_t\|$ *(thus* $h(x,V_t)$ *exists).*
- (V5') *For all* $T>0$*,* $h(\cdot,V_t)$ *satisfies*

$$\int_0^T \int_{\mathbb{R}^{n+1}} |h(x,V_t)|^2\, d\|V_t\|(x) dt < \infty. \tag{4.1}$$

- (V6') *For all* $T>0$*,* $\sup_{t\in[0,T]}\|V_t\|(\mathbb{R}^{n+1}) < \infty$.
- (V7') *For all* $0\le t_1<t_2<\infty$ *and* $\phi \in C_c^1(\mathbb{R}^{n+1}\times[0,\infty))$ *with* $\phi\ge 0$*, it holds*

$$\begin{aligned}&\|V_{t_2}\|(\phi(\cdot,t_2)) - \|V_{t_1}\|(\phi(\cdot,t_1)) \le \int_{t_1}^{t_2}\int_{\mathbb{R}^{n+1}} \frac{\partial\phi}{\partial t}(x,t)\, d\|V_t\|(x) dt \\ &+ \int_{t_1}^{t_2}\int_{\mathbb{R}^{n+1}} \Big\{\nabla\phi(x,t) - \phi(x,t) h(x,V_t)\Big\}\cdot\Big\{h(x,V_t) + u^\perp(x,t)\Big\}\, d\|V_t\|(x) dt.\end{aligned} \tag{4.2}$$

Properties of $E_i(t)$:

(E1') $E_i(0) = E_{0,i}$ *for* $i = 1, \ldots, N$.

(E2') $E_1(t), \ldots, E_N(t)$ *are mutually disjoint open sets for all* $t \geq 0$.

(E3') *Writing* $\Gamma(t) := \mathbb{R}^{n+1} \setminus \cup_{i=1}^N E_i(t)$, *we have* $\mathcal{H}^n(\Gamma(t)) < \infty$ *and* $\Gamma(t) = \cup_{i=1}^N \partial E_i(t)$ *for all* $t > 0$.

(E4') *Write* $d\mu := d\|V_t\| dt$ *and* $(\mathrm{spt}\,\mu)_t := \{x \in \mathbb{R}^{n+1} : (x,t) \in \mathrm{spt}\,\mu\}$. *Then* $\Gamma(t) = (\mathrm{spt}\,\mu)_t$ *for all* $t > 0$.

(E5') *For all* $t \geq 0$, $\|V_t\| \geq \|\nabla \chi_{E_i(t)}\|$ *for each* $i = 1, \ldots, N$ *and* $2\|V_t\| \geq \sum_{i=1}^N \|\nabla \chi_{E_i(t)}\|$.

(E6') $S(i) := \{(x,t) : x \in E_i(t), t \geq 0\}$ *is open in* $\mathbb{R}^{n+1} \times [0,\infty)$ *for* $i = 1, \ldots, N$.

(E7') *For any* $0 \leq t_1 < t_2 < \infty$, $\phi \in C_c^1(\mathbb{R}^{n+1} \times [0,\infty))$ *and* $i = 1, \ldots, N$, *we have*

$$\int_{E_i(t)} \phi(x,t)\, dx \Big|_{t=t_1}^{t_2} = \int_{t_1}^{t_2} \Big(\int_{\partial^* E_i(t)} (h+u) \cdot \nu_i\, \phi\, d\mathcal{H}^n + \int_{E_i(t)} \frac{\partial \phi}{\partial t}\, dx \Big)\, dt. \tag{4.3}$$

Here ν_i *is the outer unit-normal of the reduced boundary* $\partial^* E_i(t)$.

(E8') *Let* $\theta_t(x)$ *be the density of* $\|V_t\|$. *Then for* $\mathcal{L}^1$-*a.e.* $t \geq 0$ *and* $\mathcal{H}^n$-*a.e.* $x \in \Gamma(t)$,

(1) *if* $N \geq 3$, *then* $\theta_t(x) = 1$ *implies* $x \in \cup_{i=1}^N \partial^* E_i(t)$.

(2) *If* $N = 2$,

$$\theta_t(x) = \begin{cases} \text{odd integer for } x \in \partial^* E_1(t) (= \partial^* E_2(t)), \\ \text{even integer for } x \in \Gamma(t) \setminus \partial^* E_1(t). \end{cases} \tag{4.4}$$

We discuss the difference between the critical and subcritical cases. Overall results are similar but the subcritical case gives better results due to the availability of Huisken's monotonicity formula. With it, one can obtain an upper density ratio bound (3.4) uniform in time for any fixed time interval $[0,T]$. For the critical case, as one can see in (3.3), we only have a time-integrated L^2 bound on $\mathfrak{D}(\|V_t\|)$ which is a substantially weaker control. Thus, compared to (E2') where $E_i(t)$ is open, in (E2) we only have measurable $E_i(t)$ and a similar difference holds between (E6') and (E4). This also causes a certain weaker conclusion on (E6) compared to (E7'), where the result is reduced to the estimate (2.16) instead of the identity (4.3). Technically speaking, we do not know if $d\|V_t\|dt$ is absolutely continuous with respect to $\mathcal{H}^2$ in $\mathbb{R}^2 \times [0,\infty)$ for the critical case and this gives a difficulty in establishing the identity (4.3) in [19].

5. Proof of main theorem

For the given vector field u as in (1.4) and $m \in \mathbb{N}$, we first extend u by zero for $t \notin [0,m]$ and we shall consider the mollified vector field

$$u^{(m)}(x,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \zeta_{1/m}(x-y)\, \varrho_{1/m}(t-s)\, u(y,s)\, dyds\, \eta(|x|/m) \tag{5.1}$$

where ζ_ϵ and ϱ_ϵ are respectively 2-d and 1-d standard mollifiers, and η is in $C^\infty([0,\infty))$ with the property that $0 \leq \eta \leq 1$, $\eta(s) = 1$ for $0 \leq s \leq 1$ and $\eta(s) = 0$ for $s \geq 2$. Then $u^{(m)}$ is a C^∞ vector field with compact support. By the standard argument (see [1, Theorem

2.29 and Lemma 3.16]), one can prove for any $T > 0$ that

$$\lim_{m\to\infty} \int_0^T \|u^{(m)}(\cdot,t) - u(\cdot,t)\|^2_{W^{1,2}(\mathbb{R}^2)}\, dt = 0 \tag{5.2}$$

and

$$\sup_{t\in[0,T]} \|u^{(m)}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le \operatorname*{ess\,sup}_{t\in[0,T+1]} \|u(\cdot,t)\|_{L^2(\mathbb{R}^2)}. \tag{5.3}$$

In the remaining part of this section, we let $\{V^{(m)}\}_{t\ge 0}$ and $\{E_i^{(m)}(t)\}_{t\ge 0}$ $(i = 1,\dots,N)$ be the solution corresponding to $u = u^{(m)}$ in Theorem 4.2, and we prove that the desired solution can be obtained as a suitable limit of $V_t^{(m)}$ and $E_i^{(m)}(t)$.

Proposition 5.1. *There exists a subsequence $\{m_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ and a family of Radon measures $\{\hat\mu_t\}_{t\ge 0}$ such that $\hat\mu_t = \lim_{j\to\infty} \|V_t^{(m_j)}\|$ as measures for all $t \ge 0$.*

Proof. If we can prove the existence of such a subsequence on any $[0,T]$, then a diagonal argument shows the existence of a subsequence with the desired property. Thus we fix $T > 0$ in the following. By Proposition 3.3 with $V_t^{(m)}$, we have (3.6)-(3.8) with $c_1 = c_1(u^{(m)},T)$. Then, by (5.2) and (5.3), the right-hand sides of (3.6)-(3.8) are bounded depending only on $\|V_0\|(\mathbb{R}^2) = \mathcal{H}^1(\Gamma_0)$, $\int_0^T \|\nabla u(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}\, dt$ and $\operatorname{ess\,sup}_{t\in[0,T+1]} \|u(\cdot,t)\|_{L^2(\mathbb{R}^2)}$ for all large m. Thus, there exists a constant c_5 depending only on these quantities (and not on m) such that

$$\sup_{t\in[0,T]} \|V_t^{(m)}\|(\mathbb{R}^2) \le c_5. \tag{5.4}$$

Then arguing precisely as in the proof of (3.11), we may obtain

$$\|V_t^{(m)}\|(\mathbb{R}^2)\Big|_{t=t_1}^{t_2} + \frac{1}{4}\int_{t_1}^{t_2}\int_{\mathbb{R}^2} |h(V_t^{(m)})|^2\, d\|V_t^{(m)}\| dt \le c_6 \int_{t_1}^{t_2}\int_{\mathbb{R}^2} |\nabla u^{(m)}|^2\, dxdt, \tag{5.5}$$

where c_6 is independent of m. To proceed, we introduce the following quantities:

$$\Phi^{(m)}(t) := \|V_t^{(m)}\|(\mathbb{R}^2), \tag{5.6a}$$

$$H^{(m)}(t) := \int_0^t \int_{\mathbb{R}^2} \frac{1}{4}|h(V_s^{(m)})|^2\, d\|V_s^{(m)}\| ds, \tag{5.6b}$$

$$U^{(m)}(t) := c_6 \int_0^t \int_{\mathbb{R}^2} |\nabla u^{(m)}(x,s)|^2\, dxds, \quad U(t) := c_6 \int_0^t \int_{\mathbb{R}^2} |\nabla u(x,s)|^2\, dxds, \tag{5.6c}$$

$$\Psi_1^{(m)} := \Phi^{(m)} + H^{(m)} - U^{(m)},\ \Psi_2^{(m)} := \Phi^{(m)} - U^{(m)}. \tag{5.6d}$$

Then (5.5) shows that $\Psi_1^{(m)}$ and $\Psi_2^{(m)}$ are non-increasing functions of t. For a sequence of uniformly bounded non-increasing functions, we may extract a subsequence $\{m_j\}_{j\in\mathbb{N}}$ such that $\Psi_1^{(m_j)}$ and $\Psi_2^{(m_j)}$ converge to some non-increasing functions point-wise. More precisely, by a diagonal argument, we may choose a subsequence that converges point-wise on $\mathbb{Q}_+ = \mathbb{Q} \cap [0,\infty)$ first. The limit function is again a non-increasing function on $\mathbb{Q}_+$, which can be uniquely extended to a continuous function on $[0,\infty)$ except for a set of countable discontinuous points named $D \subset [0,\infty)$. Using again the non-increasing property, one can show the point-wise convergence on $[0,\infty)\setminus D$, and one may choose a further subsequence to have a convergence on countable D by the diagonal argument. This gives a desired subsequence $\{m_j\}_{j\in\mathbb{N}}$. Since $U^{(m_j)}$ converges to U point-wise by (5.2), we may conclude that both $\Phi^{(m_j)}$ and $H^{(m_j)}$ converge point-wise in t, and we define

$$\Phi := \lim_{j\to\infty} \Phi^{(m_j)} \text{ and } H := \lim_{j\to\infty} H^{(m_j)}. \tag{5.7}$$

Note that a non-increasing function is continuous on a co-countable set. From this and also the fact that U is a continuous function, we conclude that H and Φ are continuous on a co-countable set. By the compactness property of Radon measures, on a countable dense set $\{t(k)\}_{k\in\mathbb{N}}$ of $[0,T]$, we may further extract a subsequence (denoted by the same notation) such that

$$\|V_{t(k)}^{(m_j)}\| \text{ converges to some Radon measure } \mu_{t(k)} \text{ as } j\to\infty \text{ for each } k\in\mathbb{N}. \tag{5.8}$$

We next prove that $\mu_{t(k)}$ can be extended to a Radon measure $\hat{\mu}_t$ on a co-countable set $A\subset[0,T]$ continuously and that $\|V_t^{(m_j)}\|$ converges to $\hat{\mu}_t$ for all $t\in A$. For any $t_2>t_1$ and for any $\phi\in C_c^2(\mathbb{R}^2;\mathbb{R}_+)$, we use (4.2) and the Cauchy-Schwarz inequality to obtain

$$\|V_t^{(m)}\|(\phi)\|\Big|_{t=t_1}^{t_2} \le \int_{t_1}^{t_2}\int_{\mathbb{R}^2} -\frac{1}{2}|h(V_t^{(m)})|^2\phi + c_7\frac{|\nabla\phi|^2}{\phi} + \|\phi\|_{L^\infty}|u^{(m)}|^2\,d\|V_t^{(m)}\|(x)dt. \tag{5.9}$$

Here c_7 is a constant independent of m. Since $\sup_{\{\phi>0\}}(|\nabla\phi|^2/\phi)\le 2\|\nabla^2\phi\|_{L^\infty}$ (see [21, Lemma 3.1]), the second term on the right-hand side is bounded by $C(t_2-t_1)$ for some C independent of m. Thus the same computation in Proposition 3.3 shows

$$\|V_t^{(m)}\|(\phi)\Big|_{t=t_1}^{t_2} \le c_8\Big(t + H^{(m)}(t) + U^{(m)}(t)\Big)\Big|_{t=t_1}^{t_2}. \tag{5.10}$$

Here c_8 depends on $\|\phi\|_{C^2}$ and others but not m. Now let $t_2=t(k')>t(k)=t_1$ and m_j in place of m. Since $\|V_{t(k)}^{(m_j)}\|(\phi)$ converges to $\mu_{t(k)}(\phi)$, and also $H^{(m_j)}(t(k))$ and $U^{(m_j)}(t(k))$ converge to $H(t(k))$ and $U(t(k))$ respectively, we have

$$\mu_t(\phi)\Big|_{t=t(k)}^{t(k')} \le c_8(t+H(t)+U(t))\Big|_{t=t(k)}^{t(k')}. \tag{5.11}$$

This implies that $\mu_t(\phi)-c_8(t+H(t)+U(t))$ on the set $\{t(k)\}_{k\in\mathbb{N}}$ (recall that this set is dense in $[0,T]$) is a monotone non-increasing function, thus can be extended as a continuous function on a co-countable set of $[0,T]$. Since H is also continuous on a co-countable set and U is continuous, we may define $\hat{\mu}_t(\phi)=\lim_{j\to\infty}\mu_{t(k_j)}(\phi)$ (whenever $t(k_j)\to t$) on a co-countable set of $[0,T]$ as a continuous function. The above argument was for a fixed $\phi\in C_c^2(\mathbb{R}^2;\mathbb{R}_+)$, but we can choose a countable set $\{\phi^{(\ell)}\}_{\ell\in\mathbb{N}}\subset C_c^2(\mathbb{R}^2;\mathbb{R}_+)$ which is dense in $C_c(\mathbb{R}^2;\mathbb{R}_+)$ with respect to the norm $\|\cdot\|_{L^\infty}$, and conclude that $\hat{\mu}_t(\phi^{(\ell)})$ is similarly defined on a co-countable set of $[0,T]$ for all $\ell\in\mathbb{N}$. We name this co-countable set again as A. Next, we prove that $\lim_{j\to\infty}\|V_t^{(m_j)}\|(\phi^{(\ell)})=\hat{\mu}_t(\phi^{(\ell)})$ for all $t\in A$. To be clear, let us call an arbitrary point of A as $\tilde{t}\in A$ and let $\{t(k_n)\}_{n\in\mathbb{N}}$ be any sequence with $t(k_n)<\tilde{t}$

and $\lim_{n\to\infty} t(k_n) = \tilde{t}$. Since (5.10) is valid for any $t_2 > t_1$, choosing $t_2 = \tilde{t}$ and $t_1 = t(k_n)$, we have

$$\|V_{\tilde{t}}^{(m_j)}\|(\phi^{(\ell)}) \le \|V_{t(k_n)}^{(m_j)}\|(\phi^{(\ell)}) + c_8(t + H^{(m_j)}(t) + U^{(m_j)}(t))\Big|_{t=t(k_n)}^{\tilde{t}}. \tag{5.12}$$

By taking $\limsup_{j\to\infty}$ and using (5.7) and (5.8), we have

$$\limsup_{j\to\infty} \|V_{\tilde{t}}^{(m_j)}\|(\phi^{(\ell)}) \le \mu_{t(k_n)}(\phi^{(\ell)}) + c_8(t + H(t) + U(t))\Big|_{t=t(k_n)}^{\tilde{t}}, \tag{5.13}$$

and since H and U are continuous at $\tilde{t}$ and we define $\hat{\mu}_t(\phi^{(\ell)})$ as above, by letting $n \to \infty$, we have

$$\limsup_{j\to\infty} \|V_{\tilde{t}}^{(m_j)}\|(\phi^{(\ell)}) \le \hat{\mu}_{\tilde{t}}(\phi^{(\ell)}). \tag{5.14}$$

Similarly, let $\{t(k_n)\}_{n\in\mathbb{N}}$ be any sequence with $\tilde{t} < t(k_n)$ and $\lim_{n\to\infty} t(k_n) = \tilde{t}$. Again letting $t_2 = t(t_n)$ and $t_1 = \tilde{t}$, we have

$$\|V_{t(k_n)}^{(m_j)}\|(\phi^{(\ell)}) - c_8(t + H^{(m_j)}(t) + U^{(m_j)}(t))\Big|_{t=\tilde{t}}^{t(k_n)} \le \|V_{\tilde{t}}^{(m_j)}\|(\phi^{(\ell)}). \tag{5.15}$$

By arguing similarly and taking $\liminf_{n\to\infty}$, we have

$$\hat{\mu}_{\tilde{t}}(\phi^{(\ell)}) \le \liminf_{j\to\infty} \|V_{\tilde{t}}^{(m_j)}\|(\phi^{(\ell)}). \tag{5.16}$$

This proves the desired convergence on A, and since $\{\phi^{(\ell)}\}_{\ell\in\mathbb{N}}$ is dense in $C_c(\mathbb{R}^2;\mathbb{R}_+)$, we have the convergence of $\|V_{\tilde{t}}^{(m_j)}\|$ to $\hat{\mu}_{\tilde{t}}$ as Radon measures. On the complement of A which is countable, we may choose a further subsequence (denoted by the same notation) of $\{m_j\}$ so that $\|V_t^{(m_j)}\|$ converges to some Radon measure, which we again name $\hat{\mu}_t$. This ends the proof. $\square$

Proposition 5.2. *For a.e. $t \in [0,\infty)$, there exists a unique $V_t \in \mathbf{IV}_1(\mathbb{R}^2)$ such that $\hat{\mu}_t = \|V_t\|$, $\|\delta V_t\|$ is absolutely continuous with respect to $\|V_t\|$ and $h(V_t)$ is in L^2. Moreover, for any $T < \infty$, (2.11) holds for $h(V_t)$.*

Proof. Taking $\liminf$ in (5.5) with $m = m_j$, $t_1 = 0$, $t_2 = T$ and using Fatou's lemma, we have

$$\liminf_{j\to\infty} \int_{\mathbb{R}^2} \left|h\left(V_s^{(m_j)}\right)\right|^2 d\|V_s^{(m_j)}\| < \infty \text{ for } a.e.\ s \in [0,T]. \tag{5.17}$$

On the other hand, we have

$$\begin{aligned}\|\delta V_s^{(m_j)}\|(\mathbb{R}^2) &= \int_{\mathbb{R}^2} |h(V_s^{(m_j)})|\, d\|V_s^{(m_j)}\| \\ &\le \left(\int_{\mathbb{R}^2} |h(V_s^{(m_j)})|^2\, d\|V_s^{(m_j)}\|\right)^{\frac{1}{2}} (\|V_s^{(m_j)}\|(\mathbb{R}^2))^{\frac{1}{2}}.\end{aligned} \tag{5.18}$$

With these two assertions and (5.4), we can apply Allard's compactness theorem for integral varifolds [2, Sec. 6.4] to a.e. $s \in [0,T]$ so that there exists a subsequence $\{m_j'\} \subset \{m_j\}$ and $V_s \in \mathbf{IV}_1(\mathbb{R}^2)$ such that $V_s^{(m_j')} \xrightarrow{j\to\infty} V_s$ as varifolds. Note that the choice of the subsequence

m'_j may depend on s. On the other hand, we know that $\|V_s^{(m_j)}\| \xrightarrow{j\to\infty} \hat\mu_s$ as measures on $\mathbb{R}^2$, so we have $\hat\mu_s = \|V_s\|$. For an integral 1-varifold V_s, there exist a countably 1-rectifiable set $M_s \subset \mathbb{R}^2$ and a function $\theta_s : M_s \to \mathbb{N}$ such that $V_s = \mathbf{var}(M_s, \theta_s)$ and $\|V_s\| = \theta_s \mathcal{H}^1\lfloor_{M_s}$, which is $\hat\mu_s$. Once we know that $\hat\mu_s$ is of this form, then $\hat\mu_s$ uniquely determines V_s. So for a.e. $s \in [0,T]$, we have a unique $V_s \in \mathbf{IV}_1(\mathbb{R}^2)$ such that $\|V_s\| = \hat\mu_s$. We define $V_s \in \mathbf{V}_1(\mathbb{R}^2)$ for other $s \in [0,T]$ so that $\|V_s\| = \hat\mu_s$, for example, $V_s(\phi) = \int_{\mathbb{R}^2} \phi(x, S_0)\, d\hat\mu_s(x)$ for $\phi \in C_c(G_1(\mathbb{R}^2))$, where $S_0 \in \mathbf{G}(2,1)$ is a fixed element. [1] The resulting family $\{V_s\}_{s\in[0,T]}$ belongs to $\mathbf{IV}_1(\mathbb{R}^2)$ for a.e. $s \in [0,T]$ and satisfies the claimed convergence properties. The existence of $h(V_s) \in L^2(\|V_s\|)$ follows from the standard argument: By the varifold convergence, one can prove that, for any $g \in C^1_c(\mathbb{R}^2;\mathbb{R}^2)$,

$$
\begin{aligned}
\delta V_s(g) = \lim_{j\to\infty} \delta V_s^{(m'_j)}(g) &\le \lim_{j\to\infty} \left(\int_{\mathbb{R}^2} |h(V_s^{(m'_j)})|^2 \, d\|V_s^{(m'_j)}\|\right)^{\frac12} \left(\int_{\mathbb{R}^2} |g|^2 \, d\|V_s^{(m'_j)}\|\right)^{\frac12} \\
&= \left(\liminf_{j\to\infty} \int_{\mathbb{R}^2} |h(V_s^{(m_j)})|^2 \, d\|V_s^{(m_j)}\|\right)^{\frac12} \left(\int_{\mathbb{R}^2} |g|^2 \, d\|V_s\|\right)^{\frac12}.
\end{aligned} \tag{5.19}
$$

In the last line, we have taken a subsequence $\{m'_j\} \subset \{m_j\}$ so that the limit of $\int |h(V_s^{(m'_j)})|^2 \, d\|V_s^{(m'_j)}\|$ achieves the lim inf (left-hand side of (5.17)). The inequality (5.19) shows the absolute continuity of the total variation measure $\|\delta V_s\|$, which shows the existence of $h(V_s) \in L^2(\|V_s\|)$. By integrating (5.19) in time, by Fatou's lemma and (2.11) satisfied by $V_s^{(m)}$, we also have (2.11) for $h(V_s)$ as claimed. □

Proposition 5.3. *The property of (V3) holds for V_t.*

Proof. For a.e. s, since $h(V_s) \in L^2(\|V_s\|)$, we have the monotonicity formula (see [17, 17.7],[2, 8.1]) at each point of $\operatorname{spt}\|V_s\|$ and there exist stationary tangent cones (see [17] for the definition of tangent cone and the basic properties). Note that each tangent cone consists of half-lines with possible integer multiplicities. Thus either of the following two cases holds:

(a) There is at least one tangent cone consisting of a single line with integer multiplicity.
(b) Any tangent cone is a union of half-lines emanating from the origin (whose union is not a single line).

Consider the case (a) first, and without loss of generality, assume that the point in question is the origin $x = 0$. If the density is equal to one, then the Allard regularity theorem shows that $\operatorname{spt}\|V_s\|$ can be represented by a $C^{1,1/2}$ curve with the multiplicity equal to one in a neighborhood of the origin. Since the curvature is in L^2, we immediately deduce that it is a $W^{2,2}$ curve. Thus we may assume that the density at 0 is an integer larger than one in the following. Let $V_s^{(m'_j)}$ be the subsequence in the proof of Proposition 5.2 with uniform L^2 bound of $h(V_s^{(m'_j)})$. Let $\{r_i\}$ be a sequence converging to 0 and such that the rescaling of V_s by $1/r_i$ converges to a line L with integer multiplicity. We claim that for all sufficiently large i and j, B_{r_i} does not contain any junction of $V_s^{(m'_j)}$. Here, a "junction"

[1] Note that this is not a rectifiable varifold. However, the measure of such a set of times is zero.

means a location where curves meet with non-zero angles. For a contradiction, assume otherwise. Then, there must exist subsequences (without changing the indices) such that B_{r_i} contains at least one junction of $V_s^{(m'_j)}$ and we may also choose a further subsequence so that the Hausdorff distance between L (the limit tangent line) and $\mathrm{spt}\|V_s^{(m'_j)}\|$ in B_{2r_i} is $o(r_i)$. Then consider the change of variables $\tilde{x} = x/r_i$ and the corresponding varifolds under this change of variable as $\tilde{V}_s^{(m'_j)}$. We then have

$$\int_{B_2} |h(\tilde{V}_s^{(m'_j)})|^2 \, d\|\tilde{V}_s^{(m'_j)}\| = r_i \int_{B_{2r_i}} |h(V_s^{(m'_j)})|^2 \, d\|V_s^{(m'_j)}\| \leq r_i \int_{\mathbb{R}^2} |h(V_s^{(m'_j)})|^2 \, d\|V_s^{(m'_j)}\|$$

and the right-hand side converges to 0 as $i \to \infty$. Thus, for all sufficiently large i, the $W^{2,2}$ curves consisting of $\mathrm{spt}\|V_s^{(m'_j)}\|$ are close to straight line segments in C^1 topology. On the other hand, at a junction in B_1, there must be at least two curves emanating from there having an angle of 120 degrees. One can draw a conclusion that such a set of curves cannot be contained in a small neighborhood of a line for all sufficiently large i, and we arrive at a contradiction. Thus, for some r_i and all sufficiently large j, $V_s^{(m'_j)}$ is free of junctions in B_{r_i} with small L^2 curvatures, and can be represented as a union of graphs of functions $f_1 \leq \ldots \leq f_l$ in $W^{2,2}$ close to a line. Then one can prove that the limit V_s in B_{r_i} may be expressed similarly. This gives a desired description at a point belonging to the case (a). Next, let us consider the case (b). First note that the points of (b) are discrete point. If not, then there must be an accumulation point which must be also in (b) since the first part of the argument shows the absence of a junction in a neighborhood if it is in (a). Let $x_i \to \hat{x}$ be a sequence of converging points of (b). Then, let $r_i := |x_i - \hat{x}|$ and consider the rescaling of V_s centered at $\hat{x}$ by $1/r_i$. By the definition of (b), it converges to some tangent cone which is not a line. On the other hand, in a small neighborhood of $\tilde{x} := \lim_{i\to\infty}(x_i - \hat{x})/|x_i - \hat{x}|$ (after choosing a convergent subsequence), the support of the rescaled V_s converges to a line since the limit is a sum of half lines. Since the rescaled $V_s^{(m'_j)}$ is converging and is close to a line, the first part of the argument shows that the rescaled $V_s^{(m'_j)}$ in a small neighborhood of $\tilde{x}$ is free from junctions for all sufficiently large j, thus V_s near x_i is a point with the same description in the case of (a). This is a contradiction, so the points of (b) are discrete. At such a point, there exists a neighborhood in which the point belonging to (b) does not exist except at $x = 0$, and $\mathrm{spt}\|V_s\| \setminus \{0\}$ consists of points in (a). By the description of (a) and choosing a small neighborhood, one can argue that $\mathrm{spt}\|V_s\|$ is represented as a finite number of $W^{2,2}$ curves reaching to the junction point with small C^1 variations (note that the curvature bound gives $C^{1,1/2}$ control of the curves). The angles of intersection of these curves must be either 0, 60 or 120 degrees due to the fact that $V_s^{(m'_j)}$ is converging with the same property of junctions and in the uniformly bounded $C^{1,1/2}$ norms of curves controlled. For $N = 2$, since $V_s^{(m'_j)}$ has no junction, the V_s cannot have any point of (b). This completes the proof. $\square$

Proposition 5.4. *The function u satisfies for any $T<\infty$*

$$\lim_{j\to\infty}\int_0^T\int_{\mathbb{R}^2}|u^{(m_j)}-u|^2\,d\|V_t\|dt=\lim_{j\to\infty}\int_0^T\int_{\mathbb{R}^2}|u^{(m_j)}-u|^2\,d\|V_t^{(m_j)}\|dt=0 \tag{5.20}$$

and with $c_1(u,T)$ as in (1.5), we have (3.8) satisfied for u, i.e.,

$$\int_0^T\int_{\mathbb{R}^2}|u|^2\,d\|V_t\|dt\le 4\sqrt{c_1c_2}\|V_0\|(\mathbb{R}^2)\sqrt{1+c_2c_1\exp(c_2c_1)}\exp(c_2c_1/2).$$

Proof. For (5.20), first note that Theorem 3.2 holds true equally for $\phi\in W^{1,2}(\mathbb{R}^{n+1})$ by approximation. Thus, computing as in (3.10) and using (3.3) and (5.3),

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^2}&|u^{(m_j)}-u|^2\,d\|V_t\|dt\le\sqrt{c_2}\int_0^T\mathfrak{D}(\|V_t\|)\int_{\mathbb{R}^2}|\nabla|u^{(m_j)}-u|^2|\,dxdt\\&\le 2\sqrt{c_2}\int_0^T(\|V_t\|(\mathbb{R}^2))^{\frac12}\Big(\int_{\mathbb{R}^2}|h(V_t)|^2\,d\|V_t\|\Big)^{\frac12}\int_{\mathbb{R}^2}|u^{(m_j)}-u||\nabla(u^{(m_j)}-u)|\,dxdt\\&\le 4\sqrt{c_2}(\sup_{t\in[0,T]}\|V_t\|(\mathbb{R}^2))^{\frac12}\sup_{t\in[0,T+1]}\|u(\cdot,t)\|_{L^2(\mathbb{R}^2)}\Big(\int_0^T\int_{\mathbb{R}^2}|h(V_t)|^2\,d\|V_t\|dt\Big)^{\frac12}\\&\qquad\Big(\int_0^T\int_{\mathbb{R}^2}|\nabla(u^{(m_j)}-u)|^2\,dxdt\Big)^{\frac12}.\end{aligned}\tag{5.21}$$

By (5.2), this converges to 0 as $j\to\infty$. The same computation with respect to $V_t^{(m_j)}$ holds equally, thus we obtain (5.20). Since $u^{(m_j)}$ satisfies (3.8), the same inequality holds for u due to the convergence of $\|V_t^{(m_j)}\|\to\|V_t\|$. □

Proposition 5.5. *$\{V_t\}_{t\ge0}$ satisfies the inequality (2.14).*

Proof. We fix $T>0$, $0\le t_1<t_2\le T$ and $\phi\in C_c^2(\mathbb{R}^2\times[0,T])$ with $\phi\ge0$. Once this case is treated, the C_c^1 case can be proved by approximation. Let $\{m_j\}$ be as in Proposition 5.1. Given $\varepsilon>0$, we fix j_0 so that

$$\int_0^T\int_{\mathbb{R}^2}|u^{(m_j)}-u|^2\,d\|V_t\|dt<\varepsilon \tag{5.22}$$

when $j\ge j_0$ by using (5.20). For notational simplicity, set $g=u^{(m_{j_0})}$. Now fix $s\in[0,T]$ such that (5.17) holds, and let a subsequence $\{m_j'\}_{j\in\mathbb{N}}$ be such that

$$V_s^{(m_j')}\to V_s \text{ as varifolds,} \tag{5.23a}$$

$$\begin{aligned}&\lim_{j\to\infty}\int_{\mathbb{R}^2}\phi|h(V_s^{(m_j')})|^2-\nabla\phi\cdot g^\perp-h(V_s^{(m_j')})\cdot(\nabla\phi-g\phi)\,d\|V_s^{(m_j')}\|\\&=\liminf_{j\to\infty}\int_{\mathbb{R}^2}\phi|h(V_s^{(m_j)})|^2-\nabla\phi\cdot g^\perp-h(V_s^{(m_j)})\cdot(\nabla\phi-g\phi)\,d\|V_s^{(m_j)}\|.\end{aligned}\tag{5.23b}$$

Note that the choice of subsequence may depend on s. Using (2.4), (2.2) and (5.23a), we can deduce

$$\begin{aligned}&\lim_{j\to\infty}\int_{\mathbb{R}^2}\nabla\phi\cdot g^\perp\,d\|V_s^{(m_j')}\|+\lim_{j\to\infty}\int_{\mathbb{R}^2}h(V_s^{(m_j')})\cdot(\nabla\phi-g\phi)\,d\|V_s^{(m_j')}\|\\&\quad=\int_{\mathbb{R}^2}\nabla\phi\cdot g^\perp\,d\|V_s\|-\int_{G_1(\mathbb{R}^2)}\nabla(\nabla\phi-g\phi)\cdot S\,dV_s.\end{aligned}\tag{5.24}$$

By the similar argument for (5.19), we have

$$\int_{\mathbb{R}^2}\phi|h(V_s)|^2\,d\|V_s\|\le\lim_{j\to\infty}\int_{\mathbb{R}^2}\phi|h(V_s^{(m_j')})|^2\,d\|V_s^{(m_j')}\|,$$

thus with (5.23b) and (5.24), we find

$$\begin{aligned}&\int_{\mathbb{R}^2}\phi|h(V_s)|^2-\nabla\phi\cdot g^\perp-h(V_s)\cdot(\nabla\phi-g\phi)\,d\|V_s\|\\&\le\liminf_{j\to\infty}\int_{\mathbb{R}^2}\phi|h(V_s^{(m_j)})|^2-\nabla\phi\cdot g^\perp-h(V_s^{(m_j)})\cdot(\nabla\phi-g\phi)\,d\|V_s^{(m_j)}\|.\end{aligned}\tag{5.25}$$

Since $\phi|h|^2-|\nabla\phi||g|-|h|(|\nabla\phi|+|g|\phi)\ge-|\nabla\phi||g|-\frac{|\nabla\phi|^2}{\phi}-|g|^2\phi$ which is bounded from below by a constant depending only on g and ϕ, we may use Fatou's lemma to conclude that

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\phi|h(V_s)|^2-\nabla\phi\cdot g^\perp-h(V_s)\cdot(\nabla\phi-g\phi)\,d\|V_s\|ds\\&\le\liminf_{j\to\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\phi|h(V_s^{(m_j)})|^2-\nabla\phi\cdot g^\perp-h(V_s^{(m_j)})\cdot(\nabla\phi-g\phi)\,d\|V_s^{(m_j)}\|ds.\end{aligned}\tag{5.26}$$

Since each $V^{(m_j)}$ satisfies (4.2) with $u^{(m_j)}$,

$$\begin{aligned}&\liminf_{j\to\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\phi|h(V_s^{(m_j)})|^2-\nabla\phi\cdot g^\perp-h(V_s^{(m_j)})\cdot(\nabla\phi-g\phi)\,d\|V_s^{(m_j)}\|ds\\&=\liminf_{j\to\infty}\Big(\|V_{t_1}^{(m_j)}\|\,(\phi\,(\cdot,t_1))-\|V_{t_2}^{(m_j)}\|\,(\phi\,(\cdot,t_2))+\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\partial_t\phi\,d\|V_s^{(m_j)}\|ds\\&\qquad+\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\Big(\nabla\phi\cdot(u^{(m_j)}-g)^\perp+\phi\,h(V_s^{(m_j)})\cdot(g-u^{(m_j)})\Big)\,d\|V_s^{(m_j)}\|ds\Big)\\&=\|V_{t_1}\|\,(\phi\,(\cdot,t_1))-\|V_{t_2}\|\,(\phi\,(\cdot,t_2))+\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\partial_t\phi\,d\|V_s\|ds\\&\qquad+\liminf_{j\to\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\Big(\nabla\phi\cdot(u^{(m_j)}-g)^\perp+\phi\,h(V_s^{(m_j)})\cdot(g-u^{(m_j)})\Big)\,d\|V_s^{(m_j)}\|ds\end{aligned}\tag{5.27}$$

where in the last step we have used $\|V_s^{(m_j)}\|\to\|V_s\|$ as Radon measures for all s. For the integrals in the last line of (5.27), we may use (5.20) and (5.22) to conclude that they are bounded by a multiple of ε. Similarly, we may replace g on the first line of (5.26) by u with a similar error. By letting $\varepsilon\to 0$, this gives the desired inequality (2.14) for V_t. $\square$

Remark 5.6. For the family of varifolds $\{V_t\}_{t\geq 0}$ thus far constructed, from (V1') and from the method of construction, (V1) is satisfied. We have proved so far (V3)-(V7) and (V2) will be proved in Remark 5.8.

Proposition 5.7. *There exists a subsequence of $\{m_j\}$ (we use the same notation for this subsequence) with the following property: For each $i = 1, \ldots, N$, there exists a set $S(i) \subset \mathbb{R}^2 \times [0, \infty)$ with $\|\nabla\chi_{S(i)}\|(\mathbb{R}^2 \times [0, T]) < \infty$ for all $T > 0$ such that, writing $E_i(t) := \{x \in \mathbb{R}^2 : (x, t) \in S(i)\}$,*

$$\lim_{j\to\infty} \int_{B_R} |\chi_{E_i^{(m_j)}(t)}(x) - \chi_{E_i(t)}(x)|\, dx = 0 \tag{5.28}$$

for all $R > 0$ and for all $t \geq 0$. Moreover, (E5) holds true.

Proof. We fix $i = 1, \ldots, N$ and $T > 0$ in the following. Recall the definition $E_i^{(m)}(t)$ being the "i-th phase" corresponding to the flow with $u^{(m)}$ obtained by Theorem 4.2. Let $S^{(m)}(i)$ be the set defined in (E6') corresponding to $E^{(m)}(t)$, namely,

$$S^{(m)}(i) := \{(x, t) : x \in E_i^{(m)}(t), t \geq 0\} \tag{5.29}$$

which is a relative open set in $\mathbb{R}^2 \times [0, \infty)$. For any $g = (g^1, g^2, g^3) \in C_c^1(\mathbb{R}^2 \times (0, T); \mathbb{R}^3)$, (E7') with $t_1 = 0$ and $t_2 = T$ shows

$$\begin{aligned}\int_{S^{(m)}(i)} g^1_{x_1} + g^2_{x_2} + \frac{\partial g^3}{\partial t}\, dxdt &= \int_0^T \int_{E_i^{(m)}(t)} g^1_{x_1} + g^2_{x_2} + \frac{\partial g^3}{\partial t}\, dxdt \\ &= \int_0^T \int_{\partial^* E_i^{(m)}(t)} (g^1, g^2)\cdot \nu_i^{(m)} - (h(V_t^{(m)}) + u^{(m)})\cdot \nu_i^{(m)} g^3\, d\mathcal{H}^1 dt,\end{aligned} \tag{5.30}$$

where $\nu_i^{(m)}$ is the outer unit normal to $\partial^* E_i^{(m)}(t)$. By (E5'), for all $0 \leq t \leq T$,

$$\mathcal{H}^1(\partial^* E_i^{(m)}(t)) = \|\nabla\chi_{E_i^{(m)}(t)}\|(\mathbb{R}^2) \leq \|V_t^{(m)}\|(\mathbb{R}^2) \leq \sup_{s\in[0,T]} \|V_s^{(m)}\|(\mathbb{R}^2), \tag{5.31}$$

and again by (E5'),

$$\begin{aligned}\int_0^T \int_{\partial^* E_i^{(m)}(t)} |h(V_t^{(m)})| + |u^{(m)}|\, d\mathcal{H}^1 dt &\leq \int_0^T \int_{\mathbb{R}^2} |h(V_t^{(m)})| + |u^{(m)}|\, d\|V_t^{(m)}\| dt \\ &\leq (T \sup_{s\in[0,T]} \|V_s^{(m)}\|(\mathbb{R}^2))^{\frac12} \Big(\int_0^T \int_{\mathbb{R}^2} |h(V_t^{(m)})|^2 + |u^{(m)}|^2\, d\|V_t^{(m)}\| dt\Big)^{\frac12}.\end{aligned} \tag{5.32}$$

Since the right-hand sides of (5.31) and (5.32) are both bounded uniformly due to (3.6)-(3.8), this implies that $\|\nabla\chi_{S^{(m)}(i)}\|(\mathbb{R}^2 \times (0, T))$ is uniformly bounded. By the compactness theorem for bounded variation function applied to the subsequence $\{m_j\}$ in Proposition 5.1, we have a further subsequence (denoted by the same index) and a set of finite perimeter $S(i) \subset \mathbb{R}^2 \times (0, T)$ such that $\chi_{S^{(m_j)}(i)}$ converges to $\chi_{S(i)}$ locally in $L^1(\mathbb{R}^2 \times (0, T))$. By Fubini Theorem, we also may assume that $\chi_{E_i^{(m_j)}(t)}$ converges locally to $\chi_{E_i(t)}$ in $L^1(\mathbb{R}^2)$

and for a.e. $t \in (0,T)$. In fact, since (E7') holds equally true for $\phi \in C_c(\mathbb{R}^2)$ with $|\phi| \le 1$, for $0 \le t_1 < t_2 \le T$, we have

$$\begin{aligned} \Big| \int_{E_i^{(m)}(t_2)} \phi(x)\,dx - \int_{E_i^{(m)}(t_1)} \phi(x)\,dx \Big| &\le \int_{t_1}^{t_2} \int_{\partial^* E_i^{(m)}(t)} |h(V_t^{(m)}| + |u^{(m)}|\,d\mathcal{H}^1 dt \\ &\le c_3 (t_2 - t_1)^{\frac{1}{2}} \end{aligned} \tag{5.33}$$

where c_3 can be obtained as in (5.32) and is independent of m. For any $R > 0$, we may let $A = B_R \cap E_i^{(m)}(t_2) \setminus E_i^{(m)}(t_1)$ or $B_R \cap E_i^{(m)}(t_1) \setminus E_i^{(m)}(t_2)$ and approximate χ_A by $\phi \in C_c(\mathbb{R}^2)$ to see that (after letting $\phi \to \chi_A$ and $R \to \infty$) $|E_i^{(m)}(t_1) \triangle E_i^{(m)}(t_2)| \le 2c_3(t_2 - t_1)^{\frac{1}{2}}$. From this, one can conclude that $\chi_{E_i^{(m_j)}(t)}$ converges locally in $L^1(\mathbb{R}^2)$ for all $t \in (0,T)$ (not only a.e. t), and the limit $E_i(t)$ is $\frac{1}{2}$-Hölder continuous with respect to the Lebesgue measure. If necessary, we may define the time-slice of $S(i)$ to be $E_i(t)$ for all $t \in (0,T)$, and by a diagonal argument as $T \to \infty$, this proves the claim of (5.28) and also (E5). $\square$

Remark 5.8. By (E2'), $E_1^{(m_j)}(t), \ldots, E_N^{(m_j)}(t)$ are mutually disjoint for each $t \ge 0$. By (5.28), it follows that $E_1(t), \ldots, E_N(t)$ are mutually disjoint with respect to the Lebesgue measure. Also since $\|\nabla \chi_{E_i^{(m_j)}(t)}\| \le \|V_t^{(m_j)}\|$ (by (E5')), $\lim_{j\to\infty} \|V_t^{(m_j)}\| = \|V_t\|$, and by the lower-semicontinuity property, we have $\|\nabla \chi_{E_i(t)}\| \le \|V_t\|$ for all $t \ge 0$. Then, by [14, Proposition 29.4],

$$\frac{1}{2} \sum_{i=1}^N \|\nabla \chi_{E_i(t)}\| = \mathcal{H}^1 \lfloor_{\cup_{i=1}^N \partial^* E_i(t)} \le \|V_t\|. \tag{5.34}$$

At $t = 0$, we may argue just as in [18, Proposition 6.10]. If $\mathcal{H}^1(\cup_{i=1}^N \partial E_{0,i} \setminus \partial^* E_{0,i}) = 0$, then

$$2\|V_0\| = \sum_{i=1}^N \|\nabla \chi_{E_{0,i}}\| \le \liminf_{t\to 0+} \sum_{i=1}^N \|\nabla \chi_{E_i(t)}\| \le \liminf_{t\to 0+} 2\|V_t\| \tag{5.35}$$

while $\limsup_{t\to 0+} \|V_t\| \le \|V_0\|$ follows from (2.14). Thus $\|V_0\| = \lim_{t\to 0+} \|V_t\|$ and we have (V2).

Proposition 5.9. *Properties (E6) holds true.*

Proof. Consider a sequence of $L^1_{loc}([0,\infty))$ functions

$$f^{(m_j)}(t) := 2 \int_{\partial^* E_i^{(m_j)}(t)} |h(V_t^{(m_j)})|^2 + |u^{(m_j)}|^2 \, d\|V_t^{(m_j)}\|(x) \tag{5.36}$$

whose L^1-norm on $[0,T]$ is bounded uniformly. By the compactness theorem of Radon measure, there exists a further subsequence (denoted by the same index) and a Radon measure β on $[0,\infty)$ such that $f^{(m_j)}\,dt$ converges to β as measures. With this notation, for any $\phi \in C_c^1(\mathbb{R}^2 \times [0,\infty))$,

$$\int_{t_1}^{t_2} \int_{\partial^* E_i^{(m_j)}(t)} (|h(V_t^{(m_j)})| + |u^{(m_j)}|)\phi \le \Big(\int_{t_1}^{t_2} f^{(m_j)}\,dt \Big)^{\frac{1}{2}} \Big(\int_{t_1}^{t_2} \int_{\mathbb{R}^2} \phi^2 \, d\|V_t^{(m_j)}\| \Big)^{\frac{1}{2}} \tag{5.37}$$

where we used (E5'). Then, using (4.3) and the convergence of $f^{(m_j)}dt$, $E_i^{(m_j)}(t)$ and $\|V_t^{(m_j)}\|$ to β, $E_i(t)$ and $\|V_t\|$ respectively, we obtain (2.16) and this ends the proof. $\square$

Up to this point, (E1)-(E6) are proved. Finally,

Proposition 5.10. *The properties (E7) and (E8) hold.*

Proof. This follows from the argument in Proposition 5.3 and the resulting (V3) as follows. For a.e. $t > 0$, $V_t = \sum_{i=1}^{P(t)} \mathbf{var}(\ell_i(t), 1)$ and away from a finite number of junctions, the density of $\|V_t\|$ is precisely the number of $W^{1,1}$ curves passing through that point, all tangentially. As in the proof of Proposition 5.3 of case (a), the topology of convergence of $W^{2,2}$ curves corresponding to $V^{(m_j)}$ to the limit is locally C^1, thus the properties (E8') satisfied by $V^{(m_j)}$ holds in the limit. More precisely, suppose that $N \geq 3$ and the density of $\|V_t\|$ is 1. Then there is only one curve going through this point, thus, the number of curves of $V_t^{(m_j)}$ approaching to this single curve must be also 1. Since $E_i^{(m_j)}(t)$ $(i = 1, \ldots, N)$ converges to $E_i(t)$, and (E8')(1) shows that the approaching curves are part of the reduced boundary of $E_1^{(m_j)}(t)$ $\ldots, E_N^{(m_j)}(t)$, we can deduce that the point in question must be also in the reduced boundary of $E_1(t), \ldots, E_N(t)$. This proves the case of $N \geq 3$. If $N = 2$, then, again by the C^1 convergence of curves and the property (E8')(2), one can conclude the same property for the limit depending on the parity of the density. This ends the proof of (E7). For (E8), the unit density implies that $\theta_t(x) = 1$ for $\|V_t\|$ a.e. x and (E7) then implies that $\mathcal{H}^1(\ell_i(t) \cap \ell_j(t)) = 0$ for all $i \neq j$ and $\mathcal{H}^1(\cup_{i=1}^{P(t)} \ell_i(t) \triangle \cup_{i=1}^{N} \partial^* E_i(t)) = 0$. This immediately proves (E8). $\square$

6. Final remarks

(1) One may wonder if there exists a corresponding result for $n > 1$ that is also critical, for example, $q = 2$ and $p = n + 1$ in view of (1.7), and some appropriate condition on u itself, such as $\operatorname{ess\,sup}_{t\in[0,T]} \int_{\mathbb{R}^2} |u|^2 \, dx$ in the case of $n = 1$. Our analysis is heavily based on (1.13), and a similar strategy does not seem to work.

(2) Suppose that we have a sequence of flows $\{V_t^{(m)}\}_{t\geq 0}$ and $\{u^{(m)}(\cdot, t)\}_{t\geq 0}$ each of which satisfies the properties of Theorem 2.3 and is such that $\|V_0^{(m)}\|(\mathbb{R}^2)$ and $c_1(u^{(m)}, T)$ are uniformly bounded for each $T > 0$. It is then desirable to have a good compactness property. In the case that $u^{(m)} = 0$ (or even the subcritical case treated in [20]), one can extract a subsequence (denoted by the same index) and a limit $\{V_t\}_{t\geq 0}$ such that $\lim_{m\to\infty} \|V_t^{(m)}\| = \|V_t\|$ for all $t \geq 0$ and the limit is a Brakke flow (or flow with (1.1)). In the present case, if we can further assume that $u^{(m)}$ converges to some u strongly, namely,

$$\lim_{m\to\infty} \int_0^T \int_{\mathbb{R}^2} |\nabla u^{(m)} - \nabla u|^2 \, dx dt = 0, \tag{6.1}$$

then one can extract a subsequence such that the limit flow is a solution of (1.1). The proof of this may be carried out along the line of the proof of Theorem 2.3 in fact, since the proof is precisely the compactness of the approximate flows to the solution of (1.1). On the other hand, unless we assume this strong convergence, it is difficult to prove that the limit is a solution of (1.1).

Appendix A. Proof of Theorem 4.2

For a vector field $u \in C^1_c(\mathbb{R}^{n+1} \times [0,\infty))$, we need to change the motion law from $v = h$ to $v = h + u^\perp$ in [9, 10, 19] and we point out locations to be changed. First of all, let $N \in \mathbb{N}$ be the number of phases as in these papers, and with $\mathcal{H}^n(\Gamma_0) < \infty$, we may set the weight function (see [9, Section 3.1]) $\Omega \equiv 1$ and $c_1 = 0$. Otherwise, we may proceed with the same definitions in [9], with a slight modification provided in [19] in the definition of the volume-controlled Lipschitz deformation (see [19, Definition 3.1]). The first major modification occurs at [9, Proposition 5.7]. The function f should be changed from $f(x) := x + h_\varepsilon(x,V)\Delta t$ to

$$f(x) := x + (h_\varepsilon(x,V) + u(x))\Delta t$$

(where $u(x) = u(x,t)$ with t specified later) and the estimates [9, (5.55)] and [9, (5.56)] become

$$\left|\frac{\|f_\sharp V\|(\phi) - \|V\|(\phi)}{\Delta t} - \delta(V,\phi)(h_\varepsilon(\cdot,V) + u(\cdot))\right| \le \varepsilon^{c_2-10}, \tag{A.1}$$

$$\frac{\|f_\sharp V\|(\mathbb{R}^{n+1}) - \|V\|(\mathbb{R}^{n+1})}{\Delta t} + \frac{1}{2}\int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon}\,dx \le \varepsilon^{\frac14} + \|u\|_{C^1}\|V\|(\mathbb{R}^{n+1}), \tag{A.2}$$

and under the assumption of $\|f_\sharp V\|(\mathbb{R}^{n+1}) \le M$, [9, (5.57)] becomes

$$|\delta(V,\phi)(h_\varepsilon(\cdot,V) + u(\cdot)) - \delta(f_\sharp V,\phi)(h_\varepsilon(\cdot,f_\sharp V) + u(\cdot))| \le \varepsilon^{c_2-2n-19}, \tag{A.3}$$

and the same expression for [9, (5.58)] with $\Omega = 1$. Here $c_2 := 3n + 20$ (see [9, (5.54)]). The proof of (A.1) is the same, in essence because of the boundedness of u in C^1 (to be precise, the restriction on ε also depends on $\|u\|_{C^1}$). To obtain (A.2), we may proceed as in [9, (5.66)] with

$$\begin{aligned}\delta(V,1)(h_\varepsilon + u) &= \delta V(h_\varepsilon) + \delta V(u)\\ &\le -(1-\varepsilon^{\frac14})\int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon}\,dx + \varepsilon^{\frac14} + \|u\|_{C^1}\|V\|(\mathbb{R}^{n+1}),\end{aligned} \tag{A.4}$$

where we used [9, (5.23)] and the definition of δV. Combining (A.1) with $\phi \equiv 1$, (A.4) gives (A.2). The same proof works for (A.3) and the one corresponding to [9, (5.58)] by using the boundedness of u in C^1. The content of [9, Section 6] needs only a slight change due to u, for example, in [9, Proposition 6.1], we have $\mathcal{E}_0 := \{E_{0,1},\ldots,E_{0,N}\} \in \mathcal{OP}^N_1$, and there exists a family $\mathcal{E}_{j,l} \in \mathcal{OP}^N_1$ $(l = 0,1,\ldots,j2^{p_j})$ for each $j \in \mathbb{N}$ with

$$\mathcal{E}_{j,0} = \mathcal{E}_0 \text{ for all } j \in \mathbb{N} \tag{A.5}$$

and with the notation of (see also [9, (5.54)])

$$\Delta t_j := \frac{1}{2^{p_j}} \in (2^{-1}\varepsilon_j^{c_2}, \varepsilon_j^{c_2}], \tag{A.6}$$

we have (cf. [9, (6.3)-(6.5)])

$$\|\partial\mathcal{E}_{j,l}\|(\mathbb{R}^{n+1}) \le \mathcal{H}^1(\Gamma_0)\exp\Big(l\Delta t_j\|u\|_{C^1}\Big) + \varepsilon_j^{\frac{1}{8}} l\Delta t_j, \tag{A.7}$$

$$\begin{aligned}&\frac{\|\partial\mathcal{E}_{j,l}\|(\mathbb{R}^{n+1}) - \|\partial\mathcal{E}_{j,l-1}\|(\mathbb{R}^{n+1})}{\Delta t_j} + \frac{1}{2}\int_{\mathbb{R}^{n+1}} \frac{|\Phi_{\varepsilon_j} * \delta(\partial\mathcal{E}_{j,l})|^2}{\Phi_{\varepsilon_j} * \|\partial\mathcal{E}_{j,l}\| + \varepsilon_j}\,dx \\ &- \frac{1-j^{-5}}{\Delta t_j}\Delta_j\|\partial\mathcal{E}_{j,l-1}\|(\mathbb{R}^{n+1}) \le \varepsilon_j^{\frac{1}{8}} + \|u\|_{C^1}\|\partial\mathcal{E}_{j,l-1}\|(\mathbb{R}^{n+1}),\end{aligned} \tag{A.8}$$

$$\frac{\|\partial\mathcal{E}_{j,l}\|(\phi) - \|\partial\mathcal{E}_{j,l-1}\|(\phi)}{\Delta t_j} \le \delta(\partial\mathcal{E}_{j,l},\phi)(h_{\varepsilon_j}(\cdot,\partial\mathcal{E}_{j,l}) + u(\cdot, l\Delta t_j)) + \varepsilon_j^{\frac{1}{8}} \tag{A.9}$$

for $l = 1, 2, \ldots, j2^{p_j}$ and $\phi \in \mathcal{A}_j$. The proof of these is very similar. Starting with [9, (6.6)], we put

$$M_j := \mathcal{H}^n(\Gamma_0)\exp\Big(j\|u\|_{C^1}\Big) + 1, \tag{A.10}$$

and choose ε_j similarly so that corresponding to this M_j, [9, (6.7)] holds. Then choose an admissible Lipschitz map $f_1 \in \mathbf{E}^{vc}(\mathcal{E}_{j,l}, j)$ ([19, Definition 3.1]) as in [9, (6.9)], define $\mathcal{E}^*_{j,l+1} := (f_1)_\star\mathcal{E}_{j,l}$ and define

$$f_2(x) := x + \Delta t_j(h_{\varepsilon_j}(x, \partial\mathcal{E}^*_{j,l+1}) + u(x, (l+1)\Delta t_j)). \tag{A.11}$$

Then we define

$$\mathcal{E}_{j,l+1} := (f_2)_\star\mathcal{E}^*_{j,l+1}. \tag{A.12}$$

The rest of the proof is identical and one can deduce (A.7)-(A.9). The statement of [9, Proposition 6.4] is the same in that there exists a subsequence $\{j_l\}_{l\in\mathbb{N}}$ such that (see [9, (6.18)-(6.20)])

$$\lim_{l\to\infty}\|\partial\mathcal{E}_{j_l}(t)\|(\phi) = \mu_t(\phi) \tag{A.13}$$

for all $t \ge 0$ and for all $\phi \in C_c(\mathbb{R}^{n+1})$, and for all $T < \infty$

$$\limsup_{i\to\infty}\int_0^T\Big(\int_{\mathbb{R}^{n+1}} \frac{|\Phi_{\varepsilon_{j_l}} * \delta(\partial\mathcal{E}_{j_l}(t))|^2}{\Phi_{\varepsilon_{j_l}} * \|\partial\mathcal{E}_{j_l}(t)\| + \varepsilon_{j_l}}\,dx - \frac{1}{\Delta t_{j_l}}\Delta^{vc}_{j_l}\|\partial\mathcal{E}_{j_l}(t)\|(\mathbb{R}^{n+1})\Big)dt < \infty. \tag{A.14}$$

Moreover, for a.e. $t \in [0,\infty)$, we have

$$\lim_{l\to\infty} j_l^{2(n+1)}\Delta^{vc}_{j_l}\|\partial\mathcal{E}_{j_l}(t)\|(\mathbb{R}^{n+1}) = 0. \tag{A.15}$$

The proofs of rectifiability and integrality [9, Section 7, 8] (with slight modification as in [19]) utilize only the properties (A.14) and (A.15), so that the measure μ_t is integral for a.e. $t \ge 0$. The last part for the proof of (4.2) is similar to that in [9, Section 9] in that we have the extra term coming from u in the right-hand side. Tracing the proof of [9, Theorem 9.3], it comes down (see [9, (9.10)]) to analyzing the behavior of (with $\partial\mathcal{E}_{j_l} = \partial\mathcal{E}_{j_l}(t)$)

$$\delta(\partial\mathcal{E}_{j_l},\hat\phi)(h_{\varepsilon_{j_l}} + u_{j_l}) = \delta(\partial\mathcal{E}_{j_l})(\hat\phi(h_{\varepsilon_{j_l}} + u_{j_l})) + \int_{\mathbf{G}_1(\mathbb{R}^{n+1})} S^\perp(\nabla\hat\phi)\cdot(h_{\varepsilon_{j_l}} + u_{j_l})\,d(\partial\mathcal{E}_{j_l}) \tag{A.16}$$

as $l \to \infty$ for a.e. t. Here u_{j_l} is a piece-wise constant (only in time-direction) vector field such that

$$u_{j_l}(x,t) = u(x, k\Delta t_{j_l}) \text{ if } t \in ((k-1)\Delta t_{j_l}, k\Delta t_{j_l}]. \tag{A.17}$$

Since $u \in C^1_c(\mathbb{R}^{n+1} \times [0,\infty); \mathbb{R}^{n+1})$, u_{j_l} and ∇u_{j_l} converges to u and ∇u uniformly as $l \to \infty$. Now, the term $\delta(\partial\mathcal{E}_{j_l}, \hat{\phi})(u_{j_l})$ is bounded uniformly (in terms of $\|u\|_{C^1}$ and $\|\partial\mathcal{E}_{j_l}\|(\mathbb{R}^{n+1})$). Thus, in choosing a subsequence $\{j'_l\}_{j\in\mathbb{N}} \subset \{j_l\}_{l\in\mathbb{N}}$ in [9, (9.16)] such that

$$\limsup_{l\to\infty} \delta(\partial\mathcal{E}_{j_l}(t), \hat{\phi})(h_{\varepsilon_{j_l}} + u_{j_l}) = \lim_{l\to\infty} \delta(\partial\mathcal{E}_{j'_l}(t), \hat{\phi})(h_{\varepsilon_{j'_l}} + u_{j'_l}) \tag{A.18}$$

and proceeding similarly, one can prove that ([9, (9.19)])

$$\limsup_{l\to\infty} \int_{\mathbb{R}^{n+1}} \frac{|\Phi_{\varepsilon_{j'_l}} * \delta(\partial\mathcal{E}_{j'_l}(t))|^2}{\Phi_{\varepsilon_{j'_l}} * \|\partial\mathcal{E}_{j'_l}(t)\| + \varepsilon_{j'_l}}\, dx \le i\tilde{M}(t). \tag{A.19}$$

Here $\tilde{M}(t)$ is as defined in [9, (9.18)] and is finite for a.e. t. But for this subsequence, $\{\partial\mathcal{E}_{j'_l}(t)\}_{l\in\mathbb{N}}$ converges to $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ as varifolds due to [9, Lemma 9.1(b)], and one can conclude that

$$\lim_{l\to\infty} \delta(\partial\mathcal{E}_{j'_l}(t), \hat{\phi})(u_{j'_l}) = \delta(V_t, \hat{\phi})(u). \tag{A.20}$$

The harder part on the convergence of $\delta(\partial\mathcal{E}_{j'_l}, \hat{\phi})(h_{\varepsilon_{j'_l}})$ is identical and one can prove (4.2) for a time-independent test function ϕ, and one can proceed similarly for a time-dependent test function ϕ as well. This proves the existence of measures $\{V_t\}_{t\ge 0}$ satisfying (V1'), (V3') (for now only $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+2})$ a.e.t), (V4')-(V7'). As for (V2'), the same argument in Remark 5.8 (or [18, Proposition 6.10]) shows the claim, while (V3') in the case of $n=1$ is proved in [10], where the only ingredients needed for the proof were (A.14) and (A.15).

For $\{E_i(t)\}_{t\ge 0}$ we need modifications in [9, Section 10] and [19]. For the first one, the main point is that one needs to modify Huisken's monotonicity formula [7] appropriately due to the presence of u, which can be done easily when u is bounded. In [9, (10.2)], for $t \in [0,T]$, we replace $\hat{\rho}^R_{(y,s)}(x,t)$ by

$$\hat{\rho}^R_{(y,s)}(x,t) := \eta\Big(\frac{x-y}{R}\Big) \exp\Big(-t\|u\|^2_{L^\infty}\Big) \rho_{(y,s)}(x,t). \tag{A.21}$$

Then, proceeding as in [8, Proposition 6.2] (see the computations up to [8, (6.6)]), one can conclude that for $0 \le t_1 < t_2 \le T$,

$$\|V_t\|(\hat{\rho}^R_{(y,s)}(\cdot,t))\Big|_{t=t_1}^{t_2} \le c(n)R^{-2}(t_2-t_1) \sup_{t'\in[t_1,t_2]} R^{-n}\|V_{t'}\|(B_{2R}(y)). \tag{A.22}$$

which allows the same argument in [9, Section 10]. Another minor modification is that of [9, Lemma 10.12]. With additional bounded u, one can prove:

Lemma A.1. *For some $t \in \mathbb{R}^+$, $x \in \mathbb{R}^{n+1}$ and $r > 0$, suppose $\|V_t\|(U_r(x)) = 0$. Then for $t' \in [t, t + \frac{r^2}{2n+2r\|u\|_{L^\infty}}]$, we have $\|V_{t'}\|(U_{\sqrt{r^2-(2n+2r\|u\|_{L^\infty})(t'-t)}}(x)) = 0$.*

For completeness, we give the proof.

Proof. Without loss of generality, assume $x = 0$ and $t = 0$. Define

$$\phi(x,t) := \{r^2 - |x|^2 - (2n + 2r\|u\|_{L^\infty})t\}^4 \exp(-t\|u\|^2_{L^\infty})$$

for $|x|^2 \leq r^2 - (2n + 2r\|u\|_{L^\infty})t$ and $= 0$ otherwise. Then ϕ is C^2_c in the space variables and by the Cauchy-Schwarz inequality and the first variation formula (2.4), we have

$$\int_{\mathbb{R}^{n+1}} (\nabla\phi - h\phi) \cdot (h + u^\perp)\, d\|V_t\| \leq \int_{G_n(\mathbb{R}^{n+1})} \frac{1}{2}|u|^2\phi - S \cdot \nabla^2\phi + |u||\nabla\phi|\, dV_t(\cdot, S). \quad \text{(A.23)}$$

Writing above ϕ as $R(x,t)^4 \exp(-t\|u\|^2_{L^\infty})$, one can check that (also using $S \cdot I = n$)

$$S \cdot \nabla^2\phi = (48R^{-2}|S(x)|^2 - 8nR^{-1})\phi$$

so that (A.23) can be bounded from above by

$$\int_{\mathbb{R}^{n+1}} \frac{\|u\|^2_{L^\infty}}{2}\phi + 8nR^{-1}\phi + \|u\|_{L^\infty} 8rR^{-1}\phi\, d\|V_t\|$$

while

$$\frac{\partial\phi}{\partial t} = -\|u\|^2_{L^\infty}\phi - 8(n + r\|u\|_{L^\infty})R^{-1}\phi.$$

Thus (4.2) shows that $\|V_t\|(\phi)$ is decreasing in time and we obtain the desired conclusion. □

We can proceed exactly as in [9, Section 10] and may conclude (E1')-(E6'). For (E7') and (E8'), we need to modify the proof in [19]. Since u is C^1, the proof is similar. In [19, Section 4.1], $h + u^\perp$ in place of h is a generalized velocity in the sense of L^2 flow and the proof is identical once (4.2) is established. The proof on the existence of measure-theoretic velocities ([19, Section 4.2]) can be easily modified by replacing the motion law from h_ε to $h_\varepsilon + u$ and the same proof shows [19, Proposition 4.5] (note that u_i in the statement there is not to be confused with u in this paper). The proof for the identification of the measure-theoretic velocity in [19, Section 4.3] goes verbatim and proves (4.3). The rest of [19], Section 5 and 6, goes without change and in particular, (E8') holds true for this modified version.

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