OPTIMAL DOMAINS FOR THE CHEEGER INEQUALITY

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Abstract. In this paper we consider the scale-invariant shape functional

$$\mathcal{F}_{p,q}(\Omega)=\frac{\lambda_p^{1/p}(\Omega)}{\lambda_q^{1/q}(\Omega)},$$

where $1\le q<p\le+\infty$ and $\lambda_p(\Omega)$ (respectively $\lambda_q(\Omega)$) is the first eigenvalue of the p-Laplacian $-\Delta_p$ (respectively $-\Delta_q$) with Dirichlet boundary condition on $\partial\Omega$. We study both the maximization and the minimization problems for $\mathcal{F}_{p,q}$, and show the existence of optimal domains in $\mathbb{R}^d$, along with some of their qualitative properties. Surprisingly, the case of a bounded box D constraint

$$\max\Big\{\lambda_q(\Omega)\ :\ \Omega\subset D,\ \lambda_p(\Omega)=1\Big\},$$

leads to a problem of a different nature, for which the existence of a solution is shown through the analysis optimal capacitary measures. In the last section we list some interesting questions that, in our opinion, deserve to be investigated.

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1. Introduction

The starting point for this study is the celebrated Cheeger inequality [8], which establishes a fundamental link between spectral and geometric properties of domains. Specifically, for any open subset $\Omega\subset\mathbb{R}^d$, the inequality asserts that

$$\lambda(\Omega)\ge\frac{1}{4}h^2(\Omega) \tag{1.1}$$

where $\lambda(\Omega)$ denotes the first eigenvalue of the Dirichlet Laplacian $-\Delta$ on Ω, and $h(\Omega)$ is the Cheeger constant[1] of the domain.

The first Dirichlet eigenvalue $\lambda(\Omega)$ is defined by the Rayleigh quotient as

$$\begin{aligned}\lambda(\Omega)&=\inf\left\{\frac{\int|\nabla u|^2\,dx}{\int|u|^2\,dx}\ :\ u\in C^1_c(\Omega)\setminus\{0\}\right\}\\&=\min\left\{\frac{\int|\nabla u|^2\,dx}{\int|u|^2\,dx}\ :\ u\in H^1_0(\Omega)\setminus\{0\}\right\},\end{aligned}$$

where all integrals are taken over the entire space $\mathbb{R}^d$, with the convention that functions in $H^1_0(\Omega)$ are extended by zero outside the domain Ω.

[1] This constant should not be confused with the *Cheeger isoperimetric constant* which refers to a surface of minimal area which divides a set in two disjoint sets.

The Cheeger constant $h(\Omega)$, a geometric quantity capturing isoperimetric properties of the domain, is defined as

$$h(\Omega)=\inf\left\{\frac{P(E)}{|E|}\ :\ E\Subset\Omega\right\},$$

where $P(E)$ denotes the perimeter of the measurable set E, and $|E|$ its Lebesgue measure. Here it is implicitly understood that $P(E)=+\infty$ when the set E does not have finite perimeter. When the domain Ω is bounded and possesses a Lipschitz boundary, the Cheeger constant $h(\Omega)$ can be characterized as

$$h(\Omega)=\min\left\{\frac{P(E)}{|E|}\ :\ E\subset\Omega\right\}.$$

There exist other equivalent formulations of the Cheeger constant $h(\Omega)$, each providing a distinct perspective on its variational and geometric character. A classical expression, mirroring the Rayleigh quotient but in the L^1 setting, is given by

$$\begin{aligned}h(\Omega)&=\inf\left\{\frac{\int|\nabla u|\,dx}{\int|u|\,dx}\ :\ u\in C^1_c(\Omega)\setminus\{0\}\right\}\\&=\inf\left\{\frac{\int|\nabla u|\,dx}{\int|u|\,dx}\ :\ u\in W^{1,1}_0(\Omega)\setminus\{0\}\right\},\end{aligned}$$

where the infimum is taken over nontrivial functions in either the space of compactly supported smooth functions or in the Sobolev space $W^{1,1}_0(\Omega)$. Here and throughout the paper, for every $p\ge1$ and Ω open subset of $\mathbb{R}^d$ we denote by $W^{1,p}_0(\Omega)$ the usual Sobolev space defined as the closure of $C^\infty_c(\Omega)$ with respect to the norm $\|u\|_{W^{1,p}(\Omega)}=\|\nabla u\|_{L^p(\Omega)}+\|u\|_{L^p(\Omega)}$.

The inequality (1.1) highlights a deep connection between the geometry of a domain and the spectral behavior of the Laplacian defined on it, forming the foundation for much of the analysis that follows. Interestingly, the Cheeger constant also emerges as the limit case of the first eigenvalue of the p-Laplacian as $p\to1$. For $1\le p<+\infty$ the first Dirichlet eigenvalue of the p-Laplacian is defined by

$$\lambda_p(\Omega)=\inf\left\{\frac{\int|\nabla u|^p\,dx}{\int|u|^p\,dx}\ :\ u\in W^{1,p}_0(\Omega)\setminus\{0\}\right\},\tag{1.2}$$

so that, in this framework, the Cheeger constant satisfies $h(\Omega)=\lambda_1(\Omega)$, emphasizing its role as the $p=1$ analog of the principal eigenvalue. For a comprehensive account of the Cheeger constant and related optimization problems, we refer the reader to [15], [20], [23], [24], and [25].

For $p=+\infty$, the natural definition of the first eigenvalue λ_∞ is (see [18])

$$\lambda_\infty(\Omega)=\rho(\Omega)^{-1},$$

where $\rho(\Omega)$ denotes the so-called *inradius* of the domain Ω, defined as the largest radius of a ball that can be inscribed in Ω. Equivalently, $\rho(\Omega)$ is the maximum of the distance function to the boundary $\partial\Omega$, thereby linking spectral asymptotics and intrinsic geometry in a particularly elegant way.

If Ω is bounded, it was proved in [19, Corollary 6] that

$$\lim_{p\to 1^+} \lambda_p(\Omega) = h(\Omega)$$

and in [18, Lemma 1.5] that

$$\lim_{p\to +\infty} \lambda_p^{1/p}(\Omega) = \lambda_\infty(\Omega) = \rho(\Omega)^{-1}.$$

These two properties hold as well for unbounded sets Ω. We discuss this point in Section 2.

A broader perspective on Cheeger-type inequalities can be obtained by introducing the scale-invariant shape functional

$$\mathcal{F}_{p,q}(\Omega) = \frac{\lambda_p^{1/p}(\Omega)}{\lambda_q^{1/q}(\Omega)}, \tag{1.3}$$

which involves the first eigenvalues of the Dirichlet p- and q-Laplacians, with $1 \le q < p < +\infty$. By a slight abuse of notation, for $p = +\infty$ we interpret $\lambda_p^{1/p}(\Omega)$ to be its limit for $p \to +\infty$, so that it equals $\lambda_\infty(\Omega)$. With this convention, we define the shape functional $\mathcal{F}_{p,q}$ for every $1 \le q < p \le +\infty$. Within this framework, the classical Cheeger inequality (1.1) appears as a special instance of a more general principle. Specifically, for any $1 \le q < p \le +\infty$ and for every bounded open subset Ω of $\mathbb{R}^d$, the following inequality holds:

$$\mathcal{F}_{p,q}(\Omega) \ge \frac{q}{p} \qquad \text{for every } 1 \le q < p \le +\infty. \tag{1.4}$$

This inequality can equivalently be reformulated in terms of a monotonicity property, as follows:

the map $[1,+\infty) \ni p \mapsto p\lambda_p^{1/p}(\Omega) \in [0,+\infty)$ *is nondecreasing.*

The proof of this family of inequalities is relatively elementary and rests on a careful application of Hölder's inequality; a detailed exposition can be found in [4]. While the constant q/p in (1.4) is not optimal in general, it is known (see Theorem 1.2 of [15]) to be asymptotically sharp in high dimensions, that is, as the ambient space dimension $d \to +\infty$ (see [4] for precise asymptotic estimates).

This variational comparison not only unifies classical spectral bounds under a single conceptual framework, but also provides insight into the interplay between the domain's geometry and the behavior of eigenvalues across the full range of p-Laplacian operators. An analysis of the functional $\mathcal{F}_{p,q}$ on manifolds can be found in [10].

It is worth noting that when the domain Ω has finite Lebesgue measure, the infimum in the variational characterization of $\lambda_p(\Omega)$ given in (1.2) is indeed attained. Moreover, if $p \in (1,+\infty)$ there exists a generalized eigenfunction $u_p \in W_0^{1,p}(\Omega)$ which attains the minimum and satisfies the associated nonlinear eigenvalue PDE

$$\begin{cases} -\Delta_p u_p = \lambda_p(\Omega)|u_p|^{p-2}\, u_p, \\ u_p \in W_0^{1,p}(\Omega), \quad \|u_p\|_{L^p(\Omega)} = 1. \end{cases}$$

where $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ denotes the p-Laplacian operator.

If Ω has infinite measure, the infimum in (1.2) may not be attained so that an eigenfunction may not exist. The shape functional $\mathcal{F}_{p,q}(\Omega)$ introduced in (1.3) remains well-defined for all open sets $\Omega \subset \mathbb{R}^d$ (bounded or not) such that $\lambda_q(\Omega) \neq 0$. This generality is particularly useful when studying shape optimization problems related to the function $\mathcal{F}_{p,q}$, as we will do in the following sections.

Our primary interest lies in studying the extremal behavior - both minimization and maximization - of the shape functional $\mathcal{F}_{p,q}$. To this end, we define the associated optimal bounds:

$$\begin{aligned} m(p,q) &= \inf\left\{\mathcal{F}_{p,q}(\Omega)\ :\ \Omega \subset \mathbb{R}^d,\ \Omega \text{ open},\ \lambda_q(\Omega) > 0\right\},\\ M(p,q) &= \sup\left\{\mathcal{F}_{p,q}(\Omega)\ :\ \Omega \subset \mathbb{R}^d,\ \Omega \text{ open},\ \lambda_q(\Omega) > 0\right\}. \end{aligned} \tag{1.5}$$

A key feature of the shape functional $\mathcal{F}_{p,q}$ is its invariance under scaling transformations. Indeed, using the well-known homogeneity of the p-Laplacian eigenvalue under dilations, e.g. for $p \in [1, +\infty)$

$$\lambda_p(t\Omega) = t^{-p}\lambda_p(\Omega) \qquad \text{for every } t > 0,$$

and for $p = +\infty$, $\lambda_\infty(t\Omega) = t^{-1}\lambda_\infty(\Omega)$, it follows immediately that the functional $\mathcal{F}_{p,q}$ satisfies

$$\mathcal{F}_{p,q}(t\Omega) = \mathcal{F}_{p,q}(\Omega) \qquad \text{for every } t > 0. \tag{1.6}$$

This scaling invariance allows one to reduce the study of $\mathcal{F}_{p,q}$ to families of domains normalized with respect to either λ_p or λ_q. More precisely, we may equivalently write the extremal values as:

$$\begin{aligned} m(p,q) &= \Big(\inf\left\{\lambda_p(\Omega)\ :\ \lambda_q(\Omega) = 1\right\}\Big)^{1/p} = \Big(\sup\left\{\lambda_q(\Omega)\ :\ \lambda_p(\Omega) = 1\right\}\Big)^{-1/q}\\ M(p,q) &= \Big(\sup\left\{\lambda_p(\Omega)\ :\ \lambda_q(\Omega) = 1\right\}\Big)^{1/p} = \Big(\inf\left\{\lambda_q(\Omega)\ :\ \lambda_p(\Omega) = 1\right\}\Big)^{-1/q}. \end{aligned}$$

We now state the existence result of optimal domains for the functional $\mathcal{F}_{p,q}$.

Theorem 1.1. *Let $1 \leq q < p \leq +\infty$. Then there exist two open subsets of $\mathbb{R}^d$, denoted by $\Omega^m_{p,q}$ and $\Omega^M_{p,q}$, such that the infimum and supremum in (1.5) are attained, in the sense that:*

$$\mathcal{F}_{p,q}(\Omega^m_{p,q}) = m(p,q) \qquad \text{and} \qquad \mathcal{F}_{p,q}(\Omega^M_{p,q}) = \begin{cases} M(p,q) & \text{if } q > d\\ +\infty & \text{if } q \leq d. \end{cases}$$

In addition, $\Omega^m_{p,q}$ and $\Omega^M_{p,q}$ can be taken of class C^∞. Moreover, the quantity $M(p,q)$ is finite if and only if $q > d$.

Remark 1.2. It is worth emphasizing that the optimal domains $\Omega^m_{p,q}$ and $\Omega^M_{p,q}$ obtained in Theorem 1.1 are unbounded. As a consequence, one cannot always associate a well-defined eigenfunction to $\lambda_p(\Omega)$ or $\lambda_q(\Omega)$, since the corresponding eigenvalues may not be attained by any function in the associated Sobolev space.

A particular situation arises when $p = +\infty$, in which the functional $\mathcal{F}_{\infty,q}$ takes the explicit form

$$\mathcal{F}_{\infty,q}(\Omega) = \frac{1}{\rho(\Omega)\,\lambda_q^{1/q}(\Omega)}.$$

Thanks to the scaling property (1.6) we may, without loss of generality, impose the normalization $\rho(\Omega) = 1$, so the optimization of $\mathcal{F}_{\infty,q}(\Omega)$ reduces to the optimization of the eigenvalue $\lambda_q(\Omega)$ among all domains with fixed inradius. In particular, the minimization of $\mathcal{F}_{\infty,q}(\Omega)$ becomes a maximization problem for λ_q and the answer is straightforward: the solution is the ball. This raises the question of the possible existence of bounded minimizers of $\mathcal{F}_{p,q}(\Omega)$, at least when p is large. On the other hand, for the maximization problem of $\mathcal{F}_{p,q}(\Omega)$ we do not expect bounded optimizers. In [5], an example of a functional of the same nature, involving the ∞-torsional rigidity rather than λ_q under fixed inradius constraint, has *only* unbounded optimizers.

In several spectral optimization problems, the question of whether optimal shapes exist among all admissible domains in the Euclidean space $\mathbb{R}^d$ is both subtle and technically challenging (see for example [6], [22]). By contrast, in the present setting, the existence of an optimal domain in the entire space $\mathbb{R}^d$ can be established through a relatively straightforward argument. However, the situation becomes considerably more intricate when additional space constraints are imposed, specifically, when the admissible domains are required to lie within a fixed bounded set, that is $\Omega \subset D$ with D bounded. In such cases, a much more refined analysis is necessary, involving the use of capacitary measures, as will be discussed in detail in Section 5.

In addition, the non-uniqueness of optimal sets should be noted. Due to the definition of the functional $\mathcal{F}_{p,q}$, optimal domains can be modified locally - e.g., by removing or adding compact subsets - without affecting the value of the functional. Thus, the minimizers and maximizers should be interpreted as equivalence classes of domains with the same spectral properties.

Among the possible optimal domains $\Omega^m_{p,q}$ and $\Omega^M_{p,q}$ achieving the values $m(p,q)$ and $M(p,q)$, a particularly interesting class arises when $p > q > d$. In this regime, we are able to construct optimal domains that are complements of discrete sets in $\mathbb{R}^d$. This phenomenon is closely related to the theory of Sobolev capacities: when $p > q > d$, single points in $\mathbb{R}^d$ have positive $W^{1,q}$-capacity, and therefore their removal can influence the spectral quantities under consideration.

We formalize this observation in the following result.

Theorem 1.3. *Let $d < q < p \leq +\infty$. Then there exist open subsets $\Omega^m_{p,q}$ and $\Omega^M_{p,q}$ of $\mathbb{R}^d$ such that*

$$\mathcal{F}_{p,q}(\Omega^m_{p,q}) = m(p,q) \qquad \text{and} \qquad \mathcal{F}_{p,q}(\Omega^M_{p,q}) = M(p,q)$$

and moreover, the complements $\mathbb{R}^d \setminus \Omega^m_{p,q}$ and $\mathbb{R}^d \setminus \Omega^M_{p,q}$ are discrete sets with no accumulation points.

If $p = +\infty$, the minimizer of $\mathcal{F}_{p,q}$ is not of particular interest, since it can be build from the ball (as we will describe in the proof). In contrast, the maximization problem for $\mathcal{F}_{p,q}$, which amounts to minimizing λ_q under the constraint that $\rho(\Omega) = 1$ is much richer and more intricate. Indeed, in the two-dimensional case ($d = 2$), if $2 < q < +\infty$, we conjecture that an optimal configuration is given by the domain $\Omega^* = \mathbb{R}^2 \setminus X$, where X is a discrete set of points forming the centers of a regular hexagonal tessellation of the plane. This conjecture is supported by numerical simulations and geometric considerations, which reduce the question to an optimization problem involving triangles inscribed in a unit circle. In this simplified setting, the hexagonal arrangement emerges naturally as the most efficient configuration in minimizing λ_q under the inradius constraint (see [5]).

Remark 1.4. If instead of the full space $\mathbb{R}^d$, we require that Ω is contained in a fixed unbounded open subset $D \subset \mathbb{R}^d$, then Theorems 1.1 and 1.3 continue to hold, with the same proof, as soon as the set D satisfies the following property:

for every bounded set $A \subset D$ and for every $t, R > 0$, there exists $y \in \mathbb{R}^d$ such that $y + tA \subset D \setminus B_R$.
For instance, the above property is satisfied when D is a cone.

When an additional geometric constraint is imposed on the admissible domains - namely, requiring that Ω be contained in a fixed bounded box $D \subset \mathbb{R}^d$ - the analysis of the optimization problems becomes significantly more subtle. In this constrained setting, the existence of optimal domains requires the theory of Sobolev capacities and work within the framework of p-quasi open sets. We address this issue in detail in Section 5, where we establish the following existence result.

Theorem 1.5. *Let $1 < q < p < +\infty$. Let $D \subset \mathbb{R}^d$ be a bounded open set.*

- *Assume $\lambda_q(D) \le 1$. Then there exists an optimal p-quasi open set solving the problem*
$$\min\left\{\lambda_p(\Omega)\ :\ \Omega \subset D,\ \lambda_q(\Omega) = 1\right\}; \tag{1.7}$$
- *Assume $\lambda_p(D) \le 1$.Then there exists an optimal p-quasi open set solving*
$$\max\left\{\lambda_q(\Omega)\ :\ \Omega \subset D,\ \lambda_p(\Omega) = 1\right\}. \tag{1.8}$$

Remark 1.6. It is important to observe that, in the bounded setting, the scale-invariant formulation of the functional $\mathcal{F}_{p,q}$ no longer yields an equivalent problem. Indeed, the constraint $\Omega \subset D$ prohibits the use of homotheties, and therefore the scale invariance that plays a crucial role in the unbounded case, is lost. As a consequence, the constrained problem
$$\inf\left\{\mathcal{F}_{p,q}(\Omega)\ :\ \Omega \subset D\right\}$$
is not equivalent to either of the problems (1.7) or (1.8), and must be addressed with different techniques, yielding a distinct perspective on the problem.

Remark 1.7. The interest of Theorem 1.5 comes from the fact that we do not know whether a *bounded* minimizer of $\mathcal{F}_{p,q}(\Omega)$ in Theorem 1.1 exists or not. It is then natural to try to force the boundedness (what Theorem 1.5 does) and then to try to prove that the diameter of the optimal set is controlled independently on the box D, as soon as the box is sufficiently large. For now, such control is not available. The unboundedness of the optimal set from Theorem 1.1 does not appear to be a consequence of a technical failure of our approach. We believe that this is a genuine feature typical to this specific type of scale-invariant functionals.

The structure of the paper is as follows. In Section 2, we collect several preliminary results and technical tools that will be employed throughout the analysis. Section 3 is devoted to proving the existence of optimal domains for the unconstrained functional $\mathcal{F}_{p,q}$ and Section 4 to the proof of Theorem 1.3. In Section 5, we focus on the constrained case where the admissible domains are required to lie within a fixed bounded region D. Finally, in Section 6, we conclude by presenting a selection of open problems and directions for future investigation.

2. Preliminary results

In this section, we introduce the principal concepts that will be used in the rest of the paper. Alongside these definitions, we establish a few preliminary results that will be crucial in the subsequent analysis. For a general overview of shape optimization problems and further related details, we refer the reader to the monographs [7] and [17].

Capacity and γ_p-convergence. Let $p \in (1, +\infty)$. The p-capacity of a set $E \subset \mathbb{R}^d$, denoted by $\mathrm{cap}_p(E)$, is defined by

$$\mathrm{cap}_p(E) = \inf\left\{ \int \left(|\nabla u|^p + |u|^p\right) dx \ : \ u \in W^{1,p}(\mathbb{R}^d),\ u = 1 \text{ in a neighborhood of } E \right\}.$$

We refer the reader to [1] for an extended study of the Sobolev spaces, to [16, Chapter 2] for nonlinear capacities and to [7, Section 4.1, 48] for applications in shape optimization. If $p > d$ the capacity of a singleton is strictly positive.

When $1 < p < +\infty$, given a Sobolev function $u \in W^{1,p}(\mathbb{R}^d)$, the limit

$$\lim_{\varepsilon \to 0} \frac{1}{|B_\varepsilon(x)|} \int_{B(x,\varepsilon)} u(y)\, dy := \tilde{u}(x)$$

is defined at all points except possibly a set of p-capacity zero. The function $\tilde{u}$ is a *quasi-continuous* representative of u. Above, and throughout the paper we denote $B_r(x)$ the ball of $\mathbb{R}^d$ centered at x with radius R. If the center is the origin, we simply denote B_r. In the following, with an abuse of notation, we often identify the functions u and $\tilde{u}$, since the sets of p-capacity zero do not influence in any way the quantities involved in our framework. If $d < p < +\infty$ then every $u \in W^{1,p}(\mathbb{R}^d)$ is continuous.

Definition 2.1. *Let $1 < p < +\infty$. A set $\Omega \subset \mathbb{R}^d$ is said to be p-quasi open if there exists a function $u \in W^{1,p}(\mathbb{R}^d)$ such that $\Omega = \{\tilde{u} > 0\}$ up to sets of p-capacity zero.*

Thanks to the Sobolev embedding theorem, when $p > d$, any p-quasi open set is necessarily open. Consequently, in this regime, the notions of open and p-quasi open sets coincide.

If $1 < p < +\infty$ and if Ω is p-quasi open, one can define the Sobolev space $W_0^{1,p}(\Omega)$ as the collection of all functions $u \in W^{1,p}(\mathbb{R}^d)$ such that $\tilde{u} = 0$ *quasi-everywhere* in $\mathbb{R}^d \setminus \Omega$. When Ω is an open set this definition is equivalent to the usual one (see Chapter 4 of [7]). In particular, the first Dirichlet eigenvalue $\lambda_p(\Omega)$ is well-defined for every p-quasi open set Ω, and can be characterized variationally by the expression given in (1.2).

An essential ingredient in the analysis of existence results for optimal domains associated with the shape functional $\mathcal{F}_{p,q}$ is the notion of p-capacitary measures and the related concept of γ_p-convergence. These tools allow us to extend the needed variational principles to a broader class of objects beyond classical open sets.

Definition 2.2. *Let $1 < p \leq d$. A nonnegative Borel measure μ, possibly taking the value $+\infty$, is said to be of p-capacitary type if*

$$\mu(E) = 0 \qquad \textit{for every Borel set } E \textit{ with } p\textit{-capacity zero.}$$

Measures of p-capacitary type provide a natural generalization of p-quasi open sets. Indeed, if $\Omega \subset \mathbb{R}^d$ is a p-quasi open set, we may associate to it the Borel measure

$$\infty_{\mathbb{R}^d \setminus \Omega}(E) = \begin{cases} 0 & \text{if } \mathrm{cap}_p(E \setminus \Omega) = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

which is easily seen to be of p-capacitary type.

More generally, given a p-capacitary measure μ, one can define the first eigenvalue $\lambda_p(\mu)$ via the relaxed Rayleigh quotient:

$$\lambda_p(\mu) = \inf\left\{\frac{\int |\nabla u|^p\,dx + \int |u|^p\,d\mu}{\int |u|^p\,dx} \ :\ u \in W^{1,p}(\mathbb{R}^d)\setminus\{0\}\right\}.$$

In particular, when $\mu = \infty_{\mathbb{R}^d\setminus\Omega}$ for a p-quasi open set Ω, we recover the usual Dirichlet eigenvalue:

$$\lambda_p(\infty_{\mathbb{R}^d\setminus\Omega}) = \lambda_p(\Omega).$$

Indeed, if $\Omega \subset \mathbb{R}^d$ is a bounded p-quasi open set, then for every $u \in L^p(\mathbb{R}^d)$, we have the equivalence

$$\int_{\mathbb{R}^d} |\nabla u|^p\,dx + \int_{\mathbb{R}^d} |u|^p\,d\mu < +\infty \quad\Longleftrightarrow\quad u \in W_0^{1,p}(\Omega),$$

for the measure $\mu = \infty_{\mathbb{R}^d\setminus\Omega}$ (see [14]). On the one hand, if $\int_{\mathbb{R}^d} |\nabla u|^p\,dx < +\infty$ then $u \in W^{1,p}(\mathbb{R}^d)$ and on the other, if $\int_{\mathbb{R}^d} |u|^p\,d\mu < +\infty$, then $u = 0$ p-quasi everywhere on $\mathbb{R}^d\setminus\Omega$, for a quasi-continuous representative. This implies that $u \in W_0^{1,p}(\Omega)$.

An immediate consequence of the variational formulation is the following monotonicity property:

$$\lambda_p(\mu_1) \le \lambda_p(\mu_2) \qquad \text{whenever } \mu_1 \le \mu_2 \text{ (as measures).} \tag{2.1}$$

When dealing with shape optimization problems under domain constraints, it is natural to restrict attention to subsets of a fixed bounded open set $D \subset \mathbb{R}^d$. In this context, the class of admissible domains is given by

$$\mathcal{A}(D) = \big\{\Omega \subset D \ :\ \Omega \ p\text{-quasi open}\big\}.$$

The dependence of $\lambda_p(\Omega)$ with respect to arbitrary variations of the set Ω is, in general, not merely a matter of geometric convergence and is fully understood through the notion of γ_p-convergence.

Definition 2.3. *Let (μ_n) be a sequence of p-capacitary measures supported in D with $1 < p < +\infty$ and let μ be another p-capacitary measure on D. We say that μ_n γ_p-converges to μ, and write $\mu_n \xrightarrow{\gamma_p} \mu$, if the corresponding sequence of functionals*

$$F_n(u) = \int_D |\nabla u|^p\,dx + \int_D |u|^p\,d\mu_n \qquad u \in W_0^{1,p}(D)$$

Γ-converges in $L^p(D)$ to the functional

$$F(u) = \int_D |\nabla u|^p\,dx + \int_D |u|^p\,d\mu \qquad u \in W_0^{1,p}(D).$$

For a comprehensive treatment of Γ-convergence and its applications, we refer the reader to the classical monograph [11]. A detailed discussion of shape optimization problems, the theory of p-quasi open sets, and the framework of p-capacitary measures can be found in [14, 7] and the references therein. A relevant purely geometric convergence of domains related to the γ_p-convergence relies on the Hausdorff distance: if $A_1, A_2 \subset \mathbb{R}^d$, the Hausdorff distance between A_1, A_2 equals

$$\inf\{\varepsilon > 0 : A_1 \subseteq A_2 + B_\varepsilon, A_2 \subseteq A_1 + B_\varepsilon\}.$$

In the present setting, the key properties of the γ_p-convergence that will be relevant to our analysis are the following. We summarize below some fundamental properties of γ_p-convergence which are essential in the study of variational problems involving p-capacitary measures.

- **Simplification in the supercritical case.** When $p > d$, the notion of γ_p-convergence for a sequence (Ω_n) of open sets reduces to the Hausdorff convergence of the corresponding complements (closed sets) $\overline{D} \setminus \Omega_n$ in the ambient domain. In this regime, the distinction between open and p-quasi open sets disappears due to the Sobolev embedding. We refer the reader to [7, Section 4.8] for a full description of the γ_p-convergence.
- **Compactness.** The space of p-capacitary measures supported in D is compact with respect to the γ_p-convergence. More precisely, any sequence (μ_n) of p-capacitary measures admits a γ_p-convergent subsequence, whose limit is again a p-capacitary measure.
- **Metrizability and PDE characterization.** The space of p-capacitary measures supported in D endowed with the γ_p-convergence is metrizable. Moreover, the convergence $\mu_n \to \mu$ in the γ_p-sense is equivalent to the strong convergence in $L^p(D)$ of the corresponding solutions $w(\mu_n) \to w(\mu)$, where $w(\mu_n)$ is called the p-torsion function and solves the nonlinear PDE:
$$\begin{cases} -\Delta_p w + \mu_n w^{p-1} = 1 & \text{in } D \\ w \in W_0^{1,p}(D), \quad w \geq 0. \end{cases} \tag{2.2}$$
This correspondence allows one to define a natural metric, the so-called γ_p-distance:
$$d_{\gamma_p}(\mu, \nu) = \|w(\mu) - w(\nu)\|_{L^p}.$$
- **Continuity of the eigenvalue.** The first eigenvalue $\lambda_p(\mu)$ depends continuously on the capacitary measure μ with respect to the γ_p-convergence.
- **Characterization of γ_p-limits.** When $p \leq d$, the class of all γ_p-limits of sequences of (quasi) open sets coincides precisely with the full space of p-capacitary measures. On the other hand, for $p > d$, the situation is more rigid: the class of open sets whose complements converge in the Hausdorff sense is already compact, and hence captures all possible γ_p-limits.

  The first example illustrating this general behavior, in the subcritical regime $p \leq d$ is the construction in [9], where a sequence (Ω_n) was shown to γ_2-converge to the Lebesgue measure. The full characterization of γ_2-limits in terms of capacitary measures was established in [13], and the general result for all $p \leq d$ was later obtained in [12].

Definition 2.4. *Given a p-capacitary measure μ on D, we define the associated set Ω_μ as the region where μ is finite. More precisely,*
$$\Omega_\mu = \{w(\mu) > 0\},$$
where $w(\mu) \in W_0^{1,p}(D)$ is defined by (2.2).

The set Ω_μ is p-quasi open by construction and plays a central role in the analysis of limit configurations arising in variational problems involving p-capacitary measures.

A fundamental result, needed to prove the existence of an optimal domain Ω_{opt} for the shape functional $\mathcal{F}_{p,q}$ introduced in (1.3), is the following compatibility condition between the γ_p- and γ_q-limits of a sequence of domains.

Theorem 2.5. *Let $1 < q < p < +\infty$ and let $(\Omega_n) \subset \mathcal{A}(D)$ be a sequence of admissible domains such that*

$$\Omega_n \overset{\gamma_p}{\to} \mu \qquad \text{and} \qquad \Omega_n \overset{\gamma_q}{\to} \nu.$$

for some p- and q-capacitary measures μ and ν, respectively. Then, the measure ν vanishes on the p-quasi open set Ω_μ where μ is finite.

Proof. Let w be the solution of (2.2) associated with the measure μ, and let (w_n) be an optimal sequence for w with respect to the Γ-convergence. That is, $w_n \in W_0^{1,p}(\Omega_n)$ are the p-torsion functions on Ω_n for which we have

$$\lim_n \int |\nabla w_n|^p \, dx = \int |\nabla w|^p \, dx + \int w^p \, d\mu.$$

Now fix a parameter $\alpha > 0$ and define the auxiliary function

$$\varphi(x) = \big(w(x) - \alpha\big)^+,$$

where, as usual, x^+ denotes the positive part of x. In addition, we introduce the Lipschitz continuous function $H : \mathbb{R}^+ \to [0, 1]$ given by

$$H(s) = (s/\alpha) \wedge 1.$$

We then consider the sequence (u_n) defined by

$$u_n = \varphi \, H(w_n).$$

Note that $\varphi \in W_0^{1,p}(D)$ and $H(w_n) \in W_0^{1,p}(\Omega_n)$. The latter assertion follows from the fact that $w_n \in W_0^{1,p}(\Omega_n)$ is nonnegative and H is Lipschitz, bounded from above and satisfies $H(0) = 0$. Moreover, both functions are bounded so it follows that $u_n \in W_0^{1,p}(\Omega_n)$ for each n. Indeed, this follows from the following inequalities

$$0 \le \varphi \, H(w_n) \le w \frac{1}{\alpha} w_n \le \frac{1}{\alpha} \|w_D\|_\infty^2$$

and

$$|\nabla \varphi \, H(w_n)| \le |\nabla \varphi| + \varphi \frac{1}{\alpha} |\nabla w_n| \le |\nabla \varphi| + \|w_D\|_\infty \frac{1}{\alpha} |\nabla w_n|.$$

Moreover, by the strong convergence $w_n \to w$ in L^p and the continuity of H, it is easy to see that $u_n \to u = \varphi \, H(w)$ strongly in L^q, since the sequence is uniformly bounded and converges a.e. Applying the Γ-liminf inequality to the sequence (u_n) we obtain

$$\liminf_n \int |\nabla u_n|^q \, dx \ge \int |\nabla u|^q \, dx + \int u^q \, d\nu.$$

Recalling that $u = \varphi H(w) = \varphi = (w - \alpha)^+$ yields

$$\int |\nabla u|^q \, dx + \int u^q \, d\nu = \int |\nabla \varphi|^q \, dx + \int \varphi^q \, d\nu,$$

so that

$$\int \big((w-\alpha)^+\big)^q \, d\nu \le -\int |\nabla\varphi|^q \, dx + \liminf_n \int |\nabla u_n|^q \, dx$$
$$\le \limsup_n \int \Big[\big|H(w_n)\nabla\varphi + \varphi H'(w_n)\nabla w_n\big|^q - |\nabla\varphi|^q\Big] \, dx.$$

In order to estimate the right-hand side, we use the standard inequality

$$|b|^q - |a|^q \le C|b-a|\big(|a|^{q-1} + |b|^{q-1}\big),$$

which holds for some constant $C > 0$ depending only on q (below, the constant may change from line to line). Applying this inequality with

$$a = \nabla\varphi \qquad \text{and} \qquad b = H(w_n)\nabla\varphi + \varphi H'(w_n)\nabla w_n,$$

we obtain

$$\int \big((w-\alpha)^+\big)^q \, d\nu \le C \limsup_n \int \Big[\big|H(w_n)-1\big||\nabla\varphi| + \big|\varphi H'(w_n)\nabla w_n\big|\Big]$$
$$\cdot \Big[|\nabla\varphi|^{q-1} + \big|H(w_n)\nabla\varphi + \varphi H'(w_n)\nabla w_n\big|^{q-1}\Big] \, dx.$$

Applying Hölder's inequality, we estimate

$$\begin{aligned}
&\int \Big[\big|H(w_n)-1\big||\nabla\varphi| + \big|\varphi H'(w_n)\nabla w_n\big|\Big] \\
&\qquad \cdot \Big[|\nabla\varphi|^{q-1} + \big|H(w_n)\nabla\varphi + \varphi H'(w_n)\nabla w_n\big|^{q-1}\Big] \, dx. \\
\le\, & C\Big[\int |\nabla\varphi|^q \big|H(w_n)-1\big|^q + \big|\varphi H'(w_n)\nabla w_n\big|^q dx\Big]^{1/q} \\
&\qquad \cdot \Big[\int |\nabla\varphi|^q + \big|H(w_n)\nabla\varphi + \varphi H'(w_n)\nabla w_n\big|^q \, dx\Big]^{(q-1)/q}.
\end{aligned} \tag{2.3}$$

for some constant $C > 0$ depending only on q. Next, we apply Hölder's inequality once more to the term involving $\varphi H'(w_n)\nabla w_n$, obtaining

$$\int \big|\varphi H'(w_n)\nabla w_n\big|^q \, dx \le \Big[\int |\nabla w_n|^p \, dx\Big]^{q/p} \Big[\int \big|\varphi H'(w_n)\big|^{pq/(p-q)} dx\Big]^{(p-q)/p}.$$

Since (w_n) is an optimal sequence for w in the Γ-convergence the first factor on the right-hand side above tends to

$$\Big[\int |\nabla w|^p \, dx + \int w^p \, d\mu\Big]^{q/p},$$

and since H is Lipschitz and φ is in $L^\infty(D)$ the dominated convergence theorem implies that the second factor tends to

$$\Big[\int \big|\varphi H'(w)\big|^{pq/(p-q)} dx\Big]^{(p-q)/p},$$

which vanishes, since $\varphi H'(w) = 0$. Similarly, for the term

$$\int |\nabla\varphi|^q \big|H(w_n)-1\big|^q \, dx,$$

we observe that $H(w_n) \to H(w)$ almost everywhere, and by the definition of φ and H, we have $H(w) = 1$ wherever φ is nonzero. Thus, the dominated convergence theorem applies, yielding

$$\lim_{n\to\infty} \int |\nabla\varphi|^q |H(w_n) - 1|^q \, dx = 0.$$

Combining these observations with the fact that the second factor in (2.3) remains bounded, we conclude that

$$\int \big((w-\alpha)^+\big)^q \, d\nu = 0.$$

We deduce that $(w-\alpha)^+ = 0$ ν-almost everywhere and, since $\alpha > 0$ was chosen arbitrarily, we deduce that $w = 0$ ν-almost everywhere, which concludes the proof. $\square$

Cheeger inequality on unbounded domains. We start with a general result, establishing the asymptotic behavior of $\lambda_p(\Omega)$ when $p \to 1^+$ and $p \to +\infty$ for arbitrary open sets Ω. For the bounded case, we refer the reader to [19, Corollary 6] and [18], respectively.

Proposition 2.6. *Let $\Omega \subset \mathbb{R}^d$ be an open set. Then the following holds*

(1) *For every $p \in [1, +\infty]$ we have*

$$\lim_{R\to+\infty} \lambda_p(\Omega \cap B_R) = \lambda_p(\Omega).$$

(2) $\lim_{p\to1^+} \lambda_p(\Omega) = h(\Omega)$.
(3) $\lim_{p\to+\infty} \lambda_p^{1/p}(\Omega) = \frac{1}{\rho(\Omega)}$.

Proof. Assertion (1) follows immediately from the definition of h, λ_p and ρ. Assume Ω is unbounded; then the proof of assertion (2) is similar to [19, Corollary 6], the unboundedness of the domain being irrelevant.

Assertion (3) is proved in Corollary 6.4 of [2]. $\square$

Below, we establish a general Cheeger-type inequality.

Proposition 2.7. *Let $\Omega \subset \mathbb{R}^d$ be an open set, and let $1 \le q < p < +\infty$. Then for every q such that $\lambda_q(\Omega) > 0$ the following inequality holds:*

$$\mathcal{F}_{p,q}(\Omega) \ge \frac{q}{p}.$$

Proof. This result follows as a direct consequence of Hölder's inequality. Let $\varepsilon > 0$ and choose a function $v \in W_0^{1,p}(\Omega)$ such that

$$\lambda_p(\Omega) \le \frac{\int_\Omega |\nabla v|^p \, dx}{\int_\Omega |v|^p \, dx} \le \big(\lambda_p(\Omega)^{q/p} + \varepsilon\big)^{p/q}.$$

Without loss of generality, we can assume that $v \in C_c^\infty(\Omega)$ (by density) and $v \ge 0$ by replacing v with its absolute value. A further convolution allows us to assume that $v \in C_c^\infty(\Omega)$, $v \ge 0$. Now consider the function $v^{p/q}$ which belongs to $C_c^1(\Omega)$ so it can be used as test function for $\lambda_q(\Omega)$ in $W_0^{1,q}(\Omega)$. A direct computation yields:

$$\begin{aligned}
\lambda_q(\Omega) &\le \frac{\int_\Omega |\nabla v^{p/q}|^q\,dx}{\int_\Omega |v^{p/q}|^q\,dx}\\
&= \left(\frac{p}{q}\right)^q \frac{\int_\Omega |v|^{p-q}|\nabla v|^q\,dx}{\int_\Omega |v|^p\,dx}\\
&\le \left(\frac{p}{q}\right)^q \frac{\left(\int_\Omega |v|^p\,dx\right)^{(p-q)/p}\left(\int_\Omega |\nabla v|^p\,dx\right)^{q/p}}{\int_\Omega |v|^p\,dx}\\
&\le \left(\frac{p}{q}\right)^q \left(\lambda_p(\Omega)^{q/p}+\varepsilon\right).
\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain the desired inequality. □

Remark 2.8. We recall the following, from [3, Theorem 1.3]. Let $d < p < +\infty$. Then there exists a constant $C_{p,d} > 0$, depending only on p and d, such that for every open set $\Omega \subset \mathbb{R}^d$ the following two-sided estimate holds:

$$C_{p,d} \le \rho(\Omega)^p\,\lambda_p(\Omega) \le \lambda_p(B(0,1))$$

where $\rho(\Omega)$ denotes the inradius of Ω.

3. Existence of optimizers in $\mathbb{R}^d$

In this section we give a proof of existence of optimal domains for the shape functional $\mathcal{F}_{p,q}$, with $1 \le q < p \le +\infty$, in the entire space $\mathbb{R}^d$.

3.1. Proof of Theorem 1.1.

Proof. We prove only the existence for the maximization problem, the proof being the same in the minimization case: the only difference is to fix $\lambda_q = 1$ instead of $\lambda_p = 1$.

Fix $1 \le q < p \le +\infty$ and let (Ω_n) be a minimizing sequence of bounded open sets, which can also be taken very regular, for instance of class C^∞. By the scaling invariance of $\mathcal{F}_{p,q}$, one may assume that $\lambda_p(\Omega_n) = 1$ for every n. Since every Ω_n is bounded, one can find $R_n > 0$ such that $\Omega_n \subset B(0, R_n)$, and fixing a unit vector $\mathbf{e}$ of $\mathbb{R}^d$ we define recursively the sequence $(x_n)_n$ of vectors by

$$\begin{cases} x_{n+1} = x_n + (R_n + R_{n+1} + 1)\,\mathbf{e}\\ x_1 = 0.\end{cases}$$

The set

$$\tilde\Omega = \bigcup_{n\ge1}(x_n + \Omega_n)$$

is still an open (and smooth) subset of $\mathbb{R}^d$ and moreover, since the sets $(x_n + \Omega_n)_n$ are disjoint, we have

$$\lambda_p(\tilde\Omega) = \inf_n \lambda_p(\Omega_n) = 1, \qquad \lambda_q(\tilde\Omega) = \inf_n \lambda_q(\Omega_n) = M(p,q)^{-q}.$$

Therefore, $\tilde\Omega$ is a maximizer of $\mathcal{F}_{p,q}$.

It only remains to prove that $M(p,q)$ is finite if and only if $q > d$. This follows from a well-known capacity argument (see for example Theorems 5.3.2 and 5.5.1 of [1]); we include it here for the sake of completeness. If $q > d$, it follows from [3, Theorem 4.5] (see Remark 2.8) that for any open subset Ω of $\mathbb{R}^d$, we have

$$\lambda_p(\Omega) > 0 \iff \rho(\Omega) < +\infty \iff \lambda_q(\Omega) > 0.$$

In particular, $\lambda_p(\tilde{\Omega}) = 1$ implies $\lambda_q(\tilde{\Omega}) > 0$ which yields $M(p,q) < +\infty$.

If $q \le d < p$, since $\mathbb{R}^d \setminus \mathbb{Z}^d$ has a finite inradius, we have $\lambda_p(\mathbb{R}^d \setminus \mathbb{Z}^d) > 0$ whereas $\mathbb{Z}^d$ has zero q-capacity, so that $\lambda_q(\mathbb{R}^d \setminus \mathbb{Z}^d) = \lambda_q(\mathbb{R}^d) = 0$, which gives $M(p,q) = +\infty$.

Finally, let us consider the case $1 \le q < p \le d$. Let us first consider the case $1 < q$. By following the construction for $p = 2$ of Cioranescu and Murat [9], generalized in [12] to any $1 < q < p \le d$, one can find a sequence of radii (r_n) such that the sequence of perforated balls (B_n) defined by

$$B_n = B(0,1) \setminus \Big(\frac{1}{n}\mathbb{Z}^d \oplus B(0,r_n)\Big)$$

satisfies $\lambda_p(B_n) \to C$, while $\lambda_q(B_n) \to 0$ for some constant $C > 0$. Consequently, $\mathcal{F}_{p,q}(B_n) \to +\infty$ and so $M(p,q) = +\infty$. If $q = 1$, the result is a direct consequence of the inequality

$$\lambda_1(\Omega) \le \frac{p+1}{2}\lambda_{\frac{p+1}{2}}(\Omega)^{\frac{2}{p+1}}$$

using the previous observation. □

3.2. **The nonlinearity gap.** We now consider the *"nonlinearity gap"*

$$J_{p,q}(\Omega) = \lambda_p(\Omega) - \lambda_q(\Omega) \qquad \text{with } 1 \le q < p < +\infty, \tag{3.1}$$

and the related shape optimization problem

$$\min\big\{J_{p,q}(\Omega)\ :\ \Omega \subset \mathbb{R}^d\big\}. \tag{3.2}$$

Maximizing the gap is of little interest, since taking $\Omega = tB$ with B a ball of unit radius and $t \to 0$, the supremum of the shape functional $J_{p,q}$ is $+\infty$.

Remark 3.1. Assuming $1 \le q < p < +\infty$, the shape optimization problem (3.2) admits a smooth unbounded solution Ω_0 that is homothetic to the minimizer of the functional $\mathcal{F}_{p,q}$ obtained in Theorem 1.1. More precisely, problems (3.2) and (1.5) (for $m(p,q)$) have the same optimal domains, up to homotheties. Indeed, if in problem (3.2) we consider $t\Omega$ instead of Ω we have

$$\min_{\Omega\subset\mathbb{R}^d}\min_{t>0}\Big\{t^{-p}\lambda_p(\Omega) - t^{-q}\lambda_q(\Omega)\Big\}.$$

The optimization with respect to t can be carried out explicitely

$$\min_{\Omega\subset\mathbb{R}^d} -\frac{p-q}{p}\Big(\frac{q}{p}\Big)^{q/(p-q)}\Big(\frac{\lambda_q^{1/q}(\Omega)}{\lambda_p^{1/p}(\Omega)}\Big)^{pq/(p-q)},$$

so that the problem is reduced to the minimization of $\mathcal{F}_{p,q}$. If Ω_{opt} is a solution for $\mathcal{F}_{p,q}$ we then have

$$\Omega_0 = \bar{t}\,\Omega_{opt} \qquad \text{with } \bar{t} = \Big(\frac{p\lambda_p(\Omega_{opt})}{q\lambda_q(\Omega_{opt})}\Big)^{1/(p-q)},$$

which concludes the proof.

Remark 3.2. By the same arguments as above and under the same hypotheses we actually obtain that the generalized problem

$$\min\Big\{\Phi\big(\lambda_p(\Omega)^{\frac{1}{p}}, \lambda_q(\Omega)^{\frac{1}{q}}\big) \ :\ \Omega\subset\mathbb{R}^d,\ \text{open}\Big\}$$

admits a solution and its minimizers are homothetic to the minimizers of $\mathcal{F}_{p,q}$, for any function $\Phi:\mathbb{R}_+^*\times\mathbb{R}_+^*\to\mathbb{R}$ such that

(i) $\Phi(x,y)$ is either increasing in x or decreasing in y;
(ii) For any $x>0$, there exists $s_x^*>0$ such that $\inf_{s>0}\Phi(s\,x,\ s)=\Phi(s_x^*\,x,\ s_x^*)$.

This class of functions contains the function

$$f(x,y)=\frac{x}{y},$$

which corresponds to our main functional $\mathcal{F}_{p,q}$, it also contains the family

$$j_{\alpha,\beta}(x,y)=x^\alpha-y^\beta,$$

for any $\alpha>\beta>0$, where the nonlinearity gap $J_{p,q}$ corresponds to $j_{p,q}$.

4. Description of the optimizers

We now prove Theorem 1.3.

Proof. Let $d<q<p<+\infty$. We prove the result for the minimization problem, since the proof is identical for the maximization case. The ordering between p and q does not play any role since both are larger than d and less than $+\infty$.

Take (Ω_n) to be a minimizing sequence of bounded sets for $\mathcal{F}_{p,q}$ such that $\lambda_q(\Omega_n)=1$ for all n, and assume without loss of generality that the sequence $(\lambda_p(\Omega_n))$ is decreasing.

For simplicity, let us denote for every $R>0$ by Q_R the cuboid $(0,R)^d$. Consider now $\xi_0>0$ such that for every $u\in W^{1,q}(Q_{\xi_0})\setminus\{0\}$ which vanishes at the vertices of the cuboid, namely $u(x_1,\dots,x_d)=0$ if $x_i\in\{0,\xi_0\}$, we have

$$\frac{\int_{Q_{\xi_0}}\|\nabla u\|^q\,dx}{\int_{Q_{\xi_0}}\|u\|^q\,dx}>2. \tag{4.1}$$

Let us denote by $\tilde W^{1,q}(Q_{\xi_0})$ the space of such functions of $W^{1,q}(Q_{\xi_0})$ vanishing at the vertices of the cuboid. There are two reasons why such a choice of ξ_0 is possible in (4.1). First,

$$\inf_{v\in\tilde W^{1,q}(Q_1)\setminus\{0\}}\frac{\int_{Q_1}\|\nabla v\|^q\,dx}{\int_{Q_1}\|v\|^q\,dx}>0.$$

This is a consequence of the fact that points (namely the vertices of Q_1) have positive q-capacity and that we require v to vanish at the vertices of the cuboid Q_{ξ_0} along with the compact embedding of $W^{1q}(Q_1)$ in $L^q(Q_1)$.

Second, in view of the homogeneity of the Rayleigh quotient to rescaling of Q_1, we have

$$\inf_{v\in\tilde W^{1,q}(Q_{\xi_0})\setminus\{0\}}\frac{\int_{Q_{\xi_0}}\|\nabla v\|^q\,dx}{\int_{Q_{\xi_0}}\|v\|^q\,dx}=\frac{1}{\xi_0^q}\inf_{v\in\tilde W^{1,q}(Q_1)\setminus\{0\}}\frac{\int_{Q_1}\|\nabla v\|^q\,dx}{\int_{Q_1}\|v\|^q\,dx}.$$

For each n, we introduce $\tilde\Omega_n=(1+\frac{1}{n})^{-1/q}\Omega_n$ so that $\lambda_q(\tilde\Omega_n)=1+\frac{1}{n}$. Since each $\tilde\Omega_n$ is bounded, we can assume there exists $R_n>0$ such that R_n is an integer multiple of ξ_0 and

$B_{\xi_0} \oplus \tilde{\Omega}_n \subset Q_{R_n}$. We define the sequence $x_n \in \mathbb{R}^d$, by $x_1 = (0, \dots, 0)$, $x_n = (\alpha_n, 0, \dots, 0)$, with $\alpha_n = \sum_{i=1}^{n-1}(R_i + 1)$ for all $n \geq 2$.

We introduce the sets

$$C_k = x_k + \Big((Q_{R_k} \setminus \xi_k \mathbb{Z}^d) \cup \tilde{\Omega}_k \Big),$$

where $\xi_k < \frac{\xi_0}{2}$ is small enough such that for every $u \in W^{1,q}(C_k)$ which satisfies $u = 0$ on $(\xi_k \mathbb{Z}^d \setminus \tilde{\Omega}_k) \cap C_k$ we have

$$\frac{\int_{C_k} |\nabla u|^q \, dx}{\int_{C_k} |u|^q \, dx} \geq 1 + \frac{1}{2k}.$$

This is possible because points have positive q-capacities and $\lambda_q(\tilde{\Omega}_k) = 1 + \frac{1}{k} > 1 + \frac{1}{2k}$.

Roughly speaking, the set C_k is obtained as follows: take the cuboid Q_{R_k} and remove the points of the lattice $\xi_k \mathbb{Z}^d$, for ξ_k small. Then take the union of this set with $\tilde{\Omega}_k$. In this way, the first p and q eigenvalues of the set C_k are as close as necessary to the first p and q eigenvalues of the set $\tilde{\Omega}_k$. This relies on the fact that the γ_p- and γ_q-convergences are equivalent to the Hausdorff convergence of the complements when $d < p, q < +\infty$.

We define the set

$$A = \Big((\mathbb{R}^d \setminus \xi_0 \mathbb{Z}^d) \setminus \bigcup_{n \geq 1} (x_n + Q_{R_n}) \Big) \bigcup_{n \geq 1} C_n.$$

We claim that this set satisfies $\lambda_q(A) = 1$ and minimizes λ_p.

Indeed, on the one hand, by monotonicity to inclusions we get for every n

$$\lambda_q(A) \leq \lambda_q(\tilde{\Omega}_n) = 1 + \frac{1}{n}.$$

Consequently, $\lambda_q(A) \leq 1$. To prove that it equals 1, we consider an arbitrary function $u \in W_0^{1,q}(A)$, assumed with bounded support. The support of the function is covered by a finite number of cuboids either of the form $x_k + Q_{R_k}$, or of the form $z + Q_{\xi_0}$ for some $z \in \xi_0 \mathbb{Z}^d$ outside $\cup_k (x_k + Q_{R_k})$. Then, the algebraic inequality

$$\frac{\sum_i a_i}{\sum_i b_i} \geq \min_i \frac{a_i}{b_i}, \tag{4.2}$$

which holds for positive numbers, gives

$$\frac{\int_A |\nabla u|^q \, dx}{\int_A |u|^q \, dx} \geq \frac{\int_Q |\nabla u|^q \, dx}{\int_Q |u|^q \, dx},$$

where Q is either one of $x_k + Q_{R_k}$ or $z + Q_{\xi_0}$. In either case we get

$$\frac{\int_Q |\nabla u|^q \, dx}{\int_Q |u|^q \, dx} \geq 1 + \frac{1}{2k} \geq 1.$$

Indeed, if $Q = z + Q_{\xi_0}$, then by choice of ξ_0 we have

$$\frac{\int_Q |\nabla u|^q \, dx}{\int_Q |u|^q \, dx} \geq 2.$$

If $Q = x_k + Q_{R_k}$ then

$$\frac{\int_Q |\nabla u|^q\,dx}{\int_Q |u|^q\,dx} = \frac{\int_{C_k} |\nabla u|^q\,dx}{\int_{C_k} |u|^q\,dx} \geq 1 + \frac{1}{2k}.$$

At the same time, by monotonicity we have $\lambda_p(A) \leq \lambda_p(\tilde{\Omega}_n)$, and since $(\Omega_n)_n$ was minimizing, we get that A is a minimizer.

The case $p = +\infty$ is not fundamentally different. The minimum of $\mathcal{F}_{\infty,q}$ is achieved by a ball. We can produce an unbounded optimizer with a discrete complement as above, directly from the ball. This is of little interest.

For the maximization of $\mathcal{F}_{\infty,q}$, we take a maximizing sequence $(\Omega_n)_n$ such that $\rho(\Omega_n) = 1$. We consider $\tilde{\Omega}_n = (1 - \frac{1}{n})\Omega_n$ and build the set A as above (the same notations are used), with the parameters ξ_0 and ξ_n such that

$$\rho(\mathbb{R}^d \setminus \xi_0\mathbb{Z}^d) = \frac{1}{2} \qquad \text{and} \qquad \rho(C_k) \leq 1 - \frac{1}{2n}.$$

The set A is optimal: it satisfies the inradius constraint and

$$\lambda_q(A) \leq \inf_n \lambda_q(\tilde{\Omega}_n) = \inf_n \lambda_q(\Omega_n).$$

The proof is now complete. $\square$

5. The bounded case

In this section, we study the problem in which the admissible domains Ω are contained in a given open bounded subset D of $\mathbb{R}^d$. We consider the optimization problems (1.7) and (1.8).

5.1. **Proof of Theorem 1.5.**

Proof. We first consider the minimization problem (1.7) for some p, q such that $1 < q < p < +\infty$. Let $D \subset \mathbb{R}^d$ be an open, bounded set such that $\lambda_q(D) < 1$ (otherwise, the problem has a trivial solution if $\lambda_q(D) = 1$) and let (Ω_n) be a minimizing sequence. Since the γ_p and γ_q-convergences are compact, after possibly passing to a subsequence, we may also assume that

$$\Omega_n \xrightarrow{\gamma_p} \mu \qquad \text{and} \qquad \Omega_n \xrightarrow{\gamma_q} \nu,$$

for some p-capacitary measure μ and q-capacitary measure ν. If $p > d$, since the γ_p-convergence reduces to the Hausdorff convergence of the complements $\overline{D}\setminus\Omega_n$, the measure μ will be of the form $\mu = \infty_{\overline{D}\setminus\Omega}$ for a suitable open set Ω. By Theorem 2.5 we have $\nu = 0$ on Ω and, by the continuity of λ_p and λ_q with respect to the γ_p and γ_q-convergences respectively, we have

$$\begin{cases} \lambda_p(\Omega) = \lim_n \lambda_p(\Omega_n) = \inf\,(1.7), \\ \lambda_q(\Omega) \geq \lambda_q(\nu) = \lim_n \lambda_q(\Omega_n) = 1. \end{cases}$$

Since $\lambda_q(D) < 1$ we may choose Ω' such that $\Omega \subset \Omega' \subset D$ and $\lambda_q(\Omega') = 1$. We have $\lambda_p(\Omega') \leq \lambda_p(\Omega)$ and so Ω' is optimal.

We now consider the case $q \le d$. Again, by the continuity of λ_p and λ_q with respect to the γ_p and γ_q-convergences respectively, we have

$$\begin{cases}\lambda_p(\mu) = \lim_n \lambda_p(\Omega_n) = \inf\,(1.7),\\ \lambda_q(\nu) = \lim_n \lambda_q(\Omega_n) = 1.\end{cases}$$

By Theorem 2.5 we have that $\nu = 0$ on the set Ω_μ where μ is finite; therefore, from the monotonicity property (2.1) we deduce that

$$\begin{cases}\lambda_p(\Omega_\mu) \le \lambda_p(\mu) = \inf\,(1.7),\\ \lambda_q(\Omega_\mu) \ge \lambda_q(\nu) = 1.\end{cases}$$

As above, we may take Ω' such that $\Omega_\mu \subset \Omega' \subset D$ and $\lambda_q(\Omega') = 1$. We have $\lambda_p(\Omega') \le \lambda_p(\Omega)$, which proves the optimality of Ω'.

The existence of an optimal domain for the maximization problem (1.8) follows in exactly the same way by repeating the above argument. ☐

5.2. **The constrained nonlinearity gap.** We now consider the optimization problem for the shape functional $J_{p,q}$ introduced in (3.1) subject to the constraint $\Omega \subset D$:

$$\min\big\{J_{p,q}(\Omega)\ :\ \Omega \subset D\big\}. \tag{5.1}$$

Theorem 5.1. *For any open bounded subset D of $\mathbb{R}^d$ and p, q such that $1 < q < p < +\infty$, the shape optimization problem (5.1) admits a solution Ω^* that is a p-quasi open set.*

Proof. Let (Ω_n) be a minimizing sequence; by the compactness of γ_p and γ_q-convergences we may assume that

$$\Omega_n \xrightarrow{\gamma_p} \mu \qquad \text{and} \qquad \Omega_n \xrightarrow{\gamma_q} \nu,$$

for some p-capacitary measure μ and q-capacitary measure ν. Then the infimum in (5.1) is given by

$$\lambda_p(\mu) - \lambda_q(\nu).$$

Since for some $\Omega_0 \subset D$ we have

$$J_{p,q}(\Omega_n) \le J_{p,q}(\Omega_0)$$

and

$$\mathcal{F}_{p,q}(\Omega_n) \ge m(p,q),$$

we find that the sequences $(\lambda_p(\Omega_n))$ and $(\lambda_q(\Omega_n))$ are both bounded, and therefore the quantities $\lambda_p(\mu)$ and $\lambda_q(\nu)$ are both finite. By Theorem 2.5 we have that $\nu = 0$ on the p-quasi open set Ω_μ where μ is finite. Then

$$\lambda_p(\Omega_\mu) \le \lambda_p(\mu) \qquad \text{and} \qquad \lambda_q(\Omega_\mu) \ge \lambda_q(\nu),$$

which implies that Ω_μ is optimal for the constrained shape functional $J_{p,q}$. ☐

Remark 5.2. By repeating step by step the proof of Theorem 5.1, under the same assumptions, we obtain the existence of an optimal p-quasi open set Ω_{opt} for the shape optimization problem

$$\min\Big\{\Phi\big(\lambda_p(\Omega)^{1/p}, \lambda_q(\Omega)^{1/q}\big)\ :\ \Omega \subset D\Big\}$$

whenever $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function such that:

(i) Φ is continuous;
(ii) $\Phi(x, y)$ is increasing in x and decreasing in y;
(iii) $\lim_{t\to+\infty} \Phi(m(p, q)\, t, t) = +\infty$.

6. Further remarks and open problems

In this section, we summarize several questions that, in our opinion, deserve further investigation.

6.1. **Questions for $M_{p,q}$.** As observed in Theorem 1.1 the only interesting case is $d < q < p \le +\infty$, since $M(p, q) = +\infty$ when $q \le d$.

Question Q1. Characterizing the maximal value $M_{p,q}$ is an interesting issue.

Question Q2. In Theorem 1.1 we have shown the existence of optimal unbounded domains $\Omega^M_{p,q}$ for which $\mathcal{F}_{p,q}(\Omega^M_{p,q}) = M(p, q)$. Can there exist an optimal bounded domain? We believe this is not possible.

Question Q3. In Theorem 1.3 we have shown the existence of optimal domains $\Omega^M_{p,q}$ such that $\mathbb{R}^d \setminus \Omega^M_{p,q}$ is a discrete set. It would be interesting to show the existence of optimal domains $\Omega^M_{p,q}$ such that $\mathbb{R}^d \setminus \Omega^M_{p,q}$ is a *periodic* discrete set.

Question Q4. In the planar case $d = 2$ it would be interesting to prove (or disprove) that the domain $\mathbb{R}^2 \setminus X$ is optimal, where X is the discrete periodic set consisting of all the centers of a planar tessellation made of regular hexagons. We believe that this is true, at least in the extreme case $p = +\infty$.

6.2. **Questions for $m_{p,q}$.** In this case Theorem 1.1 gives the existence of optimal unbounded domains $\Omega^m_{p,q}$ for every $1 \le q < p \le +\infty$, and the inequality $m_{p,q} \ge q/p$ holds.

Question Q5. In this case as well, the characterization of the minimal value $m_{p,q}$ is an interesting problem.

Question Q6. It would be interesting to determine in which cases a *bounded* optimal domain $\Omega^m_{p,q}$ exists.

Question Q7. If $p = +\infty$ it is easy to see that a ball is an optimal domain for every q. Is there a threshold p^* (possibly depending on q) such that for $p \ge p^*$ a ball is still optimal?

6.3. **Questions for the problem with a bounded constraint.** In Theorem 1.5 we established the existence of optimal p-quasi open sets Ω^M_{opt} and Ω^m_{opt} solving respectively the maximization and the minimization problems (1.8) and (1.7).

Question Q8. It would be interesting to study the regularity of Ω^M_{opt} and Ω^m_{opt} in various steps:

- determine whether Ω^M_{opt} and Ω^m_{opt} are open sets;
- determine whether Ω^M_{opt} and Ω^m_{opt} are sets with finite perimeter;
- determine whether Ω^M_{opt} and Ω^m_{opt} have a higher regularity, for instance $C^{1,\alpha}$, possibly up to a small singular set.

Question Q9. In the case with a bounded constraint D the maximization problem (1.8) is slightly different from the minimization problem for $\mathcal{F}_{p,q}$. We do not know whether the latter problem admits an optimal solution.

Question Q10. We proved in Theorem 5.1 existence of a p-quasi open solution to the constraint nonlinearity gap problem

$$\min\left\{\lambda_p(\Omega)-\lambda_q(\Omega)\ :\ \Omega\subset D\right\},$$

and to its generalization

$$\min\left\{\Phi\big(\lambda_p(\Omega),\lambda_q(\Omega)\big)\ :\ \Omega\subset D\right\},$$

where $\Phi(x,y)$ is a continuous function, increasing with respect to x and decreasing with respect to y. The questions raised in *Question 8* should be investigated in this more general framework.

We also showed in Remark 3.2 that the minimizers for the problem

$$I=\min\left\{\Phi\big(\lambda_p(\Omega),\lambda_q(\Omega)\big)\ :\ \Omega\subset\mathbb{R}^d\right\},$$

are the same as the minimizers of $\mathcal{F}_{p,q}$ in $\mathbb{R}^d$ up to a rescaling, and therefore, it is possible to construct a minimizing sequence of bounded sets (Ω_n) solving the problem I. Then, the sequence (I_n) defined as

$$\forall n,\qquad I_n=\min\left\{\Phi\big(\lambda_p(\Omega),\lambda_q(\Omega)\big)\ :\ \Omega\subset B(0,n)\right\},$$

converges to I. It would be interesting to determine whether the associated sequence of solutions (Ω_n^*) converges to a global minimizer Ω^* and to characterize the properties of this selected minimizer.

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