STRATIFYING THE MODULI SPACE OF STABLE VECTOR BUNDLES BY DECOMPOSITION TYPE

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Abstract. The moduli space of μ-stable vector bundles on a normal projective variety over an algebraically closed field of characteristic $p \geq 0$ is stratified with respect to the decomposition type. On a smooth projective curve of genus at least 2 we obtain mostly sharp dimension estimates for these strata.

As an application, we obtain a dimension estimate for the closure of the prime-to-p trivializable stable bundles in the moduli space of stable vector bundles on a smooth projective curve which is sharp in characteristic 0.

1. Introduction

Recently, stability of vector bundles has been investigated under pullback. For genuinely ramified morphisms of normal projective varieties, i.e., morphisms without an étale part, stability is automatically preserved under pullback, see [5, Theorem 1.1] for the case of curves and [4, Theorem 1.2] for the general case. The behaviour of stability under pullback by étale covers was studied in [17].

In this paper we extend upon the results of [17] to obtain a stratification of the moduli space of stable bundles and to estimate the dimension of these strata over a smooth projective curve C. As an application, we study the closure of the prime-to-p trivializable vector bundles, that is, vector bundles corresponding to a representation of the prime-to-p completion of the étale fundamental group.

Consider a stable vector bundle V on C. Given an étale Galois cover $\pi : D \to C$, the pullback of V to D may fail to remain stable. However, V is still polystable on D and can decompose only in a very special way: It is a direct sum of stable vector bundles W_i such that the Galois group acts transitively on the isomorphism classes of the W_i, see [17, Lemma 3.8]. In particular, the W_i have the same rank m. Fixing the rank m of the W_i as well as the rank r and degree d of V, we obtain a stratification $Z^{s,r,d}(m, \pi)$ of the moduli space of stable vector bundles $M_C^{s,r,d}$ on C with respect to an étale Galois cover - the *decomposition stratification* with respect to the étale Galois cover $\pi : D \to C$, see Lemma 2.4. We can estimate the dimension of these strata generalizing [17, Lemma 4.4]:

Theorem 1.1 (Theorem 2.8 (ii))**.** *Let $\pi : D \to C$ be an étale Galois cover of a smooth projective curve C of genus $g_C \geq 2$. Let $r \geq 2$ and $d \in \mathbf{Z}$. Let $m \mid r$. Then we have*

$$\dim(Z^{s,r,d}(m, \pi)) \leq rm(g_C - 1) + 1.$$

Note that this stratification is dependent on the choice of the étale cover $D \to C$. To obtain a canonical stratification, recall that a vector bundle is *prime-to-p stable*

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if it remains stable on all étale Galois covers of degree prime-to-p, see [17, Definition 2.2]. There is an étale prime-to-p Galois cover $C_{r,good} \to C$ checking for prime-to-p stability of vector bundles of rank r, see [17, Theorem 1]. Iterating the cover $C_{r,good}$, we obtain an étale prime-to-p Galois cover $C_{r,split} \to C$ checking for the eventual decomposition with respect to étale prime-to-p Galois covers.

The decomposition stratification with respect to $C_{r,split} \to C$ is independent of the cover $C_{r,split} \to C$ and is called the *prime-to-p decomposition stratification*, see Definition 4.2. Fixing the degree d, we denote this stratification for $m \mid r$ by $Z^{s,r,d}(m)$. Then the dimension estimates obtained for arbitrary Galois covers are mostly sharp. This is a generalization of [17, Theorem 2].

Theorem 1.2 (Theorem 4.6 (ii))**.** *Let C be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$ and $d \in \mathbb{Z}$. Then for $m \mid r$ such that $p \nmid \frac{r}{m}$ we have*

$$\dim(Z^{s,r,d}(m)) = rm(g_C - 1) + 1.$$

As an application, we can estimate the dimension of the closure of the prime-to-p trivializable (semi)stable vector bundles. This should on the one hand be compared with a result of Ducrohet and Mehta, see [6, Corollary 5.1], asserting that in positive characteristic the étale trivializable bundles are dense in the moduli space $M_C^{ss,r,0}$ of semistable vector bundles of rank r and degree 0. On the other hand, the non-density of the prime-to-p trivializable bundles was recently obtained in [17, Corollary 3] and in rank 2 and characteristic 0 independently by Ghiasabadi and Reppen, see [7, Corollary 4.16]. Thus, it is natural to ask for a description of the closure of the prime-to-p trivializable stable vector bundles. This is closely related to the smallest prime-to-p decomposition stratum and we obtain the same dimension:

Theorem 1.3 (Theorem 5.2 (iii) and (iv))**.** *Let C be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$. Let $Z^{s,r}$ (resp. $Z^{ss,r}$) be the closure of the prime-to-p trivializable (semi)stable vector bundles in $M_C^{s,r,0}$ (resp. $M_C^{ss,r,0}$). Then we have:*

- *If $p \nmid r$, then* $\dim(Z^{s,r}) = r(g_C - 1) + 1$.
- $\dim(Z^{ss,r}) = rg_C$.

The paper is structured as follows. We begin the first section by recalling some properties of semistable vector bundles under pushforward and pullback. Then we define the decomposition stratification and estimate the dimension of the strata.

In the second section, we study stable vector bundles under étale pushforward. Pushforward under an étale cover of a smooth projective curve induces a finite morphism on the level of moduli spaces. Furthermore, we identify the big open subscheme where the direct image of a stable vector bundle is stable.

In the third section, we define the prime-to-p decomposition stratification and obtain the mostly sharp dimension estimates.

In the last section, we apply the theory developed to study the closure of the prime-to-p trivializable (semi)stable vector bundles.

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Notation. We work over an algebraically closed field k of characteristic $p \geq 0$. A variety is a separated integral scheme of finite type over k. A curve is a variety of dimension 1.

If X is a projective variety we implicitly choose an ample bundle $\mathcal{O}_X(1)$ on X. Given a finite morphism $\pi : Y \to X$ we always equip Y with the polarization $\mathcal{O}_Y(1) = \pi^*\mathcal{O}_X(1)$. By *(semi)stability* we mean μ-(semi)stability with respect to $\mathcal{O}_X(1)$. Two semistable vector bundles V and W are S-equivalent or $V \cong_S W$ if their Jordan-Hölder filtrations have the same associated graded. We denote the slope of a vector bundle V by $\mu(V)$.

We denote the moduli space of (semi)stable vector bundles of rank r and degree d on a smooth projective curve C by $M_C^{s,r,d}$ (resp. $M_C^{ss,r,d}$). In the special case of rank 1, we use the notation of the Picard scheme Pic_C^d instead.

On a projective variety X the stable vector bundles with Hilbert polynomial P form an open $M_X^{s,P}$ in the moduli space of Gieseker semistable sheaves $M_X^{G-ss,P}$.

Given a morphism $\pi : Y \to X$ of varieties and a sheaf F on X we denote the pullback π^*F also by $F_{|Y}$.

By a *cover* $Y \to X$ of varieties we mean a finite surjective separable morphism. A cover is *Galois* if the extension of function fields is Galois. A *cyclic* Galois cover is a Galois cover with cyclic Galois group. We call a cover $\pi : Y \to X$ *prime-to-p* if if its Galois closure has degree prime-to-p.

2. Stratifying by decomposition type

We define a stratification associated to the decomposition type of a stable vector bundle with respect to a fixed étale Galois cover $Y \to X$ of a normal projective variety X. Then we estimate the dimension of the strata in the curve case refining the result [17, Lemma 4.4].

For the convenience of the reader we start by recalling some properties of semistable vector bundles under pushforward and pullback:

Lemma 2.1 (Lemma 3.2.1-3.2.3, [8], see also Lemma 3.2 [17])**.** *Let $\pi : Y \to X$ be a cover of a normal projective variety X. Let V be a vector bundle on X and W be a vector bundle on Y. Then we have the following:*

- $\mu(W) = \deg(\pi)(\mu(\pi_*W) - \mu(\pi_*\mathcal{O}_Y))$.
- $\mu(V_{|Y}) = \deg(\pi)\mu(V)$.
- *V is semistable iff $V_{|Y}$ is semistable.*
- *If $V_{|Y}$ is stable, then so is V.*

If in addition $\pi : Y \to X$ is Galois, then we have:

- *If V is polystable, then $V_{|Y}$ is polystable.*
- *If π is prime-to-p, then V is polystable if and only if $V_{|Y}$ is polystable.*

In the curve case, the above lemma allows us to consider the induced morphism on the level of moduli of semistable vector bundles, that is, given a Galois cover $\pi : D \to C$ we obtain a morphism $\pi^* : M_C^{ss,r,d} \to M_D^{ss,r,\deg(\pi)d}, V \mapsto V_{|D}$. More precisely, pullback induces a morphism on the level of the semistable loci of the moduli stacks of vector bundles. Then the universal property of adequate moduli spaces, [1, Theorem 7.2.1], induces the desired morphism π^* which applies since moduli of Gieseker-semistable sheaves are constructed via geometric invariant theory (GIT) and thus are adequate moduli spaces. This morphism π^* is finite by [17, Theorem 4.2].

The higher dimensional case is more intricate as the notions of semistability and Gieseker-semistability diverge. However, we can construct the pullback morphism if we start from the stable locus. We spell this out in the proof of Lemma 2.4 below in the étale case, see also Remark 2.6 for the general one.

Regarding stability and pullback by a Galois cover, we find more structure via symmetries under the action by the Galois group. The following is the key observation for defining decomposition strata and is shown via considering the orbit of a stable direct summand under the action of the Galois group.

Lemma 2.2 (Key observation, Lemma 3.2, [17])**.** *Let $Y \to X$ be a Galois cover of a normal projective variety X. Then a stable vector bundle V on X decomposes on Y as a direct sum*

$$V_{|Y} \cong \bigoplus_{i=1}^{n} W_i^{\oplus e}$$

of pairwise non-isomorphic stable vector bundles W_i such that the Galois group acts transitively on the isomorphism classes of the W_i.

Recall that only the étale part of a cover matters for stability under pullback as the genuinely ramified part preserves stability, [4, Theorem 1.2]. Thus, we focus on étale covers in the remaining part of the paper.

Definition 2.3. Let $\pi : Y \to X$ be an étale Galois cover of a normal projective variety X. Let V be a stable vector bundle of rank r on X. The *decomposition type of V with respect to* π is the rank m of the bundles W_i in the direct sum decomposition $V_{|Y} \cong \bigoplus_{i=1}^{n} W_i^{\oplus e}$ of Lemma 2.2. The *refined decomposition type of V with respect to* π is the tuple (n, e).

Note that the refined decomposition type recovers the decomposition type as $mne = r$.

Let $P \in \mathbb{Q}[x]$ be a polynomial. Assume that the moduli space $M_X^{s,P}$ of stable vector bundles with Hilbert polynomial P is non-empty. The Hilbert polynomial determines the rank of the vector bundles in $M_X^{s,P}$ which we denote by r.

For integers $m, n, e, \geq 1$ such that $mne = r$ we define the *refined decomposition strata with respect to* π

$$Z^{s,P}(n, e, \pi) := \left\{ V \in M_X^{s,P} \,\middle|\, \begin{array}{c} V \text{ has refined decomposition} \\ \text{type } (n, e) \text{ with respect to } \pi \end{array} \right\}, \text{ and}$$

$$Z^{s,P}(\cdot \mid n, e \mid \cdot, \pi) := \bigsqcup_{(n', e')} Z^{s,P}(n', e', \pi),$$

where the union is taken over $n', e' \geq 1$ such that $n'e' = ne$ and $n' \mid n$. The *decomposition strata with respect to* π are defined as

$$Z^{s,P}(m, \pi) := \left\{ V \in M_X^{s,P} \,\middle|\, \begin{array}{c} V \text{ has decomposition type } m \\ \text{with respect to } \pi \end{array} \right\}, \text{ and}$$

$$Z^{s,P}(\cdot \mid m, \pi) := \bigsqcup_{m' \mid m} Z^{s,P}(m', \pi).$$

If X is a smooth projective curve, then the Hilbert polynomial is determined by the rank r and degree d, and we use the notations

$$Z^{s,r,d}(m,\pi), Z^{s,r,d}(\cdot \mid m,\pi), Z^{s,r,d}(n,e,\pi), \text{ and } Z^{s,r,d}(\cdot \mid n,e \mid \cdot,\pi).$$

We show that $Z^{s,P}(m,\pi)$ and $Z^{s,P}(n,e,\pi)$ form a stratification justifying the name. The main ingredient is that the direct summands appearing in the decomposition of Lemma 2.2 have the same Hilbert polynomial and that direct sums induce a morphism on the level of moduli spaces of Gieseker-semistable sheaves when the reduced Hilbert polynomials agree.

Lemma 2.4. *Let $\pi : Y \to X$ be an étale Galois cover of a normal projective variety X. Let $P \in \mathbb{Q}[x]$ be a polynomial. Assume that $M_X^{s,P}$ is non-empty and denote by r the rank of the vector bundles in $M_X^{s,P}$. Then we have the following:*

(i) $Z^{s,P}(\cdot \mid m,\pi) \subseteq M_X^{s,P}$ *is closed for $m \mid r$.*
(ii) $Z^{s,P}(\cdot \mid n,e \mid \cdot,\pi) \subseteq Z^{s,P}(m,\pi)$ *is closed for $m,n,e \geq 1$ such that $mne = r$.*

Proof. Recall that $M_X^{s,P}$ is quasi-projective as it is an open of the projective scheme $M_X^{G-ss,P}$, see [8, Theorem 4.3.4] for characteristic 0 and [11, Theorem 0.2] for positive characteristic.

Also recall that by the singular version of the Grothendieck-Riemann-Roch theorem, [2, p. 102-103], we can determine the Hilbert polynomial of a vector bundle on X via its Chern character and the homological Todd class $\tau(X)$. Since $Y \to X$ is an étale cover we also know that $\tau(X)$ pulls back to $\tau(Y)$, [16, Theorem 7.1], and we conclude that $P(V_{|Y}) = \deg(\pi)P(V)$.

We claim that for $V \in M_X^{s,P}$ the bundles W_i in the direct sum decomposition $V_{|Y} \cong \bigoplus_{i=1}^n W_i^{\oplus e}$ of Lemma 2.2 have the same Hilbert polynomial. Indeed, the Galois group G of $Y \to X$ acts transitively on the isomorphism classes of the W_i. For $\sigma \in G$ we have $P(\sigma^* W_i) = P(W_i)$ as the Hilbert polynomial is computed with respect to $\pi^*\mathcal{O}_X(1)$ which is invariant under the action of G. Thus, we have $P(W_i) = \frac{m}{r}\deg(\pi)P$ for $V \in Z^{s,P}(m,\pi)$.

Pulling back along π defines a morphism $\pi^* : M_X^{s,P} \to M_Y^{G-ss,\deg(\pi)P}$. In $M_Y^{G-ss,\deg(\pi)P}$ the set-theoretic image $\operatorname{Im}(\varphi_{n,e})$ of

$$\varphi_{n,e} : \prod_{i=1}^n M_Y^{G-ss,\frac{m}{r}\deg(\pi)P} \to M_Y^{G-ss,\deg(\pi)P}, (W_1,\dots,W_n) \mapsto \bigoplus_{i=1}^n W_i^{\oplus e}$$

is closed as it is a morphism of projective schemes, where $m,n,e \geq 1$ such that $mne = r$.

We now show (i). We claim that $Z^{s,P}(\cdot \mid m,\pi)$ is the preimage of $\operatorname{Im}(\varphi_{n,1})$ under π^* and is thus closed. Indeed, $V \in Z^{s,P}(\cdot \mid m,\pi)$ clearly lies in the preimage. If $V_{|Y} \in \operatorname{Im}(\varphi_{n,1})$ for some $V \in M_X^{s,P}$ and n such that $mn = r$, then $V_{|Y}$ is S-equivalent to $\bigoplus_{i=1}^n W_i$ for some Gieseker-semistable sheaves W_i of rank m. As $V_{|Y}$ is a direct sum of stable vector bundles with the same Hilbert polynomial, it is Gieseker-polystable. The associated graded object of a JH-filtration is unique and thus V has decomposition type m' with respect to π for some $m' \mid m$.

Similarly, (ii) is obtained from

$$(\pi^*)^{-1}(\operatorname{Im}(\varphi_{n,e})) \cap Z^{s,P}(m,\pi) = Z^{s,P}(\cdot \mid n,e \mid \cdot,\pi).$$

□

Remark 2.5. *Ordering $Z^{s,P}(m,\pi), m \mid r$, via division, we obtain a stratification of $M_X^{s,P}$ by $Z^{s,P}(m,\pi)$. Furthermore, the refined decomposition strata $Z^{m,P}(n,e,\pi)$ stratify the decomposition stratum $Z^{s,P}(m,\pi)$ for $mne = r$ if we order them via*

$$Z^{s,P}(n',e',\pi) \leq Z^{s,P}(n,e,\pi) :\Leftrightarrow n' \mid n.$$

Remark 2.6. *Let $\pi : Y \to X$ be a Galois cover of a normal projective variety X and V a stable vector bundle on X. We note that the Hilbert polynomials of the stable direct summands W_i in the decomposition $V_{|Y} \cong \bigoplus_{i=1}^n W_i^{\oplus e}$ are the same as the proof of Lemma 2.4 did not use étaleness at this stage. In particular, $V_{|Y}$ is Gieseker-semistable.*

While this observation allows us to define a pullback morphism from the stable locus $M_X^{s,P}$ to a moduli space of Gieseker-semistable sheaves on Y, it is unclear what the Hilbert polynomial(s) of the pulled back bundles are, since étaleness was essential in the application of Grothendieck-Riemann-Roch to compare the Hilbert polynomials of V and $V_{|Y}$. In principle, the Hilbert polynomial is only constant in connected families and $M_X^{s,P}$ might have more than one connected component.

2.7. **Dimension estimates.** Consider a Galois cover $\pi : D \to C$ of a smooth projective curve C with Galois group G. To estimate the dimension of the (refined) decomposition strata with respect to π we show that the decomposition of the pullback $V_{|D} \cong \bigoplus_{i=1}^n W_i^{\oplus e}$ of Lemma 2.2 can essentially be recovered from W_1. Furthermore, W_1 behaves for the dimension estimates as if it descends to an intermediate cover $D' \to C$ of degree n.

Consider the case $n = 1$. Then there is only one isomorphism class of the conjugates of W_1, i.e., W_1 is G-invariant. This does not necessarily mean that W_1 is G-equivariant, that is, admits a G-linearization and descends to C, see [17, Example 2.5] for an example distinguishing these two notions. However, W_1 descends up to a twist by a line bundle, see [17, Lemma 4.3]. The following is a generalization of the estimate obtained in [17, Lemma 4.4] and uses the same techniques in the proof.

It should also be mentioned that for a fixed étale cover $\pi : D \to C$ it was shown in [12, Proposition 2.4] that the open stratum $Z^{s,r,0}(r,\pi)$ is non-empty for rank $r \geq 2$ and degree 0.

Theorem 2.8. *Let $\pi : D \to C$ be an étale Galois cover of a smooth projective curve C of genus $g_C \geq 2$. Let $r \geq 2$ and $d \in \mathbf{Z}$. Let $m, n, e \geq 1$ such that $mne = r$. Let $r = r'$ (resp. m') be the prime-to-p part of r (resp. m). Then we have the following:*

(i) $\dim(Z^{s,r,d}(n,e,\pi)) \leq nm^2(g_C - 1) + 1$.

(ii) $\dim(Z^{s,r,d}(m,\pi)) \leq rm(g_C - 1) + 1$.

(iii) If π is a prime-to-p cover, then

$$\dim(Z^{s,r,d}(m,\pi)) \leq \frac{r'}{m'} m^2(g_C - 1) + 1.$$

(iv) If π is a prime-to-p cover and $r = p^l, l \geq 1$, then $Z^{s,r,d}(n,e,\pi)$ is empty for $n \geq 2$.

Proof. Let G denote the Galois group of $D \to C$. Note that the decomposition strata are locally closed in $M_C^{ss,r,d}$, see Lemma 2.4. Thus, the dimension of $Z^{s,r,d}(n,e,\pi)$ is the same as the dimension of its closure in $M_C^{ss,r,d}$.

(i): Consider $V \in Z^{s,r,d}(n,e,\pi)$. Then we have $V_{|D} \cong \bigoplus_{i=1}^n W_i^{\oplus e}$ for some pairwise non-isomorphic stable vector bundles W_i of rank m. Let H be the stabilizer of the isomorphism class of W_1. Then $\pi' : D' := D/H \to C$ is an intermediate cover of $D \to C$ of degree n over C as G acts transitively on the isomorphism classes of the W_i.

Fix an inclusion $\iota : W_1^{\oplus e} \to V_{|D}$. Let W be the image of

$$\bigoplus_{\sigma \in H} \sigma^* W_1^{\oplus e} \xrightarrow{\oplus \varphi_\sigma^{-1} \sigma^* \iota} V_{|D},$$

where $\varphi_\sigma : V_{|D} \xrightarrow{\sim} \sigma^* V_{|D}$ denotes the G-linearization associated to V. Then W is an H-invariant saturated subsheaf of $V_{|D}$ isomorphic to $W_1^{\oplus e}$. Thus, $W \subseteq V_{|D}$ descends to a saturated subsheaf $W' \subseteq V_{|D'}$ by [17, Lemma 2.4].

Since the action of G on the isomorphism classes of the W_i is transitive, we obtain that $V_{|D}$ is isomorphic to a direct sum of conjugates of $W'_{|D}$. As $W'_{|D}$ is semistable, so is W' by Lemma 2.1.

By [17, Lemma 4.3], there exists a line bundle L on D and a vector bundle W_1' on D' such that $(W_1')_{|D} \cong L \otimes W_1$. Then W_1' is stable. Note that $L^{\otimes em}$ descends to D' as

$$\det(W_1')_{|D} \cong L^{\otimes m} \det(W_1) \text{ and } \det(W_1)^{\otimes e} \cong \det(W')_{|D}.$$

Tensoring W_1' by a line bundle of degree 1 changes the degree of W_1' by m and we can assume that W_1' has degree $d', 0 \leq d' < m$. Fixing the degree of W_1' also fixes the degree of L. Choose a line bundle L' on D' of degree $d', 0 \leq d' < m$, and denote the moduli space of semistable vector bundles of rank m and determinant L' on D' by $M_{L'}^{ss,m}$. Denote by $P(d')$ the moduli space of line bundles on D of degree f such that their me-th power descends to D' and

$$rf = d \deg(\pi) - e \deg(\pi) d'.$$

Note that if $P(d') \neq \emptyset$, then $\dim(P(d')) = g_{D'}$ as raising a line bundle to its me-th power induces a finite morphism $\mathrm{Pic}_D^f \to \mathrm{Pic}_D^{mef}$ on the level of Picard schemes (finiteness is immediate from [13, Application 2, p. 62] and the fact that $\mathrm{Pic}_D^d \cong \mathrm{Pic}_D^0$ via tensoring by some line bundle of degree d).

Consider for a finite subset $\Sigma \subseteq G$ of cardinality n the morphism

$$\varphi_{d',\Sigma,D'} : P(d') \times M_{L'}^{ss,m} \to M_D^{ss,r,d\deg(\pi)}, (L, W_1') \mapsto \bigoplus_{\sigma \in \Sigma} \sigma^* (L \otimes (W_1')_{|D})^{\oplus e}.$$

Observe that the image $Z_{d',\Sigma,D'}$ of $\varphi_{d',\Sigma,D'}$ is closed as $\varphi_{d',\Sigma,D'}$ is a morphism of projective varieties.

The above discussion shows that

$$\pi^*(Z^{s,r,d}(n,e,\pi)) \subseteq \bigcup_{d'=0}^{m-1} \bigcup_{D' \to C} \bigcup_{\Sigma \subseteq G} Z_{d',\Sigma,D'},$$

where the union is taken over intermediate covers $D \to D' \to C$ of degree n and subsets $\Sigma \subseteq G$ of cardinality n.

We can now estimate the dimension of the refined decomposition strata. By [17, Theorem 4.2], pullback by π is a finite morphism π^* and it suffices to estimate $\dim(\pi^*(Z(n,e,\pi)))$. We have

$$\dim(Z_{d',\Sigma,D'}) \leq \dim(P(d')) + \dim(M_{L'}^{ss,m}) = m^2(g_{D'} - 1) + 1.$$

We obtain

$$\dim(Z^{s,r,d}(n,e,\pi)) = \dim(\pi^*(Z^{s,r,d}(n,e,\pi))) \leq m^2 n(g_C - 1) + 1$$

by Riemann-Hurwitz.

(ii): This is a direct consequence of (i) as

$$Z^{s,r,d}(m,\pi) = \bigsqcup_{(n,e)} Z^{s,r,d}(n,e,\pi),$$

where the union is taken over $n, e \geq 1$ such that $mne = r$. Then

$$\dim(Z^{s,r,d}(m,\pi)) = \max_{n,e} \dim(Z^{s,r,d}(n,e,\pi)) \leq rm(g_C - 1) + 1$$

as the maximum is obtained at $e = 1, n = \frac{r}{m}$.

(iii): If π is prime-to-p and $Z^{s,r,d}(m,\pi)$ is non-empty, then we claim that n is prime-to-p as well. Indeed, we have seen in (i) that $n \mid \deg(\pi)$. Then the maximum in the estimate of (ii) is obtained at $e = \frac{rm'}{r'm}, n = \frac{r'}{m'}$. Thus, we conclude

$$\dim(Z^{s,r,d}(m,\pi)) \leq \frac{r'}{m'} m^2(g_C - 1) + 1.$$

(iv): If $Z^{s,r,d}(n,e,\pi)$ is non-empty, then $n \mid \deg(\pi)$. This is impossible under the assumptions of (iv). □

3. Pushforward is finite

In this section we study the pushforward of stable vector bundles under an étale cover and the induced morphism on the level of moduli spaces. This provides the groundwork for the sharpening of the dimension estimates of Theorem 2.8 to an equality: we use pushforward as a way to construct stable vector bundles in section 4. To do so, we identify the open where the pushforward is stable and show that pushforward induces a finite morphism on the level of moduli spaces of semistable vector bundles over smooth projective curves.

The following well-known lemma describes the behaviour of semistability under pushforward along étale covers. We include the short proof for completeness.

Lemma 3.1. *Let $\pi : Y \to X$ be an étale cover of a normal projective variety X of dimension ≥ 1. Then we have the following:*

(i) $\pi_* \mathcal{O}_Y$ has slope 0.

(ii) The pushforward $\pi_* W$ of a semistable vector bundle W on Y is semistable of degree $\deg(W)$.

(iii) Let V be a semistable vector bundle on X. Then $\pi_* \mathcal{O}_Y \otimes V$ is semistable of slope $\mu(V)$.

Proof. (i): If X is a curve, then $\pi_* \mathcal{O}_Y$ has degree 0 by Riemann-Hurwitz.

We claim that the general case reduces to the curve case. By Bertini's theorem the general complete intersection curve C on X is irreducible and the same holds for $D := Y \times_X C$, see [9, Corollaire 6.11 (3)]. Furthermore, the general such C is normal, see [14, Theorem 7]. As the projection $\pi' : D = Y \times_X C \to C$ is étale, D is normal as well.

We can compute the degree of $\pi_* \mathcal{O}_Y$ on C. As π is finite, we can apply affine base change to obtain $(\pi_* \mathcal{O}_Y)_{|C} \cong \pi'_* \mathcal{O}_D$ and the result follows.

(ii): This short argument can already be found in the proof of [5, Proposition 5.1] for line bundles of degree 1 on a smooth projective curve.

Let W be a semistable vector bundle of slope μ and rank r on Y. By (i) we have $\deg \pi_* \mathcal{O}_Y = 0$. Thus, the pushforward $\pi_* W$ has slope $\mu / \deg(\pi)$. If $\pi_* W$ was not semistable, consider the maximal destabilizing subsheaf V of π_*W. By adjunction $V_{|Y} \to W$ is a non-zero morphism of semistable torsion-free sheaves contradicting

$$\mu(V_{|Y}) = \deg(\pi)\mu(V) > \deg(\pi)\mu(\pi_*W) = \mu(W).$$

(iii): Let V be a semistable vector bundle on X. Using (i) we obtain

$$\mu(V) = \mu(\pi_*(\mathcal{O}_D)) + \mu(V) = \mu(\pi_*(\mathcal{O}_D) \otimes V).$$

By the projection formula, we have $\pi_*\mathcal{O}_Y \otimes V \cong \pi_*(V_{|Y})$ which is semistable by (ii) and Lemma 2.1. □

The fact that pushforward along étale covers preserves semistability, allows us to consider the induced morphism on the level of moduli spaces. Our first observation is that this morphism is a finite morphism. As in the case of pullback, [17, Theorem 4.2], this is immediate from the uniqueness of the associated graded of a JH-filtration.

Lemma 3.2. *Let $\pi : D \to C$ be an étale cover of degree n of a smooth projective curve C of genus $g_C \geq 2$. Let $r \geq 1$ and $d \in \mathbb{Z}$. Then pushforward induces a finite morphism*

$$\pi_* : M_D^{ss,r,d} \to M_C^{ss,rn,d}, W \mapsto \pi_*W.$$

Proof. By Lemma 3.1, the pushforward of a semistable vector bundle on D is semistable on C and of the same degree. As pushforward along a finite flat morphism is well-behaved in families, we obtain a morphism on the level of moduli stacks which in induces the desired morphism on the level of moduli spaces by the universal property of adequate moduli spaces, [1, Theorem 7.2.1].

To show finiteness it suffices to show quasi-finiteness as π_* is a morphism of projective varieties. Consider $V \in M_C^{ss,rn,d}$. For W semistable on D such that $\pi_*W \cong_S V$ we find that $(\pi_*W)_{|D} \cong_S V_{|D}$. As π is affine, the counit $(\pi_*W)_{|D} \to W$ is surjective. Furthermore, $(\pi_*W)_{|D}$ is semistable of the same slope as W. We conclude that W is S-equivalent to a direct sum of stable vector bundles appearing in a JH-filtration of $V_{|D}$. As the associated graded of a JH-filtration is unique, there are only finitely many possibilities for the S-equivalence class of W. We conclude that π_* is quasi-finite and thus finite. □

We next turn our attention to stability under pushforward along an étale cover $\pi : Y \to X$. The locus where the direct image of a vector bundle is stable forms an open locus in the moduli space of stable vector bundles on Y. In the Galois case this locus is closely related to vector bundles fixed by the action of the Galois group:

Lemma 3.3. *Let $\pi : Y \to X$ be an étale Galois cover of a normal projective variety X with Galois group G. Let $P \in \mathbb{Q}[x]$. Then for $\sigma \in G \setminus \{e_G\}$*

$$U_\sigma^P := \{W \in M_Y^{G-ss,P} \mid \sigma^*W \ncong_S W\} \subseteq M_Y^{G-ss,P}$$

and $U^P := \bigcap_{\sigma \in G\setminus\{e_G\}} U_\sigma \cap M_X^{s,P}$ are open. We have the following:

(i) For $W \in U^P$ the direct image π_*W is a stable vector bundle on X.

(ii) $W \in U^P$ if and only if $W \in M_Y^{s,P}$ and W does not descend to a stable vector bundle W' on an intermediate cover $Y \to Y' \to X$ such that $Y' \ncong Y$.

On a smooth projective curve, the Hilbert polynomial is determined by the rank r and degree d and we use the notation $U^{r,d}$ instead of U^P.

If $\pi : D \to C$ is an étale Galois cover of a smooth projective curve C of genus at least 2, then for $r \geq 1$ and $d \in \mathbb{Z}$ we have the following:

(iii) $U^{r,d} \subseteq M_D^{s,r,d}$ *is big.*
(iv) If d *and* $\deg(\pi)$ *are coprime, then* $U^{r,d} = M_D^{s,r,d}$.

Note that Lemma 3.3 (i) is a generalization of [5, Proposition 5.1] as well as [3, Proposition 4.1].

Proof. Let $\sigma \in G$. To see that U_σ is open, consider the pullback square

$$\begin{tikzcd} Z_\sigma \arrow[r] \arrow[d] & M_Y^{G-ss,P} \arrow[d, "(\sigma^*, \mathrm{id})"] \\ M_Y^{G-ss,P} \arrow[r, "\Delta"] & M_Y^{G-ss,P} \times_k M^{G-ss,P}. \end{tikzcd}$$

As $M_Y^{G-ss,P}$ is projective, Δ is a closed immersion and Z_σ is closed. Clearly, Z_σ is the complement of U_σ and we find the openness.

Note that $W \in U^P$ if and only if $W \in M_Y^{s,P}$ and $\sigma^*W \ncong W$ for $\sigma \in G \setminus \{e_G\}$ as for stable vector bundles S-equivalence is the same as being isomorphic.

(i): Let $W \in U^P$. Then $V := \pi_*W$ is semistable of slope $\frac{\mu(W)}{\deg(\pi)}$ by Lemma 3.1. Consider a stable saturated subsheaf $V' \subseteq V$. Then $V'_{|Y}$ is a saturated subsheaf of $\pi^*\pi_*W \cong \bigoplus_{\sigma\in G} \sigma^*W$ and thus reflexive. Furthermore, $V'_{|Y}$ is polystable of the same slope as W. As the graded object of the JH-filtration is unique, we find that $V'_{|Y} \cong \bigoplus_{\sigma\in\Sigma} \sigma^*W$ for some $\Sigma \subseteq G$. In particular, V' is a stable vector bundle. By Lemma 2.2, G acts transitively on the isomorphism classes of the $\sigma^*W, \sigma \in \Sigma$. We conclude $V'_{|Y} = V_{|Y}$ and thus $V' = V$ as for $\sigma \in G \setminus \{e_G\}$ we have $\sigma^*W \ncong W$.

(ii): Let $W \in M_Y^{s,P}$ such that $W \notin U^P$. Then there exists $\sigma \in G \setminus \{e_G\}$ such that $\sigma^*W \cong W$. Let G' be the subgroup of G generated by σ. Then G' is cyclic and non-trivial. For cyclic covers the notions of invariance and linearization agree, see [17, Lemma 4.6]. Thus, W descends to the intermediate cover Y/G'.

Conversely, if W descends to some intermediate cover $Y \to Y'$ with Galois group G' such that $Y \ncong Y'$, then there exists W' on Y' such that $W'_{|Y} \cong W$. Thus, $\sigma'^*W \cong W$ for $\sigma' \in G'$. This concludes (ii).

(iii)-(iv): Let $r \geq 1, d \in \mathbb{Z}$ and $D \to C$ be a Galois cover of a smooth projective curve C of genus ≥ 2.

(iii): Then by the alternative description given in (ii), the open $U^{r,d}$ is obtained by removing vector bundles of rank r that pullback from an intermediate cover, i.e.,

$$U^{r,d} = \bigcap_{D \xrightarrow{\pi'} D' \to C} \left(M_D^{ss,r,d} \setminus \pi'^*(M_{D'}^{ss,r,\frac{d}{\deg(\pi')}})\right) \cap M_D^{s,r,d},$$

where the intersection is taken over intermediate covers $D' \to C$ such that $D \neq D'$. If $M_{D'}^{ss,r,\frac{d}{\deg(\pi')}}$ is non-empty, then it has dimension $\frac{r^2}{\deg(\pi')}(g_D - 1) + 1$ by Riemann-Hurwitz. Thus, we find that $U^{r,d}$ is big.

(iv): If d and $\deg(\pi)$ are coprime, then the moduli spaces $M_{D'}^{ss,r,\frac{d}{\deg(\pi')}}$ considered in (iii) are empty and we obtain (iv). $\square$

In the case of curves, the locus of bundles constructed in the previous lemma is the locus of all bundles such that the direct image is stable.

Lemma 3.4. *Let $\pi : D \to C$ be an étale Galois cover of smooth projective curves with Galois group G. Let $r \geq 1$ and d be integers. For the open*

$$U^{r,d} = \{V \in M_D^{s,r,d} \mid \sigma^* V \ncong V, \text{ for all } \sigma \in G \setminus \{e_G\}\} \subseteq M_D^{s,r,d}$$

defined in Lemma 3.3 pushforward along π induces a Cartesian diagram

$$\begin{tikzcd} M_D^{ss,r,d} \arrow[r, "\pi_*"] & M_C^{ss,\#(G)r,d} \\ U^{r,d} \arrow[r, "\pi_*"] \arrow[u, hook, "\circ"] & M_C^{s,\#(G)r,d}. \arrow[u, hook, "\circ"] \end{tikzcd}$$

In particular, $\pi_* : U^{r,d} \to M_C^{s,\#(G)r,d}$ is finite.

Proof. By Lemma 3.3, we have $U^{r,d} \subseteq \pi_*^{-1}(M_C^{s,\#(G)r,d})$. For the other inclusion, let $V \in M_D^{ss,r,d}$ such that $\pi_* V$ is stable. Then V is stable, as a subsheaf $W \subseteq V$ of the same slope yields a subsheaf $\pi_*(W) \subseteq \pi_* V$ of the same slope. By the description of $U^{r,d}$ given in Lemma 3.3 (ii), we find that it suffices to show that V does not descend to an intermediate cover $D \to D' \to C$ such that $D' \ncong D$.

Assume by way of contradiction that $V \cong V'_{|D}$ for some vector bundle V' on D', where $D \xrightarrow{\pi'} D' \to C$ is an intermediate cover such that $D \ncong D'$. Then $V' \subseteq \pi'_* V$ is a proper subsheaf of the same slope. Thus, $\pi'_* V$ is not stable and the same holds for $\pi_* V$.

Finiteness is preserved under base change and we conclude by Lemma 3.2. □

4. Prime to p decomposition strata

Let C be a smooth projective curve of genus $g_C \geq 2$. By [17, Theorem 1] there exists a prime-to-p étale Galois cover $C_{r,good} \to C$ checking for prime-to-p stability, i.e., a stable vector bundle V on C remains stable after pullback to all prime-to-p étale cover if and only if $V_{|C_{r,good}}$ is stable. We denote the open dense subscheme of prime-to-p stable bundles in $M_C^{s,r,d}$ by $M_C^{p'-s,r,d}$, see [17, Theorem 2].

This cover can be iterated to obtain a cover $C_{r,split} \to C$ checking for the eventual decomposition of a stable vector bundle of rank r on C on all étale prime-to-p covers, i.e., for a stable vector bundle V of rank r on C the polystable bundle $V_{|C_{r,split}}$ has only prime-to-p stable direct summands. This gives rise to the *prime-to-p decomposition strata*.

Similarly to the decomposition strata with respect to a cover, the dimension of the prime-to-p decomposition strata can be estimated. The main difference is that these estimates are sharp as long as the characteristic is avoided. To obtain sharp estimates we find a way to construct vector bundles with prescribed prime-to-p decomposition using cyclic covers.

4.1. **A split cover.** We begin with the definition of the cover $C_{r,split}$. This still works on normal projective varieties.

Definition 4.2. Let X be a normal projective variety. Let $r \geq 2$. We define a prime-to-p Galois cover $\pi_{r,split} : X_{r,split} \to X$ inductively as follows: Set X_1 as $X_{r,good}$. Then define for $1 \leq j < r-1$ the cover X_{j+1} as a prime-to-p Galois cover dominating $(X_j)_{l,good}$ for $l \leq r-j$ and $l \mid r$. We define $X_{r,split} := X_{r-1}$.

Let V be a stable vector bundle on X with Hilbert polynomial P. The *prime-to-p decomposition type* (or just *decomposition type*) of V is the decomposition type with respect to the cover $X_{r,split} \to X$.

We call the stratification of $M_X^{s,P}$ with respect to $\pi_{r,split}$ the *prime-to-p decomposition stratification* (or just *decomposition stratification*) and denote it by

$$Z^{s,P}(m) := Z^{s,P}(m, \pi_{r,split})$$

for $m \mid r$. If X is a smooth projective curve, the Hilbert polynomial is determined by the rank r and the degree d and we use the notation $Z^{s,r,d}(m)$ instead.

The notations $X_{r,split}$ and $Z^{s,P}(m)$ are justified as the decomposition type stays the same on any prime-to-p cover dominating $X_{r,split}$ as we show in the next lemma.

Lemma 4.3. *Let X be a normal projective variety. Let $r \geq 2$. Then a stable vector bundle of rank r on X decomposes on $X_{r,split}$ into a direct sum of prime-to-p stable vector bundles.*

In particular, the decomposition types with respect to a prime-to-p Galois cover $Y \to X$ dominating $X_{r,split} \to X$ and with respect to $X_{r,split} \to X$ agree.

Proof. Let V be stable vector bundle of rank r on X. We follow the behaviour of $V_{|X_j}$ for $j = 1, \dots, r-1$ of Definition 4.2. Let r_j be the decomposition type of V with respect to $X_j \to X$. Then we have $r_{j+1} \leq r_j$ for $1 \leq j < r-1$ and equality holds if and only if $V_{|X_j}$ is a direct sum of stable vector bundles on X_j that remain stable on X_{j+1}. By construction of X_{j+1} this is the case if and only if $V_{|X_j}$ decomposes into a direct sum of prime-to-p stable vector bundles on X_j.

As there are $r-1$ covers $X_{r-1} \to \cdots \to X_1 \to X$, we find $r_{r-1} = 1$ or $r_{r-1} = r_j$ for some $j < r-1$. In both cases $V_{|X_{r-1}} = V_{|X_{r,split}}$ decomposes into a direct sum of prime-to-p stable vector bundles.

If $Y \to X$ is a prime-to-p Galois cover dominating $X_{r,split} \to X$, consider the decomposition of $V_{|X_{r,split}} \cong \bigoplus_{i=1}^n W_i^{\oplus e}$ into pairwise non-isomorphic stable vector bundles of rank m. By the above discussion each W_i is prime-to-p stable. Thus, $V_{|Y} \cong \bigoplus_{i=1}^n (W_i)_{|Y}^{\oplus e}$ is a decomposition into stable vector bundles of rank m. $\square$

Remark 4.4. *Note that the refined decomposition type might still change and is thus not independent of the cover. For example only finitely many prime-to-p trivializable stable vector bundles of rank r become trivial on $X_{r,split}$. However, there are infinitely many prime-to-p trivializable stable bundles of rank r if the prime-to-p completion of the étale fundamental group is large enough, e.g., if $X = C$ is a smooth projective curve of genus at least* 2.

4.5. **Sharp dimension estimates.** We can now give the mostly sharp estimates of the decomposition strata. This is a generalization of [17, Theorem 2]. Since we already of the estimates from Theorem 2.8, we only need a way to construct stable bundles with prescribed prime-to-p decomposition type. This can be achieved via pushforward of the general bundle along a cyclic étale cover.

Theorem 4.6. *Let C be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$ and $d \in \mathbb{Z}$. Then for $m \mid r$ we have the following:*

- *(i)* $\dim(Z^{s,r,d}(m)) \leq (\frac{r}{m})' m^2(g_C - 1) + 1$, *where $(\frac{r}{m})'$ denotes the prime-to-p part of $\frac{r}{m}$.*
- *(ii) If $p \nmid \frac{r}{m}$, then we have* $\dim(Z^{s,r,d}(m)) = rm(g_C - 1) + 1$.

Proof. (i): The estimate follows immediately from applying Theorem 2.8 to the étale prime-to-p Galois cover $C_{r,split} \to C$.

(ii): Let $n = \frac{r}{m}$ and assume that n is prime-to-p. Then étale cyclic covers of degree n correspond to line bundles of order n. Thus, there exist such étale cyclic covers. Let $\pi : D \to C$ be étale and cyclic of degree n. Denote the Galois group by G.

By Lemma 3.3 and Lemma 3.4, the locus $U \subseteq M_D^{s,m,d}$ of stable vector bundles W on D such that $\pi_* W$ is stable is open and non-empty. Let $M_D^{p'-s,m,d}$ be the locus of prime-to-p stable vector bundles on D. Recall that $M_D^{p'-s,m,d}$ is non-empty and open, see [17, Theorem 2]. As $M_D^{s,m,d}$ is irreducible, see [15, Theorem 17], the intersection $U' := U \cap M_D^{p'-s,m,d}$ is open and non-empty as well.

We claim that for $W \in U'$ the direct image $\pi_* W$ has prime-to-p decomposition type m. Indeed, by affine base change we obtain $(\pi_* W)_{|D} \cong \bigoplus_{\sigma \in G} \sigma^* W$ which is a direct sum of prime-to-p stable vector bundles.

Pushforward induces a finite morphism $U \xrightarrow{\pi_*} M_C^{s,r,d}$, see Lemma 3.4. We obtain

$$\dim(M_D^{ss,m,d}) = \dim(U') \leq \dim(Z^{s,r,d}(m)).$$

Thus, we conclude

$$\dim(Z^{s,r,d}(m)) = rm(g_C - 1) + 1$$

by Riemann-Hurwitz and (i). □

Remark 4.7. *We note that the dimension estimates can be far from being sharp if $p \mid r$. Indeed, if $r = p^n$ for some $n \geq 1$ and d is prime-to-p, then every semistable vector bundle of rank r and degree d is prime-to-p stable. In particular, the decomposition strata $Z^{s,r,d}(m)$ are empty for $m < r$.*

5. The closure of prime to p trivializable bundles

As an application of the theory developed we study the closure of the prime-to-p trivializable stable vector bundles.

In positive characteristic the étale trivializable bundles are dense by a theorem due Ducrohet and Mehta, see [6, Corollary 5.1]. This is no longer the case if we only consider prime-to-p trivializable bundles as the general bundle remains stable on all prime-to-p covers, see [17, Theorem 2]. It is natural to ask the question what the closure Z of the prime-to-p trivializable bundles is. This is closely related to the smallest prime-to-p decomposition stratum and we can estimate the dimension of Z. This estimate is sharp if the characteristic is avoided.

Definition 5.1. Let X be a normal projective variety. We call a vector bundle V of rank r on X *prime-to-p trivializable* if there exists a prime-to-p cover $Y \to X$ such that $V_{|Y}$ is the trivial bundle of rank r.

Note that as every prime-to-p cover is dominated by a prime-to-p Galois cover, we could have alternatively required that prime-to-p trivializable vector bundles become trivial on a prime-to-p Galois cover.

As a direct consequence of [10, Proposition 1.2], representations of the prime-to-p completion of the étale fundamental group correspond to prime-to-p trivializable vector bundles.

Also recall that in rank 1 the prime-to-p trivializable line bundles are dense in Pic_C^0 if C is a smooth projective curve.

Theorem 5.2. *Let C be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$. Let $Z^{s,r}$ be the closure of the prime-to-p trivializable stable vector bundles in $M_C^{s,r,0}$ and $Z^{ss,r}$ be the closure of the prime-to-p trivializable vector bundles in $M_C^{ss,r,0}$. Then we have the following:*

- *(i)* $Z^{s,r} \subseteq Z^{s,r,0}(1)$.
- *(ii)* $\dim(Z^{s,r}) \leq r'(g_C - 1) + 1$*, where r' is the prime-to-p part of r.*
- *(iii) If $p \nmid r$, then* $\dim(Z^{s,r}) = \dim(Z^{s,r,0}(1)) = r(g_C - 1) + 1$.
- *(iv)* $\dim(Z^{ss,r}) = rg_C$.

Proof. (i): The prime-to-p decomposition type of a prime-to-p trivializable bundle is 1. Thus, we obtain $Z^{s,r} \subseteq Z^{s,r,0}(1)$ using that $Z^{s,r,0}(1)$ is closed in $M_C^{s,r,0}$, see Lemma 2.4.

(ii): This is a direct consequence of (i) and the dimension estimate for the prime-to-p decomposition strata, see Lemma 2.8 (ii).

(iii): Assume that $p \nmid r$. Let $\pi : D \to C$ be a cyclic cover of order r and Galois group $G \cong \mathbf{Z}/r\mathbf{Z}$. Consider the dense open subset $U \subseteq \mathrm{Pic}_D^0$ defined as

$$U := \{L \in \mathrm{Pic}_D^0 \mid \sigma^* L \ncong L, e_G \neq \sigma \in G\},$$

see Lemma 3.3. By Lemma 3.4, pushforward induces a finite morphism

$$\pi_* : U \to M_C^{s,r,0}, L \mapsto \pi_* L.$$

As the prime-to-p trivializable line bundles are dense in Pic_D^0 and Pic_D^0 is irreducible, we find that the prime-to-p trivializable line bundles contained in U are still dense in Pic_D^0.

Note that for a prime-to-p trivializable line bundle L its conjugates $\sigma^* L, \sigma \in G$, are prime-to-p trivializable as well. Thus, the pushforward $\pi_* L$ is prime-to-p trivializable as $(\pi_* L)_{|D} \cong \bigoplus_{\sigma \in G} \sigma^* L$. Therefore, π_* factors as $U \to Z^{s,r} \to M_C^{s,r,0}$.

The image $\pi_*(U)$ in $Z^{s,r}$ has dimension $\dim(U) = g_D$ since π_* is finite. By Riemann-Hurwitz and (ii) we conclude

$$r(g_C - 1) + 1 \leq \dim(Z^{s,r}) \leq \dim(Z^{s,r,0}(1)) \leq r(g_C - 1) + 1.$$

(iv): Observe that $Z^{ss,r}$ contains the image of the finite morphism

$$\prod_{i=1}^{r} \mathrm{Pic}_C^0 \to M_C^{ss,r,0}, (L_1, \ldots, L_r) \mapsto \bigoplus_{i=1}^{r} L_i.$$

Thus, we have $\dim(Z^{ss,r}) \geq rg_C$.

To obtain the other inequality, we claim that $Z^{ss,r}$ is the union of the images of

$$\varphi_{r_1,\ldots,r_l} : \prod_{i=1}^{l} \overline{Z^{s,r_i}} \to M_C^{ss,r,0}, (V_1, \ldots, V_l) \mapsto \bigoplus_{i=1}^{l} V_i$$

for all possible ways to write $r = \sum_{i=1}^{l} r_i$ and the closure of Z^{s,r_i} is taken in $M_C^{ss,r_i,0}$. Indeed, the image of $\varphi_{r_1,\ldots,r_l}$ is closed and so is $\bigcup_{\sum_{i=1}^{l} r_i = r} \mathrm{Im}(\varphi_{r_1,\ldots,r_l})$. Furthermore, the union contains all prime-to-p trivializable bundles of rank r.

By (ii) we have $\dim(\mathrm{Im}(\varphi_{r_1,\ldots,r_l})) \leq r(g_C - 1) + l \leq rg_C$. □

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