On the finiteness of four-body central configurations

XIANG YU [*], **SHUQIANG ZHU**

Abstract

The number of central configurations in the four body problem was proved to be finite, first by Hampton and Moeckel, then by Albouy and Kaloshin, when the masses are all positive. We prove that the four-body central configurations are finite for any four nonzero masses.

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1 Introduction

We consider the finiteness of central configurations in the four-body problem. Our approach is to extend the method due to Albouy and Kaloshin. They first used this method to show that the number of central configurations in the four-body problem is finite if the masses are all positive, and that the number of central configurations in the planar five-body problem is finite, perhaps except if the 5-tuple of positive masses belongs to a given codimension 2 subvariety of the mass space. We focus on the four-body case and obtain the finiteness result without regard to the sign of the masses, which settles the open question raised by Hampton and Moeckel in [6].

The N-body problem consists of describing the complete behavior of solutions of the Newton's equations, $\ddot{\mathbf{r}}_n = \sum_{1\le j\le N, j\ne n} \frac{m_j(\mathbf{r}_j-\mathbf{r}_n)}{|\mathbf{r}_j-\mathbf{r}_n|^3}, n = 1, \cdots, N$. Started by Newton and considered by many great mathematicians through the centuries, the problem for $N > 2$ remains largely unsolved. Central configurations arose in the study of the N-body problem. By definition, they are just special arrangement of the particles. However, they play an important role in the study of the dynamics of the N-body problem. For instance, they are related to the homographic solutions, the analysis of collision orbits, and the bifurcation of integral manifold (cf. [8, 10]).

The so-called hypothesis on finiteness of central configurations was proposed by Chazy [5] and Wintner [10], and was listed by Smale as the sixth problem on his list of problems for the 21-st century [9]: Is the number of relative equilibria (planar central configurations) finite, in the N-body problem of celestial mechanics, for any choice of positive real numbers $m_1, \cdots, m_N$ as the masses?

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Euler and Lagrange have solved this finiteness question when $N = 3$. In 2005, Hampton and Moeckel [6] studied the case of $N = 4$, and showed that for any 4-tuple of positive masses the upper bound of numbers of central configurations is 8472. Albouy and Kaloshin [1] answered positively the question in 2012 for $N = 5$, except perhaps if the 5-tuple of positive masses satisfies two given polynomial conditions. We refer the reader to the excellent review on this problem by Hampton, Moeckel [6] and Albouy, Kaloshin [1].

The outline of the method of Albouy and Kaloshin is the following: The central configurations are solutions of the algebraic system (2) in the real domain. By introducing an efficient set of coordinates, Albouy and Kaloshin embedded system (2) into the system (4). The solutions of (4) in the complex domain are called *normalized central configurations*. If there are infinitely many normalized central configurations, Albouy and Kaloshin showed that one can extract a sequence of them that forms a *singular sequence*.

These singular sequences are classified according to their limiting behaviors. This classification was done by two-colored diagrams, which consists of colored strokes and circles and each circle represents a mass. For $N = 4$, they found that the number of diagrams is 5, while the number increases to 16 for $N = 5$.

They noted that for each diagram, or equivalently each class of singular sequences, there is some algebraic constraint on the masses, and another important fact that the existence of one class of singular sequence generally implies the existence of other classes. Hence, the existence of infinitely many normalized central configurations would imply several algebraic constraints on the masses. The finiteness is asserted by showing that these constraints have no solutions. In particular, this method reduced significant amount of computations.

We investigate the finiteness question concerning central configurations for any 4-tuple of nonzero masses. This pursuit is particularly valuable for two reasons. Firstly, the inclusion of negative masses sheds light on the role of masses in the N-body problem. In the planar five-body problem, Roberts [7] (1999) has demonstrated the construction of a continuum of central configurations when negative masses are allowed. It prompts a natural inquiry into whether such a continuum is possible for the four-body problem when negative masses are permitted, an open question raised by Hampton and Moeckel [6] in 2006. Secondly, the method introduced by Albouy and Kaloshin is robust, and its applicability extends naturally to finiteness questions under weaker parameter constraints. In particular, adapted versions of this approach have been used in the four-vortex problem [11] and the five-vortex problem [13], while the original method has also been applied to the six-body problem [3, 4]. The present work further illustrates the potential of this method.

The primary discussions and results of Albouy and Kaloshin [1] center around the case of positive masses. Although some of their partial results extend to scenarios involving negative masses, the assumption of mass positiveness is crucial for their main findings. Firstly, the concept of centers of mass of clusters plays a pivotal role in their study, so it is presumed that *the subset of bodies has total mass nonzero*. Secondly, the positivity of masses simplifies the constraints in many diagrams, and, in some cases, excludes certain possibilities outright.

As our study addresses the finiteness question for any 4-tuple of nonzero masses, our approach differs somewhat. In particular, we need to modify or abandon certain rules in the classification of singular sequences (refer to Section 3.2). Consequently, more diagrams emerge, increasing the count to 11. The corresponding constraints on masses become more subtle, even for classes that also appear in Albouy and Kaloshin's classification. We conduct separate proofs for cases involving nonzero total mass and cases involving zero total mass.

The results proved here are stated in the following

Theorem 1.1. *Suppose that m_1, m_2, m_3, m_4 are real and nonzero, then there are finitely many central configurations in the four-body problem.*

In fact, we find that the number of solutions in the complex domain (i.e., the normalized central configurations) is finite, except for (up to a relabeling of the bodies) two groups of masses:

$$m_1 = m_2 = -m_3 = -m_4, \qquad m_1 = m_2, \quad m_3 = -\alpha^2 m_1, \quad m_4 = -(2-\alpha^2)m_1,$$

where α is the unique positive root of equation (51) (numerically, $\alpha \approx 1.2407$). See Theorems 5.1, 5.2, and 6.1.

The paper is structured as follows. In Section 3, we briefly review the notation of singular sequences due to Albouy and Kaloshin, the rules to classify the diagrams corresponding to singular sequences are listed. We remark that some rules are modified. In Section 4, we list all possible diagrams for the four-body problem. In Section 5 and 6, we outline the proofs of Theorem 1.1. In the Appendix, we provide comprehensive details of the proof, specifically when manual computation is straightforward. For more intricate calculations, we employed symbolic computations, and the exhaustive details can be referenced in the accompanying Mathematica notebook [12].

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2 Basic notations

A solution of the N-body problem is *homographic* if the configurations formed by the bodies remain similar to each other at all times. In the context of the N-body problem with nonzero masses, the set of homographic solutions, exhibits a richer structure compared to that of the N-body problem with positive masses. We refer to configurations that result in homographic solutions as "stationary configurations". Notably, the set of central configuration forms a subset within the broader category of stationary configurations, as expected.

Let $m_n, \mathbf{r}_n, n = 1, \cdots, N$, be the mass and position of the n-th body respectively. The equations of motion are $\ddot{\mathbf{r}}_n = A_n$, where

$$A_n = \sum_{1 \le j \le N, j \ne n} \frac{m_j(\mathbf{r}_j - \mathbf{r}_n)}{|\mathbf{r}_j - \mathbf{r}_n|^3}, n = 1, \cdots, N.$$

Definition 2.1. *The following quantities are defined. Total mass:* $m = \sum_{j=1}^N m_j$*; Moment of mass:* $M = \sum_{j=1}^N m_j \mathbf{r}_j$.

It is easy to see that configurations leading to homographic solutions satisfy the following definition.

Definition 2.2. *A configuration* $\mathbf{r} = (\mathbf{r}_1, \cdots, \mathbf{r}_N)$ *is called a stationary configuration if there exists a constant* $\lambda \in \mathbb{R}$ *such that*

$$A_j - A_k = \lambda(\mathbf{r}_j - \mathbf{r}_k), \qquad 1 \leq j, k \leq N.$$

There are several cases.

- If $\lambda = 0$ and $A_1 = \cdots = A_N = 0$, it is easy to see that

$$\mathbf{r}_j(t) = \mathbf{r}_j + t\mathbf{a}, \quad j = 1, \cdots, N,$$

  where $\mathbf{a}$ is some arbitrary nonzero constant vector, is one solution. In particular,

$$\mathbf{r}_j(t) = \mathbf{r}_j, \quad j = 1, \cdots, N,$$

  is one solution. In this case, it holds that

$$\sum_{1 \leq j < k \leq N} \frac{m_j m_k}{|\mathbf{r}_j - \mathbf{r}_k|} = -\sum_{j=1}^N m_j \mathbf{r}_j A_j = 0.$$

- If $\lambda = 0$ and $A_1 = \cdots = A_N = \mathbf{b} \neq 0$, it is easy to see that

$$\mathbf{r}_j(t) = \mathbf{r}_j + t\mathbf{a} + \frac{t^2}{2}\mathbf{b} \quad j = 1, \cdots, N,$$

  where $\mathbf{a}$ is some arbitrary nonzero constant vector, is one solution. In this case, we have

$$\sum_{j=1}^N m_j A_j = 0 \Rightarrow m = 0.$$

- If $\lambda \neq 0$ and the total mass $m \neq 0$. Define the center of mass as $\mathbf{c}_m = \frac{\sum_{j=1}^N m_j \mathbf{r}_j}{m}$. Assume that

$$A_1 - \lambda \mathbf{r}_1 = \cdots = A_N - \lambda \mathbf{r}_N = \mathbf{b}.$$

  By summing them up, we obtain $-\lambda \mathbf{c}_m = \mathbf{b}$. Let $\tilde{\mathbf{r}}_j = \mathbf{r}_j - \mathbf{c}_m$. Thus, it holds that

$$\tilde{A}_j = \lambda \tilde{\mathbf{r}}_j, \quad j = 1, \cdots, N,$$

  where $\tilde{A}_j, j = 1, \cdots, N$, is A_j evaluated at $(\tilde{\mathbf{r}}_1, \cdots, \tilde{\mathbf{r}}_N)$.

- If $\lambda \neq 0$ and the total mass $m = 0$. Assume that

$$A_1 - \lambda \mathbf{r}_1 = \cdots = A_N - \lambda \mathbf{r}_N = \mathbf{c}.$$

By summing them up, we obtain $\sum_{j=1}^N m_j \mathbf{r}_j = 0$. Let $\tilde{\mathbf{r}}_j = \mathbf{r}_j + \frac{\mathbf{c}}{\lambda}$. Thus, it holds that

$$\tilde{A}_j = \lambda \tilde{\mathbf{r}}_j, \quad j = 1, \cdots, N,$$

where $\tilde{A}_j, j = 1, \cdots, N$, is A_j evaluated at $(\tilde{\mathbf{r}}_1, \cdots, \tilde{\mathbf{r}}_N)$.

Based on the above discussions, stationary configurations can be divided into the following three classes:

Definition 2.3. *i. The stationary configuration* $\mathbf{r}$ *is an* ***equilibrium*** *if* $A_1 = \cdots = A_N = 0$.

ii. The stationary configuration $\mathbf{r}$ *is a* ***rigidly translating configuration*** *if* $A_1 = \cdots = A_N = \mathbf{b}$, *where* $\mathbf{b}$ *is some nonzero constant vector.*

iii. The stationary configuration $\mathbf{r}$ *is a* ***central configuration*** *if* $A_j = \lambda \mathbf{r}_j, j = 1, \cdots, N$, *where* λ *is some nonzero constant.*

It is important to note that our definition of a central configuration aligns with the conventional definition in [10]. **The primary emphasis of this paper lies in the finiteness of central configurations.** A configuration, denoted as $\mathbf{r}$, is considered equivalent to $\mathbf{r}'$ if the two configurations coincide under transformations involving translation, rotation, and dilation. Our counting of central configurations is conducted with respect to these equivalence classes.

3 Singular sequences and two-colored diagrams

In this section, we briefly review the basic elements of the method introduced by Albouy and Kaloshin [1], including, among others, the coordinate system, the notation of singular sequences, the two-colored diagrams, and the rules for the diagrams. We remark that it is presumed that the sum, and partial sum of the masses are nonzero in [1] and that here this hypothesis is discarded. As a result, there may be no center of mass. Hence, Rule 1c, 1d, and 1e of [1] are modified into our Rule III, IV, and V, and Rule 2a of [1] has been dropped.

3.1 Singular sequences

Recall that a planar central configuration is a solution of the following algebraic equations

$$\lambda \mathbf{r}_k = \sum_{1 \le j \le N, j \neq k} \frac{m_j (\mathbf{r}_j - \mathbf{r}_k)}{|\mathbf{r}_j - \mathbf{r}_k|^3}, \tag{1}$$

where $\mathbf{r}_k = (x_k, y_k) \in \mathbb{R}^2$, $k = 1, ..., N$. Equations (1) is invariant under rotations and dilations in the plane. Hence, we may assume that the multiplier equals 1 or -1. Set $\sigma = \pm 1, x_{kl} = x_l - x_k$, $y_{kl} = y_l - y_k$ and $r_{kl} = (x_{kl}^2 + y_{kl}^2)^{1/2} > 0$. Equations (1) becomes

$$\sigma \begin{pmatrix} x_k \\ y_k \end{pmatrix} = \sum_{j=1, j\neq k}^{N} m_j r_{jk}^{-3} \begin{pmatrix} x_{jk} \\ y_{jk} \end{pmatrix}, k = 1, ..., N, y_1 = y_2. \tag{2}$$

The last equation $y_1 = y_2$ is to remove the rotation symmetry. The system (2) is then embedded into a polynomial system in $\mathbb{C}^{2N} \times \mathbb{C}^{N(N-1)/2}$:

$$\begin{aligned} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} &= m_2 \delta_{12}^3 \begin{pmatrix} x_{21} \\ y_{21} \end{pmatrix} + m_3 \delta_{13}^3 \begin{pmatrix} x_{31} \\ y_{31} \end{pmatrix} + ... \\ &... \\ &\delta_{12}^2 (x_{12}^2 + y_{12}^2) = 1 \\ &... \\ &y_{12} = 0. \end{aligned} \tag{3}$$

Definition 3.1 (Normalized central configuration)**.** *A normalized central configuration is a solution of (3).*

Note that a central configuration, i.e., solution of equations (1) is a normalized central configuration with real x_k's, y_k's, and positive δ_{jk}'s. Though the variables of system (3) are x_k, y_k and δ_{jk}, we may use the variables $r_{jk} = 1/\delta_{jk}$ instead of the δ_{jk}'s.

Set $z_k = x_k + \mathbf{i} y_k$, $w_k = x_k - \mathbf{i} y_k, z_{jk} = z_k - z_j, w_{jk} = w_k - w_j$. Then $x_{jk}^2 + y_{jk}^2 = r_{jk}^2 = z_{jk} w_{jk}$. Set $Z_{jk} = z_{jk}^{-1/2} w_{jk}^{-3/2}, W_{jk} = z_{jk}^{-3/2} w_{jk}^{-1/2}$. Here, expressions such as $z_{jk}^{-1/2} w_{jk}^{-3/2}$ are interpreted in the same way as in [1]; namely, $z_{jk}^{-1/2} w_{jk}^{-3/2} = r_{jk}^{-1} w_{jk}^{-1}$. *The z_{jk}'s (respectively the w_{jk}'s) in the complex plane will be called z-separation (respectively w-separation).* Then the system (3) becomes

$$\begin{aligned} z_1 &= m_2 Z_{21} + m_3 Z_{31} + ... + m_N Z_{N1}, \\ w_1 &= m_2 W_{21} + m_3 W_{31} + ... + m_N W_{N1}, \\ &... \\ z_N &= m_1 Z_{1N} + m_2 Z_{2N} + ... + m_{N-1} Z_{(N-1)N}, \\ w_N &= m_1 W_{1N} + m_2 W_{2N} + ... + m_{N-1} W_{(N-1)N}, \\ z_{12} &= w_{12}. \end{aligned} \tag{4}$$

Note that we have $r_{jk} = r_{kj} = \sqrt{z_{jk} w_{jk}} = 1/\sqrt[4]{Z_{jk} W_{jk}}$ and

$$Z_{jk} = z_{jk}/r_{jk}^3, W_{jk} = w_{jk}/r_{jk}^3, Z_{jk} = -Z_{kj}, W_{jk} = -W_{kj}$$

Let $\mathcal{N} = N(N+1)/2$. Any vector $\mathcal{Q} = (z_1, z_2, \cdots, z_N, w_1, w_2, \cdots, w_N, \delta_{12}, \delta_{13}, \cdots, \delta_{(N-1)N})$, associates with two vectors in $\mathbb{C}^{\mathcal{N}}$

$$\begin{aligned} \mathcal{Z} &= (z_1, z_2, \cdots, z_N, Z_{12}, Z_{13}, \cdots, Z_{(N-1)N}), \\ \mathcal{W} &= (w_1, w_2, \cdots, w_N, W_{12}, W_{13}, \cdots, W_{(N-1)N}). \end{aligned}$$

Let $\|\mathcal{Z}\| = \max_{j=1,2,\cdots,\mathcal{N}} |\mathcal{Z}_j|$ be the modulus of the maximal component of the vector $\mathcal{Z} \in \mathbb{C}^{\mathcal{N}}$. Similarly, set $\|\mathcal{W}\| = \max_{k=1,2,\cdots,\mathcal{N}} |\mathcal{W}_k|$.

One important feature of System (4) is the symmetry: if $\mathcal{Z}, \mathcal{W}$ is a solution, so is $a\mathcal{Z}, a^{-1}\mathcal{W}$ for any $a \in \mathbb{C}\backslash\{0\}$. Thus, we can replace the normalization $z_{12} = w_{12}$ in System (4) by $\|\mathcal{Z}\| = \|\mathcal{W}\|$. Hence, the system becomes

$$\begin{aligned} z_1 &= m_2 Z_{21} + m_3 Z_{31} + ... + m_N Z_{N1}, \\ w_1 &= m_2 W_{21} + m_3 W_{31} + ... + m_N W_{N1}, \\ &... \\ z_N &= m_1 Z_{1N} + m_2 Z_{2N} + ... + m_{N-1} Z_{(N-1)N}, \\ w_N &= m_1 W_{1N} + m_2 W_{2N} + ... + m_{N-1} W_{(N-1)N}, \\ \|\mathcal{Z}\| &= \|\mathcal{W}\|. \end{aligned} \tag{5}$$

Consider a sequence $\mathcal{Q}^{(n)}$, $n = 1, 2, \cdots$, of solutions of (5), which are of course normalized central configurations. Take a sub-sequence such that the maximal component of $\mathcal{Z}^{(n)}$ is fixed, i.e., there is a $j \in \{1, 2, \cdots, \mathcal{N}\}$ that is independent of n such that $\|\mathcal{Z}^{(n)}\| = |\mathcal{Z}_j^{(n)}|$. Extract again in such a way that the sequence $\mathcal{Z}^{(n)}/\|\mathcal{Z}^{(n)}\|$ converges. Extract again in such a way that the maximal component of $\mathcal{W}^{(n)}$ is fixed. Finally, extract in such a way that the sequence $\mathcal{W}^{(n)}/\|\mathcal{W}^{(n)}\|$ converges.

If the initial sequence has the property that $\mathcal{Z}^{(n)}$ is unbounded, so is the extracted sequence. Note that $\|\mathcal{Z}^{(n)}\|$ is bounded away from zero: if the first N components of the vector $\mathcal{Z}^{(n)}$ all approach zero, so do the components of $\mathcal{W}^{(n)}$, then the denominators of the components Z_{kl}, W_{kl} approach zero, hence $\mathcal{Z}^{(n)}$ and $\mathcal{W}^{(n)}$ are unbounded.

Definition 3.2 (Singular sequence)**.** *Consider a sequence of normalized central configurations with the property that $\mathcal{Z}^{(n)}$ is unbounded. A sub-sequence extracted by the above process is called a* singular sequence*.*

Lemma 3.1 ([1])**.** *Let X be a closed algebraic subset of $\mathbb{C}^{\mathcal{N}}$ and $f : \mathbb{C}^{\mathcal{N}} \mapsto \mathbb{C}$ be a polynomial. Either the image $f(X) \in \mathbb{C}$ is a finite set, or it is the complement of a finite set. In the second case one says that f is dominating.*

We define the Newtonian potential and the moment of inertia by

$$U = \sum_{1 \le j < k \le N} m_j m_k \, \delta_{jk}, \qquad I = \sum_{j=1}^{N} m_j (x_j^2 + y_j^2).$$

When $m \neq 0$, it holds that $I = \frac{\sum_{1 \le j < k \le N} m_j m_k r_{jk}^2}{m}$.

Lemma 3.2 ([1])**.** *Consider the closed algebraic subset $\mathcal{A} \in \mathbb{C}^{2N} \times \mathbb{C}^{N(N-1)/2}$ defined by system* (3) *and the polynomial functions U and I on it. Then $U(\mathcal{A})$ and $I(\mathcal{A})$ are finite sets.*

3.2 The two-colored diagrams

For two sequences of non-zero numbers, a, b, we use $a \sim b$, $a \prec b$, $a \preceq b$, and $a \approx b$ to represent "$a/b \to 1$", "$a/b \to 0$", "a/b is bounded" and "$a \preceq b$, $a \succeq b$" respectively.

Recall that a singular sequence satisfies the property $\|\mathcal{Z}^{(n)}\| = \|\mathcal{W}^{(n)}\| \to \infty$. Set $\|\mathcal{Z}^{(n)}\| = \|\mathcal{W}^{(n)}\| = 1/\epsilon^2$, then $\epsilon \to 0$. The *two-colored diagram* were introduced to classify the singular sequences. Given a singular sequence, the indices of the bodies will be written down. The first color, called the z-color, is used to mark the maximal order components of $\mathcal{Z}$. If $z_k \approx \epsilon^{-2}$, draw a z-circle around vertex $\mathbf{k}$; If $Z_{jk} \approx \epsilon^{-2}$, draw a z-stroke between vertices $\mathbf{k}$ and $\mathbf{j}$. They consist the z-diagram. The second color, called the w-color, is used to mark the maximal order components of $\mathcal{W}$ in similar manner. Then we also have the w-diagram. The two-colored diagram is the combination of the z-diagram and the w-diagram.

If there is either a z-stroke, or a w-stroke, or both between vertex $\mathbf{k}$ and vertex $\mathbf{l}$, we say that there is an edge between them. There are three types of edges, z-edges, w-edges and zw-edges, see Figure 1.

Figure 1: On the left, vertices **1, 2** are z-circled, and a z-edge is between them; In the middle, vertices **1, 2** are z- and w-circled, and a zw-edge is between them; On the right, vertices **1, 2** are w-circled, and a w-edge is between them.

The following concepts were introduced to characterize some features of singular sequences. In the z-diagram, vertices $\mathbf{k}$ and $\mathbf{l}$ are called *z-close*, if $z_{kl} \prec \epsilon^{-2}$; a z-stroke between vertices $\mathbf{k}$ and $\mathbf{l}$ is called a *maximal z-stroke* if $z_{kl} \approx \epsilon^{-2}$; a subset of vertices are called *an isolated component of the z-diagram* if there is no z-stroke between a vertex of this subset and a vertex of its complement.

At the limit when following a singular sequence, the z_k's form *clusters*. For example, if bodies 1, 2 and 3 are such that $z_{12} \prec z_{23}$, we say that 1 clusters with 2 in z-coordinate, relatively to the bodies 1,2,3. If there is a fourth body such that $z_{24} \prec z_{12} \prec z_{23}$, we say the fourth body form a sub-cluster, e.g., together with body 2. We can write a *clustering scheme* in each coordinate. For instance, the situation considered above is simply z : 42.1...3, where three dots means the largest separation within the group, one dot means the intermediate separation, and no dot means the smallest separation.

These concepts also apply to w-diagram. A subset of bodies is called *an isolated component* if it is an isolated component of both the z-diagram and the w-diagram.

Proposition 3.1 ([1] Estimate 1). *For any (k, l), $1 \leq k < l \leq N$, we have $\epsilon^2 \preceq z_{kl} \preceq \epsilon^{-2}$, $\epsilon^2 \preceq w_{kl} \preceq \epsilon^{-2}$ and $\epsilon \preceq r_{kl} \preceq \epsilon^{-2}$. There is a zw-edge between $\mathbf{k}$ and $\mathbf{l}$ if and only if $r_{kl} \approx \epsilon$. There is a maximal z-edge between $\mathbf{k}$ and $\mathbf{l}$ if and only if $w_{kl} \approx \epsilon^2$.*

Proposition 3.2 ([1] Estimate 2). *If there is a z-stroke between $\mathbf{k}$ and $\mathbf{l}$. Then*

$$\epsilon \preceq r_{kl} \preceq 1, \ \epsilon \preceq z_{kl} \preceq \epsilon^{-2}, \ \epsilon \succeq w_{kl} \succeq \epsilon^2.$$

Under the same hypothesis the "equality cases" are characterized as follows:

$$\begin{aligned}\text{Left} : r_{kl} \approx \epsilon \Leftrightarrow z_{kl} \approx \epsilon \Leftrightarrow w_{kl} \approx \epsilon \Leftrightarrow zw-\text{edge between } k \text{ and } l,\\ \text{Right} : r_{kl} \approx 1 \Leftrightarrow z_{kl} \approx \epsilon^{-2} \Leftrightarrow w_{kl} \approx \epsilon^{2} \Leftrightarrow \text{maximal } z-\text{edge between } k \text{ and } l.\end{aligned}$$

The following twelve rules for the two-colored diagrams are valid if "z" and "w" were switched. Rules I to V are about one color, while Rules VI to XII are about two colors. Rules I and II are identical to Rules 1a and 1b in [1]. Rule III, IV and V correspond to Rule 1c, 1d and 1e in [1], but weaker due to the possibility of negative masses and clusters with zero total mass. Meanwhile, Rules VI to XII are identical to Rules 2b to 2h in [1]. Note that Rule 2a of [1], which relies on Rule 1e, has no counterpart in our case.

Rule I: At each end of any z-stroke, there is another z-stroke or/and a z-circle drawn around the name of the body. A z-circle is not isolated; a z-stroke must emanate from it. There exist at least one z-stroke in the z-diagram.

Rule II: If bodies **k** and **l** are z-close, they are both z-circled or both not z-circled.

Rule III: The moment of mass of a set of bodies forming an isolated component of the z-diagram is z-close to the origin.

Proof. Let the bodies of this isolated component be numbered $1, 2, \ldots, p$. We obtain

$$\begin{aligned}m_1 z_1 + m_2 z_2 + \cdots + m_p z_p &= \sum_{k=1}^{p} \sum_{1 \le l \le p, l \ne k} m_k m_l Z_{lk} + \sum_{k=1}^{p} \sum_{p < l \le N} m_k m_l Z_{lk}\\ &= \sum_{k=1}^{p} \sum_{p < l \le N} m_k m_l Z_{lk} \prec \epsilon^{-2}.\end{aligned}$$

□

Rule IV: Consider the z-diagram or an isolated component of it. If there is a z-circle, there is another one. The z-circled bodies can not all be z-close together except that the sum of the masses of these bodies is zero.

Proof. These are from the center of mass equation $m_1 z_1 + \cdots + m_N z_N = 0$ or Rule III $m_1 z_1 + m_2 z_2 + \cdots + m_p z_p \prec \epsilon^{-2}$ for an isolated component. □

Rule V: Consider a maximal z-stroke. At least one of the two ends is z-circled.

Proof. If the z-stroke kl is maximal, then $z_{kl} \approx \epsilon^{-2}$, and so either $z_k \approx \epsilon^{-2}$ or $z_l \approx \epsilon^{-2}$. At least one of the ends is circled. □

Rule VI: If there are two consecutive zw-edges, there is a third zw-edge closing the triangle.

Rule VII: Suppose that there is an edge from vertex 1 to vertex 2, an edge from vertex 2 to vertex 3, and that there is no edge from vertex 1 to vertex 3. Then the clustering schemes are $z : 1.2...3, w : 1...2.3$, or $z : 1...2.3, w : 1.2...3$. We say there is "skew clustering".

Rule VIII: Consider a cycle of edges, the list of z-separations corresponding to the edges, and the maximal order of the z-separations within this list. Two or more of the z-separations are of this order. The corresponding edges have the same type. If there are only two, the corresponding separations are not only of the same order, but equivalent.

Rule IX: Consider a triangle of edges in the diagram. Then the edges are of the same type (all z-edges or all w-edges or all zw-edges), all the z-separations are of the same order, all the w-separations are of the same order.

Rule X: Suppose that in the diagram there is a triangle of edges, plus a fourth vertex attached to the triangle by at least two edges, plus a fifth vertex attached to the four previous vertices by at least two edges, and so on up to a p-th vertex, $p \geq 3$. Then there is indeed an edge, of the same type, between any pair of the p vertices, all the z-separations are of the same order, all the w-separations are of the same order.

Rule XI: If four edges form a quadrilateral, then the opposite edges are of the same type.

Rule XII: Let (k_0, l_0) be such a pair of bodies that $r_{k_0 l_0} \preceq r_{kl}$ for any $k, l, 1 \leq k < l \leq n$. If $r_{k_0 l_0} \to 0$, then there is another pair of bodies (k_1, l_1) such that $r_{k_0 l_0} \approx r_{k_1 l_1}$.

Corollary 3.1 ([1]). *1. Two consecutive z-edges cannot be maximal if they are not part of a triangle of edges.*

2. Consider three vertices. There are 6, 3, 2, 1, or 0 strokes joining them. If there are three forming a triangle, they are of the same color.

3. If a zw-edge is present in the diagram, there is another one.

The *circling method* of [1] still applies. By Estimate 2 and Rule II, a z-stroke between bodies $\mathbf{k}$ and $\mathbf{l}$ forces these two vertices to be w-close, and therefore either both are w-circled or neither is. Furthermore, by Rule I, the endpoints of any chain of consecutive z-strokes must be z-circled.

4 Classification of 4-body diagrams

In this section, we identify all possible diagrams for $N = 4$. There are five diagrams for the four-body central configurations in [1]. Since fewer rules apply in our setting, more diagrams appear. In the course of the analysis we initially obtain 14 admissible diagrams. Three of them can be excluded by additional arguments, leaving 11 diagrams in total. We then classify the remaining diagrams according to whether they are compatible with nonzero total mass or zero total mass.

4.1 Initial exclusion of diagrams

In this subsection, we find 14 possible diagrams for the four-body central configurations. We adopt the approach introduced in [1]. It is to examine the *bicolored vertex*, which is a vertex which connects at least a z-stroke with at least a w-stroke, with maximal adjacent strokes. Define C to be the maximal number of strokes from a bicolored vertex.

4.1.1 No bicolored vertex

If there does not exist a bicolored vertex, then there are at most two strokes and they are isolated.

Hence, there is only *one* possible diagram, the first one in Figure 2.

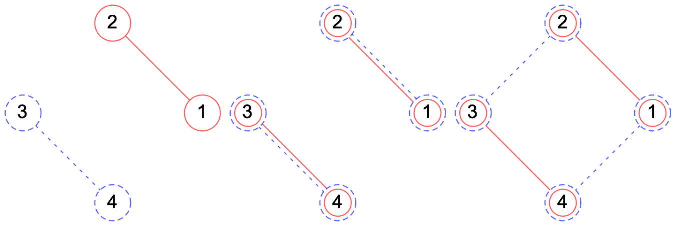


Figure 2: The first diagram is for no bicolored vertex. The next two diagrams are for $C = 2$.

4.1.2 $C = 2$

There are two cases: a zw-edge exists or not.

If it is present, it should be isolated. On the other hand, there should be another zw-edge by Corollary 3.1. Then the only possible diagram is the second one in Figure 2.

If it is not present, there are adjacent z-edges and w-edges. From any such adjacency there is no other edge. By continuing it, we see that the only diagram is the third one in Figure 2.

Hence, there are *two* possible diagrams, the second and third one in Figure 2.

4.1.3 $C = 3$

Consider a bicolored vertex with three strokes. There are two cases: a zw-edge exists or not.

If it is not present, suppose that vertex **1** connects with vertex **2** and vertex **3** by z-edges, and connects with vertex **4** by a w-edge. By Rule I, vertex **1** is w-circled, then vertex **2** and vertex **3** are also w-circled by the circling method. By Rule I again, there is a w-stroke emanating from vertex **2** and vertex **3**, which leads to a triangle with edges of different types. We exclude this case by Rule IX.

If it is present, let vertex **2** be the bicolored vertex with three strokes. Suppose it connects with vertex **1** by a zw-edge, with vertex **3** by a z-edge. By the circling method, we circle the three vertices by w-color. Then there is w-stroke from vertex **3**, which must connect to **4** by Rule IX.

There are two possibilities: either an edge between vertices **1** and **4** is present, or it is not. If this edge is absent, then we continue it to the first one of Figure 3. If such edge is present, then by Rule IX there can be no stroke between **1** and **3**, nor between **2** and **4**; in other words, these four vertices form a quadrilateral. Rule XI then implies that the admissible edges are exactly the one shown in the second diagram in Figure 3. Applying the circling method yields the last three diagrams in Figure 3.

Hence, there are *four* possible diagrams, as shown in Figure 3.

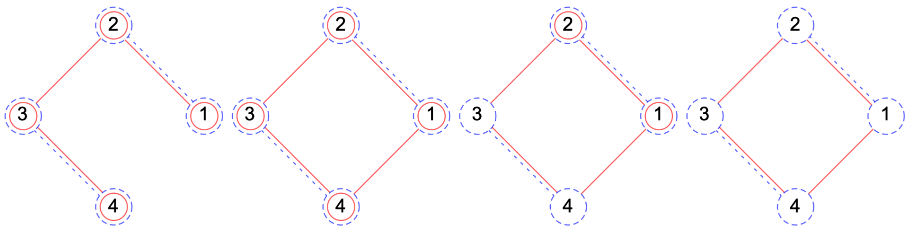


Figure 3: Four diagrams for $C = 3$. The first one and the third one will be excluded in Section 4.2.

4.1.4 $C = 4$

Consider a bicolored vertex (let us say, **1**) with four strokes.

In the first case, vertex **1** has zw-edges connected with vertex **2** and vertex **3** separately. A third zw-edge closes the triangles by Rule VI. As $C = 4$, there is no other stroke, thus vertex **4** is neither z-circled nor w-circled. If vertex **1** is not circled, we obtain the first diagram in Figure 4. If vertex **1** is circled with a single color (say, z-circled), then the admissible diagram is the second one in Figure 4. If vertex **1** is circled with both colors, the admissible diagram is the third one in Figure 4.

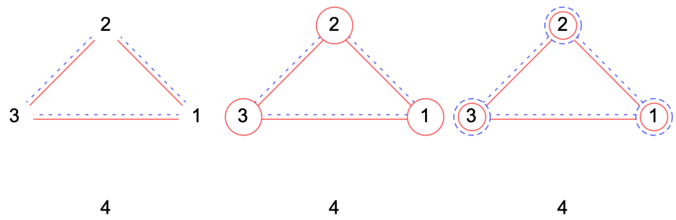


Figure 4: Three diagrams for $C = 4$. The second one will be excluded in Section 4.2.

In the second case, vertex **1** has one adjacent zw-edge connected with vertex **2**, a z-edges connected with vertex **3** and a w-edges connected with vertex **4**. Any other edge in this diagram would close a triangle, which would contradict Rule IX. By the circling method, vertex **2** is z and w-circled, and so is vertex **1**, so that vertex **3** is w-circled and vertex **4** is z-circled. Then there is a w-edge emanating from **3** by Rule I, a contradiction.

In the third case, vertex **1** has one adjacent zw-edge connected with vertex **2**, two z-edges connected with vertex **3** and **4**. If there are more strokes, it should be a z-stroke between vertex **3** and **4** by Rule IX. However, **2** is z and w-circled, and so is vertex **1** by Rule II. Then both of vertex **3** and **4** are w-circled by Rule II. Then there should be a w-stroke emanating from **3**, a contradiction.

Hence, there are *three* possible diagrams, as shown in Figure 4.

4.1.5 $C = 5$

Consider a bicolored vertex (let us say, **1**) with five strokes.

Suppose vertex **1** has one adjacent w-edge connected with vertex **4**, and two adjacent zw-edges connected with vertex **2** and vertex **3**. Then there is a fully zw-edged triangle between vertexes **1,2,3** by Rule VI. Rule IX implies that there are no more edges. By Rule I and IV, all vertices are w-circled.

There is no z-circle, otherwise all vertices are z-circled by Rule IV and II. Then there would be a z-edge emanating from vertex **4**, a contradiction.

Hence, there is only *one* possible diagram, as shown in Figure 5.

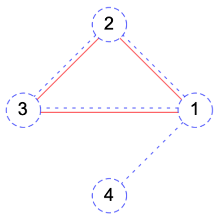


Figure 5: One diagram $C = 5$

4.1.6 $C = 6$

Consider a bicolored vertex with six strokes. Then this is a fully zw-edged diagram by Rule VI. If vertex **1** is not circled, we obtain the first diagram in Figure 6. If vertex **1** is circled with a single color (say, z-circled), then the admissible diagram is the second one in Figure 6. If vertex **1** is circled with both colors, the admissible diagram is the third one in Figure 6.

Hence, there are *three* possible diagrams, as shown in Figure 6.

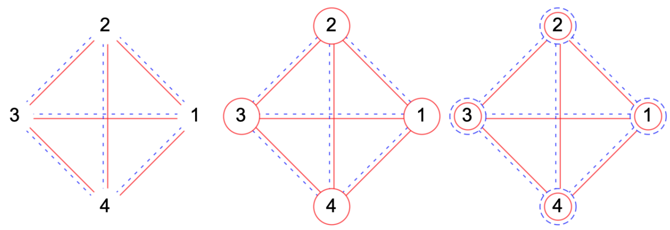


Figure 6: Three diagrams for $C = 6$

Summary: In the searching for all problematic four-body two-colored diagrams, we have found 14 of them, as shown in Figure 2-6.

4.2 Further exclusion and classifications of diagrams

Using further considerations, we exclude the first and third diagrams in Figure 3, and the second diagram in Figure 4. We then classify the remaining 11 diagrams into two lists: those compatible with a nonzero total mass, and those compatible with zero total mass.

The first and third diagrams in Figure 3 are impossible. For both diagrams, applying Rule IV, we obtain $m_1 + m_2 = 0$.

For the first diagram, it holds that

$$z_1 \sim m_2 Z_{21},\ z_2 \sim m_1 Z_{12} + m_3 Z_{32} \sim m_2 Z_{21} + m_3 Z_{32}.$$

This contradicts with $z_1 \sim z_2$, so the first diagram is impossible.

For the third diagram, if $j \in \{3,4\}, k \in \{1,2\}$, it holds that $z_{jk} \approx \epsilon^{-2}, r_{jk} \succ \epsilon$. Then $W_{jk} = z_{jk}^{-1} r_{jk}^{-1} \prec \epsilon$ and hence

$$w_{12} = m_3(W_{32} - W_{31}) + m_4(W_{42} - W_{41}) \prec \epsilon$$

This contradicts with $w_{12} \approx \epsilon$, so the third diagram is impossible.

The second diagram in Figure 4 is impossible. First, Rule IV implies that $m_1 + m_2 + m_3 = 0$ and that the total mass is not zero. The equations $w_k = \sum_{j \neq k} m_j W_{jk}, k = 1, 2, 3$, imply

$$\frac{W_{12}}{m_3} \sim \frac{W_{23}}{m_1} \sim \frac{W_{31}}{m_2} \sim a\epsilon^{-2}, \quad a \neq 0. \tag{6}$$

Note that

$$-m_4 w_4 = \sum_{j=1}^{3} m_j w_j = m_2 w_{12} + m_3 w_{13} = m_1 w_{21} + m_3 w_{23} = m_1 w_{31} + m_2 w_{32} \preceq \epsilon.$$

We claim that $w_4 \approx \epsilon$. Otherwise, the above equation leads to

$$\frac{w_{12}}{m_3} \sim \frac{w_{23}}{m_1} \sim \frac{w_{31}}{m_2} \sim b\epsilon, \quad b \neq 0.$$

Then equation (6) and the identity $W_{kl} = w_{kl}^{-1/2} z_{kl}^{-3/2}$ imply that

$$z_{12} \sim (\frac{1}{m_3^3 a^2 b})^{1/3}\epsilon,\ z_{23} \sim (\frac{1}{m_1^3 a^2 b})^{1/3}\epsilon, z_{31} \sim (\frac{1}{m_2^3 a^2 b})^{1/3}\epsilon.$$

Here, $x^{\frac{1}{3}}$ is understood as one appropriate value of the cubic root of x. Then the identity $z_{12} + z_{23} + z_{31} = 0$ implies that $\frac{1}{m_1}1^{\frac{1}{3}} + \frac{1}{m_2}1^{\frac{1}{3}} + \frac{1}{m_3}1^{\frac{1}{3}} = 0$, which contradicts with the fact $m_1 + m_2 + m_3 = 0$. Hence, it holds that $w_4 \approx \epsilon$.

Note that

$$\sum_{j=1}^{3} m_j w_{j4} = m w_4 \approx \epsilon,\ \ w_{14}, w_{24}, w_{34} \prec \epsilon^{-2},\ \ z_{14} \approx \epsilon^{-2},\ \ z_{24} - z_{14} \approx z_{34} - z_{14} \approx \epsilon.$$

Hence,

$$mI/m_4 \sim \sum_{j=1}^{3} m_j z_{j4} w_{j4} = z_{14} \sum_{j=1}^{3} m_j w_{j4} + (z_{24} - z_{14}) m_2 w_{24} + (z_{34} - z_{14}) m_3 w_{34} \approx \epsilon^{-1}.$$

This is a contradiction since the momentum of inertia I should be bounded. Hence, the second diagram in Figure 4 is impossible.

Summary: Among the 14 diagrams displayed in Figures 2–6, we have excluded three of them: the first and third diagrams in Figure 3, and the second diagram in Figure 4. Therefore, any singular sequence must converge to one of the remaining 11 diagrams.

By Rule IV, the total mass is zero for the second diagram in Figure 2, the second and fourth diagrams in Figure 3, and the second and third diagrams in Figure 6. It is straightforward to check that the total mass must be nonzero for the third diagram in Figure 4. We further claim that the total mass is also necessarily nonzero for the first diagram in Figure 4 and for the diagram in Figure 5.

The first diagram in Figure 4. Note that $r_{14}, r_{24}, r_{34} \succ \epsilon$. Without loss of generality, assume that $w_{14} \succ \epsilon$. Suppose by contradiction that $m = 0$. Then

$$0 = \sum_{j=1}^{4} m_j w_4 = \sum_{j=1}^{3} m_j w_{j4} = w_{14} \sum_{j=1}^{3} m_j + m_2 w_{21} + m_3 w_{31}.$$

Since $w_{21}, w_{31} \approx \epsilon$, it follows that $\sum_{j=1}^{3} m_j = 0$, a contradiction.

The diagram in Figure 5. It is easy to see that $w_{14} \succ \epsilon$, and $w_{21}, w_{31} \approx \epsilon$. Repeat the argument in the previous case. We then arrive at $m \neq 0$.

For the remaining three diagrams, namely, the first and third diagram in Figure 2, and the first diagram in Figure 6, the above arguments do not determine whether the total mass is zero or nonzero.

Therefore, when the total mass is nonzero, there are exactly 6 possible diagrams: the first and third diagram in Figure 2, the first and third diagrams in Figure 4, the diagram in Figure 5, and the first diagram in Figure 6. These 6 diagrams are collected in Figure 7.

If the total mass is zero, then there are exactly 8 possible diagrams: all three diagrams in Figure 2, the second and fourth diagram in Figure 3, and all three diagrams in Figure 6. These 8 diagrams are collected in Figure 10.

Note that the two lists are not disjoint: they have three diagrams in common. Consequently, although the first list contains 6 diagrams and the second contains 8 diagrams, there are only 11 distinct diagrams altogether.

5 Finiteness of central configurations with $m \neq 0$

In the first subsection, we consider the diagrams agreeable with the condition that the total mass $m \neq 0$. The constraints on the masses, as well as the orders the positions and distances, are studied. The rest of this section is devoted to prove the finiteness of central configurations with the condition that the total mass $m \neq 0$. There are two cases, whether there is some partial sum of three masses being zero or not.

The first case, where all partial sum of three masses are nonzero, is considered in the second and third subsections and we show that

Theorem 5.1. *Suppose that m_1, m_2, m_3, m_4 are real and nonzero, if $\sum_{i=1}^4 m_i \neq 0$ and $\prod_{1\leq j<k<l\leq 4}(m_j + m_k + m_l) \neq 0$, then system* (3)*, which defines the normalized central configurations in the complex domain, possesses finitely many solutions.*

The second case, where at least one partial sum of three masses is zero, is considered in the last two subsections and we show that

Theorem 5.2. *Suppose that m_1, m_2, m_3, m_4 are real and nonzero, if $\sum_{i=1}^4 m_i \neq 0$ and $\prod_{1\leq j<k<l\leq 4}(m_j + m_k + m_l) = 0$, then system* (3)*, which defines the normalized central configurations in the complex domain, possesses finitely many solutions.*

Since the momentum of inertia is a constant on a continuum of solution of system (3), the proof of the above two theorems is complete after we establish Theorem 5.3, Theorem 5.4, Theorem 5.5 and Theorem 5.6.

5.1 Problematic diagrams with nonzero total mass

Notations. From now on, $x^{1/n}$ (or $\sqrt[n]{x}$) denotes an arbitrary choice of an nth root of x, not necessarily the principal root. For example, $1^{1/2}$ may denote either of the two square roots of unity:

$$1, \quad -1.$$

We denote the set $f = 0$ by $\mathcal{V}_f$, and $f = 0, g = 0$ by $\mathcal{V}_f \bigcap \mathcal{V}_g$ or simply $\mathcal{V}_f\mathcal{V}_g$; the set $f = 0$ or $g = 0$ (i.e., $fg = 0$) by $\mathcal{V}_f \bigcup \mathcal{V}_g$. Recall that m is the total mass and it is nonzero.

Recall that only 6 diagrams are compatible with a nonzero total mass; see Figure 7.

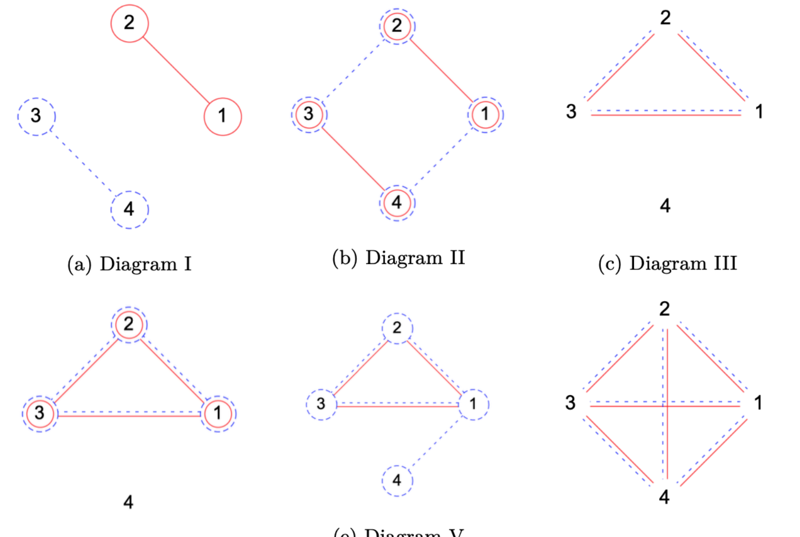
5.1.1 Diagram I

Following the argument from [1], we obtain the equation

$$\frac{m_1m_3}{(m_2m_4)^{\frac{1}{2}}} + \frac{m_2m_3}{(-m_1m_4)^{\frac{1}{2}}} + \frac{m_1m_4}{(-m_2m_3)^{\frac{1}{2}}} + \frac{m_2m_4}{(m_1m_3)^{\frac{1}{2}}} = 0,$$

or equivalently,

$$m_1m_3\sqrt{m_1m_3} + m_2m_3\sqrt{-m_2m_3} + m_1m_4\sqrt{-m_1m_4} + m_2m_4\sqrt{m_2m_4} = 0. \tag{7}$$



(a) Diagram I (b) Diagram II (c) Diagram III

(d) Diagram IV (e) Diagram V (f) Diagram VI

Figure 7: Diagrams compatible with a nonzero total mass

Note that $z_{jk}, w_{jk} \approx \epsilon^{-2}$, $j = 1, 2, k = 3, 4$. It follows

$$r_{jk} \approx \epsilon^{-2}, j = 1, 2, k = 3, 4. \tag{8}$$

and $Z_{jk}, W_{jk} \approx \epsilon^4$. Combined with the following two equations

$$\begin{gathered} w_{12} = (m_1 + m_2)W_{12} + m_3(W_{32} - W_{31}) + m_4(W_{42} - W_{41}), \\ z_{34} = (m_3 + m_4)Z_{34} + m_1(Z_{14} - Z_{13}) + m_4(Z_{24} - Z_{23}), \end{gathered}$$

we obtain

$$(m_1 + m_2)(m_3 + m_4) \neq 0, r_{12}^3 \sim m_1 + m_2, r_{34}^3 \sim m_3 + m_4. \tag{9}$$

Let us simplify equation (7). Note that equation (7) is homogeneous, and is invariant under the transformation $(m_1, m_2, m_3, m_4) \mapsto (m_2, m_1, m_4, m_3)$. Assume that the signs of the masses are the same. It suffices to consider only the case $(+, +, +, +)$. Then equation (7) becomes

$$m_1 m_3 \sqrt{m_1 m_3} + m_2 m_4 \sqrt{m_2 m_4} + i(m_2 m_3 \sqrt{m_2 m_3} + m_1 m_4 \sqrt{m_1 m_4}) = 0,$$

which is $m_1^3 m_3^3 = m_2^3 m_4^3$ and $m_2^3 m_3^3 = m_1^3 m_4^3$. Hence, it is necessary that

$$m_1 = m_2, \quad m_3 = m_4, \; m_1 m_3 > 0.$$

Assume that only one of the signs of the masses is different from the others. It suffices to consider only $(+, +, +, -)$ and $(+, -, +, +)$. In the first subcase, equation (7) is equivalent to

$$m_1 m_3 \sqrt{m_1 m_3} + m_1 m_4 \sqrt{-m_1 m_4} = 0, m_2 m_3 \sqrt{-m_2 m_3} + m_2 m_4 \sqrt{m_2 m_4} = 0.$$

Then we have $m_3 = -m_4$, which contradicts equation (9). Similarly, the second subcase is impossible neither.

Assume that only two of the signs of the masses is different from the others. It suffices to consider only $(+,+,-,-)$, $(+,-,+,-)$ and $(+,-,-,+)$. In the first subcase, equation (7) is equivalent to

$$m_1m_3\sqrt{m_1m_3}+m_2m_4\sqrt{m_2m_4}=0,\ m_2m_3\sqrt{-m_2m_3}+m_1m_4\sqrt{-m_1m_4}=0,$$

which is equivalent to

$$m_1=m_2,\ m_3=m_4,\ m_1m_3<0.$$

In the remaining subcases, equation (7) can not be simplified.

To summarize, the masses of central configurations corresponding to the first diagram belong to one of the following two sets of constraints

$$\begin{aligned}
&\mathcal{V}_{IA}[12,34]: && m_1=m_2,\ m_3=m_4;\\
&\mathcal{V}_{IB}[12,34]: && m_1m_3\sqrt{m_1m_3}+m_2m_3\sqrt{-m_2m_3}+m_1m_4\sqrt{-m_1m_4}+m_2m_4\sqrt{m_2m_4}=0,\\
& && m_1m_2<0,\ m_3m_4<0,(m_1+m_2)(m_3+m_4)\neq 0.
\end{aligned}$$

Furthermore, we claim that if $I=0$, then the masses belong to the set

$$\mathcal{I}_I[12,34]: \qquad \frac{m_1m_2}{\sqrt[3]{m_1+m_2}}+\frac{m_3m_4}{\sqrt[3]{m_3+m_4}}=0. \tag{10}$$

This is a consequence of the fact that $U=I=0$.

In fact, it is easy to see that if $I=0$, then the masses belong to the set

$$\mathcal{V}_{I0}[12,34] \quad \triangleq \quad \mathcal{V}_{IB}[12,34]\mathcal{I}_I[12,34].$$

Note that Diagram I has five other variants where the vertices are renumbered. Two of them are presented in Figure 8. We still call them Diagram I if no confusion arises. The corresponding mass sets are denoted by

$$\mathcal{V}_{IA}[ij,pk],\ \mathcal{V}_{IB}[ij,pk],\ \mathcal{I}_I[ij,pk],\ \text{and}\ \mathcal{V}_{I0}[ij,pk],$$

if there is one stroke, either z or w-stroke, between vertices **i** and **j**, and the other stroke is between vertices **p** and **k**. For instance, for both of the two diagrams in Figure 8, the corresponding mass sets are denoted by $\mathcal{V}_{IA}[13,24]$, $\mathcal{V}_{IB}[13,24]$, $\mathcal{I}_I[13,24]$, or $\mathcal{V}_{I0}[13,24]$.

5.1.2 Diagram II

By the argument from [1] and the hypothesis $m\neq 0$, the masses of central configurations corresponding to the second diagram belong to the following set

$$\mathcal{V}_{II}[13,24]: \qquad m_1m_3=m_2m_4,(m_1+m_2)(m_2+m_3)(m_3+m_4)(m_1+m_4)\neq 0.$$

By the facts $z_{12}=(m_1+m_2)Z_{12}+m_3(Z_{32}-Z_{31})+m_4(Z_{42}-Z_{41})$ and $Z_{12}=z_{12}r_{12}^{-3}$, it follows $r_{12}^3\sim(m_1+m_2)$. Similarly, we have

$$r_{12}^3\sim(m_1+m_2),r_{23}^3\sim(m_2+m_3),r_{34}^3\sim(m_3+m_4),r_{14}^3\sim(m_1+m_4). \tag{11}$$

Note that
$$r_{13} \approx \epsilon^{-2}, r_{24} \approx \epsilon^{-2}. \tag{12}$$

If $I = 0$, then the masses belong further to the set

$$\mathcal{I}_{II}[13,24]: \qquad \frac{m_1 m_2}{\sqrt[3]{m_1+m_2}} + \frac{m_2 m_3}{\sqrt[3]{m_2+m_3}} + \frac{m_3 m_4}{\sqrt[3]{m_3+m_4}} + \frac{m_1 m_4}{\sqrt[3]{m_1+m_4}} = 0. \tag{13}$$

That is, if $I = 0$, then the masses belong to the set

$$\mathcal{V}_{II0}[13,24] \quad \triangleq \quad \mathcal{V}_{II}[13,24]\mathcal{I}_{II}[13,24].$$

This is a consequence of the fact that $U = I = 0$.

Similarly, Diagram II has five other variants. Some of them are presented in the second row of Figure 9. We still call them Diagram II if no confusion arises. The corresponding mass sets are denoted by

$$\mathcal{V}_{II}[ij,pk],\ \mathcal{I}_{II}[ij,pk],\ \text{and } \mathcal{V}_{II0}[ij,pk],$$

if one diagonal is between vertices **i** and **j**, and the other is between vertices **p** and **k**. For instance, for both of the last two diagrams in the second row of Figure 9, the corresponding mass sets are denoted by $\mathcal{V}_{II}[14,23], \mathcal{I}_{II}[14,23]$ or $\mathcal{V}_{II0}[14,23]$.

5.1.3 Diagram III

Following the argument from [1], we arrive at the estimation

$$\frac{Z_{12}}{m_3} \sim \frac{Z_{23}}{m_1} \sim \frac{Z_{31}}{m_1},\ \frac{W_{12}}{m_3} \sim \frac{W_{23}}{m_1} \sim \frac{W_{31}}{m_1}.$$

Since Z_{12}, W_{12} are both of the maximal order, we assume the first set is asymptotic to $a\epsilon^{-2}$ and the second asymptotic to $b\epsilon^{-2}$, $ab \neq 0$. Multiplying the two, by $Z_{kl}W_{kl} = r_{kl}^{-4}$, we obtain

$$\frac{1}{m_3^2 r_{12}^4} \sim \frac{1}{m_1^2 r_{23}^4} \sim \frac{1}{m_2^2 r_{31}^4} \sim ab\epsilon^{-4}.$$

Then we have

$$r_{12} \sim 1^{\frac{1}{4}} \frac{\epsilon}{\sqrt{|m_3|}}c,\ r_{23} \sim 1^{\frac{1}{4}} \frac{\epsilon}{\sqrt{|m_1|}}c,\ r_{31} \sim 1^{\frac{1}{4}} \frac{\epsilon}{\sqrt{|m_2|}}c,\ c \neq 0.$$

Then by $z_{kl} = r_{kl}^3 Z_{kl}$, we have

$$z_{12} \sim 1^{\frac{1}{4}} \frac{\epsilon}{\sqrt{|m_3|}}d,\ z_{23} \sim 1^{\frac{1}{4}} \frac{\epsilon}{\sqrt{|m_1|}}d,\ z_{31} \sim 1^{\frac{1}{4}} \frac{\epsilon}{\sqrt{|m_2|}}d,\ d \neq 0. \tag{14}$$

Then the identity $z_{12} + z_{23} + z_{31} = 0$ implies that the masses belong to the set

$$\mathcal{V}_{III}[123]: \qquad \frac{1}{|m_1|^{\frac{1}{2}}} + \frac{1}{|m_2|^{\frac{1}{2}}} + \frac{1}{|m_3|^{\frac{1}{2}}} = 0.$$

It is easy to see that

$$r_{12}, r_{23}, r_{13} \approx \epsilon, r_{14}, r_{24}, r_{34} \succ \epsilon. \tag{15}$$

We discuss the distances $\{r_{j4}\}_{j=1}^{3}$ further in the following. Without loss of generality, assume that

$$z_{14} \preceq z_{24}, z_{34}, \ \ and \ z_{14} \preceq w_{14}.$$

Then $\epsilon \prec w_{14} \sim w_{24} \sim w_{34}$. For z_{14}, there are three possibilities, namely $\prec, \approx$, or $\succ \epsilon$. Note that

$$\begin{aligned}
&z_{14} \prec \epsilon, \ \Rightarrow z_{24} \approx z_{34} \approx \epsilon, \ r_{14} \prec r_{24} \approx r_{34}, \ W_{14} \succ W_{24} \approx W_{34};\\
&z_{14} \approx \epsilon, \ \Rightarrow z_{14} \approx z_{24} \approx z_{34} \approx \epsilon, \ r_{14}^2 \approx r_{24}^2 \approx r_{34}^2, \ W_{14} \approx W_{24} \approx W_{34};\\
&z_{14} \succ \epsilon, \ \Rightarrow z_{14} \sim z_{24} \sim z_{34} \succ \epsilon, \ r_{14}^2 \sim r_{24}^2 \sim r_{34}^2, \ W_{14}^2 \sim W_{24}^2 \sim W_{34}^2.
\end{aligned}$$

The masses for $z_{14} \prec \epsilon$. By $w_4 = \sum_{j=1}^{3} m_j W_{j4}$ and $z_4 = \sum_{j=1}^{3} m_j Z_{j4}$ it follows that $w_4 \sim m_1 W_{14}, z_4 \sim m_1 Z_{14}$. On the other hand, note that $\sum_{j=1}^{4} m_j w_4 = \sum_{j=1}^{3} m_j w_{j4} \preceq w_{14}$, then $w_4 \sim m_1 W_{14} \preceq w_{14}$, which implies that $r_{14} \succeq 1$. Therefore, $z_4 \sim m_1 Z_{14} \preceq z_{14}$, and $z_4, z_1 \prec \epsilon \approx z_2 \approx z_3$. Then it is easy to see that $z_{12} \sim z_2 \sim -m_3 f\epsilon, z_{13} \sim z_3 \sim m_2 f\epsilon$ and $z_{23} = z_3 - z_2 \sim (m_2 + m_3) f\epsilon$, where f is some nonzero constant. Compared with equation (14), it is necessary that

$$4|m_1| = |m_2| = |m_3|. \tag{16}$$

If $\sum_{j=1}^{3} m_j \neq 0$. The identity $\sum_{j=1}^{4} m_j w_4 = \sum_{j=1}^{3} m_j w_{j4}$ implies that $w_4 \sim \frac{\sum_{j=1}^{3} m_j}{\sum_{j=1}^{4} m_j} w_{14} \succ \epsilon$. By the equation $w_4 = \sum_{j=1}^{3} m_j W_{j4}$, it follows that

Case 1). If $z_{14} \prec \epsilon$, then $w_4 \sim m_1 W_{14} \sim \sum_{j=1}^{3} m_j w_{14}$. Then $r_{14} \approx 1$. Thus,

$$r_{14}^2 \approx 1 \prec r_{24}^2 \approx r_{34}^2 \prec \epsilon^{-1}, \ 4|m_1| = |m_2| = |m_3|;$$

Case 2). If $z_{14} \approx \epsilon$, then $w_{14} \preceq W_{14}$. Thus, $\epsilon^2 \prec r_{14}^2 \approx r_{24}^2 \approx r_{34}^2 \preceq 1$;

Case 3). If $z_{14} \succ \epsilon$, then $w_{14} \preceq W_{14}$. Thus, $\epsilon^2 \prec r_{14}^2 \approx r_{24}^2 \approx r_{34}^2 \preceq 1$.

If $\sum_{j=1}^{3} m_j = 0$. Then it is necessary that

$$\frac{1}{|m_1|^{\frac{1}{2}}} + \frac{1}{|m_2|^{\frac{1}{2}}} + \frac{1}{|m_3|^{\frac{1}{2}}} = m_1 + m_2 + m_3 = 0. \tag{17}$$

The above system has six solutions of (m_1, m_2, m_3). It is not necessary to find them for our purpose. By $\sum_{j=1}^{4} m_j w_4 = \sum_{j=1}^{3} m_j w_{j4}$ it follows that $w_4 \prec w_{14}$.

Case 1). If $z_{14} \prec \epsilon$, this is impossible since the system of equations (16) and (17) has no real solution.

Case 2). If $z_{14} \approx \epsilon$, then $\epsilon^2 \prec r_{14}^2 \approx r_{24}^2 \approx r_{34}^2 \prec \epsilon^{-1}$;

Case 3). If $z_{14} \succ \epsilon$, then we have $\epsilon^2 \prec r_{14}^2 \approx r_{24}^2 \approx r_{34}^2 \prec \epsilon^{-4}$.

Similarly, Diagram III has three other variants. Two of them are presented in the third row of Figure 9. We still call them Diagram III if no confusion arises. The corresponding mass sets are denoted by $\mathcal{V}_{III}[ijk]$, if the fully edged triangle has vertices **i**, **j**, and **k**. For instance, for the two diagrams in the third row of Figure 9, the corresponding mass sets are denoted by $\mathcal{V}_{III}[234]$ and $\mathcal{V}_{III}[134]$ respectively. To distinguish, we may refer to the first one as Diagram III with $\triangle_{234}$, the second one as Diagram III with $\triangle_{134}$ and so on.

5.1.4 Diagram IV

The masses belong to the set $\mathcal{V}_{IV}[123] : \sum_{j=1}^{3} m_j = 0$, and we have

$$r_{12}, r_{23}, r_{13} \approx \epsilon, r_{14} \sim \pm r_{24} \sim \pm r_{34} \approx \epsilon^{-2}. \tag{18}$$

Similarly, Diagram IV has three other variants. One of them is presented in the fourth row of Figure 9. We still call them Diagram IV if no confusion arises. The corresponding mass sets are denoted by $\mathcal{V}_{IV}[ijk]$, if the fully edged triangle has vertices **i**, **j**, and **k**. For instance, for the second diagram in the fourth row of Figure 9, the corresponding mass sets is denoted by $\mathcal{V}_{IV}[124]$. To distinguish, we may refer to the first one as Diagram IV with $\triangle_{123}$ and so on.

5.1.5 Diagram V

Following the argument from [1], we have

$$\frac{Z_{12}}{m_3} \sim \frac{Z_{23}}{m_1} \sim \frac{Z_{31}}{m_1} \sim a\epsilon^{-2}, \text{ and set } z_{12} = m_3, \ z_{31} = m_2,$$

with a being some nonzero constant. By $Z_{kl} = r_{kl}^{-3} z_{kl}$, we see

$$\frac{m_1}{r_{12}^3} \sim \frac{-(m_2 + m_3)}{r_{23}^3} \sim \frac{m_1}{r_{31}^3} \sim a m_1 \epsilon^{-2}.$$

Then we have

$$r_{12} \sim \epsilon^{\frac{2}{3}} c \sqrt[3]{m_1}, \ r_{23} \sim \epsilon^{\frac{2}{3}} c \sqrt[3]{-(m_2 + m_3)}, \ r_{31} \sim \epsilon^{\frac{2}{3}} c \sqrt[3]{m_1}, \ c \neq 0.$$

In this subsection and in this subsection only, $\sqrt[3]{x}$ $(x \in \mathbb{R})$ is understood as the real cubic root of x. Then the asymptotic relation $\frac{1}{m_1 r_{23}} + \frac{1}{m_2 r_{31}} + \frac{1}{m_3 r_{12}} \sim 0$ implies

$$\frac{1}{m_1 \sqrt[3]{-(m_2 + m_3)}} + \frac{1^{\frac{1}{3}}}{m_2 \sqrt[3]{m_1}} + \frac{1^{\frac{1}{3}}}{m_3 \sqrt[3]{m_1}} = 0.$$

The above equation holds if and only if

$$m_1 \sqrt[3]{-(m_2 + m_3)} = m_2 \sqrt[3]{m_1} = m_3 \sqrt[3]{m_1}, \ or \ \frac{1}{m_1 \sqrt[3]{-(m_2 + m_3)}} + \frac{1}{m_2 \sqrt[3]{m_1}} + \frac{1}{m_3 \sqrt[3]{m_1}} = 0.$$

The first case has no real solutions, and the second case reduces to

$$\mathcal{V}_V[1,23]: \qquad m_1^2(m_2+m_3)^4 = m_2^3 m_3^3,$$

which has real solutions only if $m_2m_3 > 0$.

Note that

$$\sum_{j=1}^{4} m_j w_1 + m_4 w_{14} = \sum_{j=1}^{4} m_j w_j + m_2 w_{21} + m_3 w_{31} = m_2 w_{21} + m_2 w_{31} \preceq \epsilon,$$

then $w_{14} \approx \epsilon^{-2}$ and $z_{14} \approx \epsilon^2$. It follows that

$$\sum_{j=1}^{3} m_j \neq 0, r_{12}, r_{23}, r_{13} \approx \epsilon, r_{14} \approx 1, r_{24} \approx r_{34} \approx \sqrt{\epsilon}^{-1}. \tag{19}$$

Similarly, Diagram V has 11 other variants. Some of them are presented in the fifth row of Figure 9. We still call them Diagram V if no confusion arises. The corresponding mass sets are denoted by $\mathcal{V}_V[i,jk]$, if the fully edged triangle has vertices **i**, **j**, **k** and vertex **i** is connected to the fourth vertex by one w-stroke. For instance, for the first two diagrams in the fifth row of Figure 9, the corresponding mass sets are denoted by $\mathcal{V}_V[3,14]$ and $\mathcal{V}_V[4,13]$ respectively. Both of the first two diagrams will be referred as Diagram V with $\triangle_{134}$.

5.1.6 Diagram VI

We will reach our result without discussing the mass polynomial of this diagram. Here is one remarks about this diagram. Note that the momentum of inertia I tends to zero by estimation 2. By Lemma 3.2, the momentum of inertia is constant on a continuum of central configurations, so such a singular sequence exists only on the subset $I=0$.

Proposition 5.1. *For singular sequences corresponding to Diagram V and VI, we have the following estimates*

$$r_{jk} r_{lp} \prec 1, \ r_{jk}^2 r_{lp} \preceq 1,$$

where r_{jk} and r_{lp} are any two nonadjacent distances. The same estimate holds for Diagram III, except for third case of Subsection 5.1.3, for which it is necessary that the masses satisfy equation (17)*.*

5.2 Finiteness with $m \neq 0$, $\prod_{1\leq k\leq 4}(m - m_k) \neq 0$ and $I \neq 0$

Theorem 5.3. *Suppose that m_1, m_2, m_3, m_4 are real and nonzero, if $\sum_{i=1}^{4} m_i \neq 0$, $\prod_{1\leq j<k<l\leq 4}(m_j+m_k+m_l) \neq 0$ and $I \neq 0$, then system* (3)*, which defines the normalized central configurations in the complex domain, possesses finitely many solutions.*

In this case only Diagram I, Diagram II, Diagram III and Diagram V are possible. We will say $\triangle_{jkl} \longrightarrow 0$, if we have one singular sequence corresponding to one diagram with a fully edged triangle with vertices **j, k, l**.

Proof of Theorem 5.3: It is easy to see that giving five of r_{kl}^2's, $1 \leq k < l \leq 4$, determines only finitely many geometrical configurations up to rotation. Suppose that there are infinitely many solutions of system (4) in the complex domain. Then at least two of r_{kl}^2's must take infinitely many values and thus are dominating by Lemma 3.1. Suppose that r_{kl}^2 is dominating for some $1 \leq k < l \leq 4$. There must exist a singular sequence of central configurations with $r_{kl}^2 \longrightarrow 0$, which happens only in Diagram III or Diagram V. In either case, by Proposition 5.1,

$$r_{kl}^2 r_{ij}^2 \longrightarrow 0, \ r_{ki}^2 r_{jl}^2 \longrightarrow 0, \ r_{il}^2 r_{kj}^2 \longrightarrow 0,$$

along the singular sequence. Then all the three polynomials $r_{13}^2 r_{24}^2$, $r_{14}^2 r_{23}^2$, $r_{12}^2 r_{34}^2$ are dominating. It is easy to see that there exist $\triangle_{ijk} \longrightarrow 0$. Without loss of generality, assume that $\triangle_{123} \longrightarrow 0$, so r_{12}^2, r_{23}^2 and r_{13}^2 are dominating.

There also exist singular sequences with $r_{12}^2 r_{34}^2 \longrightarrow \infty$, singular sequences with $r_{13}^2 r_{24}^2 \longrightarrow \infty$ and singular sequences with $r_{14}^2 r_{23}^2 \longrightarrow \infty$. These sequences must correspond to Diagram I or Diagram II.

Consider a singular sequence with $r_{13}^2 r_{24}^2 \longrightarrow \infty$, thus, the masses must be belong to

$$\mathcal{V}_{IA}[12, 34] \bigcup \mathcal{V}_{IA}[14, 23] \bigcup \mathcal{V}_{IB}[12, 34] \bigcup \mathcal{V}_{IB}[14, 23] \bigcup \mathcal{V}_{II}[13, 24].$$

Note that $\mathcal{V}_{IA}[12, 34] \bigcup \mathcal{V}_{IA}[14, 23] \subset \mathcal{V}_{II}[13, 24]$. Thus, the above set is the union of

$$1). \quad \mathcal{V}_{II}[13, 24]; \quad 2). \quad \mathcal{V}_{IB}[12, 34] \bigcup \mathcal{V}_{IB}[14, 23]. \tag{20}$$

Repeat the argument with $r_{14}^2 r_{23}^2$. Then the masses must belong to one of

$$1). \quad \mathcal{V}_{II}[14, 23]; \quad 2). \quad \mathcal{V}_{IB}[12, 34] \bigcup \mathcal{V}_{IB}[13, 24]. \tag{21}$$

Repeat the argument with $r_{12}^2 r_{34}^2$. Then the masses must belong to one of

$$1). \quad \mathcal{V}_{II}[12, 34]; \quad 2). \quad \mathcal{V}_{IB}[13, 24] \bigcup \mathcal{V}_{IB}[14, 23]. \tag{22}$$

Our strategy is to show that the three constraints (20), (21) and (22), together with other available constraints, can not be satisfied simultaneously.

Define

$$\begin{aligned} V[123] &\triangleq \mathcal{V}_{III}[123] \bigcup \mathcal{V}_V[1, 23] \bigcup \mathcal{V}_V[2, 13] \bigcup \mathcal{V}_V[3, 12]; \\ V[124] &\triangleq \mathcal{V}_{III}[124] \bigcup \mathcal{V}_V[1, 24] \bigcup \mathcal{V}_V[2, 14] \bigcup \mathcal{V}_V[4, 12]; \\ V[134] &\triangleq \mathcal{V}_{III}[134] \bigcup \mathcal{V}_V[1, 34] \bigcup \mathcal{V}_V[3, 14] \bigcup \mathcal{V}_V[4, 13]; \\ V[234] &\triangleq \mathcal{V}_{III}[234] \bigcup \mathcal{V}_V[2, 34] \bigcup \mathcal{V}_V[3, 24] \bigcup \mathcal{V}_V[4, 23]. \end{aligned}$$

Case 1: Three of 1) **are satisfied.** That is, $m_1 m_3 = m_2 m_4, m_1 m_4 = m_2 m_3, m_1 m_2 = m_3 m_4$. Then $m_1^2 = m_2^2 = m_3^2 = m_4^2$. Then $m_1 = m_2 = m_3 = m_4$. However, recall that $\triangle_{123} \longrightarrow 0$, consequently, the masses belong to the set $V[123]$, a contradiction.

Case 2: Two of 1) **are satisfied.** Without loss of generality, assume that the masses belong to the set $\mathcal{V}_{II}[13,24]\mathcal{V}_{II}[14,23]$, i.e., $m_1m_3 = m_2m_4, m_1m_4 = m_2m_3$. Then $m_1^2 = m_2^2, m_3^2 = m_4^2$. Then $m_1 = m_2, m_3 = m_4$. The masses also belong to the set

$$\mathcal{V}_{IB}[13,24]\bigcup\mathcal{V}_{IB}[14,23].$$

However, recall that $\triangle_{123} \longrightarrow 0$, consequently, the masses belong to the set $V[123]$, a contradiction.

Case 3: One of 1) **is satisfied.** Without loss of generality, assume that the masses belong to the set $\mathcal{V}_{II}[13,24]$, i.e., $m_1m_3 = m_2m_4$. Then the masses also belong to the set

$$\begin{aligned}&(\mathcal{V}_{IB}[12,34]\bigcup\mathcal{V}_{IB}[13,24])(\mathcal{V}_{IB}[13,24]\bigcup\mathcal{V}_{IB}[14,23])\\ &= (\mathcal{V}_{IB}[12,34]\mathcal{V}_{IB}[14,23])\bigcup\mathcal{V}_{IB}[13,24].\end{aligned}$$

Subcase 1: $\mathcal{V}_{IB}[12,34]\mathcal{V}_{IB}[14,23]$: Note that $\triangle_{123} \longrightarrow 0$. It is easy to check that $\mathcal{V}_{II}[13,24]\mathcal{V}_{IB}[12,34]\mathcal{V}_{IB}[14,23]V[123]$ is empty, a contradiction.

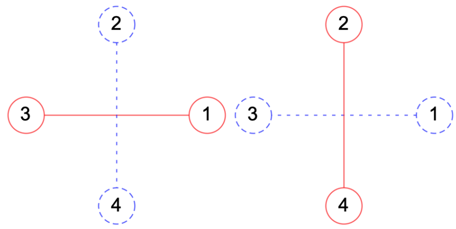


Figure 8

Subcase 2: $\mathcal{V}_{IB}[13,24]$: Except that $\triangle_{123} \longrightarrow 0$, note that in this case one of the two diagrams in Figure 8 occurs, thus r_{14}^2 and r_{34}^2 are dominating polynomials. Consider a singular sequence with $r_{14}^2 \longrightarrow 0$, then the sequence corresponds to Diagram III or V. In Diagram III, the distances not in the fully edged triangle have order $\succeq 1$ since $I \neq 0$. Thus, we have

$$\triangle_{124} \longrightarrow 0, \;\; or \; \triangle_{134} \longrightarrow 0.$$

Repeat the argument with r_{34}^2, we have

$$\triangle_{134} \longrightarrow 0, \;\; or \; \triangle_{234} \longrightarrow 0.$$

Thus, we conclude that we have

$$\triangle_{134} \longrightarrow 0, \;\; or \;\;\; \triangle_{124} \longrightarrow 0, \triangle_{234} \longrightarrow 0.$$

However, it is easy to check that both of the following two sets

$$\mathcal{V}_{II}[13,24]\mathcal{V}_{IB}[13,24]V[123]V[134], \;\; \mathcal{V}_{II}[13,24]\mathcal{V}_{IB}[13,24]V[123]V[124]V[234],$$

are empty, a contradiction.

Case 4: No 1) **is satisfied.**Then the masses belong to the set

$$\begin{aligned}&(\mathcal{V}_{IB}[12,34]\bigcup\mathcal{V}_{IB}[14,23])(\mathcal{V}_{IB}[12,34]\bigcup\mathcal{V}_{IB}[13,24])(\mathcal{V}_{IB}[13,24]\bigcup\mathcal{V}_{IB}[14,23])\\ &= (\mathcal{V}_{IB}[12,34]\mathcal{V}_{IB}[14,23])\bigcup(\mathcal{V}_{IB}[12,34]\mathcal{V}_{IB}[13,24])\bigcup(\mathcal{V}_{IB}[13,24]\mathcal{V}_{IB}[14,23]).\end{aligned}$$

Without loss of generality, assume that the masses belong to the set $\mathcal{V}_{IB}[12,34]\mathcal{V}_{IB}[13,24]$. Except that $\triangle_{123} \longrightarrow 0$, note that:

By $\mathcal{V}_{IB}[12,34]$, repeat the argument utilized above, we have two dominating polynomials, r_{14}^2, r_{24}^2. Then let $r_{14}^2 \longrightarrow 0$, we have $\triangle_{124} \longrightarrow 0,\ or\ \triangle_{134} \longrightarrow 0$. Let $r_{24}^2 \longrightarrow 0$, we have $\triangle_{124} \longrightarrow 0,\ or\ \triangle_{234} \longrightarrow 0$. Thus, we conclude that we have

$$\triangle_{124} \longrightarrow 0,\ or\ \ \triangle_{134} \longrightarrow 0, \triangle_{234} \longrightarrow 0.$$

Repeat the argument for $\mathcal{V}_{IB}[13,24]$ and the two polynomials r_{14}^2, r_{34}^2. We have

$$\triangle_{134} \longrightarrow 0,\ or\ \ \triangle_{124} \longrightarrow 0, \triangle_{234} \longrightarrow 0.$$

In short, at least two of $\triangle_{124} \longrightarrow 0$, $\triangle_{134} \longrightarrow 0$ and $\triangle_{234} \longrightarrow 0$ occur.

Subcase 1: $\triangle_{124} \longrightarrow 0$ and $\triangle_{134} \longrightarrow 0$: However, it is easy to check that

$$\mathcal{V}_{IB}[12,34]\mathcal{V}_{IB}[13,24]V[123]V[124]V[134]$$

is empty, a contradiction.

Subcase 2: $\triangle_{124} \longrightarrow 0$ and $\triangle_{234} \longrightarrow 0$: However, it is easy to check that

$$\mathcal{V}_{IB}[12,34]\mathcal{V}_{IB}[13,24]V[123]V[124]V[234]$$

is empty, a contradiction.

Subcase 3: $\triangle_{134} \longrightarrow 0$ and $\triangle_{234} \longrightarrow 0$: However, it is easy to check that

$$\mathcal{V}_{IB}[12,34]\mathcal{V}_{IB}[13,24]V[123]V[134]V[234]$$

is empty, a contradiction.

To summarize, we proved that the system (3) possesses finitely many solutions, if $\sum_{i=1}^4 m_i \neq 0$, $\prod_{1\le j<k<l\le 4}(m_j+m_k+m_l) \neq 0$ and $I \neq 0$.

$\square$

5.3 Finiteness with $m \neq 0$, $\prod_{1\le k\le 4}(m-m_k) \neq 0$ and $I = 0$

Theorem 5.4. *Suppose that m_1, m_2, m_3, m_4 are real and nonzero, if $\sum_{i=1}^4 m_i \neq 0$, $\prod_{1\le j<k<l\le 4}(m_j+m_k+m_l) \neq 0$ and $I = 0$, then system (3), which defines the normalized central configurations in the complex domain, possesses finitely many solutions.*

Proof of Theorem 5.4: Note that only Diagram I, Diagram II, Diagram III, Diagram V and Diagram VI are possible. It is easy to see that giving five of r_{kl}^2's, $1 \le k < l \le 4$, determines only finitely many geometrical configurations up to rotation. Suppose that there are infinitely many solutions of system (4) in the complex domain. Then at least two of $\{r_{kl}^2\}$'s must take infinitely many values and thus are dominating by Lemma 3.1. Suppose that r_{kl}^2 is dominating for some $1 \le k < l \le 4$. There must exist a singular sequence of central configurations with $r_{kl}^2 \longrightarrow 0$, which happens only in Diagram III, Diagram V or Diagram VI. In either case, by Proposition 5.1,

$$r_{kl}^2 r_{ij}^2 \longrightarrow 0,\ r_{ki}^2 r_{jl}^2 \longrightarrow 0,\ r_{il}^2 r_{kj}^2 \longrightarrow 0,$$

along the singular sequence. Then all the three polynomials $r_{13}^2 r_{24}^2$, $r_{14}^2 r_{23}^2$, $r_{12}^2 r_{34}^2$ are dominating. It is easy to see that there exist $\triangle_{ijk} \longrightarrow 0$ (say $\triangle_{123} \longrightarrow 0$), so r_{12}^2, r_{23}^2 and r_{13}^2 are dominating.

Then there also exist singular sequences with $r_{12}^2 r_{34}^2 \longrightarrow \infty$, singular sequences with $r_{13}^2 r_{24}^2 \longrightarrow \infty$ and singular sequences with $r_{14}^2 r_{23}^2 \longrightarrow \infty$. These sequences must correspond to Diagram I or Diagram II.

Consider a singular sequence with $r_{13}^2 r_{24}^2 \longrightarrow \infty$, thus, the masses must belong to the set

$$\mathcal{V}_{I0}[12,34] \bigcup \mathcal{V}_{I0}[14,23] \bigcup \mathcal{V}_{II0}[13,24],$$

which is the union of

$$1). \quad \mathcal{V}_{II0}[13,24]; \quad 2). \quad \mathcal{V}_{I0}[12,34] \bigcup \mathcal{V}_{I0}[14,23]. \tag{23}$$

Repeat the argument with $r_{14}^2 r_{23}^2$. Then the masses must belong to one of the sets

$$1). \quad \mathcal{V}_{II0}[14,23]; \quad 2). \quad \mathcal{V}_{I0}[12,34] \bigcup \mathcal{V}_{I0}[13,24]. \tag{24}$$

Repeat the argument with $r_{12}^2 r_{34}^2$. Then the masses belong to one of the sets

$$1). \quad \mathcal{V}_{II0}[12,34]; \quad 2). \quad \mathcal{V}_{I0}[13,24] \bigcup \mathcal{V}_{I0}[14,23]. \tag{25}$$

Our strategy is to show that the three constraints (23), (24) and (25) are incompatible.

Case 1: Three of 1) **are satisfied.** A straightforward computation shows that all the conditions are impossible.

Case 2: Two of 1) **are satisfied.** Without loss of generality, assume that the masses belong to the set

$$\mathcal{V}_{II0}[13,24]\mathcal{V}_{II0}[14,23](\mathcal{V}_{I0}[13,24] \bigcup \mathcal{V}_{I0}[14,23]).$$

However, one checks that the set is empty.

Case 3: One of 1) **is satisfied.** Without loss of generality, assume that the masses belong to the set

$$\begin{aligned} &\mathcal{V}_{II0}[13,24](\mathcal{V}_{I0}[12,34] \bigcup \mathcal{V}_{I0}[13,24])(\mathcal{V}_{I0}[13,24] \bigcup \mathcal{V}_{I0}[14,23]) \\ &= \mathcal{V}_{II0}[13,24]\mathcal{V}_{I0}[12,34]\mathcal{V}_{I0}[14,23] \bigcup \mathcal{V}_{II0}[13,24]\mathcal{V}_{I0}[13,24]. \end{aligned}$$

However, a straightforward computation shows that the set is empty.

Case 4: No 1) **is satisfied.** Then the masses belong to the set

$$\begin{aligned} &(\mathcal{V}_{I0}[12,34] \bigcup \mathcal{V}_{I0}[14,23])(\mathcal{V}_{I0}[12,34] \bigcup \mathcal{V}_{I0}[13,24])(\mathcal{V}_{I0}[13,24] \bigcup \mathcal{V}_{I0}[14,23]) \\ &= (\mathcal{V}_{I0}[12,34]\mathcal{V}_{I0}[14,23]) \bigcup (\mathcal{V}_{I0}[12,34]\mathcal{V}_{I0}[13,24]) \bigcup (\mathcal{V}_{I0}[13,24]\mathcal{V}_{I0}[14,23]). \end{aligned}$$

Without loss of generality, assume that the masses belong to the set $\mathcal{V}_{I0}[12,34]\mathcal{V}_{I0}[13,24]$. A straightforward computation shows that the set is empty.

To summarize, we proved that the system (3) possesses finitely many solutions, if $\sum_{i=1}^4 m_i \neq 0$, $\prod_{1\leq j<k<l\leq 4}(m_j+m_k+m_l) \neq 0$ and $I=0$.

$\square$

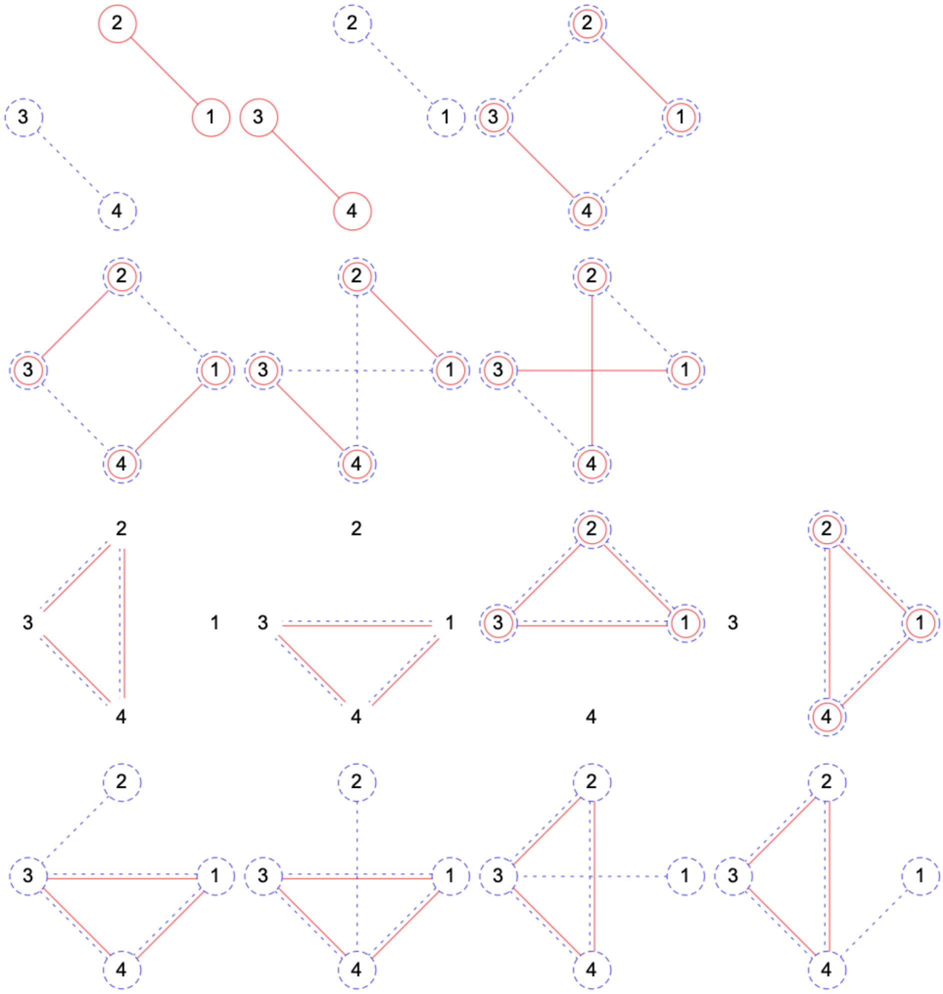


Figure 9: All possible diagrams in the level set $r_{12}^4 r_{34}^2 = const$

5.4 Finiteness of Central configurations with $m \neq 0$, $\prod_{1\leq k\leq 4}(m-m_k) = 0$ and $I \neq 0$

In this case Diagram I, Diagram II, Diagram III, Diagram IV , Diagram V and Diagram VI are all possible. First, we establish a result without regard to the value of I.

Lemma 5.1. *If some product $r_{jk}^4 r_{lm}^2$ is not dominating on the closed algebraic subset $\mathcal{A}$. Then system* (3) *possesses finitely many solutions.*

Proof of Lemma 5.1: If system (3) possesses infinitely many solutions, without loss of generality, assume that $r_{12}^4 r_{34}^2$ is not dominating on the closed algebraic subset $\mathcal{A}$. Then some level set $r_{12}^4 r_{34}^2 \equiv const \neq 0$ also includes infinitely many solutions of system (3).

In this level set, all the corresponding possible diagrams are presented in Figure 9. In particular, for Diagram III, we are in the third case of Subsection 5.1.3. More precisely, if the fully edged triangle is $\triangle_{jkl}$ and the other vertex is $\mathbf{p}$, then $r_{pj}^2 \sim r_{pk}^2 \sim r_{pl}^2 \approx \epsilon^{-1}$, and the masses belong to the set $\mathcal{V}_{III}[jkl]\mathcal{V}_{IV}[jkl]$.

Similarly, at least two of $\{r_{kl}^2\}$'s must take infinitely many values and thus are dominating. Suppose that r_{kl}^2 is dominating for some $1 \leq k < l \leq 4$. There must exist a singular sequence of central configurations with $r_{kl}^2 \longrightarrow 0$, which happens only in Diagram III, Diagram IV or Diagram V. By Proposition 5.1, $r_{12}^2 r_{34}^2 \longrightarrow 0$ along the singular sequence. Then the polynomial $r_{12}^2 r_{34}^2$ is dominating.

Consider a singular sequence with $r_{12}^2 r_{34}^2 \longrightarrow \infty$, then it is easy to see that this happens only in Diagram IV. Without loss of generality, assume that Diagram IV with $\triangle_{123}$ occurs. Note that $r_{12}^4 r_{14}^2 r_{24}^2 \longrightarrow \infty$ along a singular sequence corresponding to Diagram IV with $\triangle_{123}$, so $r_{12}^4 r_{14}^2 r_{24}^2$ is dominating. When $r_{12}^4 r_{14}^2 r_{24}^2 \longrightarrow 0$, it is easy to see that this happens only in Diagram IV with $\triangle_{124}$. Thus,

$$m_1 + m_2 + m_3 = m_1 + m_2 + m_4 = 0, \tag{26}$$

and $r_{13}^2 r_{14}^2$ is a dominating polynomial.

Consider a singular sequence with $r_{13}^2 r_{14}^2 \longrightarrow 0$. It is easy to see that this happens only in Diagram III with $\triangle_{134}$, and in the first two of Diagram V. However, it is easy to check that equation (26) is not consistent with any one of the following three sets

$$\mathcal{V}_{III}[134]\mathcal{V}_{IV}[134],\ \mathcal{V}_V[3,14],\ \mathcal{V}_V[4,13].$$

This is a contradiction. □

Hence, we assume that all $r_{jk}^4 r_{lm}^2$'s are dominating on the closed algebraic subset $\mathcal{A}$ from now on.

If Diagram IV and Diagram III (the third case of Subsection 5.1.3 with $r_{lp}^2 \succeq \epsilon^{-2}$, where vertex $\mathbf{p}$ is not in the fully edged triangle) do not occur, then $r_{jk} r_{lp} \prec 1$ holds for Diagram III, and hence the proof of Theorem 5.2 reduces to those of Theorem 5.1. Thus, we assume that either Diagram IV or Diagram III (the third case of Subsection 5.1.3 with $r_{lp}^2 \succeq \epsilon^{-2}$, where vertex $\mathbf{p}$ is not in the fully edged triangle) occurs. Then there exists some $\triangle_{ijk} \longrightarrow 0$. Without loss of generality, we assume that $\triangle_{123} \longrightarrow 0$ below. Then

$$m_1 + m_2 + m_3 = 0 \tag{27}$$

and it is easy to see that all of r_{kl}^2's, $1 \leq k < l \leq 4$, are dominating.

Now we prove the finiteness in the case of $I \neq 0$.

Theorem 5.5. *Suppose that m_1, m_2, m_3, m_4 are real and nonzero, if $\sum_{i=1}^4 m_i \neq 0$, $\prod_{1 \leq j<k<l \leq 4}(m_j + m_k + m_l) = 0$ and $I \neq 0$, then system* (3)*, which defines the normalized central configurations in the complex domain, possesses finitely many solutions.*

Proof of Theorem 5.5: Note that only Diagram I, Diagram II, Diagram III, Diagram IV and Diagram V are possible. Consider a singular sequence with $r_{14}^2 \longrightarrow 0$, then the sequence corresponds to $\triangle_{124} \longrightarrow 0$, $\triangle_{134} \longrightarrow 0$ in Diagram III, IV or V. Repeat the argument with $r_{24}^2 \longrightarrow 0$ and $r_{34}^2 \longrightarrow 0$, then it is easy to see that at least two of $\triangle_{124} \longrightarrow 0$, $\triangle_{134} \longrightarrow 0$ and $\triangle_{234} \longrightarrow 0$ occur. Without loss of generality, assume that $\triangle_{124} \longrightarrow 0$ and $\triangle_{134} \longrightarrow 0$ occur.

Corresponding to $\triangle_{124} \longrightarrow 0$ we have

$$\begin{gathered}(\frac{1}{|m_1|^{\frac{1}{2}}} + \frac{1}{|m_2|^{\frac{1}{2}}} + \frac{1}{|m_4|^{\frac{1}{2}}})(m_1 + m_2 + m_4)(m_1^2(m_2 + m_4)^4 - m_2^3 m_4^3) \\ (m_2^2(m_1 + m_4)^4 - m_1^3 m_4^3)(m_4^2(m_2 + m_1)^4 - m_2^3 m_1^3) = 0\end{gathered} \tag{28}$$

Corresponding to $\triangle_{134} \longrightarrow 0$ we have

$$\begin{gathered}(\frac{1}{|m_1|^{\frac{1}{2}}}+\frac{1}{|m_3|^{\frac{1}{2}}}+\frac{1}{|m_4|^{\frac{1}{2}}})(m_1+m_3+m_4)(m_1^2(m_3+m_4)^4-m_3^3m_4^3)\\ (m_3^2(m_1+m_4)^4-m_1^3m_4^3)(m_4^2(m_3+m_1)^4-m_3^3m_1^3)=0\end{gathered} \tag{29}$$

Let us further consider $r_{12}^4 r_{34}^2 \longrightarrow \infty$, then we are in Diagram I, Diagram II, Diagram III or Diagram IV.

If we are in Diagram I or Diagram II. Then the masses belong to the set

$$\mathcal{V}_{IA}[13,24]\bigcup\mathcal{V}_{IA}[14,23]\bigcup\mathcal{V}_{IB}[13,24]\bigcup\mathcal{V}_{IB}[14,23]\bigcup\mathcal{V}_{II}[12,34]. \tag{30}$$

However, a straightforward computation shows that masses of the above set are not consistent with equations (27), (28) and (29).

If we are in Diagram III. Then we have

$$\frac{1}{|m_1|^{\frac{1}{2}}}+\frac{1}{|m_3|^{\frac{1}{2}}}+\frac{1}{|m_4|^{\frac{1}{2}}}=0, \quad m_1+m_3+m_4=0, \tag{31}$$

or

$$\frac{1}{|m_2|^{\frac{1}{2}}}+\frac{1}{|m_3|^{\frac{1}{2}}}+\frac{1}{|m_4|^{\frac{1}{2}}}=0, \quad m_2+m_3+m_4=0. \tag{32}$$

However, a straightforward computation shows that all of these relations are not consistent with equations (27), (28) and (29).

If we are in Diagram IV. Then it is Diagram IV with $\triangle_{234}$ or $\triangle_{134}$.

Case 1: If we are in Diagram IV with $\triangle_{234}$: Then

$$m_2+m_3+m_4=0. \tag{33}$$

A straightforward computation shows that the above equation is not consistent with equations (27), (28) and (29).

Case 2: If we are in Diagram IV with $\triangle_{134}$: Then

$$m_1+m_3+m_4=0. \tag{34}$$

In this case, equation (34) is consistent with equations (27), (28) and (29) and it is necessary to include more polynomials. Let us further consider $r_{13}^4 r_{24}^2 \longrightarrow \infty$, we are in Diagram I, Diagram II, Diagram III and Diagram IV.

Subcase 1: If in Diagram I or Diagram II. Then

$$\mathcal{V}_{IA}[12,34]\bigcup\mathcal{V}_{IA}[14,23]\bigcup\mathcal{V}_{IB}[12,34]\bigcup\mathcal{V}_{IB}[14,23]\bigcup\mathcal{V}_{II}[13,24]. \tag{35}$$

However, a straightforward computation shows that masses of the above set are not consistent with equations (27), (28) and (34).

Subcase 2: If in Diagram III. We have

$$\frac{1}{|m_1|^{\frac{1}{2}}}+\frac{1}{|m_2|^{\frac{1}{2}}}+\frac{1}{|m_4|^{\frac{1}{2}}}=0, \quad m_1+m_2+m_4=0, \tag{36}$$

or equation (32). However, a straightforward computation shows that neither equation (36) nor equation (32) are consistent with equations (27), (28) and (34).

Subcase 3: If in Diagram IV. We have $\triangle_{234} \longrightarrow 0$ or $\triangle_{124} \longrightarrow 0$. If $\triangle_{234} \longrightarrow 0$, then we have $m_2 + m_3 + m_4 = 0$, not consistent with equations (27), (28) and (34).

If $\triangle_{124} \longrightarrow 0$, then $m_1 + m_2 + m_4 = 0$. By (27) and (34), the masses are

$$m_1 = -2m_2, m_3 = m_2, m_4 = m_2. \tag{37}$$

Note that $r_{23}^2 r_{24}^2 r_{34}^2 \longrightarrow \infty$, it is dominating. Let $r_{23}^2 r_{24}^2 r_{34}^2 \longrightarrow 0$, we are in Diagram III, IV and V. The masses satisfying equation (37) can not admit Diagram III and V. In Diagram IV, the fully edged triangle can only be $\triangle_{234}$, so we have $m_2 + m_3 + m_4 = 0$, which contradicts with (37).

To summarize, we proved that the system (3) possesses finitely many solutions, if $\sum_{i=1}^4 m_i \neq 0$, $\prod_{1 \leq j<k<l \leq 4}(m_j + m_k + m_l) = 0$ and $I \neq 0$.

□

5.5 Finiteness with $m \neq 0$, $\prod_{1 \leq k \leq 4}(m - m_k) = 0$ and $I = 0$

Theorem 5.6. *Suppose that m_1, m_2, m_3, m_4 are real and nonzero, if $\sum_{i=1}^4 m_i \neq 0$, $\prod_{1 \leq j<k<l \leq 4}(m_j + m_k + m_l) = 0$ and $I = 0$, then system* (3)*, which defines the normalized central configurations in the complex domain, possesses finitely many solutions.*

Proof of Theorem 5.6: Note that all six diagrams in are possible. Let us first consider $r_{12}^4 r_{34}^2 \longrightarrow \infty$, then we are in Diagram I, Diagram II, Diagram III or Diagram IV.

Then the masses belong to the set

$$\begin{gathered} \mathcal{V}_{I0}[13,24] \bigcup \mathcal{V}_{I0}[14,23] \bigcup \mathcal{V}_{II0}[12,34] \\ \bigcup (\mathcal{V}_{III}[134]\mathcal{V}_{IV}[134]) \bigcup (\mathcal{V}_{III}[234]\mathcal{V}_{IV}[234]) \bigcup \mathcal{V}_{IV}[134] \bigcup \mathcal{V}_{IV}[234] \triangleq V[12,34]. \end{gathered} \tag{38}$$

Similarly, when considering $r_{13}^4 r_{24}^2 \longrightarrow \infty$ the masses belong to the set

$$\begin{gathered} \mathcal{V}_{I0}[12,34] \bigcup \mathcal{V}_{I0}[14,23] \bigcup \mathcal{V}_{II0}[13,24] \\ \bigcup (\mathcal{V}_{III}[124]\mathcal{V}_{IV}[124]) \bigcup (\mathcal{V}_{III}[234]\mathcal{V}_{IV}[234]) \bigcup \mathcal{V}_{IV}[124] \bigcup \mathcal{V}_{IV}[234] \triangleq V[13,24]; \end{gathered} \tag{39}$$

when considering $r_{23}^4 r_{14}^2 \longrightarrow \infty$ the masses belong to the set

$$\begin{gathered} \mathcal{V}_{I0}[13,24] \bigcup \mathcal{V}_{I0}[12,34] \bigcup \mathcal{V}_{II0}[14,23] \\ \bigcup (\mathcal{V}_{III}[124]\mathcal{V}_{IV}[124]) \bigcup (\mathcal{V}_{III}[134]\mathcal{V}_{IV}[134]) \bigcup \mathcal{V}_{IV}[124] \bigcup \mathcal{V}_{IV}[134] \triangleq V[23,14]. \end{gathered} \tag{40}$$

Or

$$\begin{array}{l} V[12,34] = (\mathcal{V}_{I0}[13,24]) \bigcup (\mathcal{V}_{I0}[14,23]) \bigcup (\mathcal{V}_{II0}[12,34]) \bigcup \mathcal{V}_{IV}[134] \bigcup \mathcal{V}_{IV}[234]; \\ V[13,24] = (\mathcal{V}_{I0}[12,34]) \bigcup (\mathcal{V}_{I0}[14,23]) \bigcup (\mathcal{V}_{II0}[13,24]) \bigcup \mathcal{V}_{IV}[124] \bigcup \mathcal{V}_{IV}[234]; \\ V[23,14] = (\mathcal{V}_{I0}[13,24]) \bigcup (\mathcal{V}_{I0}[12,34]) \bigcup (\mathcal{V}_{II0}[14,23]) \bigcup \mathcal{V}_{IV}[124] \bigcup \mathcal{V}_{IV}[134]. \end{array} \tag{41}$$

Other than the above relations, recall that we have assumed that the masses belong to $\mathcal{V}_{IV}[123]$.

Case 1: If Diagram I occurs. Without loss of generality, assume that the masses belong to the set $\mathcal{V}_{I0}[13,24]$, then we claim that

$$(\mathcal{V}_{I0}[13,24]) \bigcap V[13,24] \bigcap \mathcal{V}_{IV}[123] = \emptyset. \tag{42}$$

A straightforward computation shows the claim.

Case 2: If Diagram II occurs. Without loss of generality, assume that the masses belong to the set $\mathcal{V}_{II0}[12,34]$, then we claim that

$$(\mathcal{V}_{II0}[12,34])\bigcap V[13,24]\bigcap \mathcal{V}_{IV}[123]\bigcap V[23,14]=\emptyset. \tag{43}$$

A straightforward computation shows the claim.

Therefore, Diagram I and Diagram II can not occur and the masses belong to the following set

$$\mathcal{V}_{IV}[123]\left(\mathcal{V}_{IV}[124]\mathcal{V}_{IV}[134]\bigcup \mathcal{V}_{IV}[124]\mathcal{V}_{IV}[234]\bigcup \mathcal{V}_{IV}[134]\mathcal{V}_{IV}[234]\right),$$

which contains only three solutions

$$m_2=m_3=m_4=-\tfrac{1}{2}m_1,\ m_1=m_3=m_4=-\tfrac{1}{2}m_2,\ m_1=m_2=m_4=-\tfrac{1}{2}m_3.$$

It is easy to see that the first solution is not consistent with the constraints corresponding to Diagram I, Diagram II, Diagram III or Diagram V, therefore, only Diagram IV and Diagram VI are possible now. We consider the function $r_{12}^2r_{23}^2r_{13}^2r_{14}^2r_{24}^2r_{34}^2$, then it is easy to see that it is dominating. When considering $r_{12}^2r_{23}^2r_{13}^2r_{14}^2r_{24}^2r_{34}^2\to 0$, we are in Diagram VI, thus the function $r_{12}^2r_{13}^2r_{14}^2$ is dominating. When considering $r_{12}^2r_{13}^2r_{14}^2\to\infty$, we are in Diagram IV with $\triangle_{234}\longrightarrow 0$. As a result, we have $m_2+m_3+m_4=0$, which is a contradiction.

Similarly, the other two solutions do not admit infinite solutions neither.

To summarize, we proved that the system (3) possesses finitely many solutions, if $\sum_{j=1}^4 m_j\neq 0$, $\prod_{1\leq j<k<l\leq 4}(m_j+m_k+m_l)=0$ and $I=0$.

□

6 Finiteness of central configurations with $m=0$

In the first subsection, we consider the diagrams agreeable with the condition that the total mass $m=0$. The constraints on the masses, as well as the orders the positions and distances, are studied. The rest of this section is devoted to prove the finiteness of central configurations with zero total mass.

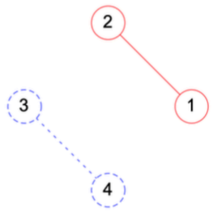
6.1 Problematic diagrams with zero total mass

Recall that only 8 diagrams are compatible with zero total mass; see Figure 10.

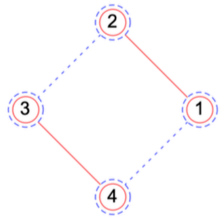
6.1.1 Diagram I

Note that the hypothesis that $m\neq 0$ is not utilized in Subsection 5.1.1, thus the estimates on the distances and mass polynomials derived there still holds for the present case. In particular, we have

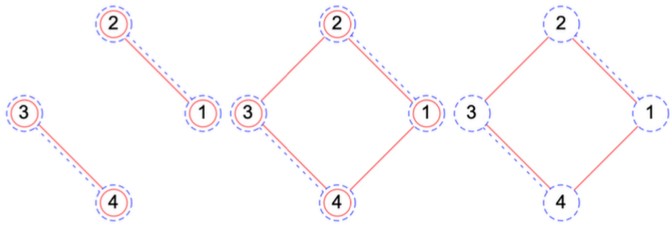
$$\begin{aligned}&(m_1+m_2)(m_3+m_4)\neq 0,\\ &r_{12}^3\sim m_1+m_2,\ r_{34}^3\sim m_3+m_4,\ r_{jk}\approx\epsilon^{-2}, j=1,2,k=3,4.\end{aligned} \tag{44}$$



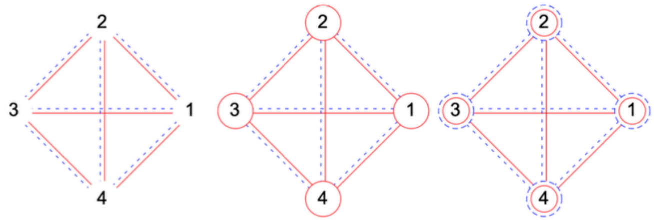
(a) Diagram I



(b) Diagram II



(c) Diagrams with two zw-edges



(d) Fully edged diagrams

Figure 10: Diagrams compatible with zero total mass

The masses corresponding to Diagram I belong to one of the following two sets

$$\begin{aligned}
&\mathcal{V}_{IA0}[12,34] :, && m_1 = m_2,\ m_3 = m_4,\ m_1 + m_3 = 0;\\
&\mathcal{V}_{IB}[12,34] : && m_1m_3\sqrt{m_1m_3} + m_2m_3\sqrt{-m_2m_3} + m_1m_4\sqrt{-m_1m_4} + m_2m_4\sqrt{m_2m_4} = 0,\\
& && m_1m_2 < 0,\ m_3m_4 < 0, (m_1+m_2)(m_3+m_4) \neq 0.
\end{aligned}$$

6.1.2 Diagram II

Following the argument from [1], the masses corresponding to Diagram II satisfy $m_1m_3 = m_2m_4$. Furthermore, we claim that

$$m_1 + m_2 = m_2 + m_3 = m_3 + m_4 = m_1 + m_4 = 0. \tag{45}$$

Indeed, if $m_1 + m_2 \neq 0$, then we have

$$z_{12} \approx \epsilon^{-2},\ m_3 + m_4 \neq 0,\ z_{34} \approx \epsilon^{-2}.$$

By the fact $z_{12} = (m_1 + m_2)Z_{12} + m_3(Z_{32} - Z_{31}) + m_4(Z_{42} - Z_{41})$, we have $z_{12} \sim (m_1 + m_2)Z_{12}$ and then $w_{12} \approx \epsilon^2$. Hence, $W_{12} = w_{12}^{-1/2} z_{12}^{-3/2} \approx \epsilon^2$. Similarly, $W_{34} \approx \epsilon^2$. We also would have $z_{13} \approx \epsilon^{-2}$ and $w_{13} \succ \epsilon$, then $W_{13} \prec \epsilon^2$. Similarly, $W_{24} \prec \epsilon^2$.

On the other hand, note $z_{13} \sim z_{12}, z_{14} \prec \epsilon$. The identity

$$0 = \sum_{j=1}^{4} m_j z_1 = \sum_{j=2}^{4} m_j z_{j1}$$

implies that $m_2 + m_3 = 0$, then $m_1 + m_4 = 0$.

Hence, we have $w_{14} = (m_1 + m_4)W_{14} + m_2(W_{24} - W_{21}) + m_3(W_{34} - W_{31}) \preceq \epsilon^2$. By Estimate 1, $w_{14} \approx \epsilon^2$. Note that $W_{14} \approx \epsilon^{-2}$. It follows that $r_{14} \approx \epsilon^{4/3}$, which contradicts with Estimate 1. Thus, equation (45) follows.

Note that

$$r_{12}, r_{23}, r_{34}, r_{14} \prec 1, r_{13} \prec \epsilon^{-2}, r_{24} \prec \epsilon^{-2}. \tag{46}$$

6.1.3 Diagrams with two zw-edges

Obviously, we have $m_1 + m_2 = m_3 + m_4 = 0$. Then the identities $0 = \sum_{i=1}^{4} m_i z_i = \sum_{i=1}^{4} m_i w_i$ imply

$$m_1 z_{12} = -m_3 z_{34}, \; m_1 w_{12} = -m_3 w_{34}. \tag{47}$$

For the last two diagrams with two zw-edges in Figure 10, we have

$$w_{23} = (m_2 + m_3)W_{23} + m_1(W_{13} - W_{12}) + m_4(W_{43} - W_{42}) \prec \epsilon^{-2},$$

which leads to

$$m_1 W_{12} \sim m_3 W_{34}. \tag{48}$$

With (47), we have $m_1^2 = m_3^2$.

For the first diagram with two zw-edges in Figure 10, we claim that at least one of z_{23}, w_{23} shall have order less than ϵ^{-2}. Otherwise, $W_{23} \approx W_{24} \approx W_{13} \approx W_{14} \approx \epsilon^4$. This contradicts with the equation

$$w_{12} = (m_1 + m_2)W_{12} + m_3(W_{32} - W_{31}) + m_4(W_{42} - W_{41}).$$

Then we have either

$$m_1 W_{12} \sim m_3 W_{34}, \; \textit{or} \; m_1 Z_{12} \sim m_3 Z_{34}. \tag{49}$$

With (47), we have $m_1^2 = m_3^2$. Therefore, the masses corresponding to Diagrams with two zw-edges satisfy

$$m_1 + m_2 = m_3 + m_4 = 0, \; m_1^2 = m_3^2. \tag{50}$$

6.1.4 Fully edged diagrams

We will reach our result without discussing the mass polynomial of this diagram.

6.2 Finiteness of Central configurations with $m = 0$

Theorem 6.1. *Suppose that m_1, m_2, m_3, m_4 are real and nonzero with $\sum_{i=1}^4 m_j = 0$, then system (3), which defines the normalized central configurations in the complex domain, possesses finitely many solutions except perhaps for, up to renumeration of the bodies, two groups of masses:*

$$m_1 = m_2 = -m_3 = -m_4,\ m_1 = m_2, m_3 = -\alpha^2 m_1, m_4 = -(2-\alpha^2)m_1,$$

where α is the unique positive root of equation (51) [1]*.*

Note that if Diagram I with $\mathcal{V}_{IA0}$, Diagram II, or Diagrams with two zw-edges occurs, then $m_k = \pm m_1, k = 2,3,4$. Thus, we assume that Diagram I with $\mathcal{V}_{IA0}$, Diagram II, and Diagrams with two zw-edges do not occur in the following discussion.
Proof of Theorem 6.1: It is easy to see that giving five of r_{kl}^2's, $1 \le k < l \le 4$, determines only finitely many geometrical configurations up to rotation. Suppose that there are infinitely many solutions of system (4) in the complex domain. Then at least two of $\{r_{kl}^2\}$'s must take infinitely many values and thus are dominating by Lemma 3.1. Suppose that r_{kl}^2 is dominating for some $1 \le k < l \le 4$. There must exist a singular sequence of central configurations with $r_{kl}^2 \longrightarrow 0$, which happens only in the fully edged diagrams. In each case, it is easy to see that all of r_{kl}^2's, $1 \le k < l \le 4$, and $r_{13}^2 r_{24}^2,\ r_{14}^2 r_{23}^2,\ r_{12}^2 r_{34}^2$ are dominating.

Consider a singular sequence with $r_{12}^2 \longrightarrow \infty$, then the masses must belong to $\mathcal{V}_{IB}[13,24] \bigcup \mathcal{V}_{IB}[14,23]$. Consider a singular sequence with $r_{13}^2 \longrightarrow \infty$, then the masses must belong to $\mathcal{V}_{IB}[12,34] \bigcup \mathcal{V}_{IB}[14,23]$. Consider a singular sequence with $r_{14}^2 \longrightarrow \infty$, then the masses must belong to $\mathcal{V}_{IB}[12,34] \bigcup \mathcal{V}_{IB}[13,24]$. Hence, at least two of $\mathcal{V}_{IB}[12,34]$, $\mathcal{V}_{IB}[13,24]$ and $\mathcal{V}_{IB}[14,23]$ occur. Without loss of generality, assume that $\mathcal{V}_{IB}[13,24]$ and $\mathcal{V}_{IB}[14,23]$ occur. That is, the masses belong to

$$\mathcal{V}_{IB}[13,24] \bigcap \mathcal{V}_{IB}[14,23] \bigcap \mathcal{V}_{m0},$$

where $\mathcal{V}_{m0}$ denotes the set $\sum_{j=1}^4 m_j = 0$.

It is easy to see $m_1, m_2 > 0, m_3, m_4 < 0$. Then a straightforward computation shows that the set $\mathcal{V}_{IB}[13,24] \bigcap \mathcal{V}_{IB}[14,23] \bigcap \mathcal{V}_{m0}$ is empty, provided that $\prod_{i \ne j}(|m_i| - |m_j|) \ne 0$. Note that

$$(m_1 + m_3)(m_2 + m_4) \ne 0,\ (m_1 + m_4)(m_2 + m_3) \ne 0,$$

Hence, assume that $m_1 = m_2$, or $m_3 = m_4$. Then $\mathcal{V}_{IB}[13,24] \bigcap \mathcal{V}_{IB}[14,23] \bigcap \mathcal{V}_{m0}$ equals

$$\{m_1 = m_2, m_3 = -\alpha^2 m_1, m_4 = -(2-\alpha^2)m_1\} \bigcup \{m_1 = m_2, m_3 = -(2-\alpha^2)m_1, m_4 = -\alpha^2 m_1\}$$
$$\bigcup \{m_1 = -\alpha^2 m_3, m_2 = -(2-\alpha^2)m_1, m_3 = m_4\} \bigcup \{m_1 = -(2-\alpha^2)m_3, m_2 = -\alpha^2 m_1, m_3 = m_4\}$$

where $\alpha \in (1, \sqrt{2})$. A straightforward computation shows that α must be the unique positive root of the following equation:

$$\alpha^{12} - 6\alpha^{10} + 2\alpha^9 + 12\alpha^8 - 12\alpha^7 - 6\alpha^6 + 24\alpha^5 - 6\alpha^4 - 18\alpha^3 + 12\alpha^2 - 7 = 0. \quad (51)$$

□

[1] Numerically, $\alpha \approx 1.2407$.

6.3 Finiteness with $m_1 = m_2 = -m_3 = -m_4$, or $m_1 = m_2, m_3 = -\alpha^2 m_1, m_4 = -(2-\alpha^2)m_1$.

For the two exceptional groups of masses listed in Theorem 6.1, instead of counting normalized central configurations in the complex domain, we restrict attention to central configurations, i.e., solutions of system (2).

Theorem 6.2. *Suppose that m_1, m_2, m_3, m_4 are real and nonzero, if $m_1 = m_2 = -m_3 = -m_4$, or $m_1 = m_2, m_3 = -\alpha^2 m_1, m_4 = -(2-\alpha^2)m_1$, where α is the unique positive root of equation (51), then system* (2) *possesses finitely many solutions.*

6.3.1 The collinear case

Proof of Theorem 6.2, Part 1: Assume that the masses are on the x-axis, and x_i is the coordinate of m_i. Then system (2) becomes

$$\begin{aligned} \sigma x_1 &= m_2 \frac{x_1 - x_2}{|x_1 - x_2|^3} + m_3 \frac{x_1 - x_3}{|x_1 - x_3|^3} + m_4 \frac{x_1 - x_4}{|x_1 - x_4|^3} \\ &\cdots \\ \sigma x_4 &= m_1 \frac{x_4 - x_1}{|x_4 - x_1|^3} + m_2 \frac{x_4 - x_2}{|x_4 - x_2|^3} + m_3 \frac{x_4 - x_3}{|x_4 - x_3|^3}, \end{aligned} \tag{52}$$

where $\sigma = \pm 1$.

If $m_1 = m_2 = -m_3 = -m_4$, without loss of generality, assume $x_1 < x_i$, $i = 2, 3, 4$. Since $\sum m_i x_i = 0$, it suffices to assume that the ordering is $x_1 < x_3 < x_4 < x_2$. A straightforward computation shows that system (52) has finitely many solutions.

If $m_1 = m_2, m_3 = -\alpha^2 m_1, m_4 = -(2-\alpha^2)m_1$, by the reflection symmetry and the fact that $m_1 = m_2$, it suffices to consider the following six orderings: $x_1 < x_2 < x_3 < x_4$, $x_1 < x_2 < x_4 < x_3$, $x_1 < x_4 < x_2 < x_3$, $x_4 < x_1 < x_2 < x_3$, $x_1 < x_3 < x_2 < x_4$, $x_1 < x_4 < x_2 < x_3$. Note that the first two orderings are impossible since $\frac{x_1+x_2}{2} \in (x_1, x_2)$, and $\frac{-m_3x_3 - m_4x_4}{2} > x_2$. The third ordering is also impossible since

$$x_4 = \frac{x_1 + x_2 + m_3 x_3}{-m_4} = \frac{(2 + m_3)x_1 + (1 + m_3)(x_2 - x_1) - m_3(x_2 - x_3)}{-m_4} < x_1.$$

For each of the last three orderings, a straightforward computation shows that system (52) has finitely many solutions.

□

6.3.2 The planar case

Recall that solving system (2) is equivalent to finding critical points of the potential restricted on the set $I = c$. For these two groups of masses, following Celli, [2], we use r_{jk} as coordinates. For system with zero total mass, there are extra constraints, namely

$$\sum_{k \neq 1} m_k r_{1k}^2 = \sum_{k \neq 2} m_k r_{2k}^2 = \sum_{k \neq 3} m_k r_{3k}^2 = \sum_{k \neq 4} m_4 r_{4k}^2 = c_0.$$

Then system (2) is equivalent to the following system

$$\begin{array}{ll} m_j m_k + 2(\lambda_j m_k + \lambda_k m_j) r_{jk}^3 = 0, & 1 \le j < k \le 4 \\ \sum_{j \neq k} m_j r_{jk}^2 = I, & 1 \le k \le 4, \end{array} \tag{53}$$

where λ_j, $1 \le j \le 4$, are the Lagrange multiplier.
Proof of Theorem 6.2, Part 2: Consider system (53).

Case 1: $m_1 = m_2 = -m_3 = -m_4$. In this case, it is easy to see that $r_{12} = r_{14} = r_{23} = r_{24}$ By imposing the normalization $r_{24} = 1$ and setting $r_{34} = t$, system (53) reduces to

$$r_{12}^2 = 4 - t^2, \ \frac{1}{(4 - t^2)^{3/2}} + \frac{1}{t^3} = 2.$$

Thus, $t \approx 0.816713, 1.825645$ and the system has only two solutions.

Case 2: $m_1 = m_2, m_3 = -\alpha^2 m_1, m_4 = -(2 - \alpha^2) m_1$. In this case, after eliminating the λ_j and I we have the relations:

$$\frac{1}{r_{12}^3} + \frac{1}{{r_{34}}^3} = \frac{1}{r_{13}^3} + \frac{1}{{r_{24}}^3} = \frac{1}{r_{14}^3} + \frac{1}{{r_{23}}^3}, \ \alpha^2 \left(r_{13}^2 - r_{23}^2\right) = (2 - \alpha^2) \left(r_{24}^2 - r_{14}^2\right).$$

It follows that $r_{13} - r_{23} = r_{14} - r_{24} = 0$, then by imposing the normalization $r_{34} = 1$, system (53) reduces to

$$\begin{array}{c} \frac{1}{r_{12}^3} = \frac{1}{r_{13}^3} + \frac{1}{r_{14}^3} - 1, \\ r_{12}^2 - (r_{13}^2 - 1)\,\alpha^2 - r_{14}^2 (4 - \alpha^2) = 0, \\ r_{13}^2 - r_{14}^2 + \alpha^2 - 1 = 0. \end{array} \tag{54}$$

The last two equations of system (54) imply $r_{12}^2 = 4 r_{14}^2 - \alpha^4$. Then r_{14} satisfies

$$\frac{1}{(4r_{14}^2 - \alpha^4)^{3/2}} = \frac{1}{(r_{14}^2 + 1 - \alpha^2)^{3/2}} + \frac{1}{r_{14}^3} - 1,$$

which has finitely many solutions. Thus, system (54) has finitely many solutions.

□

Appendix

For the theorems presented in Section 5 and Section 6, only outlines are initially provided. In this section, we furnish the complete details or guide you to where these details can be found. Without loss of generality, let $m_1 = 1$.

Computations in the proof of Theorem 5.3 can be found in [12].

Computations in the proof of Theorem 5.4

Case 1: The masses are in $\mathcal{V}_{II0}[13, 24]\mathcal{V}_{II0}[14, 23]\mathcal{V}_{II0}[12, 34]$. We have $m_2^2 = m_3^2 = m_4^2 = 1 = m_1$ and $m_i + m_j \neq 0$. Then $m_1 = m_2 = m_3 = m_4 = 1$, which does not belong to $\mathcal{I}_{II}[13, 24]$, a contradiction.

Case 2: Without loss of generality, consider only the case $\mathcal{V}_{II0}[13,24]\mathcal{V}_{II0}[14,23]\mathcal{V}_{I0}[13,24]$. First we have

$$m_1 = m_2^2 = 1,\ m_3^2 = m_4^2,\ (m_1+m_2)(m_2+m_3)(m_3+m_4) \neq 0,\ m_1m_3 < 0,\ m_2m_4 < 0.$$

Thus, we set the masses are $m_1 = m_2 = 1, m_3 = m_4 = s < 0, s \neq -1$, which does not belong to $\mathcal{I}_I[13,24]$, a contradiction.

Case 3: The masses are in either $\mathcal{V}_{II0}[13,24]\mathcal{V}_{I0}[12,34]\mathcal{V}_{I0}[14,23]$ or $\mathcal{V}_{II0}[13,24]\mathcal{V}_{I0}[13,24]$.

In the first subcase, the signs of the masses are $(+,-,+,-)$. Set $m_2 = s < 0, m_4 = t < 0$. Then $m_3 = st$. Then that the masses are in $\mathcal{I}_I[12,34]$ implies

$$\frac{s}{\sqrt[3]{1+s}} + \frac{st^2}{\sqrt[3]{t}\sqrt[3]{1+s}} = 0.$$

Then $t = -1$ and $m_1 + m_4 = 0$, hence the masses are not in $\mathcal{V}_{IB}[14,23]$, a contradiction.

In the second subcase, we have $m_1m_3 = m_2m_4$. Then the masses are in $\mathcal{I}_I[13,24]$ implies

$$m_1 + m_3 = -(m_2 + m_4),$$

which contradicts with the hypothesis that the total mass $m \neq 0$.

Case 4: The masses are in $\mathcal{V}_{I0}[12,34]\mathcal{V}_{I0}[13,24]$. We have checked that the set is empty in [12].

Computations in the proof of Lemma 5.1, Theorem 5.5 can be found in [12].

Computations in the proof of Theorem 5.6

Case 1: The masses are in $\mathcal{V}_{I0}[13,24]\mathcal{V}[13,24]\mathcal{V}_{IV}[123]$. Recall that

$$V[13,24] = (\mathcal{V}_{I0}[12,34])\bigcup(\mathcal{V}_{I0}[14,23])\bigcup(\mathcal{V}_{II0}[13,24])\bigcup\mathcal{V}_{IV}[124]\bigcup\mathcal{V}_{IV}[234].$$

We ignore the first two choices $\mathcal{V}_{I0}[12,34]\bigcup\mathcal{V}_{I0}[14,23]$, since we have showed

$$\mathcal{V}_{I0}[13,24](\mathcal{V}_{I0}[12,34]\bigcup\mathcal{V}_{I0}[14,23]) = \emptyset,$$

see **Case 4 of the proof of Theorem 5.4.** Hence, there are three possibilities.

If the masses belong to $\mathcal{V}_{I0}[13,24]\mathcal{V}_{II0}[13,24]\mathcal{V}_{IV}[123]$. Then $m_1m_3 = m_2m_4$ and the masses are in $\mathcal{I}_I[13,24]$. Then we have zero total mass, a contradiction.

If the masses belong to $\mathcal{V}_{I0}[13,24]\mathcal{V}_{IV}[124]\mathcal{V}_{IV}[123]$, then $m_3 = m_4$. Set $m_3 = s$, then $m_2 = -(s+1)$. The fact that the masses are in $\mathcal{I}_I[13,24]$ implies

$$\frac{s}{\sqrt[3]{s+1}} + s(s+1) = 0,$$

which is impossible. Similarly, the masses can not belong to $\mathcal{V}_{I0}[13,24]\mathcal{V}_{IV}[234]\mathcal{V}_{IV}[123]$.

Case 2: The masses are in $\mathcal{V}_{II0}[12,34]\mathcal{V}[13,24]\mathcal{V}_{IV}[123]\mathcal{V}[23,14]$. In fact, the set $\mathcal{V}_{II0}[12,34]\mathcal{V}[13,24]\mathcal{V}_{IV}[123]$ is empty. Since $m_2 = m_3m_4$ and $m_1 + m_2 + m_3 = 0$, we have

$$m_1 = 1,\ m_2 = \frac{-t}{1+t},\ m_3 = \frac{-1}{1+t},\ m_4 = t.$$

By Case 1, Diagram I can not occur, then the masses are in $\mathcal{V}_{II0}[13,24]\bigcup\mathcal{V}_{IV}[124]\bigcup\mathcal{V}_{IV}[234]$.

If the masses belong to $\mathcal{V}_{II0}[13,24]$, then $m_3=m_2m_4$, so we have $t=1$ (since $m_1+m_4\neq 0$). Thus, $m_2=m_3=-\frac{1}{2}$. This contradicts with the fact that the masses are in $\mathcal{I}_{II}[13,24]$, i.e.,

$$0=\frac{m_1+m_3}{\sqrt[3]{m_1+m_3}}+\frac{m_3+m_2}{\sqrt[3]{m_3+m_2}}+\frac{m_2+m_4}{\sqrt[3]{m_2+m_4}}+\frac{m_4+m_1}{\sqrt[3]{m_4+m_1}}.$$

If the masses belong to $\mathcal{V}_{IV}[124]$, then $m_3=m_4$, or $(t+1)t+1=0$, which has no real solution. Similarly, the masses can not belong to $\mathcal{V}_{IV}[234]$.

Computations in the proof of Theorem 6.1, Theorem 6.2 can be found in [12].

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Xiang Yu, Center for Applied Mathematics and KL-AAGDM, Tianjin University, Tianjin, 300072, China, xiang.zhiy@foxmail.com, yuxiang_math@tju.edu.cn

Shuqiang Zhu, School of Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China, zhusq@swufe.edu.cn