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Semigroups generated by elliptic operators in non-divergence form on unbounded domains

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Abstract: We consider an elliptic operator $\mathcal{A}$ of second order in non-divergence form on an unbounded domain $\Omega \subset \mathbb{R}^n$ which satisfies the local uniform exterior cone condition. Based on local Hölder estimates due to Krylov and Safonov we show well-posedness of the Dirichlet and Poisson problem. Then we show that the part of $\mathcal{A}$ in $L^\infty(\Omega)$ generates a holomorphic semigroup on $L^\infty(\Omega)$ whose restriction to

$$C_0(\Omega) := \left\{ u \in C\left(\overline{\Omega}\right) : u_{|\partial\Omega} = 0, \lim_{|x|\to\infty} u(x) = 0 \right\}$$

is strongly continuous at $t = 0$. In the case where Ω is bounded such results had been proved before in [ASch14].

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1 Introduction

Elliptic operators in non-divergence form are an important object of study, and optimal results for the Poisson equation and the Dirichlet problem for bounded domains satisfying merely the uniform exterior cone condition are known, see the monograph of Gilbarg-Trudinger [GT] Chapter 9.

Concerning parabolic equations in terms of generation theorems for semigroups a complete and beautiful theory is developed by Lunardi in the monograph [Lu95]. It is based on Agmon-Douglis-Nirenberg estimates which require strong regularity conditions for the boundary (C^2-boundary is supposed throughout in [Lu95]).

The aim of the paper [ASch14] was to establish generation results for bounded domains with weaker conditions at the boundary admitting in particular Lipschitz domains.
The purpose of the present paper is to extend such results to unbounded domains. Although the results on bounded domains can be used, it turns out that the task is not straightforward. Indeed, in the unbounded case also certain elliptic results are missing. In the present paper we first study the Poisson and the Dirichlet problem for elliptic operators in non-divergence form on unbounded domains. Then we show that they generate semigroups on $L^\infty(\Omega)$ and on spaces of continuous functions. As a consequence, one obtains well-posedness results for parabolic equations with Dirichlet boundary conditions under minimal regularity conditions on the boundary and the coefficients.
We now describe in more detail the results we are going to prove. The following notation and assumption will be used throughout the paper.

Let $\Omega \subset \mathbb{R}^n$ be an open (possibly unbounded) set with boundary $\partial\Omega$. **We assume throughout this paper that**

$$\Omega \text{ satisfies the local uniform exterior cone condition.} \tag{1.1}$$

This means the following: For each $R > 0$ there exists a finite, right circular cone V such that for each $x \in \partial\Omega$ satisfying $|x| \leq R$ there exists a cone V_x which is congruent to V such that $V_x \cap \overline{\Omega} = \{x\}$, cf. [GT] p.205.

We consider real coefficients $a_{ij} \in C^b(\Omega)$, such that $a_{ij} = a_{ji}$, which satisfy the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j \geq \nu|\xi|^2 \tag{1.2}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$ where $\nu > 0$ is fixed. Moreover, the coefficients $b_j, c \in L^\infty(\Omega)$ are also real, $j = 1, \ldots, n$ and $c \leq 0$.

Then we define the operator $\mathcal{A} : W^{2,n}_{loc}(\Omega) \to L^n_{loc}(\Omega)$ by

$$\mathcal{A}u = \sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j u + \sum_{j=1}^{n} b_j\partial_j u + cu.$$

In the first part we consider real spaces (Section 2-4), whereas for the study of holomorphic semigroups spaces are complex. The coefficients are real, throughout. For $\Gamma \subset \mathbb{R}^n$ we denote by $C(\Gamma)$ the space of all continuous functions from Γ to $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $C^b(\Gamma) = \{u \in C(\Gamma) : u \text{ is bounded}\}$. The space $C_0(\Omega) := \left\{u \in C\left(\overline{\Omega}\right) : u_{|\partial\Omega} = 0\,, \lim_{|x|\to\infty} u(x) = 0\right\}$ is a Banach space for the supremum norm. It coincides with the closure of $C_c^\infty(\Omega)$ in $C^b\left(\overline{\Omega}\right)$, where $C_c^\infty(\Omega)$ is the space of all function of class C^∞ on Ω with compact support.
If $f \in L^\infty(\Omega)$, we say that f **vanishes at infinity** and write $\lim_{|x|\to\infty} f(x) = 0$ or $f(x) \to 0$ as $|x| \to \infty$, if $\lim_{R\to\infty} \|f\|_{L^\infty(\Omega\setminus B_R(0))} = 0$.
Here, for $x \in \mathbb{R}^n$, $R > 0$ we denote by $B_R(x) = \{y \in \mathbb{R}^n : |x - y| < R\}$ the open ball of center x and radius $R > 0$.

Theorem 1.1 *Let $g \in C^b(\partial\Omega)$, $f \in L^\infty(\Omega)$, $\lambda > 0$. Then there exists a unique $u \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C^b\left(\overline{\Omega}\right)$ satisfying*

$$\begin{aligned} \lambda u - \mathcal{A}u &= f \quad \text{in } \Omega, \\ u_{|\partial\Omega} &= g. \end{aligned} \tag{1.3}$$

Moreover, if $f \geq 0, g \geq 0$, then $u \geq 0$.
If f and g vanish at infinity, then also u vanishes at infinity. □

If Ω is bounded, then Theorem 1.1 is proved in [ASch14] Corollary 3.4 (b), where the hypothesis on the coefficients, namely $a_{ij} \in C(\overline{\Omega})$, is not needed, and our more general hypothesis $a_{ij} \in C^b(\Omega)$ suffices. If Ω has uniform C^2-boundary, and λ is large then some version of Theorem 1.1 is [Lu95] Proposition 3.1.20.

In our proof of existence in Theorem 1.1 given here, we can use the results of [ASch14] on $\Omega \cap B_R(0)$ and pass to the limit as $R \to \infty$. But the proof of convergence is quite subtle using the local Hölder estimates due to Krylov and Safonov, see [GT] Section 9.8 and 9.9.

For the proof of uniqueness the following problem occurs which is due to Ω being unbounded. A solution $u \in C^b\left(\overline{\Omega}\right)$ which vanishes at $\partial\Omega$ need not attain a maximum. To overcome this difficulty we approximate u by functions that vanish at infinity and which also solve an elliptic equation with modified coefficients.
Uniqueness depends crucially on the assumption that $\lambda > 0$. In fact, for $\lambda = 0$, several solutions may exist if $n \geq 3$.
Concerning the parabolic problem, we consider the part A_0 of $\mathcal{A}$ in $C_0(\Omega)$ defined by

$$\begin{aligned} D(A_0) &= \left\{ u \in W^{2,n}_{loc}(\Omega) \cap C_0(\Omega) : \mathcal{A}u \in C_0(\Omega) \right\}, \\ A_0 u &:= \mathcal{A}u. \end{aligned}$$

Theorem 1.2 *The operator A_0 generates a contractive, positive C_0-semigroup T_0 on $C_0(\Omega)$.* □

Here we use the standard definition of a C_0-semigroup, see also Section 5. Concerning holomorphic semigroups we consider also semigroups which are not continuous at 0. The definition and basic properties are recalled at the beginning of Section 5. In order to give a precise statement of the next result it suffices to know the following. An operator B on a complex Banach space X generates a holomorphic semigroup S if and only if there exists $\omega \in \mathbb{R}$ such that $(\lambda - B)$ is invertible whenever $\mathrm{Re}\lambda > \omega$ and $\sup_{Re\lambda > \omega} \| \lambda(\lambda - B)^{-1} \|_{\mathcal{L}(X)} < \infty$. The semigroup S generated by B is a continuous mapping from $(0, \infty) \to \mathcal{L}(X)$ which has a holomorphic extension to a sector (see [Lu95]). It is a C_0-**semigroup** (i.e., $\lim_{t \to 0+} S(t)x = x$ for all $x \in X$) if and only if the domain $D(B)$ of B is dense in X. More information will be given in Section 5.

For our holomorphic generation results we consider $\mathbb{K} = \mathbb{C}$; that is, the spaces $L^\infty(\Omega), C_0(\Omega), W^{2,n}_{loc}(\Omega)$ consist of complex valued functions (but the coefficients of the operator $\mathcal{A}$ are real). We define the part A_∞ of $\mathcal{A}$ in $L^\infty(\Omega)$ by

$$\begin{aligned} D(A_\infty) &:= \left\{ u \in W^{2,n}_{loc}(\Omega) \cap L^\infty(\Omega) : \mathcal{A}u \in L^\infty(\Omega) \right\}, \\ A_\infty u &:= \mathcal{A}u. \end{aligned}$$

Theorem 1.3 *Assume in addition to our general assumptions that the coefficients a_{ij} are uniformly continuous. Then A_∞ generates a holomorphic semigroup T. Moreover,*

$$\| T(t) \|_{\mathcal{L}(L^\infty(\Omega))} \leq 1 \quad (t > 0)$$

and $T(t)f \geq 0$ if $f \geq 0$. □

In the case where Ω is of class C^2 and under slightly stronger assumptions on the coefficients Theorem 1.3 is proved by Lunardi [Lu95] Corollary 3.1.21. Theorem 1.3 is known for $\Omega = \mathbb{R}^n$ (see [ASch14], [Lu95]).

The complex maximum principle of [ASch14] allows us to carry over the holomorphic estimates from $\mathbb{R}^n$ to Ω. An approximation technique is needed here, see also Theorem 2.1.

In terms of parabolic differential equations Theorem 1.2 and 1.3 imply the following well-posedness result.

Corollary 1.4 *Let $u_0 \in C_0(\Omega)$. Then there exists a unique function $u \in C([0,\infty), C_0(\Omega)) \cap C^\infty((0,\infty), C_0(\Omega))$ such that*

$$u(t) \in W^{2,n}_{loc}(\Omega) \text{ and}$$
$$\dot{u}(t) = \mathcal{A}u(t), u(0) = u_0.$$

The solution is given as $u(t) = T_0(t)u_0$. □

We also determine the asymptotic behavior of the solution of the parabolic equation as $t \to \infty$.

Theorem 1.5 *In addition to the hypotheses of Theorem 1.3 assume that $\Omega \neq \mathbb{R}^n$ or $c \neq 0$. Then*

$$\lim_{t\to\infty} T(t)f = 0 \quad \text{in } L^\infty(\Omega)$$

for all $f \in L^\infty(\Omega)$ which vanish at infinity. □

The proof is based on a Tauberian theorem and a result on the image of A_∞ which we prove in Section 2.

The paper is organized as follows. We first study elliptic problems, the Poisson problem in Section 2, and the Dirichlet problem in Section 3. In Section 4 we show that A_0 generates a C_0-semigroup. It is holomorphic if the a_{ij} are uniformly continuous. This together with Theorem 1.3 is proved in Section 5. In Section 6 Theorem 1.5 on the asymptotic behavior is proved.

2 The Poisson problem for operators in non-divergence form

In this section, we solve the Poisson problem for unbounded open sets for operators in non-divergence form. We keep the assumptions of the introduction. In particular, $\mathcal{A} : W^{2,n}_{loc}(\Omega) \to L^n_{loc}(\Omega)$ is the given differential operator. In this section, we let $\mathbb{K} = \mathbb{R}$, i.e. all functions and spaces are real.

The Poisson problem was solved in [Lu95] Proposition 3.1.20 under the stronger conditions that λ is large and Ω is of class C^2.

Theorem 2.1 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an open set with (1.1) and* $\mathcal{A}$ *with (1.2). Then for any* $f \in L^\infty(\Omega)$ *and* $\lambda > 0$, *there exists a unique* $u \in C^b(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ *satisfying*

$$\begin{aligned} \lambda u - \mathcal{A}u = f \quad &in\ \Omega, \\ u = 0 \quad &on\ \partial\Omega, \end{aligned} \tag{2.1}$$

and this solution satisfies

$$\lambda \parallel u \parallel_{L^\infty(\Omega)} \leq \parallel f \parallel_{L^\infty(\Omega)} \tag{2.2}$$

and

$$u \geq 0, \quad if\ f \geq 0. \tag{2.3}$$

$\square$

In the proof and lateron we will use the following test functions.

Lemma 2.2 *There exists a function* $\eta \in C^\infty(\mathbb{R})$ *such that* $\eta(x) > 0$ *for all* $x \in \mathbb{R}$ *and that* $\eta_\delta := \eta^\delta$ *for* $\delta > 0$ *satisfies*

$$\begin{gathered} |D\eta_\delta|, |D^2\eta_\delta| \leq C\delta\eta_\delta, \\ \eta_\delta > 0, \\ \eta_\delta(t) \to 0 \quad for\ t \to \infty. \end{gathered} \tag{2.4}$$

Moreover there exist $b_{\delta j}, c_\delta \in L^\infty(\Omega)$ *satisfying*

$$\begin{gathered} |b_{\delta j} - b_j| \leq C\delta, \\ |c_\delta - c| \leq C\delta, \end{gathered} \tag{2.5}$$

such that for the operator $\mathcal{A}_\delta$ *obtained form* $\mathcal{A}$ *replacing* b_j *and* c *by* $b_{\delta j}$ *and* c_δ *one has*

$$\mathcal{A}_\delta(u\eta_\delta) = \eta_\delta \mathcal{A}u \quad for\ all\ u \in W^{2,n}_{loc}(\Omega), \delta > 0. \tag{2.6}$$

Proof:
Consider $\chi \in C^\infty([0,\infty[)$ with $\chi(t) \equiv 1$ for $t \leq 1, \chi > 0,$ and $\chi(t) = e^{-t}$ for $t \geq 2$. Using the letter C as a constant which may change from line to line, we see that $|\chi''|, |\chi'| \leq C\chi$ and $\chi(t) \to 0$ for $t \to \infty$. Next we put $\eta(x) := \chi(|x|)$ for $x \in \mathbb{R}^n$. Clearly $\eta \in C^\infty(\mathbb{R}^n)$, as $\eta \equiv 1$ on $B_1(0)$, and $\eta > 0, |D^2\eta|, |D\eta| \leq C_n\eta, \eta(x) \to 0$ for $|x| \to \infty$. Putting $\eta_\delta := \eta^\delta$ for $\delta > 0$, we see $\eta_\delta \in C^\infty(\mathbb{R}^n)$, as $\eta > 0$, and (2.4) is satisfied.

Now let $u \in W^{2,n}_{loc}(\Omega), \delta > 0$, and $u_\delta := u\eta_\delta \in W^{2,n}_{loc}(\Omega)$. Then using the summation convention, one obtains

$$\mathcal{A}u_\delta - \eta_\delta\mathcal{A}u = 2a_{ij}\partial_i u \cdot \partial_j\eta_\delta + a_{ij}u\partial_{ij}\eta_\delta + b_i u\partial_i\eta_\delta =$$

$$= (2a_{ij}\partial_j\eta_\delta/\eta_\delta)\partial_i u \cdot \eta_\delta + (a_{ij}\partial_{ij}\eta_\delta/\eta_\delta)u \cdot \eta_\delta + (b_i\partial_i\eta_\delta/\eta_\delta)u\eta_\delta.$$

Abbreviating the coefficients to

$$\begin{gathered} \tilde{b}_{\delta i} := 2a_{ij}\partial_j\eta_\delta/\eta_\delta, \\ \hat{c}_\delta := (a_{ij}\partial_{ij}\eta_\delta/\eta_\delta) + (b_i\partial_i\eta_\delta/\eta_\delta), \end{gathered}$$

we see that $|\tilde{b}_{\delta i}|, |\hat{c}_\delta| \leq C\delta$ and

$$\mathcal{A}u_\delta - \eta_\delta \mathcal{A}u = \tilde{b}_{\delta i}\partial_i u \cdot \eta_\delta + \hat{c}_\delta u_\delta = \tilde{b}_{\delta i}\partial_i u_\delta - \tilde{b}_{\delta i} u \partial_i \eta_\delta + \hat{c}_\delta u_\delta =$$

$$= \tilde{b}_{\delta i}\partial_i u_\delta + ((-\tilde{b}_{\delta i}\partial_i \eta_\delta / \eta_\delta) + \hat{c}_\delta) u_\delta.$$

Therefore putting

$$\tilde{c}_\delta := \hat{c}_\delta - \tilde{b}_{\delta i}\partial_i \eta_\delta / \eta_\delta,$$

we see that $|\tilde{c}_\delta| \leq C\delta$ and

$$\mathcal{A}u_\delta = \eta_\delta \mathcal{A}u + \tilde{b}_{\delta i}\partial_i u_\delta + \tilde{c}_\delta u_\delta. \tag{2.7}$$

Putting $b_{\delta i} := b_i - \tilde{b}_{\delta i}, c_\delta := c - \tilde{c}_\delta$ and

$$\mathcal{A}_\delta v := a_{ij}\partial_{ij} v + b_{\delta i}\partial_i v + c_\delta v = \mathcal{A}v - \tilde{b}_{\delta i}\partial_i v - \tilde{c}_\delta v,$$

we get (2.5) and by (2.7) that

$$\mathcal{A}_\delta(u\eta_\delta) = \mathcal{A}u_\delta - \tilde{b}_{\delta i}\partial_i u_\delta - \tilde{c}_\delta u_\delta = \eta_\delta \mathcal{A}u,$$

which is (2.6).

$\square$

Proof of Theorem 2.1:
Replacing $\mathcal{A}, f$ by $\mathcal{A}/\lambda, f/\lambda$, it suffices to consider $\lambda = 1$. We put $\Omega_R := \Omega \cap B_R(0)$ and see that Ω_R satisfies a uniform exterior cone condition for fixed $R > 0$.

By [ASch14] Theorem 2.4, there exists a solution $u_R \in C(\overline{\Omega_R}) \cap W^{2,q}_{loc}(\Omega)$ for any $q < \infty$ of

$$\begin{gathered} u_R - \mathcal{A}u_R = f \quad \text{in } \Omega_R, \\ u_R = 0 \quad \text{on } \partial\Omega_R. \end{gathered} \tag{2.8}$$

We first prove

$$\begin{gathered} \| u_R \|_{L^\infty(\Omega_R)} \leq \| f \|_{L^\infty(\Omega)}, \\ u_R \geq 0, \quad \text{for } f \geq 0. \end{gathered} \tag{2.9}$$

We observe for $v_R := u_R - \| f_+ \|_{L^\infty(\Omega)}$ that

$$\mathcal{A}v_R = \mathcal{A}u_R - c \| f_+ \|_{L^\infty(\Omega)} \geq u_R - f \geq u_R - f_+ \geq 0 \quad \text{in } \Omega_R \cap \{v_R > 0\}$$

since $c \leq 0$. Then by Aleksandrov's maximum principle, see [GT] Theorem 9.1,

$$\sup_{\Omega_R} v_R \leq \sup_{\partial\Omega_R} (v_R)_+ = 0,$$

hence $u_R \leq \| f_+ \|_{L^\infty(\Omega)} \leq \| f \|_{L^\infty(\Omega)}$ and (2.9) by symmetry.

Then by interior Calderon-Zygmund estimate, see [GT] Theorem 9.11, we get for any $\Omega' \subset\subset \Omega$ and R large that u_R is bounded in $W^{2,q}(\Omega')$ for any $q < \infty$, hence after passing to a subsequence $u_{R_l} \to u$ weakly in $W^{2,q}_{loc}(\Omega)$ and

$$u - \mathcal{A}u = f \quad \text{in } \Omega.$$

In the same way, we obtain by local Hölder regularity, see [GT] Corollary 9.29, that for any $x_0 \in \partial\Omega$ and R large u_R is bounded in $C^{0,\alpha}(B_\varrho(x_0) \cap \Omega)$ for appropriate $\varrho, \alpha > 0$ depending on the data and x_0, and we conclude that $u \in C^b(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$. Moreover (2.2) and (2.3) follow from (2.9).

To prove uniqueness, we consider $u \in C^b(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ with $u = 0$ on $\partial\Omega$ satisfying

$$u - \mathcal{A}u = 0 \quad \text{in } \Omega$$

and prove $u = 0$.

For $\delta > 0$ let η_δ the function from Lemma 2.2 and $u_\delta := \eta_\delta u$. Then $u_\delta \in C^b(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega), u_\delta = 0$ on $\partial\Omega$ and $\lim_{|x|\to\infty} u_\delta(x) = 0$ with (2.4). Choose δ such small that $|c - c_\delta| \leq C\delta \leq \lambda$ with (2.5). Then the coefficient of order zero of the operator $\mathcal{A}_\delta - \lambda$, see Lemma 2.2, is

$$c_\delta - \lambda \leq c_\delta - c - \lambda \leq 0,$$

as $c \leq 0$ by assumption. Moreover by (2.6) that

$$(\mathcal{A}_\delta - \lambda)u_\delta = \eta_\delta(\mathcal{A} - \lambda)u = 0.$$

If $u_\delta \not\equiv 0$, we may select after possibly switching the sign $x_k \in \Omega$ with

$$u_\delta(x_k) \to \sup_\Omega |u_\delta| > 0.$$

As by above $\lim_{|x|\to\infty, x\in\Omega} u_\delta(x) = 0$, we see that x_k is bounded, hence $x_k \to x_0 \in \overline{\Omega}$ after passing to a subsequence. Then

$$u_\delta(x_0) = \sup_\Omega |u_\delta| > 0,$$

in particular $x_0 \notin \partial\Omega$, as $u_\delta = 0$ on $\partial\Omega$, and $x_0 \in \Omega$ is an interior maximum of u_δ. As the coefficient of $\lambda - \mathcal{A}_\delta$ in front of u_δ is non-negative, we get from the strong maximum principle, see [GT] Theorem 9.6, that $u_\delta \equiv u_\delta(x_0) \neq 0$ is constant on the component of Ω containing x_0. If $\Omega \neq \mathbb{R}^n$, any component of Ω has non-empty boundary contained in $\partial\Omega$, which gives a contradiction as $u_\delta = 0 \neq u_\delta(x_0)$ on $\partial\Omega$. If $\Omega = \mathbb{R}^n$, we get a contradiction by $0 = \lim_{|x|\to\infty, x\in\Omega} u_\delta(x) = u(x_0) \neq 0$. Therefore $u_\delta \equiv 0$, hence $u \leftarrow u_\delta \equiv 0$, which proves uniqueness, and the theorem is proved.

□

Remark:
We choose the space $W^{2,n}_{loc}(\Omega) \subseteq C(\Omega)$, since the Aleksandrov maximum principle is valid with the choice of $p = n$. But by Calderon-Zygmund estimates, if $u \in W^{2,n}_{loc}(\Omega)$ and $\mathcal{A}u \in L^\infty_{loc}(\Omega)$, then $u \in W^{2,p}_{loc}(\Omega)$ for all $p < \infty$. □

In the following proposition, we add a property which will be needed later, see Theorem 4.1, and prove, if the right-hand side f of the Poisson problem (2.1) vanishes at infinity, so does the bounded solution of the Poisson problem. To simplify the notation, we introduce the space

$$L^\infty_0(\mathbb{R}^n) := \{ f \in L^\infty(\mathbb{R}^n) \mid \lim_{R\to\infty} \| f \|_{L^\infty(\mathbb{R}^n - B_R(0))} = 0 \}.$$

Proposition 2.3 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an open set with (1.1) and* $\mathcal{A}$ *with (1.2).*

For $f \in L^\infty(\Omega) \cap L_0^\infty(\mathbb{R}^n)$ *, the bounded solution* u *of the Poisson problem (2.1) vanishes at infinity as well, that is,*

$$\lim_{|x|\to\infty, x\in\Omega} u(x) = 0.$$

Proof:
Again we may assume $\lambda = 1$. First we consider $f \in L^\infty(\Omega)$ such that $\{f \neq 0\} \subseteq \Omega \cap B_{R_0}(0)$ for some $R_0 > 0$. We use again the auxiliary functions η_δ from Lemma 2.2 which satisfy (2.4) and calculate, observing that $c \leq 0, \eta_\delta > 0$, that

$$\mathcal{A}\eta_\delta \leq C(\Lambda, n)(|D^2\eta_\delta| + |\nabla\eta_\delta|) \leq C(\Lambda, n)\delta\eta_\delta.$$

For $C(\Lambda, n)\delta \leq 1/2$, we see

$$\eta_\delta - \mathcal{A}\eta_\delta \geq \frac{1}{2}\eta_\delta.$$

As $\eta_\delta > 0$ on $\mathbb{R}^n$ by (2.4), we have $\tau := \inf_{B_{R_0}(0)} \eta_\delta > 0$ and see for $M\tau/2 \geq \| f \|_{L^\infty(\Omega)}$ that

$$\frac{M}{2}\eta_\delta \geq |f| \quad \text{in } \Omega. \tag{2.10}$$

We return to the solution u_R on $\Omega_R := \Omega \cap B_R(0)$ in (2.8) and claim

$$|u_R| \leq M\eta_\delta \quad \text{on } \Omega_R. \tag{2.11}$$

Indeed putting $w_R := \pm u_R - M\eta_\delta$ on $\overline{\Omega_R}$, we see with (2.10) that

$$w_R - \mathcal{A}w_R \leq \pm f - (M/2)\eta_\delta \leq 0 \quad \text{in } \Omega_R,$$
$$w_R = \pm u_R - M\eta_\delta \leq 0 \quad \text{on } \partial\Omega_R.$$

Then by Aleksandrov's maximum principle, see [GT] Theorem 9.1,

$$\sup_{\Omega_R} w_R \leq \sup_{\partial\Omega_R} (w_R)_+ = 0,$$

hence $\pm u_R \leq M\eta_\delta$, which is (2.11).

Letting $R \to \infty$ in (2.11), we get

$$|u| \leq M\eta_\delta \quad \text{on } \Omega,$$

and hence with (2.4) that

$$\limsup_{|x|\to\infty, x\in\Omega} |u(x)| \leq \lim_{|x|\to\infty} M|\eta_\delta(x)| = 0.$$

For general $f \in L^\infty(\Omega) \cap L_0^\infty(\mathbb{R}^n)$, we choose cut-off functions $\xi_{R_0} \in C_c(B_{2R_0}(0)), \xi_{R_0} \equiv 1$ on $B_{R_0}(0), 0 \leq \xi_{R_0} \leq 1$ and consider by Theorem 2.1 the unique bounded solution of the Poisson problem

$$\lambda v_{R_0} - \mathcal{A}v_{R_0} = f\xi_{R_0} \quad \text{in } \Omega,$$
$$v_{R_0} = 0 \quad \text{on } \partial\Omega.$$

As $[f\xi_{R_0} \neq 0] \subseteq \Omega \cap B_{2R_0}(0)$, we see from above

$$\lim_{|x|\to\infty, x\in\Omega} |v_{R_0}(x)| = 0.$$

Further $w_{R_0} := u - v_{R_0} \in C^b(\overline{\Omega})$ solves the Poisson problem

$$\lambda w_{R_0} - \mathcal{A} w_{R_0} = f(1-\xi_{R_0}) \quad \text{in } \Omega,$$
$$w_{R_0} = 0 \quad \text{on } \partial\Omega.$$

As the bounded solution of the Poisson problem is unique by Theorem 2.1, we get from (2.2) that

$$\| w_{R_0} \|_{L^\infty(\Omega)} \leq \| f(1-\xi_{R_0}) \|_{L^\infty(\Omega)} \leq \| f \|_{L^\infty(\mathbb{R}^n - B_{R_0}(0))},$$

and hence together

$$\limsup_{|x|\to\infty, x\in\Omega} |u(x)| = \limsup_{|x|\to\infty, x\in\Omega} |w_{R_0}(x)| \leq \| f \|_{L^\infty(\mathbb{R}^n - B_{R_0}(0))} \quad \text{for any } R_0 > 0.$$

As $\| f \|_{L^\infty(\mathbb{R}^n - B_{R_0}(0))} \to 0$ for $R_0 \to \infty$ by assumption, we get $\lim_{|x|\to\infty, x\in\Omega} |u(x)| = 0$, and the proposition is proved.

□

This proposition enables us to consider the part A_1 of $\mathcal{A}$ in the Banach space $L^\infty(\Omega) \cap L_0^\infty(\mathbb{R}^n)$. Its domain is

$$D(A_1) := \{ u \in C_0(\Omega) \cap W^{2,n}_{loc}(\Omega) \mid \mathcal{A}u \in L^\infty(\Omega) \cap L_0^\infty(\mathbb{R}^n) \}.$$

We conclude this section by proving denseness of its image in $L^\infty(\Omega) \cap L_0^\infty(\mathbb{R}^n)$, which will be needed in Section 6.

Proposition 2.4 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set with (1.1) and $\mathcal{A}$ with (1.2).*
If $\Omega \neq \mathbb{R}^n$ or $c \neq 0 \in L^\infty(\mathbb{R}^n)$, then

$$\mathcal{A}\ D(A_1) \textit{ is dense in } L^\infty(\Omega) \cap L_0^\infty(\mathbb{R}^n).$$

Proof:
It suffices by rescaling to prove that for $f \in L^\infty(\Omega)$ satisfying $\{f \neq 0\} \subseteq \Omega \cap B_1(0)$, there exists $u_k \in D(\mathcal{A})$ with $\mathcal{A}u_k \to f$ uniformly on Ω . Actually considering the unique bounded solution $u_\lambda \in C_0(\Omega)$ of the Poisson problem

$$\begin{aligned} \lambda u_\lambda - \mathcal{A} u_\lambda &= f \quad \text{in } \Omega, \\ u_\lambda &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.12}$$

by Theorem 2.1 and Proposition 2.3, as $f \in L^\infty(\Omega) \cap L_0^\infty(\mathbb{R}^n)$ by the compact support of f , we claim

$$-\mathcal{A} u_\lambda \to f \quad \text{uniformly on } \Omega \text{ for } \lambda \searrow 0, \tag{2.13}$$

which proves the proposition.

First we conclude from (2.2) for $\tilde{u}_\lambda := \lambda u_\lambda \in C_0(\Omega)$ that

$$\| \tilde{u}_\lambda \|_{L^\infty(\Omega)} = \lambda \| u_\lambda \|_{L^\infty(\Omega)} \leq \| f \|_{L^\infty(\Omega)} . \tag{2.14}$$

Obviously

$$\begin{gathered} \lambda\tilde{u}_\lambda - \mathcal{A}\tilde{u}_\lambda = \lambda f \quad \text{in } \Omega, \\ \tilde{u}_\lambda = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.15}$$

Then as in the proof of Theorem 2.1 by interior Calderon-Zygmund estimate, see [GT] Theorem 9.11, and by local Hölder regularity, see [GT] Corollary 9.29, we get

$$\| \tilde{u}_\lambda \|_{W^{2,q}(\Omega')} \leq C_{\Omega',q} \Big(\| \tilde{u}_\lambda \|_{L^\infty(\Omega)} + \| \lambda f - \lambda\tilde{u}_\lambda \|_{L^\infty(\Omega)} \Big) \leq C_{\Omega',q} \quad \text{for } \Omega' \subset\subset \Omega, q < \infty$$

and

$$\| \tilde{u}_\lambda \|_{C^{0,\alpha}(\Omega \cap B_R(0))} \leq C_R \Big(\| \tilde{u}_\lambda \|_{L^\infty(\Omega)} + \| \lambda f - \lambda\tilde{u}_\lambda \|_{L^\infty(\Omega)} \Big) \leq C_R \quad \text{for } R \geq 1$$

and $0 < \lambda \leq 1$. After passing to a subsequence $\lambda_k \searrow 0$, we get

$$\tilde{u}_{\lambda_k} \to \tilde{u} \text{ locally uniformly in } \overline{\Omega} \tag{2.16}$$

with $\tilde{u} \in C^b(\overline{\Omega})$, and, as $\lambda\tilde{u}_\lambda, \lambda f \to 0$, by (2.15), we get by interior Calderon-Zygmund estimate, see [GT] Theorem 9.11, that $u \in W^{2,q}_{loc}(\Omega)$ and

$$\begin{gathered} -\mathcal{A}\tilde{u} = 0 \quad \text{in } \Omega, \\ \tilde{u} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.17}$$

We claim

$$\begin{gathered} \tilde{u} = 0, \\ \tilde{u}_\lambda \to 0 \quad \text{uniformly on } \Omega. \end{gathered} \tag{2.18}$$

To estimate $\tilde{u}_\lambda$ outside $B_1(0) \supseteq \{f \neq 0\}$, we observe from (2.15) that

$$\begin{gathered} \lambda\tilde{u}_\lambda - \mathcal{A}\tilde{u}_\lambda = 0 \quad \text{in } \Omega - B_1(0), \\ u_\lambda = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

If $\sup_{\Omega - B_1(0)} |\tilde{u}_\lambda| > 0$, then, as $\tilde{u}_\lambda \in C_0(\Omega)$, there exists $x_0 \in \overline{\Omega - B_1(0)}$ with $|\tilde{u}_\lambda(x_0)| \geq \sup_{\Omega - B_1(0)} |\tilde{u}_\lambda| > 0$. If $x_0 \in \Omega - \overline{B_1(0)}$, we get from the strong maximum principle, see [GT] Theorem 9.6, that $|\tilde{u}_\lambda| \equiv |\tilde{u}_\lambda(x_0)| \neq 0$ is constant on the component of Ω containing x_0 , which is a contradiction to $\tilde{u}_\lambda \in C_0(\Omega)$. Therefore $x_0 \in \partial(\Omega - B_1(0)) - \partial\Omega \subseteq \Omega \cap \partial B_1(0)$, and we conclude

$$\| \tilde{u}_\lambda \|_{C^b(\Omega - B_1(0))} \leq \| \tilde{u}_\lambda \|_{C^b(\Omega \cap \partial B_1(0))} . \tag{2.19}$$

Then we get from (2.16) that

$$\| \tilde{u} \|_{C^b(\Omega - B_1(0))} \leq \liminf_{k \to \infty} \| \tilde{u}_{\lambda_k} \|_{C^b(\Omega - B_1(0))} \leq$$

$$\leq \lim_{k\to\infty} \| \tilde{u}_{\lambda_k} \|_{C^b(\Omega\cap\partial B_1(0))} = \| \tilde{u} \|_{C^b(\Omega\cap\partial B_1(0))} . \tag{2.20}$$

Now if $\sup_\Omega |\tilde{u}| > 0$, we select $x_k \in \Omega$ with $|\tilde{u}(x_k)| \to \sup_\Omega |\tilde{u}| > 0$. Then by (2.20), we may assume that $|x_k| \leq 1$ and get after passing to a subsequence $x_k \to x_0 \in \overline{\Omega}$ with $|\tilde{u}(x_0)| = \sup_\Omega |\tilde{u}| > 0$. First this excludes $x_0 \in \partial\Omega$, as $\tilde{u} = 0$ on $\partial\Omega$. Therefore $x_0 \in \Omega$, and we get from (2.17) and the strong maximum principle, see [GT] Theorem 9.6, that $|\tilde{u}| \equiv |\tilde{u}(x_0)| \neq 0$ is constant on the component of Ω containing x_0 . If $\Omega \neq \mathbb{R}^n$, any component of Ω has non-empty boundary contained in $\partial\Omega$, which gives a contradiction as $\tilde{u} = 0 \neq \tilde{u}(x_0)$ on $\partial\Omega$. Therefore $\Omega = \mathbb{R}^n$ and

$$-c \cdot \tilde{u}(x_0) = -A\tilde{u} = 0 \quad \text{in } \mathbb{R}^n,$$

hence $c = 0 \in L^\infty(\mathbb{R}^n)$, as $\tilde{u}(x_0) \neq 0$, which again is excluded by assumption. Together we conclude $\sup_\Omega |\tilde{u}| = 0$ or likewise $\tilde{u} = 0$, which is the first part in (2.18).

This in turn implies with (2.16) that $\tilde{u}_\lambda \to \tilde{u} = 0$ locally uniformly in $\overline{\Omega}$ for $\lambda \searrow 0$, hence

$$\| \tilde{u}_\lambda \|_{C^b(\overline{\Omega\cap B_1(0)})} \to \| \tilde{u} \|_{C^b(\overline{\Omega\cap B_1(0)})} = 0$$

and with (2.19) that

$$\| \tilde{u}_\lambda \|_{C^b(\Omega)} = \| \tilde{u}_\lambda \|_{C^b(\overline{\Omega\cap B_1(0)})} \to 0,$$

which gives the second part in (2.18).

Finally the second part in (2.18) and (2.12) yield

$$-\mathcal{A}u_\lambda = f - \tilde{u}_\lambda \to f \quad \text{uniformly on } \Omega,$$

which establishes the claim (2.13), thereby proving the proposition.

□

3 The Dirichlet problem for operators in non-divergence form

In this section, we use our results on the Poisson problem to solve the Dirichlet problem. We use $C_0(\partial\Omega) := \{g \in C(\partial\Omega) : \lim_{|x|\to\infty} g(x) = 0\}$.

Theorem 3.1 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set with (1.1) and $\mathcal{A}$ with (1.2). Then for any $g \in C^b(\partial\Omega)$ and $\lambda > 0$, there exists a unique $u \in C^b(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ satisfying*

$$\begin{aligned} \lambda u - \mathcal{A}u = 0 \quad &\text{in } \Omega, \\ u = g \quad &\text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

and this solution satisfies

$$\| u \|_{C^b(\Omega)} \leq \| g \|_{C^b(\partial\Omega)} \tag{3.2}$$

$$u \geq 0, \quad \text{for } g \geq 0, \tag{3.3}$$

and

$$\lim_{|x|\to\infty, x\in\Omega} u(x) = 0, \quad \text{for } g \in C_0(\partial\Omega). \tag{3.4}$$

Proof:
Uniqueness is immediate from the uniqueness of the solution of the Poisson problem in Theorem 2.1. Moreover for any solution of the Dirichlet problem satisfying $\lim_{|x|\to\infty, x\in\Omega} u(x) = 0$, we apply the Aleksandrov's maximum principle, see [GT] Theorem 9.1, to get

$$\sup_{\Omega\cap B_R(0)} u \le \sup_{\partial(\Omega\cap B_R(0))} u_+ \le \sup_{\partial\Omega} g_+ + \sup_{\Omega\cap\partial B_R(0)} u_+,$$

hence letting $R\to\infty$, we get

$$\sup_{\Omega} u \le \sup_{\partial\Omega} g_+,$$

which gives (3.2) and (3.3) by symmetry.

For existence, we first consider the case, when there is $G\in C_c^2(\mathbb{R}^n)$ with $G = g$ on $\partial\Omega$. Then we consider by Theorem 2.1 the bounded solution of the Poisson problem

$$\lambda v - \mathcal{A}v = \lambda G - \mathcal{A}G \in L^\infty(\Omega) \quad \text{in } \Omega,$$

$$v = 0 \quad \text{on } \partial\Omega.$$

and see for $u := G - v \in C^b(\overline{\Omega}) \cap W^{2,q}_{loc}(\Omega)$ that

$$\lambda u - \mathcal{A}u = 0 \quad \text{in } \Omega,$$

$$u = g \quad \text{on } \partial\Omega,$$

hence u is a bounded solution of the Dirichlet problem.

Moreover, as $\lambda G - \mathcal{A}G$ has compact support in $\mathbb{R}^n$, we get from Proposition 2.3 that $v(x)\to 0$ for $|x|\to\infty$, hence for u as well, as G has compact support, and u satisfies (3.4) and further (3.2) and (3.3) by above.

Next for $g\in C_c(\partial\Omega)$, we extend g to $G\in C_c(\mathbb{R}^n)$ and smoothen this extension to $G_\varepsilon \in C_c^2(\mathbb{R}^n)$ with $G_\varepsilon \to G$ uniformly on $\mathbb{R}^n$. By above, there exists a solution of the Dirichlet problem

$$\lambda u_\varepsilon - \mathcal{A}u_\varepsilon = 0 \quad \text{in } \Omega,$$

$$u_\varepsilon = G_\varepsilon \quad \text{on } \partial\Omega,$$

with (3.2) - (3.4). Then clearly the differences $u_\varepsilon - u_\delta$ satisfy (3.4) as well and further (3.2) and (3.3) by above. Therefore

$$\| u_\varepsilon - u_\delta \|_{C^b(\overline{\Omega})} \le \| G_\varepsilon - G_\delta \|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{for } \varepsilon,\delta\to 0,$$

and $u_\varepsilon \to u$ uniformly on $\overline{\Omega}$, in particular $u\in C^b(\overline{\Omega})$ and $u = g$ on $\partial\Omega$. Then by interior Calderon-Zygmund estimate, see [GT] Theorem 9.11, we see further $u_\varepsilon \to u \in W^{2,q}_{loc}(\Omega)$ and $\lambda u - \mathcal{A}u = 0$ in Ω . Clearly u inherits vanishing at infinity in the sense of (3.4) from u_ε , hence satisfies (3.2) and (3.3) by above as well.

For general $g\in C^b(\partial\Omega)$, we choose cut-off functions $\xi_R \in C_c(B_{2R}(0)), \xi_R \equiv 1$ on $B_R(0), 0\le \xi_R \le 1$ and consider for $g_R := g\xi_R \in C_c(\partial\Omega)$ by above a solution of the Dirichlet problem

$$\lambda u_R - \mathcal{A}u_R = 0 \quad \text{in } \Omega,$$

$$u_R = g_R \quad \text{on } \partial\Omega,$$

with (3.2) - (3.4). We claim after passing to a subsequence

$$u_R \to u \quad \text{locally uniformly on } \overline{\Omega} \text{ for } R \to \infty. \tag{3.5}$$

If this claim is true, we see $u \in C^b(\overline{\Omega})$ and $u = g$ on $\partial\Omega$ and by interior Calderon-Zygmund estimate, see [GT] Theorem 9.11, further $u_R \to u \in W^{2,q}_{loc}(\Omega)$ and $\lambda u - \mathcal{A}u = 0$ in Ω , hence u is a solution of the Dirichlet problem. Moreover by (3.2) that

$$\| u \|_{C^b(\Omega)} \leq \liminf_{R\to\infty} \| u_R \|_{C^b(\Omega)} \leq \limsup_{R\to\infty} \| g_R \|_{C^b(\Omega)} \leq \| g \|_{C^b(\Omega)},$$

which is (3.2) for u . If $g \geq 0$, then $g_R \geq 0$, hence $u_R \geq 0$ by (3.3) and $u \geq 0$, which is (3.3) for u . If $g \in C_0(\partial\Omega)$, then $g - g_R = g(1 - \xi_R) \to 0$ uniformly on $\partial\Omega$, hence, as u_R vanish at infinity, we get by above from (3.2) for the differences $u_R - u_S$ that

$$\| u - u_R \|_{C^b(\Omega)} \leq \liminf_{S\to\infty} \| u_S - u_R \|_{C^b(\Omega)} \leq \limsup_{R\to\infty} \| g_S - g_R \|_{C^b(\Omega)} = \| g - g_R \|_{C^b(\Omega)} \cdot$$

This means $u_R \to u$ uniformly on Ω , hence u inherits vanishing at infinity in the sense of (3.4) from u_R .

It remains to prove the claim (3.5). Actually we consider $u \in C^b(\overline{\Omega}) \cap W^{2,q}_{loc}(\Omega)$ and $0 < \lambda \leq \lambda_0 < \infty, 3 \leq R < \infty$ with

$$\begin{aligned} \lambda u - \mathcal{A}u = 0 \quad & \text{in } \Omega, \\ u = g \quad & \text{on } (\partial\Omega) \cap B_R(0), \\ \| u \|_{L^\infty(\Omega)} \leq \| g \|_{L^\infty(\partial\Omega)}&, \end{aligned} \tag{3.6}$$

and we prove that the set

$$\begin{gathered} E := \{ u \in C^b(\overline{\Omega}) \cap W^{2,q}_{loc}(\Omega) \mid u \text{ satisfies (3.6) } \} \\ \text{is uniformly equicontinuous on } \Omega \cap B_{R-2}(0). \end{gathered} \tag{3.7}$$

Clearly u_R above satisfy (3.6), and (3.5) follows from (3.7) by the theorem of Arzela-Ascoli.

To prove equicontinuity, we first rescale for $B_{2\varrho}(x_0) \subseteq \Omega, 0 < \varrho \leq 1$ by putting $u_\varrho(x) := u(x_0 + \varrho x), a^\varrho_{ij}(x) := a_{ij}(x_0 + \varrho x), b^\varrho_i(x) := \varrho b_i(x_0 + \varrho x), c^\varrho(x) := \varrho^2 c(x_0 + \varrho x) \leq 0$ for $x \in B_2(0)$, and see that $a^\varrho_{ij}, b^\varrho_i, c^\varrho$ satisfy (1.2) for the same Λ, ν , as $\varrho \leq 1$. Putting $\mathcal{A}_\varrho v := a^\varrho_{ij}\partial_{ij}v + b^\varrho_i \partial_i v + c^\varrho v$, we see

$$\lambda \varrho^2 u_\varrho - \mathcal{A}_\varrho u_\varrho = 0 \quad \text{in } B_2(0)$$

and get by local Hölder regularity, see [GT] Corollary 9.24, that

$$\| u_\varrho \|_{C^{0,\alpha}(B_1(0))} \leq C\Big(\| u_\varrho \|_{L^\infty(B_2(0))} + \| \lambda u_\varrho \|_{L^\infty(B_2(0))} \Big) \leq C$$

for some $C = C(\Lambda, \nu, \Omega) < \infty$ and $\alpha = \alpha(\Lambda, \nu, \Omega) > 0$, where Λ is an upper bound of the infinity norm of the coefficients of $\mathcal{A}_\varrho$ which does not depend on $0 < \varrho \leq 1$, hence

$$[u]_{H^\alpha(B_\varrho(x_0))} \leq C \varrho^{-\alpha} \quad \text{for } B_{2\varrho}(x_0) \subseteq \Omega, 0 < \varrho \leq 1. \tag{3.8}$$

If $\Omega = \mathbb{R}^n$, this already gives uniform equicontinuity of E in $\mathbb{R}^n$. Therefore we assume $\Omega \neq \mathbb{R}^n$ or likewise $\partial\Omega \neq \emptyset$ in the following.

Next we choose a modulus of continuity σ for $g|B_R(0) \cap \partial\Omega$, that is, σ satisfies

$$|g(x) - g(y)| \leq \sigma(|x - y|) \quad \text{for } x, y \in (\partial\Omega) \cap B_R(0) \tag{3.9}$$

and $\sigma(t) \searrow 0$ for $t \searrow 0$. Then by Hölder regularity at the boundary, see [GT] Corollary 9.28, we get for any $x_0 \in (\partial\Omega) \cap B_{R-1}(0)$ that

$$osc_{B_\varrho(x)} u \leq C(\Lambda, \nu, n)\Big(\varrho^\alpha(\| u_\varrho \|_{L^\infty(\Omega \cap B_1(0))} + \| \lambda u_\varrho \|_{L^\infty(\Omega \cap B_1(0)))}) + \sigma(\sqrt{\varrho})\Big) \leq$$

$$\leq C(\varrho^\alpha + \sigma(\sqrt{\varrho})) \quad \text{for } 0 < \varrho \leq 1 \tag{3.10}$$

and some $C = C(\Lambda, \nu, n, \Omega, R) < \infty$ and $\alpha = \alpha(\Lambda, \nu, n, \Omega, R) > 0$.

Now let $x, y \in \Omega \cap B_{R-2}(0)$. In the first case, we assume $|x - y| \leq \delta d(x, \partial\Omega)^2$. We put $2\varrho := \min(1, d(x, \partial\Omega))$, in particular $B_{2\varrho}(x) \subseteq \Omega$. For $\bar{x} \in \partial\Omega \neq \emptyset$, we see

$$d(x, \partial\Omega) \leq |x - \bar{x}| \leq R + |\bar{x}| =: C_R$$

and choose $\delta > 0$ small such that

$$C_R^2\delta, \delta \leq 1/4.$$

Then

$$|x - y| \leq \delta d(x, \partial\Omega)^2 \leq C_R^2\delta \leq 1/4$$

and

$$|x - y| \leq \frac{1}{4} d(x, \partial\Omega)^2,$$

hence

$$|x - y| \leq \Big(\frac{1}{2}\min(1, d(x, \partial\Omega))\Big)^2 = \varrho^2 \leq \varrho,$$

as $\varrho \leq 1$. Therefore $y \in B_\varrho(x)$, and we get from (3.8) that

$$|u(x) - u(y)| \leq [u]_{H^\alpha(B_\varrho(x_0))} \cdot |x - y|^\alpha \leq C\varrho^{-\alpha}|x - y|^\alpha \leq C|x - y|^{\alpha/2}. \tag{3.11}$$

In the second case, when $|x - y| \geq \delta d(x, \partial\Omega)^2$, we select $x_0 \in \partial\Omega \neq \emptyset$ with $|x - x_0| = d(x, \partial\Omega)$. Clearly for $\varrho := |x - x_0| + |x - y|$, we have $x, y \in B_\varrho(x_0)$. Further for $|x - y| \leq c_0(\delta) \leq 1$ small enough

$$\varrho = |x - x_0| + |x - y| = d(x, \partial\Omega) + |x - y| \leq$$

$$\leq \sqrt{\delta^{-1}|x - y|} + |x - y| \leq C_\delta\sqrt{|x - y|} \leq 1,$$

in particular

$$|x - x_0| \leq \varrho \leq 1$$

and $x_0 \in (\partial\Omega) \cap B_{R-1}(0)$. Then we obtain from (3.10) that

$$|u(x) - u(y)| \leq osc_{B_\varrho(x_0)} u \leq C(\varrho^\alpha + \sigma(\sqrt{\varrho})) \leq C_\delta(\varrho^{\alpha/2} + \sigma(C_\delta\varrho^{1/4})). \tag{3.12}$$

Since $\varrho \leq C_\delta\sqrt{|x - y|}$ for $c_0(\delta)$ small enough, (3.11) and (3.12), show uniform equicontinuity of E on $\Omega \cap B_{R-2}(0)$, which is (3.7), and the theorem is proved.

$\square$

Now Theorem 1.1 is a consequence of Theorem 2.1 and Theorem 3.1.

As the $L^\infty-$bound for the solution of the Dirichlet problem is independent of $\lambda > 0$, we can solve the Dirichlet problem also for $\lambda = 0$, but we may loose uniqueness.

Proposition 3.2 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set with (1.1) and $\mathcal{A}$ with (1.2). Then for any $g \in C^b(\partial\Omega)$, there exists a $u \in C^b(\overline{\Omega}) \cap W^{2,q}_{loc}(\Omega)$ for all $q < \infty$ satisfying*

$$\begin{aligned} -\mathcal{A}u = 0 \quad &\textit{in } \Omega, \\ u = g \quad &\textit{on } \partial\Omega, \end{aligned} \tag{3.13}$$

which satisfies

$$\| u \|_{C^b(\Omega)} \leq \| g \|_{C^b(\partial\Omega)} \tag{3.14}$$

and

$$u \geq 0, \quad \textit{for } g \geq 0. \tag{3.15}$$

Proof:
For $\lambda > 0$, we consider by Theorem 3.1 the unique bounded solution of the Dirichlet problem

$$\lambda u_\lambda - \mathcal{A}u_\lambda = 0 \quad \text{in } \Omega,$$

$$u_\lambda = g \quad \text{on } \partial\Omega.$$

Then $\| u_\lambda \|_{L^\infty(\Omega)} \leq \| g \|_{L^\infty(\Omega)}$ by (3.2), and u_λ satisfy (3.6). Then for $0 < \lambda \leq 1$ by (3.7), the solutions u_λ are locally, uniformly equicontinuous on $\overline{\Omega}$, hence after passing to a subsequence $u_\lambda \to u$ locally uniformly in $\overline{\Omega}$ for $\lambda \searrow 0$ with $u \in C^b(\overline{\Omega})$, and, as $\lambda u_\lambda \to 0$, we get by interior Calderon-Zygmund estimate, see [GT] Theorem 9.11, that $u \in W^{2,q}_{loc}(\Omega)$ and

$$-\mathcal{A}u = 0 \quad \text{in } \Omega,$$

$$u = g \quad \text{on } \partial\Omega,$$

u is a bounded solution of the Dirichlet problem. Clearly

$$\| u_\lambda \|_{L^\infty(\Omega)} \leq \limsup_{\lambda \to 0} \| u_\lambda \|_{L^\infty(\Omega)} \leq \| g \|_{L^\infty(\Omega)},$$

which is (3.14). For $g \geq 0$, we have $u_\lambda \geq 0$ by (3.3), hence $u \geq 0$, which is (3.15), and the proposition is proved.

$\square$

As to uniqueness, we see for the Laplace operator in the exterior domain $\Omega := \mathbb{R}^n - \overline{B_1(0)}$, that the Dirichlet problem

$$\Delta u = 0 \quad \text{in } \mathbb{R}^n - \overline{B_1(0)},$$

$$u = 0 \quad \text{on } \partial B_1(0),$$

has for $n \geq 3$ the solutions

$$u(x) := \alpha(1 - |x|^{2-n}) \quad \text{for } |x| \geq 1$$

and any $\alpha \in \mathbb{R}$, in particular the solution to this Dirichlet problem is not unique. Actually, it is unique for $n = 2$. For more details, see [DL90] Chapter 2 §4 Example 13 p. 358.

4 m-dissipativity and semigroups

Let X be a real Banach space. A **semigroup** is a mapping $S : (0, \infty) \to X$ such that

$$S(t+s) = S(t)\ S(s).$$

We say that S is a C_0**-semigroup** if

$$\lim_{t \to 0+} S(t)x = x$$

for all $x \in X$. In that case, setting $S(0) := I$, the mapping $S(.)x : [0, \infty) \to X$ is continuous for all $x \in X$. If S is a C_0-semigroup, then its generator A is defined by

$$Ax = \lim_{t \to 0+} \frac{S(t)x - x}{t}$$

with domain

$$D(A) := \left\{ x \in X : \lim_{t \to 0+} \frac{S(t)x - x}{t} \text{ exists} \right\}.$$

A semigroup S is called **contractive** if $\| \ S(t) \ \| \leq 1$ for all $t > 0$. If $X = L^\infty(\Omega)$ or $X = C_0(\Omega)$ we say that S is **positive** if for $f \in X$, $f \geq 0$ implies $S(t)f \geq 0$ for all $t > 0$. Here we write $f \geq 0$ if $f(x) \geq 0$ a.e.; hence $f(x) \geq 0$ for all $x \in \Omega$ if f is continuous.

In this section we show that the part A_0 of $\mathcal{A}$ in $C_0(\Omega)$ generates a positive, contractive C_0-semigroup T_0 on the space $C_0(\Omega)$. Define the operator A_∞ on $L^\infty(\Omega)$ by

$$D(A_\infty) := \left\{ u \in C^b \left(\overline{\Omega} \right) \cap W^{2,n}_{loc}(\Omega) : u_{|\partial\Omega} = 0 \, , \, \mathcal{A}u \in L^\infty(\Omega) \right\},$$
$$A_\infty u = \mathcal{A}u \, .$$

Now Theorem 2.1 can be reformulated as follows.

Theorem 4.1 *The operator A_∞ is **m-dissipative**; i.e. for each $\lambda > 0$, $\lambda - A_\infty$ is invertible and $\| \ \lambda(\lambda - A_\infty)^{-1} \ \|_{L^\infty(\Omega)} \leq 1$. Moreover, $(\lambda - A_\infty)^{-1} \geq 0$.* □

m-dissipative operators have interesting properties, see e.g. [ABHN11] Section 3.5. If the coefficient a_{ij} are not only bounded and continuous as we are assuming so far, but even uniformly continuous on Ω, then we will see in Section 5 that A_∞ generates a holomorphic semigroup which is contractive but not a C_0-semigroup.

Next we consider the closed subspace

$$C_0(\Omega) := \left\{ u \in C \left(\overline{\Omega} \right) : u_{|\partial\Omega} = 0, \lim_{|x| \to \infty} u(x) = 0 \right\}$$

of $L^\infty(\Omega)$. Denote by A_0 the part of A_∞ in $C_0(\Omega)$. Then

$$D(A_0) = \left\{ u \in C_0(\Omega) \cap W^{2,n}_{loc}(\Omega) : \mathcal{A}u \in C_0(\Omega) \right\},$$
$$A_0 u := \mathcal{A}_\infty u \, .$$

Theorem 4.2 *The operator A_0 generates a positive contractive C_0-semigroup $(T_0(t))_{t \geq 0}$ on $C_0(\Omega)$.* □

It follows from the results in Section 2 that A_0 is m-dissipative. The main point in the proof of Theorem 4.2 is to show that $D(A_0)$ is dense in $C_0(\Omega)$. For that we need local Hölder continuity which is a consequence of [GT] Corollary 9.24 and which we used before.

Proof of Theorem 4.2

a) Let $\lambda > 0$. Given $f \in C_0(\Omega)$, by Theorem 2.1 there exists a unique $u \in C^b\left(\overline{\Omega}\right) \cap W^{2,n}_{loc}(\Omega)$ such that $u_{|\partial\Omega} = 0$ and $\lambda u - \mathcal{A}u = f$. It follows from Proposition 2.3 that $\lim_{|x|\to\infty, x\in\Omega} u(x) = 0$. Thus $u \in C_0(\Omega)$. This shows that $(0,\infty) \subset \varrho(A_0)$ (the resolvent set of A_0). Moreover by (2.3), $\| \lambda R(\lambda, A_0) \| \leq 1$, where $R(\lambda, A_0) = (\lambda - A_0)^{-1}$.

b) We show that $D(A_0)$ is dense in $C_0(\Omega)$. Let $u \in C^\infty_c(\Omega)$. We show that $u \in \overline{D(A_0)}$. Since $C^\infty_c(\Omega)$ is dense in $C_0(\Omega)$ this proves the claim.

Let $u \in C^\infty_c(\Omega)$, $u - \mathcal{A}u =: f \in L^\infty(\Omega)$. Then f has compact support. Consider a mollifier $(\varrho_\varepsilon)_{\varepsilon>0}$ $C^\infty_c(\mathbb{R}^n)$; i.e. $\varrho_\varepsilon \geq 0, \int_{\mathbb{R}^n} \varrho_\varepsilon \, dx = 1, \operatorname{supp} \varrho_\varepsilon \subset B_\varepsilon(0)$. Let $f_\varepsilon = \varrho_\varepsilon * f$. Then $f_\varepsilon \in C^\infty_c(\Omega), \| f_\varepsilon \|_\infty \leq \| f \|_\infty$ and $f_\varepsilon \to f$ in $L^n(\Omega)$ as $\varepsilon \to 0$. By a) there exist $u_\varepsilon \in D(A_0)$ such that $u_\varepsilon - A_0 u_\varepsilon = f_\varepsilon$.

We show that $u_\varepsilon \to u$ as $\varepsilon \to 0$ uniformly on Ω. Let $K \subset \Omega$ be compact and $\varepsilon_0 > 0$ such that $\operatorname{supp} f_\varepsilon \subset K$ for all $\varepsilon \in (0, \varepsilon_0]$. Choose $R > 0$ such that $K \subset B_R(0)$, and let $\Omega_R := \Omega \cap B_R(0)$.

Claim: There exist $\varrho > 0, \alpha > 0$ such that $\| u_\varepsilon \|_{C^\alpha(B_\varrho(x))} \leq$ const for all $\varepsilon \in (0, \varepsilon_0], x \in \overline{\Omega}_R$. In fact,

$$\| u_\varepsilon \|_{L^\infty(\Omega)} \leq \| f_\varepsilon \|_{L^\infty(\Omega)} \leq \| f \|_{L^\infty(\Omega)}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Choose $\varrho > 0$ such that $\operatorname{dist}(K, \partial\Omega) > 2\varrho$. There exist $x_1, \ldots, x_m \in K$ such that $K \subset \bigcup_{j=1}^m B_\varrho(x_j)$. Then there exists $\delta > 0$ such that for each pair $z_1, z_2 \in K$ satisfying $|z_1 - z_2| \leq \delta$ there exists $j \in \{1 \ldots m\}$ such that $z_1, z_2 \in B_\varrho(x_j)$.

By local Hölder regularity [GT] Corollary 9.24, for each $j \in \{1, \ldots, m\}$ we find $\alpha > 0$, $C > 0$ such that $|u_\varepsilon(x) - u_\varepsilon(y)| \leq C|x-y|^\alpha$ whenever $x, y \in B_\varrho(x_j)$, $0 < \varepsilon \leq \varepsilon_0$. Thus we find $\alpha > 0, C > 0$ such that $|u_\varepsilon(z_1) - u_\varepsilon(z_2)| \leq C|z_1 - z_2|^\alpha$ for all $z_1, z_2 \in K$ such that $|z_1 - z_2| \leq \delta$ and for all $0 < \varepsilon \leq \varepsilon_0$. Since $\| u_\varepsilon \|_{L^\infty(\Omega)} \leq c_1$ for all $0 < \varepsilon \leq \varepsilon_0$ and some $c_1 > 0$ one has for $|z_1 - z_2| > \delta$, $|u_\varepsilon(z_1) - u_\varepsilon(z_2)| \leq 2c_1 \leq (2c_1/\delta^\alpha)|z_1 - z_2|^\alpha$. Thus $\sup_{0<\varepsilon\leq\varepsilon_0} \| u_\varepsilon \|_{C^\alpha(K)} < \infty$. As a consequence there exists a sequence $\varepsilon_m \to 0$ such that u_{ε_m} converges uniformly on K as $m \to \infty$. Observe that $u_\varepsilon - \mathcal{A}u_\varepsilon = 0$ on $\Omega \backslash K$. Moreover, $\lim_{|x|\to\infty, x\in\Omega} u_\varepsilon(x) = 0$.

It follows from the Aleksandrov maximum principle that

$$\| u_{\varepsilon_l} - u_{\varepsilon_m} \|_{L^\infty(\Omega\backslash K)} = \| u_{\varepsilon_l} - u_{\varepsilon_m} \|_{L^\infty(\partial(\Omega\backslash K))} =$$

$$= \| u_{\varepsilon_l} - u_{\varepsilon_m} \|_{L^\infty(\partial K)} \to 0 \quad \text{as } l, m \to \infty.$$

Here we used that $u_{\varepsilon|\partial\Omega} = 0$. We have shown that u_{ε_m} converges uniformly on Ω as $m \to \infty$, say to $w \in C_0(\Omega)$.

By the Calderon-Zygmund estimate [GT] Theorem 9.11, for $\Omega' \subset\subset \Omega$ there exists $C > 0$ such that

$$\| v \|_{W^{2,n}(\Omega')} \leq C \| v - \mathcal{A}v \|_{L^n(\Omega')} .$$

This shows that u_{ε_m} converges in $W^{2,n}_{loc}(\Omega)$ as $m \to \infty$. Thus $w \in W^{2,n}_{loc}(\Omega)$ and $w - \mathcal{A}w = f$. It follows from Theorem 2.1 that $w = u$. Thus $u = \lim_{m\to\infty} u_{\varepsilon_m} \in \overline{D(A_0)}$.

c) The Hille-Yosida Theorem implies that A_0 generates a contractive C_0-semigroup T_0 on $C_0(\Omega)$. Theorem 2.1 shows that for $0 \leq f \in C_0(\Omega), R(\lambda, A_0)f \geq 0$ for all $\lambda > 0$. Thus $T_0(t)f = \lim_{m\to\infty} \left(I - \frac{t}{m}A_0\right)^{-m} f \geq 0$. We have shown that the semigroup T_0 is positive.

□

We show next that the semigroup T_0 is irreducible (see [Na86] B-III Definition 3.1).

Proposition 4.3 *If Ω is connected, the semigroup T_0 is **irreducible**; i.e. if $0 \leq f \in C_0(\Omega)$, then for all $x \in \Omega$ there exists $t > 0$ such that $(T_0(t)f)(x) > 0$.*

Proof:
Let $\lambda > 0, 0 \leq f \in C_0(\Omega), f \neq 0, u = R(\lambda, A_0)f$. We have to show that $u(x) > 0$ for all $x \in \Omega$ (see [Na86] B-III Definition 3.1). We know that $u \geq 0$.

Assume that there exists $x_0 \in \Omega$ such that $u(x_0) = 0$. Then u attains a non-positive, interior minimum, hence by the strong maximum principle [GT] Theorem 9.11 and since $u \in W^{2,n}_{loc}(\Omega)$, we get that $u \equiv 0$ is constant on the connected Ω, a contradiction to $f \neq 0$.

□

Proposition 4.3 implies that $\sigma(A_0) \neq \emptyset$ and that $s(A_0) = \sup\{Re\lambda : \lambda \in \sigma(A_0)\} \in \sigma(A_0)$, see [Na86] C-III Theorem 3.7.

5 Holomorphic semigroups

In this section we show that the C_0-semigroup generated by A_0 is holomorphic. We also consider the space $L^\infty(\Omega)$ on which we obtain a holomorphic semigroup which is not strongly continuous at $t = 0$. In this section the underlying field is $\mathbb{C}$, i.e. the spaces $L^p(\Omega), C^b(\Omega), C_0(\Omega), W^{2,p}_{loc}(\Omega)$ are all complex-valued, but the coefficients are all real valued as before.

We use the following terminology. Let X be a complex Banach space. A **holomorphic semigroup** is a holomorphic mapping $S : \Sigma_\theta \to \mathcal{L}(X)$ such that $S(z_1 + z_2) = S(z_1)S(z_2)$ for all $z_1, z_2 \in \Sigma_\theta$,

$$\sup_{z\in\Sigma_\theta,|z|\leq 1} \| S(z) \| < \infty \quad (\textbf{local bondedness})$$

and such that for $x \in X, S(t)x = 0$ for all $t > 0$ implies $x = 0$ (**non-degeneracy**). Here $\theta \in (0, \pi/2]$ and Σ_θ is the sector $\Sigma_\theta := \left\{re^{i\alpha} : r > 0, |\alpha| < \theta\right\}$. To make the angle precise we say that S is a **holomorphic semigroup of angle θ**. As a consequence, there exist $M \geq 0,\ \omega \in \mathbb{R}$ such that

$$\| S(z) \| \leq Me^{\omega|z|} \quad (z \in \Sigma_\theta). \tag{5.1}$$

Moreover, there exists a unique operator B on X such that $\{\lambda \in \mathbb{C} : Re\lambda > \omega\} \subset \varrho(B)$ and

$$R(\lambda, B) = \int_0^\infty e^{-\lambda t} S(t)\, \mathrm{d}t \quad (Re\lambda > \omega).$$

We call B the **generator** of S. Here, $\varrho(B) := \{\lambda \in \mathbb{C} : \lambda - B : D(B) \to X$ is bijective and $(\lambda - B)^{-1} \in \mathcal{L}(X)\}$ denotes the **resolvent set** of B. For $\lambda \in \varrho(B), R(\lambda, B) := (\lambda - B)^{-1}$ is the **resolvent** of B at λ. The set $\sigma(B) = \mathbb{C}\backslash\varrho(B)$ is the **spectrum** of B. In fact one has even $\Sigma_{\theta+\pi/2} \subset \varrho(B)$ and $R(\lambda, B) = \int_0^\infty e^{-\lambda t} S(t)\ \mathrm{d}t$ whenever $\lambda \in \Sigma_{\theta+\pi/2}$. By [Lu95] Proposition 2.1.1 (c), we have $S(t)X \subset D(B)$ and $S'(t) = BS(t)$ for all $t > 0$. Moreover, S is a C_0**-semigroup** if and only if $\overline{D(B)} = X$. In that case $\lim_{z\to 0, z\in\Sigma_\theta} S(z)x = x$ for all $x \in X$.

Remark (non-degeneracy): The non-degeneracy condition is supposed to make sure that B is an operator (and not just a relation). The C_0-condition implies trivially non-degeneracy. Since S is holomorphic, S is non-degenerate if and only if $S(z)$ is injective for all (equivalently one) $z \in \Sigma_\theta$. In fact, if $S(z_0)x = 0$, then $S(t_0 + t)x = S(t)S(z_0)x = 0$ for all $t > 0$. By the uniqueness theorem this implies that $S(z)x = 0$ for all $z \in \Sigma_\theta$. Hence $x = 0$, if S is non-degenerate.

Generators of holomorphic semigroups can be characterized as follows.

Proposition 5.1 *An operator B on X generates a holomorphic semigroup S if and only if there exists $\omega \in \mathbb{R}$ such that*

$$\lambda \in \varrho(B) \textit{ whenever } Re\lambda > \omega \textit{ and } \sup_{Re\lambda>\omega} \| \lambda R(\lambda, B) \| < \infty.$$

In that case

$$S(t) = \lim_{m\to\infty} \left(I - \frac{t}{m}B\right)^{-m} \tag{5.2}$$

in $\mathcal{L}(X)$ for all $t > 0$. □

This follows from [ABHN11] Corollary 3.7.12 and Proposition 3.7.4. The Euler formula (5.2) follows from Widder's Theorem [ABHN11] Theorem 1.7.7 (cf. the proof of [ABHN11] Corollary 3.3.6).

For later purposes we give the following result on invariant subspaces. If B is an operator on X and Y a subspace of X, the **part** B_Y of B **in** Y is defined by

$$D(B_Y) := \{y \in D(B) \cap Y : By \in Y\},$$
$$B_Y y := By.$$

Proposition 5.2 *Let B be the generator of a holomorphic semigroup $S : \Sigma_\theta \to \mathcal{L}(X)$ where $0 < \theta \leq \pi/2$ such that $\| S(t) \| \leq Me^{\omega t}(t > 0)$ and let Y be a closed subspace of X. If*

$$R(\lambda, A)Y \subset Y \quad \textit{for } \lambda > \omega,$$

then $S(z)Y \subset Y$ for all $z \in \Sigma_\theta$ and $S_Y(t) := S(z)_{|Y}$ defines a holomorphic semigroup S_Y on Y whose generator is B_Y.

Proof:
By the definition of B_Y, we have $(\omega, \infty) \subset \varrho(B_Y)$ and $R(\lambda, B_Y) = R(\lambda, B)_{|Y}$ for all $\lambda > \omega$. Let $y \in Y$. Then by (5.2), $S(t)y = \lim_{n\to\infty}(I - \frac{t}{n}B)^{-n}y \in Y$ for all $t > 0$. We show that even $S(z)Y \subset Y$ for all $z \in \Sigma_\theta$. Denote by $q : X \to X/Y$ the quotient map. Then

$$q \circ S : \Sigma_\theta \to \mathcal{L}(X, X/Y)$$

is holomorphic. Since $q \circ S_{|(0,\infty)} = 0$, it follows from the uniqueness theorem for holomorphic functions that $q(S(z)) = 0$ for all $z \in \Sigma_\theta$. Thus $S(z)Y \subset Y$ for all $z \in \Sigma_\theta$. It is clear that S_Y is a holomorphic semigroup of angle θ on Y. Moreover, for $y \in Y$, we have

$$R(\lambda, B_Y)y = R(\lambda, B)y = \int_0^\infty e^{-\lambda t} S(t)y \, \mathrm{d}t = \int_0^\infty e^{-\lambda t} S_Y(t)y \, \mathrm{d}t.$$

Thus B_Y is the generator of S_Y.

□

Now we consider the domain Ω and the real coefficients satisfying the hypotheses formulated in the introduction. But in the present and following sections **we assume in addition that the $\mathbf{a_{ij}} : \boldsymbol{\Omega} \to \mathbb{R}$ are uniformly continuous.** Throughout this section we consider complex spaces. As before the operator $\mathcal{A} : W^{2,n}_{loc}(\Omega) \to L^n_{loc}(\Omega)$ is given by $\mathcal{A}u = \sum a_{ij}\partial_i\partial_j u + \sum b_j \partial_j u + cu$ and we define its part A_∞ in $L^\infty(\Omega)$ by

$$D(A_\infty) := \left\{ u \in C^b(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega) : u_{|\partial\Omega} = 0, \ \mathcal{A}u \in L^\infty(\Omega) \right\},$$
$$A_\infty u = \mathcal{A}u.$$

Theorem 5.3 *The operator A_∞ generates a holomorphic semigroup T on $L^\infty(\Omega)$. Moreover*

$$\| T(t) \|_{\mathcal{L}(L^\infty(\Omega))} \leq 1$$

for all $t > 0$ and $T(t)f \geq 0$ for all $t > 0$ if $0 \leq f \in L^\infty(\Omega)$. □

For the proof we need some preparation. The following result is valid for arbitrary complex Banach lattices E, F. We need it for $E = F = L^\infty(\Omega)$ and for $E = C^b(\partial\Omega)$ and $F = C^b(\Omega)$.

Lemma 5.4 *Let $S : E \to F$ be linear such that*

$$f \geq 0 \Rightarrow Sf \geq 0 \quad \text{and} \quad \| Sf \|_F \leq \| f \|_E \quad \text{if} \quad 0 \leq f \in E.$$

Then $\| S \| \leq 1$.

Proof:
For any $f \in E, |f| = \sup_{\theta \in \mathbb{R}} Re(e^{i\theta} f)$. Hence $|Sf| \leq S|f|$. Thus $\| Sf \| = \| |Sf| \| \leq \| S|f| \| \leq \| |f| \| = \| f \|$.

□

Let $\lambda > 0$. As a consequence of Theorem 2.1, $(0, \infty) \subset \varrho(A_\infty)$. Since $R(\lambda, A)f \geq 0$ and $\| \lambda R(\lambda, A_\infty)f \|_{L^\infty(\Omega)} \leq \| f \|_{L^\infty(\Omega)}$ if $0 \leq f \in L^\infty(\Omega)$, it follows from Lemma 5.4 that

$$\| \lambda R(\lambda, A) \|_{\mathcal{L}(L^\infty(\Omega))} \leq 1 \quad (\lambda > 0).$$

We have shown that A_∞ is m-dissipative and $R(\lambda, A_\infty) \geq 0$ for $\lambda > 0$.

Lemma 5.5 *Let B be an m-dissipative operator on a complex Banach space X. Then $(\lambda - B)$ is invertible and $\| (\lambda - B)^{-1} \| \leq 1/Re\lambda$ whenever $Re\lambda > 0$.*

Proof:
By [ABHN11] Corollary 3.4.6 there exists $S : (0,\infty) \to \mathcal{L}(X)$ satisfying $S(0) = 0$ and

$$\| S(t) - S(s) \| \leq |t - s|$$

such that

$$(\lambda - A)^{-1} = \int\limits_0^\infty \lambda e^{-\lambda t} S(t) \, \mathrm{d}t$$

if $Re\lambda > 0$. Let $x \in X, x' \in X'$ such that $\| x \| \leq 1, \| x' \| \leq 1$. Then there exists $h \in L^\infty(0,\infty)$ such that $\| h \|_{L^\infty(0,\infty)} \leq 1$ and

$$\langle S(t)x, x' \rangle \ = \int\limits_0^t h(s) \, \mathrm{d}s \quad (t > 0).$$

Integration by parts yields

$$\left|\langle (\lambda - A)^{-1}x, x' \rangle\right| = \left| \int\limits_0^\infty e^{-\lambda t} h(t) \, \mathrm{d}t \right| \leq \int\limits_0^\infty e^{-Re\lambda t} \| h \|_\infty \, \mathrm{d}t \leq \frac{1}{Re\lambda}.$$

This proves the claim.

□

For the proof of Theorem 5.3 we will use known holomorphy estimates on $\mathbb{R}^n$. For that we extend the coefficients to $\mathbb{R}^n$.

Lemma 5.6 *There exist uniformly continuous, bounded extensions $\tilde{a}_{ij} : \mathbb{R}^n \to \mathbb{R}$ of a_{ij} such that*

$$\sum_{i,j=1}^n \tilde{a}_{ij}(x)\xi_i\xi_j \geq \min\left\{\frac{\nu}{2}, 1\right\} |\xi|^2$$

for all $x \in \mathbb{R}^n, \xi \in \mathbb{R}^n$.

Proof:
By a result of [McS34] Theorem 2 there exist uniformly continuous, bounded extensions $b_{ij} : \mathbb{R}^n \to \mathbb{R}$ of a_{ij}. Let $\epsilon = (\nu/2)\ (1/n^2)$. There exists $\delta > 0$ such that $|x - y| < \delta$ implies $|b_{ij}(x) - b_{ij}(y)| < \epsilon$. Choose a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(r) = 1$ for $r \leq 0, \varphi(r) = 0$ for $r \geq \delta$ and $0 < \varphi(r) < 1$ if $0 < r < \delta$. Let $\eta(x) = \varphi(d(x,\Omega))$. Then $\eta : \mathbb{R}^n \to \mathbb{R}^n$ is uniformly continuous and $\eta(x) = 1$ if $x \in \Omega, \eta(x) = 0$ if $x \notin \Omega_\delta$ and $0 < \eta(x) < 1$ if $x \in \Omega_\delta \backslash \Omega$, where $\Omega_\delta = \{x \in \mathbb{R}^n :\ \mathrm{dist}(\Omega, x) < \delta\}$.
Let

$$\tilde{a}_{ij}(x) = \eta(x)b_{ij}(x) + (1 - \eta(x))\delta_{ij}.$$

Then $\tilde{a}_{ij}$ is uniformly continuous on $\mathbb{R}^n, \tilde{a}_{ij}(x) = a_{ij}(x)$ for $x \in \Omega$.
For $x \in \Omega, \sum \tilde{a}_{ij}(x)\xi_i\xi_j \geq v|\xi|^2$, for $x \notin \Omega_\delta, \sum \tilde{a}_{ij}(x)\xi_i\xi_j = |\xi|^2$. For $x \in \Omega_\delta\backslash\Omega$ we choose

$y \in \Omega$ s.t. $|x-y| < \delta$. Then $|b_{ij}(x) - a_{ij}(y)| < \epsilon = (\nu/2)\ (1/n^2)$.
Hence

$$\sum_{i,j=1}^{n} \tilde{a}_{ij}\xi_i\xi_j = \sum_{i,j=1}^{n} \eta(x)(b_{ij}(x) - a_{ij}(y)\xi_i\xi_j + \sum_{i,j=1}^{n} \eta(x)a_{ij}(y)\xi_i\xi_j + (1-\eta(x))|\xi|^2 \geq$$

$$\geq -\epsilon \sum_{i,j=1}^{n} \frac{\xi_i\xi_j}{|\xi|^2}|\xi|^2\eta(x) + \eta(x)\nu|\xi|^2 + (1-\eta(x))|\xi|^2.$$

□

We extend b_j and c_j by 0 on $\mathbb{R}^n \backslash \Omega$. Define $\mathcal{B} : W^{2,n}_{loc}(\mathbb{R}^n) \to L^n_{loc}(\mathbb{R}^n)$ by

$$\mathcal{B}u := \sum_{i,j=1}^{n} \tilde{a}_{ij}(x)\partial_i\partial_j u + \sum_{i=1}^{n} b_i\partial_i u + cu,$$

and then define the operator B_∞ on $L^\infty(\Omega)$ by

$$D(B_\infty) = \left\{ u \in W^{2,n}_{loc}(\mathbb{R}^n) \cap L^\infty(\Omega); \mathcal{B}u \in L^\infty(\Omega) \right\},$$
$$B_\infty u = \mathcal{B}u.$$

Perturbing a result of Lunardi [Lu95] it was shown in [ASch14] Theorem 4.6 that B_∞ generates a holomorphic semigroup on $L^\infty(\mathbb{R}^n)$; i.e. there exist $\omega \in \mathbb{R}, M \geq 0$ such that $(\lambda - B_\infty)$ is invertible and

$$\| \lambda(\lambda - B_\infty)^{-1} \| \leq M \tag{5.3}$$

whenever $Re\lambda > \omega$.
It is here that we need the assumption that the coefficients a_{ij} are uniformly continuous.

Proof of Theorem 5.3:
Let $Re\lambda > \omega \geq 0$ where $\omega \geq 0$ is such that (5.3) holds. Let $f \in L^\infty(\Omega)$. Since A_∞ is m-dissipative, by Lemma 5.5, there exists a unique $u \in C^b(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ such that $\lambda u - \mathcal{A}u = f$, and $u_{|\partial\Omega} = 0$. By (5.3) there exists a unique $v \in C^b(\mathbb{R}^n) \cap W^{2,n}_{loc}(\mathbb{R}^n)$ such that $\lambda v - \mathcal{A}v = f$, where we extend f by 0 outside Ω. Moreover, $\| \lambda v \|_{L^\infty(\mathbb{R}^n)} \leq M \| f \|_{L^\infty(\Omega)}$. Let $w = u - v \in C^b(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$. Then $\lambda w - \mathcal{A}w = 0$. We want to show that

$$\| w \|_{C^b(\overline{\Omega})} \leq \| w \|_{C^b(\partial\Omega)}.$$

For that we modify w as in Lemma 2.2 using the same notation. For $\delta > 0$ small, let $w_\delta = \eta^\delta w$. Then $\lambda w_\delta - \mathcal{A}_\delta w_\delta + |c_\delta| w_\delta = |c_\delta| w_\delta$. Since $\lim_{|x|\to\infty} w_\delta(x) = 0$, there exists $x_0 \in \overline{\Omega}$ such that

$$|w_\delta(x_0)| = \sup_{x\in\overline{\Omega}} |w_\delta(x)| > 0 \quad (\text{if } w \neq 0).$$

Assume that $x_0 \in \Omega$. Then by [ASch14] Lemma 4.2,

$$Re(\overline{w_\delta(x_0)} \cdot (\lambda w_\delta(x_0) - |c_\delta| w_\delta(x_0))) = Re(\overline{w_\delta(x_0)} \cdot (\mathcal{A}_\delta w_\delta - |c_\delta| w_\delta)(x_0)) \leq 0$$

Hence

$$(Re\lambda)|w_\delta(x_0)|^2 \leq |c_\delta|(w_\delta(x_0))^2 \leq \delta C |w_\delta(x_0)|^2.$$

Choosing $\delta > 0$ so small that $\delta C < Re\lambda$, we deduce that $w_\delta(x_0) = 0$, a contradiction. Thus $x_0 \in \partial\Omega$. Consequently,

$$\| w_\delta \|_{C^b(\overline{\Omega})} \leq \| w_\delta \|_{C^b(\partial\Omega)}$$

for $\delta > 0$ small. Let $x \in \Omega$. Then

$$|w_\delta(x)| \leq \| w_\delta \|_{C^b(\partial\Omega)} \leq \| w \|_{C^b(\partial\Omega)} .$$

Sending $\delta \to 0_+$, we deduce that $|w(x)| \leq \| w \|_{C^b(\partial\Omega)}$. Hence $\| w \|_{C^b(\Omega)} \leq \| w \|_{C^b(\partial\Omega)} = \| v \|_{C^b(\partial\Omega)}$. Then

$$\begin{aligned} \| \lambda u \|_{C^b(\Omega)} &= \| \lambda\omega + \lambda v \|_{C^b(\Omega)} \leq \| \lambda\omega \|_{C^b(\Omega)} + \| \lambda v \|_{C^b(\Omega)} \leq \\ &\leq \| \lambda v \|_{C^b(\partial\Omega)} + \| \lambda v \|_{C^b(\Omega)} \leq 2M \| f \|_{L^\infty(\Omega)} . \end{aligned}$$

We have shown that $\| \lambda R(\lambda, A_\infty) \|_{\mathcal{L}(L^\infty(\Omega))} \leq 2M$ for $Re\lambda > \omega$. Thus A_∞ generates a holomorphic semigroup T on $L^\infty(\Omega)$. Since A_∞ is dissipative, it follows from [ABHN11] Proposition 3.7.16 that

$$\| T(t) \|_{\mathcal{L}(L^\infty(\Omega))} \leq 1 \quad \text{for all } t \in (0, \infty).$$

For $\lambda > 0$ we have

$$0 \leq (\lambda - A_\infty)^{-1} = \int_0^\infty e^{-\lambda t} T(t) \, \mathrm{d}t.$$

By (5.2) this implies that $T(t) \geq 0$.

$\square$

Next we consider the closed subspace

$$C_0(\Omega) := \left\{ u \in C^b(\overline{\Omega}) : u_{|\partial\Omega} = 0, \lim_{|x| \to \infty, x \in \Omega} u(x) = 0 \right\}$$

of $L^\infty(\Omega)$. By A_0 we denote the part of A_∞ in $C_0(\Omega)$, i.e.

$$D(A_0) := \left\{ u \in W^{2,n}_{loc}(\Omega) \cap C_0(\Omega) : \mathcal{A}u \in C_0(\Omega) \right\},$$
$$A_0 u := \mathcal{A}u.$$

Then, as a consequence of Lemma 5.4 and Theorem 4.2, the operator A_0 generates a positive contractive C_0-semigroup T_0 on $C_0(\Omega)$.

Corollary 5.7 *Consider $\theta \in (0, \pi/2]$ such that the holomorphic semigroup T generated by A_∞ has angle θ. Then $T(z)\, C_0(\Omega) \subset C_0(\Omega)$ for all $z \in \Sigma_\theta$ and $T_{|C_0(\Omega)} : \Sigma_\theta \to \mathcal{L}(C_0(\Omega))$ is a holomorphic C_0-semigroup whose generator is A_0.*

Proof:
By Proposition 2.3, $R(\lambda, A_\infty)\, C_0(\Omega) \subset C_0(\Omega)$ for all $\lambda > 0$. Now the claim follows from Proposition 5.2.

$\square$

Corollary 1.4 from the introduction is an immediate consequence of Corollary 5.7 and well-known properties of holomorphic C_0-semigroups, see for example [ABHN11] Corollary 3.7.21.

The semigroup T_0 is positivity-improving in the following sense.

Proposition 5.8 *Let $0 \leq f \in C_0(\Omega)$, $f \neq 0$. Then $(T_0(t)f)(x) > 0$ for all $x \in \Omega$, $t > 0$.*

Proof:
Since T_0 is irreducible by Proposition 4.3 the claim follows from holomorphy by a result of Majewski and Robinson [Na86] C-III Theorem 3.2 b.

$\square$

We may also consider the closed subspace

$$E_1 := L^\infty(\Omega) \cap L_0^\infty(\mathbb{R}^n) = \left\{ u \in L^\infty(\Omega) : \lim_{|x|\to\infty} u(x) = 0 \right\}.$$

By the same proof and with the same notations as in Corollary 5.7 one obtains the following.

Corollary 5.9 *One has $T(z)E_1 \subset E_1$ for all $z \in \Sigma_\theta$ and the restriction T_1 of T to E_1 defines a holomorphic semigroup on E_1. Its generator A_1 is given by*

$$D(A_1) := \left\{ u \in W^{2,n}_{loc}(\Omega) \cap C_0(\Omega) : \mathcal{A}u \in E_1 \right\},$$
$$A_1 u := \mathcal{A}u.$$

$\square$

We conclude this section commenting on the case where Ω is bounded. Then Theorem 5.3 also holds if Ω is merely Wiener regular, but does not necessarily satisfy the uniform exterior cone condition if in addition one assumes that the coefficients a_{ij} are Dini-continuous, see [ASch14] Theorem 4.1. The proof was based on elliptic results by Krylov. Those have been generalized recently by H. Dong, D-H Kim and S. Kim [DKK25].

6 Asymptotic behavior as t $\to \infty$

We keep the assumptions of Section 5. In particular, all spaces are complex and T is the holomorphic semigroup generated by A_∞ on $L^\infty(\Omega)$. Recall that $c \in L^\infty(\Omega)$ is the coefficient of order 0 of $\mathcal{A}$ and $c \leq 0$. Recall also that we assume that the coefficients a_{ij} are supposed to be uniformly continuous. Our aim is to prove the following result.

Theorem 6.1 *Assume that $\Omega \neq \mathbb{R}^n$ or $c \neq 0$. Let $f \in L^\infty(\Omega)$ such that $\lim_{R\to\infty} \| f \|_{L^\infty(\Omega \setminus B_R(0))} = 0$. Then*

$$\lim_{t\to\infty} \| T(t)f \|_{L^\infty(\Omega)} = 0.$$

□

Remark 6.2 *If $\Omega = \mathbb{R}^n$ and $c = 0$, then $T(t)1_\Omega = 1_\Omega$ for all $t > 0$ so that the assertion of Theorem 6.1 does not hold for $f = 1_\Omega$.* □

We will use the following abstract result which is based on the complex Tauberian theorem [ABHN11] Corollary 4.4.10.

Theorem 6.3 *Let S be a holomorphic semigroup on a Banach space X with generator B. Assume that*

(a) $\sup_{t>0} \| S(t) \| < \infty$ *and*

(b) $\sigma(B) \cap i\mathbb{R} \subset \{0\}$.

Then $\lim_{t\to\infty} S(t)x = 0$ for all $x \in \overline{ran}\, B$.

Proof:
For $2 \le s < t$,

$$\| S(t) - S(s) \| = \left\| \int_s^t \frac{\mathrm{d}}{\mathrm{d}r} S(r)\ \mathrm{d}r \right\| = \left\| \int_s^t BS(r)\ \mathrm{d}r \right\| =$$

$$= \left\| \int_s^t S(r-1)BS(1)\ \mathrm{d}r \right\| \le M \| BS(1) \| \, |t-s|.$$

Thus S is Lipschitz continuous on $[2,\infty)$ and in particular slowly oscillating (see [ABHN11] Definition 4.2.1).
Now let $x = By$ where $y \in D(B)$, $f(t) = S(t)x$. Then f is slowly oscillating and

$$\int_0^t f(s)\ \mathrm{d}s = \int_0^t S(s)By\ \mathrm{d}s = S(t)y - y.$$

Thus

$$\sup_{t>0} \left\| \int_0^t f(r)\ \mathrm{d}r \right\| < \infty.$$

Now by [ABHN11] Corollary 4.4.10, the spectral hypothesis (b) implies that

$$\lim_{t\to\infty} f(t) = 0.$$

Since S is bounded on $(0,\infty)$, it follows that $\lim_{t\to\infty} S(t)x = 0$ for all $x \in \overline{ran}\, B$.

□

Corollary 6.4 *More generally, the following holds under the assumptions (a) and (b) of Theorem 6.3. Let $x \in X$. Then*

$$\begin{array}{lll} \lim_{t\to\infty} S(t)x = 0 & & \textit{if and only if} \\ S(t_0)x \in \overline{ran}\, B & \textit{for some } t_0 > 0 & \textit{if and only if} \\ S(t)x \in \overline{ran}\, B & \textit{for all } t > 0. & \end{array}$$

Proof:
If $S(t_0)x \in \overline{ran}\, B$, then Theorem 6.3 implies that

$$\lim_{t\to\infty} S(t)x = \lim_{t\to\infty} S(t)S(t_0)x = 0.$$

Conversely, assume that $\lim_{t\to\infty} S(t)x = 0$. Let $x' \in X'$ such that $x'|ran\, B = 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle x', S(t)x\rangle = \langle x', BS(t)x\rangle = 0$$

for all $t > 0$. Hence $\langle x', S(\cdot)x\rangle$ is constant. Since $\lim_{t\to\infty} S(t)x = 0$, it follows that $\langle x', S(t)x\rangle = 0$ for all $t > 0$. Thus $S(t)x \in \overline{ran}\, B$ by the Hahn-Banach Theorem.

□

For the proof of Theorem 6.1 we use the following result of Greiner.

Theorem 6.5 (Greiner) *Let B be an operator on $L^\infty(\Omega)$ (or more generally on a complex Banach lattice) such that $(0,\infty) \subset \varrho(B)$,*

$$\sup_{\lambda>0} \| \lambda R(\lambda, B) \| < \infty,$$

*and $R(\lambda, B) \geq 0$ for all $\lambda > 0$. Then $\sigma(B) \cap i\mathbb{R}$ is **cyclic**; i.e., if $i\eta \in \sigma(B)$, where $\eta \in \mathbb{R}$, then $i\eta k \in \sigma(B)$ for all $k \in \mathbb{Z}$.*

Proof:
This is [Na86] C-III Remark 2.15 (a).

□

Proof of Theorem 6.1:
Since $\| \lambda R(\lambda, A_\infty) \| \leq 1$ for $\lambda > 0$, the hypotheses of Greiner's Theorem are satisfied. Since A_∞ generates a holomorphic semigroup one has $\omega + \Sigma_{\theta+\pi/2} \subset \varrho(A_\infty)$ for some $\omega \in \mathbb{R}$, $\theta \in (0, \pi/2]$. Thus $\sigma(A_\infty) \cap i\mathbb{R}$ is bounded. By Greiner's Theorem this implies that $\sigma(A_\infty) \cap i\mathbb{R} \subset \{0\}$. Let $f \in L^\infty(\Omega) \cap L^\infty_0(\mathbb{R}^n)$. Then by Proposition 2.4, $f \in \overline{ran}(A_\infty)$. It follows from Theorem 6.3 that $\lim_{t\to\infty} T(t)f = 0$.

□

Corollary 6.6 *Assume that $c \neq 0$ or $\Omega \neq \mathbb{R}^n$. Then the semigroup T_0 is **stable**; i.e.*

$$\lim_{t\to\infty} T_0(t)f = 0 \quad \text{in } C_0(\Omega)$$

for all $f \in C_0(\Omega)$. □

Recall that we assume here that the coefficients a_{ij} are uniformly continuous on Ω. This was needed for the proof of the holomorphy of T_0. We do not know whether Corollary 6.6 is true if the a_{ij} are merely bounded and continuous as in Section 4. It would be enough to show that $\sigma(A_0) \cap i\mathbb{R}$ is bounded. We also do not know whether Corollary 6.6 is true for $\Omega = \mathbb{R}^n$ if $c = 0$. It is so, if $\mathcal{A}$ has constant coefficients as can be seen by the explicit formula for T_0 in that case. But in general, if $c = 0$ and $\Omega = \mathbb{R}^n$, then we do not know whether $ranA_0$ is dense in $C_0(\Omega)$.

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