LOG CALABI–YAU PAIRS OF BIRATIONAL COMPLEXITY ZERO

JOSHUA ENWRIGHT, FERNANDO FIGUEROA, AND JOAQUÍN MORAGA

Abstract. In this article, we study the geometry of log Calabi–Yau pairs (X, B) of index one and birational complexity zero. Firstly, we propose a conjecture that characterizes such pairs (X, B) in terms of their dual complex and the rationality of their log canonical places. Secondly, we show that for these pairs the open set $X \setminus B$ is divisorially covered by open affine subvarieties which are isomorphic to open subvarieties of algebraic tori. We introduce and study invariants that measure the number of these open subvarieties of algebraic tori. Thirdly, we study boundedness properties of log Calabi–Yau pairs of index one and birational complexity zero. For instance, in dimension 2 we prove that such pairs are affinely bounded.

Contents

1. Introduction

Calabi–Yau varieties are one of the three building blocks of smooth projective varieties. Log Calabi–Yau pairs (X, B) are geometric objects that appear as degenerations of Calabi–Yau varieties. The *complexity* of a log Calabi–Yau pair (X, B) is defined to be

$$c(X, B) := \dim X + \dim \mathrm{Cl}_{\mathbb{Q}}(X) - |B|,$$

where $|B|$ is the sum of the coefficients of B. The complexity of a log Calabi–Yau pair is non-negative and it determines whether the log Calabi–Yau pair is toric (see [3, Theorem 1.2]). The *birational complexity*, denoted by $c_{\mathrm{bir}}(X, B)$, of a log Calabi–Yau pair (X, B), is the infimum among the complexities of birational models of (X, B). The birational complexity has recently been introduced by Mauri and the third author as a proxy to study the dual complex of log Calabi–Yau pairs [22]. Log Calabi–Yau pairs of index one[1] and birational complexity zero are precisely those that admit birational toric models [22, Theorem 1.6]. In this article, we study the geometry of log Calabi–Yau pairs of index one and birational complexity zero.

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[1]The *index* of a log Calabi–Yau pair (X, B) is the smallest positive integer m for which $m(K_X + B) \sim 0$.

1.1. **Birational complexity zero.** In this subsection, we discuss a fundamental conjecture and a structural theorem for log Calabi–Yau pairs of birational complexity zero.

In what follows, we write Σ^n for the sum of the coordinate hyperplanes of the n-dimensional projective space $\mathbb{P}^n$. Thus, the log pair $(\mathbb{P}^n, \Sigma^n)$ is a log Calabi–Yau pair of index one. We propose the following conjecture that characterizes log Calabi–Yau pairs admitting toric models, i.e., log Calabi–Yau pairs of birational complexity zero and index one.

Conjecture 1.1. *Let (X, B) be a log Calabi–Yau pair of dimension n. Assume that the following conditions are satisfied:*

(1) *we have $\mathcal{D}(X, B) \simeq_{\mathrm{PL}} S^{n-1}$; and*
(2) *each dlt stratum of (X, B) is rational.*

Then, we have $c_{\mathrm{bir}}(X, B) = 0$. In particular, we have a crepant birational map $(\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$.

In the previous conjecture, the dlt strata of (X, B) are the strata of a dlt modification of the pair (see Definition 2.6). Let us emphasize that due to [9, Corollary 4.(i)] and [20, Proposition 3], Conjecture 1.1.(1) implies that $K_X + B \sim 0$. This explains the last line in the statement of the conjecture. Conjecture 1.1 is known in dimension 2 by the work of Gross, Hacking, and Keel [15, Proposition 1.3]. Furthermore, Conjecture 1.1 is known for $X \simeq \mathbb{P}^3$ by the work of Ducat [7, Theorem 1.2]. It is worth mentioning that the previous conjecture fails if we drop any of its hypotheses. In [16, §3], Kaloghiros gives an example of a log Calabi–Yau pair (X, B) with $\mathcal{D}(X, B) \simeq_{\mathrm{PL}} S^2$ and X is birationally superrigid. In particular, the pair (X, B) does not have a toric model, so its birational complexity is positive. In [22, Theorem 1.18], the authors give examples, in each dimension $n \geqslant 3$, of log Calabi–Yau pairs (X_n, B_n) of dimension n, birational complexity n, rational dlt strata, and dual complex $\mathcal{D}(X_n, B_n) \simeq_{\mathrm{PL}} \mathbb{P}^{n-1}_{\mathbb{R}}$.

We turn to state a structural theorem for log Calabi–Yau pairs of index one and birational complexity zero. Toric varieties are covered by algebraic tori with monomial gluing functions. Cluster varieties can be regarded as generalizations of toric varieties. A cluster variety admits a big open subset that is covered by algebraic tori. In the case of cluster varieties, the gluing functions are not necessarily monomial, however, they preserve the volume function of the tori. The following theorem is a generalization of the previous facts to the case of log Calabi–Yau pairs of birational complexity zero. As it is expected, in this broader setting, one needs to consider more general affine varieties to perform these coverings.

Theorem 1.2. *Let (X, B) be a log Calabi–Yau pair of dimension n, index one and birational complexity zero. Then, there is a finite set $\{U_i\}_{i\in I}$ of open affine subvarieties of $X\backslash B$ satisfying the two following conditions:*

(1) *the set $\{U_i\}_{i\in I}$ divisorially covers $X\backslash B$;*[2] *and*
(2) *each U_i is isomorphic to an open subset of $\mathbb{G}_m^n$.*

The open sets in Theorem 1.2 appear as $\phi(\mathbb{G}_m^n \backslash \mathrm{Ex}(\phi))$ where $\phi\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$ is a crepant birational map. We say that a finite set $\{\phi_i\}_{i\in I}$ of crepant birational maps $\phi_i\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$ *divisorially covers* (X, B) if the open affines $U_i := \phi_i(\mathbb{G}_m^n \backslash \mathrm{Ex}(\phi_i))$ divisorially cover $X\backslash B$ (see Definition 2.21).

Theorem 1.2 states that whenever $c_{\mathrm{bir}}(X, B) = 0$ the variety $X\backslash B$ is divisorially covered by open subsets $U \subseteq \mathbb{G}_m^n$. The geometry of U is determined by its complement on $\mathbb{G}_m^n$. We turn to study this complement in the following subsection.

[2] We say that an algebraic variety V is *divisorially covered* by the open sets $U_1, \ldots, U_k$ if $\operatorname{codim}_V \left(V \backslash \bigcup_{i=1}^k U_i\right) \geqslant 2$.

1.2. **The torus exceptional degree.** In this subsection, we study the affine open subvarieties covering log Calabi–Yau pairs of birational complexity zero. Let (X,B) be a log Calabi–Yau pair and let $\varphi\colon (\mathbb{P}^n,\Sigma^n) \dashrightarrow (X,B)$ be a crepant birational map. We define the *torus exceptional degree* of φ to be

$$\mathcal{T}(\varphi) := \deg_{\mathbb{P}^n}\left(\overline{\mathrm{Ex}^1(\varphi)|_{\mathbb{G}_m^n}}\right)$$

where $\mathrm{Ex}^1(\varphi)$ is the reduced divisorial part of the exceptional locus of φ. Note that the torus exceptional degree is zero if and only if ϕ induces an embedding of a big open subset of the algebraic torus $\mathbb{G}_m^n$. A big open subset of an algebraic torus will be called an *almost torus*. The *torus exceptional degree* of (X,B) is defined to be:

$$\mathcal{T}(X,B) := \min\{\mathcal{T}(\varphi) \mid \ \varphi\colon (\mathbb{P}^n,\Sigma^n) \dashrightarrow (X,B) \text{ is a crepant birational map }\}.$$

In the case that the set in the definition is empty, we will set $\mathcal{T}(X,B)=\infty$. If (X,B) is a log Calabi–Yau pair of index one, then the torus exceptional degree $\mathcal{T}(X,B)$ is finite if and only if $c_{\mathrm{bir}}(X,B)=0$. The *total torus exceptional degree* is defined to be:

$$\mathcal{T}_t(X,B) = \min\left\{\sum_{i\in I}\mathcal{T}(\varphi_i) \,\middle|\, \{\varphi\}_{i\in I} \text{ is a set of crepant birational maps divisorially covering } (X,B)\right\}.$$

In Subsection 2.3, we expand on the aforementioned definitions. The following gives a structural theorem for the torus exceptional degree and the total torus exceptional degree.

Theorem 1.3. *Let (X,B) be a log Calabi–Yau pair of index one. The following statements hold:*
1. *we have $\mathcal{T}_t(X,B) \geqslant \mathcal{T}(X,B) \geqslant 0$;*
2. *the value $\mathcal{T}_t(X,B)$ is finite if and only if (X,B) has birational complexity zero;*
3. *$\mathcal{T}(X,B)=0$ if and only if (X,B) is divisorially covered by finitely many almost tori; and*
4. *If X is a $\mathbb{Q}$-factorial Fano variety and $\mathcal{T}(X,B)=0$, then the log Calabi–Yau pair (X,B) is covered by finitely many algebraic tori.*

Theorem 1.3.(3) was first proved by Corti in [4, Theorem 23]. We use analogous methods and write our proof for the sake of completeness. In dimension two, we prove that the torus exceptional degree is either infinite or bounded above by 13.

Theorem 1.4. *Let (X,B) be a log Calabi–Yau surface of index one and birational complexity zero. Then, we have $\mathcal{T}(X,B) \leqslant 13$.*

The previous theorem states that log Calabi–Yau surfaces of birational complexity zero admit affine charts of the form $\phi(\mathbb{G}_m^2\backslash\mathrm{Ex}(\phi))$ where $\phi\colon \mathbb{G}_m^2 \dashrightarrow X\backslash B$ is a birational map and the degree of $\mathrm{Ex}(\phi)$ is bounded above. In Example 6.2, we show that there is a sequence log Calabi–Yau pairs (X_n,B_n) of dimension 2, index one, and birational complexity zero, for which the sequence $\mathcal{T}_t(X_n,B_n)$ diverges to infinity. Theorem 1.4 will be a consequence of the following explicit result for Gorenstein del Pezzo surfaces of rank one.

Theorem 1.5. *Let X be a Gorenstein del Pezzo surface of Picard rank one. Let (X,B) be a log Calabi–Yau pair of index one and birational complexity zero. Then, we have $\mathcal{T}(X,B)\in\{0,1,\dots,13\}$.*

We propose the following conjecture which predicts that the torus exceptional degree is bounded above in terms of the dimension.

Conjecture 1.6. *Let n be a positive integer. There exists a positive integer t_n, only depending on n, satisfying the following. Let (X,B) be a log Calabi–Yau pair of dimension n, index one, and birational complexity zero. Then, we have $\mathcal{T}(X,B)\leqslant t_n$.*

1.3. **Connections with boundedness.** In this subsection, we draw a connection between log Calabi–Yau pairs of birational complexity zero and boundedness of algebraic varieties. First, we introduce the concept of *affinely bounded* families of projective varieties.

Definition 1.7. Let $\mathcal{F}$ be a family of projective varieties. We say that $\mathcal{F}$ is *affinely bounded* if there exists a scheme T of finite type over the ground field $\mathbb{K}$ and a finite type affine morphism $\mathcal{U} \to T$ such that every element $X \in \mathcal{F}$ admits an open affine set isomorphic to $\mathcal{U}_t$ for some $t \in T$.

Bounded families of projective varieties are affinely bounded. In Example 6.5, we recall that the converse is not true. Further, affinely bounded families are birationally bounded. In Example 6.6, we show that there are families of rational varieties that are not affinely bounded. As an application of Theorem 1.4, we obtain the following theorem.

Theorem 1.8. *Let $\mathcal{F}_2$ be the family of projective surfaces X satisfying the following conditions:*

- *there is a log Calabi–Yau pair (X, B) with $K_X + B \sim 0$; and*
- *we have $c_{\rm bir}(X, B) = 0$.*

Then the family $\mathcal{F}_2$ is affinely bounded.

We expect the previous theorem to generalize to higher dimensions.

Conjecture 1.9. *Let $\mathcal{F}_n$ be the family of n-dimensional projective varieties X satisfying the following conditions:*

- *there exists a log Calabi–Yau pair (X, B) with $K_X + B \sim 0$; and*
- *we have $c_{\rm bir}(X, B) = 0$.*

Then the family $\mathcal{F}_n$ is affinely bounded.

1.4. **The cluster type.** In this subsection, we introduce the concept of cluster type log Calabi–Yau pairs and prove some structural theorems about these pairs. We say that a log Calabi–Yau pair (X, B) is *of cluster type* if $\mathcal{T}(X, B) = 0$. By Theorem 1.3, for a log Calabi–Yau pair (X, B) of cluster type the open subvariety $X \backslash B$ is covered by finitely many almost tori. In this case, the *cluster type* of (X, B), denoted by $\mathcal{C}(X, B)$, is the smallest number of almost tori needed to divisorially cover $X \backslash B$. If a log Calabi–Yau pair is not of cluster type, then we set $\mathcal{C}(X, B) = \infty$. The following theorem characterizes log Calabi–Yau toric pairs among pairs of cluster type.

Theorem 1.10. *Let (X, B) be a log Calabi–Yau pair. Then, we have that $\mathcal{C}(X, B) \geqslant 1$. Furthermore, the equality holds if and only if the pair (X, B) is a toric log Calabi–Yau pair.*

In Example 6.7, we show that for every $c \geqslant 1$ there exists a log Calabi–Yau surface (X, B) of cluster type with $\mathcal{C}(X, B) = c$. One observes that the log Calabi-Yau surfaces constructed in Example 6.7 are also of large complexity, arising by performing a large number of non-toric blow ups of a toric pair. The following theorem puts this observation in context.

Theorem 1.11. *Let (X, B) be a log Calabi-Yau pair of index one and cluster type. Then*

$$\mathcal{C}(X, B) \leqslant c(X, B) + 1.$$

In a related direction, the following theorem states that positivity of the boundary also imposes bounds on the cluster type, at least in the case of surfaces.

Theorem 1.12. *Let X be a Fano surface and let (X, B) be a log Calabi–Yau pair. Then we have $\mathcal{C}(X, B) \in \{1, 2, \infty\}$.*

The previous theorem states that Fano surfaces of cluster type are quite similar to Fano toric surfaces. In particular, for every Fano surface X of cluster type, we can write

$$X = \mathbb{G}_m^2 \cup_\phi \mathbb{G}_m^2 \sqcup_{i=1}^{j} \mathbb{G}_{m,i} \sqcup_{i=1}^{k} \{x_i\}$$

where $\phi^*\Omega = \Omega$ with $\Omega = \frac{dt_1 \wedge dt_2}{t_1 t_2}$ and the $x_1, \dots, x_k \in X$ are closed points. Another pleasant property shared between cluster type surfaces and toric surfaces is a tight bound on the number of singular points in terms of Picard rank. It is shown in [1, Theorem 1.1] that a log Del Pezzo surface of Picard rank one has at most 4 singular points, and in [17, Corollary 1.8.1] it is shown that a log Del Pezzo of arbitrary Picard rank ρ has at most $2\rho + 4$ singular points. For cluster type surfaces, we prove the following:

Theorem 1.13. *Let (X, B) be a log Calabi-Yau surface of index one and cluster type. Then*

$$|X_{sing}| \leqslant \rho(X) + 2.$$

1.5. **Connections with cluster varieties.** We proceed to explain the connection of cluster type varieties with previous concepts and results from the literature. Cluster varieties are often defined as the spectra of cluster algebras as in the work of Fomin and Zelevensky (see, e.g., [12], [11]). Recently, Corti defined a generalized notion of cluster varieties (see Definition [5, Definition 2]). If (X, B) is a cluster type pair, then the variety $U := X \backslash B$ is a cluster variety in the sense of Corti. Indeed, if (X, B) is a cluster type pair with $j \colon \mathbb{G}_m^n \dashrightarrow X \backslash B$ a small birational map and $p \colon (Y, B_Y) \to (X, B)$ a $\mathbb{Q}$-factorial dlt modification, then the triple $(Y, B_Y, p^{-1} \circ j)$ defines a cluster variety $U_Y := Y \backslash B_Y \simeq U$ in the sense of Corti. However, the variety $U := X \backslash B$ may not be a cluster variety in the usual sense. We use the notion of cluster type pairs and cluster type varieties to resolve this ambiguity.

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2. Preliminaries

In this section, we collect some preliminaries about log Calabi–Yau pairs, birational transformations, and bounded families. We work over an algebraically closed field $\mathbb{K}$ of characteristic zero. The least number of generators of a group G, will be denoted as $d(G)$. We denote by Σ^n the sum of the coordinate hyperplanes of $\mathbb{P}^n$.

2.1. **Log Calabi–Yau pairs.** In this subsection, we recall some basic concepts related to log Calabi–Yau pairs.

Definition 2.1. A *log pair* is a couple (X, B) where X is a normal quasi-projective variety and B is an effective $\mathbb{Q}$-divisor for which $K_X + B$ is $\mathbb{Q}$-Cartier. Let $\pi \colon Y \to X$ be a projective birational morphism from a normal variety and $E \subset Y$ be a prime divisor. The *log discrepancy* of (X, B) at E, denoted by $a_E(X, B)$ is the rational number

$$1 - \operatorname{coeff}_E(B_Y) \text{ where } K_Y + B_Y = \pi^*(K_X + B).$$

The log discrepancy $a_E(X, B)$ depends only on the divisorial valuation induced by E on X and not in the model Y. We say that a log pair (X, B) is *log canonical* (resp. *Kawamata log terminal*) if all its log discrepancies are non-negative (resp. positive). We abbreviate log canonical (resp. Kawamata log terminal) by lc (resp. klt). A *non-klt place* of (X, B) is a prime divisor E over X for which $a_E(X, B) \leqslant 0$. If (X, B) is lc, then the non-klt places are also called log canonical places. A *log canonical center* of (X, B) is the image on X of a log canonical place.

Definition 2.2. Let (X, B) be a log pair. We say that (X, B) is *canonical* if all its log discrepancies are at least 1. A divisor E over X for which $a_E(X, B) = 1$ holds is called a *canonical place* of the log pair (X, B). We adopt the previous terminology regardless of whether (X, B) is a canonical pair or not.

Definition 2.3. Let (X, B) be a log canonical pair. We say that (X, B) is *divisorially log terminal* (or dlt for short) if there exists an open subset $U \subset X$ satisfying the following conditions:

(1) the pair (U, B_U) is log smooth,
(2) every log canonical center of (X, B) intersects U and is a stratum of $\lfloor B_U \rfloor$.

Definition 2.4. Let (X, B) be a log pair. A *dlt modification* $p\colon Y \to X$ of (X, B) is a projective birational morphism satisfying the following conditions:

(1) the pair (Y, B_Y) is dlt, where $B_Y = \mathrm{Ex}(p) + p_*^{-1}(B \wedge B_{red})$,
(2) every p-exceptional divisor is a non-klt place of (X, B), and
(3) we have that $K_Y + B_Y$ is nef over X,

In the previous setting, we may also say that $p\colon (Y, B_Y) \to (X, B)$ is a *dlt modification*.

We recall the following lemma (see, e.g., [10, Theorem 2.9]).

Lemma 2.5. *Let (X, B) be a log pair. Then, there exists a dlt modification $p\colon (Y, B_Y) \to (X, B)$.*

Definition 2.6. Let (X, B) be a log pair. A *dlt stratum* of (X, B) is either:

- a stratum of $\lfloor B_Y \rfloor$, or
- the variety Y,

where $(Y, B_Y) \to (X, B)$ is a dlt modification of (X, B). We will refer to the collection of all dlt strata among all dlt modifications of (X, B) as the *dlt strata* of (X, B).

We will need the following lemma.

Lemma 2.7. *Let (X, B) be a log canonical pair. Let P be a divisor on X. There is an $\epsilon > 0$ and a dlt modification of $(X, B + \epsilon P)$ extracting only log canonical places of (X, B).*

Proof. Let $q\colon Z \to X$ be a log resolution of $(X, B + P)$. Log discrepancies are continuous with respect to the boundary divisor. Hence, for $\epsilon > 0$ small enough, for every q-exceptional divisor E, $a_E(X, B + \epsilon P) \leqslant 0$ if and only if $a_E(X, B) = 0$. Thus, every non-klt place of $(X, B + \epsilon P)$ is a log canonical place of (X, B). $\square$

Definition 2.8. A projective log pair (X, B) is said to be *log Calabi–Yau* if $K_X + B \equiv 0$ and (X, B) is lc.

For a log Calabi–Yau pair (X, B) it is known that $K_X + B \sim_{\mathbb{Q}} 0$ (see, e.g., [13, Theorem 1.5]).

Definition 2.9. The *index* of a log Calabi–Yau pair (X, B) is the smallest positive integer m for which $m(K_X + B) \sim 0$.

Definition 2.10. Let (X, B) and (Y, B_Y) be two log pairs and let $\pi\colon X \dashrightarrow Y$ be a birational map. We say that π is *crepant* for (X, B) and (Y, B_Y) if the following condition holds. There exists a common resolution $p\colon Z \to X$ and $q\colon Z \to Y$ for which $p^*(K_X + B) = q^*(K_Y + B_Y)$. In this case, we also say that $\pi\colon (X, B) \to (Y, B_Y)$ is a *crepant birational map* and that (Y, B_Y) is *crepant birational equivalent* to (X, B). In the previous setting, we write $(X, B) \simeq_{\mathrm{bir}} (Y, B_Y)$.

If (X, B) is a log Calabi–Yau pair, then every log pair (Y, B_Y) which is crepant birational equivalent to (X, B) is also log Calabi–Yau. Furthermore, they have the same index (see, e.g. [9, Lemma 3.1]).

Definition 2.11. Let (X, B) be a log Calabi–Yau pair of dimension n. The *coregularity* of (X, B), denoted by $\operatorname{coreg}(X, B)$, is the dimension of the smallest log canonical center in any dlt modification of (X, B). If (X, B) is klt, then we set $\operatorname{coreg}(X, B) = n$. Thus, we have $\operatorname{coreg}(X, B) \in \{0, \ldots, n\}$.

We finish this subsection by collecting some lemmata. The following lemma is a straightforward application of the connectedness of log canonical centers, see, e.g., [10, Theorem 1.2].

Lemma 2.12. *Let (X, B) be a log Calabi–Yau surface with $\operatorname{coreg}(X, B) = 0$ and $K_X + B \sim 0$. Let $p\colon X \to Y$ be a projective birational morphism contracting the irreducible curve C. Assume that the curve C is not contained in B. Then, the curve C intersects B in at most one point.*

The following lemma is known to the experts.

Lemma 2.13. *Let (X, B) be a log Calabi–Yau surface of index one. Let $f\colon X \to Y$ be a birational contraction. Then we can factor f as $h \circ g$ where g is a birational contraction that only contracts canonical places of (X, B) and h is a birational contraction that only contracts log canonical places of (X, B).*

2.2. **Birational complexity.** In this subsection, we recall the concepts of complexity and birational complexity.

Definition 2.14. Let (X, B) be a log pair. The *complexity* of (X, B), denoted by $c(X, B)$, is defined to be

$$\dim X + \dim \operatorname{Cl}_{\mathbb{Q}}(X) - |B|,$$

where $|B|$ is the sum of the coefficients of B.

The complexity of a log Calabi–Yau pair is non-negative. Furthermore, if the complexity is zero, then $(X, \lfloor B \rfloor)$ is toric (see [3, Theorem 1.2]).

Definition 2.15. Let (X, B) be a log Calabi–Yau pair. The *birational complexity* of (X, B), denoted by $c_{\mathrm{bir}}(X, B)$, is defined to be:

$$\min\{c(Y, B_Y) \mid (Y, B_Y) \simeq_{\mathrm{bir}} (X, B)\}.$$

The minimum in the previous definition exists due to [22, Lemma 2.31].

The following theorem is due to Mauri and the third author (see [22, Theorem 1.6]).

Theorem 2.16. *Let (X, B) be a log Calabi–Yau pair of dimension n and index one. We have $c_{\mathrm{bir}}(X, B) = 0$ if and only if $(X, B) \simeq_{\mathrm{bir}} (\mathbb{P}^n, \Sigma^n)$.*

The previous theorem states that a log Calabi–Yau pair admits a birational toric model if and only if it has birational complexity zero.

2.3. **Torus exceptional degree.** In this subsection, we introduce the concept of torus exceptional degree and related invariants.

Definition 2.17. Let (X, B) be a log Calabi–Yau pair. Let $\phi\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$ be a crepant birational map. The *torus exceptional degree* of ϕ is defined to be

$$\mathcal{T}(\phi) := \deg_{\mathbb{P}^n}\left(\overline{\operatorname{Ex}^1(\phi)|_{\mathbb{G}_m^n}}\right)$$

where $\operatorname{Ex}^1(\phi)$ denotes the reduced sum of the exceptional divisors of ϕ. The torus exceptional degree is a non-negative integer.

For instance, for every toric birational map $\phi\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (T, B_T)$ we have $\mathcal{T}(\phi) = 0$ although the exceptional locus of ϕ may contain the coordinate hyperplanes as exceptional divisors. Indeed, in this case, we have that $\phi|_{\mathbb{G}_m^n}$ is an isomorphism.

Definition 2.18. Let (X, B) be a log Calabi–Yau pair. The *torus exceptional degree* of (X, B) is defined to be

$$\mathcal{T}(X, B) := \min\{\mathcal{T}(\phi) \mid \phi\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B) \text{ is a crepant birational map}\}.$$

If the set in the previous definition is empty, then we set $\mathcal{T}(X, B) = \infty$.

Given a crepant birational map $\phi\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$, we can produce crepant birational maps to (X, B) of arbitrarily large torus exceptional degree by precomposing with Cremona transformations. On the other hand, finding crepant birational maps with small torus exceptional degree is related to proving affine boundedness of the pairs (X, B). This is why we are mostly concerned with minimizing this invariant. The torus exceptional degree $\mathcal{T}(\phi)$ is minimized precisely when $X \backslash B$ admits an affine open which is isomorphic to a big open subset of an algebraic torus. This motivates the following definition.

Definition 2.19. An *almost torus* is a big open subset of an algebraic torus $\mathbb{G}_m^n$.

If (X, B) is a log Calabi–Yau pair with $\mathcal{T}(X, B) = 0$, then there is an open embedding $U \hookrightarrow X \backslash B$ of an almost torus U. Now, we turn to introduce some definitions and invariants that allow us to understand how far is $X \backslash B$ from being covered by almost tori.

Definition 2.20. Let X be an algebraic variety. Let $\{U_i\}_{i \in I}$ be a finite set of open subvarieties $U_i \subset X$. We say that the set $\{U_i\}_{i \in I}$ *divisorially covers* X if

$$\operatorname{codim}_X(X \backslash \cup_{i \in I} U_i) \geqslant 2.$$

It is clear that a set that divisorially covers X admits a finite subset that divisorially covers X.

Definition 2.21. Let (X, B) be a log Calabi–Yau pair. Let $\{\phi_i\}_{i \in I}$ be a set of crepant birational maps $\phi_i\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$. We say that the set $\{\phi_i\}_{i \in I}$ *divisorially covers* (X, B) if the open subvarieties $\phi_i(\mathbb{G}_m^n \backslash \operatorname{Ex}(\phi_i))$ divisorially cover $X \backslash B$. In the previous setting, we also say that (X, B) is *divisorially covered* by the crepant birational maps $\{\phi_i\}_{i \in I}$.

Definition 2.22. Let (X, B) be a log Calabi–Yau pair. The *total torus exceptional degree* is defined to be

$$\mathcal{T}_t(X, B) := \min \left\{ \sum_{i \in I} \mathcal{T}(\phi) \;\middle|\; \text{the crepant birational maps } \{\phi_i\}_{i \in I} \text{ divisorially cover } (X, B) \right\}.$$

The following lemma will be used in Example 6.2 to show that there are examples of log Calabi–Yau surfaces (X, B) for which $\mathcal{T}_t(X, B)$ is finite but arbitrarily large.

Lemma 2.23. *Let X be a rational projective variety. Let $\{D_i\}_{i \in \mathbb{N}}$ be an infinite sequence of prime divisors. Set*

$$r_k := \min \left\{ \dim H_1 \left(X^{\mathrm{reg}} \backslash \cup_{i \in I} D_i, \mathbb{Q} \right) \mid I \subset \mathbb{N} \text{ and } |I| = k \right\}.$$

Then, we have $\lim_{k \to \infty} r_k = \infty$.

Proof. Let $\phi\colon \mathbb{P}^n \dashrightarrow X$ be a birational map. We can find a hypersurface $H \subset \mathbb{P}^n$ such that ϕ restricts to an embedding $\mathbb{P}^n \backslash H \hookrightarrow X$. Let $P_1, \dots, P_\ell$ be the prime components of $X^{\mathrm{reg}} \backslash \phi(\mathbb{P}^n \backslash H)$. Set

$$s_k := \min \left\{ \dim H_1 \left((\mathbb{P}^n \backslash H) \backslash \cup_{i \in I} \phi^* D_i, \mathbb{Q} \right) \mid I \subset \mathbb{N} \text{ and } |I| = k \right\}.$$

By [6, Proposition 4.1.3], we know that $\lim_{k\to\infty} s_k = \infty$. On the other hand, for every finite subset $I \subset \mathbb{N}$, we have

$$\dim H_1\left(X^{\mathrm{reg}} \backslash \cup_{i\in I} D_i, \mathbb{Q}\right) - \dim H_1\left((\mathbb{P}^n \backslash H) \backslash \cup_{i\in I} \phi^* D_i, \mathbb{Q}\right) \leqslant \ell.$$

Then, the statement follows. ☐

2.4. **Boundedness.** In this subsection, we prove the following lemma related to affinely bounded families of normal varieties.

Lemma 2.24. *Let $\mathcal{F}$ be a family of normal varieties which is affinely bounded. Then, there exists a constant $r := r(\mathcal{F})$, only depending on $\mathcal{F}$, satisfying the following. Let $X \in \mathcal{F}$, then $d(\pi_1(X^{\mathrm{reg}})) \leqslant r$.*

Proof. Let $\mathcal{A} \to T$ be a bounded family of affine varieties such that for every $X \in \mathcal{F}$ there is an open embedding $\mathcal{A}_t \hookrightarrow X$. Then, the family $\mathcal{A}^{\mathrm{reg}} := \{U^{\mathrm{reg}} \mid U \in \mathcal{A}\}$ is also bounded. Let $\mathcal{A}^{\mathrm{reg}} \to W$ be its bounding family. By Verdier Theorem [26, Corollaire 5.1], we can stratify the morphism $\mathcal{A}^{\mathrm{reg}} \to W$ into finitely many locally trivial fibrations $\mathcal{A}_i^{\mathrm{reg}} \to W_i$. We conclude that the family of groups $\{\pi_1(\mathcal{A}_t^{\mathrm{reg}})/\simeq\}_{t\in T}$ is a finite set.

In particular, $d(\pi_1(\mathcal{A}_t^{\mathrm{reg}}))$ is bounded above in terms of $\mathcal{F}$, meaning that $d(\pi_1(\mathcal{A}_t^{\mathrm{reg}})) \leqslant r(\mathcal{F})$ holds for every $t \in T$. Let $X \in \mathcal{F}$. Then there is an open embedding $\mathcal{A}_t \hookrightarrow X$ for some $t \in T$. Therefore, we have a surjective homomorphism

$$\pi_1(\mathcal{A}_t^{\mathrm{reg}}) \to \pi_1(X^{\mathrm{reg}}).$$

Thus, we have $d(\pi_1(X^{\mathrm{reg}})) \leqslant r(\mathcal{F})$. ☐

The previous lemma will be used in Example 6.6 to show an example of a birationally bounded family that is not affinely bounded.

2.5. **Cluster type.** In this subsection, we introduce the concept of cluster type log Calabi–Yau pairs and prove a lemma.

Definition 2.25. Let $\phi\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$ be a crepant birational map. We say that ϕ is *of cluster type* if $\mathcal{T}(\phi) = 0$. Equivalently, a crepant birational map is said to be of cluster type if it induces an embedding of an almost torus in $X \backslash B$.

Definition 2.26. Let (X, B) be a log Calabi–Yau pair. We say that (X, B) is *of cluster type* if $\mathcal{T}(X, B) = 0$. In other words, a log Calabi–Yau pair (X, B) is said to be *of cluster type* if there exists a crepant birational map $\phi\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$ which is of cluster type.

Definition 2.27. Let (X, B) be a log Calabi–Yau pair. The *cluster type* of (X, B), denoted by $\mathcal{C}(X, B)$, is the least number of almost tori that divisorially cover $X \backslash B$. If $X \backslash B$ is not divisorially covered by almost tori, then we set $\mathcal{C}(X, B) = \infty$.

Following the previous definitions, we may say that a log Calabi–Yau pair (X, B) is of cluster type $\mathcal{C}(X, B)$. Theorem 1.3 states that a log Calabi–Yau pair (X, B) is of cluster type if and only if $X \backslash B$ is divisorially covered by finitely many almost tori.

Lemma 2.28. *Let (X, B) be a log Calabi–Yau pair of index one and dimension n. Let $\phi\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$ be a cluster type crepant birational map. Then, there is a commutative diagram of projective birational maps:*

$$\begin{tikzcd}
(T, B_T) \arrow[d, "p_1"'] & (X_0, B_0) \arrow[l, "q"] \arrow[d, dashed, "p_2"] \\
(\mathbb{P}^n, \Sigma^n) & (X, B) \arrow[l, dashed, "\phi^{-1}"]
\end{tikzcd}$$

where the following conditions are satisfied:

- *p_1 is a toric projective birational morphism,*
- *p_2 is a birational contraction map that only extracts log canonical places of (X, B), and*
- *q is a projective birational morphism that only extracts canonical places of (T, B_T).*

We may choose T to be smooth and X_0 to be $\mathbb{Q}$-factorial. Furthermore, if $n = 2$ then we may choose the above such that p_2 is a projective morphism.

Proof. We begin by choosing some toric projective birational morphism $p_1\colon (T, B_T) \to (\mathbb{P}^n, \Sigma^n)$ extracting all those log canonical places which have divisorial center on X but not on $\mathbb{P}^n$. Using [24, Lemma 2.37] to replace (T, B_T) by a higher model if necessary, we may assume that T is smooth and that the indeterminacy locus of $\phi \circ p_1$ contains no toric strata. As T is smooth, the log canonical centers of the toric pair (T, B_T) are precisely the toric strata. Thus, by [23, Corollary 1] we may find a projective birational morphism $q\colon (X_0, B_0) \to (T, B_T)$ with the following properties:

(1) q extracts precisely those canonical places which have divisorial center on X but not on T,
(2) all q-exceptional divisors are $\mathbb{Q}$-Cartier.

It follows from condition (2) and the smoothness of T that X_0 is $\mathbb{Q}$-factorial. It follows from condition (1) that

$$p_2 = \phi \circ p_1 \circ q\colon (X_0, B_0) \dashrightarrow (X, B)$$

is a birational contraction map; in particular, if $n = 2$ then p_2 cannot have nonempty indeterminacy locus. By construction, p_2 contracts only log canonical places. ☐

3. Birational complexity zero

In this section, we show a structural theorem for varieties of birational complexity zero. In the first subsection, we study affine varieties that divisorially cover log Calabi–Yau pairs of birational complexity zero. In the second subsection, we study the birational complexity of Gorenstein del Pezzo surfaces of rank one.

3.1. **Log Calabi–Yau pairs.** In this subsection, we give a proof of Theorem 1.2. First, we will need the following lemma.

Lemma 3.1. *Let (X, B) be a toric log Calabi–Yau pair of dimension n. Let E be a divisorial valuation exceptional over X with $a_E(X, B) = 1$. Then, there exists a crepant birational map $\phi\colon (Y, B_Y) \dashrightarrow (X, B)$ satisfying the following conditions:*

(1) *we have $c(Y, B_Y) = 1$,*
(2) *the center of E on Y is a prime divisor E_Y,*
(3) *the variety Y is $\mathbb{Q}$-factorial and admits a fibration $p\colon Y \to Z$ to a variety of dimension $n - 1$,*

(4) *there is a prime component* F_Y *of* $p^{-1}(p(E_Y))$ *that dominates* $p(E_Y)$*, is not contained in* B_Y*, and it is different from* E_Y*, and*
(5) *the pair* $(Y, B_Y + \epsilon F_Y)$ *is dlt for* $\epsilon > 0$ *small enough.*

Furthermore, the birational map $\phi\colon Y \dashrightarrow X$ *only extracts* E_Y *and log canonical places of* (X, B)*.*

Proof. First, we argue that in some log resolution (T, B_T) of (X, B) the center of E on T is a prime divisor of a prime component of B_T. Since $\mathbb{G}_m^n$ is smooth, for every toric modification $(X_i, B_i) \to (X, B)$ the center of E on X_i must be contained in B_i. We construct a sequence of toric birational morphisms as follows: we set $(X_0, B_0) := (X, B)$, we let (X_1, B_1) be a toric log resolution of (X_0, B_0), and for each $i \geqslant 1$ we let (X_{i+1}, B_{i+1}) be the blow up of the smallest stratum of B_i containing $c_E(X_i)$. This process terminates precisely when $c_E(X_i)$ is contained in a prime component of B_i and is not contained in any stratum of smaller dimension. If we are able to perform at least m blow ups without this process terminating, then it follows from [19, Lemma 2.29] that $a_E(X) \geqslant m$. We conclude that this process must terminate after finitely many steps. Thus, we can find a log resolution (T, B_T) of (X, B) where the center of E is a prime divisor of a prime component of B_T. By applying Lemma 2.7 to (T, B_T) and any non-toric prime divisor $P \subset T$ containing $c_E(T)$, we can further assume that the center $c_E(T)$ contains no toric strata. Note that $(T, B_T) \to (X, B)$ is a toric birational map so $T \to X$ only extracts log canonical places of (X, B).

Let $\Sigma_T \subset N_{\mathbb{Q}}$ be the fan associated to T. Let $Q \subset B_T$ be the unique prime component that contains $c_E(T)$. Let $\rho_Q \in \Sigma_T(1)$ be the ray corresponding to the torus invariant divisor Q. Let $N_{0,\mathbb{Q}} \subset N_{\mathbb{Q}}$ be an orthogonal complementary subspace to the ray generated by ρ_Q. Let $\{e_1, \dots, e_{n-1}\}$ be an orthogonal basis of $N_{0,\mathbb{Q}}$. Consider the fan

$$\Sigma_0 := \{\pm e_1, \dots, \pm e_{n-1}, \pm \rho_Q\}.$$

Then $X(\Sigma_0) \simeq (\mathbb{P}^1)^n$, the center of Q on $X(\Sigma_0)$ is a torus invariant divisor, and this torus invariant divisor is a section for a fibration $(\mathbb{P}^1)^n \to (\mathbb{P}^1)^{n-1}$. Let Σ_1 be a common refinement of Σ_T and Σ_0. Let T_1 be the projective toric variety associated to Σ_1. Then, we have a projective birational morphism $T_1 \to T$ and a fibration $p_{T_1}\colon T_1 \to Z := (\mathbb{P}^1)^{n-1}$ such that the strict transform of Q on T_1 dominates Z. We let (T_1, B_{T_1}) be the log pull-back of (T, B_T) to T_1. Observe that $T_1 \to X$ is a toric birational morphism, so it only extracts log canonical places of (X, B).

Let $Y_0 \to T_1$ be a log resolution of (T_1, B_{T_1}) that extracts E. Let E_{Y_0} be the center of E on Y_0. Let B_{Y_0} be the strict transform of B_{T_1} on Y_0. Let R_{Y_0} be the reduced exceptional of $Y_0 \to T_1$ minus E. Then, the pair $(Y_0, B_{Y_0} + R_{Y_0})$ is log smooth. Note that

$$K_{Y_0} + B_{Y_0} + R_{Y_0} \sim_{\mathbb{Q}} \sum_{\substack{P \subseteq Y_0 \\ P \neq E_{Y_0}}} a_P(T_1, B_{T_1}) P.$$

We call Ω_{Y_0} the effective divisor on the right side of the previous $\mathbb{Q}$-linear equivalence. Note that Ω_{Y_0} contains the support of all the prime divisors exceptional over T_1 that have positive log discrepancy with respect to (T_1, B_{T_1}) except for E_{Y_0}. The divisor Ω_{Y_0} is degenerate over T_1 in the sense of [21, Definition 2.9]. We run a $(K_{Y_0} + B_{Y_0} + R_{Y_0})$-MMP over T_1 with scaling of an ample divisor. After finitely many steps all the prime components of Ω_{Y_0} are contracted. Note that this MMP terminates after these components are contracted. We let Y_1 be the model obtained by this MMP. Let B_{Y_1} be the push-forward of B_{Y_0} on Y_1. Let E_{Y_1} be the push-forward of E_{Y_0} on Y_1. Then, the pair (Y_1, B_{Y_1}) satisfies the following conditions:

(i) the pair (Y_1, B_{Y_1}) is $\mathbb{Q}$-factorial and dlt, and
(ii) the birational morphism $Y_1 \to T_1$ only extracts E_{Y_1} and log canonical places of (T_1, B_{T_1}).

Condition (ii) implies that $c(Y_1, B_{Y_1}) = 1$. Let $p_{Y_1}\colon Y_1 \to Z$ be the induced fibration and let E_{Y_1} be the center of E on Y_1. Then we have that $p_{Y_1}^{-1}(p_{Y_1}(E_{Y_1}))$ has at least two prime components: E_{Y_1} and the strict transform of $p_{T_1}^{-1}(p_{Y_1}(E_{Y_1}))$. Thus, the log pair (Y_1, B_{Y_1}) satisfies conditions (1)-(4). Let F_{Y_1} be a prime component of $p_{Y_1}^{-1}(p_{Y_1}(E_{Y_1}))$ that dominates $p_{Y_1}(E_{Y_1})$, is not contained in B_{Y_1}, and is different from E_{Y_1}. By [18, Theorem 3.1], we can take a minimal dlt modification $(Y, B_Y + \epsilon F_Y)$ of $(Y_1, B_{Y_1} + \epsilon F_{Y_1})$. By Lemma 2.7, for $\epsilon > 0$ small enough all the non-klt places of $(Y_1, B_{Y_1} + \epsilon F_{Y_1})$ are log canonical places of (Y_1, B_{Y_1}). In particular, the pair (Y, B_Y) is a dlt modification of (Y_1, B_{Y_1}). Thus, we have $c(Y, B_Y) = 1$. Conditions (1)-(4) are clearly preserved on (Y, B_Y). We conclude that the pair (Y, B_Y) satisfies the conditions (1)-(5). Note that the birational map $\phi\colon Y \dashrightarrow X$ only extracts E_Y and log canonical places of (X, B). ☐

Now, we turn to prove the first theorem of this article.

Proof of Theorem 1.2. Let (X, B) be a log Calabi–Yau pair of dimension n, index one, and birational complexity zero. By [22, Theorem 1.6], there is a crepant birational map $\phi\colon (T, B_T) \dashrightarrow (X, B)$ where (T, B_T) is a toric log Calabi–Yau pair. Let

$$U_1 := \phi(T \backslash (B_T \cup \mathrm{Ex}(\phi))).$$

Then U_1 is isomorphic to an open subvariety of $T\backslash B_T \simeq \mathbb{G}_m^n$. As (T, B_T) has no log canonical centers along $T\backslash B_T$, we conclude that ϕ induces an embedding $U_1 \hookrightarrow X\backslash B$. Let $E_2, \dots, E_k$ be the $(n-1)$-dimensional prime components of $(X\backslash B)\backslash U_1$.

For each $i \in \{2, \dots, k\}$, we argue that there exists a crepant birational map $\phi_i\colon (T_i, B_{T_i}) \dashrightarrow (X, B)$, from a toric log Calabi–Yau pair (T_i, B_{T_i}) such that the center of E_i on T_i is a divisor intersecting $T_i\backslash B_{T_i}$. This implies that the divisor E_i intersects

$$U_i := \phi_i(T_i \backslash (B_{T_i} \cup \mathrm{Ex}(\phi_i))).$$

Thus, the open affines $U_1, \dots, U_k$ cover $X\backslash B$ divisorially.

It suffices to show that the statement holds for $E := E_2$. By assumption, the divisor E gives a divisorial valuation exceptional over T. Note that $a_E(T, B_T) = 1$. We can apply Lemma 3.1 to the toric log Calabi–Yau pair (T, B_T) and the valuation E. We obtain a crepant birational map $\phi\colon (Y, B_Y) \dashrightarrow (T, B_T)$ satisfying the following conditions:

(1) we have $c(Y, B_Y) = 1$,
(2) the center of E on Y is a prime divisor E_Y,
(3) the variety Y is $\mathbb{Q}$-factorial and admits a fibration $p\colon Y \to Z$ to a variety of dimension $n-1$,
(4) there is a prime component F_Y of $p^{-1}(p(E_Y))$ that dominates $p(E_Y)$, is not contained in B_Y, and is different from E_Y, and
(5) the pair $(Y, B_Y + \epsilon F_Y)$ is dlt for $\epsilon > 0$ small enough.

By condition (5), we can run a $(K_Y + B_Y + \epsilon F_Y)$-MMP over Z with scaling of an ample divisor. Note that $K_Y + B_Y \sim_{\mathbb{Q}} 0$ so $K_Y + B_Y + \epsilon F_Y \sim_{\mathbb{Q}} \epsilon F_Y$. By condition (4), we know that the divisor F_Y is degenerate over Z, so this MMP terminates after contracting F_Y. Let T_1 be the model obtained after contracting F_Y. Let B_{T_1} be the push-forward of B_Y on T_1. Since F_Y is not contained in B_Y, the complexity drops precisely by one when we contract this divisor. Hence, we get $c(T_1, B_{T_1}) = 0$, so (T_1, B_{T_1}) is a toric log Calabi–Yau pair. Further, the center of E on T_1 is a divisor that intersects $T_1\backslash B_{T_1}$. This finishes the proof. ☐

3.2. **Log Calabi–Yau Surfaces.** In this subsection we study the birational complexity of log Calabi–Yau pairs (X, B), where X is a Gorenstein del Pezzo surface of Picard rank 1.

Lemma 3.2. *Let $(X;x)$ be a D-type or E-type singularity. Then the only* 1*-complement of $(X;x)$ is the trivial one, i.e., $B=0$.*

Proof. Let $p\colon Y\to X$ be a minimal resolution. For each $E\subset Y$ exceptional over X we have $a_E(X)=1$. Assume that $(X,B;x)$ is a 1-complement and $B\neq 0$ through $x\in X$. Then $a_E(X,B)$ is integral and strictly less than 1 for each $E\subset Y$ exceptional over X. Thus, $a_E(X,B)=0$ for every such a curve. We conclude that $p^*(K_X+B)=K_Y+B_Y+E$ where E is the reduced exceptional divisor. Here, B_Y is the strict transform of B on Y. As $(X;x)$ is either D-type or E-type, there is a curve $C\subset E$ that intersects at least 3 other prime components E_1,E_2, and E_3 of E. Note that K_Y+B_Y+E is log canonical and trivial over X. Performing adjunction to C we observe

$$(K_Y+B_Y+E)|_C=K_C+E_1|_C+E_2|_C+E_3|_C+B_Y|_C.$$

From the previous formula, we get that $(K_Y+B_Y+E)\cdot C>0$ leading to a contradiction. □

By [27, Theorem 1.2], there is an explicit list of the possible singularities Gorenstein del Pezzo surfaces of Picard rank 1. Moreover these are up to isomorphism uniquely determined by their singularity type, except for the cases with singularities in $\{E_8,E_7+A_1,E_6+A_2,2D_4\}$.

For the case of singularities $2D_4$, there is an infinite family of Gorenstein del Pezzo surfaces of Picard rank 1, we will label any member of this family by $X(2D_4)$.

For each of the singularities $\{E_8,E_7+A_1,E_6+A_2\}$, there are two isomorphism classes of Gorenstein del Pezzo surfaces of Picard rank 1, and they can be obtained as contractions from rational elliptic surfaces. For each one of these singularities there is a case where the elliptic surface does not have an I_1 singular fiber (see [2, Table 4.1]), these cases will be labeled $X_1(-)$. The other case will be labeled $X(-)$ as with any other singularity that has a unique member.

The main statement of this subsection is the following theorem:

Theorem 3.3. *Let X be a Gorenstein del Pezzo surface of rank* 1*. Then there exists a* 1*-complement B of X with $c_{\rm bir}(X,B)=0$, if and only if X is not isomorphic to $X(2D_4),X_1(E_8),X_1(E_7+A_1),X_1(E_6+A_2)$.*

In [25, Example 3.13], it is shown that the varieties of type $X(2D_4)$ do not admit a 1-complement of coregularity 0, hence a fortiori they cannot have a 1-complement with birational complexity 0. This is also true for the remaining 3 varieties in the statement of Theorem 3.3

Lemma 3.4. *Let X be one of $X_1(E_8),X_1(E_7+A_1),X_1(E_6+A_2)$, then X admits no* 1*-complement of coregularity* 0*.*

Proof. Assume that B is such that $K_X+B\sim 0$ and $\operatorname{coreg}(X,B)=0$. Let $x\in X$ be the E-type singularity, and $f:Y\to X$ the minimal resolution of X. Let B_Y be the strict transform of B.

Let $E_1,E_2,\dots E_k$ be the exceptional curves over x of $f:Y\to X$, where $E_2,\dots E_k$ form a chain, and E_1 intersects only E_4. Here k can be 6, 7 or 8. By Lemma 3.2, B cannot contain x, therefore B_Y cannot intersect any E_i. By [2, Table II], there exists a unique sequence of blow downs $\varphi:Y\to\mathbb{P}^2$ that contracts every E_i, except for E_1. Let E_1' and B' be the images of E_1 and B_Y by φ. In $\mathbb{P}^2$, B' is of degree 3 and E_1' is a line.

By the description of the maps in [2, Table II], in $\varphi:Y\to\mathbb{P}^2$ there is a unique point in E_1' over which blow ups are performed, three of these are performed over the strict transforms of E_1', whose exceptional divisors are the strict transforms of E_2, E_3 and E_4. If any irreducible component of B' were to have order of contact with E_1' of 1 or 2, it would imply that B_Y would intersect E_2 or E_3 respectively. Therefore B' can have only one irreducible component, hence B is also irreducible. Since $\operatorname{coreg}(X,B)=0$, B must have nodal singularities. This would imply that a fiber in the rational elliptic surface from which X can be obtained (see [2, Table 4.1]) is a nodal curve, a contradiction

□

To prove Theorem 3.3, we will study some special configurations of curves in $\mathbb{P}^2$ that get mapped to the exceptional divisors of minimal resolutions of these surfaces.

Lemma 3.5. *Let X be a Gorenstein del Pezzo surface of Picard rank one, whose singularity type is $\{A_2 + A_1, A_4, D_5, E_6, E_7, E_8, D_6{+}A_1, D_8, E_7{+}A_1, 3A_2, A_2{+}A_5, A_2{+}E_6, D_4{+}3A_1, 2A_1{+}D_6, A_1{+}2A_3, D_5{+}A_3, 4A_2\}$ and let $(\mathbb{P}^2, C)$ be the projective plane with a nodal cubic. Then, there exists a crepant birational map $(\mathbb{P}^2, C) \dashrightarrow (X, B)$, where (X, B) is log canonical.*

Proof. We will start by describing some maps from $\mathbb{P}^2$ to each of these surfaces.

Let C be a nodal cubic in $\mathbb{P}^2$ with a flex P and let L be a line tangent to C at P. If we blow up $3, 4, 5, 6, 7$ or 8 times over P along the strict transform of C, we end up with the minimal resolutions of $X(A_1 + A_2)$, $X(A_4)$, $X(D_5)$, $X(E_6)$, $X(E_7)$ or $X(E_8)$, respectively.

Let C be a nodal cubic in $\mathbb{P}^2$ with a flex P, L_1 be the line tangent to C at P and L_2 another line passing through P, tangent to C at some other point Q. If we blow up 5 times over P along the strict transform of C, and we blow up over Q two times along the strict transform of C, we end up with the minimal resolution of $X(D_6 + A_1)$. If we blow up once more over Q at the strict transform of C, we end up with the minimal resolution of $X(D_8)$.

Let C be a nodal cubic in $\mathbb{P}^2$ with a flex P, L_1 be the line tangent to C at P and L_2 another line passing through P, tangent to C at some other point Q. If we blow up 6 times over P and 2 times over Q along the strict transform of C, we end up with the minimal resolution of $X(E_7 + A_1)$.

Let C be a nodal cubic in $\mathbb{P}^2$ with two flexes P_1 and P_2, and L_1 and L_2 be the lines tangent to C at P_1 and P_2 respectively. If we blow up 3 times over P_1 and 3 times over P_2 along the strict transform of C, we end up with the minimal resolution of $X(3A_2)$. If we furthermore blow up over P_1 along the strict transform of C once more, we end up with the minimal resolution of $X(A_5 + A_2)$. If we furthermore blow up over P_1 along the strict transform of C once more, we end up with the minimal resolution of $X(E_6 + A_2)$.

Let C be a nodal cubic in $\mathbb{P}^2$ with a flex P and node N, let L_1 be the line tangent to C at P, let L_2 be a line passing through P tangent to C at Q and let L_3 be the line passing through P and N. If we blow up 3 times over P and 2 times over Q along the strict transform of C. And we blow up twice over N along the strict transform of L_3, then we have the minimal resolution of $X(D_4 + 3A_1)$. If we blow up once more over Q along the strict transform of C, then we have the minimal resolution of $X(2A_1 + D_6)$.

Let C be a nodal cubic in $\mathbb{P}^2$ with a flex P, let L_1 be the line tangent to C at P, let L_2 be a line passing through P and tangent to C at Q_1, and let L_3 be a line passing through Q_1 and tangent to C at Q_2. If we blow up 3 times over P, 2 times over Q_1 and 2 times over Q_2 along the strict transform of C, then we have the minimal resolution of $X(A_1 + 2A_3)$. If we blow up once more over Q_2 along the strict transform of C, then we have the minimal resolution of $X(D_5 + A_3)$.

In any of these previous cases let X be the Gorenstein del Pezzo surface of rank 1. If we take the pair $(X, \hat{C})$, where $\hat{C}$ is the strict transform of C via the maps described. The described maps from $(\mathbb{P}^2, C)$ to $(X, \hat{C})$ are crepant birational.

Let C be a nodal cubic in $\mathbb{P}^2$, with node N and two flexes P_1 and P_2, and L_1 and L_2 be the lines tangent to C at P_1 and P_2 respectively. If we blow up 2 times over N, 3 times over P_1 and 3 times over P_2 along C, then we have the minimal resolution of $X(4A_2)$.

In this final case, let B be the image on X of the (-1)-curve on the minimal resolution that maps to N in $\mathbb{P}^2$. Then the described map from $(\mathbb{P}^2, C)$ to (X, B) is crepant birational. □

Lemma 3.6. *Let X be a Gorenstein del Pezzo surface of Picard rank one, whose singularity type is $X(2A_4)$ and let $(\mathbb{P}^2, Q + L)$ be the projective plane with a conic and a transversal line. Then, there exists a crepant birational map $(\mathbb{P}^2, Q + L) \dashrightarrow (X, B)$, where (X, B) is log canonical.*

Proof. Let Q be a conic with secant line L_1. Write P_1 and P_2 for the points of intersection between Q and L_1. Let L_2 be a line tangent to Q at P_2, and let L_3 be a line tangent to Q at a point $P_3 \in Q \backslash L_1$. Write P_4 for the point of intersection between L_3 and L_1. Blow up once at P_1 and once at P_4. Blow up three times above P_2 along L_2 and three times above P_3 along Q. We obtain the minimal resolution of $X(2A_4)$.

As $(\mathbb{P}^2, Q + L)$ is crepant birational to $(\mathbb{P}^2, \Sigma^2)$, via one standard Cremona transformation, we obtain the desired result. □

Lemma 3.7. *Let X be a Gorenstein del Pezzo surface of Picard rank one, whose singularity type is $\{A_8, A_7 + A_1, A_1 + A_2 + A_5, 2A_1 + A_3, A_1 + A_5, A_7\}$ and let $(\mathbb{P}^2, L_1 + L_2 + L_3)$ be the projective plane with the three coordinate axes. Then, there exists a crepant birational map $(\mathbb{P}^2, L_1 + L_2 + L_3) \dashrightarrow (X, B)$, where (X, B) is log canonical.*

Proof. Let L_1, L_2, L_3 be three lines in $\mathbb{P}^2$ where P_1 is the intersection of L_1 and L_3 and P_2 is the intersection of L_2 and L_3. Blow up 3 times over P_1 at the strict transform of L_1 and 3 times over P_2 at the strict transform of L_1. Next, blow up one interior point of each of the two exceptional (-1)-curves. We obtain the minimal resolution of $X(A_8)$.

Let L_1, L_2, L_3 be three lines in $\mathbb{P}^2$. Write P_1 for the point of intersection between L_1 and L_2, P_2 for the the point of intersection between L_2 and L_3, and Q for a point in $L_1 \backslash (L_2 \cup L_3)$. Blow up twice over Q along L_1 and three times over P_2 along L_3. Blow up twice over P_1 along L_2. Finally, blow up once at an interior point of the exceptional (-1)-curve over P_1. We obtain the minimal resolution of $X(A_7 + A_1)$.

Let L_1, L_2, L_3 be three lines in $\mathbb{P}^2$ where P_1 is the intersection of L_1 and L_3. Let Q_1 and Q_2 be points in L_1 and L_2 respectively, but in no other L_i. If we blow up 3 times over P_1 along the strict transform of L_3, we blow up 2 times over Q_1 along the strict transform of L_1 and we blow up 3 times over Q_2 along the strict transform of L_2, then we have the minimal resolution of $X(A_1 + A_2 + A_5)$

Let L_1, L_2, L_3 be three lines in $\mathbb{P}^2$ where P_1 is the intersection of L_1 and L_3 and P_2 is the intersection of L_2 and L_3. If we blow up 3 times over P_1 along the strict transform of L_1 and we blow up 2 times over P_2 along the strict transform of L_3, then we have the minimal resolution of $X(2A_1 + A_3)$.

If furthermore we blow up once over P_1 at a point in the (-1)-curve, not in any (-2)-curve, then we have the minimal resolution of $X(A_1 + A_5)$.

If furthermore we blow up once more over P_2 at a point in the (-1)-curve, not in any (-2)-curve, then we have the minimal resolution of $X(A_7)$.

□

Proof of Theorem 3.3. By [27, Theorem 1.2], the surfaces in Lemmas 3.5, 3.6, and 3.7, are all the Gorenstein del Pezzo surface of rank 1, other than those listed in the statement of the theorem. Therefore, all the required surfaces admit a 1-complement B with $c_{\mathrm{bir}}(X, B) = 0$. By Lemma 3.4 and [25, Example 3.13], the surfaces listed in the statement of the Theorem do not admit a 1-complement with coregularity 0, those a fortiori they cannot admit a 1-complement of birational complexity 0. □

4. Torus exceptional degree

In this section, we prove several statements about the torus exceptional degree of log Calabi–Yau pairs. First, we are concerned with the general picture, and then we focus on log Calabi–Yau surfaces.

4.1. **Log Calabi–Yau pairs.** In this subsection, we prove the positivity of the torus exceptional degree and the characterization of log Calabi–Yau pairs with torus exceptional degree zero.

Proof of Theorem 1.3. Let (X, B) be a log Calabi–Yau pair of dimension n and index one. By definition, we have that $\mathcal{T}_t(X, B) \geqslant \mathcal{T}(X, B) \geqslant 0$. This proves (1). If $\mathcal{T}_t(X, B)$ is finite, then the log Calabi–Yau pair admits a crepant birational map $(\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$ so its birational complexity is zero. On the other hand, assume that (X, B) has birational complexity zero. Then, by the proof of Theorem 1.2, there is a finite set $\{\phi_i\}_{i\in I}$ of crepant birational maps $\phi_i\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$ that divisorially cover (X, B). Thus, we get

$$\mathcal{T}_t(X, B) \leqslant \sum_{i\in I} \mathcal{T}(\phi_i),$$

and the right-hand side is finite. This proves (2).

We turn to prove (3). If (X, B) is divisorially covered by almost tori, then there are birational maps

$$\varphi_i\colon \mathbb{P}^n \dashrightarrow X,$$

that induce embeddings $\varphi_i\colon \mathbb{G}_m^n\backslash Z_i \hookrightarrow X\backslash B$ where Z_i is a closed subvariety of $\mathbb{G}_m^n$ of codimension at least 2. Let $p_i\colon Y_i \to \mathbb{P}^n$ and $q_i\colon Y_i \to X$ give a resolution of the birational map ϕ_i. Let $\Delta_i := p_{i*}q_i^*B$. Since $\mathbb{G}_m^n\backslash Z_i \hookrightarrow X\backslash B$ is an embedding, we conclude that Δ_i is supported on $\mathbb{P}^n\backslash\mathbb{G}_m^n$. This means that Δ_i is supported on the coordinate hyperplanes. As $(\mathbb{P}^n, \Delta_i)$ is a sub log Calabi–Yau pair, we conclude that $\Delta_i = \Sigma^n$ for each i. Thus, we have a finite set of crepant birational maps

$$\phi_i\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B),$$

that divisorially cover (X, B). Furthermore, for each i, we have $\mathcal{T}(\phi_i) = 0$. Indeed, the birational maps $\phi_i\colon \mathbb{P}^n \dashrightarrow X$ are small in the algebraic torus. We conclude that $\mathcal{T}(X, B) = \mathcal{T}_t(X, B) = 0$.

Now, assume that $\mathcal{T}(X, B) = 0$. This means that there exists a crepant birational map

$$\phi\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B),$$

such that $\mathcal{T}(\phi) = 0$. In other words, no exceptional divisor of ϕ intersects the algebraic torus $\mathbb{G}_m^n$. Let

$$U_1 := \phi(\mathbb{P}^n\backslash(\Sigma^n \cup \mathrm{Ex}(\phi))).$$

Then U_1 is an almost torus that admits an embedding $U_1 \hookrightarrow X\backslash B$. Let $E_2, \dots, E_k$ be the divisorial prime components of $(X\backslash B)\backslash U_1$.

For each $i \in \{2, \dots, k\}$, we show that there exists a crepant birational map $\phi_i\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$ such that the center of E_i on $\mathbb{P}^n$ is a divisor intersecting $\mathbb{G}_m^n$ and ϕ_i is of cluster type, i.e., induces a small birational map on $\mathbb{G}_m^n$. This would imply that the open sets

$$U_i := \phi_i(\mathbb{P}^n\backslash(\Sigma^n \cup \mathrm{Ex}(\phi_i)))$$

are almost tori that cover $X\backslash B$ divisorially.

It suffices to show the statement for $E := E_2$. The divisor E is exceptional over the domain $\mathbb{P}^n$ of ϕ. Then, we can apply Lemma 3.1 to the pair $(\mathbb{P}^n, \Sigma^n)$ and the valuation E. We obtain a crepant birational map $\phi\colon (Y, B_Y) \dashrightarrow (X, B)$ satisfying the following conditions:

(1) we have $c(Y, B_Y) = 1$,
(2) the center of E on Y is a prime divisor E_Y,

(3) the variety Y is $\mathbb{Q}$-factorial and admits a fibration $p\colon Y \to Z$ to a variety of dimension $n-1$,
(4) there is a prime component F_Y of $p^{-1}(p(E_Y))$ that dominates E_Y, is not contained in B_Y, and it is different from E_Y, and
(5) the pair $(Y, B_Y + \epsilon F_Y)$ is dlt for $\epsilon > 0$ small enough.

Furthermore, we know that the crepant birational map $(Y, B_Y) \dashrightarrow (\mathbb{P}^n, \Sigma^n)$ only extracts E_Y and log canonical places of $(\mathbb{P}^n, \Sigma^n)$. By condition (5), we can run a $(K_Y + B_Y + \epsilon F_Y)$-MMP over Z. This MMP terminates with a toric log Calabi–Yau pair (T, B_T) due to conditions (1) and (3). Let E_T be the center of E on T. Note that $(T, B_T) \dashrightarrow (\mathbb{P}^n, \Sigma^n)$ is a crepant birational map that only extracts E_T and log canonical places of $(\mathbb{P}^n, \Sigma^n)$. Since the center of E_T on X is a divisor, we conclude that the crepant birational map $f\colon (T, B_T) \dashrightarrow (X, B)$ is small on $T \backslash B_T$. Let $g\colon (T, B_T) \dashrightarrow (\mathbb{P}^n, \Sigma^n)$ be any crepant toric birational map to $(\mathbb{P}^n, \Sigma^n)$. Toric birational maps are isomorphisms on the algebraic torus $\mathbb{G}_m^n$. We conclude that the crepant birational map

$$\phi_2 := g^{-1} \circ f\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$$

is of cluster type and the center of E on its domain $\mathbb{P}^n$ is a divisor that intersects $\mathbb{G}_m^n$ non-trivially. Hence, the almost tori $U_2 := \phi_2(\mathbb{P}^n \backslash (\Sigma^n \cup \operatorname{Ex}(\phi_2)))$ intersects the divisor E_2 non-trivially. This finishes the proof of (3).

Now, we turn to prove (4). Assume that X is Fano and $\mathcal{T}(X, B) = 0$. By (3), we know that (X, B) is divisorially covered by finitely many almost tori. It suffices to show that, in this case, such almost tori are indeed algebraic tori. Let $\{\phi_i\}_{i \in I}$ be a finite set of crepant birational maps $\phi_i\colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B)$ of cluster type such that $V_i := \phi_i(\mathbb{G}_m^n \backslash \operatorname{Ex}(\phi_i))$ divisorially cover $X \backslash B$. Let $P_1, \dots, P_k$ be the prime exceptional divisors of ϕ_i^{-1} that are not contained in B. As X is Fano, the variety $X \backslash B$ is affine, and so $U_i := X \backslash \operatorname{supp}(B + m_1 P_1 + \dots + m_k P_k)$ is affine, where m_i is the Cartier index of P_i. The birational map ϕ_i^{-1} induces a small birational map between the affine variety U_i and the affine variety $\mathbb{G}_m^n$. Note that V_i and U_i are small birational. So, ϕ_i^{-1} induces an isomorphism between U_i and $\mathbb{G}_m^n$. We conclude that $X \backslash B$ is divisorially covered by finitely many algebraic tori. □

4.2. **Log Calabi–Yau surfaces.** In this subsection, we study the torus exceptional degree of log Calabi–Yau surfaces. In order to prove Theorem 1.5, we first establish some lemma regarding particular birational maps in $\mathbb{P}^2$.

Lemma 4.1. *Let C be a nodal cubic in $\mathbb{P}^2$ with node N and two other points P_1 and P_2. Let L_i be some lines, each one passing through at least one of the points P_1, P_2 or N. Let Q_j be some conics, each one passing through at least two of the points P_1, P_2 or N. There exists a crepant birational map $\varphi : (\mathbb{P}^2, \Sigma^2) \dashrightarrow (\mathbb{P}^2, C)$, with $\tau(\varphi) \leqslant 3$, such that the strict transform of each L_i is empty or a curve of degree at most 2, and the strict transforms of each Q_j is empty or a curve of degree at most 4.*

Proof. Let φ_1 be the standard Cremona transformation with centers at N, P_1, P_2. Let L_i', Q_j and C' be the strict transforms of L_i, Q_j and C respectively. By the choice of centers, C' is a conic, L_i' are lines or points and Q_j' have degrees 1 or 2. Let P the strict transform of the line P_1P_2, and N_1, N_2 be the images of the points over N lying on the strict transform of C. Let φ_2 be the standard Cremona transformation with centers N_1, N_2, P. Let C'', L_i'' and Q_j'' be the strict transforms of C', L_i' and Q_j'. By the choice of centers C'' is a line and the degrees of L_i'' and Q_j'' are at most 2 and 4 respectively. Let E_{N_1} and E_{N_2} be the images of the blow ups at N_1 and N_2.

The birational map $\varphi_1^{-1} \circ \varphi_2^{-1} : (\mathbb{P}^2, C'' + E_{N_1} + E_{N_2}) \dashrightarrow (\mathbb{P}^2, C)$ is crepant birational, with $\tau(\varphi_1^{-1} \circ \varphi_2^{-1}) = 3$, satisfying the desired properties. □

Lemma 4.2. *Let Q be a conic in $\mathbb{P}^2$ and L a line transversal to Q. Let C_i be some curves in $\mathbb{P}^2$. There exists a crepant birational map $\phi : (\mathbb{P}^2, \Sigma^2) \dashrightarrow (\mathbb{P}^2, Q + L)$, with $\tau(\phi) = 1$, such that the strict transform of each C_i is empty or a curve of degree at most twice the degree of C_i.*

Proof. Let P_1, P_2 be the intersection points of Q and L, and let P_3 be another point in Q. Let φ be the standard Cremona transformation center at P_1, P_2, P_3. Then φ^{-1} satisfies the desired properties. $\square$

Proof of Theorem 1.5. As a Gorenstein del Pezzo surface of Picard rank one that admits a 1-complement of birational complexity zero, X must be one of the surfaces appearing in the hypotheses of Lemmas 3.5, 3.6 or 3.7. Denote by $f: Y \to X$ the minimal resolution of X. The pair (Y, B_Y) induced from (X, B) via log pullback has B_Y effective, hence (Y, B_Y) is a log Calabi–Yau pair of index one. Given a crepant birational map $\phi: (\mathbb{P}^2, \Sigma^2) \dashrightarrow (Y, B_Y)$, we can compute $\mathcal{T}(f \circ \phi)$ as the sum of $\mathcal{T}(\phi)$ and the degrees of the strict transforms on $\mathbb{P}^2$ of the (-2)-curves on Y. We will use this observation to bound $\mathcal{T}(X, B)$ by considering ϕ that factor as $\phi = g^{-1} \circ \varphi$ for $\varphi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as in Lemmas 4.1 and 4.2 and $g: Y \to \mathbb{P}^2$ a birational morphism as in [2, Table II].

Let us establish some notation. Given a singularity type Γ appearing in the hypotheses of Lemmas 3.5, 3.6 or 3.7, we will denote by $X(\Gamma)$ the Gorenstein del Pezzo surface of Picard rank one with that singularity type and by $Y(\Gamma)$ its minimal resolution. We will refer to morphisms in [2, Table II] via triples of the form $(\Gamma, ((-1)\text{-curves to be contracted}), (-2)\text{-curves to be contracted.})$ For example, the unique morphism listed in the case $\Gamma = A_3 + 2A_1$ will be referred to as $(A_3 + 2A_1, (15, 35), 235)$. We will denote by D_i the (-2)-curves on the surfaces $Y(\Gamma)$, numbered as in [2, Table II]. We will denote by $E_{i_1 \ldots i_k}$ a (-1)-curve on a surface $Y(\Gamma)$ that meets the (-2)-curves $D_{i_1}, \ldots, D_{i_k}$ and no others.

We separate in cases, depending on the isomorphism class of X.

(1) The surface X is isomorphic to one of $X(A_2 + A_1), X(A_4), X(D_5), X(E_6), X(E_7)$ or $X(E_8)$.

Let $g: Y(A_2 + A_1) \to \mathbb{P}^2$ be the morphism $(A_2 + A_1, (13), 12)$. The morphisms $(A_4, (2), 234)$, $(D_5, (3), 2345)$, $(E_6, (6), 23456)$, $(E_7, (7), 234567)$ and $(E_8, (8), 2345678)$ factor through this morphism in such a way that every (-2)-curve has image on $Y(A_2 + A_1)$ either a (-2)-curve or the (-1)-curve E_{13}. Thus, it suffices to analyze only the case $X = X(A_2 + A_1)$.

Write $Y = Y(A_2 + A_1)$ and denote by (Y, B_Y) the log Calabi–Yau pair induced from (X, B) via log pullback. Write $B_{\mathbb{P}^2} = g_* B_Y$. The curve D_3 is the only (-2)-curve on Y that is not g-exceptional, and its image under g is a line L. If B_Y does not contain D_3, then $B_{\mathbb{P}^2}$ must be a nodal cubic meeting the line L in exactly one point. It follows from Lemma 4.1 that $\mathcal{T}(X, B) \leqslant 5$ in this case. If B_Y contains D_3, then it follows from Lemmas 4.1 and 4.2 that $\mathcal{T}(X, B) \leqslant 3$.

(2) The surface X is isomorphic to one of $X(D_6 + A_1)$ or $X(D_8)$.

Let $g: Y(D_6 + A_1) \to \mathbb{P}^2$ be the morphism $(D_6 + A_1, (67, 3), 12356)$. The morphism $(D_8, (2, 7), 123567)$ factors through this morphism in such a way that every (-2)-curve has image on $Y(D_6 + A_1)$ either a (-2)-curve or the (-1)-curve E_{67}. Thus, it suffices to analyze only the case $X = X(D_6 + A_1)$.

Write $Y = Y(A_2 + A_1)$ and denote by (Y, B_Y) the log Calabi–Yau pair induced from (X, B) via log pullback. Write $B_{\mathbb{P}^2} = g_* B_Y$. The curves D_4 and D_7 are the only (-2)-curves on Y that are not g-exceptional, and their images under g are lines $L_1 = g_* D_4$ and $L_2 = g_* D_7$. If B_Y does not contain D_7, then $B_{\mathbb{P}^2}$ must be a nodal cubic meeting L_1 and L_2 at their point of intersection. It follows from Lemma 4.1 that $\mathcal{T}(X, B) \leqslant 7$ in this case. If B_Y contains D_7, then $B_{\mathbb{P}^2}$ consists of either L_2 and a conic or L_1, L_2 and a third line. We have $\mathcal{T}(X, B) = 0$ in the latter case, and $\mathcal{T}(X, B) \leqslant 3$ in the former case by Lemma 4.2.

(3) We have $X = X(E_7 + A_1)$.

Write $Y = Y(E_7 + A_1)$, let $g\colon Y \to \mathbb{P}^2$ be the morphism $(E_7 + A_1, (78, 2), 234568)$, and denote by (Y, B_Y) the log Calabi–Yau pair induced from (X, B) via log pullback. Write $B_{\mathbb{P}^2} = g_* B_Y$. The curves D_1 and D_7 are the only (-2)-curves on Y that are not g-exceptional, and their images under g are lines $L_1 = g_* D_1$ and $L_2 = g_* D_7$. If B_Y does not contain D_1, then $B_{\mathbb{P}^2}$ must be a nodal cubic meeting L_1 and L_2 at their point of intersection. It follows from Lemma 4.1 that $\mathcal{T}(X, B) \leqslant 7$ in this case. If B_Y contains D_1, then $B_{\mathbb{P}^2}$ consists of either L_1 and a conic or L_1, L_2 and a third line. We have $\mathcal{T}(X, B) = 0$ in the latter case, and $\mathcal{T}(X, B) \leqslant 3$ in the former case by Lemma 4.2.

(4) The surface X is isomorphic to one of $X(3A_2)$, $X(A_2 + A_5)$, $X(A_2 + E_6)$ or $X(A_8)$.

Let $g\colon Y(3A_2) \to \mathbb{P}^2$ be the morphism $(3A_2, (16, 23, 45), 246)$. The morphisms $(A_2+A_5, (17, 56, 3), 2357)$, $(A_2+E_6, (27, 68, 1)12458)$ and $(A_8, (18, 3, 6), 13467)$ factor through this morphism in such a way that every (-2)-curve has image on $Y(3A_2)$ either a (-2)-curve or one of the (-1)-curves E_{23} or E_{45}. Thus, it suffices to analyze only the case $X = X(3A_2)$.

Write $Y = Y(3A_2)$ and denote by (Y, B_Y) the log Calabi–Yau pair induced from (X, B) via log pullback. Write $B_{\mathbb{P}^2} = g_* B_Y$. The curves D_1, D_3 and D_5 are the only (-2)-curves on Y that are not g-exceptional, and their images L_1, L_2 and L_3 on $\mathbb{P}^2$ form a triangle of lines. If B_Y contains none of the (-1)-curves E_{16}, E_{23} or E_{45}, then $B_{\mathbb{P}^2}$ is a nodal cubic passing through each of the points of intersection between pairs of L_1, L_2 and L_3. Choosing any two of these points, it follows from Lemma 4.1 that $\mathcal{T}(X, B) \leqslant 9$. If B_Y contains at least one of the (-1)-curves E_{16}, E_{23} or E_{45}, then $B_{\mathbb{P}^2}$ is a triangle of lines containing at least two of L_1, L_2 and L_3. We have $\mathcal{T}(X, B) \leqslant 1$ in this case.

(5) The surface X is isomorphic to one of $X(D_4 + 3A_1)$ or $X(D_6 + 2A_1)$.

Let $g\colon Y(D_4 + 3A_1) \to \mathbb{P}^2$ be the morphism $(D_4 + 3A_1, (15, 37, 46), 1267)$. The morphism $(D_6 + 2A_1, (17, 38, 5), 12568)$ factors through this morphism in such a way that every (-2)-curve has image on $Y(D_4 + 3A_1)$ either a (-2)-curve or the (-1)-curve E_{46}. Thus, it suffices to analyze only the case $X = X(D_4 + 3A_1)$.

Write $Y = Y(D_4 + 3A_1)$ and denote by (Y, B_Y) the log Calabi–Yau pair induced from (X, B) via log pullback. Write $B_{\mathbb{P}^2} = g_* B_Y$. The curves D_3, D_4 and D_5 are the only (-2)-curves on Y that are not g-exceptional, and their images L_1 L_2 and L_3 on $\mathbb{P}^2$ are three lines meeting in a single point. If B_Y contains none of the (-1)-curves E_{15}, E_{37} or E_{46}, then $B_{\mathbb{P}^2}$ is a nodal cubic passing through the common intersection point of L_1, L_2 and L_3. It follows from Lemma 4.1 that $\mathcal{T}(X, B) \leqslant 9$ in this case. If B_Y contains at least one of the (-1)-curves E_{15}, E_{37} or E_{46}, then $B_{\mathbb{P}^2}$ must be a triangle of lines containing two of L_1, L_2 and L_3. We have $\mathcal{T}(X, B) \leqslant 1$ in this case.

(6) The surface X is isomorphic to one of $X(A_1 + 2A_3)$, $X(D_5 + A_3)$ or $X(A_7 + A_1)$.

Let $g\colon Y(A_1 + 2A_3) \to \mathbb{P}^2$ be the morphism $(A_1 + 2A_3, (14, 57, 36), 1267)$. The morphisms $(D_5 + A_3, (16, 38, 4), 14578)$ and $(A_7 + A_1, (28, 17, 4), 34678)$ factor through this morphism in such a way that every (-2)-curve has image on $Y(A_1 + 2A_3)$ either a (-2)-curve or one of the (-1)-curves E_{36} or E_{57}. Thus, it suffices to analyze only the case $X = X(A_1 + 2A_3)$.

Write $Y = Y(A_1 + 2A_3)$ and denote by (Y, B_Y) the log Calabi–Yau pair induced from (X, B) via log pullback. Write $B_{\mathbb{P}^2} = g_* B_Y$. The curves D_3, D_4 and D_5 are the only (-2)-curves on Y that are not g-exceptional, and their images $L_1 = g_* D_3$ $L_2 = g_* D_4$ and $L_3 = g_* D_5$ on $\mathbb{P}^2$ form a triangle of lines. If B_Y contains neither of the (-1)-curves E_{14} or E_{36}, then $B_{\mathbb{P}^2}$ is a nodal cubic passing through the points $L_1 \cap L_2$ and $L_1 \cap L_3$. It follows from Lemma 4.1 that $\mathcal{T}(X, B) \leqslant 9$ in this case. If B_Y contains at least on of the (-1)-cuves E_{14} or E_{36}, then $B_{\mathbb{P}^2} = L_1 + L_2 + L_3$. We have $\mathcal{T}(X, B) = 0$ in this case.

(7) We have $X = X(4A_2)$.

Write $Y = Y(4A_2)$, let $g\colon Y \to \mathbb{P}^2$ be the morphism $(4A_2, (138, 145, 12), 25678)$, and denote by (Y, B_Y) the log Calabi–Yau pair induced from (X, B) via log pullback. Write $B_{\mathbb{P}^2} = g_* B_Y$. The curves D_1, D_3 and D_4 are the only (-2)-curves on Y that are not g-exceptional. The curve $C = g_* D_1$ is a nodal cubic, and $L_1 = g_* D_3$ and $L_2 = g_* D_4$ are lines tangent to C at flex points. If B_Y contains D_1, then $B_{\mathbb{P}^2} = C$. We have $\mathcal{T}(X, B) \leqslant 7$ in this case by Lemma 4.1. If B_Y contains at least one of D_3 or D_4, then B_Y is a triangle of lines containing $L_1 + L_2$. We have $\mathcal{T}(X, B) \leqslant 3$ in this case.

Suppose that B_Y contains none of D_1, D_3 and D_4. Then $B_{\mathbb{P}^2}$ is a nodal cubic passing through the node of C and both of the points $C \cap L_1$ and $C \cap L_2$. Let $\varphi_1\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a standard Cremona transformation centered at points $p_1, p_2, p_3 \in \mathbb{P}^2$, at least one of which is the node of $B_{\mathbb{P}^2}$ and at least two of which are the points $C \cap L_1$ and $C \cap L_2$. Let $(\mathbb{P}^2, B'_{\mathbb{P}^2})$ be the log Calabi–Yau pair induced on the codomain of φ_1 by $(\mathbb{P}^2, B_{\mathbb{P}^2})$. Thus, $B'_{\mathbb{P}^2}$ is a conic and a secant line. Write $C' = (\varphi_1)_* C$, $L'_1 = (\varphi_1)_* L_1$ and $L'_2 = (\varphi_1)_* L_2$. Note that L'_1 and L'_2 are lines and that $\deg(C') \leqslant 4$. Let $\varphi_2\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a standard Cremona transformation chosen as in the proof of Lemma 4.1. Let $(\mathbb{P}^2, B''_{\mathbb{P}^2})$ be the log Calabi–Yau pair induced on the codomain of φ_2 by $(\mathbb{P}^2, B'_{\mathbb{P}^2})$. Thus, $B''_{\mathbb{P}^2}$ is a triangle of lines. Write $C'' = (\varphi_2)_* C'$, $L''_1 = (\varphi_2)_* L'_1$ and $L''_2 = (\varphi_2)_* L'_2$. Then $\deg(C'') \leqslant 6$ and L''_1 and L''_2 both have degree at most 2. It follows from the proof of Lemma 4.1 that $\mathcal{T}(\varphi_1^{-1} \circ \varphi_2^{-1}) \leqslant 3$. Thus, we see that $\mathcal{T}(X, B) \leqslant 13$ in this case.

(8) We have $X = X(2A_4)$.

Write $Y = Y(2A_4)$, let $g\colon Y \to \mathbb{P}^2$ be the morphism $(A_4, (16, 58), 123478)$, and denote by (Y, B_Y) the log Calabi–yau pair induced from (X, B) via log pullback. Write $B_{\mathbb{P}^2} = g_* B_Y$. The curves D_5 and D_6 are the only (-2)-curves on Y that are not g-exceptional. The curve $Q = g_* D_6$ is a conic, and the curve $L = g_* D_5$ is a secant line to Q. If B_Y contains at least one of D_5 or D_6, then $B_{\mathbb{P}^2} = Q + L$. We have $\mathcal{T}(X, B) \leqslant 1$ by Lemma 4.2 in this case. If B_Y contains neither D_5 nor D_6, then $B_{\mathbb{P}^2}$ is a nodal cubic meeting Q in two points, one of which is a point of $L \cap Q$. It follows from Lemma 4.1 that $\mathcal{T}(X, B) \leqslant 9$ in this case.

(9) We have $X = X(A_5 + A_3 + A_1)$.

Write $Y = Y(A_5 + A_3 + A_1)$, let $g\colon Y \to \mathbb{P}^2$ be the morphism $(A_5 + A_3 + A_1, (27, 38, 15), 45678)$, and denote by (Y, B_Y) the log Calabi–yau pair induced from (X, B) via log pullback. Write $B_{\mathbb{P}^2} = g_* B_Y$. The curves D_1, D_2 and D_3 are the only (-2)-curves on Y that are not g-exceptional, and their images $L_1 = g_* D_1$, $L_2 = g_* D_2$ and $L_3 = g_* D_3$ on $\mathbb{P}^2$ form a triangle of lines. If B_Y contains at least one of D_1, D_2 or D_3, then $B_{\mathbb{P}^2} = L_1 + L_2 + L_3$. We have $\mathcal{T}(X, B) = 0$ in this case. If B_Y contains none of D_1, D_2 or D_3, then $B_{\mathbb{P}^2}$ is a nodal cubic meeting L_2 and passing through the point $L_1 \cap L_3$. It follows from Lemma 4.1 that $\mathcal{T}(X, B) \leqslant 9$ in this case.

(10) The surface X is isomorphic to one of $X(2A_1 + A_3)$, $X(A_1 + A_5)$ or $X(A_7)$.

Let $g\colon Y(2A_1 + A_3) \to \mathbb{P}^2$ be the morphism $(2A_1 + A_3, (15, 34), 235)$. The morphisms $(A_1 + A_5, (16, 4), 2346)$ and $(A_7, (2, 6), 23467)$ factor through this morphism in such a way that every (-2)-curve has image on $Y(2A_1 + A_3)$ either a (-2)-curve or one of the (-1)-curves E_{15} or E_{34}. Thus, it suffices to analyze only the case $X = X(2A_1 + A_3)$.

Write $Y = Y(A_1 + 2A_3)$ and denote by (Y, B_Y) the log Calabi–Yau pair induced from (X, B) via log pullback. Write $B_{\mathbb{P}^2} = g_* B_Y$. The curves D_1 and D_4 are the only (-2)-curves that are not g-exceptional, and their images $L_1 = g_* D_1$ and $L_2 = g_* D_2$ on $\mathbb{P}^2$ are lines. If B_Y doesn't contain E_{34}, then $B_{\mathbb{P}^2}$ is a nodal cubic meeting both L_1 and L_2. It follows from Lemma 4.1 that $\mathcal{T}(X, B) \leqslant 7$

in this case. If B_Y contains E_{34}, then $B_{\mathbb{P}^2}$ is a triangle of lines that contains $L_1 + L_2$. We have $\mathcal{T}(X, B) = 0$ in this case.

□

Now, we turn to prove Theorem 1.4. First, we state a lemma which is a consequence of [14, Lemma 2.12].

Lemma 4.3. *Let (X, B) be a dlt log Calabi–Yau surface of index one. If $f\colon X \to Y$ is a composite of divisorial contractions, then $B_Y = f_*B$ is contained in the smooth locus of Y.*

Proof of Theorem 1.4. Let (X, B) be a log Calabi–Yau surface of index one. Let $f\colon (X', B') \to (X, B)$ be a dlt modification. Then X' is canonical and B' is contained in the smooth locus of X'. We may run a $K_{X'}$-MMP to obtain

$$X' \xrightarrow{g} X'' \xrightarrow{\pi} Z,$$

with g a composite of divisorial contractions and π a Mori fiber space. We note that X'' is canonical. By Lemma 4.3, $B'' = g_*B'$ is contained in the smooth locus of X''. Note that for any crepant birational map $\phi : (\mathbb{P}^2, \Sigma^2) \dashrightarrow (X, B)$ we have

$$\mathcal{T}(\phi) \leqslant \mathcal{T}(g \circ f^{-1} \circ \phi)$$

since g is a morphism and f only extracts log canonical places of (X, B). It follows that we have an inequality

$$\mathcal{T}(X, B) \leqslant \mathcal{T}(X'', B''),$$

and thus it suffices to consider index one log Calabi–Yau surfaces (X, B) satisfying the following:

(1) X is canonical,
(2) X is equipped with a Mori fiber space $\pi\colon X \to Z$, and
(3) $B \subset X_{reg}$ and $B \neq 0$.

If Z is a point, then X is a rank one Gorenstein del Pezzo surface. In this case, the statement follows from Theorem 1.5.

From now on, we assume that $Z \simeq \mathbb{P}^1$ and π has irreducible fibers. We have $B \cdot F = 2$ for every fiber F of π, and since B is contained in the smooth locus of X we have that $B \cdot F_{red}$ is an integer for each fiber F. Since π can only have multiple fibers it follows that either F is contained in the smooth locus of X or that $B \cdot F_{red} = 1$. In particular, there are no singular fibers that contain a node of B. Thus, after possibly performing an elementary transformation centered at some node of B, we may assume that B has at least two components and that at least one of them is a fiber. If B has at least three components, then at least one component B_0 satisfies $B_0 \cdot F = 1$ for each fiber F. But this means there can be no multiple fibers, and so X must be smooth. In particular, X is a Hirzebruch surface. After performing at most one additional elementary transformation centered at a node of B, we may assume that B contains two fibers of π and hence is the toric boundary for some toric structure on X. In the previous process, we only extracted log canonical places of (X, B), so $\mathcal{T}(X, B) = 0$ in this case.

To conclude, we consider the case in which B has only two components. Both components are isomorphic to $\mathbb{P}^1$, one component B_1 is a fiber of π, and the other component B_2 is a 2-section. Each singular fiber meets B_2 in precisely one point, hence it corresponds to a ramification point of the morphism $B_2 \to \mathbb{P}^1$ induced by π. By Riemann-Hurwitz, there can be at most two singular fibers. We resolve the singularities of X and run a relative MMP over $\mathbb{P}^1$. We obtain a crepant birational map

$$\begin{tikzcd} (X,B) \arrow[rr, dashed, "g"] \arrow[dr, "\pi"'] & & (X',B') \arrow[dl, "\pi'"] \\ & \mathbb{P}^1. & \end{tikzcd}$$

over $\mathbb{P}^1$, where X' is a Hirzebruch surface and B' is a 1-complement consisting of a fiber B'_1 of π' and a 2-section B'_2. Write F_1 and F_2 for the two fibers of π' meeting B'_2 at ramification points of the induced morphism $B'_2 \to \mathbb{P}^1$. Observe that F_1 and F_2 are canonical places of (X,B) extracted on X'.

We note that X' is either $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow up $\mathbb{F}_1$ of $\mathbb{P}^2$ at a point. Indeed, let $C \subset X'$ be a section with $C^2 = -n$, $n \geqslant 0$. Since C is distinct from both B'_1 and B'_2 and since $B'_1 \cdot C = 1$, we have

$$2 - n = -K_{X'} \cdot C = B' \cdot C \geqslant 1,$$

hence $n \leqslant 1$. If $X' \cong \mathbb{P}^1 \times \mathbb{P}^1$, then we may perform an elementary transformation centered at a node of B' to obtain a 1-complement of $\mathbb{F}_1$. with two components. Thus, we may assume that $X' \cong \mathbb{F}_1$. Note that the negative section on X' does not meet the 2-section B'_2. Blowing down the negative section, we obtain a complement B'' on $\mathbb{P}^2$ consisting of a line B''_1 and a conic B''_2 intersecting transversely, and the images of F_1 and F_2 are tangent lines to B''_2. We perform a standard Cremona transformation centered at the points of intersection between B''_2 and F_1, B''_2 and F_2, and one of the intersection points of B''_1 and B''_2. We obtain a crepant toric model for (X,B) whose exceptional locus consists of at most four lines, namely two of the exceptional divisors of the Cremona together with some subset of the strict transforms of F_1 and F_2. It follows in this case that $\mathcal{T}(X,B) \leqslant 4$. $\square$

Proof of Theorem 1.8. By Theorem 1.4 any projective surface in $\mathcal{F}_2$ has a log pair structure (X,B), with $\mathcal{T}(X,B) \leqslant 13$. This implies that there exists a birational map $\varphi : \mathbb{G}_m^2 \dashrightarrow X \backslash B$, with $\mathrm{Ex}(\varphi)$ of degree at most 13. Thus there exists an open subset of X isomorphic to $\mathbb{A}^2 \backslash D$, where D is a curve of degree at most 15. These open subsets are fibers of the morphism:

$$\mathrm{Spec}(\mathbb{K}[x, y, a_{0,0}, a_{0,1} \dots, a_{15,15}]_{\Sigma_{i,j} a_{i,j} x^i y^j}) \to \mathrm{Spec}(\mathbb{K}[a_{0,0}, a_{0,1} \dots, a_{15,15}]).$$

Thus $\mathcal{F}_2$ is affinely bounded. $\square$

5. Cluster type

In this section, we prove several statements about the cluster type of a log Calabi–Yau pair. First, we characterize log Calabi–Yau toric pairs as those having cluster type one. Secondly, we study the cluster type of Fano surfaces.

5.1. **Log Calabi–Yau pairs.** In this subsection, we prove the characterization of log Calabi–Yau toric pairs using the cluster type as well as the bound relating cluster type and complexity.

Proof of Theorem 1.10. Let (X,B) be a log Calabi–Yau pair. By definition the cluster type $\mathcal{C}(X,B)$ of (X,B) is either infinite or a finite number at least one.

If (X,B) is a toric log Calabi–Yau pair, then $X \backslash B \simeq \mathbb{G}_m^n$ and so $\mathcal{C}(X,B) = 1$. Now, assume that $\mathcal{C}(X,B) = 1$. Then, there exists a crepant birational map

$$\phi \colon (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X,B)$$

that is of cluster type and $U := \phi(\mathbb{P}^n \backslash (\Sigma^n \cup \mathrm{Ex}(\phi)))$ divisorially covers $X \backslash B$. In other words, every exceptional divisor of ϕ^{-1} is contained in B. Thus, there exists a crepant birational morphism $\psi \colon (T, B_T) \to (\mathbb{P}^n, \Sigma^n)$, that only extracts log canonical places of $(\mathbb{P}^n, \Sigma^n)$ such that the induced birational map $\varphi\colon (T, B_T) \dashrightarrow (X, B)$ is a birational contraction. Henceforth, we get a commutative diagram

$$\begin{tikzcd} (T, B_T) \arrow[d, "\psi"] \arrow[dr, dashed, "\varphi"] & \\ (\mathbb{P}^n, \Sigma^n) \arrow[r, dashed, "\phi"] & (X, B). \end{tikzcd}$$

Every log canonical place over a toric pair is a toric valuation. Hence, the log Calabi–Yau pair (T, B_T) is a toric log Calabi–Yau pair. We conclude that (X, B) is toric, since φ is a birational contraction [3, Lemma 2.3.2] . □

Proof of Theorem 1.11. Begin by fixing some cluster type crepant birational map

$$\phi : (\mathbb{P}^n, \Sigma^n) \dashrightarrow (X, B).$$

By applying Lemma 2.28 we obtain a commutative diagram of projective birational maps

$$\begin{tikzcd} (T, B_T) \arrow[d, "p_1"] & (X_0, B_0) \arrow[l, "q"'] \arrow[d, dashed, "p_2"] \\ (\mathbb{P}^n, \Sigma^n) & (X, B) \arrow[l, dashed, "\phi^{-1}"'] \end{tikzcd}$$

where the following conditions are satisfied:

- p_1 is a toric projective birational morphism,
- p_2 is a birational contraction map that only extracts log canonical centers of (X, B), and
- q is a projective birational morphism that only extracts canonical centers of (T, B_T).

Write $E_1, \dots, E_m \subset X_0$ for the prime q-exceptional divisors. For each $i \in \{1, \dots, m\}$, we argue that there exists a cluster type crepant birational map $\phi_i\colon (T_i, B_{T_i}) \dashrightarrow (X, B)$ from a toric log Calabi-Yau pair (T_i, B_{T_i}) such that the center of E_i on T_i is a divisor intersecting $T_i \backslash B_{T_i}$. The maps $\phi, \phi_1, \dots, \phi_m$ will then give a divisorial covering of $X \backslash B$ by almost tori. Since $m = c(X_0, B_0) = c(X, B)$, the desired bound will then follow.

For $i \in \{1, \dots, m\}$, apply Lemma 3.1 to the toric log Calabi-Yau pair (T, B_T) and the valuation E_i. We obtain a crepant birational map $\psi_i\colon (Y_i, B_{Y_i}) \dashrightarrow (T, B_T)$ satisfying the following conditions:

(1) we have $c(Y_i, B_{Y_i}) = 1$,
(2) the center of E_i on Y_i is a prime divisor E_{Y_i},
(3) the variety Y_i is $\mathbb{Q}$-factorial and admits a fibration $p_i\colon Y_i \to Z_i$ to a variety of dimension $n-1$,
(4) there is a prime component F_{Y_i} of $p_i^{-1}(p_i(E_{Y_i}))$ that dominates E_{Y_i}, is not contained in B_{Y_i}, and is different from E_{Y_i}, and
(5) the pair $(Y_i, B_{Y_i} + \epsilon_i F_{Y_i})$ is dlt for $\epsilon_i > 0$ small enough,
(6) $\psi_i\colon Y_i \dashrightarrow T$ only extracts E_{Y_i} and log canonical places of (T, B_T).

By condition (5), we can run a $(K_{Y_i} + B_{Y_i} + \epsilon_i F_{Y_i})$-MMP over Z_i with scaling of an ample divisor. By condition (4), we know that the divisor F_{Y_i} is degenerate over Z_i, so this MMP terminates after contracting F_{Y_i}. Let T_i be the model obtained after contracting F_{Y_i}, and write $q_i\colon Y_i \dashrightarrow T_i$ for the resulting map. Let B_{T_i} be the push-forward of B_{Y_i} by q_i. Since F_{Y_i} is not contained in B_{Y_i}, the complexity drops precisely by one when we contract this divisor. It follows that $c(T_i, B_{T_i}) = 0$, hence that (T_i, B_{T_i}) is a toric log Calabi–Yau pair. Further, the center of E_i on T_i is a divisor that intersects $T_i \backslash B_{T_i}$. Finally, we note by condition (6) the only canonical place contracted by $\psi^{-1} \circ q_i^{-1} \colon T_i \dashrightarrow T$ is E_i. But E_i is extracted again by the cluster type $p_2 \circ q^{-1}$, so it follows that

$$\phi_i = p_2 \circ q^{-1} \circ \psi^{-1} \circ q_i^{-1} \colon (T_i, B_{T_i}) \dashrightarrow (X, B)$$

is a cluster type map. This completes the proof. □

5.2. **Fano surfaces of cluster type.** In this subsection, we study the cluster type of Fano surfaces. Our main result is that a Fano surface has cluster type either one, two, or infinite. In addition, we prove a bound on the size of the singular set of a cluster type surface in terms of its Picard rank.

Proposition 5.1. *Let X be a surface of cluster type. Then X has only A-type singularities.*

Proof. Let $\phi \colon (\mathbb{P}^2, \Sigma^2) \dashrightarrow (X, B)$ be a crepant birational map of cluster type. By Lemma 3.2, we know that X has A-type singularities along B. Thus, it suffices to show that X only has A-type singularities along $X \backslash B$. Let $B_1, \dots, B_k$ be the components of B. Let $C_1, \dots, C_j$ be the divisors on X exceptional over $\mathbb{P}^2$ that are not contained in B. Let $p_1 \colon (T, B_T) \to (\mathbb{P}^2, \Sigma^2)$ be a toric crepant birational morphism satisfying the following conditions:

- the center of each B_i on T is a divisor, and
- every divisor C_i can be obtained from T by a sequence of blow ups at points of B_T that are not zero strata.

Let $t_1, \dots, t_s$ be the centers of the curves C_i in T. Let $p \colon X_0 \to T$ be a projective birational morphism obtained by consecutively blowing up the points $t_1, \dots, t_s$ such that each C_i is indeed a divisor on X_0. Let (X_0, B_0) be the crepant pull-back of (T, B_T) to X_0. For each $i \in \{1, \dots, s\}$ the divisor $p^{-1}(t_i)$ is a chain $E_{i,1} \cup \dots \cup E_{i,j_i}$ of rational curves such that only E_{i,j_i} intersects B_0. We have a crepant projective birational morphism $q \colon (X_0, B_0) \to (X, B)$. Observe that, as ϕ is of cluster type, the projective birational morphism q can only contract curves on $B_0 \cup \bigcup_{i=1}^{j} \left(\bigcup_{k=1}^{j_i} E_{i,k} \right)$. Let $x \in X \backslash B$ be a singular point and let $F = q^{-1}(x)$. If F intersects B_0, then x is contained in B and hence $(X; x)$ is an A-type singularity. If F does not intersect B_0, then F is contained in $\bigcup_{i=1}^{j} \left(\bigcup_{k=1}^{j_i} E_{i,k} \right)$. We conclude that F is contained in a disjoint union of chains of rational curves. As F is connected, it must be a chain of rational curves itself so $(X; x)$ is an A-type singularity. □

Lemma 5.2. *Let X be a Fano surface and (X, B) be a complement. Let $\phi \colon (\mathbb{P}^2, \Sigma^2) \dashrightarrow (X, B)$ be a crepant birational map of cluster type. Let $C_1, \dots, C_k$ be the curves of X that are not contained on B and are exceptional over $\mathbb{P}^2$. Then for $i \neq j$ the curves C_i and C_j can only intersect along B.*

Proof. Let $\phi \colon (\mathbb{P}^2, \Sigma^2) \dashrightarrow (X, B)$ be a crepant birational map of cluster type. Let $B_1, \dots, B_\ell$ be the components of B. Let $p_1 \colon (T, B_T) \to (\mathbb{P}^2, \Sigma^2)$ be a toric crepant birational morphism satisfying the following conditions:

- the center on T of each B_i is a divisor,
- every C_i can be obtained from T by a sequence of blow ups at points of B_T that are not zero strata.

Let $t_1, \dots, t_s$ be the centers of the curves C_i's in T. Let $p \colon X_0 \to T$ be a projective birational morphism obtained by consecutively blowing up the points t_i's such that each C_i is a divisor on X_0. We may choose p minimal with this property. For each $i \in \{1, \dots, s\}$ the divisor $p^{-1}(t_i)$ is a chain $E_i := E_{i,1} \cup \dots \cup E_{i,\ell_i}$ of rational curves. Note that only E_{i,ℓ_i} intersects B_0. We have a crepant birational morphism $q \colon (X_0, B_0) \to (X, B)$. Since ϕ is of cluster type, the morphism q only contracts curves in $B_0 \bigcup_{i=1}^{s} \left(\bigcup_{k=i}^{\ell_i} E_{i,j} \right)$. As we chose

p to be minimal, the curves E_{i,ℓ_i} is not contracted. Since B is ample the image of each $E_{i,j}$ on X is either a point or a curve intersecting B. Hence, all the curves $E_{i,1},\dots,E_{i,\ell_i-1}$ must be contracted. Otherwise, their images would not intersect B in X, leading to a contradiction of the Fano condition. As the chains $E_1,\dots,E_s$ are disjoint, we conclude that the curves $\phi(E_{1,j_1}),\dots,\phi(E_{s,\ell_s})$ can only intersect on X along B. By construction, we conclude that the curves $\phi(E_{1,j_1}),\dots,\phi(E_{s,\ell_s})$ coincide with the curves $C_1,\dots,C_j$, up to reordering. This finishes the proof. $\square$

Proof of Theorem 1.12. If (X,B) is not a cluster type pair, then $\mathcal{C}(X)=\infty$ and we are done. Hence, from now on, we assume that (X,B) is of cluster type and we aim to show that either $\mathcal{C}(X,B)=1$ or $\mathcal{C}(X,B)=2$.

Let X be a Fano surface and (X,B) be a cluster type log Calabi–Yau pair. Let $\phi\colon(\mathbb{P}^2_\phi,\Sigma^2)\dashrightarrow(X,B)$ be a crepant birational map of cluster type. Let $C_1,\dots,C_k$ be the curves of X that are not contained in B and are exceptional over $\mathbb{P}^2_\phi$. If $k=0$, then $\phi_1(\mathbb{G}_m^2)$ divisorially covers $X\backslash B$ and $\mathcal{C}(X,B)=1$. From now on, we assume that $k\geqslant 1$. For each crepant birational map $\varphi\colon(\mathbb{P}^2_\varphi,\Sigma^2)\dashrightarrow(X,B)$ of cluster type, we denote by $M(\varphi)$ the number of curves in $\{C_1,\dots,C_k\}$ that are not exceptional over $\mathbb{P}^2_\varphi$. If there exists such a φ with $M(\varphi)=k$, then we are done. Indeed, in this case, the almost tori $\phi(\mathbb{G}_m^2)$ and $\varphi(\mathbb{G}_m^2)$ divisorially cover $X\backslash B$ and hence $\mathcal{C}(X,B)=2$.

Let $\psi\colon(\mathbb{P}^2_\psi,\Sigma^2)\dashrightarrow(X,B)$ be the crepant birational map of cluster type that maximizes $M(\psi)$. Assume that $M(\psi)=s<k$. We may assume that the curves $C_1,\dots,C_s$ are non-exceptional over $\mathbb{P}^2_\psi$ and the curves $C_{s+1},\dots,C_k$ are exceptional over $\mathbb{P}^2_\psi$. Let $p_1\colon(T,B_T)\to(\mathbb{P}^2_\psi,\Sigma^2)$ be a toric projective birational morphism satisfying the following conditions:

(i) every component of B is a divisor on T,
(ii) every ψ^{-1}-exceptional curve can be obtained from T by blowing up a sequence of points of B_T that are not zero strata, and
(iii) If $t\in T$ is the center of a ψ^{-1}-exceptional curve, then t is contained on a prime component B_i of B_T that is the section of a toric fibration $f_t\colon T\to\mathbb{P}^1$.

By assumption, the curve C_{s+1} is exceptional over $\mathbb{P}^2_\psi$ and so it is exceptional over T. Thus C_{s+1} is a ψ^{-1}-exceptional curve. Note that C_{s+1} is not contained in B so condition (ii) applies. Henceforth, the curve C_{s+1} can be obtained from T by blowing up consecutively a point t_{s+1} of B_T that is not a zero strata. Furthermore, by condition (iii), we know that there is a prime component B_{s+1} of B_T containing t_{s+1} and a toric fibration $f_{s+1}\colon T\to\mathbb{P}^1$ for which B_{s+1} is a section. Let $p\in\mathbb{G}_m\subset\mathbb{P}^1$ be the image of t_{s+1} in $\mathbb{P}^1$ via f_{s+1}. Let F_{s+1} be the fiber of f_t over t_{s+1}. There are two cases: either F_{s+1} agrees with one of the curves C_i with $i\in\{1,\dots,s\}$ or it does not. In the previous sentence, for simplicity, we are identifying the C_i's with their strict transforms in T. In what follows, we arise to a contradiction in either case.

First, assume that $F_{s+1}=C_1$. Let $p\colon X_0\to T$ be a projective birational map obtained by a sequence of blow ups of points of B_T that are not strata such that $q\colon X_0\to X$ is a morphism. We may assume that p is minimal with the previous property. Let $x_1,\dots,x_r$ be the points blown up by p. We may assume $x_1=t_{s+1}$. For each i the curve $p^{-1}(x_i)=E_{i,1}\cup\dots E_{i,\ell_i}$ is a chain of rational curves. By the minimality of p, only E_{i,ℓ_i} intersects B_0. By the cluster type assumption of ψ, we know that $\mathrm{Ex}(p)$ is contained in $\cup_{i=1}^r p^{-1}(x_i)\cup B_0$. As B is ample, all the curves $E_{1,1},\dots,E_{1,s_1-1}$ must be contracted by p. We conclude that $p(E_{1,s_1})$ and $p(F_{s+1})$ intersect along $X\backslash B$. However, by construction, we have $p(E_{1,s_1})=C_{s+1}$ and $p(F_{s+1})=C_1$. We conclude that C_1 and C_{s+1} intersect along $X\backslash B$. This contradicts Lemma 5.2.

Now, we assume that F_{s+1} is not among the curves $C_1,\dots,C_k$. We can perform a weighted blow up at t_{s+1} that only extracts C_{s+1}. Let $X_1\to X_0$ be such a blow up. Let (X_1,B_1) be the crepant pull-back of (X_0,B_0). Note that we still have a fibration $X_1\to\mathbb{P}^1$ and the strict transform of F_{s+1} is degenerate over

$\mathbb{P}^1$. Thus, there is a projective birational morphism $X_1 \to X_2$ that only contracts F_{s+1}. Let (X_2, B_2) be the log Calabi–Yau pair obtained by pushing-forward B_1 to X_2. By construction, the log Calabi–Yau pair (X_2, B_2) is toric and the curves $C_1, \dots, C_{s+1}$ are divisors on X_2 not contained in B_2. As (X_2, B_2) is toric, we have a commutative diagram of crepant birational maps:

$$\begin{tikzcd}
(X_2, B_2) \arrow[r, dashed, "\pi_1"] \arrow[d, dashed, "\pi_2"'] & (X, B) \\
(\mathbb{P}^2_\xi, \Sigma^2) \arrow[ur, dashed, "\xi"'] &
\end{tikzcd}$$

where π_1 is a toric birational map and ξ is of cluster type. As π_1 is a toric birational map, it is an isomorphism on $\mathbb{G}_m^2$. In particular, the center of the curves $C_1, \dots, C_{s+1}$ are divisors on $\mathbb{P}^2_\xi$. We conclude that $M(\xi) = s + 1 > M(\psi)$. This contradicts the minimality of ψ.

We conclude that $M(\phi) = k$ and so $\mathcal{C}(X, B) = 2$. This finishes the proof. □

Proof of Theorem 1.13. Begin by fixing some cluster type crepant birational map

$$\phi\colon (\mathbb{P}^2, \Sigma^2) \dashrightarrow (X, B).$$

By applying Lemma 2.28 we obtain a commutative diagram of projective birational maps

$$\begin{tikzcd}
(T, B_T) \arrow[d, "p_1"] & (X_0, B_0) \arrow[l, "q"'] \arrow[d, "p_2"] \\
(\mathbb{P}^2, \Sigma^2) & (X, B) \arrow[l, dashed, "\phi^{-1}"']
\end{tikzcd}$$

where the following conditions are satisfied:

- p_1 is a toric projective birational morphism,
- p_2 is a projective birational morphism that only extracts log canonical centers of (X, B),
- q is a projective birational morphism that only extracts canonical centers of (T, B_T), and
- T is smooth and X_0 is $\mathbb{Q}$-factorial.

The nodes of B account for at most $|B|$ singular points of X. There is at most one singular point contained in the image on X of any q-exceptional curve on X_0; there are $\rho(X_0/T) = c(X_0, B_0) = c(X, B)$ such curves. Putting these together, we obtain

$$\begin{aligned} |X_{sing}| &\leqslant |B| + c(X, B) \\ &= \rho(X) + 2. \end{aligned}$$

This completes the proof. □

6. Examples and questions

In this section, we collect some examples and questions. We start with some examples of log Calabi–Yau pairs that exhibit some pathological behavior with respect to the torus exceptional degree.

Example 6.1. In this example, we show a log Calabi–Yau surface (X, B) of index one and birational complexity zero that is not of cluster type. Let $(\mathbb{P}^1 \times \mathbb{P}^1, Q + F)$ where Q is a general curve of multi-degree $(2, 1)$ and F is a fiber of the projection onto the first component. Then, the pair $(\mathbb{P}^1 \times \mathbb{P}^1, Q + F)$ is a log Calabi–Yau pair. Let F' be a fiber that is tangent to Q at a single point x. Let p be the image on $\mathbb{P}^1$ of x. We construct a new log Calabi–Yau pair following the next steps:

(1) we blow up X at x,
(2) we blow up the intersection of the two components of the fiber over p,
(3) we recursively blow up n times the intersection of the strict transform of Q and the fiber over p, and
(4) we blow down all the (-2)-curves contained in the fiber over p.

By doing so, we obtain a new log Calabi–Yau pair (X_n, B_n) of index one with a Mori fiber space to $\mathbb{P}^1$. Since $(\mathbb{P}^1 \times \mathbb{P}^1, Q + F) \simeq_{\mathrm{bir}} (\mathbb{P}^2, \Sigma^2)$, we have that (X_n, B_n) has birational complexity zero. Furthermore, the surface X_n has a D_n singularity. By Proposition 5.1, we conclude that (X_n, B_n) is not of cluster type.

Example 6.2. In this example, we show that there is a sequence of log Calabi–Yau surfaces (X_n, B_n) of index one and birational complexity zero for which the sequence $\mathcal{T}_t(X_n, B_n)$ diverges to infinity.

Let (X, B) be a log Calabi–Yau pair of index one, birational complexity zero, that is not of cluster type. For instance, we can consider the log Calabi–Yau surface constructed in Example 6.1. Let p be a point on B that is not a zero-dimensional strata. Let (X_n, B_n) be the log Calabi–Yau pair obtained by recursively blowing up p, and then blowing up n times the intersection of the strict transform of B with the previously introduced (-1)-curve. We let $E_1 \cup \dots E_{n+1}$ be the chain of rational curves that is the exceptional divisor of $X_n \to X$. Assume by the sake of contradiction that $\mathcal{T}_t(X_n, B_n) \leqslant k$ for some constant k. For each crepant birational map $\phi\colon (\mathbb{P}^2, \Sigma^2) \dashrightarrow (X_n, B_n)$, we have $\mathcal{T}(\phi) \geqslant 1$. Thus, we conclude that for each n the log Calabi–Yau pair (X_n, B_n) is divisorially covered by $\{\phi_{n,i}\}_{i \in I_n}$ with $|I_n| \leqslant k$. For each n and i, we let

$$|\phi_{n,i}| = |\{j \in \{1, \dots, n+1\} \mid \phi_{n,i}(\mathbb{G}_m^2 \backslash \mathrm{Ex}(\phi_{n,i}) \cap E_j \neq \varnothing\}|.$$

Observe that

$$\sum_{i \in I_n} |\phi_{n,i}| \geqslant n + 1.$$

Since $|I_n| \leqslant k$, for each n, there is one of the $\phi_{n,i}$'s for which $|\phi_{n,i}| \geqslant \lfloor \frac{n+1}{k} \rfloor$. Let's assume that it is $\phi_{n,1}$. Set $V_n := \phi_{n,1}(\mathbb{G}_m^2 \backslash \mathrm{Ex}(\phi_{n,1}))$. Let $X_n \to X_n'$ be the projective birational contraction over X that contracts all the curves E_j that intersect V_n trivially. Let B_n' be the push-forward of B_n on X_n'. Thus, we get a commutative diagram

$$\begin{tikzcd}
 & (X_n, B_n) \arrow[d, "\pi_n"] \\
 & (X_n', B_n') \arrow[d, "\pi_n'"] \\
(\mathbb{P}^2, \Sigma^2) \arrow[uur, dashed, "\phi_{n,1}"] \arrow[ur, dashed, "\phi_{n,1}"'] \arrow[r, dashed] & (X, B).
\end{tikzcd}$$

Let $C_{n,1}, \dots, C_{n,j_n}$ be closure on X_n' of the prime components of

$$(X_n' \backslash B_n') \backslash V_n.$$

Since $\rho(X) = 2$, we get $|j_n| \geqslant \lfloor \frac{n+1}{k} \rfloor - 1$. By Lemma 2.24, since $\mathcal{T}(\phi_{n,1}) \leqslant k$, we conclude that

$$\dim H_1(V_n, \mathbb{Q}) \leqslant f(k)$$

for certain constant $f(k)$ that only depends on k. Since all the exceptional divisors of π_n' are mapped to a smooth point in X, we get

$$\dim H_1(V_n \backslash \mathrm{Ex}(\pi_n'), \mathbb{Q}) - \dim H_1(V_n, \mathbb{Q}) \leqslant 1.$$

In particular, we have

$$\dim H_1(V_n \backslash \mathrm{Ex}(\pi_n'), \mathbb{Q}) \leqslant f(k) + 1.$$

Let U_n be the image on X of $V_n \backslash \mathrm{Ex}(\pi'_n)$. Then, as observed above we have $\dim H_1(U_n, \mathbb{Q}) \leqslant f(k) + 1$. Note that

$$U_n = X \backslash \left(B \cup_{j=1}^{j_n} \pi'_{n*} C_{n,j} \right).$$

The sequence of divisors $\{\pi'_{n*} C_{n,j}\}$ is infinite, as j_n diverges to infinite. Thus, we get a contradiction of Lemma 2.23 as $\dim H_1(U_n, \mathbb{Q})$ is bounded above.

In Theorem 1.8, we show that there exists a bounded family of affine surfaces $\mathcal{A} \to T$ such that for every log Calabi–Yau surface (X, B) of index one and birational complexity zero there is an embedding $\mathcal{A}_t \hookrightarrow X \backslash B$ for a suitable $t \in T$. The previous example shows that, in general, there is no upper bound k such that every such $X \backslash B$ can be divisorially covered by at most k open affines of the form $\mathcal{A}_t$. One may wonder if the log pair (X, B) can be divisorially covered by an arbitrary number of open affines of the form $\mathcal{A}_t$. This motivates the following definition.

Definition 6.3. Let (X, B) be a log Calabi–Yau pair of index one and birational complexity zero. We define the *covering torus exceptional degree* of (X, B), denoted by $\mathcal{T}_c(X, B)$, to be

$$\min \left\{ \max_{i \in I} \mathcal{T}(\phi_i) \mid \{\phi_i\}_{i \in I} \text{ is a set of crepant birational maps divisorially covering } (X, B) \right\}.$$

From the definitions, it is clear that a log Calabi–Yau pair (X, B) satisfies:

$$\mathcal{T}_t(X, B) \geqslant \mathcal{T}_c(X, B) \geqslant \mathcal{T}(X, B).$$

While we expect $\mathcal{T}(X, B)$ to be bounded above in terms of the dimension (see Conjecture 1.6), we know that $\mathcal{T}_t(X, B)$ can be arbitrarily large even for surfaces (see Example 6.2). This leads to the following question.

Question 6.4. Let (X, B) be a log Calabi–Yau pair of dimension n and birational complexity zero. Is $\mathcal{T}_c(X, B)$ bounded above in terms of the dimension?

Now, we turn to discuss some examples of bounded, affinely bounded, and birationally bounded families.

Example 6.5. In this example, we show a family of varieties that is affinely bounded but is not bounded. Let $n \geqslant 2$. The set $\mathcal{T}_n$ of n-dimensional projective toric varieties is affinely bounded. Indeed, for every n-dimensional projective toric variety X we have an open embedding $\mathbb{G}_m^n \hookrightarrow X$. On the other hand, the family $\mathcal{T}_n$ is not bounded as we can find an infinite sequence of smooth projective toric varieties X_r with $\rho(X_r) = r$.

Example 6.6. In this example, we show a set of varieties that is birationally bounded but is not affinely bounded. Let $\mathcal{R}_2$ be the set of all projective rational surfaces. The set $\mathcal{R}_2$ is birationally bounded. We show that is not affinely bounded. Consider $\pi_1 \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ the projection to the first component. Let $\{p_i\}_{i \in \mathbb{N}}$ be an infinite sequence of points in $\mathbb{P}^1$. Let X_k be the projective surface obtained as follows. We blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at $q_i \in \pi_1^{-1}(p_i)$ for each $i \in \{1, \dots, k\}$ and then over each p_i we blow up the intersection of the (-1)-curve and the strict transform of the fiber. Finally, we obtain X_k by contracting all the (-2)-curves contained in the fibers of the fibration to $\mathbb{P}^1$. By doing so, we obtain a projective surface X_k with $2k$ singularities of type A_1. Furthermore, X_k admits a Mori fiber space onto $\mathbb{P}^1$ with precisely $2k$ fibers of multiplicity 2. By [8, Lemma 3.13], we conclude that there is a surjective homomorphism

$$\pi_1^{\mathrm{reg}}(X_k) \to \pi_1 \left(\mathbb{P}^1, \frac{1}{2}(p_1 + \cdots + p_k) \right) \simeq \langle \gamma_1, \dots, \gamma_k \mid \gamma_1 \cdots \gamma_k, \gamma_1^2, \dots, \gamma_k^2 \rangle.$$

Then the rank of $\pi_1^{\mathrm{reg}}(X_k)$ diverges as k goes to infinite. Thus, the sequence of varieties $\{X_k\}_{k \in \mathbb{N}}$ is birationally bounded but it is not affinely bounded.

Finally, we explain an example of a log Calabi–Yau surface of large cluster type.

Example 6.7. In this example, we show that for every integer $n \geqslant 0$ there exists a log Calabi–Yau pair (X, B) of dimension two with $\mathcal{C}(X, B) = n + 1$.

Consider the log Calabi–Yau surface (X_n, B_n) obtained as follows. Start from $(\mathbb{P}^1 \times \mathbb{P}^1, B)$ where B is the toric boundary. Pick a point $p \in B$ which is not a zero stratum. Let B_1 be the unique prime component of B containing p. Blow up p and then recursively blow up $(n-1)$ times the intersection of the strict transform of B with the previously extracted (-1)-curve. By doing so, we get a log Calabi–Yau pair (X_n, B_n) that admits a fibration $f\colon X_n \to \mathbb{P}^1$. Furthermore, the fibers $f^{-1}(0)$ and $f^{-1}(\infty)$ are contained in the support of B_n. All fibers of f are isomorphic to $\mathbb{P}^1$, except a single fiber, which is a chain of rational curves all of which are (-2)-curves except for (-1)-curves at either end. We may assume that this unique reducible fiber is $f^{-1}(1)$ and write $f^{-1}(1) = E_0 \cup E_1 \cup \cdots \cup E_n$. Let $\phi\colon (\mathbb{P}^2, \Sigma^2) \dashrightarrow (X_n, B_n)$ be a crepant birational map of cluster type. We apply Lemma 2.28 and obtain a commutative diagram of projective birational maps

$$\begin{tikzcd} (T_n, B_{T_n}) \arrow[d, "p_1"] & (X_{n,0}, B_{n,0}) \arrow[l, "q"'] \arrow[d, "p_2"] \\ (\mathbb{P}^2, \Sigma^2) & (X_n, B_n) \arrow[l, dashed, "\phi^{-1}"'] \end{tikzcd}$$

where the following conditions are satisfied:

- p_1 is a toric projective birational morphism,
- p_2 is a projective birational morphism that only extracts log canonical centers of (X, B), and
- q is a projective birational morphism that only extracts canonical centers of (T_n, B_{T_n}).

By abuse of notation, we write E_i for the strict transform of E_i on $X_{n,0}$. Note that p_2 only extracts log canonical centers of (X_n, B_n). Let $f_0\colon X_{n,0} \to \mathbb{P}^1$ be the induced fibration on $X_{n,0}$. The fibers of f_0 are either rational curves intersecting $B_{n,0}$ twice, are contained in $B_{n,0}$, or are equal to $E_0 \cup \cdots \cup E_n$. Furthermore, every curve in $X_{n,0}$ which is horizontal over $\mathbb{P}^1$ is either contained in $B_{n,0}$ or intersects $B_{n,0}$ at least twice. Indeed, every curve on $X_{n,0}$ which is horizontal over $\mathbb{P}^1$ must intersect $f_{n,0}^{-1}(0)$ and $f_{n,0}^{-1}(\infty)$ and both fibers are contained in $B_{n,0}$. By Lemma 2.12, we conclude that all the exceptional divisors of q are vertical over $\mathbb{P}^1$. Since every fiber $f_0^{-1}(t)$, with $t \in \mathbb{G}_m$ and $t \neq 1$, intersects $B_{n,0}$ twice, we conclude that every exceptional divisor of q must be contained in $E_0 \cup \cdots \cup E_n$. Since (T, B_T) is toric, by a complexity computation, we conclude that q must contract n of the curves $E_0, \ldots, E_n$. Thus, for every crepant birational map $\phi\colon (\mathbb{P}^2, \Sigma^2) \dashrightarrow (X_n, B_n)$ of cluster type, the image $\phi(\mathbb{G}_m^2)$ can intersect at most one of the curves $E_0 \cup \cdots \cup E_n$. Hence, we conclude that at least $n+1$ almost tori are required to cover $X_n \backslash B_n$ divisorially. This means that $\mathcal{C}(X_n, B_n) \geqslant n+1$. On the other hand, by contracting precisely n of the curves $E_0, \ldots, E_n$, we observe that $n+1$ almost tori cover $X_n \backslash B_n$ divisorially. Thus, we get $\mathcal{C}(X_n, B_n) = n + 1$.

The previous example shows that the cluster type of a log Calabi–Yau surface can be an arbitrary positive integer. In contrast, Theorem 1.12 implies that $\mathcal{C}(X, B) \leqslant 2$ for a cluster type log Calabi–Yau pair (X, B) whenever X is a Fano surface. It is natural to expect similar behavior in higher dimensions. This motivates the following question.

Question 6.8. Let X be a n-dimensional Fano variety. Let (X, B) be a log Calabi–Yau pair of cluster type. Can we bound $\mathcal{C}(X, B)$ above in terms of n?

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UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, USA
Email address: jlenwright1@math.ucla.edu

Department of Mathematics, Northwestern University, Evanston, Il 60208, USA
Email address: fernando.figueroa@northwestern.edu

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, USA
Email address: jmoraga@math.ucla.edu