GEOMETRIC ANALYSIS OF A MIXED ELLIPTIC-PARABOLIC CONFORMALLY INVARIANT BOUNDARY VALUE PROBLEM

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Abstract. We treat the blow-up behavior of α-Dirac-harmonic maps. Such maps arise from a version of the nonlinear supersymmetric σ-model of QFT, and the difficulties arise from the conformal invariance of the problem and the coupling of a second order elliptic (or parabolic) system for a map with a first order elliptic system for a spinor along this map. The solutions are called Dirac-harmonic maps. Building upon the pioneering work of Sacks-Uhlenbeck, we develop a general spectrum of methods (Pohozaev identity, three circle method, blow-up analysis, energy identities, energy decay estimates etc.) for conformally invariant variational problems at this particularly challenging example. We study the refined blow-up behaviour and asymptotic analysis for a sequence of α-Dirac harmonic maps from a compact Riemann surface with smooth boundary into a compact Riemannian manifold with uniformly bounded energy. We prove generalized energy identities and novel energy decay estimates. This enables us to present a very detailed blow-up analysis.

1. Introduction

The nonlinear supersymmetric σ-model of quantum field theory contains the basic mathematical ingredients of other such problems, like supersymmetric Yang-Mills, that are more difficult to analyze. Still, the mathematical analysis of this σ-model is challenging. This is caused by the two main ingredients, nonlinearity and supersymmetry. The nonlinearity is difficult because of the conformal invariance of the problem. Supersymmetry is difficult because it needs non-commuting variables. In our program, we want to disentangle these two difficulties and concentrate on the first, the nonlinearity. Therefore, we modify the model in such a way that we can replace the non-commuting by ordinary, commuting variables. This leads us to a system of coupled nonlinear partial differentiel equations for the stationary points of the action functional. We have called the solutions of theese equations Dirac-harmonic maps. Dealing with the conformal invariance then brings us into the realm of geometric analysis. In fact, perhaps the most important core of geometric analysis consists of problems that are invariant under a non-compact group of local symmetries, like, as in our case, the conformal group. Such problems, in particular, do not satisfy a Palais-Smale condition. Therefore, the standard methods of the calculus of variations and nonlinear PDEs usually don't apply. The Palais-Smale condition for such problems, however, only barely fails, and more precisely, we are typically looking at limit cases of this condition. And this then requires a sophisticated asymptotic analysis that controls the formation of singularities and characterizes or even classifies the possible singularities. While the techniques employed need to carefully exploit the particular structure at

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hand, the techniques developed for one particular problem sometimes generalize to others, because the fundamental phenomena involving loss of compactness, blow-up of solutions, energy identities, are similar.

A key insight was that the loss of compactness is caused by blow-up solutions that in many important cases could be completely classified. This was first achieved for harmonic maps from Riemann surfaces by Sacks-Uhlenbeck [30] and for constant mean curvature surfaces by Wente [34]. In both cases, the variational problem is conformally invariant, that is, depends only on the conformal structure of the underlying Riemann surface, but not on the choice of a metric in a conformal class. The blow-up solutions then live on the 2-sphere and represent minimal or constant mean curvature spheres. Also, by Noether's theorem, the conformal invariance leads to a conserved quantity, which here means that a certain quadratic differential is holomorphic. This holomorphic quadratic differential then on one hand helps to solve the regularity issue of solutions and on the other hand implies that the blow-up solutions have a special structure, because on the 2-sphere, every holomorphic quadratic differential vanishes. In the sequel, we shall focus on the situation for harmonic maps, because that is the situation that we shall generalize in this paper.

Also, in this situation, two general schemes have been developed that provided systematic approaches. One scheme is a perturbation method, approaching the functional in question as a limit of functionals that do satisfy the Palais-Smale condition. The other is a parabolic method, turning the elliptic Euler-Lagrange equations into parabolic equations and to analyze the formation of singularities along the flow or as time goes to infinity. The perturbation method was introduced in [30]. As already said, the perturbed energy functional, obtained by raising the integrand to some power $\alpha > 1$, satisfies the Palais-Smale condition, so that variational methods can be applied to obtain so called α-harmonic maps as critical points of this perturbed functional. By studying the limit of a sequence of α-harmonic maps as α goes to 1, one can prove some existence result for harmonic maps. Alternatively, one can apply the heat flow method. A global weak solution of the harmonic map heat flow from closed surfaces or surfaces with boundary which is smooth except for finitely many singularities was established in [33, 3]. In general, such a flow may develop singularities already at some finite time, see for example Chang-Ding-Ye [4]. Both the α-perturbation method and the heat flow method allow for a precise analysis of the singularities and their formation. In the α-perturbation case, the approximating functionals are explicit, and in the heat flow method situation, the formation of singularities is controlled by an explicit parabolic equation.

Once these schemes were established, a general asymptotic analysis for the formation of singularities for harmonic maps from Riemann surfaces could be developed, and we now understand this issue in considerable detail. In particular, precise energy identities are known, meaning that the possible loss of energy in the limit of a sequence is precisely captured by the energies carried by the singularities, and there is no connecting neck between the limit and the singularities that might carry a nontrivial amount of energy. For the energy identity of a minimax sequence of certain approximating harmonic maps, see [17, 22]. For a sequence of minimizing α-harmonic maps, it was proved by Chen-Tian [6] that the necks converges to some limit geodesics of finite length which implies that the energy identity holds true. The energy identity for a sequence of α-harmonic maps into certain special target manifolds was investigated in [25, 24]. For a general blow-up sequence of α-harmonic maps, however, the energy identity is in general not necessary true, see the counter example constructed by Li-Wang [27]. Nevertheless, in [26], they investigated the general blow-up situation for

α-harmonic maps and established a generalized energy identity and studied the asymptotic behaviors of the necks, the limit of which are geodesics in the target but in general not necessary of finite lengths.

Such advanced theories cannot succeed by analytical tools alone. They require a very precise and careful utilization of the geometric and algebraic structure of the particular setting. On the other hand, however, the approach should not be too specialized but rather also bring out some more general features, and in particular, develop methods of a more general scope. For this reason, we have systematically turned to another area, quantum field theory, where variational problems with noncompact symmetries naturally arise that have a rich and subtle geometric and algebraic structure and that have lead to spectacular mathematical insights. As an example of such a problem, motivated by the nonlinear supersymmetric sigma model [12], Dirac-harmonic maps are defined as solutions of a system consisting of a harmonic map type equation coupled with a Dirac type equation. They were introduced and first studied in [7, 8]. They combine and generalize harmonic maps and harmonic spinors, both of which have been widely and extensively studied, but only separately and not coupled as here.

In this paper, on one hand, we shall build upon the pioneering work of Sacks-Uhlenbeck [30] and on the other hand use structures and identities from quantum field theory to deal with a refined blow-up analysis corresponding to the existence problem of Dirac-harmonic maps considered in the previous paper [20]. We shall use a heat flow method that links a parabolic system for a harmonic map type functional with a first order elliptic system that solves a nonlinear Dirac equation along the flow. As already mentioned, in general, such a flow does not exist globally, but may blow up at finite time, even in the simpler case of harmonic maps [4]. To overcome this difficulty, we study the corresponding elliptic-parabolic system for α-Dirac harmonic maps, that is, the heat flow for a Sacks-Uhlenbeck type functional with a non-linear Dirac equation along the flow. Of course, the hardest difficulties that we need to overcome then emerge for $\alpha \to 1$.

The resulting variational problems therefore combines also essentially all the difficulties that one has to face in geometric analysis when dealing with problems with a noncompact local symmetry group. And since these difficulties are intertwined, they cannot be solved with the individual tools that have been developed for each of them in isolation, but we also need to extend the existing methods substantially. In some sense, this is the culmination of the work of ourselves and many colleagues over several years. Let us first informally describe the difficulties and point out our strategies to overcome them. We consider the functional

$$L(\phi,\psi) = \frac{1}{2}\int_M \Big(|d\phi|^2 + \langle \psi, \not{D}\psi\rangle_{\Sigma M\otimes\phi^\star TN}\Big)dM, \tag{1.1}$$

where ϕ is a map from a Riemann surface M with smooth boundary ∂M into a Riemannian manifold N, and ψ is a spinor along that map. $\not{D}$ is a nonlinear Dirac operator that depends on ϕ, and through ϕ also on the geometry of N. Therefore, the two fields ϕ and ψ are coupled. While the precise notation will be explained shortly, hopefully this already suffices to understand the essential structure of the problem. So, here are the difficulties and how we address them:

(1) The functional is conformally invariant, and since the conformal group is noncompact, the Palais-Smale condition is violated. To handle this, there exist of course well established

schemes. In particular, following Sacks-Uhlenbeck [30], we can modify (1.1) as

$$L_\alpha(\phi,\psi) = \frac{1}{2}\int_M \left\{(1+|d\phi|^2)^\alpha + \langle \psi, \not{D}\psi\rangle\right\} dM, \tag{1.2}$$

for $\alpha > 1$. This functional then satisfies Palais-Smale, and we can study the limit $\alpha \searrow 1$.

(2) However, neither (1.1) nor (1.2) is bounded from below, because of the spinor term, as the spectrum of the Dirac operator is neither bounded from below nor from above. Therefore, classical variational methods (like direct minimization procedure or minimax scheme) don't apply to get the existence of critical points of (1.2), let alone (1.1).

(3) As already indicated, we shall use heat flow methods as an alternative to variational methods. But here the difficulty is that while the Euler-Lagrange equation for ϕ is second order and can therefore be easily parabolized, the equation for ψ is first order (it looks simple, $\not{D}\psi = 0$, but this is nonlinear, because the operator depends on ϕ, as already pointed out).

(4) Our solution consists in studying a new elliptic-parabolic system [10, 18, 19, 20] under appropriate boundary conditions that carries the nonlinear Dirac equation along as an elliptic side constraint for a harmonic type flow. To control that elliptic constraint, sharp estimates for solutions of Dirac equations are needed [10, 20].

(5) In order to then control the limit $\alpha \searrow 1$, we need to carry out a blow-up analysis, because singularities may form and so-called bubbles may split off in the limit, see [20]. While for the original Sacks-Uhlenbeck scheme for harmonic maps, this procedure is well established, here we encounter new difficulties due to the limiting behavior of the spinor part.

(6) Also, the bubbles that split off need not be connected to the remaining solution. In order to show that in the limit, energy is only carried by that remaining solution and the bubbles, we need to control the necks that connect the forming bubbles with the rest for $\alpha > 1$, but disappear in the limit $\alpha \searrow 1$. In fact, the precise analysis reveals that they converge to geodesic lines. When such a geodesic is of finite length, it appears as the image of a cylinder of infinite length and therefore carries no energy. To show this, we shall apply three circle type method in the spirit of Simon [31] and Qing-Tian [29], see also Colding-De Lellis-Minicozzi [11]. In our situation, we first need to derive a new Pohozaev identity. Incidentally, the validity of a Pohozaev identity, rather than the stronger condition of conformal invariance, is what is equivalent to the removability of singularities in geometric variational problems [21], and therefore, our emphasis of such an identity fits well into the general scheme.

(7) Nevertheless, we cannot directly use a classical technique like the three-circle method (or alternative methods developed for harmonic maps), because in that scheme, we would have to deal with two additional terms that are not uniformly bounded in L^2 (in fact they are even not uniformly bounded in L^p for any $1 < p < 2$), causing new obstruction. In technical terms, to overcome this difficulty, we need to control the decay of the tangential energies of both the map part and the spinor part on the neck domain as $\alpha \searrow 1$. This is one of the main technical achievements of the present paper. Also, we obtain a new estimate for the energy decay of the spinor.

(8) After we having obtained the decay estimates of the tangential energies by applying the new three-circle method, it is still unclear and difficult to derive the estimates of the radial energies, especially for the spinor part. Here, we will apply a Hardy type inequality method, which seems to be very useful for the study of first order type equations. Combining this with a iteration argument, then we shall obtain an optimal decay estimate for the weighted energy

of the spinor on the neck domain. With this in hand, by using a Pohozaev type identity for the map part, we will get the decay estimate for this map part.

For more detailed explanations about the new difficulties and challenges mentioned above, see discussions after the main results Theorem 2.2 and Theorem 2.3 in Section 2.

The rest of this paper is organized as follows. In Section 2, we introduce our functional in precise technical terms and state and explain our main results. In Section 3, we shall then prove some basic lemmas for α-Dirac-harmonic maps which will be used in this paper, such as the small energy regularity, the energy gap theorem and a new Pohozaev type identity. Then we establish the three circle theorem for a system of integro-differential equations which can be applied to the α-Dirac-harmonic map system. In Section 4, we prove our first main Theorem 2.2 about generalized energy identities. The limit behavior of α-Dirac-harmonic necks is studied in Section 5 and Theorem 2.3 is proved in this section.

2. Summary and main results

Let M be a compact Riemann surface with smooth boundary ∂M, equipped with a Riemannian metric g and with a fixed spin structure, ΣM be the spinor bundle over M and $\langle\cdot,\cdot\rangle_{\Sigma M}$ be the natural Hermitian inner product on ΣM. Choosing a local orthonormal basis $e_\gamma, \gamma = 1,2$ on M, the usual Dirac operator is defined as $\partial\!\!\!/ := e_\gamma \cdot \nabla_{e_\gamma}$, where ∇ is the spin connection on ΣM and $\cdot$ is the Clifford multiplication. This multiplication is skew-adjoint:

$$\langle X\cdot\psi, \varphi\rangle_{\Sigma M} = -\langle \psi, X\cdot\varphi\rangle_{\Sigma M}$$

for any $X \in \Gamma(TM)$, $\psi,\ \varphi \in \Gamma(\Sigma M)$.

Let ϕ be a smooth map from M to another compact Riemannian manifold (N,h) with dimension $n \geq 2$. Denote ϕ^*TN the pull-back bundle of TN by ϕ and then we get the twisted bundle $\Sigma M\otimes\phi^*TN$. There is a natural metric $\langle\cdot,\cdot\rangle_{\Sigma M\otimes\phi^*TN}$ on $\Sigma M\otimes\phi^*TN$ induced from the metrics on ΣM and ϕ^*TN. Likewise, there is a natural connection $\widetilde{\nabla}$ on $\Sigma M\otimes\phi^*TN$ induced from the connections on ΣM and ϕ^*TN. Let ψ be a section of the bundle $\Sigma M\otimes\phi^*TN$. In local coordinates, it can be written as

$$\psi = \psi^i \otimes \partial_{y^i}(\phi),$$

where each ψ^i is a usual spinor on M and ∂_{y^i} is the nature local basis on N. Then $\widetilde{\nabla}$ becomes

$$\widetilde{\nabla}\psi = \nabla\psi^i\otimes\partial_{y^i}(\phi) + (\Gamma^i_{jk}\nabla\phi^j)\psi^k\otimes\partial_{y^i}(\phi), \tag{2.1}$$

where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection of N. The Dirac operator along the map ϕ is defined by

$$D\!\!\!\!/\,\psi := e_\alpha\cdot\widetilde{\nabla}_{e_\alpha}\psi.$$

We consider the following functional

$$L(\phi,\psi) \;=\; \frac{1}{2}\int_M \Big(|d\phi|^2 + \langle\psi, D\!\!\!\!/\,\psi\rangle_{\Sigma M\otimes\phi^\star TN}\Big)dM,$$

where $dM = dvol_g$.

The functional $L(\phi,\psi)$ is conformally invariant, see [8]. That is, for any conformal diffeomorphism $f : M \to M$, setting

$$\widetilde{\phi} = \phi\circ f \qquad and \qquad \widetilde{\psi} = \lambda^{-1/2}\psi\circ f.$$

where λ is the conformal factor of the conformal map f, *i.e.* $f^*g = \lambda^2 g$, there holds $L(\widetilde{\phi}, \widetilde{\psi}) = L(\phi, \psi)$. Critical points (ϕ, ψ) are called Dirac-harmonic maps from M to N.

The Euler-Lagrange equations of the functional L are

$$\left(\Delta_g \phi^i + \Gamma^i_{jk} g^{\alpha\beta} \phi^j_\alpha \phi^k_\beta\right) \frac{\partial}{\partial y^i}(\phi(x)) = R(\phi, \psi), \tag{2.2}$$

$$\not{D}\psi = 0, \tag{2.3}$$

where $\Delta_g := \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\beta}(\sqrt{g} g^{\beta\gamma} \frac{\partial}{\partial x^\gamma})$ is the Laplacian operator with respect to the Riemannian metric g, $R(\phi, \psi)$ is defined by

$$R(\phi, \psi) = \frac{1}{2} R^m_{lij}(\phi(x)) \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle \frac{\partial}{\partial y^m}(\phi(x)).$$

Here R^m_{lij} is the Riemann curvature tensor of the target manifold (N, h).

By Nash's embedding theorem, we embed N isometrically into some $\mathbb{R}^K$. Then, critical points (ϕ, ψ) of the functional L satisfy the following extrinsic Euler-Lagrange equations

$$\Delta_g \phi = -A(\phi)(d\phi, d\phi) + Re(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi); \psi)), \tag{2.4}$$

$$\not{\partial}_g \psi = \mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi), \tag{2.5}$$

where $\not{\partial}$ is the usual Dirac operator on (M, g), A is the second fundamental form of N in $\mathbb{R}^K$, and

$$\mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi) := (\nabla \phi^i \cdot \psi^j) \otimes A(\partial_{y^i}, \partial_{y^j}),$$

$$Re(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi); \psi)) := P(A(\partial_{y^l}, \partial_{y^j}); \partial_{y^i}) Re(\langle \psi^i, d\phi^l \cdot \psi^j \rangle).$$

Here $P(\xi; \cdot)$ denotes the shape operator, defined by $\langle P(\xi; X), Y \rangle = \langle A(X, Y), \xi \rangle$ for $X, Y \in \Gamma(TN)$, and $Re(z)$ denotes the real part of $z \in \mathbb{C}$.

For $p > 1$, we denote

$$W^{1,p}(M, N) := \left\{\phi \in W^{1,p}(M, \mathbb{R}^K) \mid \phi(x) \in N, \ a.e. \ x \in M\right\},$$

$$W^{1,p}(\Sigma M \otimes \phi^* TN) := \left\{\psi \in W^{1,p}(\Sigma M \otimes \phi^* \mathbb{R}^K) \mid \psi(x) \text{ is along the map } \phi, \ a.e. \ x \in M\right\}.$$

Here $\psi \in \Gamma(\Sigma M \otimes \phi^* TN)$ is along the map ϕ and should be understood as a K-tuple of spinors $(\psi^1, ..., \psi^K)$ satisfying

$$\sum_{i=1}^K \nu_i \psi^i(x) = 0$$

for any normal vector $\nu = (\nu_1, ..., \nu_K) \in \mathbb{R}^K$ at $\phi(x)$. For more details and background on Dirac-harmonic maps, we refer to [7, 8, 36, 9, 32].

The blow-up theory for sequences of Dirac-harmonic maps including the energy identity and the no neck property, i.e., bubble tree convergence, was systematically explored in [7, 35, 28]. However, the existence problem for Dirac-harmonic maps turns out to be difficult. Since the functional $L(\phi, \psi)$ does not have a lower bound due to the fact that the second term in L does not have a fixed sign, classical variational methods and heat flow methods developed for harmonic maps cannot be applied directly. In order to study the existence problem, a modified heat flow approach for Dirac-harmonic maps from spin Riemann surface with smooth boundary was introduced in [10], and the short time existence of a solution was shown. This is a heat flow that couples a parabolic second order equation for the map with a first order elliptic equation for the spinor. That is, the solution of the first order

Dirac type equation is carried along a harmonic map type flow. We call this flow the Dirac-harmonic map flow, which is:

$$\begin{cases}\partial_t\phi = \Delta_g\phi + A(d\phi, d\phi) + Re\left(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi);\psi)\right), & in\ \ M\times[0,T),\\ \partial\!\!\!/_g\psi = \mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi), & in\ \ M\times[0,T),\end{cases}$$

with the following boundary-initial data:

$$\begin{cases}\phi(x,t) = \varphi(x), & on \quad \partial M\times[0,T];\\ \phi(x,0) = \phi_0(x), & in \quad M;\\ \mathbf{B}\psi(x,t) = \mathbf{B}\psi_0(x), & on \quad \partial M\times[0,T];\\ \phi_0(x) = \varphi(x), & on \quad \partial M.\end{cases} \tag{2.6}$$

where $\mathbf{B} = \mathbf{B}^\pm$ is the chiral boundary operator defined as follows:

$$\begin{aligned}\mathbf{B}^\pm : L^2(\partial M, \Sigma M\otimes\phi^*TN|_{\partial M}) &\to L^2(\partial M, \Sigma M\otimes\phi^*TN|_{\partial M})\\ \psi &\mapsto \frac{1}{2}\left(Id \pm \overrightarrow{n}\cdot G\right)\cdot\psi,\end{aligned}$$

where $\overrightarrow{n}$ is the outward unit normal vector field on ∂M, $G = ie_1\cdot e_2$ is the chiral operator defined using a local orthonormal frame $\{e_\gamma\}_{\gamma=1}^2$ on M and satisfying:

$$G^2 = Id, \quad G^* = G, \quad \nabla G = 0, \quad G\cdot X = -X\cdot G,$$

for any $X\in\Gamma(TM)$. One can also take $\mathbf{B}$ to be the MIT bag boundary operator $\mathbf{B}^\pm_{MIT}$ as considered in [10]. See e.g. [16, 2] for more detailed discussions on these boundary operators.

Furthermore, the existence of a global weak solution to the Dirac-harmonic map flow in dimension two under certain boundary-initial constraint was obtained in [18]. By studying the limit behaviour as time approaches infinity, we proved the existence results of Dirac-harmonic maps with Dirichlet-chiral boundary condition in a given homotopy class under certain boundary-initial constraint in [18]. A technical difficulty stems from the fact that along the Dirac-harmonic map flow considered in [18], we can not control the energy of the spinor field $||\psi||_{L^4(M)}$ as time approaches the first singular time $T_1 > 0$, even for the L^1-norm. This is the main difficulty and why we need to impose the boundary-initial constraint in [18] in order to obtain a global weak solution to the Dirac-harmonic map flow and show some existence results by letting t go to infinity.

In order to get rid of the above mentioned boundary-initial constraint and study the general existence problem, in [20], we have therefore introduced the following functional

$$L_\alpha(\phi,\psi) = \frac{1}{2}\int_M\left\{(1+|d\phi|^2)^\alpha + \langle\psi, D\!\!\!\!/\,\psi\rangle\right\}dM, \tag{2.7}$$

where $\alpha > 1$ is a constant. Critical points $(\phi_\alpha,\psi_\alpha)$ of L_α are called *α-Dirac-harmonic maps* from M to N. They satisfy the Euler-Lagrange equations [20]

$$\Delta_g\phi = -(\alpha-1)\frac{\nabla_g|\nabla_g\phi|^2\nabla_g\phi}{1+|\nabla_g\phi|^2} - A(d\phi,d\phi) + \frac{Re\left(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi);\psi)\right)}{\alpha(1+|\nabla_g\phi|^2)^{\alpha-1}}, \tag{2.8}$$

$$\partial\!\!\!/_g\psi = \mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi). \tag{2.9}$$

When the spinor field vanishes, the functional L_α reduces to Sacks-Uhlenbeck's perturbed one [30].

Since the functional L_α is not bounded from below, the classical Ljusternik-Schnirelman theory may not be applied here to obtain critical points and hence the existence problem of α-Dirac-harmonic map is subtle. To overcome this difficulty, we introduced a new parabolic-elliptic system in [20] under appropriate boundary value conditions, i.e., the so called *α-Dirac-harmonic map flow* with Dirichlet-chiral boundary data:

$$\partial_t\phi = \Delta_g\phi + (\alpha-1)\frac{\nabla_g|\nabla_g\phi|^2\nabla_g\phi}{1+|\nabla_g\phi|^2} + A(d\phi, d\phi) - \frac{Re\left(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi);\psi)\right)}{\alpha(1+|\nabla_g\phi|^2)^{\alpha-1}}, \tag{2.10}$$

$$\not{\partial}_g\psi = \mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi), \tag{2.11}$$

with the boundary-initial data (2.6). The advantage of the α-Dirac-harmonic map flow over the Dirac-harmonic map flow of [10, 18] is explained in [20]. There, we showed that for any $\alpha > 1$ sufficiently close to 1, there is a global regular solution to this new flow and hence as time goes to infinity, one gets a limit α-Dirac-harmonic map $(\phi_\alpha, \psi_\alpha) : M \to N$ with Dirichlet-chiral boundary data $(\phi_\alpha, \mathbf{B}\psi_\alpha)|_M = (\varphi, \mathbf{B}\psi_0)$ and with uniformly bounded energy

$$\int_M (1+|d\phi_\alpha|^2)^\alpha dM + \int_M |\psi_\alpha|^4 dM \le C(M, N, \|\nabla\phi_0\|_{L^{2\alpha}(M)}, \|\mathbf{B}\psi_0\|_{L^2(\partial M)}).$$

The second important step in [20] then is to perform the blow-up analysis for above sequence of α-Dirac-harmonic maps with Dirichlet-chiral boundary data and with uniformly bounded energy. We can show that the weak limit is just the desired Dirac-harmonic map (note that the nontrivial boundary condition ensures that the map part and the spinor part are both nontrivial). Precisely, we proved the following

Theorem 2.1 (**Theorem 1.2 in** [20])**.** *Let (M, g) be a smooth compact spin Riemann surface with smooth boundary ∂M, (N, h) an n-dimensional smooth compact Riemannian manifold. Let $\lambda \in (0, 1)$ and*

$$(\phi_{\alpha_k}, \psi_{\alpha_k}) \in C^{2+\lambda}(M, N) \times C^{1+\lambda}(M, \Sigma M \otimes \phi^*_{\alpha_k}TN)$$

with $\alpha_k \searrow 1$ be a sequence of α_k-Dirac-harmonic maps with uniformly bounded energy, i.e.

$$\int_M (1+|d\phi_\alpha|^2)^\alpha dM + \int_M |\psi_\alpha|^4 dM \le \Lambda$$

and with Dirichlet-chiral boundary data $(\phi_{\alpha_k}, \mathbf{B}\psi_{\alpha_k})|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$ where $\varphi \in C^{2+\lambda}(\partial M, N)$, $\psi_0 \in C^{1+\lambda}(\partial M, \Sigma M \otimes \varphi^*TN)$. Then there exists a finite blow-up set $\mathbf{S} = \{p_1, ..., p_I\}$ such that, by passing to a subsequence, there exists a smooth Dirac-harmonic map

$$(\phi, \psi) \in C^{2+\lambda}(M, N) \times C^{1+\lambda}(M, \Sigma M \otimes \phi^*TN)$$

with Dirichlet-chiral boundary data $(\phi, \mathbf{B}\psi)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$ such that $(\phi_{\alpha_k}, \psi_{\alpha_k}) \to (\phi, \psi)$ weakly in $W^{1,2}(M) \times L^4(M)$ and strongly in $C^2_{loc}(M \setminus \mathbf{S}) \times C^1_{loc}(M \setminus \mathbf{S})$.

In this paper, we shall carry out the most difficult step and study the finer blow-up behavior and asymptotic analysis for a sequence of α-Dirac-harmonic maps near the interior blow-up points. More precisely, we shall investigate the limit behaviour of the Dirac-harmonic necks appearing during the blow-up process.

Since in general, multiple bubbles can split off at a blow-up point and the functional L_α is not conformally invariant, we shall consider the following more general α-energy functionals[1]

$$L_{\alpha,\sigma_\alpha}(\phi,\psi)=\frac{1}{2}\int_{D_1(0)}\left\{(\sigma_\alpha+|\nabla_{g_\alpha}\phi|^2)^\alpha+\sigma_\alpha^{1-\alpha}\langle\psi,\not{D}\psi\rangle\right\}dvol_{g_\alpha},\ \alpha>1,$$

where $g_\alpha=e^{\varphi_\alpha}((dx^1)^2+(dx^2)^2)$, $\varphi_\alpha\in C^\infty(D_1)$, $\varphi_\alpha(0)=0$, φ_α converges smoothly to $\varphi_0\in C^\infty(D_1)$, $0<\sigma_\alpha\leq 1$ satisfying

$$0<\beta_0\leq\liminf_{\alpha\searrow 1}\sigma_\alpha^{\alpha-1}\leq 1.$$

Critical points of L_{α,σ_α} are also called *α-Dirac-harmonic maps*[2], which satisfy the following Euler-Lagrange equations

$$\Delta_{g_\alpha}\phi = -(\alpha-1)\frac{\nabla_{g_\alpha}|\nabla_{g_\alpha}\phi|^2\nabla_{g_\alpha}\phi}{\sigma_\alpha+|\nabla_{g_\alpha}\phi|^2}-A(d\phi,d\phi)+\frac{Re\left(P(\mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi);\psi)\right)}{\alpha(\sigma_\alpha+|\nabla_{g_\alpha}\phi|^2)^{\alpha-1}},\tag{2.12}$$

$$\not{\partial}_{g_\alpha}\psi = \mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi).\tag{2.13}$$

As (2.13) is conformally invariant, it is easy to see that the equations (2.12) and (2.13) are equivalent to [3]

$$\Delta\phi+(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi|^2\nabla\phi}{\sigma_\alpha+|\nabla_{g_\alpha}\phi|^2}+A(\phi)(d\phi,d\phi)-\frac{Re\left(P(\mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi);\psi)\right)}{\alpha(\sigma_\alpha+|\nabla_{g_\alpha}\phi|^2)^{\alpha-1}}=0,\tag{2.14}$$

$$\begin{aligned}div\left\{(\sigma_\alpha+|\nabla_{g_\alpha}\phi|^2)^{\alpha-1}\nabla\phi\right\}+(\sigma_\alpha+|\nabla_{g_\alpha}\phi|^2)^{\alpha-1}A(\phi)(d\phi,d\phi)&\\ -\frac{1}{\alpha}Re\left(P(\mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi);\psi)\right)=0&\end{aligned}\tag{2.15}$$

and

$$\not{\partial}\psi=\mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi),\tag{2.16}$$

where $\Delta=\frac{\partial^2}{(\partial x^1)^2}+\frac{\partial^2}{(\partial x^2)^2}$ and ∇ are the derivatives and $\not{\partial}$ is the Dirac operator corresponding to the standard Euclidean metric. e_γ is a local orthonormal basis with respect to the standard Euclidean metric which is different from (2.12) and (2.13), but for simplicity of notation, we still use it. Although the original considered object is a sequence of α-Dirac-harmonic maps on a domain surface with a general Riemannian metric (see (2.12) and (2.13)), for the purposes of this paper, where in the end the difficulties boil down to elliptic estimates, we always work with the equations (2.14),

[1]One can check that a rescaled α-Dirac-harmonic map, e.g. $\left(\phi_\alpha(\lambda_\alpha x),\lambda_\alpha^{\alpha-1}\sqrt{\lambda_\alpha}\psi_\alpha(r_\alpha x)\right)$ is locally a critical point of this functional, we refer to Section 4 (the proof of Theorem 1.4) in [20] for details. One can also see the beginning of section 2 in [26] for the analogous case of α-harmonic maps.

[2]In our pervious paper [20], critical points of L_{α,σ_α} are called *general α-Dirac-harmonic maps* and we established appropriate analytical background for this kind of maps, see Lemma 4.1 and Lemma 4.2 in [20]. In this paper, for simplicity of notations, these maps are collectively called α-Dirac-harmonic maps.

[3]More precisely, $(\phi\circ Id, e^{\frac{\varphi_\alpha}{2}}\psi\circ Id)$ satisfies (2.14)-(2.16), where $Id:(D_1(0),(dx^1)^2+(dx^2)^2)\to(D_1(0),g_\alpha)$ is a conformal map defined by $Id(x)=x$. For simplicity of notation, we identify $(\phi\circ Id, e^{\frac{\varphi_\alpha}{2}}\psi\circ Id)$ with (ϕ,ψ) in the sequel. Although the energy $\int_M(1+|d\phi|^2)^\alpha dM$ is not conformally invariant for $\alpha>1$, the limit $\lim_{\alpha\searrow 1}\int_M\left((1+|d\phi|^2)^\alpha-1\right)dM$ is conformally invariant. Combining this with the fact that $\int_M|\psi|^4dM$ is conformally invariant, this identification is indeed legitimate for the questions we are concerned with in this paper.

(2.15) and (2.16) whose domain metric is the standard Euclidean metric, where g_α can be viewed as a smooth function on $(D_1(0), (dx^1)^2 + (dx^2)^2)$.

Before presenting our results, we shall first give a general description of the blow-up procedure and the bubbling phenomena for α-Dirac-harmonic maps. For the case of α-harmonic maps, see [30, 26] .

Denote

$$E_{\alpha,\sigma_\alpha}(\phi) = \int_M (\sigma_\alpha + |d\phi|^2)^\alpha dM,\ E(\phi) = \int_M |d\phi|^2 dM,\ E(\psi) = \int_M |\psi|^4 dM,$$
$$E_\alpha(\phi) = \int_M (1 + |d\phi|^2)^\alpha dM,\ E(\phi,\psi) = \int_M (|d\phi|^2 + |\psi|^4) dM.$$

Consider a sequence of α-Dirac-harmonic maps $\{(\phi_\alpha,\psi_\alpha)\} : M \to N$ with Dirichlet-chiral boundary data $(\phi_\alpha, \mathbf{B}\psi_\alpha)|_M = (\varphi, \mathbf{B}\psi_0)$ and with uniformly bounded energy [4]

$$E_{\alpha,\sigma_\alpha}(\phi_\alpha) + E(\psi_\alpha) \le \Lambda.$$

From Theorem 1.2 and Theorem 1.4 in [20], we know that, by passing to a subsequence, $(\phi_\alpha,\psi_\alpha)$ converges smoothly to some limit Dirac-harmonic map $(\phi,\psi) : M \to N$ with Dirichlet-chiral boundary data $(\phi, \mathbf{B}\psi)|_M = (\varphi, \mathbf{B}\psi_0)$ away from at most finitely many blow-up points $\mathbf{S} = \{x_i\}_{i=1}^I$ as $\alpha \searrow 1$. Moreover, we show that at each blow-up point, that is, when the energy of the map concentrates, after suitable rescaling, a bubble, namely, a nontrivial Dirac-harmonic sphere splits off.

By the classical blow-up theory for α-harmonic maps [30, 26], it is well known that at most finitely many bubbles can occur at a given blow-up point and the necks connecting the weak limit map and the bubbles or one bubble to the next are all converging to geodesics, i.e., a bubble tree construction holds. Here, we shall see that this phenomenon also holds for α-Dirac-harmonic maps. See Section 4 for the constructions of the first and the second bubble (if multiple bubbles occur at the blow-up point). The rest bubbles (if any) can be constructed by a standard induction argument similar to the works by Li-Wang [26] and Chen-Li [5].

More precisely, for a fixed blow-up point $x_i,\ 1 \le i \le I$, we may assume there are k_i bubbles occurring at this point, i.e. there are a sequence of points $\{x_\alpha^{ij}\}$, $j = 1, ..., k_i$, and a sequence of positive numbers $\{\lambda_\alpha^{ij}\}$ with $x_\alpha^{ij} \to x_i$, $\lambda_\alpha^{ij} \to 0$ as $\alpha \searrow 1$ and one of the following two alternatives holds true: if $1 \le j_1, j_2 \le k_i$ and $j_1 \ne j_2$,

(A1) for any fixed $R > 0$, $D^M_{R\lambda_\alpha^{ij_1}}(x_\alpha^{ij_1}) \cap D^M_{R\lambda_\alpha^{ij_2}}(x_\alpha^{ij_2}) = \emptyset$, whenever α is sufficiently close to 1.

(A2) $\frac{\lambda_\alpha^{ij_1}}{\lambda_\alpha^{ij_2}} + \frac{\lambda_\alpha^{ij_2}}{\lambda_\alpha^{ij_1}} = \infty$, as $\alpha \searrow 1$.

Moreover, the following two rescaled fields[5]

$$\sigma_\alpha^{ij} := \phi_\alpha(x_\alpha^{ij} + \lambda_\alpha^{ij} x), \quad \xi_\alpha^{ij} := (\lambda_\alpha^{ij})^{\alpha-1} \sqrt{\lambda_\alpha^{ij}}\, \psi_\alpha(x_\alpha^{ij} + \lambda_\alpha^{ij} x)$$

converge in $C^k_{loc}(\mathbb{R}^2 \setminus \{p_1^{ij}, ..., p_{s_j}^{ij}\})$ to a nontrivial Dirac-harmonic map (σ^{ij}, ξ^{ij}) defined on $\mathbb{R}^2$, which can be conformally extended to a nontrivial Dirac-harmonic map from $\mathbb{S}^2$. See the beginning of

[4]Here and in the sequel, for simplicity of notations, when talking about a sequence of $(\phi_\alpha,\psi_\alpha)$ for $\alpha \searrow 1$, we mean the sequence of $(\phi_{\alpha_k},\psi_{\alpha_k})$ for a given sequence of $\alpha_k \searrow 1$.

[5]Let us explain the transformation of the spinor. It can be seen as a linear transformation (i.e. $\lambda_\alpha^{\alpha-1}\psi_\alpha$) composed with a conformal transformation (i.e. $\sqrt{\lambda_\alpha}\psi_\alpha(x_\alpha + \lambda_\alpha x)$). Since α-Dirac-harmonic maps are not conformally invariant, in order to get unified bubble equations, we need an additional factor $\lambda_\alpha^{\alpha-1}$ in the scale.

Section 4. Define two types of quantities:

$$\mu_{ij} = \liminf_{\alpha\searrow 1}(\lambda_\alpha^{ij})^{2-2\alpha}, \quad \nu_{ij} = \liminf_{\alpha\searrow 1}(\lambda_\alpha^{ij})^{-\sqrt{\alpha-1}}. \tag{2.17}$$

It is easy to see that $\nu_{ij} \in [1,\infty]$. Also, we can see that there exists a positive constant $\mu_{max} \geq 1$ such that $\mu_{ij} \in [1,\mu_{max}]$. In fact, for simplicity of notations, we may assume there is only one blow-up point denoted by $x \in M$, and there are k_1 bubbles occurring at this point, i.e., there are a sequence of points $\{x_\alpha^j\}$ and a sequence of positive numbers $\{\lambda_\alpha^j\}$, $1 \leq j \leq k_1$ satisfying (A1) or (A2). Without loss of generality, we assume λ_α^1 is the smallest one, i.e. $\frac{\lambda_\alpha^1}{\lambda_\alpha^j} \leq C < \infty$ for all $j = 2,...,k_1$, as $\alpha \searrow 1$. We just need to show that

$$\mu_1 = \liminf_{\alpha\searrow 1}(\lambda_\alpha^1)^{2-2\alpha} \leq \mu_{max}.$$

By applying the standard blow-up argument for α-Dirac-harmonic maps (see the beginning of Sec. 4 or Sec. 4 in [20]), we have:

$$(\sigma_\alpha^1,\xi_\alpha^1) := \left(\phi_\alpha(x_\alpha^1+\lambda_\alpha^1 x),(\lambda_\alpha^1)^{\alpha-1}\sqrt{\lambda_\alpha^1}\psi_\alpha(x_\alpha^1+\lambda_\alpha^1 x)\right) \to (\sigma^1,\xi^1) \text{ in } C^k_{loc}(\mathbb{R}^2),$$

where (σ^1,ξ^1) can be conformally extended to a nontrivial Dirac-harmonic sphere. Therefore, we have

$$\begin{aligned}\Lambda \geq \lim_{R\to\infty}\lim_{\alpha\searrow 1}\int_{D_{\lambda_\alpha^1 R}(x_\alpha^1)}|\nabla_{g_\alpha}\phi_\alpha|^{2\alpha}dvol_{g_\alpha} &= \lim_{R\to\infty}\lim_{\alpha\searrow 1}(\lambda_\alpha^1)^{2-2\alpha}\int_{D_R(0)}|\nabla_{g_\alpha}\sigma_\alpha^1|^{2\alpha}dvol_{g_\alpha(x_\alpha^1+\lambda_\alpha^1 x)}\\ &= \lim_{R\to\infty}\mu_1\int_{D_R(0)}|\nabla\sigma^1|^2dx = \mu_1 E(\sigma^1).\end{aligned}$$

By the energy gap property for Dirac-harmonic spheres (see Lemma 3.3), we have

$$\mu_1 \leq \frac{\Lambda}{E(\sigma^1)} \leq \frac{\Lambda}{\epsilon_1}. \tag{2.18}$$

Now, we state our first main result about the generalized energy identities for α-Dirac-harmonic maps. In this paper, we shall only consider the case of interior blow-up. The case of boundary blow-up will be considered in the future.

Theorem 2.2. *Under the assumptions of Theorem 2.1, assume* $\mathbf{S} \cap \partial M = \emptyset$, *i.e. all the blow-up points are the interior points, then there are finitely many bubbles: a finite set of Dirac-harmonic spheres* $(\sigma_i^l,\xi_i^l) : S^2 \to N$, $l = 1,...l_i$, *where* $l_i \geq 1$, $i = 1,...,I$, *such that, the following generalized energy identities hold:*

$$\lim_{k\to\infty} E_{\alpha_k}(\phi_{\alpha_k}) = E(\phi) + |M| + \sum_{i=1}^{I}\sum_{l=1}^{l_i}\mu_{il}^2 E(\sigma_i^l),$$

$$\lim_{k\to\infty} E(\psi_{\alpha_k}) = E(\psi) + \sum_{i=1}^{I}\sum_{l=1}^{l_i}\mu_{il}^2 E(\xi_i^l),$$

where the quantities μ_{il} *are defined as in* (2.17).

Furthermore, we shall show that the map parts of the α-Dirac-harmonic necks appearing during the interior blow-up process converge to geodesics in the target manifold N and then derive the length formula of these neck geodesics. More precisely, we have

Theorem 2.3. *Under the same assumptions as in Theorem 2.2, assume* $\mathbf{S} = \{x_1\}$ *and there is only one bubble in* $D_r^M(x_1) \subset M$ *for some* $r > 0$*, for the sequence* $\{(\phi_{\alpha_k}, \psi_{\alpha_k})\}$*, denoted by* (σ^1, ξ^1)*, which is a Dirac-harmonic sphere. Let*

$$\nu^1 = \liminf_{\alpha \searrow 1} (\lambda_\alpha^1)^{-\sqrt{\alpha-1}}. \tag{2.19}$$

Then the map part of the Dirac-harmonic neck appearing during the blow-up process is converging to a geodesic in the target manifold N. Moreover, we have the following alternatives:

(1) *when* $\nu^1 = 1$*, the set* $\phi(D_r^M(x_1)) \cup \sigma^1(S^2)$ *is a connected set in N;*

(2) *when* $\nu^1 \in (1, \infty)$*, then the set* $\phi(D_r^M(x_1))$ *and* $\sigma^1(S^2)$ *are connected by a geodesic of length*
$$L = \sqrt{\frac{E(\sigma^1)}{\pi}} \log \nu^1;$$

(3) *when* $\nu^1 = \infty$*, the map part of the Dirac-harmonic neck contains at least an infinite length curve which is a geodesic in N;*

Remark 2.4. Although Theorem 2.3 is stated and proved for the case that only one single bubble occurs at a blow-up point, by following the arguments in Section 5, it is not hard to extend these results to the case of multiple bubbles occurring at the blow-up point. In fact, the multiple bubble case will be more complicated. For example, if we have three bubbles: (σ^1, ξ^1), (σ^2, ξ^2), (σ^3, ξ^3) with the blow-up positions and radii $(x_\alpha, \lambda_\alpha^i)$, $i = 1, 2, 3$, i.e.

$$\left(\phi_\alpha(x_\alpha + \lambda_\alpha^i x), (\lambda_\alpha^i)^{\alpha-1}\sqrt{\lambda_\alpha^i}\psi_\alpha(x_\alpha + \lambda_\alpha^i x)\right) \rightharpoonup (\sigma^i, \xi^i)$$

weakly in $W^{1,2}_{loc}(\mathbb{R}^2) \times L^4_{loc}(\mathbb{R}^2)$, satisfying $\frac{\lambda_\alpha^1}{\lambda_\alpha^2} \to 0$, $\frac{\lambda_\alpha^2}{\lambda_\alpha^3} \to 0$ and $\nu^1, \nu^2, \nu^3 < \infty$, then $\phi(D_\delta^M(x_\alpha))$ and $\sigma^3(S^2)$ are connected by a geodesic with length

$$L = \sqrt{\frac{E(\sigma^1) + E(\sigma^2) + E(\sigma^3)}{\pi}} \log \nu^3,$$

and $\sigma^3(S^2)$ and $\sigma^2(S^2)$ are connected by a geodesic of length

$$L = \sqrt{\frac{E(\sigma^1) + E(\sigma^2)}{\pi}} \log \frac{\nu^2}{\nu^3},$$

$\sigma^2(S^2)$ and $\sigma^1(S^2)$ are connected by a geodesic of length

$$L = \sqrt{\frac{E(\sigma^1)}{\pi}} \log \frac{\nu^1}{\nu^2}.$$

A key step in proving Theorem 2.3 is to establish certain decay estimates of the tangential energies of both the map part and the spinor part on the neck domain as $\alpha \searrow 1$. In the absence of the spinor part, i.e., when ϕ_α are α-harmonic maps, this is achieved by Li-Wang [26] where the authors used some idea in [13] to derive a differential inequality on the neck. However, this kind of technique requires the special structure of harmonic map type equations and hence cannot be applied to the α-Dirac-harmonic map systems, where α-harmonic map type equations are coupled with some first order Dirac type equations.

Recall that another powerful tool in deriving exponential decay on the long cylinders is the three circle type method. For maps $\{\phi_n\}$ with L^2-uniformly bounded tension fields $\tau(\phi_n)$, or equivalently, solutions to the harmonic map system up to some error terms $\tau(\phi_n)$ that are L^2-uniformly bounded, the corresponding three circle method was derived in [29], which used a special case of the three circle theorem due to [31] to show that the tangential energy of the sequence of solutions on the long cylinder decays exponentially. In fact, the condition that the error terms $\tau(\phi_n)$ are L^2-uniformly bounded ensures the following decay estimate:

$$\||x-x_n|\cdot\tau(\phi_n)\|_{L^2(D_{2t}(x_n)\setminus D_t(x_n))}\leq Ct,$$

which is crucial in [29] to get the exponential decay.

However, in view of the equation (2.12) for the map part ϕ_α in the α-Dirac-harmonic map system, the error term contains the following two terms: the second derivative term

$$(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2}, \tag{2.20}$$

and the curvature term

$$\frac{Re\left(P(\mathcal{A}(d\phi_\alpha(e_\gamma),e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right)}{\alpha(\sigma_\alpha+|\nabla_g\phi_\alpha|^2)^{\alpha-1}}, \tag{2.21}$$

both of which are not uniformly bounded in L^2 anymore and hence the results in [29] cannot be applied.

Therefore, we need to develop new methods. Instead of the schemes just discussed, in the present paper, to overcome the new difficulties occurring, we shall first establish a type of three circle theorem for the integro-differential equations corresponding to α-Dirac-harmonic map type systems, see Theorem 3.8. Then, we shall derive the decay of the tangential energies of both the map part and the spinor part on the neck domain, see Lemma 5.2. To achieve this, we shall handle the two error terms separately. For the second derivative term (2.20), although it is not uniformly bounded in L^p for any $p>1$ and the exponential decay will not hold [6], however, our new observation is that one can still get some decay at the speed of $\alpha-1$. For the curvature term (2.21), we need to decompose it into several forms which can be used in the three circle theorem. We will see that the proof is still subtle. In particular, our result applies to the case of α-harmonic maps, i.e. when $\psi_\alpha\equiv 0$ and hence leads to decay estimates for α-harmonic map type systems, which is still new in the existing literature. Moreover, in the case of α-harmonic maps, by applying the results in Lemma 5.2, we can show that the tangential energies decay at a speed of $\alpha-1$, which is an improvement of the speed of $\sqrt{\alpha-1}$ given by Proposition 4.2 in [26].

Differently from the case of Dirac-harmonic maps [28, 19] where the decay of the tangential energy of the spinor is sufficient, in order to study the limit behaviour of α-Dirac-harmonic necks, here we need to get the decay of some weighted energy of the spinor as $\alpha\searrow 1$. Compared to the cases of Dirac-harmonic maps [28, 19], here, however, the exponential decay of the spinor's energy will not hold, since the map part ϕ_α satisfies α-harmonic map system rather than the harmonic map system.

[6] one can see that it is controlled by $(\alpha-1)|\nabla^2\phi_\alpha|$ and by Lemma 3.1, we have

$$\||x-x_\alpha|^{2(1-\frac{1}{p})}|\nabla^2\phi_\alpha|\|_{L^p(D_{2t}(x_\alpha)\setminus D_t(x_\alpha))}\leq C\|\nabla\phi_\alpha\|_{L^2(D_{4t}(x_\alpha)\setminus D_{\frac{t}{2}}(x_\alpha))}.$$

To overcome this new difficulty, we shall need new observations. We shall prove the energy $\|\psi_\alpha\|_{L^4}$ of the spinor decays at a speed of $(\alpha-1)^\gamma$ for any $\gamma \in (0, \frac{1}{3})$, more precisely, the following holds:

$$\lim_{\alpha \searrow 1} \frac{1}{(\alpha-1)^{4\gamma}} \int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)} |\psi_\alpha|^4 dx = 0. \tag{2.22}$$

See Lemma 5.4 and Lemma 5.5. Such kind of decay estimate plays a key role in the present paper, as it is not only important in the proof of Theorem 2.3 (see Proposition 5.9) which will be discussed later. In technical terms, the above decay estimate (2.22) is achieved by applying a Hardy-type inequality to derive a differential inequality on the neck to get the decay of some weighted energy of a spinor.

Notations: Δ_g, ∇_g and $\not\partial_g$ are operators corresponding to the Riemannian metric g. Δ, ∇ and $\not\partial$ operators corresponding to the standard Euclidean metric. $dM = dvol_g$ and $dx = dx^1dx^2$ are the volume forms with respect to g and standard Euclidean metric respectively. Throughout the paper, the uppercase letter C denotes a positive constant which may vary from line to line.

3. A Pohozaev type identity and a three circle theorem

In this section, we shall first prove several basic lemmas for α-Dirac-harmonic maps, for instance, the small energy regularity, the energy gap theorem and a new Pohozaev type identity. Secondly, we shall establish a new type of three circle theorem for α-Dirac-harmonic map system.

Lemma 3.1. *For any $1 < p < \infty$, there exist two positive constants $\epsilon_0 > 0$ and $\alpha_0 > 1$ depending only on g and N, such that if $(\phi_\alpha, \psi_\alpha) : (D_1(0), g_\alpha) \to N$ is a sequence of α-Dirac-harmonic map with $E(\phi_\alpha, \psi_\alpha) \le \Lambda$ and*

$$E(\phi_\alpha) \le \epsilon_0^2, \ 1 \le \alpha \le \alpha_0,$$

where $g_\alpha = e^{\varphi_\alpha}((dx^1)^2 + (dx^2)^2)$ and $\varphi_\alpha(0) = 0$, $\varphi_\alpha \to \varphi_0$ in $C^\infty(D_1(0))$ as $\alpha \to 1$, then there hold:

$$\|\nabla\phi_\alpha\|_{W^{1,p}(D_{1/2}(0))} \le C(p, g, \Lambda, N)\|\nabla\phi_\alpha\|_{L^2(D_1(0))}, \ \ \|\psi_\alpha\|_{W^{1,p}(D_{1/2}(0))} \le C(p, g, \Lambda, N)\|\psi_\alpha\|_{L^4(D_1(0))}, \tag{3.1}$$

$$\|\nabla\phi_\alpha\|_{C^1(D_{1/2}(0))} \le C(g, \Lambda, N)\|\nabla\phi_\alpha\|_{L^2(D_1(0))}, \ \ \|\psi_\alpha\|_{C^0(D_{1/2}(0))} \le C(g, \Lambda, N)\|\psi_\alpha\|_{L^4(D_1(0))}. \tag{3.2}$$

Proof. Since $(\phi_\alpha, \psi_\alpha)$ satisfy (2.14) and (2.16), according to Lemma 4.1 in [20], we have that (3.1) holds. By Sobolev embedding, for some $0 < \delta < 1$, there holds

$$\|\nabla\phi_\alpha\|_{C^\delta(D_{3/4}(0))} \le C(g, \Lambda, N)\|\nabla\phi_\alpha\|_{L^2(D_1(0))}, \ \|\psi_\alpha\|_{C^\delta(D_{3/4}(0))} \le C(g, \Lambda, N)\|\psi_\alpha\|_{L^4(D_1(0))}.$$

To estimate $\|\nabla\phi_\alpha\|_{C^1}$, by (2.14), noting that ϕ_α satisfies the following elliptic equation that

$$\begin{aligned}\Delta\phi_\alpha + (\alpha-1)\frac{g_\alpha^{\kappa\beta}\frac{\partial\phi_\alpha}{\partial x^\kappa}\frac{\partial\phi_\alpha}{\partial x^\gamma}}{\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2}\frac{\partial^2\phi_\alpha}{\partial x^\beta\partial x^\gamma} &= -A(\phi_\alpha)(d\phi_\alpha, d\phi_\alpha) + \frac{Re\left(P(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha); \psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_g\phi_\alpha|^2)^{\alpha-1}} \\ &\quad - (\alpha-1)\frac{\frac{\partial g_\alpha^{\kappa\beta}}{\partial x^\gamma}\frac{\partial\phi_\alpha}{\partial x^\kappa}\frac{\partial\phi_\alpha}{\partial x^\beta}\frac{\partial\phi_\alpha}{\partial x^\gamma}}{\sigma_\alpha + |\nabla_g\phi_\alpha|^2} \\ &:= f(\phi_\alpha, \psi_\alpha, \nabla\phi_\alpha),\end{aligned}$$

when $\alpha - 1$ is sufficiently small, using the standard schauder estimates of elliptic operator, we get

$$\|\nabla\phi_\alpha\|_{C^1(D_{1/2}(0))} \le C(\|f\|_{C^\delta(D_{3/4}(0))} + \|\nabla\phi_\alpha\|_{C^0(D_{3/4}(0))}) \le C(g, \Lambda, N)\|\nabla\phi_\alpha\|_{L^2(D_1(0))}.$$

□

As an application of Lemma 3.1, we have

Lemma 3.2. *Let $D_1(0) \subset \mathbb{R}^2$ be the unit disk, $g_\alpha = e^{\varphi_\alpha(x)}((dx^1)^2 + (dx^2)^2)$ and $g = e^{\varphi_0(x)}((dx^1)^2 + (dx^2)^2)$ be a family of metrics on $D_1(0)$, where $\varphi_\alpha \in C^\infty(D_1)$, $\varphi_\alpha(0) = 0$ and $\varphi_\alpha \to \varphi_0$ in $C^\infty(D_1)$ as $\alpha \searrow 1$. Let $(\phi_\alpha, \psi_\alpha) \in C^\infty(D_1(0), N)$ be a sequence of α-Dirac-harmonic maps with uniformly bounded energy $E_{\alpha,\sigma_\alpha}(\phi_\alpha) + E(\psi_\alpha) \leq \Lambda$ and with $0 < \beta_0 \leq \lim_{\alpha\searrow 1}(\sigma_\alpha)^{\alpha-1} \leq 1$, then there exists a positive β_1 independent of α, such that*

$$\beta_0 \leq \liminf_{\alpha\searrow 1} \|(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\|_{C^0(D)} \leq \limsup_{\alpha\searrow 1} \|(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\|_{C^0(D)} \leq \beta_1.$$

Proof. Without loss of generality, we assume 0 is the only energy concentration point for the sequence $\{(\phi_\alpha, \psi_\alpha)\}$. Then by the blow-up process described in Section 2, we can get finite bubbles at this point, i.e. there exist a positive sequence $\lambda^i_\alpha \to 0$ and a sequence of points $x^i_\alpha \to 0$, $i = 1, ..., I$, as $\alpha \searrow 1$, which satisfy (A1) or (A2). Also, without loss of generality, we assume λ^1_α is the smallest one, i.e. $\limsup_{\alpha\searrow 1} \frac{\lambda^1_\alpha}{\lambda^j_\alpha} \leq C$ for $j = 2, ..., I$. By a standard blow-up argument, we know $(\phi_\alpha(x_0 + \lambda^1_\alpha x), (\lambda^1_\alpha)^{\alpha-1}\sqrt{\lambda^1_\alpha}\psi_\alpha(x_0 + \lambda^1_\alpha x))$ has no energy concentration points for all points $x_0 \in D_{\frac{1}{2}}$. By Lemma 3.1, we have

$$|\nabla\phi_\alpha|(x) \leq C \; for\ any\ x \in D_1(0) \setminus D_{\frac{1}{2}(0)},$$

since 0 is the only energy concentration point and

$$|\nabla\phi_\alpha|(x) \leq \frac{C}{\lambda^1_\alpha} \; for\ any\ x \in D_{\frac{1}{2}(0)}.$$

Thus, we obtainn

$$\|(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\|_{C^0(D)} \leq C(1 + (\lambda^1_\alpha)^{2-2\alpha}) \leq C(1 + \mu_{max}).$$

Then the conclusion of the lemma follows immediately. □

The following energy gap result is a small improvement of the one given in Theorem 3.1. in [7].

Lemma 3.3 (Energy gap)**.** *There exists an $\epsilon_1 = \epsilon_1(N) > 0$ such that, if $(\phi, \psi) : S^2 \to N$ is a smooth Dirac-harmonic map satisfying*

$$\int_{S^2} |d\phi|^2 < \epsilon_1^2.$$

Then both ϕ and ψ are trivial.

Proof. **Step1.** Claim: $\|\psi\|_{L^{4/3}(S^2)} \leq C\|\partial\!\!\!/\psi\|_{L^{4/3}(S^2)}$, where ψ is a spinor on S^2 and $C > 0$ is a universal constant.

In fact, if not, then there exists a sequence of spinors $\{\psi_k\}$ on S^2 such that

$$\|\psi_k\|_{L^{4/3}(S^2)} > k\,\|\partial\!\!\!/\psi_k\|_{L^{4/3}(S^2)}.$$

Without loss of generality, we assume $\|\psi_k\|_{L^{4/3}(S^2)} = 1$, then we have

$$\|\partial\!\!\!/\psi_k\|_{L^{4/3}(S^2)} < \frac{1}{k}. \tag{3.3}$$

By standard elliptic estimates, we get

$$\|\psi_k\|_{W^{1,4/3}(S^2)} \leq C.$$

Thus, there exists a subsequence of $\{\psi_k\}$ (we still denote it by $\{\psi_k\}$) and $\eta \in W^{1,4/3}(S^2)$ satisfying

$$\psi_k \to \eta \quad weakly \quad in \quad W^{1,4/3}(S^2) \quad and \quad strongly \quad in \quad L^{4/3}(S^2). \tag{3.4}$$

Combining this with $\|\psi_k\|_{L^{4/3}(S^2)} = 1$ and the inequality (3.3), we get $\|\eta\|_{L^{4/3}(S^2)} = 1$ and

$$\|\partial\!\!\!/\eta\|_{L^{4/3}(S^2)} = 0. \tag{3.5}$$

So, $\eta \equiv 0$ since there is no nontrivial harmonic spinor on S^2. This is a contradiction.

Step2. By standard elliptic estimates, we have

$$\begin{aligned}\|\psi\|_{L^4(S^2)} &\leq C\|\psi\|_{W^{1,4/3}(S^2)}\\ &\leq C(\|\partial\!\!\!/\psi\|_{L^{4/3}(S^2)} + \|\psi\|_{L^{4/3}(S^2)})\\ &\leq C\|\partial\!\!\!/\psi\|_{L^{4/3}(S^2)} \leq C\||d\phi||\psi|\|_{L^{4/3}(S^2)} \leq C\|d\phi\|_{L^2(S^2)}\|\psi\|_{L^4(S^2)} \leq C\epsilon_1\|\psi\|_{L^4(S^2)}.\end{aligned}$$

Taking $\epsilon_1 > 0$ sufficiently small, then we have $\psi \equiv 0$. Therefore,

$$\begin{aligned}\|d\phi\|_{W^{1,4/3}(S^2)} &\leq C\|\Delta\phi\|_{L^{4/3}(S^2)}\\ &\leq C\||d\phi|^2\|_{L^{4/3}(S^2)}\\ &\leq C\|d\phi\|_{L^2(S^2)}\|d\phi\|_{L^4(S^2)} \leq C\|d\phi\|_{L^2(S^2)}\|d\phi\|_{W^{1,4/3}(S^2)} \leq C\epsilon_1\|d\phi\|_{W^{1,4/3}(S^2)}.\end{aligned}$$

Thus ϕ has to be a constant map. □

Now we state a new Pohozaev type identity for α-Dirac-harmonic maps.

Lemma 3.4. *Let (D, g_α) be a unit disk in $\mathbb{R}^2$ equipped with a metric $g_\alpha = e^{\varphi_\alpha}\left((dx^1)^2 + (dx^2)^2\right)$, where $\varphi_\alpha \in C^\infty(D)$. If $(\phi_\alpha, \psi_\alpha)$ is a critical point of $L_{\alpha,\sigma_\alpha}(\phi, \psi)$, then for any $0 < t < 1$, there holds*

$$\begin{aligned}&(1-\frac{1}{2\alpha})\int_{\partial D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\frac{\partial\phi_\alpha}{\partial r}|^2 - \frac{1}{2\alpha}\int_{\partial D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2\\ &= (1-\frac{1}{\alpha})\frac{1}{t}\int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx\\ &\quad + \frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha} + \frac{1}{2\alpha}\int_{\partial D_t}\langle\psi_\alpha, r^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle\\ &\quad + (1-\frac{1}{\alpha})\frac{1}{2t}\int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 r\frac{\partial\varphi_\alpha}{\partial r}dx\\ &\quad - \frac{\sigma_\alpha}{\alpha t}\int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}(1+\frac{r}{2}\frac{\partial\varphi_\alpha}{\partial r})e^{\varphi_\alpha}dx.\end{aligned} \tag{3.6}$$

Here, $dx = dx^1dx^2$.

Proof. Multiplying (2.15) by $r\frac{\partial\phi_\alpha}{\partial r}$, we have

$$0 = \int_{D_t} div\left\{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\nabla\phi_\alpha\right\} r\frac{\partial\phi_\alpha}{\partial r}dx - \int_{D_t}\left\langle\frac{1}{\alpha}Re\left(P(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right), r\frac{\partial\phi_\alpha}{\partial r}\right\rangle dx. \tag{3.7}$$

On one hand, integrating by parts, by using the fact that $r\frac{\partial}{\partial r} = \sum_{\beta=1}^{2} x^\beta \frac{\partial}{\partial x^\beta}$, we have

$$\begin{aligned}
&\int_{D_t} div\left\{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\nabla\phi_\alpha\right\} r\frac{\partial\phi_\alpha}{\partial r}dx \\
&= \int_{\partial D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1} r|\frac{\partial\phi_\alpha}{\partial r}|^2 - \int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\nabla\phi_\alpha\nabla(x^\beta\frac{\partial\phi_\alpha}{\partial x^\beta})dx \\
&= \int_{\partial D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1} r|\frac{\partial\phi_\alpha}{\partial r}|^2 - \int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx \\
&\quad - \int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\frac{1}{2}r\frac{\partial}{\partial r}|\nabla\phi_\alpha|^2 dx.
\end{aligned} \tag{3.8}$$

Noting that $|\nabla\phi_\alpha|^2 = e^{\varphi_\alpha}|\nabla_{g_\alpha}\phi_\alpha|^2$, we get

$$\begin{aligned}
&- \int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\frac{1}{2}r\frac{\partial}{\partial r}|\nabla\phi_\alpha|^2 dx \\
&= - \int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\frac{1}{2}r\frac{\partial}{\partial r}|\nabla_{g_\alpha}\phi_\alpha|^2 e^{\varphi_\alpha}dx - \int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx \\
&= -\frac{1}{2\alpha}\int_{D_t} r\frac{\partial}{\partial r}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha} e^{\varphi_\alpha}dx - \int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx.
\end{aligned} \tag{3.9}$$

Integrating by parts yields that

$$\begin{aligned}
&- \frac{1}{2\alpha}\int_{D_t} r\frac{\partial}{\partial r}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha} e^{\varphi_\alpha}dx \\
&= -\frac{1}{2\alpha}\int_{\partial D_t} r(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha} e^{\varphi_\alpha} + \frac{1}{2\alpha}\int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha} div\{xe^{\varphi_\alpha}\}dx \\
&= -\frac{1}{2\alpha}\int_{\partial D_t} r(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 - \frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t} r(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1} e^{\varphi_\alpha} \\
&\quad + \frac{1}{\alpha}\int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha} e^{\varphi_\alpha}dx + \frac{1}{2\alpha}\int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha} r\frac{\partial\varphi_\alpha}{\partial r}e^{\varphi_\alpha}dx.
\end{aligned} \tag{3.10}$$

By (3.8), (3.9) and (3.10), we obtain

$$
\begin{aligned}
&\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}r|\frac{\partial\phi_\alpha}{\partial r}|^2-\frac{1}{2\alpha}\int_{\partial D_t}r(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\\
&\quad-\int_{D_t}div\left\{(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\nabla\phi_\alpha\right\}r\frac{\partial\phi_\alpha}{\partial r}dx\\
&=\frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t}r(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}+\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx-\frac{1}{\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}e^{\varphi_\alpha}dx\\
&\quad+\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx-\frac{1}{2\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}r\frac{\partial\varphi_\alpha}{\partial r}e^{\varphi_\alpha}dx\\
&=\frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t}r(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}+(1-\frac{1}{\alpha})\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx\\
&\quad-\frac{\sigma_\alpha}{\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}dx+\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx\\
&\quad-\frac{1}{2\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}r\frac{\partial\varphi_\alpha}{\partial r}e^{\varphi_\alpha}dx\\
&=\frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t}r(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}+(1-\frac{1}{\alpha})\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx\\
&\quad-\frac{\sigma_\alpha}{\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}(1+\frac{r}{2}\frac{\partial\varphi_\alpha}{\partial r})e^{\varphi_\alpha}dx+(1-\frac{1}{\alpha})\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_{D_t}div\left\{(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\nabla\phi_\alpha\right\}r\frac{\partial\phi_\alpha}{\partial r}dx\\
&=\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}r|\frac{\partial\phi_\alpha}{\partial r}|^2-\frac{1}{2\alpha}\int_{\partial D_t}r(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\\
&\quad-\frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t}r(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}-(1-\frac{1}{\alpha})\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx\\
&\quad+\frac{\sigma_\alpha}{\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}(1+\frac{r}{2}\frac{\partial\varphi_\alpha}{\partial r})e^{\varphi_\alpha}dx-(1-\frac{1}{\alpha})\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx.
\end{aligned}
\tag{3.11}
$$

On the other hand, according to Proposition 2.2 in [19] that

$$\langle\psi,\widetilde{\nabla}_X(\not{D}\psi)-\not{D}(\widetilde{\nabla}_X\psi)\rangle=2\left\langle Re\left(P(\mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi);\psi)\right),\nabla_X\phi\right\rangle$$

whenever $[X,e_\gamma]=0,\ \gamma=1,2$, where $[\cdot,\cdot]$ is the Lie bracket and a well-known fact that

$$\int_M\langle\psi,\not{D}\omega\rangle=\int_M\langle\not{D}\psi,\omega\rangle-\int_{\partial M}\langle\overrightarrow{n}\cdot\psi,\omega\rangle$$

where $\overrightarrow{n}$ is the outward unit normal vector field on ∂M, using the equation $\not{D}\psi_\alpha = 0$, we get

$$
\begin{aligned}
&-\int_{D_t}\left\langle Re\left(P(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right), r\frac{\partial\phi_\alpha}{\partial r}\right\rangle\\
&=\frac{1}{2}\int_{D_t}\langle x^\beta\psi_\alpha, \not{D}\widetilde{\nabla}_{\frac{\partial}{\partial\beta}}\psi_\alpha\rangle\\
&=\frac{1}{2}\int_{D_t}\langle\not{D}(x^\beta\psi_\alpha), \widetilde{\nabla}_{\frac{\partial}{\partial\beta}}\psi_\alpha\rangle dx-\frac{1}{2}\int_{\partial D_t}\langle\overrightarrow{n}\cdot x^\beta\psi_\alpha, \widetilde{\nabla}_{\frac{\partial}{\partial\beta}}\psi_\alpha\rangle dx\\
&=-\frac{1}{2}\int_{D_t}\langle\psi_\alpha, \not{D}\psi_\alpha\rangle dx+\frac{1}{2}\int_{D_t}\langle\not{D}\psi_\alpha, r\widetilde{\nabla}_{\frac{\partial}{\partial r}}\psi_\alpha\rangle dx+\frac{1}{2}\int_{\partial D_t}\langle\psi_\alpha, r\frac{\partial}{\partial r}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial r}}\psi_\alpha\rangle\\
&=\frac{1}{2}\int_{\partial D_t}\langle\psi_\alpha, r\frac{\partial}{\partial r}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial r}}\psi_\alpha\rangle=-\frac{1}{2}\int_{\partial D_t}\langle\psi_\alpha, r^{-1}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle,
\end{aligned}
$$

Then the conclusion of the lemma follows immediately from the above two formulas. □

As a direct corollary of Lemma 3.2 and Lemma 3.4, we have the following Pohozaev type estimate for α-Dirac-harmonic maps.

Corollary 3.5. *Let (D, g_α) be a unit disk in $\mathbb{R}^2$ equipped with a metric $g_\alpha = e^{\varphi_\alpha}\left((dx^1)^2+(dx^2)^2\right)$, where $\varphi_\alpha(0)=0$ and $\varphi_\alpha\to\varphi_0\in C^\infty(D)$ smoothly. If $(\phi_\alpha,\psi_\alpha)$ is a critical point of $L_{\alpha,\sigma_\alpha}(\phi,\psi)$ where $0<\beta_0<\lim_{\alpha\searrow 1}\sigma_\alpha^{\alpha-1}\le 1$ and $E_{\alpha,\sigma_\alpha}(\phi_\alpha)\le\Lambda$, then for any $0<t<1$, we have*

$$
\begin{aligned}
&(1-\frac{1}{2\alpha})\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\frac{\partial\phi_\alpha}{\partial r}|^2-\frac{1}{2\alpha}\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2\\
&=(1-\frac{1}{\alpha})\frac{1}{t}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx\\
&\quad+\frac{1}{2\alpha}\int_{\partial D_t}\langle\psi_\alpha, r^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle+O(t)+O(\alpha-1).
\end{aligned}
\tag{3.12}
$$

In the end of this section, we will establish the three circle theorem for α-Dirac-harmonic type system. Let us first state the three circle theorem for harmonic functions (see e.g. [31, 29, 28]).

Theorem 3.6. *There exists a constant $L>0$, such that if u is a nontrivial smooth harmonic function defined in $[(i-1)L,(i+2)L]\times S^1$ that satisfies*

$$\int_{\{iL\}\times S^1}ud\theta=\int_{\{(i+1)L\}\times S^1}ud\theta=0,$$

then

$$\|u\|^2_{L^2([iL,(i+1)L]\times S^1)}<\frac{1}{2}(e^{-L}\|u\|^2_{L^2([(i-1)L,iL]\times S^1)}+e^{-L}\|u\|^2_{L^2([(i+1)L,(i+2)L]\times S^1)}).\tag{3.13}$$

Next, we prove an L^2 interior estimate for the following integro-differential equations corresponding to the α-Dirac-harmonic map type system.

Lemma 3.7. *Suppose $u \in W^{2,2}(D_4 \setminus D_1)$, $v \in W^{1,2}(D_4 \setminus D_1)$ and satisfies*

$$\Delta u = A^1 u + A^2 \nabla u + A^3 v + \frac{1}{2\pi}\int_0^{2\pi} A^4 u + A^5 \nabla u + A^6 v d\theta + f_1, \tag{3.14}$$

$$\partial\!\!\!/ v = B^1 u + B^2 \nabla u + B^3 v + \frac{1}{2\pi}\int_0^{2\pi} B^4 u + B^5 \nabla u + B^6 v d\theta + f_2, \tag{3.15}$$

where

$$\sum_{i=1}^{6}(\|A^i\|_{L^\infty(D_4\setminus D_1)} + \|B^i\|_{L^\infty(D_4\setminus D_1)}) \leq \Lambda \text{ and } \sum_{i=1}^{2}\|f_i\|_{L^2(D_4\setminus D_1)} \leq \Lambda. \tag{3.16}$$

Then there holds

$$\|u\|_{W^{2,2}(D_3\setminus D_2)} + \|v\|_{W^{1,2}(D_3\setminus D_2)} \leq C(\Lambda)\left(\|u\|_{L^2(D_4\setminus D_1)} + \|v\|_{L^2(D_4\setminus D_1)} + \sum_{i=1}^{2}\|f_i\|_{L^2(D_4\setminus D_1)}\right). \tag{3.17}$$

Proof. The proof is similar to Lemma 3.1 in [28] or Lemma 3.2 in [19].

Denote $B_\sigma = D_{3+\sigma} \setminus D_{2-\sigma}$, $0 < \sigma < 1$. Let $\sigma' = \frac{\sigma+1}{2}$. Take a cut-off function $\eta(x) = \eta(|x|)$ with compact support in $B_{\sigma'}$ satisfying $\eta(x) \equiv 1$ in B_σ and $|\nabla\eta| \leq \frac{4}{(1-\sigma)}$ and $|\Delta\eta| \leq \frac{16}{(1-\sigma)^2}$. Computing directly, we get

$$\begin{aligned}\Delta(\eta u) &= \eta\Delta u + 2\nabla\eta\nabla u + \Delta\eta u\\ &= (2\nabla\eta + \eta A^2)\nabla u + (\Delta\eta + \eta A^1)u + \eta A^3 v + \eta f_1 + \eta\cdot\frac{1}{2\pi}\int_0^{2\pi} A^4 u + A^5\nabla u + A^6 v d\theta.\end{aligned}$$

By the standard elliptic estimate, we have

$$\|\eta u\|_{W^{2,2}(D_4)} \leq C\left(\frac{\|\nabla u\|_{L^2(B_{\sigma'})}}{1-\sigma} + \frac{\|u\|_{L^2(B_{\sigma'})} + \|v\|_{L^2(B_{\sigma'})}}{(1-\sigma)^2} + \|\eta f_1\|_{L^2(D_4)}\right). \tag{3.18}$$

We now introduce seminorms as in [19, 28] and define for $j = 0, 1, 2$

$$\Xi_j = \sup_{0\leq\sigma\leq 1}(1-\sigma)^j\|D^j u\|_{L^2(B_\sigma)}.$$

Multiplying (3.18) by $(1-\sigma)^2$ and noting that $1-\sigma' = \frac{1-\sigma}{2}$, we have

$$\Xi_2 \leq C\left(\Xi_1 + \Xi_0 + \|v\|_{L^2(D_4\setminus D_1)} + \|f_1\|_{L^2(D_4\setminus D_1)}\right). \tag{3.19}$$

Since Ξ_j satisfy an interpolation inequality

$$\Xi_1 \leq \epsilon\Xi_2 + \frac{C}{\epsilon}\Xi_0 \tag{3.20}$$

for any $\epsilon > 0$, where $C > 0$ is a universal constant. Using (3.20) in (3.19), we get

$$\Xi_2 \leq C\left(\|u\|_{L^2(D_4\setminus D_1)} + \|v\|_{L^2(D_4\setminus D_1)} + \|f_1\|_{L^2(D_4\setminus D_1)}\right),$$

this is

$$\|D^2 u\|_{L^2(B_\sigma)} \leq \frac{C}{(1-\sigma)^2}\left(\|u\|_{L^2(D_4\setminus D_1)} + \|v\|_{L^2(D_4\setminus D_1)} + \|f_1\|_{L^2(D_4\setminus D_1)}\right).$$

Taking $\sigma = \frac{1}{2}$, it follows

$$\|u\|_{W^{2,2}(B_{1/2})} \leq C\left(\|u\|_{L^2(D_4\setminus D_1)} + \|v\|_{L^2(D_4\setminus D_1)} + \|f_1\|_{L^2(D_4\setminus D_1)}\right). \tag{3.21}$$

Similarly, we can compute

$$\partial\!\!\!/(\eta v) = \eta B^1 u + \eta B^2 \nabla u + (\eta B^3 + \nabla\eta)v + \eta\frac{1}{2\pi}\int_0^{2\pi} B^4 u + B^5\nabla u + B^6 v d\theta + \eta f_2.$$

By the first order elliptic estimate and choosing a new cut-off function η suitably, we have

$$\begin{aligned}\|v\|_{W^{1,2}(B_{\frac{1}{4}})} &\le C(\|\nabla u\|_{L^2(B_{\frac{1}{2}})} + \|u\|_{L^2(B_{\frac{1}{2}})} + \|v\|_{L^2(B_{\frac{1}{2}})} + \|f_2\|_{L^2(B_{\frac{1}{2}})})\\ &\le C(\|u\|_{L^2(D_4\setminus D_1)} + \|v\|_{L^2(D_4\setminus D_1)} + \sum_{i=1}^{2}\|f_i\|_{L^2(D_4\setminus D_1)}).\end{aligned}$$

Then it is easy to see that the conclusion of the lemma follows immediately. □

Denote $P_i := D_{e^{(i+1)L}r_2}\setminus D_{e^{iL}r_2}$ and

$$F_i(u,v) := \int_{P_i}\frac{1}{|x|^2}|u|^2dx + \int_{P_i}\frac{1}{|x|}|v|^2dx,$$

where $L>0$ is the constant in Theorem 3.6.

We have the following three circle theorem for the integro-differential equations corresponding to α-Dirac-harmonic map type systems:

Theorem 3.8. *Suppose $u\in W^{2,2}(P_{i-1}\cup P_i\cup P_{i+1})$, $v\in W^{1,2}(P_{i-1}\cup P_i\cup P_{i+1})$ satisfy equations* (3.14) *and* (3.15). *Then there exists a positive constant $\delta_0>0$, such that if*

$$\max_{i-1,i,i+1}(\||x|f_1\|^2_{L^2(P_j)} + \||x|^{\frac{1}{2}}f_2\|^2_{L^2(P_j)}) \le \delta_0 F_i(u,v), \tag{3.22}$$

and

$$\begin{aligned}&|x|^2(|A^1|+|A^4|) + |x|^{\frac{3}{2}}(|A^3|+|A^6|+|B^1|+|B^4|)\\ &+ |x|(|A^2|+|A^5|+|B^3|+|B^6|) + |x|^{\frac{1}{2}}(|B^2|+|B^5|)\le\delta_0,\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}&|\int_0^{2\pi}u(e^{iL}r_2,\theta)d\theta|^2 + |\int_0^{2\pi}u(e^{(i+1)L}r_2,\theta)d\theta|^2\\ &+ e^{iL}r_2|\int_0^{2\pi}v(e^{iL}r_2,\theta)d\theta|^2 + e^{(i+1)L}r_2|\int_0^{2\pi}v(e^{(i+1)L}r_2,\theta)d\theta|^2 \le \delta_0 F_i(u,v),\end{aligned} \tag{3.24}$$

then, there hold

(a) *$F_{i+1}(u,v)\le e^{-L}F_i(u,v)$ implies $F_i(u,v)\le e^{-L}F_{i-1}(u,v)$;*

(b) *$F_{i-1}(u,v)\le e^{-L}F_i(u,v)$ implies $F_i(u,v)\le e^{-L}F_{i+1}(u,v)$;*

(c) *either $F_i(u,v)\le e^{-L}F_{i-1}(u,v)$ or $F_i(u,v)\le e^{-L}F_{i+1}(u,v)$.*

Proof. One can find that the condition (3.23) is in fact stronger than (3.12) in [19], so the conclusions of theorem follow immediately from Theorem 3.3 in [19]. □

As a direct application of the above three circle theorem, we can get the following decay lemma:

Lemma 3.9. *Let $\delta_0 > 0$ be the constant in Theorem 3.8. Let $u \in W^{2,2}(D_{e^{(l+1)L}r_2} \setminus D_{r_2})$ and $v \in W^{1,2}(D_{e^{(l+1)L}r_2} \setminus D_{r_2})$, for some integer $l > 1$, satisfy equations* (3.14) *and* (3.15) *and assume the followings hold:*

$$\begin{aligned}&|x|^2(|A^1|+|A^4|)+|x|^{\frac{3}{2}}(|A^3|+|A^6|+|B^1|+|B^4|)\\ &+|x|(|A^2|+|A^5|+|B^3|+|B^6|)+|x|^{\frac{1}{2}}(|B^2|+|B^5|)\leq\delta_0,\end{aligned}\tag{3.25}$$

and

$$\int_{\partial D_r} u = \int_{\partial D_r} v = 0.$$

Then for any $1 \leq i \leq l$, there holds

$$F_i(u,v)\leq C\max_{j=1,\dots,l}(\||x|f_1\|^2_{L^2(P_j)}+\||x|^{\frac{1}{2}}f_2\|^2_{L^2(P_j)})+C\big(F_0(u,v)+F_l(u,v)\big)(e^{-(l-i)L}+e^{-iL}).\tag{3.26}$$

Proof. Denote the set

$$\mathbb{I}:=\left\{1\leq i\leq l-1\Big|\max_{i-1,i,i+1}(\||x|f_1\|^2_{L^2(P_j)}+\||x|^{\frac{1}{2}}f_2\|^2_{L^2(P_j)})\geq\delta_0F_i(u,v)\right\}.\tag{3.27}$$

If $i \in \mathbb{I}$ then it is easy to see that (3.26) holds. If $i \notin \mathbb{I}$, by (c) of Theorem 3.8, we have

$$F_i(u,v)\leq e^{-L}F_{i-1}(u,v)\ or\ F_i(u,v)\leq e^{-L}F_{i+1}(u,v).$$

If $F_i(u,v)\leq e^{-L}F_{i-1}(u,v)$, we shall consider the following two cases:

Case 1-1: if there exists $1 \leq i_1 < i$ such that $i_1 \in \mathbb{I}$ and for any $i_1 < j < i$, there holds $j \notin \mathbb{I}$, then by (a) in Theorem 3.8, we will get

$$F_i(u,v)\leq e^{-(i-i_1)L}F_{i_1}(u,v)\leq C\max_{i_1-1,i_1,i_1+1}(\||x|f_1\|^2_{L^2(P_j)}+\||x|^{\frac{1}{2}}f_2\|^2_{L^2(P_j)}).$$

Case 1-2: if such constant i_1 does not exist, i.e. $j \notin \mathbb{I},\ j = 2,...,i$, then by (a) in Theorem 3.8, we will get

$$F_i(u,v)\leq e^{-iL}F_0(u,v).$$

Similarly, if $F_i(u,v)\leq e^{-L}F_{i+1}(u,v)$, then

Case 2-1: if there exists $i < i_2 < l-1$ such that $i_2 \in \mathbb{I}$ and for any $i < j < i_2$, there holds $j \notin \mathbb{I}$, then by (b) in Theorem 3.8, we will get

$$F_i(u,v)\leq e^{-(i_2-i)L}F_{i_2}(u,v)\leq C\max_{i_2-1,i_2,i_2+1}(\||x|f_1\|^2_{L^2(P_j)}+\||x|^{\frac{1}{2}}f_2\|^2_{L^2(P_j)}).$$

Case 2-2: if such constant i_2 does not exist, i.e. $j \notin \mathbb{I},\ j = i,...,l-1$, then by (b) in Theorem 3.8, we will get

$$F_i(u,v)\leq e^{-(l-i)L}F_l(u,v).$$

Thus, we obtain

$$F_i(u,v)\leq C\max_{j=1,\dots,l}(\||x|f_1\|^2_{L^2(P_j)}+\||x|^{\frac{1}{2}}f_2\|^2_{L^2(P_j)})+Ce^{-iL}F_0(u,v)$$

or

$$F_i(u,v)\leq C\max_{j=1,\dots,l}(\||x|f_1\|^2_{L^2(P_j)}+\||x|^{\frac{1}{2}}f_2\|^2_{L^2(P_j)})+Ce^{-(l-i)L}F_l(u,v).$$

Then the conclusion of the lemma follows immediately. □

Corollary 3.10. *Under the assumptions of Lemma 3.9, we have*

$$\begin{aligned}&\|\nabla u\|_{L^2(P_i)}+\|\nabla v\|_{L^{\frac{4}{3}}(P_i)}\\ &\le C\max_{j=1,...,l}(\||x|f_1\|_{L^2(P_j)}+\||x|^{\frac12}f_2\|_{L^2(P_j)})+C\big(F_0^{1/2}(u,v)+F_l^{1/2}(u,v)\big)(e^{-\frac12(l-i)L}+e^{-\frac12 iL}).\end{aligned}\tag{3.28}$$

Proof. By the interior estimate in lemma 3.7 and the standard scaling argument, we have

$$\begin{aligned}&\|\nabla u\|_{L^2(P_i)}+\|\nabla v\|_{L^{\frac{4}{3}}(P_i)}\\ &\le C(F_i^{\frac12}(u,v)+F_{i-1}^{\frac12}(u,v)+F_{i+1}^{\frac12}(u,v)+\||x|f_1\|_{L^2(P_{i-1}\cup P_i\cup P_{i+1})}+\||x|^{\frac12}f_2\|_{L^2(P_{i-1}\cup P_i\cup P_{i+1})})\\ &\le C\max_{j=1,...,l}(\||x|f_1\|_{L^2(P_j)}+\||x|^{\frac12}f_2\|_{L^2(P_j)})+C\big(F_0^{1/2}(u,v)+F_l^{1/2}(u,v)\big)(e^{-\frac12(l-i)L}+e^{-\frac12 iL}).\end{aligned}$$

□

4. Generalized energy identities for α-Dirac-harmonic maps

In this section, we shall prove our first main result Theorem 2.2. We first consider the following simpler case of a single interior blow-up point.

Theorem 4.1. *Let $D_1(0)\subset\mathbb{R}^2$ be the unit disk, let $g_\alpha=e^{\varphi_\alpha(x)}((dx^1)^2+(dx^2)^2)$ and $g=e^{\varphi_0(x)}((dx^1)^2+(dx^2)^2)$ be a family of metrics on $D_1(0)$, where $\varphi_\alpha\in C^\infty(D_1)$, $\varphi_\alpha(0)=0$ and $\varphi_\alpha\to\varphi_0$ in $C^\infty(D_1)$ as $\alpha\searrow 1$. Let $(\phi_\alpha,\psi_\alpha)\in C^\infty(D_1(0),N)$ be a sequence of α-Dirac-harmonic maps satisfying*

(a) $\sup_\alpha(E_{\alpha,\sigma_\alpha}(\phi_\alpha)+E(\psi_\alpha))\le\Lambda$, *and* $0<\beta_0\le\lim_{\alpha\searrow1}(\sigma_\alpha)^{\alpha-1}\le 1$,

(b) $(\phi_\alpha,\psi_\alpha)\to(\phi,\psi)$ *strongly in* $C^\infty_{loc}(D\setminus\{0\})$ *as* $\alpha\searrow 1$.

Then there exist a subsequence of $(\phi_\alpha,\psi_\alpha)$ (still denoted by $(\phi_\alpha,\psi_\alpha)$) and a nonnegative integer L such that, for any $i=1,...,L$, there exist a sequence of points x^i_α, positive numbers λ^i_α and a nonconstant Dirac-harmonic sphere (σ^i,ξ^i) such that:

(1) $x^i_\alpha\to 0$, $\lambda^i_\alpha\to 0$, *as* $\alpha\searrow 1$*;*

(2) $\lim_{\alpha\searrow1}\left(\frac{\lambda^i_\alpha}{\lambda^j_\alpha}+\frac{\lambda^j_\alpha}{\lambda^i_\alpha}+\frac{|x^i_\alpha-x^j_\alpha|}{\lambda^i_\alpha+\lambda^j_\alpha}\right)=\infty$ *for any* $i\neq j$*;*

(3) (σ^i,ξ^i) *is the weak limit of* $\left(\phi_\alpha(x^i_\alpha+\lambda^i_\alpha x),(\lambda^i_\alpha)^{\alpha-1}\sqrt{\lambda^i_\alpha}\psi_\alpha(x^i_\alpha+\lambda^i_\alpha x)\right)$ *in* $W^{1,2}_{loc}(\mathbb{R}^2)\times L^4_{loc}(\mathbb{R}^2)$.

(4) ***Generalized energy identities:*** *we have*

$$\lim_{\delta\to0}\lim_{\alpha\searrow1}E_{\alpha,\sigma_\alpha}(\phi_\alpha,D_\delta(0))=\sum_{i=1}^L\mu_i^2E(\sigma^i),\tag{4.1}$$

$$\lim_{\delta\to0}\lim_{\alpha\searrow1}E(\psi_\alpha,D_\delta(0))=\sum_{i=1}^L\mu_i^2E(\xi^i),\tag{4.2}$$

where $\mu_i=\lim_{\alpha\searrow1}(\lambda^i_\alpha)^{2-2\alpha}$.

Proof of Theorem 4.1. By our assumptions, we know that 0 is the only blow-up point of the sequence $\{(\phi_\alpha, \psi_\alpha)\}$ in D, *i.e.*

$$\liminf_{\alpha\searrow 1} E(\phi_\alpha, \psi_\alpha; D_r) \geq \frac{\epsilon_0^2}{8} \text{ for all } r > 0 \tag{4.3}$$

where $\epsilon_0 > 0$ is the constant in Lemma 3.1. By the standard blow-up argument, we can assume that there exist sequences $x_\alpha \to 0$ and $\lambda_\alpha \to 0$ such that

$$E(\phi_\alpha, \psi_\alpha; D_{\lambda_\alpha}(x_\alpha)) = \sup_{\substack{x\in D, r\leq\lambda_\alpha \\ D_r(x)\subset D}} E(\phi_\alpha, \psi_\alpha; D_r(x)) = \frac{\epsilon_0^2}{32}. \tag{4.4}$$

We first recall the process of constructing the first bubble for a sequence of α-Dirac-harmonic maps (one can also refer to the proof of Theorem 1.4 in [20]). Setting

$$(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x)) = \left(\phi_\alpha(x_\alpha + \lambda_\alpha x),\ \sqrt{\lambda_\alpha}\psi_\alpha(x_\alpha + \lambda_\alpha x)\right), \tag{4.5}$$

by (2.14) , it is easy to see that, for any $R > 0$, $(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x))$ lives in $D_R(0) \subset \mathbb{R}^2$ for α close to 1 and satisfies

$$\begin{cases} \Delta\widetilde{u}_\alpha & = -(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\widetilde{u}_\alpha|^2\nabla\widetilde{u}_\alpha}{\sigma_\alpha\lambda_\alpha^2+|\nabla_{g_\alpha}\widetilde{u}_\alpha|^2} - A(d\widetilde{u}_\alpha, d\widetilde{u}_\alpha) + \frac{Re\left(P(\mathcal{A}(d\widetilde{u}_\alpha(e_\gamma), e_\gamma\cdot\widetilde{v}_\alpha);\widetilde{v}_\alpha)\right)}{\alpha(\sigma_\alpha+\lambda_\alpha^{-2}|\nabla_{g_\alpha}\widetilde{u}_\alpha|^2)^{\alpha-1}}, \\ \not{\partial}\widetilde{v}_\alpha & = \mathcal{A}(d\widetilde{u}_\alpha(e_\gamma), e_\gamma\cdot\widetilde{v}_\alpha), \end{cases} \tag{4.6}$$

where $g_\alpha(x) = e^{\varphi_\alpha(x_\alpha+\lambda_\alpha x)}((dx^1)^2 + (dx^2)^2)$ and we used the fact that the second equation, i.e. the equation for the spinor part, is conformally invariant.

Although $(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x))$ is not a α-Dirac-harmonic map, from the proof of Lemma 3.1 (or the proof of Lemma 4.1 and Lemma 4.2 in [20]), one can easy check that the small energy regularity result still holds for $(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x))$, i.e. the conclusions of Lemma 3.1 also hold for $(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x))$. Combing this with (4.12), we know there exists a subsequence of $\{\alpha\}$ (still denoted by the same symbols) and $(\widetilde{\sigma}, \widetilde{\xi}) \in W^{2,2}_{loc}(\mathbb{R}^2) \times W^{1,2}_{loc}(\mathbb{R}^2)$, such that $E(\widetilde{\sigma}, \widetilde{\xi}; D_1(0)) = \frac{\epsilon_0^2}{32}$ and

$$(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x)) \rightharpoonup (\widetilde{\sigma}, \widetilde{\xi}) \textit{ weakly in } W^{2,2}_{loc}(\mathbb{R}^2) \times W^{1,2}_{loc}(\mathbb{R}^2), \tag{4.7}$$

$$(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x)) \to (\widetilde{\sigma}, \widetilde{\xi}) \textit{ in } W^{1,2}_{loc}(\mathbb{R}^2) \times L^4_{loc}(\mathbb{R}^2). \tag{4.8}$$

Next, we prove the following **Claim 1:**

$$1 \leq \liminf_{\alpha\searrow 1} \lambda_\alpha^{2(1-\alpha)} \leq \limsup_{\alpha\searrow 1} \lambda_\alpha^{2(1-\alpha)} \leq \mu_{max} < \infty. \tag{4.9}$$

To show this claim, we just need to prove that

$$\limsup_{\alpha\searrow 1} \lambda_\alpha^{2(1-\alpha)} < \infty.$$

In fact, if it does not hold, then there exists a subsequence $\alpha_j \to 1$ such that

$$\lim_{j\to\infty} \lambda_{\alpha_j}^{2(1-\alpha_j)} := \mu_1 = \infty.$$

By (4.6) and (4.7), it is easy to see that $\widetilde{\sigma}:\mathbb{R}^2\to N$ is a harmonic map such that $\widetilde{u}_{\alpha_j}\to\widetilde{\sigma}$ in $C^1_{loc}(\mathbb{R}^2)$ as $j\to\infty$. Then we have

$$\Lambda \geq \lim_{R\to\infty}\lim_{j\to\infty}\int_{D_{\lambda_{\alpha_j}R}(x_{\alpha_j})}|\nabla_{g_{\alpha_j}}\phi_{\alpha_j}|^{2\alpha_j}dvol_{g_{\alpha_j}} = \lim_{R\to\infty}\lim_{j\to\infty}(\lambda_{\alpha_j})^{2-2\alpha_j}\int_{D_R(0)}|\nabla_{g_{\alpha_j}}\widetilde{u}_{\alpha_j}|^{2\alpha_j}dvol_{g_{\alpha_j}(x_{\alpha_j}+\lambda_{\alpha_j}x)}$$
$$= \lim_{R\to\infty}\mu_1\int_{D_R(0)}|\nabla\widetilde{\sigma}|^2dx = \mu_1E(\widetilde{\sigma}).$$

which is a contradiction to the fact that $E(\widetilde{\sigma})\geq\overline{\epsilon}>0$ which follows from well known energy gap theorem of harmonic sphere, since $\widetilde{\sigma}:\mathbb{R}^2\to N$ is a nontrivial harmonic map with finite energy and hence it can be conformally extended to a harmonic sphere. Thus, **Claim 1**, i.e. (4.9) holds true.

Now setting

$$u_\alpha(x) := \phi_\alpha(x_\alpha+\lambda_\alpha x),\ v_\alpha(x) := \lambda_\alpha^{\alpha-1}\sqrt{\lambda_\alpha}\psi_\alpha(x_\alpha+\lambda_\alpha x),$$

since the equation for the spinor part is also invariant by multiplying a constant to the spinor, it is easy to see that (u_α, v_α) satisfies

$$\begin{cases}\Delta u_\alpha &= -(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}u_\alpha|^2\nabla u_\alpha}{\sigma_\alpha\lambda_\alpha^2+|\nabla_{g_\alpha}u_\alpha|^2} - A(du_\alpha,du_\alpha) + \lambda_\alpha^{2(1-\alpha)}\frac{Re\left(P(\mathcal{A}(du_\alpha(e_\gamma),e_\gamma\cdot v_\alpha);v_\alpha)\right)}{\alpha(\sigma_\alpha+\lambda_\alpha^{-2}|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}},\\ \partial\!\!\!/ v_\alpha &= \mathcal{A}(du_\alpha(e_\gamma),e_\gamma\cdot v_\alpha).\end{cases} \tag{4.10}$$

By (4.9), it is easy to see that $(u_\alpha(x), v_\alpha(x))$ is an α-Dirac-harmonic map (by replacing σ_α with $\sigma_\alpha\lambda_\alpha^2$) living in a region which tends to $\mathbb{R}^2$ as $\alpha\searrow 1$. Moreover, for any $x\in\mathbb{R}^2$, when α is sufficiently close to 1, by (4.4) and the fact that $\lambda_\alpha^{\alpha-1}\leq 1$, we have

$$E(u_\alpha,v_\alpha;D_1(x))\leq\frac{\epsilon_0^2}{32}. \tag{4.11}$$

According to Lemma 3.1, there exist a subsequence of (u_α,v_α) (we still denote it by (u_α,v_α)) and a Dirac-harmonic map $(\sigma^1,\xi^1)\in W^{1,2}(\mathbb{R}^2,N)$ such that

$$\lim_{\alpha\searrow 1}u_\alpha(x)=\sigma^1(x)\text{ in }W^{1,2}_{loc}(\mathbb{R}^2)\ \textit{and}\ \lim_{\alpha\searrow 1}v_\alpha(x)=\xi^1(x)\text{ in }L^4_{loc}(\mathbb{R}^2).$$

Besides, by (4.4) and (4.9), we know

$$E(\sigma^1,\xi^1;D_1(0)) = \lim_{\alpha\searrow 1}E(\phi_\alpha;D_{\lambda_\alpha}(x_\alpha)) + \lim_{\alpha\searrow 1}\lambda_\alpha^{4(\alpha-1)}E(\psi_\alpha;D_{\lambda_\alpha}(x_\alpha))$$
$$\geq \lim_{\alpha\searrow 1}\lambda_\alpha^{4(\alpha-1)}E(\phi_\alpha,\psi_\alpha;D_{\lambda_\alpha}(x_\alpha)) \geq \frac{1}{\mu_{max}^2}\frac{\epsilon_0^2}{32}.$$

By the standard theory of harmonic maps, (σ^1,ξ^1) can be extended to a nontrivial Dirac-harmonic sphere. We call the above nontrivial Dirac-harmonic sphere (σ^1,ξ^1) the first bubble.

By a standard induction argument as in the case of approximate harmonic maps with L^2-uniformly bounded tension fields [14], we only need to prove the theorem in the case where there is only one bubble. Under the "one bubble" assumption, we first make the following:

Claim 2: For any $\epsilon>0$, there exist $\delta>0$ and $R>0$ such that

$$\int_{D_{8t}(x_\alpha)\setminus D_t(x_\alpha)}|\nabla\phi_\alpha|^2dx + \left(\int_{D_{8t}(x_\alpha)\setminus D_t(x_\alpha)}|\psi_\alpha|^4dx\right)^{\frac{1}{2}} \leq \epsilon^2 \text{ for any } t\in(\frac{1}{4}\lambda_\alpha R, 2\delta) \tag{4.12}$$

when $\alpha-1$ is small enough.

The proof of this claim is now standard and follows from a contradiction argument. In fact, if (4.12) is not true, then we can find $\overline{\epsilon} > 0$, $t_\alpha \to 0$, such that $\lim_{\alpha\searrow 1} \frac{t_\alpha}{\lambda_\alpha} = \infty$ and

$$E(\phi_\alpha, \psi_\alpha; D_{8t_\alpha}(x_\alpha) \setminus D_{t_\alpha}(x_\alpha)) \geq \overline{\epsilon} > 0. \tag{4.13}$$

Setting

$$u'_\alpha(x) := \phi_n(x_\alpha + t_\alpha x), v'_\alpha(x) := t_\alpha^{\alpha-1}\sqrt{t_\alpha}\psi_\alpha(x_\alpha + t_\alpha x),$$

then from the above arguments, we know that (u'_α, v'_α) is also a α-Dirac-harmonic map. Furthermore, it is easy to see that 0 is an energy concentration point for (u'_α, v'_α) and (σ^1, ξ^1) is also a bubble for (u'_α, v'_α). To construct the second bubble, we have to consider the following two cases:

(a) (u'_α, v'_α) has no other energy concentration points except 0.

By Lemma 3.1, passing to a subsequence, we may assume that (u'_α, v'_α) converges to a Dirac-harmonic map $(\sigma^2, \xi^2) : \mathbb{R}^2 \to N$ strongly in $W^{1,2}_{loc}(\mathbb{R}^2 \setminus \{0\}) \times L^4_{loc}(\mathbb{R}^2 \setminus \{0\})$ as $\alpha \searrow 1$. In particular, we have

$$E(\sigma^2, \xi^2; D_8 \setminus D_1) = \lim_{\alpha\searrow 1} E(u'_\alpha, v'_\alpha; D_8 \setminus D_1) \geq \frac{1}{\mu^2_{max}}\overline{\epsilon}.$$

According to the standard theory of Dirac-harmonic maps, we know that (σ^2, ξ^2) is a nontrivial Dirac-harmonic sphere. This is the second bubble. This is a contradiction to the "one bubble" assumption.

(b) (u'_α, v'_α) has another energy concentration point $p \neq 0$.

Without loss of generality, we may assume that p is the only blow-up point in $D_r(p)$ for some small $r > 0$. By the standard theory of blow-up analysis, there exist $x'_\alpha \to p$ and $\lambda'_\alpha \to 0$ such that

$$E(u'_\alpha, v'_\alpha; D_{\lambda'_\alpha}(x'_\alpha)) = \sup_{\substack{x\in D_r(p), s\leq r_n \\ D_s(x)\subset D_r(p)}} E(u'_\alpha, v'_\alpha; D_s(x)) = \frac{\epsilon_0^2}{32}. \tag{4.14}$$

From the process of constructing the first bubble, we know that there exists a nontrivial Dirac-harmonic sphere (σ^2, ξ^2) such that

$$(u'_\alpha(x'_\alpha + \lambda'_\alpha x), (\lambda'_\alpha)^{\alpha-1}\sqrt{\lambda'_\alpha}v'_\alpha(x'_\alpha + \lambda'_\alpha x)) \to (\sigma^2, \xi^2) \text{ in } W^{1,2}_{loc}(\mathbb{R}^2) \times L^4_{loc}(\mathbb{R}^2)$$

as $\alpha \searrow 1$. This is

$$\Big(\phi_\alpha(x_\alpha + t_\alpha x'_\alpha + t_\alpha\lambda'_\alpha x), (t_\alpha\lambda'_\alpha)^{\alpha-1}\sqrt{t_\alpha\lambda'_\alpha}\psi_\alpha(x_\alpha + t_\alpha x'_\alpha + t_\alpha\lambda'_\alpha x)\Big) \to (\sigma^2, \xi^2) \text{ in } W^{1,2}_{loc}(\mathbb{R}^2) \times L^4_{loc}(\mathbb{R}^2)$$

as $\alpha \searrow 1$. By (4.14), (σ^2, ξ^2) is nontrivial. Therefore, we again get the second bubble contradicting the "one bubble" assumption. So, we proved **Claim 2** (4.12).

To proceed, we need to establish a series of auxiliary lemmas and we leave the rest of the proof of Theorem 4.1 to the end of this section. □

Without loss of generality, we always assume $\delta = e^{k_\alpha L}\lambda_\alpha R$ where $L > 0$ is the constant in Theorem 3.6 and k_α is an integer which goes to infinity as $\alpha \searrow 1$. For simplicity of notation, we still denote $P_i = D_{e^{(i+1)L}\lambda_\alpha R}(x_\alpha) \setminus D_{e^{iL}\lambda_\alpha R}(x_\alpha)$.

Firstly, we show the generalized energy identity for the spinor part.

Lemma 4.2. *Under the assumption of Theorem 4.1, if there is no energy concentration for the sequence $(\phi_\alpha, \psi_\alpha)$ in the region $D_\delta(x_\alpha) \setminus D_{\lambda R}(x_\alpha)$, i.e.* (4.12) *holds, we have*

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\searrow 1}\left(\|\psi_\alpha\|_{L^4(D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha))} + \|\nabla\psi_\alpha\|_{L^{\frac{4}{3}}(D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha))}\right) = 0.$$

Proof. Firstly we use a finite decomposition argument that is similar to the one in [35] to decompose $\Sigma := D_\delta(x_\alpha) \setminus D_{\lambda_\alpha R}(x_\alpha)$ into finite parts

$$\Sigma = \cup_{j=1}^{s_\alpha} Q_j, \quad Q_j := \cup_{i=m_{j-1}+1}^{m_j} P_i, \quad 0 = m_0 < m_1 <, ..., < m_{s_\alpha} = k_\alpha$$

such that $s_\alpha \leq S$ and

$$E(\phi_\alpha, \psi_\alpha; Q_j) \leq \frac{1}{C_1(N)}, \quad j = 1, ..., s_\alpha, \tag{4.15}$$

where $C_1(N) > 0$ is a constant depending only on N to be determined later and S is a uniform integer for all $\alpha - 1$ small enough.

From (4.12), for any $\epsilon << \frac{1}{2C_1(N)}$, we have

$$E(\phi_\alpha, \psi_\alpha; P_i) < C(L)\epsilon < \frac{1}{2C_1(N)}, \quad i = 1, ..., k_\alpha$$

when $\alpha - 1$ is small.

If

$$E(\phi_\alpha, \psi_\alpha; \Sigma) \leq \frac{1}{C_1(N)},$$

let $m_1 = k_\alpha$ and then $Q_1 = \Sigma$. Otherwise, we can choose an integer $1 \leq m_1 < k_\alpha$ such that

$$\frac{1}{2C_1(N)} < E(\phi_\alpha, \psi_\alpha; Q_1) \leq \frac{1}{C_1(N)} \quad \text{and} \quad E(\phi_\alpha, \psi_\alpha; Q_1 \cup P_{m_1+1}) > \frac{1}{C_1(N)}.$$

This is the first step of the division. Inductively, suppose that m_j is chosen such that $E(\phi_\alpha, \psi_\alpha; Q_j) \leq \frac{1}{C_1(N)}$. If

$$E(\phi_\alpha, \psi_\alpha; \cup_{i=m_j+1}^{k_\alpha} P_i) \leq \frac{1}{C_1(N)},$$

let $m_{j+1} = k_\alpha$, thus $Q_{j+1} = \cup_{i=m_j+1}^{k_\alpha} P_i$. If not, then similar to the first step, we can find $m_j < m_{j+1} < k_\alpha$ such that

$$\frac{1}{2C_1(N)} < E(\phi_\alpha, \psi_\alpha; Q_{j+1}) \leq \frac{1}{C_1(N)} \quad \textit{and} \quad E(\phi_\alpha, \psi_\alpha; Q_{j+1} \cup P_{m_{j+1}+1}) > \frac{1}{C_1(N)}.$$

Since $E(\phi_\alpha, \psi_\alpha)$ is uniformly bounded by Λ, we will finish our division after at most $S = [2C_1(N)\Lambda] + 1$ steps. So we have finished the division.

Take a cut-off function $\eta \in C_0^\infty(D_{e^{(m_j+2)L}\lambda_\alpha R}(x_\alpha) \setminus D_{e^{m_{j-1}L}\lambda_\alpha R}(x_\alpha))$ such that $0 \leq \eta \leq 1$ and

$$\eta|_{D_{e^{(m_j+1)L}\lambda_\alpha R}(x_\alpha)\setminus D_{e^{(m_{j-1}+1)L}\lambda_\alpha R}(x_\alpha)} \equiv 1$$

and

$$|\nabla\eta| \leq \frac{C}{e^{m_jL}\lambda_\alpha R} \quad on \quad D_{e^{(m_j+2)L}\lambda_\alpha R}(x_\alpha) \setminus D_{e^{(m_j+1)L}\lambda_\alpha R}(x_\alpha) \quad and$$

$$|\nabla\eta| \leq \frac{C}{e^{m_{j-1}L}\lambda_\alpha R} \quad on \quad D_{e^{(m_{j-1}+1)L}\lambda_\alpha R}(x_\alpha) \setminus D_{e^{m_{j-1}L}\lambda_\alpha R}(x_\alpha).$$

By the standard elliptic estimates, we have

$$\begin{aligned}
\|\eta\psi_\alpha\|_{W^{1,4/3}(D_1)} &\le C\|\eta\partial\!\!\!/\psi_\alpha + \nabla\eta\cdot\psi_\alpha\|_{L^{\frac{4}{3}}(D_1)} \\
&\le \frac{1}{4}C(N)\||d\phi_\alpha||\eta\psi_\alpha|\|_{L^{\frac{4}{3}}(D_1)} + C\||\nabla\eta||\psi_\alpha|\|_{L^{\frac{4}{3}}(D_1)} \\
&\le \frac{1}{4}C(N)\|d\phi_\alpha\|_{L^2(D_{e^{(m_j+2)L}\lambda_\alpha R}(x_\alpha)\setminus D_{e^{m_{j-1}L}\lambda_\alpha R}(x_\alpha))}\|\eta\psi_\alpha\|_{L^4(D_1)} + C\|\nabla\eta\psi_\alpha\|_{L^{\frac{4}{3}}(P_{m_{j-1}}\cup P_{m_j+1})} \\
&\le \frac{1}{4}C(N)\frac{2}{\sqrt{C_1(N)}}\|\eta\psi_\alpha\|_{L^4(D_1)} + C\|\nabla\eta\|_{L^2(P_{m_{j-1}}\cup P_{m_j+1})}\|\psi_\alpha\|_{L^4(P_{m_{j-1}}\cup P_{m_j+1})},
\end{aligned}$$

where the last inequality is from (4.15) and (4.12). Then, taking $C_1(N) = C^2(N) + 1$, by (4.12) and Sobolev embedding, for any $\epsilon > 0$, we have

$$\|\psi_\alpha\|_{L^4(Q_j)} + \|\nabla\psi_\alpha\|_{L^{4/3}(Q_j)} \le C\|\psi_\alpha\|_{L^4(P_{m_{j-1}}\cup P_{m_j+1})} \le C\epsilon,$$

when $\alpha - 1,\ \delta,\ \frac{1}{R}$ are small enough.

So,

$$\|\psi_\alpha\|_{L^4(\Sigma)} + \|\nabla\psi_\alpha\|_{L^{4/3}(\Sigma)} \le \sum_{j=1}^{s_\alpha}(\|\psi_\alpha\|_{L^4(Q_j)} + \|\nabla\psi_\alpha\|_{L^{4/3}(Q_j)}) \le CS\epsilon,$$

which implies the conclusion of the lemma immediately. □

As a corollary of Lemma 4.2, we get the following generalized energy identity for the spinor part.

Corollary 4.3. *Under the assumptions of Lemma 4.2, we have*

$$\lim_{\delta\to 0}\lim_{\alpha\searrow 1} E(\psi_\alpha, D_\delta(0)) = \mu_1^2 E(\xi^1).$$

Proof. By Lemma 4.2, we have

$$\begin{aligned}
&\lim_{\delta\to 0}\lim_{\alpha\searrow 1} E(\psi_\alpha, D_\delta(0)) \\
&= \lim_{\delta\to 0}\lim_{\alpha\searrow 1}\Big(E(\psi_\alpha, D_\delta(0)\setminus D_{\frac{\delta}{2}}(x_\alpha)) + E(\psi_\alpha, D_{\frac{\delta}{2}}(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)) + E(\psi_\alpha, D_{\lambda_\alpha R}(x_\alpha))\Big) \\
&= \lim_{\alpha\searrow 1} E(\psi_\alpha, D_{\lambda_\alpha R}(x_\alpha)) = \lim_{\alpha\searrow 1}\lambda_\alpha^{4-4\alpha}\int_{D_R(0)}|\nu_\alpha|^4 dx = \mu_1^2 E(\xi^1).
\end{aligned}$$

□

Next, we will show that there is no concentration for some kind of stronger weighted energy of the spinor part. The proof is based on some Hardy-type inequality on $\mathbb{R}^2$.

Lemma 4.4. *Under the assumption of Lemma 4.2, there holds*

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\to 1}\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)}\frac{|\psi_\alpha|^2}{|x - x_\alpha|}dx = 0. \tag{4.16}$$

Proof. Without loss of generality, we may assume $x_\alpha = 0$.

The key of the proof is the following Hardy-type inequality on $\mathbb{R}^2$: for any $f \in C_0^\infty(\mathbb{R}^2\setminus\{0\})$, there holds

$$\|\frac{f}{|x|}\|_{L^1(\mathbb{R}^2)} \le \|\nabla f\|_{L^1(\mathbb{R}^2)} \tag{4.17}$$

where the constant 1 is the best possible constant (for a simple proof, see [1]).

We choose the cut-off function $\eta \in C_0^\infty(D_\delta \setminus D_{\lambda_\alpha R})$ such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on $D_{\frac{1}{2}\delta} \setminus D_{2\lambda_\alpha R}$ and

$$|\nabla \eta| \le \frac{C}{\delta} \quad on \quad D_\delta \setminus D_{\frac{1}{2}\delta} \quad and \quad |\nabla \eta| \le \frac{C}{\lambda_n R} \quad on \quad D_{2\lambda_\alpha R} \setminus D_{\lambda_\alpha R}.$$

Taking $f = \eta|\psi_\alpha|^2$ in the inequality (4.17), we have

$$\begin{aligned}
\|\eta \frac{|\psi_\alpha|^2}{|x|}\|_{L^1(\mathbb{R}^2)} &\le \|\nabla(\eta|\psi_\alpha|^2)\|_{L^1(\mathbb{R}^2)} \\
&\le \|2\eta\psi_\alpha \nabla \psi_\alpha\|_{L^1(\mathbb{R}^2)} + \|\nabla\eta|\psi_\alpha|^2\|_{L^1(D_\delta \setminus D_{\lambda_\alpha R})} \\
&\le \|\psi_\alpha\|_{L^4(D_\delta\setminus D_{\lambda_\alpha R})}\|\nabla\psi_\alpha\|_{L^{\frac{4}{3}}(D_\delta\setminus D_{\lambda_\alpha R})} + C\frac{1}{\delta}\||\psi_\alpha|^2\|_{L^1(D_\delta\setminus D_{\frac{1}{2}\delta})} + C\frac{1}{\lambda_\alpha R}\||\psi_\alpha|^2\|_{L^1(D_{2\lambda_\alpha R}\setminus D_{\lambda_\alpha R})} \\
&\le \|\psi_\alpha\|_{L^4(D_\delta\setminus D_{\lambda_\alpha R})}\|\nabla\psi_\alpha\|_{L^{\frac{4}{3}}(D_\delta\setminus D_{\lambda_\alpha R})} + C\|\psi_\alpha\|^2_{L^4(D_\delta\setminus D_{\frac{1}{2}\delta})} + C\|\psi_\alpha\|^2_{L^4(D_{2\lambda_\alpha R}\setminus D_{\lambda_\alpha R})} \\
&\le C\epsilon
\end{aligned}$$

where the last inequality is from (4.12) and Lemma 4.2. Thus,

$$\int_{D_{\frac{1}{2}\delta}\setminus D_{2\lambda_\alpha R}} \frac{|\psi_\alpha|^2}{|x|} dx \le C\epsilon.$$

Combining this with (4.12) again, we get the conclusion of the lemma. □

As an application of Lemma 3.1, we have the following lemma.

Lemma 4.5. *Under the assumption of Lemma 4.2, for any $\lambda_\alpha R \le t_1 \le t_2 \le \delta$, there hold*

$$|x - x_\alpha||\nabla\phi_\alpha| + |x - x_\alpha|^2|\nabla^2\phi_\alpha| + \sqrt{|x - x_\alpha|}|\psi_\alpha| \le C\epsilon,\ \forall\ x \in D_{t_2}(x_\alpha) \setminus D_{t_1}(x_\alpha) \tag{4.18}$$

and

$$\int_{D_{t_2}(x_\alpha)\setminus D_{t_1}(x_\alpha)} |\nabla^2\phi_\alpha||\phi_\alpha - \phi_\alpha^*| dx \le C \int_{D_{4t_2}(x_\alpha)\setminus D_{\frac{1}{2}t_1}(x_\alpha)} |\nabla\phi_\alpha|^2 dx, \tag{4.19}$$

where $\phi_\alpha^*(r) = \frac{1}{2\pi}\int_0^{2\pi} \phi_\alpha(r,\theta)d\theta$ and $C > 0$ is independent of α.

Proof. For any $t_1 \le t \le t_2$, it is easy to see that $\left(\phi_\alpha(x_\alpha + tx), t^{\alpha-1}\sqrt{t}\psi_\alpha(x_\alpha + tx)\right)$ is a α-Dirac-harmonic map which is defined on $D_2(0) \setminus D_{\frac{1}{4}}(0)$. By assumption (4.12) and Lemma 3.1, we have

$$\begin{aligned}
t\|\nabla\phi_\alpha(x_\alpha + tx)\|_{L^\infty(D_1(0)\setminus D_{\frac{1}{2}}(0))} + t^2\|\nabla^2\phi_\alpha(x_\alpha + tx)\|_{L^\infty(D_1(0)\setminus D_{\frac{1}{2}}(0))} &\le C\|\nabla\phi_\alpha\|_{L^2(D_{2t}(x_\alpha)\setminus D_{\frac{t}{4}}(x_\alpha))}, \\
t^{\alpha-1}\sqrt{t}\|\psi_\alpha(x_\alpha + tx)\|_{L^\infty(D_1(0)\setminus D_{\frac{1}{2}}(0))} &\le Ct^{\alpha-1}\|\psi_\alpha\|_{L^4(D_{2t}(x_\alpha)\setminus D_{\frac{t}{4}}(x_\alpha))}.
\end{aligned}$$

Noting that

$$\frac{C}{\sqrt{\mu_{max}}} \le (\lambda_\alpha R)^{\alpha-1} \le t^{\alpha-1} \le 1$$

where $\mu_{max} = \frac{\Lambda}{\epsilon_1}$ (see (2.18)), we obtain

$$t\|\nabla\phi_\alpha(x_\alpha + tx)\|_{L^\infty(D_1(0)\setminus D_{\frac{1}{2}}(0))} + t^2\|\nabla^2\phi_\alpha(x_\alpha + tx)\|_{L^\infty(D_1(0)\setminus D_{\frac{1}{2}}(0))} + \sqrt{t}\|\psi_\alpha(x_\alpha + tx)\|_{L^\infty(D_1(0)\setminus D_{\frac{1}{2}}(0))}$$
$$\leq C\left(\|\nabla\phi_\alpha\|_{L^2(D_{2t}(x_\alpha)\setminus D_{\frac{t}{4}}(x_\alpha))} + \|\psi_\alpha\|_{L^4(D_{2t}(x_\alpha)\setminus D_{\frac{t}{4}}(x_\alpha))}\right),$$

which implies (4.18).

For (4.19), denoting I_α is a positive integer such that

$$2^{I_\alpha - 1}t_1 \leq t_2 \leq 2^{I_\alpha}t_1,$$

then according to Lemma 3.1, we have

$$\begin{aligned}\int_{D_{t_2}(x_\alpha)\setminus D_{t_1}(x_\alpha)} |\nabla^2\phi_\alpha||\phi_\alpha - \phi_\alpha^*|dx &\leq \sum_{i=1}^{I_\alpha}\int_{D_{2^it_1}(x_\alpha)\setminus D_{2^{i-1}t_1}(x_\alpha)} |\nabla^2\phi_\alpha||\phi_\alpha - \phi_\alpha^*|dx \\ &\leq C\sum_{i=1}^{I_\alpha}(2^it_1)^2\|\nabla^2\phi_\alpha\|_{L^\infty(D_{2^it_1}(x_\alpha)\setminus D_{2^{i-1}t_1}(x_\alpha))}2^it_1\|\nabla\phi_\alpha\|_{L^\infty(D_{2^it_1}(x_\alpha)\setminus D_{2^{i-1}t_1}(x_\alpha))} \\ &\leq C\sum_{i=1}^{I_\alpha}\int_{D_{2^{i+1}t_1}(x_\alpha)\setminus D_{2^{i-2}t_1}(x_\alpha)} |\nabla\phi_\alpha|^2dx \leq C\int_{D_{4t_2}(x_\alpha)\setminus D_{\frac{t_1}{2}}(x_\alpha)} |\nabla\phi_\alpha|^2dx.\end{aligned}$$

□

To estimate the energy of the map part, we first prove that the tangential energy on the neck domain is converging to zero.

Lemma 4.6. *Under the assumptions of Lemma 4.2, there holds:*

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\searrow 1}\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)} |x - x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx = 0;$$

Proof. Denote

$$\phi_\alpha^*(r) = \frac{1}{2\pi}\int_0^{2\pi}\phi_\alpha(r,\theta)d\theta.$$

Then by Lemma 3.1, we have

$$\begin{aligned}\|\phi_\alpha - \phi_\alpha^*\|_{L^\infty(D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha))} &= \sup_{\lambda_\alpha R\leq t\leq\delta}\|\phi_\alpha - \phi_\alpha^*\|_{D_{2t}(x_\alpha)\setminus D_t(x_\alpha)} \\ &\leq \sup_{\lambda_\alpha R\leq t\leq\delta}\|\phi_\alpha\|_{Osc(D_{2t}(x_\alpha)\setminus D_t(x_\alpha))} \\ &\leq C\sup_{\lambda_\alpha R\leq t\leq\delta}\|\nabla\phi_\alpha\|_{L^2(D_{4t}(x_\alpha)\setminus D_{\frac{1}{2}t}(x_\alpha))} \leq C\epsilon,\end{aligned}$$

where the last inequality follows from (4.12).

Using equation (2.14), we have

$$\begin{aligned}\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)} -\Delta\phi_\alpha(\phi_\alpha - \phi_\alpha^*)dx = \int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}\Big[&(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2} + A(\phi_\alpha)(d\phi_\alpha, d\phi_\alpha) \\ &- \frac{Re\left(P(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_g\phi_\alpha|^2)^{\alpha-1}}\Big](\phi_\alpha - \phi_\alpha^*)dx\end{aligned}$$

On the one hand, integrating by parts, we get

$$\begin{aligned}
left &= \int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)} -\Delta\phi_\alpha(\phi_\alpha-\phi^*_\alpha)dx\\
&= \int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)} (|\nabla\phi_\alpha|^2-\frac{\partial\phi_\alpha}{\partial r}\frac{\partial\phi^*_\alpha}{\partial r})dx - \int_{\partial(D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha))}\frac{\partial\phi_\alpha}{\partial r}(\phi_\alpha-\phi^*_\alpha)dx\\
&\geq \int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)} (|\nabla\phi_\alpha|^2-|\frac{\partial\phi_\alpha}{\partial r}|^2)dx - \int_{\partial(D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha))}\frac{\partial\phi_\alpha}{\partial r}(\phi_\alpha-\phi^*_\alpha).
\end{aligned}$$

On the other hand, by Lemma 3.2 and Lemma 4.5, there holds

$$\begin{aligned}
right &\leq C\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)} (\alpha-1)|\nabla^2\phi_\alpha||\phi_\alpha-\phi^*_\alpha|dx + C\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}|\nabla\phi_\alpha|^2|\phi_\alpha-\phi^*_\alpha|dx\\
&\quad + C\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}|\nabla\phi_\alpha||\psi_\alpha|^2|\phi_\alpha-\phi^*_\alpha|dx\\
&\leq C(\alpha-1)\int_{D_{2\delta}(x_\alpha)}|\nabla\phi_\alpha|^2dx + C\epsilon\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}(|\nabla\phi_\alpha|^2+|\psi_\alpha|^4)dx \leq C((\alpha-1)+\epsilon).
\end{aligned}$$

Combing these together, we arrived

$$\begin{aligned}
\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx &= \int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}(|\nabla\phi_\alpha|^2-|\frac{\partial\phi_\alpha}{\partial r}|^2)dx\\
&\leq \int_{\partial(D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha))}\frac{\partial\phi_\alpha}{\partial r}(\phi_\alpha-\phi^*_\alpha) + C((\alpha-1)+\epsilon).
\end{aligned}$$

As for the boundary term, by trace theory, we have

$$\begin{aligned}
\int_{\partial D_\delta(x_\alpha)}\frac{\partial\phi_\alpha}{\partial r}(\phi_\alpha-\phi^*_\alpha) &\leq C\epsilon\int_{\partial D_\delta(x_\alpha)}|\frac{\partial\phi_\alpha}{\partial r}|\\
&\leq C\epsilon(\|\nabla\phi_\alpha\|_{L^2(D_\delta(x_\alpha)\setminus D_{\frac{1}{2}\delta}(x_\alpha))} + \delta\|\nabla^2\phi_\alpha\|_{L^2(D_\delta(x_\alpha)\setminus D_{\frac{1}{2}\delta}(x_\alpha))})\\
&\leq C\epsilon\|\nabla\phi_\alpha\|_{L^2(D_{2\delta}(x_\alpha)\setminus D_{\frac{1}{4}\delta}(x_\alpha))} \leq C\epsilon,
\end{aligned}$$

where the third inequality follows from Lemma 3.1.

Similarly,

$$\int_{\partial D_{\lambda_\alpha R}(x_\alpha)}\frac{\partial\phi_\alpha}{\partial r}(\phi_\alpha-\phi^*_\alpha) \leq C\epsilon.$$

Then the conclusion of the lemma follows immediately. □

Combining Lemma 4.6 with Lemma 3.2, we get

Lemma 4.7. *Under the assumptions of Lemma 4.2, there holds*

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\searrow 1}\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx = 0.$$

Denote

$$F_\alpha(t) = \int_{D_{\lambda^t_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx,$$

$$F_{r,\alpha}(t)=\int_{D_{\lambda_\alpha^t}(x_\alpha)\setminus D_{\lambda_\alpha^{t_0}}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\frac{\partial\phi_\alpha}{\partial r}|^2dx$$

and

$$F_{\theta,\alpha}(t)=\int_{D_{\lambda_\alpha^t}(x_\alpha)\setminus D_{\lambda_\alpha^{t_0}}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx,\ 0<t\le t_0<1.$$

By Corollary 3.5, for $t\in[\epsilon,t_0]$, we have

$$\begin{aligned}&(1-\frac{1}{2\alpha})\frac{d}{dt}F_{r,\alpha}(t)-\frac{1}{2\alpha}\frac{d}{dt}F_{\theta,\alpha}\\&=\frac{\alpha-1}{\alpha}\log\lambda_\alpha F_\alpha(t)+\frac{1}{2\alpha}\lambda_\alpha^t\log\lambda_\alpha\int_{\partial D_{\lambda_\alpha^t}(x_\alpha)}\langle\psi_\alpha,|x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle+O(\lambda_\alpha^t\log\lambda_\alpha).\end{aligned}$$

Then

$$\begin{aligned}(1-\frac{1}{2\alpha})F_{r,\alpha}(t)-\frac{1}{2\alpha}F_{\theta,\alpha}=&\frac{1}{2}\int_{t_0}^t[-\frac{1}{\alpha}\log\lambda_\alpha^{2(1-\alpha)}F_\alpha(t)+O(\lambda_\alpha^t\log\lambda_\alpha)]dt\\&+\frac{1}{2\alpha}\int_{D_{\lambda_\alpha^t}(x_\alpha)\setminus D_{\lambda_\alpha^{t_0}}(x_\alpha)}\langle\psi_\alpha,|x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle.\end{aligned}\tag{4.20}$$

It is easy to check that

$$\frac{1}{2}\int_{t_0}^t[-\frac{1}{\alpha}\log\lambda_\alpha^{2(1-\alpha)}F_\alpha(t)+O(\lambda_\alpha^t\log\lambda_\alpha)]dt$$

is compact in $C^0([\epsilon,t_0])$. By Lemma 4.2,

$$\frac{1}{2\alpha}\int_{D_{\lambda_\alpha^t}(x_\alpha)\setminus D_{\lambda_\alpha^{t_0}}(x_\alpha)}\langle\psi_\alpha,|x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle\to 0\ in\ C^0([\epsilon,t_0]).$$

Combing these with the fact that $F_{\theta,\alpha}(t)\to 0$ by Lemma 4.7, we know that the sequences $(1-\frac{1}{2\alpha})F_{r,\alpha}(t)-\frac{1}{2\alpha}F_{\theta,\alpha}$, $\{F_\alpha(t)\}$, $\{F_{r,\alpha}(t)\}$ and $\{F_{\theta,\alpha}(t)\}$ are compact in the $C^0([\epsilon,t_0])$ topology for any $\epsilon>0$. Thus, there exist two functions F and F_r which belong to $C^0([\epsilon,t_0])$ such that

$$F_\alpha\to F\quad and\quad F_{r,\alpha}\to F_r\quad in\ C^0([\epsilon,t_0])$$

as $\alpha\searrow 1$.

Lemma 4.8. *The functionals $F(t)$ and $F_r(t)$ satisfy the following relation:*

$$F_r(t)=\mu^{t_0-t}F(t_0)-F(t_0)\ for\ any\ 0<t\le t_0<1.\tag{4.21}$$

Moreover, we have:

$$\lim_{t_0\to 1^-}F(t_0)=\mu E(\sigma^1(x)),\tag{4.22}$$

where $\mu:=\lim_{\alpha\searrow 1}(\lambda_\alpha)^{2-2\alpha}$.

Proof. With the help of Lemma 4.6 and equality (4.20), the proof of this lemma is similar to Lemma 3.4 in [26]. Noting that

$$|\widetilde{\nabla}_{\partial\theta}\psi_\alpha|\le C(|\frac{\partial\psi_\alpha}{\partial\theta}|+|\frac{\partial\phi_\alpha}{\partial\theta}||\psi_\alpha|),$$

we have

$$
\begin{aligned}
&\left|\frac{1}{2\alpha}\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)}\langle\psi_\alpha, |x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle\right| \\
&\le C\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)}(|\psi_\alpha||x-x_\alpha|^{-1}|\frac{\partial\psi_\alpha}{\partial\theta}| + |\psi_\alpha|^2|x-x_\alpha|^{-1}|\frac{\partial\phi_\alpha}{\partial\theta}|) \\
&\le C\|\psi_\alpha\|_{L^4(D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha))}\||x-x_\alpha|^{-1}|\frac{\partial\psi_\alpha}{\partial\theta}|\|_{L^{\frac{4}{3}}(D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha))} \\
&\quad + C\|\psi_\alpha\|^2_{L^4(D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha))}\||x-x_\alpha|^{-1}|\frac{\partial\phi_\alpha}{\partial\theta}|\|_{L^2(D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha))} \to 0
\end{aligned}
\tag{4.23}
$$

as $\alpha \searrow 1$, $R \to \infty$, $\delta \to 0$, where we used Lemma 4.2.

Letting $\alpha \searrow 1$ in (4.20) and using (4.23), we get

$$F_r(t) = -\log\mu\int_{t_0}^{t} F(s)ds \; for \; any \; 0 < t < t_0. \tag{4.24}$$

Since

$$F_\alpha(t) = F_{r,\alpha}(t) + F_{\theta,\alpha}(t) + F_\alpha(t_0),$$

letting $\alpha \searrow 1$, Lemma 4.7 yields

$$F(t) = F_r(t) + F(t_0).$$

Then,

$$F_r(t) = -\log\mu\int_{t_0}^{t} F(s)ds = -\log\mu\int_{t_0}^{t}(F_r(s) + F(t_0))\,ds,$$

which implies $F_r(t) \in C^1$ and

$$\frac{d}{dt}F_r(t) = -\log\mu(F_r(t) + F(t_0)).$$

Thus,

$$F_r(t) = \mu^{t_0-t}F(t_0) - F(t_0).$$

For equation (4.22), by Corollary 3.5, integrating (3.12) from $\lambda_\alpha R$ to $\lambda_\alpha^{t_0}$, we have

$$\begin{aligned}
&F_\alpha(t_0) - \int_{D_{\lambda_\alpha R}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx \\
&\leq C \int_{D_{\lambda_\alpha^{t_0}}(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2 dx \\
&\quad + C\left|\int_{D_{\lambda_\alpha^{t_0}}(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)} \langle\psi_\alpha, |x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle dx\right| + C\int_{\lambda_\alpha R}^{\lambda_\alpha^{t_0}} \frac{\alpha-1}{r}dr + C(\lambda_\alpha^{t_0} - \lambda_\alpha R) \\
&\leq C \int_{D_{\lambda_\alpha^{t_0}}(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2 dx \\
&\quad + C\left|\int_{D_{\lambda_\alpha^{t_0}}(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)} \langle\psi_\alpha, |x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle dx\right| \\
&\quad + C\left((t_0-1)(\alpha-1)\log\lambda_\alpha - (\alpha-1)\log R\right) + C(\lambda_\alpha^{t_0} - \lambda_\alpha R).
\end{aligned} \tag{4.25}$$

Combining this with Lemma 4.7 and (4.23), we get

$$\begin{aligned}
\lim_{t_0\to 1^-} F(t_0) &= \lim_{R\to\infty}\lim_{\alpha\searrow 1}\int_{D_{\lambda_\alpha R}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx \\
&= \lim_{R\to\infty}\lim_{\alpha\searrow 1}\int_{D_R(0)} (\sigma_\alpha\lambda_\alpha^2 + |\nabla_{g_\alpha(x_\alpha+\lambda_\alpha x)}u_\alpha|^2)^{\alpha-1}(\lambda_\alpha)^{2-2\alpha}|\nabla u_\alpha|^2 dx \\
&= \mu\lim_{R\to\infty}\int_{D_R(0)} |\nabla\sigma^1|^2 dx = \mu E(\sigma^1(x)).
\end{aligned}$$

□

Lemma 4.9. *Under the assumptions of Lemma 4.2, for any $t \in (0,1)$, there holds*

$$\lim_{\alpha\searrow 1}\int_{D_{\lambda_\alpha^t}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx = \mu^{2-t}E(\sigma^1).$$

Proof. By Lemma 4.7 and Lemma 4.8, we have

$$\lim_{\alpha\searrow 1}\int_{D_{\lambda_\alpha^t}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx = F_r(t) + F(t_0) = \mu^{t_0-t}F(t_0).$$

Then the conclusion of the lemma follows from letting $t_0 \nearrow 1$ and Lemma 4.8. □

In the end of this section, we complete the proof of Theorem 4.1.

Proof of Theorem 4.1. By Corollary 3.5, we have

$$
\begin{aligned}
&\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx\\
&\le C\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx\\
&\quad+C\left|\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)}\langle\psi_\alpha,|x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle dx\right|+C\int_{\lambda_\alpha^t}^{\delta}\frac{\alpha-1}{r}dr+C(\delta-\lambda_\alpha^t)\\
&\le C\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx\\
&\quad+C\left|\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)}\langle\psi_\alpha,|x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\partial_\theta}\psi_\alpha\rangle dx\right|\\
&\quad+C\left((\alpha-1)\log\delta-t(\alpha-1)\log\lambda_\alpha\right)+C(\delta-\lambda_\alpha^t).
\end{aligned}\tag{4.26}
$$

Combining (4.26) with (4.23), we have

$$\lim_{\delta\to0}\lim_{t\to0}\lim_{\alpha\searrow1}\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx=0.$$

By Lemma 4.9, we get

$$
\begin{aligned}
&\lim_{\delta\to0}\lim_{\alpha\searrow1}\int_{D_\delta(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx\\
&=\lim_{\delta\to0}\lim_{t\to0}\lim_{\alpha\searrow1}\left(\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx+\int_{D_{\lambda_\alpha^t}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx\right)\\
&=\mu^2E(\sigma^1).
\end{aligned}
$$

Combining the above equality with Corollary 4.3, we finished the proof of Theorem 4.1. □

Proof of Theorem 2.2. It is easy to see that Theorem 2.2 is a consequence of Theorem 4.1. □

5. Decay estimates and refined neck analysis

In this section, we shall study the asymptotic behaviour of the necks and prove Theorem 2.3. Firstly, we state the main theorem in this section.

Theorem 5.1. *Under the assumptions of Theorem 4.1, if we assume that there is only one bubble* (σ^1,ξ^1)*, which is a Dirac-harmonic sphere, let* $\nu=\lim_{\alpha\searrow1}(\lambda_\alpha)^{-\sqrt{\alpha-1}}$. *We have*

(1) *when* $\nu=1$*, the set* $\phi(D_1(0))\cup\sigma^1(S^2)$ *is a connected set in* N*;*

(2) *when* $\nu\in(1,\infty)$*, the set* $\phi(D_1(0))$ *and* $\sigma^1(S^2)$ *are connected by a geodesic* γ *in* N *with length*

$$L(\gamma)=\sqrt{\frac{E(\sigma^1)}{\pi}}\log\nu;$$

(3) *when $\nu=\infty$, the neck contains at least an infinite length geodesic curve.*

There are two main crucial steps in proving Theorem 5.1. The first one is to apply the three-circle method developed for the class of integro-differential equations considered in Section 3 to the general α-Dirac-harmonic map system to get some decay of tangential neck energies of both the map part and the spinor part. Here, we get a decay at the speed of $\alpha-1$ rather than the exponential decay as in the (approximate) Dirac-harmonic map case [28, 19]. The key point is to write the general α-Dirac-harmonic system into the special forms given in our three-circle result - Theorem 3.8. The treatment of the two bad error terms mentioned in Section 2, namely, the second derivative term (2.20) and the curvature term (2.21), is complicated and subtle, see Lemma 5.2. The second step is to derive the decay of some weighted neck energy of the spinor part. The key point here is to apply some Hardy-type inequality to derive a differential inequality on the neck domain, see Lemma 5.4 and Lemma 5.5. This step is tricky.

As a first step towards Theorem 5.1, by applying the three circle theorem given in Section 3, we shall establish the following decay estimate for the tangential energies of both the map part and the spinor part:

Lemma 5.2. (Decay of tangential energies) *If there is no energy concentration for the sequence $(\phi_\alpha,\psi_\alpha)$ in the region $D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)$, i.e.* (4.12) *holds, we have*

$$\left(\int_{P_i}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx\right)^{\frac12}+\left(\int_{P_i}|x-x_\alpha|^{-\frac43}|\frac{\partial\psi_\alpha}{\partial\theta}|^{\frac43}dx\right)^{\frac34}\le\left((\alpha-1)+e^{-\frac{iL}{2}}+e^{-\frac{(k_\alpha-i)L}{2}}\right)o(\alpha,\delta,R),$$

where $P_i=D_{e^{(i+1)L}\lambda_\alpha R}(x_\alpha)\setminus D_{e^{iL}\lambda_\alpha R}(x_\alpha)$ and $\lim_{\delta\to0}\lim_{R\to\infty}\lim_{\alpha\searrow1}o(\alpha,\delta,R)=0$.

Proof. Since (4.12) holds, by (4.18), we have

$$|x-x_\alpha||\nabla\phi_\alpha|+|x-x_\alpha|^2|\nabla^2\phi_\alpha|+\sqrt{|x-x_\alpha|}|\psi_\alpha|\le C\epsilon,\ \forall\ x\in D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha). \tag{5.1}$$

Denote

$$\phi_\alpha^*(r)=\frac{1}{2\pi}\int_0^{2\pi}\phi_\alpha(r,\theta)d\theta,\quad \psi_\alpha^*(r)=\frac{1}{2\pi}\int_0^{2\pi}\psi_\alpha(r,\theta)d\theta,$$

and

$$u=\phi_\alpha-\phi_\alpha^*,\ v=\psi_\alpha-\psi_\alpha^*.$$

Next, we use the same method as in [28] to compute the equation for $(\phi_\alpha-\phi_\alpha^*,\psi_\alpha-\psi_\alpha^*)$. By equation (2.14), we have

$$\begin{aligned}\Delta\phi_\alpha^* &= \frac{1}{2\pi}\int_0^{2\pi}\Delta\phi_\alpha d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}-(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2}-A(\phi_\alpha)(d\phi_\alpha,d\phi_\alpha)+\frac{Re\left(P(\mathcal{A}(d\phi_\alpha(e_\gamma),e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right)}{\alpha(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}}d\theta\\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

Computing directly, we have

$$\begin{aligned}
-\mathrm{II} &= \frac{1}{2\pi}\int_0^{2\pi} A(\phi_\alpha)(d\phi_\alpha, d\phi_\alpha) - A(\phi_\alpha^*)(d\phi_\alpha, d\phi_\alpha) + A(\phi_\alpha^*)(d\phi_\alpha, d\phi_\alpha) - A(\phi_\alpha^*)(d\phi_\alpha^*, d\phi_\alpha^*) \\
&\quad + A(\phi_\alpha^*)(d\phi_\alpha^*, d\phi_\alpha^*) d\theta \\
&= A(\phi_\alpha^*)(d\phi_\alpha^*, d\phi_\alpha^*) + \frac{1}{2\pi}\int_0^{2\pi} A^4(\phi_\alpha - \phi_\alpha^*) + A^5\nabla(\phi_\alpha - \phi_\alpha^*) d\theta,
\end{aligned}$$

where A^i may change from line to line and it just stands for some expression satisfying

$$|A^4| \leq C(N)(|d\phi_\alpha|^2 + |d\phi_\alpha||\psi_\alpha|^2), \ |A^5| \leq C(N)(|d\phi_\alpha| + |\psi_\alpha|^2).$$

Similarly,

$$\begin{aligned}
\mathrm{III} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{Re\left(P(\phi_\alpha)(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha); \psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} - \frac{Re\left(P(\phi_\alpha^*)(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha); \psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} d\theta \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{Re\left(P(\phi_\alpha^*)(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha); \psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} - \frac{Re\left(P(\phi_\alpha^*)(\mathcal{A}(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha); \psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} d\theta \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{Re\left(P(\phi_\alpha^*)(\mathcal{A}(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha); \psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} - \frac{Re\left(P(\phi_\alpha^*)(\mathcal{A}(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha^*); \psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} d\theta \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{Re\left(P(\phi_\alpha^*)(\mathcal{A}(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha^*); \psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} - \frac{Re\left(P(\phi_\alpha^*)(\mathcal{A}(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha^*); \psi_\alpha^*)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} d\theta \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} Re\left(P(\phi_\alpha^*)(\mathcal{A}(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha^*); \psi_\alpha^*)\right)\left(\frac{1}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} - \frac{1}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha^*|^2)^{\alpha-1}}\right) d\theta \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} Re\left(P(\phi_\alpha^*)(\mathcal{A}(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha^*); \psi_\alpha^*)\right)\left(\frac{1}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha(x)}\phi_\alpha^*|^2)^{\alpha-1}} - \frac{1}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha(x_\alpha)}\phi_\alpha^*|^2)^{\alpha-1}}\right) d\theta \\
&\quad + \frac{Re\left(P(\phi^*)(\mathcal{A}(\phi^*)(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha^*); \psi_\alpha^*)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha(x_\alpha)}\phi_\alpha^*|^2)^{\alpha-1}} \\
&= \frac{Re\left(P(\phi^*)(\mathcal{A}(\phi^*)(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha^*); \psi_\alpha^*)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha(x_\alpha)}\phi_\alpha^*|^2)^{\alpha-1}} + \frac{1}{2\pi}\int_0^{2\pi} h_1 d\theta \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} A^4(\phi_\alpha - \phi_\alpha^*) + A^5\nabla(\phi_\alpha - \phi_\alpha^*) + \frac{1}{2\pi} Re\int_0^{2\pi} A^6(\psi_\alpha - \psi_\alpha^*) d\theta,
\end{aligned}$$

where

$$|A^6| \leq C(N)|d\phi_\alpha||\psi_\alpha|,$$

$$h_1 = Re\left(P(\phi_\alpha^*)(\mathcal{A}(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha^*); \psi_\alpha^*)\right)\left(\frac{1}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha(x)}\phi_\alpha^*|^2)^{\alpha-1}} - \frac{1}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha(x_\alpha)}\phi_\alpha^*|^2)^{\alpha-1}}\right)$$

and we used the estimate

(5.2)

$$Re\left(P(\phi_\alpha^*)(\mathcal{A}(\phi_\alpha^*)(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha^*); \psi_\alpha^*)\right)\left(\frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} - \frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha^*|^2)^{\alpha-1}}\right) = A^5\nabla(\phi_\alpha - \phi_\alpha^*).$$

In fact, if $|\nabla_{g_\alpha}\phi_\alpha| = |\nabla_{g_\alpha}\phi^*_\alpha|$, then the estimate holds immediately for $A^5 = 0$. If $|\nabla_{g_\alpha}\phi_\alpha| \neq |\nabla_{g_\alpha}\phi^*_\alpha|$, without loss of generality, we assume $|\nabla_{g_\alpha}\phi^*_\alpha| \leq |\nabla_{g_\alpha}\phi_\alpha|$. Then

$$
\begin{aligned}
0 &\leq \left(\frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi^*_\alpha|^2)^{\alpha-1}} - \frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}}\right)\\
&= \frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}}\left([(1+\frac{|\nabla_{g_\alpha}\phi_\alpha|^2 - |\nabla_{g_\alpha}\phi^*_\alpha|^2}{\sigma_\alpha + |\nabla_{g_\alpha}\phi^*_\alpha|^2})^\alpha - 1](\sigma_\alpha + |\nabla_{g_\alpha}\phi^*_\alpha|^2) + |\nabla_{g_\alpha}\phi^*_\alpha|^2 - |\nabla_{g_\alpha}\phi_\alpha|^2\right)\\
&\leq \frac{|\nabla_{g_\alpha}\phi_\alpha|^2 - |\nabla_{g_\alpha}\phi^*_\alpha|^2}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}}, \qquad (5.3)
\end{aligned}
$$

where in the last estimate we used the inequality

$$(1+x)^\alpha - 1 \leq 2x,\ \forall x \in [0, b_\alpha]$$

when $\alpha - 1$ is small enough, where b_α satisfies $\limsup_{\alpha\searrow 1}(1+b_\alpha)^{\alpha-1} \leq C < \infty$. Taking

$$b_\alpha = \frac{|\nabla_{g_\alpha}\phi_\alpha|^2 - |\nabla_{g_\alpha}\phi^*_\alpha|^2}{\sigma_\alpha + |\nabla_{g_\alpha}\phi^*_\alpha|^2}$$

and noting that, by Lemma 3.2, there holds

$$
\begin{aligned}
\limsup_{\alpha\searrow 1}(1+\frac{|\nabla_{g_\alpha}\phi_\alpha|^2 - |\nabla_{g_\alpha}\phi^*_\alpha|^2}{\sigma_\alpha + |\nabla_{g_\alpha}\phi^*_\alpha|^2})^{\alpha-1} &\leq \limsup_{\alpha\searrow 1}(1+\frac{|\nabla_{g_\alpha}\phi_\alpha|^2}{\sigma_\alpha})^{\alpha-1}\\
&= \limsup_{\alpha\searrow 1}\sigma_\alpha^{1-\alpha}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1} \leq C < \infty,
\end{aligned}
$$

it is easy to see that (5.3) follows immediately.

According to (5.3), we can write that

$$\left(\frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi^*_\alpha|^2)^{\alpha-1}} - \frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}}\right) = \widetilde{A}^5\frac{|\nabla_{g_\alpha}\phi_\alpha|^2 - |\nabla_{g_\alpha}\phi^*_\alpha|^2}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}}$$

where $0 < \widetilde{A}^5 \leq 1$ and then

$$
\begin{aligned}
&Re\left(P(\phi^*_\alpha)(\mathcal{A}(\phi^*_\alpha)(d\phi^*_\alpha(e_\gamma), e_\gamma\cdot\psi^*_\alpha);\psi^*_\alpha)\right)\left(\frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi^*_\alpha|^2)^{\alpha-1}} - \frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}}\right)\\
&= Re\left(P(\phi^*_\alpha)(\mathcal{A}(\phi^*_\alpha)(d\phi^*_\alpha(e_\gamma), e_\gamma\cdot\psi^*_\alpha);\psi^*_\alpha)\right)\widetilde{A}^5\frac{|\nabla_{g_\alpha}\phi_\alpha|^2 - |\nabla_{g_\alpha}\phi^*_\alpha|^2}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}}\\
&= Re\left(P(\phi^*_\alpha)(\mathcal{A}(\phi^*_\alpha)(d\phi^*_\alpha(e_\gamma), e_\gamma\cdot\psi^*_\alpha);\psi^*_\alpha)\right)\widetilde{A}^5\frac{\langle\nabla_{g_\alpha}\phi_\alpha, \nabla_{g_\alpha}(\phi_\alpha - \phi^*_\alpha)\rangle - \langle\nabla_{g_\alpha}\phi^*_\alpha, \nabla_{g_\alpha}(\phi_\alpha - \phi^*_\alpha)\rangle}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}}\\
&= A^5\nabla(\phi_\alpha - \phi^*_\alpha),
\end{aligned}
$$

which implies that (5.2) holds.

Therefore, we get

$$
\begin{aligned}
\Delta\phi^*_\alpha = &-A(\phi^*_\alpha)(d\phi^*_\alpha, d\phi^*_\alpha) + \frac{Re\left(P(\phi^*)(\mathcal{A}(\phi^*)(d\phi^*_\alpha(e_\gamma), e_\gamma\cdot\psi^*_\alpha);\psi^*_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha(x_\alpha)}\phi^*_\alpha|^2)^{\alpha-1}} - \frac{1}{2\pi}\int_0^{2\pi}(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2}d\theta\\
&+ \frac{1}{2\pi}\int_0^{2\pi}h_1 d\theta + \frac{1}{2\pi}\int_0^{2\pi}A^4(\phi_\alpha - \phi^*_\alpha) + A^5\nabla(\phi_\alpha - \phi^*_\alpha) + \frac{1}{2\pi}Re\int_0^{2\pi}A^6(\psi_\alpha - \psi^*_\alpha)d\theta.
\end{aligned}
$$

Using the same method, we get

$$\begin{aligned}\Delta(\phi_\alpha-\phi_\alpha^*) &= A^1(\phi_\alpha-\phi_\alpha^*)+A^2\nabla(\phi_\alpha-\phi_\alpha^*)+Re(A^3(\psi_\alpha-\psi_\alpha^*))\\ &\quad+\frac{1}{2\pi}\int_0^{2\pi}A^4(\phi_\alpha-\phi_\alpha^*)+A^5\nabla(\phi_\alpha-\phi_\alpha^*)+Re(A^6(\psi_\alpha-\psi_\alpha^*))d\theta\\ &\quad+h_1-\frac{1}{2\pi}\int_0^{2\pi}h_1d\theta-(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2}+\frac{1}{2\pi}\int_0^{2\pi}(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2}d\theta,\end{aligned}\tag{5.4}$$

and

$$\begin{aligned}\not\partial(\psi_\alpha-\psi_\alpha^*) &= B^1(\phi_\alpha-\phi_\alpha^*)+B^2\nabla(\phi_\alpha-\phi_\alpha^*)+B^3(\psi_\alpha-\psi_\alpha^*)\\ &\quad+\frac{1}{2\pi}\int_0^{2\pi}B^4(\phi_\alpha-\phi_\alpha^*)+B^5\nabla(\phi_\alpha-\phi_\alpha^*)+B^6(\psi_\alpha-\psi_\alpha^*)d\theta,\end{aligned}\tag{5.5}$$

where $A^i,\ B^i,\ i=1,...,6$ satisfy

$$|A^1|+|A^4|\le C(N)(|d\phi_\alpha|^2+|d\phi_\alpha||\psi_\alpha|^2),\ |A^3|+|A^6|+|B^1|+|B^4|\le C(N)|d\phi_\alpha||\psi_\alpha|$$
$$|A^2|+|A^5|+|B^3|+|B|^6\le C(N)(|d\phi_\alpha|+|\psi_\alpha|^2),\ |B^2|+|B^5|\le C(N)|\psi_\alpha|.$$

By (5.3) and Lemma 3.2, we have

$$\begin{aligned}|h_1| &\le C|\nabla\phi_\alpha^*||\psi_\alpha^*|^2\left|\frac{|\nabla_{g_\alpha(x)}\phi_\alpha^*|^2-|\nabla_{g_\alpha(x_\alpha)}\phi_\alpha^*|^2}{(\sigma_\alpha+|\nabla\phi_\alpha^*|^2)^\alpha}\right|\\ &= C|\nabla\phi_\alpha^*||\psi_\alpha^*|^2\left|\frac{\left(g_\alpha^{\beta\gamma}(x)-g_\alpha^{\beta\gamma}(x_\alpha)\right)\frac{\partial\phi_\alpha^*}{\partial x^\beta}\frac{\partial\phi_\alpha^*}{\partial x^\gamma}}{(\sigma_\alpha+|\nabla\phi_\alpha^*|^2)^\alpha}\right|\\ &\le C|\nabla\phi_\alpha^*||\psi_\alpha^*|^2\left|\frac{|x-x_\alpha||\nabla\phi_\alpha^*|^2}{(\sigma_\alpha+|\nabla\phi_\alpha^*|^2)^\alpha}\right|\le C|x-x_\alpha||\nabla\phi_\alpha^*||\psi_\alpha^*|^2,\end{aligned}$$

which implies

$$\begin{aligned}\||x-x_\alpha|h_1\|_{L^2(P_i)} &\le \||x-x_\alpha|^2|\nabla\phi_\alpha^*||\psi_\alpha^*|^2\|_{L^2(P_i)}\\ &\le e^{iL}\lambda_\alpha Ro(\alpha,\delta,R)=e^{-(k_\alpha-i)L}\delta o(\alpha,\delta,R)=e^{-(k_\alpha-i)L}o(\alpha,\delta,R),\end{aligned}\tag{5.6}$$

where we used the fact that

$$|x-x_\alpha||\nabla\phi_\alpha^*|+\sqrt{|x-x_\alpha|}|\psi_\alpha^*|\le\frac{1}{2\pi}\int_0^{2\pi}\left(|x-x_\alpha||\nabla\phi_\alpha|+\sqrt{|x-x_\alpha|}|\psi_\alpha|\right)d\theta=o(\alpha,\delta,R),$$

which follows from (5.1).

Note that

$$\begin{aligned}&|x|^2(|A^1|+|A^4|)+|x|^{\frac{3}{2}}(|A^3|+|A^6|+|B^1|+|B^4|)\\ &+|x|(|A^2|+|A^5|+|B^3|+|B^6|)+|x|^{\frac{1}{2}}(|B^2|+|B^5|)\le C\epsilon.\end{aligned}\tag{5.7}$$

Substituting $u=\phi_\alpha-\phi_\alpha^*$, $v=\psi_\alpha-\psi_\alpha^*$, $f_2=0$ and

$$f_1=h_1-\frac{1}{2\pi}\int_0^{2\pi}h_1d\theta-(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2}+\frac{1}{2\pi}\int_0^{2\pi}2(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2}d\theta$$

in Corollary 3.10 and noting that

$$\begin{aligned}\||x-x_\alpha|f_1\|_{L^2(P_i)} &\leq C(\alpha-1)\||x-x_\alpha||\nabla^2\phi_\alpha|\|_{L^2(P_i)}+C\||x-x_\alpha||h_1|\|_{L^2(P_i)}\\ &\leq C(\alpha-1)\|\nabla\phi_\alpha\|_{L^2(P_{i-1}\cup P_i\cup P_{i+1})}+C\||x-x_\alpha||h_1|\|_{L^2(P_i)}\\ &\leq \Big((\alpha-1)+e^{-(k_\alpha-i)L}\Big)o(\alpha,\delta,R),\ i=1,...,k_\alpha,\end{aligned}$$

where we used (5.6) and Lemma 3.1, we obtain the energy decay in the θ-direction,

$$\begin{aligned}&\|r^{-1}\frac{\partial\phi_\alpha}{\partial\theta}\|_{L^2(P_i)}+\|r^{-1}\frac{\partial\psi_\alpha}{\partial\theta}\|_{L^{\frac{4}{3}}(P_i)}\\ &\leq \|\nabla u\|_{L^2(P_i)}+\|\nabla v\|_{L^{\frac{4}{3}}(P_i)}\\ &\leq C(\alpha-1)\max_{j=1,...,k_\alpha}\|\nabla\phi_\alpha\|_{L^2(P_j)}+C(F_0^{1/2}(u,v)+F_{k_\alpha}^{1/2}(u,v))(e^{-\frac{1}{2}(k_\alpha-i)L}+e^{-\frac{1}{2}iL})\\ &\leq C\Big((\alpha-1)+e^{-\frac{1}{2}(k_\alpha-i)L}+e^{-\frac{1}{2}iL}\Big)o(\alpha,\delta,R),\end{aligned}\tag{5.8}$$

where we used the fact that

$$F_0^{1/2}(u,v)+F_{k_\alpha}^{1/2}(u,v)=o(\alpha,\delta,R),$$

which follows from the Poincaré inequality and (4.12). We finished the proof of this lemma. □

As a direct corollary of Lemma 5.2, we have

Corollary 5.3. *Under the assumption of Lemma 5.2, assume $\nu>1$, then for $t_\alpha\in[t_1,t_2]$ where $0<t_1\leq t_2<1$, we have*

$$\lim_{\alpha\searrow 1}\frac{1}{\alpha-1}\left(\||x-x_\alpha|^{-1}|\frac{\partial\phi_\alpha}{\partial\theta}|\|_{L^2(D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha))}+\||x-x_\alpha|^{-1}|\frac{\partial\psi_\alpha}{\partial\theta}|\|_{L^{\frac{4}{3}}(D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha))}\right)=0,$$

where $K>1$ is any fixed positive constant.

Proof. Let $i_1,i_2\in[0,k_\alpha]$ be two integers such that

$$e^{i_1L}\lambda_\alpha R\leq\frac{1}{K}\lambda_\alpha^{t_\alpha}<e^{(i_1+1)L}\lambda_\alpha R\ \text{ and }\ e^{i_2L}\lambda_\alpha R\leq K\lambda_\alpha^{t_\alpha}<e^{(i_2+1)L}\lambda_\alpha R.$$

Then we have

$$e^{-i_1L}\leq KRe^L\lambda_\alpha^{1-t_\alpha},\ e^{-(k_\alpha-i_2)L}\leq\frac{K}{\delta}\lambda_\alpha^{t_\alpha},\ i_2-i_1\leq\frac{2\log K}{L}.$$

By Lemma 5.2, we have

$$\begin{aligned}&\frac{1}{\alpha-1}\left(\||x-x_\alpha|^{-1}|\frac{\partial\phi_\alpha}{\partial\theta}|\|_{L^2(D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha))}+\||x-x_\alpha|^{-1}|\frac{\partial\psi_\alpha}{\partial\theta}|\|_{L^{\frac{4}{3}}(D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha))}\right)\\ &\leq\frac{1}{\alpha-1}\sum_{i=i_1}^{i_2}\left(\||x-x_\alpha|^{-1}|\frac{\partial\phi_\alpha}{\partial\theta}|\|_{L^2(P_i)}+\||x-x_\alpha|^{-1}|\frac{\partial\psi_\alpha}{\partial\theta}|\|_{L^{\frac{4}{3}}(P_i)}\right)\\ &\leq\frac{1}{\alpha-1}\left((\alpha-1)(i_2-i_1)+e^{-\frac{i_1L}{2}}+e^{-\frac{(k_\alpha-i_2)L}{2}}\right)o(\alpha,\delta,R)\\ &\leq\frac{1}{\alpha-1}\left((\alpha-1)\frac{2\log K}{L}+\sqrt{KRe^L}\lambda_\alpha^{\frac{1-t_\alpha}{2}}+\sqrt{\frac{K}{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\right)o(\alpha,\delta,R).\end{aligned}$$

Since $\nu > 1$, then $\lambda_\alpha \le e^{-\frac{C}{\sqrt{\alpha-1}}}$ and the conclusion of the corollary follows immediately from the above inequality. □

Next, combining the Hardy type inequality (4.17) with Lemma 5.2, we shall derive the following new decay estimate of the weighted energy of spinor part on the neck region, which is crucial in the proofs of Theorem 2.3 (e.g. Proposition 5.9).

Lemma 5.4. **(Decay of spinor)** *Under the assumption of Lemma 5.2, assume $\nu > 1$, then for $t_\alpha \in [t_1, t_2]$ where $0 < t_1 \le t_2 < 1$, we have*

$$\lim_{\alpha\searrow 1} \frac{1}{\alpha-1}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)} \frac{|\psi_\alpha|^2}{|x-x_\alpha|}dx = 0, \tag{5.9}$$

which implies

$$\lim_{\alpha\searrow 1} \frac{1}{\alpha-1}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)} |\psi_\alpha|^4 dx = 0, \tag{5.10}$$

where $K > 1$ is any fixed positive constant.

Proof. We divide the proof into two steps.

Step 1 We prove

$$\lim_{\alpha\searrow 1} \frac{1}{\sqrt{\alpha-1}}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)} \frac{|\psi_\alpha|^2}{|x-x_\alpha|}dx = 0. \tag{5.11}$$

For simplicity of notation, we also assume $x_\alpha = 0$. Define

$$f(t) := \int_{D_{e^t\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}}} \frac{|\psi_\alpha|^2}{|x|}dx,\ t\in[0, 8\log\frac{1}{\alpha-1}].$$

Since $\nu > 1$, then $\lambda_\alpha \le e^{-\frac{C}{\sqrt{\alpha-1}}}$ and it is easy to see that

$$e^t \le (\frac{1}{\alpha-1})^8 \le \lambda_\alpha^{-\epsilon},$$

where $\epsilon < \min\{t_1, 1-t_2\}$.

For any $\rho > 0$, taking the cut-off function $\eta \in C_0^\infty(D_{e^t\lambda_\alpha^{t_\alpha}+\rho} \setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})$ such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on $D_{e^t\lambda_\alpha^{t_\alpha}} \setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}}$ and $|\nabla\eta| \le \frac{2}{\rho}$. Taking $f = \eta|\psi_\alpha|^2$ in the Hardy inequality (4.17), we get

$$\begin{aligned}
\|\eta\frac{|\psi_\alpha|^2}{|x|}\|_{L^1(\mathbb{R}^2)} &\le \|\nabla(\eta|\psi_\alpha|^2)\|_{L^1(\mathbb{R}^2)}\\
&\le \|2\eta\psi_\alpha\nabla\psi_\alpha\|_{L^1(\mathbb{R}^2)} + \|\nabla\eta|\psi_\alpha|^2\|_{L^1(\mathbb{R}^2)}\\
&\le \|2\eta\psi_\alpha\frac{1}{|x|}\frac{\partial\psi_\alpha}{\partial\theta}\|_{L^1(\mathbb{R}^2)} + \|2\eta\psi_\alpha\frac{\partial\psi_\alpha}{\partial r}\|_{L^1(\mathbb{R}^2)} + \|\nabla\eta|\psi_\alpha|^2\|_{L^1(\mathbb{R}^2)}.
\end{aligned}$$

On the one hand, we know

$$\frac{\partial\psi_\alpha}{\partial r} = \frac{\partial}{\partial r}\cdot\frac{1}{|x|}\frac{\partial}{\partial\theta}\cdot\frac{1}{|x|}\frac{\partial\psi_\alpha}{\partial\theta} + \frac{\partial}{\partial r}\cdot\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha) \tag{5.12}$$

from equation (2.13). So, we have

$$|2\eta\psi_\alpha \frac{\partial \psi_\alpha}{\partial r}| \leq |2\eta\psi_\alpha \frac{1}{|x|}\frac{\partial \psi_\alpha}{\partial \theta}| + C|\eta||d\phi_\alpha||\psi_\alpha|^2.$$

On the other hand, by inequality (4.18), we have

$$|x||d\phi_\alpha| + \sqrt{|x|}|\psi_\alpha| \leq C\epsilon \quad on \quad D_\delta \setminus D_{\lambda_\alpha R}. \tag{5.13}$$

Combining these, we get

$$\begin{aligned}
\|\eta\frac{|\psi_\alpha|^2}{|x|}\|_{L^1(\mathbb{R}^2)} &\leq 4\|\eta\psi_\alpha\frac{1}{|x|}\frac{\partial\psi_\alpha}{\partial\theta}\|_{L^1(\mathbb{R}^2)} + C\|\eta|d\phi_\alpha||\psi_\alpha|^2\|_{L^1(\mathbb{R}^2)} + \|\nabla\eta|\psi_\alpha|^2\|_{L^1(\mathbb{R}^2)} \\
&\leq 4\|\eta\psi_\alpha\frac{1}{|x|}\frac{\partial\psi_\alpha}{\partial\theta}\|_{L^1(\mathbb{R}^2)} + C\epsilon\|\eta\frac{|\psi_\alpha|^2}{|x|}\|_{L^1(\mathbb{R}^2)} + \|\nabla\eta|\psi_\alpha|^2\|_{L^1(\mathbb{R}^2)}.
\end{aligned}$$

Taking $\epsilon > 0$ sufficiently small such that $C\epsilon \leq \frac{1}{2}$, we have

$$\begin{aligned}
\|\eta\frac{|\psi_\alpha|^2}{|x|}\|_{L^1(\mathbb{R}^2)} &\leq 8\|\psi_\alpha\frac{1}{|x|}\frac{\partial\psi_\alpha}{\partial\theta}\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})} + 2\|\nabla\eta|\psi_\alpha|^2\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})} \\
&\leq 8\|\psi_\alpha\|_{L^4(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}\||x|^{-1}\frac{\partial\psi_\alpha}{\partial\theta}\|_{L^{\frac{4}{3}}(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})} \\
&\quad + \frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})} + \frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\sigma})} \\
&\leq C\||x|^{-1}\frac{\partial\psi_\alpha}{\partial\theta}\|_{L^{\frac{4}{3}}(D_{e^{t+1}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-(t+1)}\lambda_\alpha^{t_\alpha}})} + \frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})} + \frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}.
\end{aligned} \tag{5.14}$$

where ρ is small enough such that $D_{e^t\lambda_\alpha^{t_\alpha}+\rho} \setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho} \subset D_{e^{t+1}\lambda_\alpha^{t_\alpha}} \setminus D_{e^{-(t+1)}\lambda_\alpha^{t_\alpha}}$.

Let $i_1, i_2 \in [0, k_\alpha]$ be two integers such that

$$e^{i_1 L}\lambda_\alpha R \leq e^{-(t+1)}\lambda_\alpha^{t_\alpha} < e^{(i_1+1)L}\lambda_\alpha R \ \ and \ \ e^{i_2 L}\lambda_\alpha R \leq e^{(t+1)}\lambda_\alpha^{t_\alpha} < e^{(i_2+1)L}\lambda_\alpha R.$$

Then we have

$$e^{-i_1 L} \leq Re^{(t+1+L)}\lambda_\alpha^{1-t_\alpha} \leq CR(\alpha-1)^{-8}\lambda_\alpha^{1-t_\alpha}, \ e^{-(k_\alpha - i_2)L} \leq \frac{e^{t+1}}{\delta}\lambda_\alpha^{t_\alpha} \leq C\frac{(\alpha-1)^{-8}}{\delta}\lambda_\alpha^{t_\alpha},$$

and

$$i_2 - i_1 \leq \frac{2(1+t)}{L} + 1 \leq C\log\frac{1}{\alpha - 1}.$$

Combing this with (5.14) and Lemma 5.2, we get

$$
\begin{aligned}
&\|\eta\frac{|\psi_\alpha|^2}{|x|}\|_{L^1(\mathbb{R}^2)}\\
&\le \sum_{i=i_1}^{i_2}\left((\alpha-1)+e^{-\frac{iL}{2}}+e^{-\frac{(k_\alpha-i)L}{2}}\right)o(\alpha,\delta,R)+\frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})}+\frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}\\
&\le \left((\alpha-1)(i_2-i_1)+e^{-\frac{i_1L}{2}}+e^{-\frac{(k_\alpha-i_2)L}{2}}\right)o(\alpha,\delta,R)+\frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})}+\frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}\\
&\le \left((\alpha-1)\log\frac{1}{\alpha-1}+\sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}}+\frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\right)o(\alpha,\delta,R)\\
&\quad+\frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})}+\frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}
\end{aligned}
$$

Letting $\rho\to 0$, we get

$$
\begin{aligned}
f(t)=&\int_{D_{e^t\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}}}\frac{|\psi_\alpha|^2}{|x|}dx\\
&\le\left((\alpha-1)\log\frac{1}{\alpha-1}+\sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}}+\frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\right)o(\alpha,\delta,R)+4e^t\lambda_\alpha^{t_\alpha}\int_{\partial D_{e^t\lambda_\alpha^{t_\alpha}}}\frac{|\psi_\alpha|^2}{|x|}\\
&\quad+4e^{-t}\lambda_\alpha^{t_\alpha}\int_{\partial D_{e^{-t}\lambda_\alpha^{t_\alpha}}}\frac{|\psi_\alpha|^2}{|x|}\\
&\le\left((\alpha-1)\log\frac{1}{\alpha-1}+\sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}}+\frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\right)o(\alpha,\delta,R)+4f'(t).
\end{aligned}
$$

This is

$$
(e^{-\frac{1}{4}t}f(t))'\ge -e^{-\frac{1}{4}t}\left((\alpha-1)\log\frac{1}{\alpha-1}+\sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}}+\frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\right)o(\alpha,\delta,R). \tag{5.15}
$$

Integrating the above ODE from K to $8\log\frac{1}{\alpha-1}$, we get

$$
\begin{aligned}
f(K)&\le C(\alpha-1)^2f(8\log\frac{1}{\alpha-1})+\left((\alpha-1)\log\frac{1}{\alpha-1}+\sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}}+\frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\right)o(\alpha,\delta,R)\\
&\le\left((\alpha-1)^2+(\alpha-1)\log\frac{1}{\alpha-1}+\sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}}+\frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\right)o(\alpha,\delta,R).
\end{aligned}
$$

where the last inequality follows from Lemma 4.4 since

$$
f(8\log\frac{1}{\alpha-1})\le\int_{D_\delta\setminus D_{\lambda_\alpha R}}\frac{|\psi_\alpha|^2}{|x|}.
$$

Then (5.11) follows immediately, which implies

$$
\lim_{\alpha\searrow 1}\frac{1}{\sqrt{\alpha-1}}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}|\psi_\alpha|^4dx=0. \tag{5.16}
$$

Step 2 We prove the conclusions of the lemma.

Using the decay estimate (5.16) (i.e. $\|\psi_\alpha\|_{L^4(P_i)} = o\Big((\alpha-1)^{\frac{1}{8}}\Big)$) in (5.14) and repeat above process again, we have

$$\begin{aligned}
\|\eta\frac{|\psi_\alpha|^2}{|x|}\|_{L^1(\mathbb{R}^2)} \leq{}& 8\|\psi_\alpha\|_{L^4(D_{e^{t+1}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-(t+1)}\lambda_\alpha^{t_\alpha}-\rho})}\||x|^{-1}\frac{\partial\psi_\alpha}{\partial\theta}\|_{L^{\frac{4}{3}}(D_{e^{t+1}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-(t+1)}\lambda_\alpha^{t_\alpha}-\rho})} \\
&+\frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})}+\frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})} \\
\leq{}& \sum_{i=i_1}^{i_2}8\|\psi_\alpha\|_{L^4(P_i)}\sum_{i=i_1}^{i_2}\||x|^{-1}\frac{\partial\psi_\alpha}{\partial\theta}\|_{L^{\frac{4}{3}}(P_i)} \\
&+\frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})}+\frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})} \\
\leq{}& (\alpha-1)^{\frac{1}{8}}(i_2-i_1)\Big((\alpha-1)(i_2-i_1)+e^{-\frac{i_1L}{2}}+e^{-\frac{(k_\alpha-i_2)L}{2}}\Big)o(\alpha,\delta,R) \\
&+\frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})}+\frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})} \\
\leq{}& \Big((\alpha-1)^{1+\frac{1}{8}}(\log\frac{1}{\alpha-1})^2+\sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}}+\frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\Big)o(\alpha,\delta,R) \\
&+\frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})}+\frac{4}{\rho}\||\psi_\alpha|^2\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}.
\end{aligned}$$

Letting $\rho\to 0$ and similar to deriving (5.15), we have

$$(e^{-\frac{1}{4}t}f(t))' \geq -e^{-\frac{1}{4}t}\Big((\alpha-1)^{1+\frac{1}{8}}(\log\frac{1}{\alpha-1})^2+\sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}}+\frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\Big)o(\alpha,\delta,R). \tag{5.17}$$

Integrating the above ODE from K to $8\log\frac{1}{\alpha-1}$, we get

$$f(K)\leq C(\alpha-1)^2f(8\log\frac{1}{\alpha-1})+\Big((\alpha-1)^{1+\frac{1}{8}}(\log\frac{1}{\alpha-1})^2+\sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}}+\frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\Big)o(\alpha,\delta,R),$$

which implies

$$\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)^{1+\frac{1}{16}}}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}\frac{|\psi_\alpha|^2}{|x-x_\alpha|}dx=0. \tag{5.18}$$

Thus we proved (5.9). The inequality (5.10) is a consequence of (5.9) and (4.18). □

From the proof of Lemma 5.4, we can actually get a better decay estimate.

Lemma 5.5. (Improved decay of spinor) *Under the assumption of Lemma 5.4, for any $\beta\in(0,\frac{4}{3})$, we have*

$$\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)^\beta}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}\frac{|\psi_\alpha|^2}{|x-x_\alpha|}dx=0,$$

$$\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)^\beta}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}|\psi_\alpha|^4dx=0.$$

Proof. Set

$$\gamma := \sup\left\{\beta \geq 1 \ \left|\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)^{\beta}}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}\frac{|\psi_\alpha|^2}{|x-x_\alpha|}dx = 0\right.\right\}.$$

By estimate (5.18), it ie easy to see that $\gamma > 1$. Next, we **claim:** $\gamma = \frac{4}{3}$.

In fact, if not, then $\gamma < \frac{4}{3}$ and there exists a small constant $\epsilon' > 0$ such that $\gamma < \frac{4}{3}(1-\frac{5}{4}\epsilon')$. Taking $\beta = \gamma - \epsilon' > 1$, then

$$\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)^{\beta}}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}\frac{|\psi_\alpha|^2}{|x-x_\alpha|}dx = 0. \tag{5.19}$$

Using the new decay estimate (5.19) and repeating the process of **Step 2** in the proof of Lemma 5.4 again, we have

$$f(K) \leq C(\alpha-1)^2 f(8\log\frac{1}{\alpha-1}) + \left((\alpha-1)^{1+\frac{\beta}{4}}(\log\frac{1}{\alpha-1})^2 + \sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}} + \frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\right)o(\alpha,\delta,R),$$

which implies

$$\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)^{1+\frac{\beta}{4}-\epsilon'}}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}\frac{|\psi_\alpha|^2}{|x-x_\alpha|}dx = 0. \tag{5.20}$$

Since $\gamma < \frac{4}{3}(1-\frac{5}{4}\epsilon')$, it is easy to see that $1+\frac{\beta}{4}-\epsilon' > \gamma$. This is a contradiction to the definition of γ. □

It is interesting to ask whether the constant $\frac{4}{3}$ in the decay estimate in Lemma 5.5 is the best constant to characterise the decay of the spinor or not?

As a corollary of Lemma 5.4 and Lemma 3.1, we get

Corollary 5.6. *Assume $\nu > 1$. Let $0 < t_1 \leq t_2 < 1$, by passing to a subsequence, there holds*

$$\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)^{\frac{1}{4}}}\|\sqrt{|x-x_\alpha|}|\psi_\alpha|\|_{C^0(D_{\lambda_\alpha^{t_1}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_2}}(x_\alpha))} = 0.$$

Proof. For any $t \in [\lambda_\alpha^{t_2}, \lambda_\alpha^{t_1}]$, by Lemma 3.1 (or see the proof of Lemma 4.5), we have

$$\|\sqrt{|x-x_\alpha|}|\psi_\alpha|\|_{C^0(D_{2t}(x_\alpha)\setminus D_t(x_\alpha))} \leq C\|\psi_\alpha\|_{L^4(D_{4t}(x_\alpha)\setminus D_{\frac{1}{2}t}(x_\alpha))}.$$

Then the conclusion of the corollary follows immediately from Lemma 5.4. □

By Lemma 5.2 (or Corollary 5.3), we have

Lemma 5.7. *Under the assumption of Lemma 5.2, suppose that $\nu > 1$ and t_α is a positive number such that $0 < t_1 \leq t_\alpha \leq t_2 < 1$, then we have:*

(1)

$$\frac{1}{\sqrt{\alpha-1}}\left(\phi_\alpha(x_\alpha+\lambda_\alpha^{t_\alpha}x) - \phi_\alpha(x_\alpha+(\lambda_\alpha^{t_\alpha},0))\right) \to \overrightarrow{a}\log|x|$$

strongly in $C^j_{loc}(\mathbb{R}^2\setminus\{0\},\mathbb{R}^K)$ for any integer j, where $\overrightarrow{a} \in T_yN$ is a vector in $\mathbb{R}^K$ with

$$|\overrightarrow{a}| = \mu^{1-\lim_{\alpha\searrow 1}t_\alpha}\sqrt{\frac{E(\sigma^1)}{\pi}}.$$

(2)

$$\left\|\frac{1}{\sqrt{\alpha-1}}\left(\phi_\alpha-\frac{1}{2\pi}\int_0^{2\pi}\phi_\alpha d\theta\right)\right\|_{C^0(D_{\lambda_\alpha^{t_1}}(x_\alpha)\backslash D_{\lambda_\alpha^{t_2}}(x_\alpha))}$$
$$+\left\|\frac{|x-x_\alpha|}{\sqrt{\alpha-1}}\nabla\left(\phi_\alpha-\frac{1}{2\pi}\int_0^{2\pi}\phi_\alpha d\theta\right)\right\|_{C^0(D_{\lambda_\alpha^{t_1}}(x_\alpha)\backslash D_{\lambda_\alpha^{t_2}}(x_\alpha))}\to 0.$$

Proof. Set

$$u_\alpha(x)=\phi_\alpha(x_\alpha+\lambda_\alpha^{t_\alpha}x),\quad w_\alpha(x)=(\lambda_\alpha^{t_\alpha})^{\alpha-1}\sqrt{\lambda_\alpha^{t_\alpha}}\psi_\alpha(x_\alpha+\lambda_\alpha^{t_\alpha}x)$$

and

$$v_\alpha(x)=\frac{1}{\sqrt{\alpha-1}}\left(\phi_\alpha(x_\alpha+\lambda_\alpha^{t_\alpha}x)-\phi_\alpha(x_\alpha+(\lambda_\alpha^{t_\alpha},0))\right).$$

By (4.26), (4.23), Lemma 3.2 and Lemma 5.2, we have

$$\int_{D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\backslash D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha)}|\nabla\phi_\alpha|^2dx$$
$$\leq C\int_{D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\backslash D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx$$
$$\leq C\int_{D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\backslash D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx$$
$$+C|\int_{D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\backslash D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha)}\langle\psi_\alpha,|x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\partial_\theta}\psi_\alpha\rangle dx|+C\int_{2^{-k}\lambda_\alpha^{t_\alpha}}^{2^k\lambda_\alpha^{t_\alpha}}\frac{\alpha-1}{r}dr+C(k)\lambda_\alpha^{t_\alpha}$$
$$\leq C\||x-x_\alpha|^{-1}\frac{\partial\phi_\alpha}{\partial\theta}\|^2_{L^2(D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\backslash D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha))}+\||\psi_\alpha\|_{L^4(D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\backslash D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha))}\||x-x_\alpha|^{-1}\frac{\partial\psi_\alpha}{\partial\theta}\|_{L^{\frac{4}{3}}(D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\backslash D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha))}$$
$$+C\||\psi_\alpha\|^2_{L^4(D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\backslash D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha))}\||x-x_\alpha|^{-1}\frac{\partial\phi_\alpha}{\partial\theta}\|_{L^2(D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\backslash D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha))}+C(k)(\alpha-1+\lambda_\alpha^{t_\alpha})$$

(5.21)
$$\leq C(k)(\alpha-1),$$

where we used the fact that

$$\lambda_\alpha=o((\alpha-1)^m)$$

for any $m>0$ since $\nu>1$.

By Lemma 3.1, we have

$$\|\nabla u_\alpha\|_{C^0(D_{2^k}(0)\backslash D_{2^{-k}}(0))}+\|\nabla^2u_\alpha\|_{C^0(D_{2^k}(0)\backslash D_{2^{-k}}(0))}\leq C(k)\sqrt{\alpha-1}. \tag{5.22}$$

Then,

$$\|\nabla v_\alpha\|_{C^0(D_{2^k}(0)\backslash D_{2^{-k}}(0))}+\|\nabla^2v_\alpha\|_{L^\infty(D_{2^k}(0)\backslash D_{2^{-k}}(0))}\leq C(k).$$

Noting that $v_\alpha(1,0)=0$, thus

$$\|v_\alpha\|_{C^2(D_{2^k}(0)\backslash D_{2^{-k}}(0))}\leq C(k).$$

Since v_α satisfies the following equation

$$\Delta v_\alpha+\sqrt{\alpha-1}O(|\nabla v_\alpha|^2)+(\alpha-1)O(|\nabla^2v_\alpha|)+O(|w_\alpha|^2)O(|\nabla v_\alpha|)=0$$

and by (4.18), there holds

$$\lim_{\alpha\searrow 1}\|w_\alpha\|_{C^0(D_{2^k}(0)\setminus D_{2^{-k}}(0))}=0,$$

then there exists a subsequence of v_α (without changing the notation) such that

$$v_\alpha\to v_0\quad in\quad C^1_{loc}(\mathbb{R}^2\setminus\{0\})$$

where v_0 satisfies $v_0(1,0)=0$ and

$$\Delta v_0=0.$$

Moreover, Corollary 5.3 tells us that $v_0(x)=v_0(|x|)$. Set

$$v_0=\overrightarrow{a}\log r=(a_1,...,a_K)\log r.$$

By Corollary 3.5 and Corollary 5.3, we have

$$\begin{aligned}
&\frac{1}{\alpha-1}\int_{D_{2\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_\alpha}}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx\\
&=\frac{2}{2\alpha-1}\int_{\lambda_\alpha^{t_\alpha}}^{2\lambda_\alpha^{t_\alpha}}\frac{1}{t}F_\alpha(\log_{\lambda_\alpha}t)dt+\frac{1}{(2\alpha-1)(\alpha-1)}\int_{D_{2\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_\alpha}}(x_\alpha)}\langle\psi_\alpha,|x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle+o(1)\\
&=\frac{2}{2\alpha-1}\log 2F_\alpha(t_\alpha)+o(1)\to 2\log 2F(\lim_{\alpha\searrow 1}t_\alpha),
\end{aligned}$$

where we used the fact that

$$\lambda_\alpha^{t_\alpha}=o((\alpha-1)^m)$$

for any $m>0$ since $\nu>1$.

On the other hand, there also holds

$$\begin{aligned}
\frac{1}{\alpha-1}\int_{D_{2\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_\alpha}}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx&=\int_{D_2(0)\setminus D_1(0)}\left(\sigma_\alpha+|\nabla_{g_\alpha}v_\alpha|^2\frac{\alpha-1}{\lambda_\alpha^{2t_\alpha}}\right)^{\alpha-1}|\nabla v_\alpha|^2dx\\
&\to 2\pi\log 2|\overrightarrow{a}|^2\mu^{\lim_{\alpha\searrow 1}t_\alpha}.
\end{aligned}$$

Therefore,

$$|\overrightarrow{a}|^2=\frac{1}{\pi}\mu^{-\lim_{\alpha\searrow 1}t_\alpha}F(\lim_{\alpha\searrow 1}t_\alpha)=\frac{E(\sigma^1)}{\pi}\mu^{2-2\lim_{\alpha\searrow 1}t_\alpha},$$

where the last equality follows from Lemma 4.9 and we proved the statement (1).

For statement (2), if it was false, then there would exist $t_\alpha\in[t_1,t_2]$ and $\theta_\alpha\in[0,2\pi]$ such that

$$\frac{1}{\sqrt{\alpha-1}}\left(\lambda_\alpha^{t_\alpha}|\nabla\phi_\alpha(\lambda_\alpha^{t_\alpha},\theta_\alpha)-\nabla\phi_\alpha^*(\lambda_\alpha^{t_\alpha})|+|\phi_\alpha(\lambda_\alpha^{t_\alpha},\theta_\alpha)-\phi_\alpha^*(\lambda_\alpha^{t_\alpha})|\right)\to b\neq 0, \tag{5.23}$$

where

$$\phi_\alpha^*:=\frac{1}{2\pi}\int_0^{2\pi}\phi_\alpha d\theta.$$

Let $\lim_{\alpha\searrow 1}\theta_\alpha=\theta_0$, then it is obvious that

$$|\nabla v_0(1,\theta_0)-\nabla v_0^*(1)|+|v_0(1,\theta_0)-v_0^*(1)|=0,$$

where v_0^* is the average of v_0 similar to the definition of ϕ_α^*.

However, by (5.23) and the fact that $v_\alpha \to v_0$ in $C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$, we have

$$\begin{aligned} b &= \lim_{\alpha \searrow 1} \frac{1}{\sqrt{\alpha - 1}} \left(\lambda_\alpha^{t_\alpha} |\nabla \phi_\alpha(\lambda_\alpha^{t_\alpha}, \theta_\alpha) - \nabla \phi^*_\alpha(\lambda_\alpha^{t_\alpha})| + |\phi_\alpha(\lambda_\alpha^{t_\alpha}, \theta_\alpha) - \phi^*_\alpha(\lambda_\alpha^{t_\alpha})|\right) \\ &= \lim_{\alpha \searrow 1} |\nabla v_\alpha(1, \theta_\alpha) - \nabla v^*_\alpha(1)| + \lim_{\alpha \searrow 1} |v_\alpha(1, \theta_\alpha) - v^*_\alpha(1)| \\ &= |\nabla v_0(1, \theta_0) - \nabla v^*_0(1)| + |v_0(1, \theta_0) - v^*_0(1)| = 0, \end{aligned}$$

which is a contradiction. So, the statement (2) holds and we finished the proof of the lemma. □

Corollary 5.8. *Under the assumption of Lemma 5.7, we have:*

$$\int_{\lambda_\alpha^t}^{2\lambda_\alpha^t} \frac{1}{\sqrt{\alpha - 1}} |\frac{\partial \phi_\alpha}{\partial r}| dr \to \log 2 \mu^{1-t} \sqrt{\frac{E(\sigma^1)}{\pi}} \quad \textit{in } C^0([t_1, t_2])$$

and

$$\begin{aligned} \frac{1}{\sqrt{\alpha - 1}} (r |\frac{\partial \phi_\alpha}{\partial r}|)(\lambda_\alpha^t, \theta) &\to \mu^{1-t} \sqrt{\frac{E(\sigma^1)}{\pi}} \quad \textit{in } C^0([t_1, t_2]), \\ \frac{1}{\sqrt{\alpha - 1}} (r^{-1} |\frac{\partial \phi_\alpha}{\partial \theta}|)(\lambda_\alpha^t, \theta) &\to 0 \quad \textit{in } C^0([t_1, t_2]). \end{aligned}$$

Proof. We just prove the second claim, since the the proofs of the other two claims are similar. In fact, if it was false, then there would exist $t_\alpha \in [t_1, t_2]$ and $\theta_\alpha \in [0, 2\pi]$ such that

$$\left| \frac{1}{\sqrt{\alpha - 1}} (r |\frac{\partial \phi_\alpha}{\partial r}|)(\lambda_\alpha^{t_\alpha}, \theta_\alpha) - \mu^{1-t_\alpha} \sqrt{\frac{E(\sigma^1)}{\pi}} \right| \geq b > 0. \tag{5.24}$$

However, by Lemma 5.7, we have

$$\frac{\lambda_\alpha^{t_\alpha}}{\sqrt{\alpha - 1}} \frac{\partial \phi_\alpha}{\partial r}(\lambda_\alpha^{t_\alpha}, \theta_\alpha) \to \overrightarrow{a},$$

where $|\overrightarrow{a}| = \mu^{1 - \lim_{\alpha \searrow 1} t_\alpha} \sqrt{\frac{E(\sigma^1)}{\pi}}$. This yields the following

$$\frac{1}{\sqrt{\alpha - 1}} (r |\frac{\partial \phi_\alpha}{\partial r}|)(\lambda_\alpha^{t_\alpha}, \theta_\alpha) \to \mu^{1 - \lim_{\alpha \searrow 1} t_\alpha} \sqrt{\frac{E(\sigma^1)}{\pi}},$$

which contradicts to (5.24). We finished the proof of this corollary. □

For $0 < t_1 < t_2 < 1$, define the following curve:

$$\omega_\alpha(r) := \frac{1}{2\pi} \int_0^{2\pi} \phi_\alpha(r, \theta) d\theta : [\lambda_\alpha^{t_2}, \lambda_\alpha^{t_1}] \to \mathbb{R}^K,$$

which is denoted by γ_α, where (r, θ) is the polar coordinate around the point x_α.

Denote

$$\ddot{\omega}_\alpha := (\frac{d}{dr})^2 \omega_\alpha, \ \dot{\omega}_\alpha := \frac{d}{dr} \omega_\alpha.$$

By a direct computation, we have

$$
\begin{aligned}
\ddot{\omega}_\alpha &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial^2}{\partial r^2}\phi_\alpha(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Delta\phi_\alpha(r,\theta)d\theta - \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{r}\frac{\partial}{\partial r}\phi_\alpha(r,\theta)d\theta \\
&= -\frac{1}{2\pi}\int_0^{2\pi} A(\phi_\alpha)(d\phi_\alpha, d\phi_\alpha)d\theta - \frac{1}{2\pi}\int_0^{2\pi} O(|\nabla\phi_\alpha||\psi_\alpha|^2)d\theta + \frac{\alpha-1}{2\pi}\int_0^{2\pi} O(|\nabla^2\phi_\alpha|)d\theta - \frac{1}{r}\dot{\omega}_\alpha.
\end{aligned}
\tag{5.25}
$$

Proposition 5.9. *After passing to a subsequence, the sequence of the curves $\{\gamma_\alpha\}$ in $\mathbb{R}^N$, which is defined by ω_α, and parameterized by its arc length, converges to a curve $\omega \subset N$ as $\alpha \searrow 1$, where ω is a geodesic on N, i.e.*

$$\frac{d^2\omega}{ds^2} + A(\omega)(\frac{d\omega}{ds}, \frac{d\omega}{ds}) = 0.$$

Proof. Denote the induced second fundamental form of γ_α in $\mathbb{R}^K$ by A_{γ_α}. Set

$$G_\alpha := -\ddot{\omega}_\alpha - \frac{\dot{\omega}_\alpha}{r}.$$

By Lemma 3.1, it is easy to see that γ_α converges to some curve on N denoted by γ.

Next, we will show that γ is just a geodesic on N. Let s be the arc length parameter of γ_α, i.e.

$$s(r) = \int_{\lambda_\alpha^{t_\alpha}}^{r} |\dot{\omega}_\alpha(\tau)|d\tau,$$

where $t_\alpha \in [t_1, t_2]$.

By Lemma 5.7 and Corollary 5.8, for any $t_1 \in (0,1)$, we have

$$
\begin{aligned}
\int_{\lambda_\alpha^{t_\alpha}}^{\lambda_\alpha^{t_1}} |\dot{\omega}_\alpha|dr &\le \int_{\lambda_\alpha^{t_\alpha}}^{\lambda_\alpha^{t_1}} \frac{\sqrt{\alpha-1}}{r}\mu\sqrt{\frac{E(\sigma^1)}{\pi}}dr \\
&= (t_1 - t_\alpha)\sqrt{\alpha-1}\log\lambda_\alpha\mu\sqrt{\frac{E(\sigma^1)}{\pi}} \to (\lim_{\alpha\searrow 1} t_\alpha - t_1)\log\nu\mu\sqrt{\frac{E(\sigma^1)}{\pi}}
\end{aligned}
$$

as $\alpha \searrow 1$. So, if $\nu = \infty$, then for any $s \in (0,\infty)$,

$$\omega_\alpha|_{[0,s]} \subset \omega_\alpha|_{[\lambda_\alpha^{t_2}, \lambda_\alpha^{t_1}]}.$$

If $\nu \in (1,\infty)$, then $\mu = 1$ and noting that

$$
\begin{aligned}
\int_{\lambda_\alpha^{t_\alpha}}^{\lambda_\alpha^{t_1}} |\dot{\omega}_\alpha|dr &\ge \int_{\lambda_\alpha^{t_\alpha}}^{\lambda_\alpha^{t_1}} \frac{\sqrt{\alpha-1}}{r}\sqrt{\frac{E(\sigma^1)}{\pi}}dr \\
&= (t_1 - t_\alpha)\sqrt{\alpha-1}\log\lambda_\alpha\sqrt{\frac{E(\sigma^1)}{\pi}} \to (\lim_{\alpha\searrow 1} t_\alpha - t_1)\log\nu\sqrt{\frac{E(\sigma^1)}{\pi}}
\end{aligned}
$$

as $\alpha \searrow 1$, there exists $t_1' \in (0, t_1)$ such that

$$\omega_\alpha|_{[0,s]} \subset \omega_\alpha|_{[\lambda_\alpha^{t_2}, \lambda_\alpha^{t_1'}]}$$

whenever $s \in \left(0, (\lim_{\alpha\searrow 1} t_\alpha - t_1)\log\nu\sqrt{\frac{E(\sigma^1)}{\pi}}\right)$.

Computing directly, we obtain

$$\begin{aligned}
\frac{d^2\omega_\alpha}{ds^2} &= A_{\gamma_\alpha}(\omega_\alpha(s))(\frac{d\omega_\alpha}{ds}, \frac{d\omega_\alpha}{ds}) = \frac{1}{|\dot\omega_\alpha|^2}A_{\gamma_\alpha}(\omega_\alpha(t))(\frac{d\omega_\alpha}{dt}, \frac{d\omega_\alpha}{dt}) \\
&= \frac{1}{|\dot\omega_\alpha|^2}(\ddot\omega_\alpha - \frac{\langle\ddot\omega_\alpha, \dot\omega_\alpha\rangle}{|\dot\omega_\alpha|^2}\dot\omega_\alpha) = \frac{1}{|\dot\omega_\alpha|^2}(-G_\alpha + \frac{\langle G_\alpha, \dot\omega_\alpha\rangle}{|\dot\omega_\alpha|^2}\dot\omega_\alpha) \\
&= -\frac{1}{|\dot\omega_\alpha|^2}\frac{1}{2\pi}\int_0^{2\pi} A(\phi_\alpha)(\nabla\phi_\alpha, \nabla\phi_\alpha)d\theta + \frac{1}{|\dot\omega_\alpha|^2}\frac{1}{2\pi}\left(\int_0^{2\pi} O(|\nabla\phi_\alpha||\psi_\alpha|^2) + (\alpha-1)O(|\nabla^2\phi_\alpha|)d\theta\right) \\
&\quad + \frac{\dot\omega_\alpha}{|\dot\omega_\alpha|^4}\frac{1}{2\pi}\int_0^{2\pi}\langle A(\phi_\alpha)(\nabla\phi_\alpha, \nabla\phi_\alpha), \dot\omega_\alpha\rangle d\theta \\
&\quad - \frac{\dot\omega_\alpha}{|\dot\omega_\alpha|^4}\frac{1}{2\pi}\int_0^{2\pi}\langle O(|\nabla\phi_\alpha||\psi_\alpha|^2) + (\alpha-1)O(|\nabla^2\phi_\alpha|), \dot\omega_\alpha\rangle d\theta := \mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV}.
\end{aligned} \tag{5.26}$$

On the one hand, by Lemma 5.7 and using the fact that

$$\langle A(\phi_\alpha)(\nabla\phi_\alpha, \nabla\phi_\alpha), \frac{\partial\phi_\alpha}{\partial r}\rangle = 0,$$

we have

$$\begin{aligned}
\mathbf{III} &= \frac{\dot\omega_\alpha}{|\dot\omega_\alpha|^4}\frac{1}{2\pi}\int_0^{2\pi}\langle A(\phi_\alpha)(\nabla\phi_\alpha, \nabla\phi_\alpha), \dot\omega_\alpha - \frac{\partial\phi_\alpha}{\partial r}\rangle d\theta \\
&\le \left\|\frac{|x - x_\alpha|}{\sqrt{\alpha-1}}\nabla(\phi_\alpha - \phi_\alpha^*)\right\|_{C^0(D_{\lambda_\alpha^{t_1}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_\alpha}}(x_\alpha))} \to 0
\end{aligned} \tag{5.27}$$

as $\alpha \searrow 1$, where we used the fact from Lemma 5.7 and Corollary 5.8 that

$$|\dot\omega_\alpha| \ge C\frac{\sqrt{\alpha-1}}{|x - x_\alpha|} \text{ and } |\nabla\phi_\alpha| \le C\frac{\sqrt{\alpha-1}}{|x - x_\alpha|}.$$

On the other hand, by (5.22) and Corollary 5.6, Corollary 5.8, we have

$$\mathbf{II} + \mathbf{IV} \le C\left(\frac{|x - x_\alpha||\nabla\phi_\alpha|}{\sqrt{\alpha-1}}\frac{|x - x_\alpha||\psi_\alpha|^2}{\sqrt{\alpha-1}} + \sqrt{\alpha-1}\frac{|x - x_\alpha|^2|\nabla^2\phi_\alpha|}{\sqrt{\alpha-1}}\right) \to 0 \tag{5.28}$$

as $\alpha \searrow 1$.

Therefore, we obviously see that

$$|\frac{d^2\omega_\alpha}{ds^2}| \le C(N).$$

Thus, $\phi_\alpha(s)$ will converge locally to a smooth vector valued function from $[0, s]$ into $\mathbb{R}^K$, denoted by $\omega(s)$, in the sense of C^1, i.e. $\gamma_\alpha|_{[\lambda_\alpha^{t_2}, \lambda_\alpha^{t_1}]}$ converges locally to the curve γ.

Next, we will show that γ is a geodesic.

By Lemma 5.7 and Corollary 5.8, we obtain

$$\begin{aligned}
-\mathbf{I} &= \frac{1}{|\dot{\omega}_\alpha|^2}\frac{1}{2\pi}\int_0^{2\pi} A(\phi_\alpha)(\nabla\phi_\alpha,\nabla\phi_\alpha)d\theta \\
&= \frac{1}{|\dot{\omega}_\alpha|^2}\frac{1}{2\pi}\left(\int_0^{2\pi}(A(\phi_\alpha)-A(\omega_\alpha))(\nabla\phi_\alpha,\nabla\phi_\alpha)+A(\omega_\alpha)(\nabla(\phi_\alpha-\omega_\alpha),\nabla\phi_\alpha)d\theta\right) \\
&\quad+\frac{1}{|\dot{\omega}_\alpha|^2}\frac{1}{2\pi}\int_0^{2\pi} A(\omega_\alpha)(\nabla\omega_\alpha,\nabla(\phi_\alpha-\omega_\alpha))d\theta+\frac{1}{|\dot{\omega}_\alpha|^2}\frac{1}{2\pi}\int_0^{2\pi} A(\omega_\alpha)(\nabla\omega_\alpha,\nabla\omega_\alpha)d\theta \\
&= \frac{1}{|\dot{\omega}_\alpha|^2}A(\omega_\alpha)(\nabla\omega_\alpha,\nabla\omega_\alpha)+O\left(\frac{|x-x_\alpha||\nabla(\phi_\alpha-\omega_\alpha)|}{\sqrt{\alpha-1}}\right)+O(|\phi_\alpha-\omega_\alpha|). \qquad (5.29)
\end{aligned}$$

Combining (5.27), (5.28), (5.29) with Lemma 5.7, letting $\alpha \searrow 1$, we have

$$\frac{d\omega}{ds}(s)-\frac{d\omega}{ds}(0)=-\int_0^s A(\omega)(\frac{d\omega}{ds},\frac{d\omega}{ds})$$

which implies

$$\frac{d^2\omega}{ds^2}=-A(\omega)(\frac{d\omega}{ds},\frac{d\omega}{ds}).$$

We finished the proof of this proposition. □

Now we can prove our main theorem in this section.

Proof of Theorem 5.1. By Proposition 5.9, we just need to compute the length of the geodesic. Without loss of generality, we assume $\lambda_\alpha^{t_1}=2^{m_\alpha}\lambda_\alpha^{t_2}$ for some integer

$$m_\alpha=\frac{t_1-t_2}{\log 2}\log\lambda_\alpha,$$

which tends to infinity as $\alpha \searrow 1$. Then it is sufficient to consider the following three cases.

Case (3): $\nu=\infty$.

By Corollary 5.8, there holds

$$L(\gamma_\alpha|_{D_{2^{k+1}\lambda_\alpha^{t_2}}(x_\alpha)\setminus D_{2^k\lambda_\alpha^{t_2}}(x_\alpha)})\geq\sqrt{\alpha-1}\left(\sqrt{\frac{E(\sigma^1)}{\pi}}\log 2+o(1)\right).$$

Then

$$L(\gamma_\alpha)\geq Cm_\alpha\sqrt{\alpha-1}\geq -C\sqrt{\alpha-1}\log\lambda_\alpha\to\infty.$$

Case (2): $\nu\in(1,\infty)$.

First, we prove the following equalities

$$\lim_{\delta\to 0}\lim_{t\to 0}\lim_{\alpha\searrow 1} Osc_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)}\phi_\alpha=0 \tag{5.30}$$

and

$$\lim_{R\to\infty}\lim_{t\to 1}\lim_{\alpha\searrow 1} Osc_{D_{\lambda_\alpha^t}(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)}\phi_\alpha=0. \tag{5.31}$$

In fact, by Lemma 5.2, there holds

$$\left(\int_{P_i}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx\right)^{\frac{1}{2}}+\left(\int_{P_i}|x-x_\alpha|^{-\frac{4}{3}}|\frac{\partial\psi_\alpha}{\partial\theta}|^{\frac{4}{3}}dx\right)^{\frac{3}{4}}\leq\left((\alpha-1)+e^{-\frac{iL}{2}}+e^{-\frac{(k_\alpha-i)L}{2}}\right)o(\alpha,\delta,R).$$

where $P_i=D_{e^{(i+1)L}\lambda_\alpha R}\setminus D_{e^{iL}\lambda_\alpha R}$, $0\leq i\leq k_\alpha:=[\frac{\delta}{\lambda_\alpha R}]-1$.

By (5.21), we obtain

$$\begin{aligned}\int_{P_i}|\nabla\phi_\alpha|^2dx &\leq C\||x-x_\alpha|^{-1}\frac{\partial\phi_\alpha}{\partial\theta}\|^2_{L^2(P_i)}+\|\|\psi_\alpha\|_{L^4(P_i)}\||x-x_\alpha|^{-1}\frac{\partial\psi_\alpha}{\partial\theta}\|_{L^{\frac{4}{3}}(P_i)}\\ &\quad+C\|\|\psi_\alpha\|^2_{L^4(P_i)}\||x-x_\alpha|^{-1}\frac{\partial\phi_\alpha}{\partial\theta}\|_{L^2(P_i)}+C(\alpha-1+\lambda_\alpha^{t_\alpha})\\ &\leq(e^{-\frac{iL}{2}}+e^{-\frac{(k_\alpha-i)L}{2}})o(\alpha,\delta,R)+C(\alpha-1+e^{iL}\lambda_\alpha R).\end{aligned}\tag{5.32}$$

According to Lemma 3.1, we get

$$Osc_{P_i}\phi_\alpha\leq(e^{-\frac{iL}{4}}+e^{-\frac{(k_\alpha-i)L}{4}})o(\alpha,\delta,R)+C(\sqrt{\alpha-1}+e^{\frac{iL}{2}}\sqrt{\lambda_\alpha R}).\tag{5.33}$$

Let $i_t\in[1,k_\alpha]$ be the positive integer such that

$$e^{i_tL}\lambda_\alpha R\leq\lambda_\alpha^t<e^{(i_t+1)L}\lambda_\alpha R.$$

Then $k_\alpha-i_t\leq C\log\frac{\delta}{\lambda_\alpha^t}$ and

$$Osc_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)}\phi_\alpha\leq\sum_{i=i_t}^{k_\alpha}Osc_{P_i}\phi_\alpha\leq o(\alpha,\delta,R)+C(\sqrt{\alpha-1}\log\frac{\delta}{\lambda_\alpha^t}+\sqrt{\delta})$$

and (5.30) follows immediately. The proof of (5.31) is similar.

Secondly, since $\nu\in(1,\infty)$ implies $\mu=1$, by Corollary 5.8, there holds

$$L(\gamma_\alpha|_{D_{2^{k+1}\lambda_\alpha^{t_2}}(x_\alpha)\setminus D_{2^k\lambda_\alpha^{t_2}}(x_\alpha)})=\sqrt{\alpha-1}\left(\sqrt{\frac{E(\sigma^1)}{\pi}}\log 2+o(1)\right).$$

Then

$$L(\gamma)=\lim_{\alpha\searrow 1}\sqrt{\alpha-1}m_\alpha\left(\sqrt{\frac{E(\sigma^1)}{\pi}}\log 2+o(1)\right)=(t_2-t_1)\sqrt{\frac{E(\sigma^1)}{\pi}}\log\nu.$$

Case (1): $\nu=1$.

By (5.33), we have

$$Osc_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)}\phi_\alpha\leq\sum_{i=1}^{k_\alpha}Osc_{P_i}\phi_\alpha\leq o(\alpha,\delta,R)+C(\sqrt{\alpha-1}\log\frac{\delta}{\lambda_\alpha R}+\sqrt{\delta})$$

which implies that

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\searrow 1}Osc_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)}\phi_\alpha=0.$$

We finished the proof of Theorem 5.1. □

Proof of Theorem 2.3. It is easy to see that Theorem 2.3 is a consequence of Theorem 5.1. □

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