# $C^1$-REGULARITY OF MINIMA OF DINI-TYPE VARIATIONAL INTEGRALS

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ABSTRACT. We study the interior gradient regularity of local minimizers of non-differentiable variational integrals, focusing on the borderline regime in which the dependence on the unknown variable $u \mapsto f(p,u,x)$ is merely Dini continuous. Classical regularity theory guarantees $C^{1,\alpha}$-smoothness under Hölder continuity assumptions on the $u$-dependence of the integrand. Our main result shows that local minimizers are in fact of class $C^1_{\text{loc}}(\Omega)$ under the weaker assumption that the $u$-dependence of the integrand (and the coefficients) is Dini continuous. The proof introduces a new non-homogeneous scaling and normalization scheme that propagates oscillation decay across scales without relying on homogeneity or classical excess-decay arguments. The method is intrinsically nonlinear and flexible, and is expected to extend to more general growth conditions, systems, and free boundary problems with non-homogeneous dependence on the state variable.

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## 1. Introduction

The gradient regularity of local minima of non-differentiable (or non-smooth) energy functionals of the type

$$F(u,\Omega) := \int_\Omega f(Du,\ u,\ x)\,dx \tag{1.1}$$

has been a central topic in the calculus of variations and in its applications to free boundary problems, phase transitions, and geometric analysis since the early 1980s. The interest in such results stems from the fact that control of the gradient $\nabla u$ not only yields finer information on weak solutions of the associated (singular) Euler–Lagrange equations, but also underpins critical geometric properties essential for free boundary regularity analysis, stability, and geometric asymptotics of the models. This has motivated a large and diverse literature.

For smooth integrands, foundational contributions of Morrey, De Giorgi, Nash, and Moser (among others) show that minima (and even critical points) of elliptic variational integrals enjoy interior Hölder continuity and, under suitable structural assumptions, $C^{1,\alpha}$-regularity.

In the non-smooth case, where the integrand in (1.1) may fail to be differentiable in the unknown $u$, regularity issues become more delicate, since no classical Euler–Lagrange equation is available in general. A landmark result in this direction, due to Giaquinta and Giusti [20], shows that if

$$(x,u) \mapsto f(z,x,u)$$

is Hölder continuous in the lower-order variables and $f$ has quadratic growth in $z = Du$, then any local minimizer is of class $C^{1,\alpha}$. Since then, an extensive literature has developed addressing a broad range of phenomena: degenerate elliptic problems (see, e.g., [30, 31]), non-uniformly elliptic and "double-phase" models ([7, 11, 12, 13], among many others), as well as numerous free-boundary and geometric applications (cf. [1, 3, 4, 14, 23, 26, 27, 36, 38, 33, 34, 35, 37] for some recent contributions).

Despite this progress, a common structural feature remains: to obtain full gradient regularity of a minimizer, one usually assumes a quantitative Hölder continuity in the dependence of the integrand on the lower-order variables, and in particular a Hölder condition in the $u$–variable. From a methodological standpoint, most existing approaches rest on two synergistic ingredients:

(i) $C^{1,\alpha}$ a priori estimates for solutions of the associated homogeneous (or frozen-coefficient) Euler–Lagrange equation, often of the form $\operatorname{div}(\partial_z f) = 0$;
(ii) a homogeneous decay estimate (or excess-decay lemma) for the minimizer (or solution) that ensures a geometric improvement of the gradient oscillation across scales, yielding gradient Hölder continuity.

These two ingredients work in tandem when the dependence on $u$ and $x$ can be "frozen" at small scales and the resulting homogeneous problems enjoy $C^{1,\alpha}$ regularity estimates. If, however, the $u$-dependence is strongly non-homogeneous or if the homogeneous equation fails to satisfy $C^{1,\alpha}$ estimates, classical arguments break down.

In particular, when no additional regularity is assumed on the $(x, u)$-dependence of the integrand, the best one can typically guarantee is continuity of the minimizer (say $C^{0,\varepsilon}_{\text{loc}}$), but no gradient continuity. Under mere continuity in $(x, u)$, minimizers are known to be locally $C^{0,\alpha}$ for all $0 < \alpha < 1$, but the gradient may remain unbounded or discontinuous. See, for example, [10, 31] and, in a free boundary context, [5] for borderline Lipschitz and $C^1$ regularity in one-phase variational free boundary problems involving bounded or continuous $u$-dependence.

Several more recent works push these boundaries, particularly regarding the dependence on the spatial variable $x$. For instance, in [9], the authors obtain gradient continuity (i.e. continuity of $\nabla u$) for local minimizers of scalar functionals of the form

$$\int_\Omega f(x, u, Du)\, dx + \int_\Omega u\, \mu\, dx,$$

under the borderline assumption $\mu \in L(n, 1)$ (a Lorentz space), combined with Dini continuity in the $x$-variable of $f$. However, the result still assumes Hölder continuity in the $u$-variable of $f$. Similarly, in [8], the authors prove $C^1$-regularity of minimizers for non-autonomous functionals under Dini or log-Dini conditions on the coefficients in $x$, but with no dependence on $u$.

Thus, the case where the integrand only enjoys a mild (non-Hölder) modulus of continuity in the $u$-variable remains largely unexplored.

In this paper we focus on quadratic-growth variational integrals of the form

$$J(u, \Omega) = \int_\Omega |\nabla u|^2 + f(u)\, dx,$$

and address the following question: *can one obtain full interior $C^1$-regularity of a local minimizer $u$ if the integrand fails to be Hölder continuous on its $u$-dependence?*

Our main result answers this question in the affirmative in the setting of non-autonomous quadratic functionals with Dini structure.

**Theorem 1.** *Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in H^1(\Omega)$ be a local minimizer of*

$$J(u, \Omega) := \int_\Omega \langle A(x)\nabla u, \nabla u\rangle + f(u)\, dx,$$

*where $A(x)$ is symmetric and $\Lambda$-uniformly elliptic, i.e.*

$$\mathrm{Spec}(A(x)) \subset [\Lambda^{-1}, \Lambda] \quad \textit{for some } \Lambda \geq 1.$$

*Assume that $A(x)$ is 1-Dini-continuous, that is,*

$$\|A(x) - A(y)\|_{\mathrm{Sym}(n)} \leq \tilde{\omega}(|x - y|),$$

*for some modulus of continuity $\tilde{\omega}$ verifying*

$$\int_0^1 \frac{\tilde{\omega}(\lambda)}{\lambda}\, d\lambda < +\infty.$$

*Assume further that $f$ has subcritical growth and that, for all $s, t \in \mathbb{R}$,*

$$|f(s) - f(t)| \leq \omega(|s - t|),$$

*where $\omega$ is a $1/2$-Dini modulus of continuity, i.e.*

$$\int_0^1 \frac{\sqrt{\omega(\lambda)}}{\lambda}\, d\lambda < +\infty.$$

*Then $u \in C^1_{\mathrm{loc}}(\Omega)$, with local estimates depending only on $n$, $\Lambda$, and $\omega$.*

The key novelty of Theorem 1 lies in establishing full interior $C^1$-regularity of local minimizers under a merely Dini-type control on the $u$-dependence (and on the spatial dependence of $A(x)$), completely dispensing with any Hölder assumption in the $u$-variable.

The proof introduces a new, intrinsically nonlinear methodology, inspired by techniques from free boundary problems, where non-homogeneous and highly unbalanced scalings are intrinsic to the geometry of solutions. The iterative renormalization strategy employed here bears some conceptual resemblance to the nonlinear scaling ideas developed in [2] for handling fully nonlinear equations with arbitrary degeneracy laws. From a heuristic perspective, one may also compare our approach with the scaling method introduced in [39] and refined in [40] to obtain higher regularity at critical points. Nevertheless, the scheme introduced in the present paper is of a fundamentally different nature: it is designed specifically for non-differentiable variational integrals and relies crucially on the Dini-type control in the zero-order term. In particular, the nonlinear normalization and transmission-of-oscillation mechanism developed here appear to be new and intrinsic to this setting.

In the core of our analysis is a *non-homogeneous scaling and normalization scheme* that enables the transmission of oscillation-decay information across successive, non-homogeneous scales while preserving the structural properties of the functional. Each iteration combines harmonic approximation with a nonlinear normalization step that rebalances the dependence on the unknown $u$. This allows us to propagate the smallness of the potential oscillation across scales and to derive a quantitative affine approximation principle, which in turn yields a modulus of differentiability for local minimizers. In essence, the argument replaces the classical homogeneous excess-decay framework with a nonlinear, scale-adaptive iteration. The approach resolves a natural question regarding $C^1$-regularity under Dini-type continuity and paves the way for broader applications. We expect the ideas developed here to extend naturally to problems with general $p$-growth, to elliptic systems (yielding partial $C^1$-regularity), and to certain classes of free boundary models with non-homogeneous dependence on the state variable.

The remainder of the paper is organized as follows. In Section 2 we gather the structural preliminaries and auxiliary results required for the analysis. Section 3 establishes the key quantitative approximation and decay estimates that form the core of the argument. In Section 4 we develop the non-homogeneous scaling and normalization scheme and complete the proof of Theorem 1.

## 2. Preliminary results

In this section we collect several auxiliary results and structural properties that will be repeatedly used in the proof of Theorem 1. We first consider a slightly more general class of variational integrals with explicit dependence on the spatial variable. We then recall the $C^1$-regularity theory for $A$-harmonic functions under Dini continuity assumptions on the coefficients, and finally discuss the scaling behavior of such functionals.

### 2.1. **An extended class of functionals.**

It is convenient to introduce a broader class of quadratic-growth functionals, given by

$$\tilde{J}(u, \Omega) := \int_{\Omega} \langle A(x)\nabla u, \nabla u \rangle + f(x, u)\, dx, \tag{2.1}$$

where $A(x)$ is a symmetric, $\Lambda$-uniformly elliptic matrix, i.e.

$$\Lambda^{-1}|\xi|^2 \le \langle A(x)\xi, \xi \rangle \le \Lambda |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n,\ x \in \Omega.$$

Our key structural assumption is a uniform Dini-type continuity of $f$ with respect to $u$, uniformly in $x$. Namely, we assume that there exists a nonnegative, increasing modulus $\omega(\cdot)$ such that for all $s, t \in \mathbb{R}$ and a.e. $x \in \Omega$,

$$|f(x, s) - f(x, t)| \le \omega(|s - t|)\,, \tag{2.2}$$

and that $\sqrt{\omega}$ is a Dini modulus, i.e.

$$\int_0^1 \frac{\sqrt{\omega(r)}}{r}\, dr < +\infty.$$

Theorem 1 remains valid for local minimizers of (2.1), provided $A$ itself is Dini continuous in $x$. This level of continuity on both the coefficients and the potential represents a natural threshold for the gradient continuity of local minima.

### 2.2. **Basic estimates and boundedness of minimizers.**

By standard arguments in the calculus of variations (see, for instance, [22]), local minimizers of (2.1) are locally bounded whenever the potential $f(x, u)$ exhibits subcritical growth in $u$; that is,

$$|f(x, u)| \le C(1 + |u|^q), \qquad q < 2^* = \frac{2n}{n-2} \quad \text{for } n > 2.$$

In particular, if $u$ is a local minimizer of $\tilde{J}$ in $B_2$, then there exists a constant $C_0 = C(n, \lambda, \Lambda, \|u\|_{L^2(B_2)}) > 0$ such that

$$\|u\|_{L^\infty(B_1)} \le C_0. \tag{2.3}$$

Once the $L^\infty$ bound is established, one may regard $f$ as a bounded potential on the range $[0, C_0]$,

$$\|f\|_{L^\infty(\mathbb{R}\times[0,C_0])} \le C_1,$$

and the De Giorgi–Nash–Moser theory applies to yield local Hölder continuity of $u$. By a refinement of the classical oscillation-decay argument, one can even obtain a Log-Lipschitz modulus of continuity for local minimizers, with constants depending only on $n$, the ellipticity ratio $\Lambda$, and $\|u\|_{L^2(B_1)}$, see for instance [5].

2.3. **Comparison with $A$-harmonic replacements.** A fundamental step in our analysis is the comparison of a local minimizer $u$ with its $A$-harmonic replacement. In contrast with the homogeneous case, the potential term $f(x,u)$ introduces a non-homogeneous component in the scaling, so that the comparison must be carried out more carefully. The role of the $A$-harmonic replacement is to provide a "frozen" approximation of $u$ at each scale. The main difficulty is to transfer oscillation decay across scales in a non-homogeneous fashion.

We recall the following regularity result for $A$-harmonic functions, which serves as our linear benchmark.

**Theorem 2** ($C^1$ regularity for $A$-harmonic functions)**.** *Let $h \in H^1(B_r)$ be a weak solution of*

$$\operatorname{div}(A(x)\nabla h) = 0 \quad \textit{in } B_r.$$

*Assume that A is $\Lambda$-uniformly elliptic and Dini-continuous in x. Then $h \in C^1(B_{r/2})$ and its gradient is uniformly continuous. More precisely, there exists a modulus of continuity $\varrho$, depending only on n, $\Lambda$, and the Dini modulus of A, such that*

$$|\nabla h(x) - \nabla h(y)| \le C(n,\Lambda)\, \|h\|_{L^2(B_r)}\, \varrho(|x-y|), \qquad x, y \in B_{r/2}.$$

**Remark 2.1.** *Theorem 2 goes back to classical work of Lieberman, see* [29, Theorem 5.1]*. The corresponding result for linear systems was obtained by Y. Y. Li in* [28]*, answering a question posed by H. Brezis; see also* [15]*. See also* [16, 25] *for a corresponding non-linear potential theory. The connection between Dini continuity of the coefficients and $C^1$-regularity is by now well understood. For instance, in* [15] *the authors show that weak solutions to the conormal derivative problem for divergence-form elliptic equations are $C^1$ up to the boundary, assuming Dini mean oscillation of the leading coefficients and a $C^1$ boundary with Dini-continuous derivatives. Related phenomena linking Dini-type assumptions and fine regularity also appear in* [6, 18, 24, 17]*. The estimate in Theorem 2 is essentially optimal and plays a decisive role in the nonlinear iteration scheme developed later.*

2.4. **Non-homogeneous scaling of the functional.** We conclude this section by discussing the scaling properties of $\tilde{J}$ and introducing the *non-homogeneous scaling transformation* that underlies our iteration argument. Unlike the classical homogeneous scaling used for autonomous integrands, the potential term $f(x,u)$ couples the scaling in both the spatial and the $u$-variables.

**Proposition 2.1** (Scaling property)**.** *Let $u \in H^1(B_1)$ be a local minimizer of $\tilde{J}$ defined in (2.1), and let $B_r(x_0) \subset B_1$. Denote by h the A-harmonic replacement of u in $B_r(x_0)$. Given constants $0 < a < r$, $b > 0$, and $c \in \mathbb{R}$, define $v\colon B_1 \to \mathbb{R}$ by*

$$v(x) := \frac{(u-h)(ax+x_0)+c}{b}.$$

*Then v is a local minimizer of the rescaled energy functional*

$$\tilde{J}_{a,b,c}(v) = \int_{B_1} \langle A(ax+x_0)\nabla v, \nabla v\rangle + \frac{a^2}{b^2}\, f(ax+x_0,\, bv(x) + h(ax+x_0) - c)\, dx.$$

*Proof.* The proof follows by a direct change of variables. Setting $y = ax + x_0$, we write

$$u(y) = b\,v\Big(\tfrac{y-x_0}{a}\Big) + h(y) - c. \tag{2.4}$$

Differentiating with respect to $x$ gives

$$a\,\nabla u(ax + x_0) = b\,\nabla v(x) + a\,\nabla h(ax + x_0).$$

Substituting this into the definition of $\tilde{J}(u)$ and performing the change of variables $y = ax + x_0$ yields

$$\tilde{J}(u) = a^{n+2}b^{-2}\int_{B_1} \langle A(ax{+}x_0)\nabla v(x), \nabla v(x)\rangle + \frac{b^2}{a^2} f(ax + x_0,\ bv(x) + h(ax + x_0) - c)\ dx,$$

which gives the desired functional and proves the claim. □

In light of Proposition 2.1, for any given constants $a, b, c > 0$ and $x_0 \in B_{1/2}$, we define the *scaled potential*

$$\tilde{f}(z,t) = \tilde{f}_{a,b,c}(z,t) := \frac{a^2}{b^2}\, f(az + x_0,\ bt + h(az + x_0) - c)\,. \tag{2.5}$$

From assumption (2.2), one immediately obtains that $\tilde{f}$ inherits a modulus from $f$:

$$\Big|\tilde{f}(z,t) - \tilde{f}(z,s)\Big| \le \frac{a^2}{b^2}\,\omega\,(b|t-s|)\,, \qquad \text{for all } s,t \in \mathbb{R}. \tag{2.6}$$

The key idea will be to construct sequences of scaling parameters as to ensure that the Dini structure of the original integrand gets transmitted through the non-homogeneous scaling. This property will be essential in the iterative excess-decay argument developed in the next section.

## 3. Quantitative $L^\infty$ approximation by $C^1$ functions

A central ingredient in our argument is the quantitative comparison between a local minimizer $u$ and its $A$-harmonic replacement $h$. Such comparisons play a fundamental role in nonlinear regularity theory, bridging the variational structure of the problem with the linear regularity available for $A$-harmonic functions.

In this section, we first establish uniform $L^\infty$-bounds for the difference $(u - h)$. We then show that, when the potential $f(x,u)$ is sufficiently close to a constant, the minimizer $u$ is quantitatively close to its harmonic replacement. These results provide the foundation for the non-homogeneous scaling and iteration scheme of Section 4.

3.1. **$L^2$–$L^\infty$ control.** Let $u$ be a local minimizer of the energy functional (2.1) in $B_1$, and let $h$ denote its $A$-harmonic replacement in $B_{1/2}$. The difference $u - h$ behaves, in a precise sense, as a *Q-minimizer* of a uniformly elliptic quadratic functional in the sense of Giaquinta–Giusti [21]; see also [22]. More concretely, one verifies that $u - h$ is a $Q$-minimum of

$$\mathscr{D}(v) := \int_{B_{1/2}} (|\nabla v|^2 + 1)\,dx,$$

for some $Q > 0$ depending only on the ellipticity constant and $\|u\|_{H^1(B_1)}$. Hence $(u - h)$ inherits the uniform regularity theory available for $Q$-minima, including quantitative integral-to-supremum estimates, see also [19], as well as [32] for regularity in free boundary problems.

**Lemma 3.1** ($L^2$–$L^\infty$ estimate for $Q$-minima)**.** *Let $u$ be a local minimizer of (2.1) in $B_1$, and let $h$ be its $A$-harmonic replacement in $B_{1/2}$. Fixed $0 < \lambda < 1/2$, there exists a constant $C = C(n, \Lambda, \|u\|_{H^1(B_1)}, \lambda) > 0$ such that*

$$\|u - h\|_{L^\infty(B_\lambda)} \le C\,\|u - h\|_{L^2(B_{1/2})}. \tag{3.1}$$

*Proof.* The claim follows from standard energy decay and local boundedness estimates for $Q$-minima (see, e.g., [21, 22]). □

3.2. **Quantitative $L^\infty$ closeness.** We now refine Lemma 3.1 by showing that when the potential term $f(x, u)$ has sufficiently small oscillation, the minimizer $u$ must be uniformly close to its harmonic replacement $h$. This provides a first quantitative step towards the full $C^1$-regularity result.

**Lemma 3.2** (Quantitative approximation)**.** *Let $u$ be a local minimizer of (2.1) in $B_1$, with $\|u\|_{L^\infty(B_1)} \le 1$. Fix $\lambda < 1/2$ and $\varepsilon > 0$ (say $\lambda = 1/4$ and $\varepsilon = 1/100$). There exists a constant $0 < \delta \lll 1$, (depending only on $n$, $\Lambda$ and the choices for $0 < \lambda < 1/2$, and $\varepsilon > 0$) such that if $\omega(2) \le \delta$ on $B_2$, then*

$$\sup_{B_\lambda} |u - h| \le \lambda^{1+\varepsilon}, \tag{3.2}$$

*where $h$ denotes the $A$-harmonic replacement of $u$ in $B_{1/2}$.*

*Proof.* Since $u$ is a minimizer of (2.1), it competes with its $A$-harmonic replacement $h$ in $B_{1/2}$. By minimality of $u$,

$$\int_{B_{1/2}} \left(\langle A(x)\nabla u, \nabla u\rangle - \langle A(x)\nabla h, \nabla h\rangle\right) dx \le \int_{B_{1/2}} [f(x, h) - f(x, u)]\, dx. \tag{3.3}$$

Using that $u - h = 0$ on $\partial B_{1/2}$ and that $h$ is $A$-harmonic, we have

$$\int_{B_{1/2}} \langle A(x)\nabla(u - h), \nabla(u - h)\rangle\, dx = \int_{B_{1/2}} \left(\langle A(x)\nabla u, \nabla u\rangle - \langle A(x)\nabla h, \nabla h\rangle\right) dx. \tag{3.4}$$

By ellipticity of $A$,

$$\Lambda^{-1} \int_{B_{1/2}} |\nabla(u - h)|^2\, dx \le \int_{B_{1/2}} \langle A(x)\nabla(u - h), \nabla(u - h)\rangle\, dx. \tag{3.5}$$

Since $\|u\|_\infty, \|h\|_\infty \le 1$, we have $|u - h| \le 2$ in $B_{1/2}$. By the Dini continuity of $f$ in $u$, assumption (2.2) yields

$$\int_{B_{1/2}} |f(x, h) - f(x, u)|\, dx \le \int_{B_{1/2}} \omega(|h - u|)\, dx \le |B_{1/2}|\delta. \tag{3.6}$$

Combining (3.3)–(3.6), we deduce

$$\int_{B_{1/2}} |\nabla(u - h)|^2\, dx \le C\,\Lambda\,\delta.$$

By the Poincaré inequality,

$$\int_{B_{1/2}} |u-h|^2\,dx \;\le\; C\,\Lambda\,\delta.$$

Finally, applying Lemma 3.1 and then fixing some $\lambda < 1/2$, we obtain

$$\sup_{B_\lambda} |u-h| \le C_1\,\Lambda\,\delta.$$

Choosing $\delta > 0$ sufficiently small so that $C_1\Lambda\delta \le \lambda^{1+\varepsilon}$ gives (3.2) and concludes the proof. □

**Remark 3.1.** *Lemma 3.2 can be viewed as a quantitative counterpart of the classical "freezing" argument in regularity theory, adapted here to a non-homogeneous setting. It shows that small oscillations of the potential term $f(x,u)$ yield a quantitative $L^\infty$-approximation of the minimizer by a universally controlled $C^1$ function, namely its $A$-harmonic replacement. In this sense, the lemma establishes that uniform smallness of the oscillation of the integrand translates into smallness of the deviation from a harmonic profile, paving the way for an iterative oscillation-decay mechanism.*

The next section develops this idea further. We introduce a *non-homogeneous scaling and normalization algorithm* specifically designed to preserve the structural assumptions of Lemma 3.2 across successive scales. This iterative scheme allows us to propagate oscillation decay through a hierarchy of rescaled versions of the functional. At each step, the normalization acts through a nonlinear adjustment of the scaled integrand, keeping the problem within the smallness regime. Carrying out this algorithm inductively yields a uniform decay of the oscillation and, ultimately, the proof of Theorem 1. The Dini structure of the modulus of continuity of $f$ will be crucial to ensure the convergence of a sequence of approximating hyperplanes.

## 4. A non-homogeneous scaling algorithm

We now begin the proof of Theorem 1. The argument relies on an iterative renormalization and scaling scheme that captures oscillation decay in the absence of homogeneous structure. The core idea is to construct, at successive scales, a sequence of uniformly controlled $C^1$ functions approximating the given local minimizer, with the associated potentials remaining small in a quantitative sense so that Lemma 3.2 applies at each step.

4.1. **Smallness regime.** Let $x_0 \in B_{1/2}$ be fixed. We show that $u$ is differentiable at $x_0$, and that $\nabla u$ is continuous in a neighborhood of $x_0$. Without loss of generality, assume $x_0 = 0$ and $u(0) = 0$. After a normalization and a rescaling (which do not affect the Dini character of the coefficients), we may assume that $u$ is a normilzed (i.e. $\|u\|_\infty \le 1$) local minimizer of (2.1) satisfying

$$\omega(2) < \delta,$$

where $\delta > 0$ is the smallness constant from Lemma 3.2. Denote by $h_0$ the $A$-harmonic replacement of $u$ in $B_{1/2}$. Lemma 3.2 ensures that

$$\sup_{B_\lambda} |u - h_0| \le \lambda^{1+\varepsilon}, \tag{4.1}$$

for some universal constants $0 < \lambda \ll 1$ and $\varepsilon > 0$. The maximum principle yields $\|h_0\|_{L^\infty(B_{1/2})} \le 1$.

The remainder of the proof builds upon (4.1) through an iterative renormalization scheme. At each step, we rescale and normalize the deviation from the harmonic replacement, obtaining a new function that again satisfies the hypotheses of Lemma 3.2. This process produces a quantitative oscillation decay and ultimately leads to the $C^1$-regularity of $u$.

4.2. **First renormalization.** Define the rescaled function

$$v_1(x) := \frac{(u - h_0)(\lambda x)}{\mu_1 \lambda},$$

for some $\lambda^\varepsilon \le \mu_1 < 1$ to be chosen through an algorithm to be detailed below. By (4.1), $\|v_1\|_{L^\infty(B_1)} \le 1$. By Proposition 2.1, $v_1$ is a local minimizer of the rescaled functional

$$J_1(v) := \int_{B_1} \langle A(\lambda x)\nabla v, \nabla v\rangle + \tilde{f}_1(x, v)\, dx,$$

where

$$\tilde{f}_1(z, t) = \mu_1^{-2} f(\lambda z, \mu_1 \lambda t + h_0(\lambda z)). \tag{4.2}$$

Since $|h_0| \le 1$ and $|t| \le 1$, the argument of $f$ in (4.2) remains bounded by 2. We now define the selection rule for choosing $\lambda^\varepsilon \le \mu_1 < 1$.

- If
$$\omega_{\lambda^\varepsilon}(2) \le \delta,$$
then set
$$\mu_1 := \lambda^\varepsilon.$$
- Otherwise, that is, if
$$\omega_{\lambda^\varepsilon}(2) > \delta,$$
we determine $\mu_1$ by solving the threshold equation
$$\omega_{\mu_1}(2) = \delta.$$
Indeed, by the Dini continuity of $f$ in the $u$-variable,
$$|\tilde{f}_1(z, t) - \tilde{f}_1(z, s)| \le \mu_1^{-2}\, \omega(\mu_1 \lambda |t - s|) =: \omega_{\mu_1}(|t - s|).$$
For $\mu_1$ sufficiently close to 1, we have
$$\omega_{\mu_1}(2) \approx \omega(2\lambda) < \delta.$$
Therefore, since $\omega_{\lambda^\varepsilon}(2) > \delta$, continuity allows us to choose $\mu_1 \in [\lambda^\varepsilon, 1)$ such that
$$\omega_{\mu_1}(2) = \mu_1^{-2}\, \omega(2\mu_1 \lambda) = \delta. \tag{4.3}$$

With this choice, $v_1$ satisfies the smallness condition in Lemma 3.2, and hence there exists an $A$-harmonic function $h_1$ in $B_{1/2}$ such that

$$\sup_{B_\lambda} |v_1 - h_1| \le \lambda^{1+\varepsilon}.$$

4.3. **Iterative construction.** We now repeat the procedure. Define

$$v_2(x) := \frac{(v_1 - h_1)(\lambda x)}{\mu_2 \lambda},$$

for some $\lambda^\varepsilon \le \mu_2 < 1$ to be determined. Then $v_2$ minimizes a new functional

$$J_2(v) = \int_{B_1} \langle A(\lambda^2 x)\nabla v, \nabla v\rangle + \tilde{f}_2(x, v)\, dx,$$

where

$$\tilde{f}_2(z,t) = \mu_2^{-2} \tilde{f}_1(\lambda z,\, \mu_2 \lambda t + h_1(\lambda z)) = (\mu_1\mu_2)^{-2} f(\mu_1\mu_2\lambda^2 t + \mu_1 \lambda h_1(\lambda z) + h_0(\lambda^2 z)).$$

By the continuity of $f$,

$$|\tilde{f}_2(z,t) - \tilde{f}_2(z,s)| \le (\mu_1\mu_2)^{-2}\, \omega(\mu_1\mu_2\lambda^2|t-s|) =: \omega_{\mu_2}(|t-s|).$$

Arguing as in the first step, we select $\mu_2$ either as $\mu_2 = \lambda^\varepsilon$ or so that

$$\omega_{\mu_2}(2) = (\mu_1\mu_2)^{-2}\omega(\mu_1\mu_2\lambda^2) = \delta,$$

ensuring that $v_2$ satisfies the assumptions of Lemma 3.2. Therefore, there exists an $A$-harmonic function $h_2$ such that

$$\sup_{B_\lambda} |v_2 - h_2| \le \lambda^{1+\varepsilon}.$$

4.4. **Recursive iteration and summability.** Iterating the construction, we define recursively

$$v_{k+1}(x) := \frac{(v_k - h_k)(\lambda x)}{\mu_{k+1}\lambda}, \qquad \lambda^\varepsilon \le \mu_{k+1} < 1,$$

and denote $\pi_k := \prod_{i=1}^k \mu_i$. A direct computation shows that $\tilde{f}_k$ can be written as

$$\tilde{f}_k(z,t) = \pi_k^{-2} f\left(\pi_k \lambda^k t + \sum_{j=0}^{k-1} \pi_j \lambda^{j+1} h_{k-1-j}(\lambda^{j+1} z)\right).$$

Each $\tilde{f}_k$ inherits a modulus of continuity

$$|\tilde{f}_k(z,t) - \tilde{f}_k(z,s)| \le \pi_k^{-2}\, \omega(\pi_k \lambda^k |t-s|).$$

Once $\mu_1, \ldots, \mu_k$ have been chosen, we select $\mu_{k+1}$ either to be $\lambda^\varepsilon$ or else so as to satisfy

$$\pi_{k+1}^{-2}\, \omega(\pi_{k+1}\lambda^{k+1}) = \delta. \tag{4.4}$$

Since $\lambda^\varepsilon \le \mu_i < 1$, it follows that $\lambda^{k\varepsilon} \le \pi_k \le 1$, and by Dini integrability of $\omega$,

$$\sum_{k=1}^\infty \pi_k \le \frac{C}{\lambda\sqrt{\delta}} \int_0^1 \frac{\sqrt{\omega(t)}}{t}\, dt < \infty.$$

This quantitative control shows that the cumulative corrections introduced by the iterative normalization remain summable, i.e. $(\pi_k)_{k=1}^{\infty} \in \ell_1$.

4.5. **Convergence of affine approximations.** At each iteration step, we obtain an $A$-harmonic function $h_k$ satisfying

$$\sup_{B_\lambda} |v_k - h_k| \le \lambda^{1+\varepsilon}. \tag{4.5}$$

Rewriting this information in terms of the original minimizer $u$ yields

$$\sup_{B_\lambda} |u(\lambda^k x) - \mathscr{H}_k(x)| \le \lambda^{1+\varepsilon}\lambda^k \pi_k, \tag{4.6}$$

where

$$\mathscr{H}_k(x) := h_0(\lambda^k x) + \pi_1 \lambda h_1(\lambda^{k-1}x) + \pi_2\lambda^2 h_2(\lambda^{k-2}x) + \cdots + \pi_k \lambda^k h_k(x). \tag{4.7}$$

For notation convenience, declare $\pi_0 := 1$. In the sequel define

$$b_k := \sum_{i=0}^{k} \pi_i \lambda^i h_i(0) \in \mathbb{R}, \qquad \text{and} \qquad A_k := \sum_{i=0}^{k} \pi_i \nabla h_i(0) \in \mathbb{R}^n. \tag{4.8}$$

Next we consider the affine approximating function

$$L_k(x) := A_k \cdot x + b_k. \tag{4.9}$$

Since $\sum_k \pi_k < \infty$, both sequences $(A_k)$ and $(b_k)$ are Cauchy and hence converge to some $A_\infty$ and $b_\infty$, respectively. We define the limiting affine function

$$L_\infty(x) = A_\infty \cdot x + b_\infty. \tag{4.10}$$

We now use $L_k$ to approximate $\mathscr{H}_k(x)$ in a controlled fashion. We initially note that, since each $h_i$ is $A$-harmonic in $B_{1/2}$ and uniformly bounded in $L^\infty$, Theorem 2 implies that they are uniformly $C^1$, i.e.

$$\sup_{B_r} \left| h_j(x) - [h_j(0) + \nabla h_j(0)x] \right| \le C r \varrho(r), \tag{4.11}$$

for some modulus of continuity $\varrho$. We then can estimate, for all $j = 0, 1, \cdots n$,

$$\left| h_j(\lambda^{k-j}x) - [h_j(0) + \lambda^{k-j}\nabla h_j(0)\cdot x] \right| \le C\lambda^{k-j}\varrho(\lambda^{k-j}). \tag{4.12}$$

Multiplying (4.12) by $\pi_j \lambda^j$ and summing up from $j = 0$ to $j = k$ we obtain

$$\left| \mathscr{H}_k(x) - L_k(\lambda^k x) \right| \le C\lambda^k \sum_{j=0}^{k} \pi_j \varrho(\lambda^{k-j}). \tag{4.13}$$

**Oscillation estimate.** Let $0 < r < 1$ and choose $k \in \mathbb{N}$ such that $\lambda^{k+1} < r \le \lambda^k$. We estimate the deviation of $u$ from its limiting affine approximation $L_\infty$. Using (4.6), (4.13), and the convergence of $(L_k)$, we find

(4.14)
$$\begin{aligned} \sup_{B_r} |u - L_\infty| &\le \sup_{B_{\lambda^k}} |u - L_\infty| \\ &\le \sup_{B_{\lambda^k}} \left| u - \mathscr{H}_k(\lambda^{-k}x) \right| + \sup_{B_{\lambda^k}} \left| \mathscr{H}_k(\lambda^{-k}x) - L_k(x) \right| + \sup_{B_{\lambda^k}} |L_k - L_\infty| . \end{aligned}$$

The first term in (4.14) is controlled by (4.6):

$$\sup_{B_{\lambda^{k+1}}} \left|u(x) - \mathscr{H}_k(\lambda^{-k}x)\right| = \sup_{B_\lambda}\left|u(\lambda^k x) - \mathscr{H}_k(x)\right| \le \lambda^{1+\varepsilon}\lambda^k \pi_k. \tag{4.15}$$

For the second term, we employ (4.13)

$$\sup_{B_{\lambda^k}}\left|\mathscr{H}_k(\lambda^{-k}x) - L_k(x)\right| \le C\lambda^k \sum_{j=0}^{k} \pi_j \varrho(\lambda^{k-j}). \tag{4.16}$$

Finally, for the third term, the convergence of the affine approximations implies

$$\sup_{B_{\lambda^k}}|L_k - L_\infty| \le C\,\lambda^k\left(\sum_{j=k}^{\infty}\pi_j + \sum_{j=k}^{\infty}\lambda^j\pi_j\right). \tag{4.17}$$

Combining (4.15)–(4.17), we obtain

$$\sup_{B_r}|u - L_\infty| \le C\,\lambda^k\left(\lambda^{1+\varepsilon}\pi_k + \sum_{j=0}^{k}\pi_j\varrho(\lambda^{k-j}) + \sum_{j=k}^{\infty}\pi_j + \sum_{j=k}^{\infty}\lambda^j\pi_j\right). \tag{4.18}$$

Define

$$\alpha_k := \lambda^{1+\varepsilon}\pi_k + \sum_{j=0}^{k}\pi_j\varrho(\lambda^{k-j}) + \sum_{j=k}^{\infty}\pi_j + \sum_{j=k}^{\infty}\lambda^j\pi_j,$$

and notice $\lim_{k\to 0}\alpha_k = 0$. Back to the oscillation control, since $\lambda^{k+1} < r \le \lambda^k$, we have $\lambda^k \le r/\lambda$, and therefore

$$\sup_{B_r}|u - L_\infty| \le \frac{C}{\lambda}\, r\cdot\varsigma(r), \tag{4.19}$$

where

$$\varsigma(r) := \alpha_{\lfloor \log_\lambda r\rfloor}; \tag{4.20}$$

a modulus of continuity.

**Conclusion.** Since both $\varrho$ and $\varsigma$ are moduli of continuity, (4.19) shows that $u$ admits an affine approximation $L_\infty$ at the origin with a quantitative modulus of differentiability depending only on the ellipticity constants and the Dini modulus $\omega$ of the integrand. Hence,

$$u \in C^1_{\mathrm{loc}}(\Omega),$$

and Theorem 1 follows.

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