

EXTREMAL EFFECTIVE CURVES AND NON-SEMIAMPLE LINE BUNDLES ON $\overline{M}_{g,n}$

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ABSTRACT. We develop a new method for establishing extremality in the closed cone of effective curves on the moduli space of curves, and determine the extremality of many boundary 1-strata. Using this result, together with a general criterion for non-semiampleness that extends Keel’s argument, we show that a substantial portion of the cone of nef divisors of $\overline{M}_{g,n}$ is not semiample. As an application, we construct the first explicit example of a non-contractible extremal ray of the closed cone of effective curves on $\overline{M}_{3,n}$, which becomes contractible in positive characteristic. This highlights the dependence of the geometry of $\overline{M}_{g,n}$ on the characteristic of the base field. Our method relies on two main ingredients: (1) the construction of a new collection of nef divisors on $\overline{M}_{g,n}$, and (2) the identification of a tractable inductive structure on the Picard group, arising from Knudsen’s construction of $\overline{M}_{g,n}$.

1. INTRODUCTION

In this paper, we investigate the birational geometry of $\overline{M}_{g,n}$, the moduli space of stable, n -pointed genus g curves, by introducing a new method to establish extremality in the cone of effective curves. This approach provides new insights into the dual cone of nef divisors and, in particular, illuminates the subtle nature of semiample divisors on $\overline{M}_{g,n}$.

By Stein factorization, the investigation of morphisms from $\overline{M}_{g,n}$ to projective varieties can be reduced to the study of projective contractions of $\overline{M}_{g,n}$. Here, a projective contraction means a morphism to a projective variety $f : X \rightarrow Y$ such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Mori theory offers a framework for such analysis by associating to each contraction $f : X \rightarrow Y$ of projective varieties the relative cone of curves $\overline{NE}_1(f)$, which corresponds to a face of the closed cone of curves $\overline{NE}_1(X)$. The cone $\overline{NE}_1(f)$ always determines the morphism f , and in certain favorable situations—such as when X is a log Fano variety over a field of characteristic zero—this theory yields a well-behaved bijection between contractions and faces.

Two fundamental issues can obstruct a correspondence between faces of the effective cone of curves and projective contractions. First, $\overline{NE}_1(X)$ may be non-polyhedral, with infinitely many faces, some of which consist only of limits of curve classes. Second, it may happen that not every face of $\overline{NE}_1(X)$ corresponds to a contraction, since the cone of nef divisors may contain divisors that are not semiample. While conjecturally the first issue will not arise on $\overline{M}_{g,n}$ (cf. Conjecture 2.1), our first main theorem shows that the second phenomenon is especially pronounced for $\overline{M}_{g,n}$: a large portion of the nef cone consists of non-semiample divisors.

Theorem 1.1. Assume that the characteristic of the base field is 0, and that either $g = 2$, $n \geq 2$, or $g \geq 3$, $n \geq 1$. Then there is a codimension $\lfloor \frac{g}{2} \rfloor$ face F of the nef cone of $\overline{M}_{g,n}$ such that its general element is not semiample. More precisely, there is a codimension one linear subspace E of F such that any nef divisor in $F \setminus E$ is not semiample.

If $g = 2$ or 3, then F has codimension one, and for all g , the codimension is constant as $n \rightarrow \infty$. This indicates that the non-semiample region of the nef cone occupies a non-negligible portion of it. Theorem 1.1 implies the following, more directly related to the second issue:

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Corollary 1.2. Assume that the characteristic of the base field is 0 and $g \geq 3$, $n \geq 1$. Then there exists a dimension $\lfloor \frac{g}{2} \rfloor$ face C of $\overline{\text{NE}}_1(\overline{\text{M}}_{g,n})$ which is not contractible, i.e. there is no projective contraction of $\overline{\text{M}}_{g,n}$ whose relative cone of curves is C .

Hence, in particular, there exists a noncontractible extremal ray of $\overline{\text{NE}}_1(\overline{\text{M}}_{3,n})$. However, by Theorem 11.1, this extremal ray of $\overline{\text{NE}}_1(\overline{\text{M}}_{3,n})$ is contractible in positive characteristic, and the contraction admits a very simple description. This highlights the dependence of the geometry of $\overline{\text{M}}_{g,n}$ on the characteristic of the base field. We will return to this point at the end of the introduction and in Section 12.

The construction of nef but non-semiample line bundles on $\overline{\text{M}}_{g,n}$ dates back at least to Keel [32], who proved that in characteristic 0, ψ -classes are not semiample in general. In addition, we prove in Theorem 4.1 that ψ -classes are semiample on $\overline{\text{M}}_{1,n}$, and thus we completely determine the cases in which ψ -classes are semiample (Corollary 4.2). Moreover, in Section 6, we explain that they play a significant role in contracting F-curves on $\overline{\text{M}}_{1,n}$ (cf. Remark 6.7).

To prove Theorem 1.1, we utilize a non-semiample criterion, Theorem 3.1. The proof of Theorem 3.1 follows Keel's method, employing arguments similar to those of [9]. One major difficulty in the proof of Theorem 1.1 lies in computing the dimension of the space of nef divisors satisfying the criterion of Theorem 3.1. Surprisingly, this difficulty is intimately related to the first issue of Mori theory discussed above.

The irreducible components of 1-dimensional boundary strata on $\overline{\text{M}}_{g,n}$, so-called **F-curves**, are classified into six types in [26]. The **F-conjecture** asserts that $\overline{\text{NE}}_1(\overline{\text{M}}_{g,n})$ is generated by F-curves, and hence is polyhedral. There has been extensive work on the F-conjecture (e.g., [19–21, 24, 26, 33, 36, 38]), but still it remains wide open. We will recall the relevant background in Section 2, including the notation for F-curves.

We define an extremal ray of a cone to be a **regular extremal** if the corresponding face of the dual cone has codimension 1. Note that if the cone is polyhedral, then every extremal ray is regular, but this is not true in general (cf. Remark 6.2). This notion was implicitly used in [41] to prove that the cone of moving curves of $\overline{\text{M}}_{g,n}$, for $g, n \geq 2$, is non-polyhedral (and hence the cone of pseudo-effective divisors is also non-polyhedral). This was done by constructing a non-regular extremal ray [41, Theorem 1.1].

The following result shows that many extremal F-curves are indeed regular, providing further evidence for the F-conjecture. Since the exact statements use notation from Section 2, we provide here only abbreviated versions of the theorems. For the full statements, we refer to Theorem 10.1 and Theorem 10.2.

Theorem 1.3. Assume that the characteristic of the base field is not equal to 2. Then:

- (1) Type 1 and type 4 F-curves on $\overline{\text{M}}_{g,n}$ span regular extremal rays of $\overline{\text{NE}}_1(\overline{\text{M}}_{g,n})$, whereas type 2 F-curves do not.
- (2) Apart from three exceptional families, each of which spans a regular extremal ray, no type 3 F-curve spans an extremal ray of $\overline{\text{NE}}_1(\overline{\text{M}}_{g,n})$.
- (3) Knudsen-type F-curves (cf. Definition 10.11) are regular extremal. In particular, every F-curve on $\overline{\text{M}}_{0,n}$ and $\overline{\text{M}}_{1,n}$ is regular extremal.
- (4) There exist three additional families of type 6 F-curves, each of which spans a regular extremal ray.

In characteristic 2, the statements concerning Type 3 curves remain valid. Moreover, if [26, Theorem 0.3] holds in characteristic 2, then all of the statements follow.

In the same vein, our method can be applied to study the regular extremality of F-curves for small genus.

Theorem 1.4. Assume that the characteristic of the base field is not equal to 2. Then:

- (1) Every type 5 or type 6 F-curve on $\overline{\text{M}}_{2,n}$ is regular extremal.
- (2) Apart from one exceptional family (resp. two exceptional families), every type 5 and type 6 F-curve on $\overline{\text{M}}_{3,n}$ (resp. $\overline{\text{M}}_{4,n}$) is regular extremal.

Moreover, if [26, Theorem 0.3] holds in characteristic 2, then all of the statements follow.

The dependence on the assumption that the characteristic is not equal to 2 arises from the fact that [26], and consequently many results on the F-conjecture, rely on this assumption. As stated above, if one can establish the relevant results on the F-conjecture in characteristic 2, then our results can be extended to characteristic 2 as well. Moreover, by Corollary 6.6, every F-curve on $\overline{M}_{0,n}$ and $\overline{M}_{1,n}$ is regular extremal, regardless of the characteristic.

As suggested by the statements, F-curves of type 5 and 6 are more subtle than the others. For type 5 curves, we have an explicit conjecture (Conjecture 10.10) describing which of them are regular extremal. In contrast, for type 6 curves, no conjectural description is currently available. We refer to the beginning of Section 10.3 for a detailed discussion.

These theorems can be understood in the context of the general problem of understanding extremal effective cycles. The study of extremal effective cycles on $\overline{M}_{g,n}$ is a well-developed area, with many important results as in [4, 7, 8, 40, 43, 49]. We refer to [6] for a survey of these results. However, many findings focus on relatively low-codimension cases. In contrast, this paper investigates the situation for extremal curves. In this setting, since many boundary strata are known to be non-extremal, distinguishing extremal from non-extremal curves is already a nontrivial task.

Theorem 1.3 and Theorem 1.4 require a construction of sufficiently many nef divisors on $\overline{M}_{g,n}$ that contract a fixed curve and span a codimension one subspace. In some simpler cases, such as $g = 0$ or $g = 1$ (see Section 6), ψ -classes and their pullbacks do the job. However, this approach fails for higher genus. One needs other nef divisors, and even with enough nef divisors, computing the dimension of their span becomes challenging.

The two main advances of this paper are the construction of new nef divisors on $\overline{M}_{g,n}$ (see Section 7), and the development of a new induction scheme to verify the dimension of the span of these divisors (see Section 9).

The new nef divisors introduced in Section 7, referred to here as **semigroup kappa divisors**, are certain sums of the κ class and boundary classes (cf. Definition 7.1). By examining their intersection with F-curves [26, Theorem 2.1] and boundary restriction, we prove that they are nef in Theorem 7.5. These divisors are particularly useful for contracting certain F-curves, as the intersection number of the kappa class with any F-curve is 1.

To verify that the set of nef divisors we construct spans a codimension-1 subspace, we employ a new two-step induction argument based on Knudsen's construction of $\overline{M}_{g,n}$, which we describe in detail in Section 9. In [10, 12], the author observed a particularly tractable inductive structure on the Picard group of $\overline{M}_{g,n}$ for $g \leq 1$, using Knudsen's construction. In this paper, we extend this observation to arbitrary genus in Theorem 8.1. A crucial point is that the relative cone arising from Knudsen's construction is a simplicial cone generated by F-curves. This enables explicit computation of the dimension of the span of certain nef divisors.

Finally, we note that Theorem 1.3 and Theorem 1.4 hold in almost arbitrary characteristic, whereas Theorem 1.1 and Corollary 1.2 hold only over fields of characteristic zero. Moreover, as shown in [32] and also in Corollary 1.2, there exist divisors that are semiample, as well as morphisms that occur only over fields of positive characteristic. This demonstrates that the geometry of $\overline{M}_{g,n}$ depends strongly on the characteristic of the base field, and raises the question of providing a modular description of such morphisms in positive characteristic (cf. Section 12.3).

One may compare this with the complete subvariety problem for the moduli space of abelian varieties A_g [27, 34], where the maximal-dimensional complete subvariety of A_g is described in terms of the p -rank, a structure that exists only in positive characteristic. Just as this problem highlights the role of the p -rank, we expect that the answer to the question raised in Section 12.3 will likewise shed further light on the essential differences between $\overline{M}_{g,n}$ in positive and in zero characteristic. In particular, since Theorem 11.1 provides an example of a contraction of $\overline{M}_{3,n}$ with a remarkably simple description that exists only in positive characteristic, it is an interesting problem to find a modular interpretation of this morphism. Such a description could potentially reveal new structural features of curves in positive characteristic.

1.1. Structure of the paper. In Section 2, we review the basics of divisors and curves on $\overline{\mathcal{M}}_{g,n}$. After this, the paper is divided into two parts: semiample and extremality. We first explain the semiample part. In Section 3, we present a non-semiample criterion (Theorem 3.1), which reduces Theorem 1.1 to extremality of certain F-curves. After establishing the extremality of these in Section 10, Theorem 1.1 and Corollary 1.2 are proved in Section 11. We further discuss the (non-)semiample of certain divisors on $\overline{\mathcal{M}}_{g,n}$ in Section 4 and Section 5. We hope that these sections contribute to the literature on semiample divisors on $\overline{\mathcal{M}}_{g,n}$.

The remainder of the paper is devoted to the extremality of F-curves. In Section 6, we provide an explicit solution to this problem for $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{1,n}$, together with a discussion of some basic convex geometry. This proof serves as a model argument that we generalize in later sections. Section 7 and Section 8 are devoted to describing two main tools for proving extremality: a new family of nef line bundles (Theorem 7.5) and an exact sequence (Theorem 8.1). Using these tools, we develop a new method for proving regular extremality of curves in Section 9. This method allows us to establish the extremality of certain F-curves in Section 10. Using this result, we prove Theorem 1.1 and Corollary 1.2 in Section 11. Further discussion and open problems are given in Section 12.

1.2. Notations and Conventions. Throughout, any Picard group $\text{Pic}(X)$, the cone of nef divisors $\text{Nef}(X)$, Chow group $A_d(X)$, curve class $[C]$, and divisors/line bundles will be considered over \mathbb{Q} . Thus, unless otherwise stated, these terms will refer to their \mathbb{Q} -versions, such as the \mathbb{Q} -Picard group, \mathbb{Q} -divisors, and so on. For a variety X , $\rho(X)$ denotes the Picard number of X . $\overline{\mathcal{M}}_{g,n}$ will denote the moduli stack of stable curves, while $\overline{\mathcal{M}}_{g,n}$ will refer to its coarse moduli space. We denote by $[n]$ the set $\{1, 2, \dots, n\}$, and by π_S the projection map $\overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,S^c}$ that forgets the marked points indexed by $S \subseteq [n]$. For any cone $N \subseteq \mathbb{R}^n$, we define $N \otimes \mathbb{R}$ to be the subspace of \mathbb{R}^n spanned by N .

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2. PRELIMINARIES

In this section, we review background on curves and divisors on $\overline{\mathcal{M}}_{g,n}$. Recall that $\overline{\mathcal{M}}_{g,n}$ denotes the moduli space, while $\overline{\mathcal{M}}_{g,n}$ denotes the moduli stack. The base field is an arbitrary algebraically closed field.

Let $\partial^r \overline{\mathcal{M}}_{g,n}$ denote the locus of stable curves with at least r nodes. Then $\partial^r \overline{\mathcal{M}}_{g,n}$ is a pure codimension r subvariety of $\overline{\mathcal{M}}_{g,n}$, and its irreducible components are called **codimension r boundary strata**. In particular, if $r = 3g - 4 + n$, then $\partial^r \overline{\mathcal{M}}_{g,n}$ is pure of dimension 1, and its irreducible components are called **F-curves**. A divisor on $\overline{\mathcal{M}}_{g,n}$ is **F-nef** if it intersects all F-curves non-negatively.

Conjecture 2.1. [F-conjecture, [26, 33]] F-curves generate $\overline{\text{NE}}_1(\overline{\mathcal{M}}_{g,n})$. Equivalently, a divisor on $\overline{\mathcal{M}}_{g,n}$ is nef if and only if it is F-nef.

By a sequence of papers, including [19–21, 24, 26, 33, 36, 38], the F-conjecture is known to hold for $g+n \leq 8$ or for $g \leq 44$ with $n = 0$, when the characteristic of the base field is not equal to 2. In characteristic 2, the methods of [21, 36] (resp. [16]) prove the conjecture for $g = 0$, $n \leq 8$ (resp. $n = 0, g = 2, 3$). For later use, in Proposition 7.16, we verify the F-conjecture in arbitrary characteristic for $\overline{\mathcal{M}}_{1,2}$, $\overline{\mathcal{M}}_{2,1}$, $\overline{\mathcal{M}}_{2,2}$, and $\overline{\mathcal{M}}_{3,1}$.

One significant advantage of Conjecture 2.1 is that, as we will explain in Theorem 2.2, the cone of F-nef divisors admits a very explicit description. To this end, we will provide a more detailed description of F-curves and divisors on $\overline{M}_{g,n}$, following [26].

From now on, we identify F-curves with their classes in $A_1(\overline{M}_{g,n})$, i.e. up to numerical equivalence. There are six types of F-curves (cf. [26, Theorem 2.2]), which can be described as images of certain gluing maps. Note that, with our definition, the following description gives only the numerical class of an F-curve, since there is some freedom in choosing the fixed stable curves. If we choose these stable curves to be maximally degenerate, then we recover the F-curves themselves.

Type 1: Let $i : \overline{M}_{1,1} \rightarrow \overline{M}_{g,n}$ be the map that attaches a fixed semistable curve C of genus $g - 1$ with $n + 1$ marked points and then stabilizes. The image F_1 of i is the **F-curve of type 1**.

Type 2: Let $i : \overline{M}_{0,4} \rightarrow \overline{M}_{g,n}$ be the map that attaches a fixed semistable curve C of genus $g - 3$ with $n + 4$ marked points to the four marked points on curves parametrized by $\overline{M}_{0,4}$, and then stabilizes. The image F_2 of i is the **F-curve of type 2**.

Type 3: Choose natural numbers $g_1 + g_2 = g - 2$ and a decomposition $I_1 \sqcup I_2 = [n]$. Fix semistable curves C_1 (resp. C_2) of genus g_1 (resp. g_2) with $|I_1| + 1$ (resp. $|I_2| + 3$) marked points. Let $i : \overline{M}_{0,4} \rightarrow \overline{M}_{g,n}$ be the map that attaches C_1 to the first point and C_2 to the remaining three points, and then stabilizes. The image $F_3^{g_1}(I_1)$ of i is an **F-curve of type 3**. Note that g_2 and I_2 are determined by g_1 and I_1 .

Type 4: Choose natural numbers $g_1 + g_2 = g - 2$ and a decomposition $I_1 \sqcup I_2 = [n]$. Fix semistable curves C_i of genus g_i with $|I_i| + 2$ marked points for $i = 1, 2$. Let $i : \overline{M}_{0,4} \rightarrow \overline{M}_{g,n}$ be the map that attaches C_1 and C_2 to two of the four points and then stabilizes. The image $F_4^{g_1}(I_1)$ of i is an **F-curve of type 4**.

Type 5: Choose natural numbers $g_1 + g_2 + g_3 = g - 1$ and a decomposition $I_1 \sqcup I_2 \sqcup I_3 = [n]$. Fix semistable curves C_i of genus g_i with $|I_i| + 1$ marked points for $i = 1, 2$, and C_3 of genus g_3 with $|I_3| + 2$ marked points. Let $i : \overline{M}_{0,4} \rightarrow \overline{M}_{g,n}$ be the map that attaches C_1 and C_2 to one point each, and C_3 to the remaining two points, and then stabilizes. The image $F_5^{g_1, g_2}(I_1, I_2)$ of i is an **F-curve of type 5**.

Type 6: Choose natural numbers $g_1 + g_2 + g_3 + g_4 = g$ and a decomposition $I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4 = [n]$. Fix semistable curves C_i of genus g_i with $|I_i| + 1$ marked points. Let $i : \overline{M}_{0,4} \rightarrow \overline{M}_{g,n}$ be the map that attaches each C_i to one of the four points and then stabilizes. The image $F_6^{g_1, g_2, g_3, g_4}(I_1, I_2, I_3, I_4)$ of i is an **F-curve of type 6**.

While the actual curve on $\overline{M}_{g,n}$ depends on the choice of semistable curves, its numerical class remains independent. For a pictorial description of these curves, we refer to [26, Figure (2.3)]. Note that if $g = 0$, then only type 6 curves exist; if $g = 1$, then types 1, 5, and 6 exist; if $g = 2$, then all types except type 2 exist; and if $g \geq 3$, then F-curves of all types exist.

Note that the λ -class λ , the ψ -classes ψ_i , and the boundary classes $\delta_{i,I}$ generate the \mathbb{Q} -Picard group of $\overline{M}_{g,n}$. Furthermore, if $g \geq 3$, they form a basis of the \mathbb{Q} -Picard group. For relations among these classes, we refer to [3, Theorem 4]. Following [26], we express a divisor D as a linear combination:

$$D = a\lambda - b_{\text{irr}}\delta_{\text{irr}} + \sum_{i=1}^n b_{0,i}\psi_i - \sum b_{i,I}\delta_{i,I}.$$

Theorem 2.2. [26, Theorem 2.1] The intersections of F-curves with the divisor D are as follows:

- (1) $D \cdot F_1 = a - 12b_{\text{irr}} + b_{1,\emptyset}$.
- (2) $D \cdot F_2 = b_{\text{irr}}$.
- (3) $D \cdot F_3^i(I) = b_{i,I}$.
- (4) $D \cdot F_4^i(I) = 2b_{\text{irr}} - b_{i,I}$.
- (5) $D \cdot F_5^{i,j}(I, J) = b_{i,I} + b_{j,J} - b_{i+j, I \cup J}$.
- (6) $D \cdot F_6^{i,j,k,l}(I, J, K, L) = b_{i,I} + b_{j,J} + b_{k,K} + b_{l,L} - b_{i+j, I \cup J} - b_{i+k, I \cup K} - b_{i+l, I \cup L}$.

3. NON-SEMIAMPLeness CRITERION

Theorem 3.1. (char $k = 0$) Let $g, n \geq 2$ and L be a line bundle on $\overline{\mathcal{M}}_{g,n}$ such that

$$L = a\lambda - b_{\text{irr}}\delta_{\text{irr}} + \sum_i b_{0,i}\psi_i - \sum_{i,I} b_{i,I}\delta_{i,I}.$$

Assume that there exist $i, j \in [n]$ such that

- (1) $b_{0,\{i,j\}} = 0$,
- (2) $b_{0,i} \neq b_{0,j}$, and
- (3) $b_{0,k} = 0$ for every $k \neq i, j$.

Then L is not semiample.

The proof is almost identical to that of [9, Theorem 4.6]. However, we will reproduce the proof here since a more specific circumstance is considered and a different language is used there. The proof is based on Keel's counterexample [31, Section 3].

Proof. First, we will consider the case of $n = 2$. In this case, condition (3) is vacuous. Since $\text{Pic}(\overline{\mathcal{M}}_{g,n}) \simeq \text{Pic}(\overline{\mathcal{M}}_{g,n})$, we will consider L as a line bundle on $\overline{\mathcal{M}}_{g,n}$. Let C be a projective smooth curve of genus g . Then, the projection $\pi_1 : C \times C \rightarrow C$ with the diagonal embedding $\Delta : C \rightarrow C \times C$ can be considered as a family of genus g curves with a marked point. Hence, there exists a corresponding morphism

$$\begin{array}{ccc} C \times C & \xrightarrow{u} & \overline{\mathcal{M}}_{g,2} \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ C & \longrightarrow & \overline{\mathcal{M}}_{g,1} \end{array}$$

Consider u^*L . Note that the image of u does not intersect any boundary divisors except $\delta_{0,\{1,2\}}$. Since $b_{0,\{1,2\}} = 0$, the boundary divisors do not contribute to u^*L . Moreover, since the image of the composition map $C \times C \rightarrow \overline{\mathcal{M}}_{g,2} \rightarrow \overline{\mathcal{M}}_g$ is a point corresponding to C , we have $u^*\lambda = 0$. Hence, $u^*L = b_{0,1}u^*\psi_1 + b_{0,2}u^*\psi_2$. Since the image of $u \circ \Delta$ is contained in $\Delta_{0,\{1,2\}}$, we have $\Delta^*u^*L = 0$.

Now assume that L is a semiample line bundle. Then u^*L is also a semiample line bundle. Let $f : C \times C \rightarrow \mathbb{P}^n$ be a morphism such that $f^*\mathcal{O}(1) = u^*L^{\otimes m}$. Since $\Delta^*u^*L = 0$, the map f contracts the image of Δ . Hence, there exists an effective divisor T on $C \times C$ such that $\Delta \cap T = \emptyset$ and $u^*L^{\otimes m} = [T]$. In particular, the restriction of $u^*L^{\otimes m}$ to any infinitesimal neighborhood of Δ is trivial. Let Δ_2 be the second-order infinitesimal neighborhood of Δ .

By the definition of ψ classes (cf. [9, Proposition 3.9]), we have $u^*\psi_i = \pi_i^*\Omega_C^1(\Delta)$ for $i = 1, 2$. By [32, Lemma 3.5], we have $u^*\psi_1|_{\Delta_2} = -u^*\psi_2|_{\Delta_2}$. Hence,

$$u^*L^{\otimes m}|_{\Delta_2} = m(b_{0,1} - b_{0,2})u^*\psi_1|_{\Delta_2} = m(b_{0,1} - b_{0,2})\pi_1^*\Omega_C^1(\Delta)|_{\Delta_2}.$$

Since $b_{0,1} \neq b_{0,2}$, this is not a trivial line bundle by [32, Lemma 3.4]. This leads to a contradiction. Hence, L is not a semiample line bundle.

Now consider the general case. We may assume that $i = 1$ and $j = n$. Then there exist a projective smooth curve C of genus g , a projective smooth curve D , and morphisms $s_1, \dots, s_{n-1} : D \rightarrow C$ such that the trivial family of curves $\pi_1 : D \times C \rightarrow D$ with sections s_1, \dots, s_{n-1} forms a nontrivial family of genus g curves with $n - 1$ marked points. The construction uses elliptic curves. See [50] or [9, proof of Corollary 4.8].

As in the case of $n = 2$, we have an induced morphism $u : D \times C \rightarrow \overline{\mathcal{M}}_{g,n}$,

$$\begin{aligned} u^*L &= b_{0,1}u^*\psi_1 + b_{0,n}u^*\psi_n - \sum_{i=2}^{n-1} b_{0,\{i,n\}}u^*\delta_{0,\{i,n\}} \\ &= b_{0,1}\pi_1^*s_1^*\Omega_C^1(s_1) + b_{0,n}\pi_2^*s_2^*\Omega_C^1(s_2) - \sum_{i=2}^{n-1} b_{0,\{i,n\}}[s_i] \end{aligned}$$

where we identify $s_i : D \rightarrow D \times C$ with its image, and $s_1^* u^* L = 0$. Assume that L is a semiample line bundle. Then, by the same argument as above, there exists an effective divisor T on $D \times C$ such that $u^* L^{\otimes m} = [T]$ for some $m > 0$ and $T \cap s_1 = \emptyset$.

Consider $f = (s_1, \text{id}) : D \times C \rightarrow C \times C$. Let R be the effective divisor $T + \sum_{i=2}^n [s_i]$, and define $R' := f_* R$, $U := C \times C \setminus R'$, and $V := f^{-1}(U)$. Since $f^{-1}(\Delta) = s_1$ and $s_1 \cap R = \emptyset$, we have $\Delta \subseteq U$. Since $R \cap V = \emptyset$, we obtain

$$u^* L|_V = b_{0,1} \pi_1^* s_1^* \Omega_C^1(s_1)|_V + b_{0,n} \pi_2^* \Omega_C^1(s_1)|_V = (f|_V)^*(b_{0,1} \pi_1^* \Omega_C^1(\Delta) + b_{0,n} \pi_2^* \Omega_C^1(\Delta)).$$

On the other hand, since $u^* L^{\otimes m} = [T]$ and $T \cap V = \emptyset$, we conclude that $u^* L^{\otimes m}|_V$ is trivial. By a standard push-pull argument (e.g., [9, Lemma 4.7]), there exists $d > 0$ such that

$$db_{0,1} \pi_1^* \Omega_C^1(\Delta) + db_{0,n} \pi_2^* \Omega_C^1(\Delta) = 0 \quad \text{on } U.$$

Since $\Delta \subseteq U$, its second-order infinitesimal neighborhood Δ_2 is also contained in U , so its restriction to Δ_2 is also zero. However, by [31, Lemma 3.5], we have

$$(db_{0,1} \pi_1^* \Omega_C^1(\Delta) + db_{0,n} \pi_2^* \Omega_C^1(\Delta))|_{\Delta_2} = d(b_{0,1} - b_{0,n}) \pi_1^* \Omega_C^1(\Delta)|_{\Delta_2},$$

which is nonzero by [31, Lemma 3.4] and condition (2). This leads to a contradiction. Hence, L is not semiample. \square

The condition $b_{0,i} \neq b_{0,j}$ is essential: see Proposition 5.13.

4. SEMIAMPLENESS OF ψ CLASSES ON $\overline{M}_{1,n}$

Theorem 4.1. ψ_i 's are semiample on $\overline{M}_{1,n}$.

Proof. We will use induction on n . If $n = 1$, then $\overline{M}_{1,1} = \mathbb{P}^1$ and ψ_1 is nef, so ψ_1 is semiample. If $n = 2$, there are several ways to prove this. There exists a surjective map $f : \overline{M}_{0,5} \rightarrow \overline{M}_{1,2}$, which sends $(C, \{P_i\}_{i=1}^5)$ to (D, Q_1, Q_2) , where D is a double cover of C ramified at P_2, P_3, P_4, P_5 , Q_2 is the inverse image of P_2 , and Q_1 is an inverse image of P_1 . See, for example, [42, Section 4.1]. Since $\overline{M}_{1,2}$ is normal, a line bundle L on $\overline{M}_{1,2}$ is semiample if and only if $f^* L$ is. Since $\overline{M}_{0,5}$ is a Fano variety, any nef line bundle is semiample. Therefore, any nef line bundle on $\overline{M}_{1,2}$ is semiample. Alternatively, by [42, Proposition 4.1.5], for the natural map $\alpha : \overline{M}_{1,2} \rightarrow \overline{M}_2$ corresponding to Δ_{irr} , $\psi_1 = \alpha^*(12\lambda - \delta_{\text{irr}})$. By [25, Corollary 4.3], $12\lambda - \delta_{\text{irr}}$ is semiample, hence ψ_1 is also semiample.

Now assume ψ_i 's are semiample on $\overline{M}_{1,n-1}$ and $n \geq 3$. It is enough to prove that ψ_1 is semiample on $\overline{M}_{1,n}$. Let π_{n-1} and $\pi_n : \overline{M}_{1,n} \rightarrow \overline{M}_{1,n-1}$ be the projection maps forgetting the $(n-1)$ th and n th points, respectively. Then

$$\psi_1 = \pi_{n-1}^* \psi_1 + \delta_{0, \{1, n-1\}} = \pi_n^* \psi_1 + \delta_{0, \{1, n\}}.$$

By the induction hypothesis, $\pi_{n-1}^* \psi_1$ and $\pi_n^* \psi_1$ are semiample. Hence, the stable base locus $\mathbf{B}(\psi_1)$ is contained in $\Delta_{0, \{1, n-1\}}$ and $\Delta_{0, \{1, n\}}$. However, their intersection is trivial, hence $\mathbf{B}(\psi_1) = \emptyset$. Therefore, ψ_1 is semiample. \square

Corollary 4.2. The ψ -classes are semiample on $\overline{M}_{g,n}$ if and only if one of the following holds:

- (1) The characteristic of the base field is positive.
- (2) $g \leq 1$.
- (3) $(g, n) = (2, 1)$.

Proof. Case (1) is proved in [32, Theorem 0.4]; see also [9, Proposition 3.9]. The $g = 0$ case of (2) follows from the existence of Kapranov's construction [29, 30], especially [30, Proposition 2.8]. We have just proved the $g = 1$ case. (3) follows from [42, Corollary 4.6.2], where Rulla proved that any nef line bundle on $\overline{M}_{2,1}$ is semiample. Now consider the characteristic zero case, excluding (2) and (3). If $g \geq 2$ and $n \geq 2$, then Theorem 3.1 (or the original argument in [32]) implies that the ψ_i 's are not semiample. For $g \geq 3$, let $f : \overline{M}_{g-1,3} \rightarrow \overline{M}_{g,1}$ be the map

corresponding to Δ_{irr} . Then $f^*\psi_1 = \psi_1$, and this is not semiample. Hence, ψ_1 is not semiample if $g \geq 3$. \square

Question 4.3. What is the contraction of $\overline{\mathcal{M}}_{1,n}$ corresponding to ψ_i ?

Regarding Question 4.3, there are many other modular compactifications of $\mathcal{M}_{1,n}$, for example [22, 46–48]. One needs to identify the correct compactification corresponding to ψ_i , construct a map from $\overline{\mathcal{M}}_{1,n}$ and prove the projectivity of its coarse moduli space.

5. EXAMPLES

In this section, we present examples of semiample and non-semiample nef divisors. Throughout this section, we assume that the characteristic of the base field is zero.

Example 5.1. On $\overline{\mathcal{M}}_{2,2}$, the following divisors are the set of extremal rays of the nef cone:

$$\begin{aligned} &\lambda, 12\lambda - \delta_{\text{irr}}, \psi_1, \psi_2, \pi_1^*\psi, \pi_2^*\psi, \\ &\psi_1 + \psi_2 + \delta_{1,\{1\}}, \psi_1 + \psi_2 - 2\delta_{0,\{1,2\}} + \delta_{1,\{1\}} \\ &\delta_{\text{irr}} + 10\psi_1 + 10\psi_2 + 12\delta_{1,\emptyset} + 2\delta_{1,\{1\}} = 10(\lambda + \psi_1 + \psi_2 + \delta_{1,\emptyset}) \end{aligned}$$

Note that on $\overline{\mathcal{M}}_{2,n}$, $10\lambda = \delta_{\text{irr}} + 2\delta_1$. Since the F-conjecture is verified in this case, the nef cone coincides with the cone of divisors that intersect all F-curves nonnegatively. This cone is computed using the Sage code available in [11, fnefcone].

Theorem 5.2. There exists a 5-dimensional subcone of the nef cone of $\overline{\mathcal{M}}_{2,2}$ whose generic element is not semiample.

Proof. Consider the following divisors:

$$\lambda, 12\lambda - \delta_{\text{irr}}, \psi_1, \psi_2, \psi_1 + \psi_2 + \delta_{1,\{1\}}, \lambda + \psi_1 + \psi_2 + \delta_{1,\emptyset}.$$

A generic element of the cone generated by these elements satisfies the condition of Theorem 3.1, and hence is not semiample. It is straightforward to see that these elements generate a 5-dimensional space, spanned by $\psi_1, \psi_2, \delta_{\text{irr}}, \delta_{1,\emptyset}, \delta_{1,\{1\}}$. \square

Since the Picard number of $\overline{\mathcal{M}}_{2,2}$ is 6, this represents the largest possible such cone. Hence, this provides an instance of the cone described in Theorem 1.1 within $\overline{\mathcal{M}}_{2,2}$.

Proposition 5.3. If D is a divisor on $\overline{\mathcal{M}}_{3,1}$ of the form

$$a\lambda - b_{\text{irr}}\delta_{\text{irr}} + b_{0,1}\psi_1 - b_{1,\emptyset}\delta_{1,\emptyset}$$

for some $b_{0,1} \neq b_{1,\emptyset}$, then D is not semiample.

Proof. Let $f : \overline{\mathcal{M}}_{2,2} \rightarrow \overline{\mathcal{M}}_{3,1}$ be the morphism obtained by attaching a genus one curve at a marked point. By assumption, the pullback f^*D satisfies the condition of Theorem 3.1. Therefore, f^*D is not semiample, and it follows that D is also not semiample. \square

Using Proposition 5.3, we can directly verify Theorem 1.1 and Corollary 1.2 for $\overline{\mathcal{M}}_{3,1}$.

Example 5.4. There exists no contraction of $\overline{\mathcal{M}}_{3,1}$ whose relative cone of curves is the extremal ray spanned by $F_3^1(1)$. Suppose, for contradiction, that such a contraction exists and let D be the corresponding semiample divisor on $\overline{\mathcal{M}}_{3,1}$. Since D contracts $F_3^1(1)$, it must be of the form described in Proposition 5.3. Then, by Theorem 3.1, it must satisfy $b_{0,1} = b_{1,\emptyset}$. Under the additional assumption that $D \cdot F_3^1(1) = 0$, this is precisely equivalent to $D \cdot F_5^{2,0}(\emptyset, 1) = 0$. Thus, D must also contract $F_5^{2,0}(\emptyset, 1)$.

Example 5.5. On $\overline{\mathcal{M}}_{3,1}$, the following divisors are the set of extremal rays of the F-nef cone: $\psi_1, \lambda, 12\lambda - \delta_{\text{irr}}, 10\lambda - \delta_{\text{irr}} - 2\delta_{1,\emptyset} - 2\delta_{1,\{1\}}, 10\lambda - \delta_{\text{irr}} + 2\psi_1 - 2\delta_{1,\emptyset}, 11\lambda - \delta_{\text{irr}} + 3\psi_1 - \delta_{1,\emptyset} - 2\delta_{1,\{1\}}$. They generate the nef cone by Proposition 7.16. This is computed by the same method as in Example 5.1.

Theorem 5.6. There exists a 4-dimensional subcone of the nef cone of $\overline{M}_{3,1}$ whose generic element is not semiample.

Proof. Consider the following divisors:

$$\psi_1, \lambda, 12\lambda - \delta_{\text{irr}}, 10\lambda - \delta_{\text{irr}} + 2\psi_1 - 2\delta_{1,\emptyset}.$$

A generic element of the cone generated by these elements satisfies the condition of Proposition 5.3, and hence is not semiample. It is straightforward to see that these elements generate a 4-dimensional space, spanned by $\lambda, \psi_1, \delta_{\text{irr}}, \delta_{1,\emptyset}$. \square

Again, this is the largest possible such cone, since the Picard number of $\overline{M}_{3,1}$ is 5. This naturally leads to the following question, which is partially answered by Theorem 1.1.

Question 5.7. Let ρ denote the Picard number of $\overline{M}_{g,n}$, with $g \geq 3$ and $n \geq 1$, or $g = 2$ and $n \geq 2$. Does there exist a $(\rho - 1)$ -dimensional subcone of the nef cone, whose general element is not semiample?

By the following observation, this is related to the extremality of F-curves.

Proposition 5.8. Let C be a pointed n -dimensional polyhedral cone in \mathbb{R}^n , and let C^* be its dual. Then $v \in C$ spans an extremal ray if and only if the dimension of $C^* \cap v^\perp$ is exactly $n - 1$.

One important caveat is that, since it is not known whether the cone of curves on $\overline{M}_{g,n}$ is polyhedral, extremality alone is generally insufficient to draw definitive conclusions. To address this, we introduce the notion of a regular extremal ray in Definition 6.1.

From the perspective of Proposition 5.8, Theorem 5.2 corresponds to the fact that $F_3^0([2])$ spans an extremal ray in $\overline{M}_{2,2}$, while Theorem 5.6 reflects the extremality of $F_3^1(1)$ in $\overline{M}_{3,1}$. The connection between non-semiample and F-curves arises from Theorem 3.1: assumptions (1) and (3) in that theorem are equivalent to the condition that the divisor contracts certain F-curves. In particular, assumption (1) corresponds to the condition $L \cdot F_3^0(\{i, j\}) = 0$, and (3) is an empty condition when $n = 2$. This naturally raises the question of whether these F-curves are extremal. Even when $n > 2$, we follow a similar strategy as in Proposition 5.3, by pulling back the divisor to the case $n = 2$. Therefore, in what follows, we focus on analyzing the extremality of F-curves. Indeed, in [4], the author raised the following conjecture:

Conjecture 5.9. [4, Conjecture 1.3] All boundary strata in $\overline{M}_{g,n}$ are extremal.

However, this conjecture is not true for F-curves, i.e. one-dimensional boundary strata. The easiest example is the type 2 F-curve. If $g \geq 3$, then by the explicit formula Theorem 2.2,

$$[F_2] = \frac{1}{2} ([F_3(i, I)] + [F_4(i, I)])$$

for any $1 \leq i \leq g - 2$. Moreover, by Theorem 10.6, all type 3 curves except those listed in Theorem 1.3 are not extremal. This provides further examples of non-extremal F-curves.

As an illustrative example, we examine the extremality of F-curves in $\overline{M}_4, \overline{M}_{3,1}$, and $\overline{M}_{2,2}$ using the computer program [11, extray]. This program determines which F-curves are extremal in the cone of F-curves. The program runs in a few seconds on the author's laptop. By the discussion in Section 2, we know that for these spaces, this cone coincides with the cone of curves, so extremality in the cone of F-curves is equivalent to extremality in the cone of curves. Note that, for the sake of time complexity, we exclude type 2 and type 3 F-curves not listed in Theorem 1.3 from the program, since they are already known to be non-extremal.

Example 5.10. On \overline{M}_4 , the following divisors are the set of extremal rays of the nef cone:

$$\lambda, 12\lambda - \delta_{\text{irr}}, 10\lambda - \delta_{\text{irr}} - 2\delta_1, 10\lambda - \delta_{\text{irr}} - 2\delta_1 - 2\delta_2, 21\lambda - 2\delta_{\text{irr}} - 3\delta_1 - 4\delta_2$$

Note that $21\lambda - 2\delta_{\text{irr}} - 3\delta_1 - 4\delta_2$ contracts the type 6 F-curve. This is computed by the same method as in Example 5.1.

Here is a summary of the extremality of F-curves:

	extremal?	Relation
F_1	Y	-
F_2	N	$\frac{1}{2} ([F_3^1(\emptyset)] + [F_4^1(\emptyset)])$
$F_3^1(\emptyset)$	N	$\frac{1}{2} ([F_5^{1,1}(\emptyset, \emptyset)] + [F_3^2(\emptyset)])$
$F_3^2(\emptyset) = F_5^{1,2}(\emptyset, \emptyset)$	Y	-
$F_4^1(\emptyset)$	Y	-
$F_4^2(\emptyset)$	Y	-
$F_5^{1,1}(\emptyset, \emptyset)$	N	$\frac{1}{2} ([F_6^{1,1,1,1}(\emptyset, \emptyset, \emptyset, \emptyset)] + [F_3^2(\emptyset)])$
$F_6^{1,1,1,1}(\emptyset, \emptyset, \emptyset, \emptyset)$	Y	-

Table 1: F-curves on \overline{M}_4 .

We have a similar table for $\overline{M}_{3,1}$ and $\overline{M}_{2,2}$.

	extremal?	Relation
F_1	Y	-
F_2	N	$\frac{1}{2} ([F_3^1(\emptyset)] + [F_4^1(\emptyset)])$
$F_3^0(\{1\})$	N	$\frac{1}{2} ([F_5^{0,1}(\{1\}, \emptyset)] + [F_5^{0,2}(\{1\}, \emptyset)])$
$F_3^1(\emptyset)$	N	$\frac{1}{2} ([F_3^1(\{1\})] + [F_5^{1,1}(\emptyset, \emptyset)])$
$F_3^1(\{1\}) = F_5^{1,1}(\{1\}, \emptyset)$	Y	-
$F_4^1(\emptyset) = F_4^2(\{1\})$	Y	-
$F_4^1(\{1\}) = F_4^2(\emptyset)$	Y	-
$F_5^{0,1}(\{1\}, \emptyset)$	N	$\frac{1}{3} ([F_6^{1,1,1,0}(\emptyset, \emptyset, \emptyset, \{1\})] + 2[F_3^0(\{1\})])$
$F_5^{0,2}(\{1\}, \emptyset)$	Y	-
$F_5^{1,1}(\emptyset, \emptyset)$	Y	-
$F_6^{1,1,1,0}(\emptyset, \emptyset, \emptyset, \{1\})$	Y	-

Table 2: F-curves on $\overline{M}_{3,1}$.

	extremal?	Relation
F_1	Y	-
$F_3^0(\{1\})$	N	$\frac{1}{2} ([F_5^{0,1}(\{1\}, \emptyset)] + [F_5^{0,1}(\{1\}, \{2\})])$
$F_3^0(\{2\})$	N	$\frac{1}{2} ([F_5^{0,1}(\{2\}, \emptyset)] + [F_5^{0,1}(\{2\}, \{1\})])$
$F_3^0(\{1, 2\}) = F_5^{0,1}(\{1, 2\}, \emptyset)$	Y	-
$F_4^1(\emptyset)$	Y	-
$F_4^1(\{1\})$	Y	-
$F_5^{0,0}(\{1\}, \{2\})$	Y	-
$F_5^{0,1}(\{1\}, \emptyset)$	Y	-
$F_5^{0,1}(\{1\}, \{2\})$	Y	-
$F_5^{0,1}(\{2\}, \emptyset)$	Y	-
$F_5^{0,1}(\{2\}, \{1\})$	Y	-
$F_6^{1,1,0,0}(\emptyset, \emptyset, \{1\}, \{2\})$	Y	-

Table 3: F-curves on $\overline{M}_{2,2}$.

Remark 5.11. The previous tables, together with a result [4, Theorem 1.1], might suggest the conjecture that all type 6 F-curves are extremal. However, this is an illusion. On \overline{M}_7 ,

$$[F_6^{1,1,2,3}(\emptyset, \emptyset, \emptyset, \emptyset)] = \frac{1}{2} ([F_6^{1,1,1,4}(\emptyset, \emptyset, \emptyset, \emptyset)] + [F_6^{1,2,2,2}(\emptyset, \emptyset, \emptyset, \emptyset)]).$$

This counterexample arises from the observation in [37, Table 1], which is also mentioned in [8, Remark 6.3] and [4, Remark 4.6]. The key point is that not every extremal ray of $\overline{M}_{0,g+n}$ remains extremal in the quotient $\overline{M}_{0,g+n}/S_g$.

From now on, we will prove the semiample-ness of certain line bundles on $\overline{M}_{2,2}$. The following lemma will be useful for this purpose.

Lemma 5.12. Let $f : X \rightarrow Y$ be a contraction of projective varieties. Let L be a semiample \mathbb{Q} -line bundle on X . If L intersects trivially with $\overline{NE}_1(f)$, then L descends to Y , i.e., $L \in f^*\text{Pic}(Y)$.

Proof. Let $f' : X \rightarrow Y'$ be the contraction corresponding to L . Then, by [15, Proposition 1.14(b)], f' factors through f . Since $L \in f'^*\text{Pic}(Y')$, it follows that $L \in f^*\text{Pic}(Y)$. \square

Proposition 5.13. For any $\epsilon > 0$, $D_\epsilon = \epsilon(12\lambda - \delta_{\text{irr}}) + \psi_1 + \psi_2$ is semiample on $\overline{M}_{2,2}$.

Proof. First, we will prove D_1 is semiample. By [25, Corollary 4.3], $12\lambda - \delta_{\text{irr}}$ is semiample for $g \leq 11$. Since $\psi_1 + \psi_2 - 2\delta_{0,\{1,2\}}$ and $12\lambda - \delta_{\text{irr}}$ are semiample by Corollary 4.2, the stable base locus of D_1 is contained in $\Delta_{0,\{1,2\}}$. Moreover, for $\xi : \overline{M}_{2,2} \rightarrow \overline{M}_3$,

$$D_1 = \xi^*(12\lambda - \delta_{\text{irr}}) + \delta_{1,1}$$

Hence, the stable base locus is contained in $\Delta_{1,1}$. However, this does not intersect with $\Delta_{0,\{1,2\}}$. Hence, D_1 is semiample.

Let $f : \overline{M}_{2,2} \rightarrow X$ be the contraction corresponding to D_1 . Among the F-curves, D_1 contracts only F_1 and $F_3^0([2])$, by Theorem 2.2. Hence, D_1 lies in the interior of the face of the nef cone that intersects F_1 and $F_3^0([2])$ trivially, i.e., the cone generated by

$$12\lambda - \delta_{\text{irr}}, \psi_1, \psi_2, \psi_1 + \psi_2 + \delta_{1,\{1\}}, \delta_{\text{irr}} + 10\psi_1 + 10\psi_2 + 12\delta_{1,\emptyset} + 2\delta_{1,\{1\}}.$$

Let $F = f^*\text{Nef}(X)$ be the pullback of the nef cone of X , which is a subcone of the described cone. Since D_1 is semiample, D_ϵ is also semiample for $\epsilon \geq 1$. As D_ϵ contracts only F_1 and $F_3^0([2])$, it intersects trivially with $\overline{NE}_1(f)$. Thus, by Lemma 5.12, D_ϵ descends to X . Therefore, D_ϵ descends to X for any $\epsilon > 0$. Moreover, for $\epsilon > 0$, D_ϵ is a nef line bundle that contracts only F_1 and $F_3^0([2])$, and is thus ample on X , hence semiample on $\overline{M}_{2,2}$. \square

Proposition 5.14. For any $\epsilon > 0$ and $0 < a < 1$, $D_{a,\epsilon} = \epsilon(12\lambda - \delta_{\text{irr}}) + \psi_1 + \psi_2 + a\delta_{1,\{1\}}$ is semiample on $\overline{M}_{2,2}$.

Proof. By Proposition 5.13, the stable base locus of $D_{a,\epsilon}$ is contained in $\Delta_{1,\{1\}}$. Let $\xi : \overline{M}_{2,2} \rightarrow \overline{M}_3$ be the clutching map. Note that, by [42, Proposition 3.3.6], any nef line bundle on \overline{M}_3 is semiample. In particular, the divisor $10\lambda - \delta_{\text{irr}} - 2\delta_1$ is semiample on \overline{M}_3 , and thus

$$\begin{aligned} \xi^*(10\lambda - \delta_{\text{irr}} - 2\delta_1) &= 10\lambda - \delta_{\text{irr}} + \psi_1 + \psi_2 - \delta_{1,\{1\}} - 2\delta_{1,\emptyset} - 2\delta_{0,\{1,2\}} \\ &= \psi_1 + \psi_2 + \delta_{1,\{1\}} - 2\delta_{0,\{1,2\}} \end{aligned}$$

is semiample. Here, we used the relation $10\lambda - \delta_{\text{irr}} - 2\delta_1 = 0$ on $\overline{M}_{2,n}$ (cf. [3, Theorem 4]). Then,

$$D_{a,\epsilon} = (\epsilon(12\lambda - \delta_{\text{irr}}) + (1-a)(\psi_1 + \psi_2)) + a(\psi_1 + \psi_2 + \delta_{1,\{1\}} - 2\delta_{0,\{1,2\}}) + 2a\delta_{0,\{1,2\}},$$

so the stable base locus of $D_{a,\epsilon}$ is contained in $\Delta_{0,\{1,2\}}$. Since $\Delta_{0,\{1,2\}}$ and $\Delta_{1,\{1\}}$ are disjoint, the stable base locus of $D_{a,\epsilon}$ is empty, and thus $D_{a,\epsilon}$ is semiample. \square

Proposition 5.15. On $\overline{M}_{2,2}$, any line bundle contained in the interior of the cone generated by

$$12\lambda - \delta_{\text{irr}}, \psi_1 + \psi_2, \psi_1 + \psi_2 + \delta_{1,\{1\}}, \delta_{\text{irr}} + 10\psi_1 + 10\psi_2 + 12\delta_{1,\emptyset} + 2\delta_{1,\{1\}}$$

is semiample.

Proof. Let $f : \overline{M}_{2,2} \rightarrow X$ be the contraction described in the proof of Proposition 5.13, i.e., the contraction corresponding to F_1 and $F_3^0([2])$. Since $\epsilon(12\lambda - \delta_{\text{irr}}) + \psi_1 + \psi_2$, $\epsilon(12\lambda - \delta_{\text{irr}}) + \psi_1 + \psi_2 + a\delta_{1,\{1\}}$, and $12\lambda - \delta_{\text{irr}}$ are semiample line bundles that contract F_1 and $F_3^0([2])$, they descend to X by Lemma 5.12. In particular, the Picard rank of X is at least 3.

By Theorem 3.1, $f^*\text{Pic}(X)$ is strictly smaller than the face of the nef cone contracting F_1 and $F_3^0([2])$ (cf. proof of Proposition 5.13). Hence, the Picard rank of X is exactly 3. Therefore, $f^*\text{Pic}(X)$ is the intersection of the nef cone with the subspace spanned by $12\lambda - \delta_{\text{irr}}$, $\psi_1 + \psi_2$, and $\delta_{1,\{1\}}$, which is precisely the cone described in the proposition. Its interior corresponds to pullback of ample line bundles on X , hence these are semiample. \square

Proposition 5.16. On $\overline{M}_{2,2}$, any line bundle contained in the interior of the cone generated by

$$\lambda, 12\lambda - \delta_{\text{irr}}, \psi_1 + \psi_2, \psi_1 + \psi_2 + \delta_{1,\{1\}}, \delta_{\text{irr}} + 10\psi_1 + 10\psi_2 + 12\delta_{1,\emptyset} + 2\delta_{1,\{1\}}$$

is semiample.

Proof. The proof is similar to the proof of Proposition 5.15, but uses the contraction corresponding to $(12 + \epsilon)\lambda - \delta_{\text{irr}} + \psi_1 + \psi_2$, which only contracts $F_3^0([2])$. \square

Remark 5.17. (1) The contraction associated with F_1 is studied in [13, 14] and is known to have good properties, as F_1 is K_X -negative.

(2) Unlike the case of $g = 3$ stated in Corollary 1.2, there exists a contraction that contracts only the F-curve $F_3^0([2])$. The key difference is that, in the case $g \geq 3$, the codimension-one linear subspace spanning the face in Theorem 1.1 is itself a face. However, in the case of $g = 2$, this is not true—the linear subspace lies in the interior of the cone.

(3) The contractions appearing in Proposition 5.15 and Proposition 5.16 have an interesting property: if we denote the contraction by $f : \overline{M}_{2,2} \rightarrow X$, then

$$f^*\text{Pic}(X) \subsetneq \overline{NE}_1(f)^\perp.$$

In particular, these contractions differ in nature from those considered in [12], [10, Section 4], or Section 8, where the equality $\overline{NE}_1(f)^\perp = f^*\text{Pic}(X)$ holds. This serves as an explicit example illustrating why the condition $\overline{NE}_1(f)^\perp = f^*\text{Pic}(X)$ is closely related to the semiampleness of divisors, as discussed in [12].

6. THE CASE OF $\overline{M}_{0,n}$ AND $\overline{M}_{1,n}$

Here, we analyze the extremality of F-curves for genus 0 and 1, which is easier to prove and more explicit than the general genus case. We begin with some definitions and a lemma.

Definition 6.1. Let C be a proper cone (i.e., a cone that does not contain a full straight line and has nonempty interior) in an n -dimensional Euclidean space. For any subset $S \subseteq C$, the **index of extremality** is defined as

$$I(S) = n - \dim S^\perp,$$

where S^\perp is the face of the dual cone C^* orthogonal to S . A face $F \subseteq C$ is said to be **regular** if $\dim F = I(F)$. Moreover, if F is a ray then it is said to be **regular extremal**.

Remark 6.2. For any face F , $I(F) \geq \dim F$. If C is a polyhedral cone, it is straightforward to verify that every face is regular. However, this property does not hold in general. Consider the region

$$D = \{x^2 + (|y| + 1)^2 \leq 4\}$$

and let C be the 3-dimensional cone over D . In this setting, every ray on the boundary of C is extremal, but only the rays corresponding to $(\pm\sqrt{3}, 0)$ are regular extremal.

Note that some previous papers have already implicitly utilized similar notions. For example, [41] proved that the closed cone of moving curves of $\overline{M}_{g,n}$ for $g, n \geq 2$ is not polyhedral (or equivalently, that the closed cone of pseudoeffective divisors is not polyhedral) by constructing an extremal ray F of the moving cone with $2 \leq I(F) \leq n$.

Lemma 6.3. Let X and Y be projective, normal, \mathbb{Q} -factorial varieties, and let $f : X \rightarrow Y$ be a morphism. Assume the following conditions hold:

(1) $\ker f^*$ is spanned by nef line bundles.

(2) Any nef line bundle on X is the pullback of a nef line bundle on Y .

Then f_* maps any (regular) face of $\overline{\text{NE}}_1(X)$ to a (regular) face of $\overline{\text{NE}}_1(Y)$. In particular, for any face F of $\overline{\text{NE}}_1(X)$, $I(F) = I(f_*F)$.

Proof. Note that (2) implies that the induced map $f^* : \text{Pic}(Y) \rightarrow \text{Pic}(X)$ is surjective, or, equivalently, that $f_* : \overline{\text{NE}}_1(X) \rightarrow \overline{\text{NE}}_1(Y)$ is injective. This establishes that $I(F) \leq I(f_*F)$ for any face F . Moreover, (1) implies that $f_*A_1(X)_{\mathbb{R}} \cap \overline{\text{NE}}_1(Y)$ forms a face of $\overline{\text{NE}}_1(Y)$. By (2), we have

$$f_*(A_1(X)_{\mathbb{R}}) \cap \overline{\text{NE}}_1(Y) = f_*\overline{\text{NE}}_1(X).$$

Thus, f_* maps faces to faces. It remains to verify the second assertion.

Condition (1) ensures that there exist $\rho(Y) - \rho(X)$ linearly independent nef line bundles on Y that vanish on X . Furthermore, for any face F , there are $\rho(X) - I(F)$ linearly independent nef line bundles on X that intersect F trivially. By (2), we can extend these to nef line bundles on Y . Consequently, we obtain $\rho(Y) - I(F)$ independent line bundles that intersect f_*F trivially. In particular, this yields the inequality $I(f_*F) \leq I(F)$. Since we previously established that $I(F) \leq I(f_*F)$, it follows that $I(F) = I(f_*F)$, completing the proof of the second assertion. \square

Remark 6.4. Lemma 6.3 is a straightforward variant of [8, Proposition 2.5] and [4, Lemma 2.7]. However, since it considers both the effective cone and its dual, Lemma 6.3 has the advantage of establishing extremality within the closed cone of pseudoeffective cycles, not only the (possibly non-closed) cone of effective cycles.

Let $f_1 : \overline{M}_{0,n} \rightarrow \overline{M}_{0,n+1}$ and $f_2 : \overline{M}_{1,n} \rightarrow \overline{M}_{1,n+1}$ be the maps attaching a 3-pointed, genus 0 stable curve to a marked point, and let $f_3 : \overline{M}_{0,n} \rightarrow \overline{M}_{1,n-1}$ be the map attaching an elliptic curve to a marked point. Note that f_3 is a special case of the flag map, $F : \overline{M}_{0,g+n}/S_g \rightarrow \overline{M}_{g,n}$, defined and studied in [26].

Theorem 6.5. The maps f_1 and f_2 satisfy the conditions of Lemma 6.3. If the characteristic of the base field is not equal to 2, then the same holds for f_3 and F .

Proof. Note that f_3 is a special case of F , so it suffices to prove the claim for f_1 , f_2 , and F .

(1) First, consider f_1 . It is well known that $\rho(\overline{M}_{0,n}) = 2^{n-1} - \binom{n}{2} - 1$ (e.g., [3]). Hence, it is enough to produce

$$\rho(\overline{M}_{0,n+1}) - \rho(\overline{M}_{0,n}) = 2^{n-1} - n$$

linearly independent nef divisors on $\overline{M}_{0,n+1}$ which intersect $\Delta_{0,[n-1]}$ trivially. Let

$$P = \{S \subseteq [n-1] \mid |S^c| \geq 2\},$$

and for $S \in P$, define $\psi_S := \pi_S^* \psi_{n+1}$. These divisors are nef and intersect $\Delta_{0,[n-1]}$ trivially. It remains to prove that they are linearly independent.

Assume

$$D = \sum_{S \in P} a_S \psi_S = 0.$$

We will prove that $a_S = 0$ for all S by induction on $|S^c|$. If $|S^c| = 2$, i.e., $S^c = \{p, q\}$, then

$$F_6^{0,0,0}(p, q, n, \{p, q, n\}^c) \cdot D = a_S = 0.$$

Assume that $a_S = 0$ for every $|S^c| \leq m$. Then for any S such that $|S^c| = m+1$,

$$F_6^{0,0,0}(S^c \cup \{n+1\}, S_1, S_2, S_3) \cdot D = a_S = 0$$

for any nonempty S_1 , S_2 , and S_3 . Therefore, the ψ_S are linearly independent, and f_1 satisfies condition (1).

The proof for f_2 is similar. We have

$$\rho(\overline{M}_{1,n+1}) - \rho(\overline{M}_{1,n}) = 2^n - 1.$$

Define P as above, and for $S \subsetneq [n-1]$, let $\psi_S^1 = \pi_S^* \psi_n$ and $\psi_S^2 = \pi_S^* \psi_{n+1}$. Then, it suffices to show that the divisors ψ_S^1, ψ_S^2 for $S \neq [n-1]$, together with $\pi_{[n-1]}^* \psi$, are linearly independent.

Note that on $\overline{M}_{1,n}$, we have $\psi_1 = \psi_2$. The proof is similar to the case of f_1 , so we omit it.

Now consider F . By [26, Theorem 4.7], if the characteristic of the base field is not 2, there exists a nef divisor D_{GKM} on $\overline{M}_{g,n}$ such that $F^* D_{\text{GKM}} = 0$ and D_{GKM} intersects F-curves of type 1–5 positively. Hence, $D_{\text{GKM}} \in \ker F^*$, and for any divisor $D \in \ker F^*$ and sufficiently large r , the divisor $D + rD_{\text{GKM}}$ is nef by [26, Theorem 0.3]. Therefore, (1) holds.

(2) The maps f_1 and f_2 are sections of the projection map; hence, condition (2) is automatic for them. For F , this condition is proved in [26, Theorem 0.7] if the characteristic of the base field is not equal to 2. \square

Corollary 6.6. Any F-curve on $\overline{M}_{0,n}$ or $\overline{M}_{1,n}$ spans a regular extremal ray of the closed cone of curves. Moreover, for each such F-curve, there exists a contraction of $\overline{M}_{0,n}$ or $\overline{M}_{1,n}$ whose relative cone of curves is precisely the ray spanned by the F-curve.

Proof. On $\overline{M}_{0,n}$, every F-curve arises as the image of the fundamental class of $\overline{M}_{0,4}$ under a composition of maps $f_1 : \overline{M}_{0,n} \rightarrow \overline{M}_{0,n+1}$ for various values of n . By Lemma 6.3 and Theorem 6.5, each such class spans an extremal ray.

We now consider $\overline{M}_{1,n}$ for general characteristic. Here, we can only use the maps f_2 . Note that the type 1 F-curve is the pushforward of a curve on $\overline{M}_{1,1}$ via a sequence of f_2 's, and hence is regular extremal. Any type 5 or type 6 F-curve is the pushforward, along a sequence of f_2 's, of one of the following:

- (1) $F_5^{0,0}(\{1\}, \{2\})$ on $\overline{M}_{1,2}$,
- (2) $F_5^{0,0}(\{1\}, \{2\})$ on $\overline{M}_{1,3}$,
- (3) $F_6^{0,0,0,1}(\{1\}, \{2\}, \{3\}, \emptyset)$ on $\overline{M}_{1,3}$,
- (4) $F_6^{0,0,0,1}(\{1\}, \{2\}, \{3\}, \{4\})$ on $\overline{M}_{1,4}$.

Hence, it is enough to prove that these curves are regular extremal. For (1) and (3), note that they are contracted by all projections. Hence,

$$\sum_{i=1}^n \pi_i^* \text{Pic}(\overline{M}_{1,n-1})$$

contracts them. By [10, Lemma 4.3], this is a codimension 1 subspace. Therefore, they are regular extremal.

For (2) and (4), these curves are contracted by divisors in

$$\sum_{i=1}^{n-1} \pi_i^* \text{Pic}(\overline{M}_{1,n-1}).$$

This space has codimension 2, which can be shown using [10, Theorem 4.1] or via direct computation, at least for $n = 3, 4$. Moreover, they are also contracted by ψ_n , which is not contained in the above space by [10, Lemma 4.3]. Therefore, they are also regular extremal.

For the second assertion, we proceed by induction on n . The base cases are well known. As shown in the proof of Theorem 6.5, the kernels of f_1^* and f_2^* are generated by ψ classes and their pullbacks. Hence, by Theorem 4.1, they are generated by semiample divisors. Moreover, by the same reasoning as in Theorem 6.5, any semiample divisor on $\overline{M}_{0,n}$ or $\overline{M}_{1,n}$ is the pullback of a semiample divisor via f_1^* or f_2^* . Consequently, by the same argument used in Lemma 6.3, the orthogonal complement F^\perp is spanned by semiample divisors. Therefore, the product of the corresponding morphisms yields the desired contraction. \square

We will provide another proof of Corollary 6.6 in Section 10.3.

Remark 6.7. By the proof of Corollary 6.6, for any F-curve on $\overline{M}_{0,n}$ or $\overline{M}_{1,n}$, its orthogonal complement is spanned by either pullbacks of ψ -classes or ample line bundles along projection maps. Therefore, on $\overline{M}_{0,n}$, the corresponding morphism can be constructed as the product of all

contractions of the form $f_{\text{Kap}} \circ \pi_S$ that contract F , where f_{Kap} denotes Kapranov's construction [29, 30] associated with the class ψ_i . An analogous argument reduces the construction of corresponding morphisms in genus one to Question 4.3.

7. NEW FAMILIES OF NEF DIVISORS ON $\overline{M}_{g,n}$

Here, we introduce variants of the κ class, which form a new family of nef divisors on $\overline{M}_{g,n}$. We refer to the beginning of Section 9 for a motivation for constructing such divisors.

Note that κ is an ample divisor on $\overline{M}_{g,n}$ and satisfies

$$\kappa = 12\lambda - \delta + \psi$$

by Mumford's formula. Define

$$B_{g,n} = \{\delta_{i,I} \mid 0 \leq i \leq g, I \subseteq [n], |I| \geq 1 \text{ if } i = 0 \text{ and } I \neq [n] \text{ if } i = g\},$$

i.e., the set of all boundary classes except δ_{irr} , including $\delta_{0,i} = -\psi_i$. Note that we identify $\delta_{i,I}$ and δ_{g-i,I^c} .

Definition 7.1. A subset $D \subseteq B_{g,n}$ is called a **semigroup of boundary divisors** (semigroup for short, if there is no confusion) if it satisfies the condition:

$$\delta_{i,I}, \delta_{j,J} \in D, \quad i + j \leq g, \quad I \cap J = \emptyset \Rightarrow \delta_{i+j, I \cup J} \in D.$$

when it makes sense, i.e. if $(i + j, I \cup J) \neq (0, \emptyset), (g, [n])$.

The corresponding **semigroup kappa divisor** is defined as

$$\kappa_{g,n}(D) := \kappa + \sum_{\delta_{i,I} \in D} \delta_{i,I}.$$

Remark 7.2. We identify $\delta_{i,I}$ with δ_{g-i,I^c} , so the condition for a semigroup must also hold under this identification. For example, since $\delta_{3,\emptyset} = \delta_{g-3,[n]}$, the inclusion $\delta_{2,\emptyset}, \delta_{3,\emptyset} \in D$ implies not only $\delta_{5,\emptyset} \in D$ but also that $\delta_{g-1,[n]} = \delta_{1,\emptyset} \in D$.

Due to this symmetry, it is sometimes cumbersome to check whether a set of boundary divisors forms a semigroup. The following lemma is helpful for this purpose.

Lemma 7.3. Let $D \subseteq B_{g,n}$ and $p \in [n]$. Then D is a semigroup if and only if the following holds: for any $\delta_{i,I}, \delta_{j,J} \in D$ such that $p \notin I, J$,

- (1) If $I \cap J = \emptyset$ and $i + j \leq g$, then $\delta_{i+j, I \cup J} \in D$.
- (2) If $I \subseteq J$, $i \leq j$, and $(i, I) \neq (j, J)$, then $\delta_{j-i, J \setminus I} \in D$.

Proof. First, assume that D is a semigroup. Since $p \notin I \cup J$, condition (1) follows directly from the semigroup property. For (2), note that under the given assumptions, we have $\delta_{g-j, J^c} = \delta_{j,J} \in D$, so by the semigroup condition, it follows that $\delta_{j-i, J \setminus I} \in D$.

Conversely, assume that $\delta_{i,I}, \delta_{j,J} \in D$ with $i + j \leq g$, $I \cap J = \emptyset$, and $(i + j, I \cup J) \neq (0, \emptyset), (g, [n])$. If $p \notin I \cup J$, then $\delta_{i+j, I \cup J} \in D$ by (1). If $p \in I \cup J$, we may assume without loss of generality that $p \in J$ and $p \notin I$. Then $\delta_{g-j, J^c} = \delta_{j,J} \in D$, and since $i \leq g - j$ and $I \subsetneq J^c$, condition (2) implies $\delta_{g-j-i, J^c \setminus I} = \delta_{i+j, I \cup J} \in D$. Hence D is a semigroup. \square

Now we will prove that semigroup kappa divisors are nef. The following well-known lemma is useful in general.

Lemma 7.4. For the boundary divisor

$$\Delta_{i,I} \simeq \overline{M}_{i,I+1} \times \overline{M}_{g-i,I^c+1},$$

choose the attaching maps $\theta_1 : \overline{M}_{i,I+1} \rightarrow \overline{M}_{g,n}$ and $\theta_2 : \overline{M}_{g-i,I^c+1} \rightarrow \overline{M}_{g,n}$. A divisor D on $\overline{M}_{g,n}$ is nef (resp. semiample, ample) on $\Delta_{i,I}$ if and only if $\theta_1^* D$ and $\theta_2^* D$ are nef (resp. semiample, ample).

Proof. First, we show that

$$\mathrm{Pic}(\overline{M}_{i,I+1}) \times \mathrm{Pic}(\overline{M}_{g-i,I^c+1}) \simeq \mathrm{Pic}(\overline{M}_{i,I+1} \times \overline{M}_{g-i,I^c+1}).$$

Choose a prime l different from the characteristic of the base field. It suffices to prove this after tensoring with \mathbb{Q}_l . By [39, Theorem 0.1], we have

$$H_{\acute{e}t}^1(\overline{M}_{g,n}, \mathbb{Q}_l) = 0 \quad \text{and} \quad H_{\acute{e}t}^2(\overline{M}_{g,n}, \mathbb{Q}_l) \simeq \mathrm{Pic}(\overline{M}_{g,n}) \otimes_{\mathbb{Q}} \mathbb{Q}_l$$

for any g, n . Therefore, the statement follows from the Künneth formula. Hence, on $\Delta_{i,I}$ we have

$$D = \pi_1^* \theta_1^* D + \pi_2^* \theta_2^* D.$$

The desired equivalence is then immediate. \square

Theorem 7.5. $\kappa_{g,n}(D)$'s are nef. Moreover, in positive characteristic, they are semiample.

Proof. Step 1. Let D be a semigroup on $\overline{M}_{g,n}$ such that $\delta_{0,j} \in D$. Then there exists a semigroup D' on $\overline{M}_{g,n-1}$ such that $\kappa_{g,n}(D) = \pi_j^* \kappa_{g,n-1}(D')$.

Note that the assumption that $\delta_{0,j}$ exists implies $n > 0$. We may assume that $j = n$. Define a set of (boundary) divisors on $\overline{M}_{g,n-1}$:

$$D' := \{\delta_{i,I} \mid I \subseteq [n-1], \delta_{i,I} \in D\}.$$

First, we will prove that this is a well-defined semigroup. We need to show that if $\delta_{i,I} \in D'$, so $I \subseteq [n-1]$ and $\delta_{i,I} \in D$ as a divisor on $\overline{M}_{g,n}$, then $\delta_{g-i, [n-1] \setminus I} \in D'$, i.e., $\delta_{g-i, [n-1] \setminus I} \in D$ as a divisor on $\overline{M}_{g,n}$. Since $\delta_{i,I} \in D$, we have $\delta_{g-i, [n] \setminus I} \in D$. Also, by $\delta_{0,n} \in D$, it follows from the semigroup condition that $\delta_{g-i, [n-1] \setminus I} \in D$. Therefore, D' is well-defined. The semigroup condition for D' then follows directly from the condition for D .

Now let

$$\tilde{D} = \{(i, I) \mid 0 \leq i \leq g, I \subseteq [n-1], (i, I) \neq (0, \emptyset), (g, [n-1]) \text{ and } \delta_{i,I} \in D'\}.$$

Note that $(i, I) \mapsto \delta_{i,I}$ is a two-to-one map from \tilde{D} to D' , and

$$D = \left\{ \delta_{i,I} \mid (i, I) \in \tilde{D} \right\} \sqcup \{\delta_{0,n}\} = \left\{ \delta_{i, I \cup \{n\}} \mid (i, I) \in \tilde{D} \right\} \sqcup \{\delta_{0,n}\}.$$

This is straightforward, since if $\delta_{i,I} \in D$ and $(i, I) \neq (0, n), (g, [n-1])$, then exactly one of (i, I) or $(g-i, I^c)$ lies in \tilde{D} , depending on whether I contains n or not. Therefore,

$$\begin{aligned} \pi_n^* \kappa_{g,n-1}(D') &= \pi_n^* \kappa + \frac{1}{2} \sum_{(i,I) \in \tilde{D}} \pi_n^* \delta_{i,I} = \kappa - \psi_n + \frac{1}{2} \sum_{(i,I) \in \tilde{D}} (\delta_{i,I} + \delta_{i, I \cup \{n\}}) \\ &= \kappa + \frac{1}{2} \left(\sum_{(i,I) \in \tilde{D}} \delta_{i,I} - \psi_n \right) + \frac{1}{2} \left(\sum_{(i,I) \in \tilde{D}} \delta_{i, I \cup \{n\}} - \psi_n \right) = \kappa_{g,n}(D) \end{aligned}$$

so the conclusion follows. We have used [3, Lemma 1(i)] in the computation.

Step 2. Let D be a semigroup on $\overline{M}_{g,n}$ and let $\theta : \overline{M}_{g',n'+1} \rightarrow \overline{M}_{g,n}$ be a map attaching a fixed stable curve. Then $\theta^* \kappa_{g,n}(D)$ is also a semigroup kappa divisor.

The map θ is a composition of attaching maps that either increase the number of marked points (i.e., $n' < n$) or attach an elliptic tail. Hence, it suffices to prove the statement for such attaching maps.

First, assume $n' < n$. Then, by [3, Lemma 1],

$$\theta^* \kappa_{g,n}(D) = \kappa + \sum_{i \leq g', I \subseteq [n']}$$

Thus, it is enough to show that $D' = \{\delta_{i,I} \mid i \leq g', I \subseteq [n']\}$ is a semigroup on $\overline{M}_{g',n'+1}$. It is straightforward to verify that D' satisfies the two conditions of Lemma 7.3 with $p = n' + 1$, using the corresponding conditions for D .

Second, let $\theta : \overline{M}_{g-1, n+1} \rightarrow \overline{M}_{g, n}$ be the map attaching an elliptic curve to the $(n+1)$ -th point. Then, by [3, Lemma 1],

$$\theta^* \delta_{i, I} = \begin{cases} \delta_{i, I} + \delta_{i-1, I \cup \{n+1\}} & \text{if } 1 \leq i \leq g-1, \\ \delta_{i, I} & \text{if } i = 0, \\ \delta_{i-1, I \cup \{n+1\}} & \text{if } i = g. \end{cases}$$

We choose the representative of D as

$$\tilde{D} := \{(i, I) \mid n \notin I, \delta_{i, I} \in D\},$$

so for any element $x \in D$, there exists a unique $(i, I) \in \tilde{D}$ such that $x = \delta_{i, I}$. Then it suffices to show that

$$\begin{aligned} D' &= \left\{ \delta_{i, I} \mid i \leq g-1, (i, I) \in \tilde{D} \right\} \sqcup \left\{ \delta_{i-1, I \cup \{n+1\}} \mid 1 \leq i, (i, I) \in \tilde{D} \right\} \\ &= \left\{ \delta_{i, I} \mid i \leq g-1, (i, I) \in \tilde{D} \right\} \sqcup \left\{ \delta_{g-i, [n] \setminus I} \mid 1 \leq i, (i, I) \in \tilde{D} \right\} \end{aligned}$$

is a semigroup on $\overline{M}_{g-1, n+1}$. Note that, as a divisor on $\overline{M}_{g, n}$, we have $\delta_{g-i, [n] \setminus I} = \delta_{i, I} \in D$. Hence, the two conditions of Lemma 7.3 for D' with $p = n+1$ follow from the corresponding conditions for D .

Step 3. Proof of the theorem

Consider the first assertion. We will use induction on $3g-3+n$, i.e., the dimension of $\overline{M}_{g, n}$. The base cases are $(g, n) = (0, 4)$ and $(1, 1)$. For $(g, n) = (1, 1)$, any semigroup kappa divisor is either κ or $\kappa - \psi_1$, both of which are nef. For $(g, n) = (0, 4)$, we prove a stronger result: any semigroup kappa divisor intersects type 6 F-curves nonnegatively.

By the formula Theorem 2.2, the intersection of κ with any F-curve is 1. Choose a type 6 F-curve $F_6^{g^1, g^2, g^3, g^4}(I_1, I_2, I_3, I_4)$ on $\overline{M}_{g, n}$. If 0 (resp. 1, 2, 3, 4) among the δ_{g_i, I_i} are contained in D , then at least 0 (resp. 0, 1, 3, 3) among $\delta_{0, I_1 \cup I_2}$, $\delta_{0, I_1 \cup I_3}$, $\delta_{0, I_1 \cup I_4}$ are contained in D by the definition of a semigroup. Hence, the intersection of $\sum_{\delta_{i, I} \in D} \delta_{i, I}$ with the F-curve is at least -1 . Therefore, the intersection of $\kappa_{g, n}(D)$ with any type 6 F-curve is nonnegative.

Now we proceed with the induction step. If D contains $\delta_{0, i}$ for some i , then by Step 1, it is the pullback of a semigroup kappa divisor along the projection map π_i . Hence, $\kappa_{g, n}(D)$ is nef by the induction hypothesis. Now assume that D contains no $\delta_{0, i}$'s. Then κ is ample and $\sum_{\delta_{i, I} \in D} \delta_{i, I}$ is effective. Thus, it is enough to prove that $\kappa_{g, n}(D)|_{\Delta_{i, I}}$ is nef for every boundary divisor of form $\Delta_{i, I}$.

To prove that $\kappa_{g, n}(D)$ is nef on this divisor, by Lemma 7.4, it is enough to show that $\theta^* \kappa_{g, n}(D)$ is nef for any attaching map $\theta : \overline{M}_{g', n'+1} \rightarrow \overline{M}_{g, n}$. By Step 2, this pullback is also a semigroup kappa divisor. By the induction hypothesis, it is nef. This completes the proof of the first assertion.

Now consider the second assertion. Again, we use induction. Since any nef line bundle on $\overline{M}_{0, 4}$ and $\overline{M}_{1, 1}$ is semiample, the base cases follow from the first assertion. Now consider the induction step. If D contains $\delta_{0, i}$ for some i , then the same argument applies and shows the semiampleness. If D contains no $\delta_{0, i}$'s, then $\kappa_{g, n}(D)$ is the sum of an ample and an effective divisor, so it is big. Since it is big and nef, we can apply [32]. The exceptional locus of $\kappa_{g, n}(D)$ is contained in the union of boundary divisors of the form $\Delta_{i, I}$. Hence, it suffices to prove that $\kappa_{g, n}(D)|_{\Delta_{i, I}}$ is semiample. The proof is exactly the same as in the nef case. \square

Note that the role of kappa divisors in this proof is threefold: (1) they provide a nef divisor that is stable under pullback by clutching maps, (2) they ensure that the divisors $\kappa_{g, n}(D)$ are F-nef, and (3) the formula [3, Lemma 1(i)]. This naturally raises the question of whether one can replace κ with other divisors. The second property is not essential, as one may instead restrict attention to those divisors that are already F-nef. This leads to the following.

Theorem 7.6. Let $(L_{g,n})_{2g-2+n>0}$ be a family of nef line bundles such that (1) for any clutching map $\theta : \overline{M}_{g',n'+1} \rightarrow \overline{M}_{g,n}$, we have $\theta^* L_{g,n} = L_{g',n'+1}$, and (2) $\pi_i^* L_{g,n} = L_{g,n+1} - \psi_i$. Then

$$L_{g,n}(D) := L_{g,n} + \sum_{\delta_{i,I} \in D} \delta_{i,I}$$

is nef whenever it is F-nef. Moreover, if $L_{g,n}$ is big and nef for every g, n satisfying $2g-2+n > 0$, then $L_{g,n}(D)$ is semiample in positive characteristic whenever it is F-nef.

For example, one may take any nef divisor that is the sum of the κ -class and a linear combination of λ and $12\lambda - \delta_{\text{irr}}$. In particular, the log canonical divisor with respect to the boundary

$$K_{\overline{\mathcal{M}}_{g,n}} + \delta = 13\lambda - \delta + \psi = \kappa + \lambda$$

(see, for example, [22]) satisfies this condition. We omit the proof, as it proceeds identically to that of Theorem 7.5.

- Example 7.7.**
- (1) If $I \neq \emptyset, [n]$, or $I = \emptyset$ and $2i > g$, or $I = [n]$ and $2i < g$, then $\kappa + \delta_{i,I}$ is a nef divisor on $\overline{M}_{g,n}$.
 - (2) If $(\delta_{i,I}, \delta_{j,J}, \delta_{k,K})$ is a triple such that each divisor satisfies the condition of (1), $i+j+k = g$ and $I \sqcup J \sqcup K = [n]$, then $\kappa + \delta_{i,I} + \delta_{j,J} + \delta_{k,K}$ is nef.
 - (3) $\kappa + \sum \delta_{i,\emptyset}$ is nef. Moreover, $\kappa + \sum_{i \text{ even}} \delta_{i,\emptyset}$ is also nef. However, $\kappa + \sum_{i \text{ odd}} \delta_{i,\emptyset}$ is not nef in general. Similarly, for any $d \in \mathbb{Z}_{>0}$, $\kappa + \sum_{d|i} \delta_{i,\emptyset}$ is nef.
 - (4) Let $I, J \subseteq [n]$ be such that $I \setminus J, J \setminus I$ and $I \cap J$ are nonempty. Then, for any $i, j \leq g$, the pair $(\delta_{i,I}, \delta_{j,J})$ is a semigroup if $I \cup J \neq [n]$ or $i + j < g$. Hence, $\kappa + \delta_{i,I} + \delta_{j,J}$ is nef in this case.

Remark 7.8. Any symmetric semigroup kappa divisor on $\overline{M}_{0,n}$ is of the following form: for a positive integer $d \mid n$,

$$D_{d,n} := \kappa + \sum_{d||I} \delta_{0,I}.$$

For $d = 2, 3$, this is related to other divisors that have already been studied: type A, level 1 affine coinvariant divisors [17] and type A parafermion divisors [5].

Let $F_{1,1,i}$ be an F-curve of the form $F_6^{0,0,0,0}(I_1, I_2, I_3, I_4)$ where $|I_1| = |I_2| = 1$ and $|I_3| = i$, viewed as a class in $A_1(\overline{M}_{0,n}/S_n)$. Since we are considering the symmetrized situation, the class does not depend on the exact choice of the I_j 's. Note that, by [1, Corollary 2.2], these classes form a basis of $A_1(\overline{M}_{0,n}/S_n)$. Hence, to compare symmetric divisors, it is enough to compute their intersections with the $F_{1,1,i}$'s. Using Theorem 2.2, we obtain

$$D_{2,n} \cdot F_{1,1,i} = \begin{cases} 0 & \text{if } 2 \mid i, \\ 4 & \text{if } 2 \nmid i, \end{cases}$$

and

$$D_{3,n} \cdot F_{1,1,i} = \begin{cases} 3 & \text{if } i \equiv 2 \pmod{3}, \\ 0 & \text{otherwise.} \end{cases}$$

By comparing these with [17, Proposition 5.2] and [5, Propositions 4.41, 4.42, 4.48, 4.49], we find that $D_{2,n}$ (resp. $D_{3,n}$) coincides, up to a positive constant, with the \mathfrak{sl}_2 (resp. \mathfrak{sl}_3) level 1 affine coinvariant divisor, and with the \mathfrak{sl}_r level 2 (resp. level 3) parafermion coinvariant divisor. Hence, by [17, Lemma 2.5], they are also semiample in characteristic zero.

The notion underlying this is that of divisors arising from symmetric functions, as defined in [20]. Symmetric semigroup kappa divisors on $\overline{M}_{0,n}$, level 1 affine coinvariant divisors, and certain parafermion coinvariant divisors are all examples of divisors from symmetric functions. They lie in the same abelian group but arise from different symmetric functions, and in some simple cases, they coincide up to a positive constant.

As we will see in Section 10, semigroup kappa divisors are useful for constructing nef divisors that contract certain F-curves. In view of Corollary 6.6, Remark 7.8, and the semiampleness of semigroup kappa divisors in characteristic p , the following questions naturally arise:

- Question 7.9.** (1) In characteristic zero, which semigroup kappa divisors are semiample?
 (2) In positive characteristic, is there a modular interpretation of the morphism defined by semigroup kappa divisors?

From now on, we will list some sporadic new nef divisors, which will be used in Section 10.

Proposition 7.10 (char $k \neq 2$). For any $0 \leq \epsilon_{i,I} \leq \frac{1}{4}$,

$$\kappa + \sum_{i,I} \epsilon_{i,I} \delta_{i,I}$$

is nef, where $\delta_{i,I}$'s include $\delta_{0,i}$'s.

Proof. First, we will prove that this divisor is F-nef. The intersection of κ with any F-curve is 1. By Theorem 2.2, the intersection of $\sum_{i,I} \epsilon_{i,I} \delta_{i,I}$ with an F-curve is at least the sum of four terms of the form $-\epsilon_{i,I}$, hence at least -1 . Therefore, the total intersection is nonnegative, and the divisor is F-nef.

We will first prove that $\kappa - \frac{1}{4} \sum_{i \in I} \psi_i$ is nef, where $I \subseteq [n]$. Note that this is a special case of the divisor in the statement. Let $F : \overline{M}_{0,g+n} \rightarrow \overline{M}_{g,n}$ be the flag map. Then, by [3, Lemma 1], the pullback of the divisor along F is a divisor of the same form. Since this divisor is F-nef, by [26, Theorem 0.3], it is enough to check the genus 0 case.

Since the ψ_i are nef, it is enough to show that $\kappa - \frac{1}{4} \psi$ is nef on $\overline{M}_{0,g}$. Let $F : \overline{M}_{0,g} \rightarrow \overline{M}_g$ be the flag map, and consider the divisor $\kappa + \frac{1}{4} \delta_1$. By [3, Lemma 1], we have $F^*(\kappa + \frac{1}{4} \delta_1) = \kappa - \frac{1}{4} \psi$, so it suffices to show that $\kappa + \frac{1}{4} \delta_1$ is nef. Since this is a special case of the divisor in the statement, it is F-nef. Using $\kappa = 12\lambda - \delta$ on \overline{M}_g , this follows from [24, Corollary 5.4].

Now we prove the general assertion. As in the proof of Theorem 7.5, we use induction on $3g - 3 + n$. The base case follows from the F-nefness.

Consider the induction step. Since $\kappa - \frac{1}{4} \sum_{i \in I} \psi_i$ is nef, the divisor in the statement is a sum of a nef divisor and an effective sum of boundary divisors of the form $\Delta_{i,I}$. Hence, it is enough to show that the restriction to $\Delta_{i,I}$ is nef. This follows from Lemma 7.4: by [3, Lemma 1], the pullback of the divisor along an attaching map θ is of the same form, and the induction hypothesis applies. \square

Now, let $\delta_{k,K}$ be a boundary divisor satisfying the condition of Example 7.7 (1) and

$$B_{k,K}^0 := \{\delta_{j,J} \mid J \subseteq K^c, j \leq g - k, J = \emptyset \Rightarrow k < j, J = K^c \Rightarrow g - 2k > j\}.$$

This set has a natural involution given by $\delta_{j,J} \mapsto \delta_{g-k-j, [n] \setminus (K \cup J)}$. Let $B_{k,K}$ be the quotient of $B_{k,K}^0$ by this involution, and let $\pi : B_{k,K}^0 \rightarrow B_{k,K}$ be the quotient map.

Proposition 7.11 (char $k \neq 2$). For any subset $B \subseteq B_{k,K}$,

$$D_B := \kappa + \delta_{k,K} + \frac{1}{4} \sum_{\pi(\delta_{i,I}) \in B} \delta_{i,I}$$

is nef.

Proof. By Proposition 7.10, D_B is a sum of a nef line bundle and $\delta_{k,K}$. Hence, it is enough to show that $D_B|_{\Delta_{k,K}}$ is nef. Let

$$\theta_1 : \overline{M}_{k,K \cup \{p\}} \rightarrow \overline{M}_{g,n} \quad \text{and} \quad \theta_2 : \overline{M}_{g-k, K^c \cup \{p\}} \rightarrow \overline{M}_{g,n}$$

be the relevant attaching maps. By Lemma 7.4, it is enough to show that $\theta_1^* D_B$ and $\theta_2^* D_B$ are nef. We have

$$\theta_1^* D_B = \kappa - \psi_p = \pi_p^* \kappa,$$

which is nef. Also,

$$\theta_2^* D_B = \kappa - \psi_p + \frac{1}{4} \sum_{\pi(\delta_{i,I}) \in B} \delta_{i,I} = \pi_p^* \left(\kappa + \frac{1}{4} \sum_{\delta_{i,I} \in B} \delta_{i,I} \right),$$

and by Proposition 7.10, this is also nef. Therefore, D_B is nef. \square

Proposition 7.12 (char $k \neq 2$). On $\overline{M}_{g,1}$ with $g \geq 3$,

$$D := \kappa + \delta_{1,\{1\}} + \frac{1}{4} (\delta_{1,\emptyset} + \delta_{g-2,\emptyset})$$

is nef.

Proof. Most of the proof is the same as in Proposition 7.11. The difference is that $\theta_1^* D$ is not simply $\kappa - \psi_p$. In this case,

$$\theta_1^* D = \kappa - \psi_p + \frac{1}{4} \delta_{1,\emptyset}$$

which is nef on $\overline{M}_{1,2}$. Hence, D is also nef. \square

Proposition 7.13.

$$D := \kappa + \frac{2}{3} \delta_{1,\emptyset} + \frac{1}{3} \delta_{2,\emptyset}$$

is a nef divisor on $\overline{M}_{3,n}$.

Proof. First, we will prove that this is F-nef. Since the intersection of κ with any F-curve is 1, by the explicit intersection formula in Theorem 2.2, it is straightforward to see that the intersection of D with any F-curve of types 1–4 is nonnegative. By a similar argument, we can show that the intersection with a type 5 F-curve is also nonnegative, except in the case $F_5^{1,1}(\emptyset, \emptyset)$. Since the coefficient of $\delta_{2,\emptyset}$ is $\frac{1}{3}$, a direct computation gives $D \cdot F_5^{1,1}(\emptyset, \emptyset) = 0$.

Now consider the type 6 case. The main factor is, among the δ_{g_i, I_i} 's, how many of them are $\delta_{1,\emptyset}$ or $\delta_{2,\emptyset}$. The problematic case occurs when 2 or 3 among them are $\delta_{1,\emptyset}$. In these cases, we can explicitly compute the intersection number using Theorem 2.2, and the result is 0 in both cases. Hence, D is F-nef.

Since κ is ample, it is enough to show that $D|_{\Delta_{1,\emptyset}}$ and $D|_{\Delta_{2,\emptyset}}$ are nef. Let

$$\theta_1 : \overline{M}_{1,1} \rightarrow \overline{M}_{3,n}, \quad \theta_2 : \overline{M}_{2,n+1} \rightarrow \overline{M}_{3,n}, \quad \theta_3 : \overline{M}_{2,1} \rightarrow \overline{M}_{3,n}, \quad \theta_4 : \overline{M}_{1,n+1} \rightarrow \overline{M}_{3,n}$$

be the relevant attaching maps. By Lemma 7.4, it suffices to prove that $\theta_i^* D$ is nef for $i = 1, 2, 3, 4$. Since D is F-nef, each $\theta_i^* D$ is also F-nef. In particular, $\theta_1^* D$ and $\theta_3^* D$ are nef by the known cases of the F-conjecture.

We have

$$\theta_2^* D = \kappa - \frac{2}{3} \psi_{n+1} + \frac{1}{3} \delta_{2,\emptyset} + \frac{1}{3} \delta_{1,\{n+1\}}$$

on $\overline{M}_{2,n+1}$, by [3, Lemma 1]. Since $\kappa - \psi_{n+1} = \pi_{n+1}^* \kappa$ is nef, it is enough to show that the restriction of $\theta_2^* D$ to $\Delta_{2,\emptyset}$ and $\Delta_{1,\{n+1\}}$ is nef. As in the previous paragraph, there are four pullbacks to consider, but two of them are trivially nef by the known cases of the F-conjecture. The two other nontrivial cases are

$$\alpha_1 : \overline{M}_{0,n+2} \rightarrow \overline{M}_{2,n+1}, \quad \alpha_2 : \overline{M}_{1,n+1} \rightarrow \overline{M}_{2,n+1}.$$

Then we have

$$\alpha_1^* \theta_2^* D = \kappa - \frac{2}{3} \psi_{n+1} - \frac{1}{3} \psi_{n+2}, \quad \alpha_2^* \theta_2^* D = \kappa - \frac{1}{3} \psi_{n+1}.$$

Since the divisors $\kappa - \psi_i$ are nef, both of these are nef. Hence, $\theta_2^* D$ is nef.

Next, we have

$$\theta_4^* D = \kappa - \frac{2}{3} \psi_{n+1} + \frac{1}{3} \delta_{1,\emptyset}$$

on $\overline{M}_{1,n+1}$. By the same argument as above, it is enough to show that $\beta^*\theta_4^*D$ is nef for

$$\beta : \overline{M}_{0,n+2} \rightarrow \overline{M}_{1,n+1}.$$

Since

$$\beta^*\theta_4^*D = \kappa - \frac{2}{3}\psi_{n+1} - \frac{1}{3}\psi_{n+2},$$

this is also nef. Hence, θ_4^*D is nef, and therefore D is nef. □

Proposition 7.14 (char $k \neq 2$).

$$D := \kappa + \frac{1}{2}\delta_{1,\emptyset} + \frac{1}{2}\delta_{2,\emptyset}$$

is a nef divisor on $\overline{M}_{4,n}$.

Proof. Step 1. On $\overline{M}_{0,n}$ with $n \geq 4$,

$$D_0 := \kappa - \frac{1}{2} \sum_{i=1}^4 \psi_i + \frac{1}{2} \sum_{\{i,j\} \subset [4]} \delta_{0,\{i,j\}}$$

is nef.

Since the case $n = 4$ is straightforward, we may assume $n \geq 5$. For any $\{i, j\} \subset [4]$,

$$\kappa - \psi_i - \psi_j + \delta_{0,\{i,j\}} = \pi_i^* \pi_j^* \kappa$$

is nef. By averaging these divisors, we find that

$$\kappa - \frac{1}{2} \sum_{i=1}^4 \psi_i + \frac{1}{6} \sum_{\{i,j\} \subset [4]} \delta_{0,\{i,j\}}$$

is nef. Hence, for any $\Delta_{0,\{i,j\}} \simeq \overline{M}_{0,([n] \setminus \{i,j\}) \cup p}$, it suffices to prove that $D_0|_{\Delta_{0,\{i,j\}}}$ is nef. Without loss of generality, we may assume $(i, j) = (3, 4)$. On $\overline{M}_{0,([n] \setminus \{3,4\}) \cup p}$, $D_0|_{\Delta_{0,\{3,4\}}}$ is

$$\kappa - \frac{1}{2}(\psi_1 + \psi_2 + \psi_p) + \delta_{0,\{1,2\}} = \frac{1}{2}\pi_1^*\pi_2^*\kappa + \frac{1}{2}\pi_p^*\kappa,$$

which is nef.

Step 2. D is F-nef.

Since the intersection of κ with any F-curve is 1, and the coefficients of the boundary divisors are $\frac{1}{2}$, it is straightforward to verify that they intersect F-curves of types 1 to 5 non-negatively by Theorem 2.2. For a type 6 F-curve $F_6^{g_1, g_2, g_3, g_4}(I_1, I_2, I_3, I_4)$, the result depends on the number of $\delta_{1,\emptyset}$ and $\delta_{2,\emptyset}$ among the δ_{g_i, I_i} 's. If there are at most two such divisors, then the intersection is trivially non-negative. The remaining cases are $F_6^{1,1,1,1}(\emptyset, \emptyset, \emptyset, \emptyset)$ and $F_6^{0,1,1,2}([n], \emptyset, \emptyset, \emptyset)$, where a direct computation shows that the intersection is non-negative.

Step 3. D is nef.

By Step 2 and [26, Theorem 0.3], it is enough to prove that F^*D is nef. By [3, Lemma 1],

$$F^*D = \kappa - \frac{1}{2} \sum_{i=1}^4 \psi_i + \frac{1}{2} \sum_{\{i,j\} \subset [4]} \delta_{0,\{i,j\}},$$

which is nef by Step 1. □

Proposition 7.15.

$$D := \kappa - \frac{1}{2}\psi_1 + \frac{1}{2}\delta_{2,\emptyset} + \frac{1}{2}\delta_{4,\emptyset} + \frac{1}{2}\delta_{1,\{1\}}$$

is nef on $\overline{M}_{4,n}$ for $n \geq 2$.

Proof. Since

$$\kappa - \frac{1}{2}\psi_1 + \frac{1}{2}\delta_{2,\emptyset} + \frac{1}{2}\delta_{4,\emptyset} = \frac{1}{2}\pi_1^*\kappa + \frac{1}{2}(\kappa + \delta_{2,\emptyset} + \delta_{4,\emptyset}),$$

which is nef by Example 7.7 (3), it suffices to show that $D|_{\Delta_{1,\{1\}}}$ is nef. Let

$$\theta_1 : \overline{M}_{1,\{1,p\}} \rightarrow \overline{M}_{4,n}, \quad \theta_2 : \overline{M}_{3,\{1\}^c \cup p} \rightarrow \overline{M}_{4,n}$$

be the relevant attaching maps. By Lemma 7.4, it is enough to prove that θ_i^*D is nef for $i = 1, 2$. We have

$$\theta_1^*D = \kappa - \frac{1}{2}\psi_1 - \frac{1}{2}\psi_p = \frac{1}{2}(\pi_1^*\kappa + \pi_p^*\kappa)$$

and

$$\theta_2^*D = \kappa + \frac{1}{2}\delta_{2,\emptyset} - \frac{1}{2}\psi_p = \frac{1}{2}(\pi_p^*\kappa + \kappa + \delta_{2,\emptyset}),$$

since $n \geq 2$. Both are nef by Example 7.7. \square

For future reference, we record the following.

Proposition 7.16. The F-conjecture holds for $\overline{M}_{1,2}$, $\overline{M}_{2,1}$, $\overline{M}_{2,2}$ and $\overline{M}_{3,1}$. Moreover, if the base field has positive characteristic, then any nef line bundle on $\overline{M}_{1,2}$ and $\overline{M}_{2,1}$ is semiample.

Note that the first part follows from [26] if the base field has characteristic not equal to 2.

Proof. On $\overline{M}_{1,2}$ (resp. $\overline{M}_{2,1}$), the cone of F-nef divisors is generated by $\lambda, 12\lambda - \delta_{\text{irr}}$ (resp. $\lambda, 12\lambda - \delta_{\text{irr}}, \psi_1$). These are nef regardless of the characteristic.

Consider $\overline{M}_{2,2}$. The extremal rays of the cone of F-nef divisors are given in Example 5.1. The divisors in the first line are nef by the same argument as above. Since

$$\psi_1 + \psi_2 - 2\delta_{0,\{1,2\}} = \pi_1^*\psi + \pi_2^*\psi,$$

the second line is a sum of nef divisors and $\delta_{1,\{1\}}$. Hence it is enough to prove this is nef on $\Delta_{1,\{1\}}$. Since the F-conjecture holds for $\overline{M}_{1,2}$, F-nefness implies nefness on $\Delta_{1,\{1\}}$ by Lemma 7.4. For

$$D = \lambda + \psi_1 + \psi_2 + \delta_{1,\emptyset},$$

it is again enough, by Lemma 7.4, to show that θ^*D is nef, where $\theta : \overline{M}_{1,3} \rightarrow \overline{M}_{2,2}$. We compute

$$\theta^*D = \lambda + \psi_1 + \psi_2 - \psi_3 + \delta_{1,\emptyset}.$$

Note that $\psi_1 = \psi_2$ on $\overline{M}_{1,2}$. Therefore,

$$\psi_2 - \delta_{0,\{1,2\}} = \pi_1^*\psi_2 = \pi_1^*\psi_3 = \psi_3 - \delta_{0,\{1,3\}},$$

so

$$\theta^*D = \lambda + \psi_1 + \delta_{0,\{1,2\}} - \delta_{0,\{1,3\}} + \delta_{1,\emptyset} = \lambda + \pi_3^*\psi_1 + \delta_{0,\{1,2\}} + \delta_{1,\emptyset}.$$

Since $\lambda + \pi_3^*\psi_1$ is nef and the F-conjecture holds for $\overline{M}_{1,2}$, this divisor is nef by Lemma 7.4.

Now consider $\overline{M}_{3,1}$. Note that [16] proves the F-conjecture for \overline{M}_3 . In particular,

$$10\lambda - \delta_{\text{irr}} - 2\delta_{1,\emptyset} - 2\delta_{1,\{1\}} = \pi^*(10\lambda - \delta_{\text{irr}} - 2\delta_1)$$

is nef. Hence, the first four divisors in Example 5.5 are nef, and it remains to show that the following two divisors are also nef:

$$10\lambda - \delta_{\text{irr}} + 2\psi_1 - 2\delta_{1,\emptyset}, \quad 11\lambda - \delta_{\text{irr}} + 3\psi_1 - \delta_{1,\emptyset} - 2\delta_{1,\{1\}}.$$

We compute:

$$10\lambda - \delta_{\text{irr}} + 2\psi_1 - 2\delta_{1,\emptyset} = (10\lambda - \delta_{\text{irr}} - 2\delta_{1,\emptyset} - 2\delta_{1,\{1\}}) + 2\psi_1 + 2\delta_{1,\{1\}},$$

so it suffices to prove that its restriction to $\Delta_{1,\{1\}}$ is nef. This follows from Lemma 7.4 and the F-conjecture for $\overline{M}_{1,2}$ and $\overline{M}_{2,1}$.

Moreover, we have:

$$11\lambda - \delta_{\text{irr}} + 3\psi_1 - \delta_{1,\emptyset} - 2\delta_{1,\{1\}} = \lambda + (10\lambda - \delta_{\text{irr}} - 2\delta_{1,\emptyset} - 2\delta_{1,\{1\}}) + 3\psi_1 + \delta_{1,\emptyset}.$$

Again, this follows from Lemma 7.4 and the F-conjecture for $\overline{M}_{1,1}$ and $\overline{M}_{2,2}$.

We now consider the second assertion. Since λ corresponds to the Satake compactification, it is always semiample on \overline{M}_g for $g \geq 2$ and on $\overline{M}_{1,1}$. Also, by [44], the divisor $12\lambda - \delta_{\text{irr}}$ is semiample on \overline{M}_g for $g \geq 2$. Note that λ is zero on genus 0, and $12\lambda - \delta_{\text{irr}}$ is zero on genus 0 and 1. Since λ and $12\lambda - \delta_{\text{irr}}$ on $\overline{M}_{g,n}$ are pullbacks of those on \overline{M}_g or $\overline{M}_{1,1}$, they are semiample in general. Therefore, the second assertion follows from this and [32]. \square

8. DIVISORS ON KNUDSEN'S CONSTRUCTION

The main purpose of this section is to prove the exact sequence in Theorem 8.1, which will be used in Section 9. As before, we refer to the beginning of Section 9 for the motivation behind this exact sequence.

The **Knudsen construction** is the map

$$f_{g,n}^{\text{Knu}} : \overline{M}_{g,n} \rightarrow \overline{M}_{g,n-1} \times_{\overline{M}_{g,n-2}} \overline{M}_{g,n-1},$$

induced by the two forgetful maps π_{n-1} and π_n . This map was originally defined in [35]. For any $g \geq 0$ and $n \geq 2$ such that $2g - 2 + n > 0$, we define $F_{g,n}^{\text{Knu}}$ to be

$$\left\{ F_6^{0,0,g_1,g_2}(\{n-1\}, \{n\}, I_1, I_2) \mid g_1 + g_2 = g, I_i \neq \emptyset \text{ if } g_i = 0 \right\} \cup \left\{ F_5^{0,0}(\{n-1\}, \{n\}) \right\}.$$

This is exactly the set of F-curves contracted by $f_{g,n}^{\text{Knu}}$.

The following theorem, which is the main theorem of this section, will play a crucial role in developing a method for proving regular extremality of curves in Section 9.

Theorem 8.1. The sequence

$$0 \rightarrow \text{Pic}(\overline{M}_{g,n-2}) \xrightarrow{(\pi_n^*, -\pi_{n-1}^*)} \text{Pic}(\overline{M}_{g,n-1}) \times \text{Pic}(\overline{M}_{g,n-1}) \xrightarrow{\pi_n^* + \pi_{n-1}^*} \text{Pic}(\overline{M}_{g,n}) \rightarrow \mathbb{Q}^{F_{g,n}^{\text{Knu}}} \rightarrow 0.$$

is exact, where the last map is given by the intersection pairing. Moreover, the image of $\pi_n^* + \pi_{n-1}^*$ is the same as the image of $f_{g,n}^{\text{Knu}*}$.

Remark 8.2. In general, $f_{g,n}^{\text{Knu}}$ is not a contraction and admits a Stein factorization

$$\overline{M}_{g,n} \rightarrow X_{g,n}^{\text{Knu}} \rightarrow \overline{M}_{g,n-1} \times_{\overline{M}_{g,n-2}} \overline{M}_{g,n-1},$$

where $X_{g,n}^{\text{Knu}}$ is the coarse moduli space of $\overline{M}_{g,n-1} \times_{\overline{M}_{g,n-2}} \overline{M}_{g,n-1}$, as in [10, Theorem 4.5]. As in [12], Theorem 8.1 characterizes the Picard group of $X_{g,n}^{\text{Knu}}$ as the image of $\pi_n^* + \pi_{n-1}^*$.

The following theorem is the key step of Theorem 8.1. Indeed, a much weaker version of this theorem would suffice, but we will prove a stronger version in order to establish Corollary 8.6.

Theorem 8.3. (char $k \neq 2$) For any $C \in F_{g,n}^{\text{Knu}}$, there exists a nef divisor D_C such that $D_C \cdot C > 0$ and $D_C \cdot C' = 0$ for every $C' \in F_{g,n}^{\text{Knu}} \setminus \{C\}$.

Proof. If $g = 0$, then this is proved in [12, Corollary 3.15]. Assume $g \geq 1$. If C is the type 5 curve in $F_{g,n}^{\text{Knu}}$, then

$$D = \pi_{\{n-1,n\}}^* \left(\kappa + \sum_{1 \leq i \leq g} \delta_{i,\emptyset} \right)$$

works, where $\pi_{\{n-1,n\}} : \overline{M}_{g,n} \rightarrow \overline{M}_{g,2}$. Note that D is nef by Example 7.7, and $D_C \cdot C = 2$.

Now assume that C is type 6. Then C is the image of a curve C_0 in $F_{0,g+n}^{\text{Knu}}$ under the flag map $F : \overline{M}_{0,g+n} \rightarrow \overline{M}_{g,n}$. By [12, Corollary 3.15], there exists a nef divisor on $\overline{M}_{0,g+n}$ which intersects C_0 nontrivially and contracts all other curves in $F_{0,g+n}^{\text{Knu}}$. By symmetrizing this with the S_g -action and using [26, Theorem 0.7], there exists a nef divisor D_0 on $\overline{M}_{g,n}$ such that $D_0 \cdot C \neq 0$ and contracts all other type 6 curves in $F_{g,n}^{\text{Knu}}$.

Let D_{GKM} be a divisor from [26, Section 4]. This is a nef line bundle on $\overline{\mathcal{M}}_{g,n}$ such that $F^*D_{\text{GKM}} = 0$ and it intersects nontrivially with all F-curves of types 1–5. Choose a constant c such that $D_0 - cD_{\text{GKM}}$ contracts the type 5 curve in $F_{g,n}^{\text{Knu}}$.

Now, choose a sufficiently ample divisor A on $\overline{\mathcal{M}}_{g,n-1}$ and let

$$D := D_0 - cD_{\text{GKM}} + \pi_{n-1}^*A + \pi_n^*A.$$

Note that D satisfies all the conditions of the statement except possibly nefness. Since F-curves not in $F_{g,n}^{\text{Knu}}$ are not contracted by both π_{n-1} and π_n , this is F-nef for a sufficiently ample divisor A . Moreover, since $F^*D_{\text{GKM}} = 0$, we have F^*D nef. Therefore, by [26, Theorem 0.7], D is nef. \square

Corollary 8.4. $F_{g,n}^{\text{Knu}}$ is linearly independent.

Proof. If $\text{char } k \neq 2$, then this follows from Theorem 8.3. This also implies the characteristic 2 case, since the Picard group of the moduli space of curves does not depend on the characteristic. \square

The following lemma is well known. See, for example, [3].

- Lemma 8.5.**
- (1) $\rho(\overline{\mathcal{M}}_{0,n}) = 2^{n-1} - \binom{n}{2} - 1$.
 - (2) $\rho(\overline{\mathcal{M}}_{1,n}) = 2^n - n$.
 - (3) $\rho(\overline{\mathcal{M}}_2) = 2$ and $\rho(\overline{\mathcal{M}}_{2,n}) = 3 \cdot 2^{n-1}$ for $n \geq 1$.
 - (4) If $g \geq 3$, then $\rho(\overline{\mathcal{M}}_g) = \lfloor \frac{g}{2} \rfloor + 2$ and $\rho(\overline{\mathcal{M}}_{g,n}) = (g+1) \cdot 2^{n-1} + 1$ for $n \geq 1$.

Proof of Theorem 8.1. The case of $g = 0$ (resp. $g = 1$) is proved in [12, Theorem 3.17] (resp. [10, Corollary 4.9]). Hence, we will assume that $g \geq 2$. The surjectivity of the last map follows from Corollary 8.4.

Now, we will prove that the sequence

$$\text{Pic}(\overline{\mathcal{M}}_{g,n-1}) \times \text{Pic}(\overline{\mathcal{M}}_{g,n-1}) \xrightarrow{\pi_n^* + \pi_{n-1}^*} \text{Pic}(\overline{\mathcal{M}}_{g,n}) \rightarrow \mathbb{Q}^{F_{g,n}^{\text{Knu}}}$$

is exact. Since the last map is surjective, it suffices to prove that there exists a set T consisting of $|F_{g,n}^{\text{Knu}}|$ divisors such that $T \cup \text{Im } \pi_{n-1}^* \cup \text{Im } \pi_n^*$ spans $\text{Pic}(\overline{\mathcal{M}}_{g,n})$. Note that, by [3, Lemma 1 (i), (ii), (iii)], it is enough to show that every boundary divisor is contained in the span of $T \cup \text{Im } \pi_{n-1}^* \cup \text{Im } \pi_n^*$. Let V_T be the span of $T \cup \text{Im } \pi_{n-1}^* \cup \text{Im } \pi_n^*$.

First, consider the case of $n = 2$. In this case, $|F_{g,n}^{\text{Knu}}| = \lfloor \frac{g}{2} \rfloor + 1$. Define

$$T := \left\{ \delta_{i, \{1,2\}} \mid 0 \leq i \leq \lfloor \frac{g}{2} \rfloor \right\}.$$

Then, by [3, Lemma 1 (iv)], it is straightforward to see that V_T contains all of the boundary divisors, so $V_T = \text{Pic}(\overline{\mathcal{M}}_{g,n})$.

Next, assume $n \geq 3$. In this case, $|F_{g,n}^{\text{Knu}}| = (g+1) \cdot 2^{n-3}$. Define

$$T := \{ \delta_{i,I} \mid I \subseteq [n-1], n-1, n-2 \in I \} \cup \{ \psi_n \}$$

Then, it is straightforward to see that $|T| = (g+1) \cdot 2^{n-3}$. First, consider $\delta_{i,I}$ for $I \subseteq [n]$, such that $n-1, n \in I$. If $(i, I) = (0, \{n-1, n\})$, then $\delta_{i,I} = \psi_n - \pi_{n-1}^* \psi_n$, so $\delta_{i,I} \in V_T$. Assume $(i, I) \neq (0, \{n-1, n\})$. If $n-2 \notin I$, then $\delta_{i, I \setminus \{n-1\}} \in T$, so

$$\delta_{i,I} = \pi_{n-1}^* \delta_{i, I \setminus \{n-1\}} - \delta_{i, I \setminus \{n-1\}} \in V_T$$

and if $n-2 \in I$, then $\delta_{i, I \setminus \{n\}} \in T$, so

$$\delta_{i,I} = \pi_n^* \delta_{i, I \setminus \{n\}} - \delta_{i, I \setminus \{n\}} \in V_T.$$

Hence, $\delta_{i,I} \in V_T$ for all such I . Now, again by [3, Lemma 1 (iv)], it is straightforward that this implies $\delta_{i,I} \in V_T$ for any (i, I) . Therefore, $V_T = \text{Pic}(\overline{\mathcal{M}}_{g,n})$. This completes the proof of the

exactness of $\text{Pic}(\overline{\mathcal{M}}_{g,n-1}) \times \text{Pic}(\overline{\mathcal{M}}_{g,n-1}) \xrightarrow{\pi_n^* + \pi_{n-1}^*} \text{Pic}(\overline{\mathcal{M}}_{g,n}) \rightarrow \mathbb{Q}^{F_{g,n}^{\text{Knu}}}$

Now consider the exactness of the full sequence. Since π_i is a contraction, the pullback map π_i^* is injective. Therefore, $\text{Pic}(\overline{M}_{g,n-2}) \xrightarrow{(\pi_n^*, -\pi_{n-1}^*)} \text{Pic}(\overline{M}_{g,n-1}) \times \text{Pic}(\overline{M}_{g,n-1})$ is also injective. Hence, the theorem follows from the following elementary computation:

$$|F_{g,n}^{\text{Knu}}| = \rho(\overline{M}_{g,n-2}) - 1 = \rho(\overline{M}_{g,n}) - \rho(\overline{M}_{g,n-1}) - \rho(\overline{M}_{g,n-1}) + \rho(\overline{M}_{g,n-2})$$

for $g \geq 3$, and

$$|F_{2,n}^{\text{Knu}}| = \rho(\overline{M}_{2,n-2}) = \rho(\overline{M}_{2,n}) - \rho(\overline{M}_{2,n-1}) - \rho(\overline{M}_{2,n-1}) + \rho(\overline{M}_{2,n-2})$$

which follows directly from Lemma 8.5. \square

We observe that this establishes the extremality of F-curves in $F_{g,n}^{\text{Knu}}$. The following result follows directly from Theorem 8.1 and Theorem 8.3.

Corollary 8.6 (char $k \neq 2$). $\overline{NE}_1(f_{g,n}^{\text{Knu}})$ is a simplicial cone generated by $F_{g,n}^{\text{Knu}}$. Moreover, any element of $F_{g,n}^{\text{Knu}}$ is a regular extremal ray.

9. METHOD FOR PROVING REGULAR EXTREMALITY

Here, we describe the method for proving the regular extremality of F-curves, using Section 7 and Section 8. Let F be a curve whose regular extremality we wish to prove. Our goal is to construct a set of nef divisors T that contract F and span a codimension 1 subspace of the Picard group. This is precisely what was done in the proof of Corollary 6.6. What we proved there is essentially the following: if we let

$$T := \{\pi_S^* \psi_i \mid \pi_S^* \psi_i \cdot F = 0\},$$

then T spans a codimension 1 subspace. We aim to apply the same strategy for curves on $\overline{M}_{g,n}$ in general. However, there are two main obstacles to this method:

- (1) We do not have enough nef divisors. Pullbacks of ψ -classes alone are insufficient.
- (2) Even with enough nef divisors, it is difficult to compute the dimension of their span.

Section 7 addresses issue (1); we will soon see that the divisors constructed there are well-suited for contracting F-curves. Issue (2) will be resolved by an inductive argument using Theorem 8.1. Instead of proving the regular extremality of a single curve F , we prove the regular extremality of a family of curves on $\overline{M}_{g,n}$ —with fixed g and varying n —that is stable under the projection maps, to use the induction. Then, the regular extremality of $\pi_* F$ and Theorem 8.1 reduce the problem to computing the intersection of divisors contracting F with $F_{g,n}^{\text{Knu}}$. This is formalized by the following lemma.

Lemma 9.1. Fix $g, n \in \mathbb{N}$. Let $C_{n+2} \subseteq \overline{NE}_1(\overline{M}_{g,n+2})$ be a subset, $C_{n+1}^i := \pi_{n+i,*} C_{n+2}$ for $i = 1, 2$, and $C_n := \pi_{n+2,*} C_{n+1}^1 = \pi_{n+1,*} C_{n+1}^2$. Let $N_{n+2}, N_{n+1}^1, N_{n+1}^2, N_n$ denote the set of nef divisors that intersect $C_{n+2}, C_{n+1}^1, C_{n+1}^2, C_n$ trivially. Assume the following conditions hold:

- (1) $I(C_n), I(C_{n+1}^1)$ and $I(C_{n+1}^2)$ coincide with the dimension of the subspace spanned by C_n, C_{n+1}^1 , and C_{n+1}^2 .
- (2) The image of the intersection pairing $N_{n+2} \rightarrow \mathbb{R}^{F_{g,n+2}^{\text{Knu}}}$ spans $\mathbb{R}^{F_{g,n+2}^{\text{Knu}}}$.

Then $I(C_{n+2}) \leq I(C_{n+1}^1) + I(C_{n+1}^2) - I(C_n)$. If, moreover, $I(C_{n+1}^2) = I(C_n)$, then $I(C_{n+2}) = I(C_{n+1}^1)$ and this coincides with the dimension of the subspace spanned by C_{n+2} .

Here and in Lemma 9.2, we use \mathbb{R} -coefficients instead of \mathbb{Q} -coefficients. This is because these results can be applied to prove the extremality of limits of curve classes in $\overline{NE}_1(\overline{M}_{g,n})$, not only of curve classes themselves, although in this paper we use only curve classes.

Proof. This is a direct consequence of Theorem 8.1. We have an exact sequence

$$0 \rightarrow \text{Pic}(\overline{M}_{g,n})_{\mathbb{R}} \rightarrow \text{Pic}(\overline{M}_{g,n+1})_{\mathbb{R}} \times \text{Pic}(\overline{M}_{g,n+1})_{\mathbb{R}} \rightarrow \text{Pic}(\overline{M}_{g,n+2})_{\mathbb{R}} \rightarrow \mathbb{R}^{F_{g,n+2}^{\text{Knu}}} \rightarrow 0.$$

We first compute the dimension of the image of

$$(N_{n+1}^1 \otimes \mathbb{R}) \times (N_{n+1}^2 \otimes \mathbb{R}) \rightarrow N_{n+2} \otimes \mathbb{R} \subseteq \text{Pic}(\overline{M}_{g,n+2})_{\mathbb{R}}.$$

By the condition of (1), $N_n \otimes \mathbb{R}$ coincides with the subspace of \mathbb{R} -divisors that intersect C_n trivially. Therefore, by the exact sequence above, the kernel of this map is $N_n \otimes \mathbb{R}$. Hence, the dimension of the image is

$$\dim N_{n+1}^1 + \dim N_{n+1}^2 - \dim N_n = 2\rho(\overline{M}_{g,n+1}) - \rho(\overline{M}_{g,n}) - (I(C_{n+1}^1) + I(C_{n+1}^2) - I(C_n)).$$

By (2), there exists a $|F_{g,n+2}^{\text{Knu}}|$ -dimensional subspace of $N_{n+2} \otimes \mathbb{R}$ that is independent from the image of $(N_{n+1}^1 \otimes \mathbb{R}) \times (N_{n+1}^2 \otimes \mathbb{R})$. Hence, the dimension of $N_{n+2} \otimes \mathbb{R}$ is at least

$$\begin{aligned} & 2\rho(\overline{M}_{g,n+1}) - \rho(\overline{M}_{g,n}) + |F_{g,n+2}^{\text{Knu}}| - (I(C_{n+1}^1) + I(C_{n+1}^2) - I(C_n)) \\ &= \rho(\overline{M}_{g,n+2}) - (I(C_{n+1}^1) + I(C_{n+1}^2) - I(C_n)). \end{aligned}$$

Hence $I(C_{n+2}) \leq I(C_{n+1}^1) + I(C_{n+1}^2) - I(C_n)$. Moreover, since π_{i*} on A_1 is surjective, by (1), the dimension of the subspace spanned by C_{n+2} is at least $I(C_{n+1}^1)$. Hence, the codimension of N_{n+2} is at least $I(C_{n+1}^1)$. Therefore, $I(C_{n+2}^2) = I(C_n)$ implies $I(C_{n+2}) = I(C_{n+1}^1)$. The last assertion is straightforward since the dimension of such a subspace is $\leq I(C_{n+2})$. \square

Hence, while using the induction, the main part of the proof is to check the second condition. However, this remains a nontrivial task, since $F_{g,n}^{\text{Knu}}$ has roughly $g \cdot 2^{n-3}$ elements, which is still a large number when n is large. Nevertheless, we have not yet fully exploited the inductive structure. In Lemma 9.1, we only used the projections π_{n+1} and π_{n+2} , while ignoring the other n projections. The second condition of Lemma 9.1 can be made easier to verify by taking these into account. To facilitate this, we now introduce some notation.

For $n \geq 2$, we will denote

$$V_{g,n} = \begin{cases} \mathbb{R}^{F_{g,n}^{\text{Knu}}} & \text{if } n = 2, \\ \bigoplus_{0 \leq j \leq \lfloor \frac{g}{2} \rfloor} \mathbb{R}e_j & \text{if } g \text{ is odd or } n \text{ is even } \neq 2, \\ \bigoplus_{0 \leq j < \lfloor \frac{g}{2} \rfloor} \mathbb{R}e_j & \text{otherwise.} \end{cases}$$

Also, we define $E : \mathbb{R}^{F_{g,n}^{\text{Knu}}} \rightarrow V_{g,n}$ by the identity map when $n = 2$ and

$$E(h) = \sum_j \sum_{F_6^{0,0,j,g-j}(\{n-1\}, \{n\}, I, I^c)} (-1)^{|I|} h(F_6^{0,0,j,g-j}(\{n-1\}, \{n\}, I, I^c)) e_j,$$

where we consider h as a function from $F_{g,n}^{\text{Knu}}$ to \mathbb{R} , if $n \neq 2$. Note that E is defined as a direct sum of linear functions, each corresponding to j . The definition of E may appear technical, but the underlying idea is simple: the kernel of the map

$$E : \mathbb{R}^{F_{g,n+2}^{\text{Knu}}} \rightarrow V_{g,n+2}$$

is precisely the image of

$$\bigoplus_{1 \leq i \leq n} \pi_i^* : \bigoplus_{1 \leq i \leq n} \mathbb{R}^{F_{g,n+1}^{\text{Knu}}} \rightarrow \mathbb{R}^{F_{g,n+2}^{\text{Knu}}}$$

where π_i^* is the dual of the pushforward $\pi_{i,*} : F_{g,n+2}^{\text{Knu}} \rightarrow F_{g,n+1}^{\text{Knu}} \cup \{0\}$. More precisely,

$$\pi_i^*(h)(F) = \begin{cases} h(\pi_{i,*}(F)) & \text{if } \pi_{i,*}(F) \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Thus, the inductive strategy allows us to verify the statement after composing with E . This can be formalized as follows.

Lemma 9.2. Let $C_{n+2} \subseteq \overline{NE}_1(\overline{M}_{g,n+2})$ be a subset, and define $C_{n+1}^i := \pi_{i*} C_{n+2}$ for $1 \leq i \leq n$. Let N_{n+2}, N_{n+1}^i be defined as before. If

- (1) For each $1 \leq i \leq n$, the image of the intersection pairing $N_{n+1}^i \rightarrow \mathbb{R}^{F_{g,n+1}^{\text{Knu}}}$ spans $\mathbb{R}^{F_{g,n+1}^{\text{Knu}}}$.

(2) The image of the composition of the intersection pairing with E ,

$$f : N_{n+2} \rightarrow \mathbb{R}^{F_{g,n+2}^{\text{Knu}}} \rightarrow V_{g,n+2},$$

spans $V_{g,n+2}$,

then the image of the intersection pairing $N_{n+2} \rightarrow \mathbb{R}^{F_{g,n+2}^{\text{Knu}}}$ spans $\mathbb{R}^{F_{g,n+2}^{\text{Knu}}}$.

Proof. Note that π_i for $1 \leq i \leq n$ maps $\mathbb{R}^{F_{g,n+2}^{\text{Knu}}}$ to $\mathbb{R}^{F_{g,n+1}^{\text{Knu}}}$. By (1), the span of the image of $N_{n+2} \rightarrow \mathbb{R}^{F_{g,n+2}^{\text{Knu}}}$ contains the image of $\pi_i^* : \mathbb{R}^{F_{g,n+1}^{\text{Knu}}} \rightarrow \mathbb{R}^{F_{g,n+2}^{\text{Knu}}}$ for $1 \leq i \leq n$. Therefore, it is enough to prove that

$$N_{n+2} \rightarrow \mathbb{R}^{F_{g,n+2}^{\text{Knu}}} \rightarrow \mathbb{R}^{F_{g,n+2}^{\text{Knu}}} / \left(\bigoplus_{i=1}^n \pi_i^* \mathbb{R}^{F_{g,n+1}^{\text{Knu}}} \right)$$

is surjective, which reduces to (2) once we show that the kernel of E is the image of $\bigoplus_{1 \leq i \leq n} \pi_i^*$.

If $n = 0$, there is nothing to prove. Assume $n > 0$. Note that in the definition of $V_{g,n}$, as well as in the maps E and π_i^* , every morphism respects the decomposition indexed by j , which parametrizes the genus occurring in type 6 curves, and the type 5 curve. Hence, it is enough to prove the statement within each component of the decomposition of $F_{g,n+1}^{\text{Knu}}$, $F_{g,n+2}^{\text{Knu}}$, and $V_{g,n}$.

If $j \neq \frac{g}{2}$, then the statement follows from

$$\begin{aligned} & \pi_{i*}^{-1} \left(F_6^{0,0,j,g-j}(\{n+1\}, \{n+2\}, I, J) \right) = \\ & \left\{ F_6^{0,0,j,g-j}(\{n+1\}, \{n+2\}, I \cup \{i\}, J), F_6^{0,0,j,g-j}(\{n+1\}, \{n+2\}, I, J \cup \{i\}) \right\} \end{aligned}$$

where $\pi_{i*} : F_{g,n+2}^{\text{Knu}} \rightarrow F_{g,n+1}^{\text{Knu}}$. The statement for the type 5 curve is automatic

$$\pi_{i*}^{-1} \left(F_5^{0,0}(\{n+1\}, \{n+2\}) \right) = \left\{ F_5^{0,0}(\{n+1\}, \{n+2\}) \right\}.$$

Now assume $g = 2j$. Then, there is one additional relation:

$$F_6^{0,0,j,j}(\{n+1\}, \{n+2\}, I, J) = F_6^{0,0,j,j}(\{n+1\}, \{n+2\}, J, I).$$

If n is even, then this relation, together with the formula for

$$\pi_{i*}^{-1} \left(F_6^{0,0,j,g-j}(\{n+1\}, \{n+2\}, I, J) \right),$$

shows that the image of $\bigoplus_{1 \leq i \leq n} \pi_i^*$ is surjective in this component, which completes the proof. If n is odd, then this new relation lies in the image of $\bigoplus_{1 \leq i \leq n} \pi_i^*$, so the situation is the same as in the case $g \neq 2j$. Hence, the statement also holds by the same argument. \square

Note that the dimension of $V_{g,n}$ is $\lfloor \frac{g}{2} \rfloor + 1$, regardless of the value of n . This illustrates how significantly Lemma 9.2 simplifies the entire inductive process.

We will implicitly apply the following useful computation while proving theorems.

Lemma 9.3. Let $f : \text{Pic}(\overline{M}_{g,n}) \rightarrow V_{g,n}$ be the natural extension of the map f in Lemma 9.2, i.e., the composition of the pairing map $\text{Pic}(\overline{M}_{g,n}) \rightarrow \mathbb{R}^{F_{g,n}^{\text{Knu}}}$ and $E : \mathbb{R}^{F_{g,n+2}^{\text{Knu}}} \rightarrow V_{g,n}$. Then

$$f(\kappa) = f(\psi_i) = -e_0, \quad \text{and} \quad f(\delta_{i,I}) = (-1)^{|I|+1} e_i \quad \text{for} \quad 0 \leq i \leq \left\lfloor \frac{g}{2} \right\rfloor.$$

Proof. If $i \neq 1$, then the coefficient of e_i in $f(\kappa)$ is 0, since the intersection of any F-curve with κ is 1 and

$$\sum_{I \subseteq [n]} (-1)^{|I|} = 0.$$

The coefficient is -1 for e_0 because there is no $F_6^{0,0,0,g}(n-1, n, \emptyset, [n-2])$. The case of ψ_i is handled in the same way as that of κ . The last assertion follows directly from Theorem 2.2 by a case-by-case analysis with respect to $|I \cap \{n-1, n\}|$. \square

Note that if g is even and n is odd, then there is no $e_{\lfloor \frac{g}{2} \rfloor}$. However, for convenience, we will extend the notation by including $e_{\lfloor \frac{g}{2} \rfloor}$ with the convention $e_{\lfloor \frac{g}{2} \rfloor} = 0$. Lemma 9.3 still holds under this convention.

10. EXTREMALITY OF F-CURVES

In this section, we assume that the characteristic of the base field is not equal to 2, unless stated otherwise. This section is devoted to proving Theorem 1.3 and Theorem 1.4 using Section 7 to Section 9. Before we begin, as promised, we provide a detailed statement of Theorem 1.3 and Theorem 1.4.

- Theorem 10.1.** (1) Types 1 and 4 F-curves on $\overline{M}_{g,n}$ span regular extremal rays of $\overline{NE}_1(\overline{M}_{g,n})$, while type 2 F-curves do not.
- (2) Apart from the following three exceptional cases, no type 3 F-curve spans an extremal ray of $\overline{NE}_1(\overline{M}_{g,n})$.
- (a) $F_3^0([n])$ on $\overline{M}_{2,n}$ for $n \geq 1$,
 - (b) $F_3^1([n])$ on $\overline{M}_{3,n}$ for $n \geq 0$,
 - (c) $F_3^2(\emptyset)$ on \overline{M}_4 .
- In the three exceptional cases, the corresponding curve spans a regular extremal ray.
- (3) Knudsen-type F-curves (cf. Definition 10.11) are regular extremal. Consequently, every F-curve on $\overline{M}_{0,n}$ and $\overline{M}_{1,n}$ is regular extremal.
- (4) The following type 6 F-curves are regular extremal:
- (a) $F_6^{1,1,1,g-3}(\emptyset, \emptyset, \emptyset, [n])$,
 - (b) $F_6^{1,1,2,g-4}(\emptyset, \emptyset, \emptyset, [n])$ for $n \neq 0$,
 - (c) $F_6^{0,1,1,g-2}(i, \emptyset, \emptyset, [n] \setminus i)$ for all $i \in [n]$.

- Theorem 10.2.** (1) Any type 5 or type 6 F-curve on $\overline{M}_{2,n}$ is regular extremal.
- (2) Except $F_5^{0,1}(I, \emptyset)$, every type 5 or type 6 F-curve on $\overline{M}_{3,n}$ is regular extremal.
- (3) Except for the following cases, every type 5 or type 6 F-curve on $\overline{M}_{4,n}$ is regular extremal:
- (a) $F_5^{i,1}(I, \emptyset)$, $i = 0, 1$.
 - (b) $F_5^{2,1}([n], \emptyset)$ and $F_5^{0,2}([n], \emptyset)$.

10.1. Type 1, 2 and 4. In this case, the proof of (non-)extremality is relatively straightforward and follows as a direct application of [26].

Theorem 10.3. The type 1 F-curve on $\overline{M}_{g,n}$, $g \geq 1$, is regular extremal.

Proof. Let L_1, \dots, L_ρ be a basis of $\text{Pic}(\overline{M}_{g,n})$ consisting of nef line bundles. Consider the set of line bundles

$$T = \left\{ L_i - \frac{L_i \cdot [F_1]}{12} \lambda \mid 1 \leq i \leq \rho \right\}.$$

Since $\lambda \cdot F_1 = 12$, each line bundle $L_i - \frac{L_i \cdot [F_1]}{12} \lambda$ intersects F_1 trivially. Moreover, since λ intersects all other F-curves trivially, it follows that $L_i - \frac{L_i \cdot [F_1]}{12} \lambda$ is F-nef. For the flag map $F : \overline{M}_{0,g+n} \rightarrow \overline{M}_{g,n}$, the pullback of $L_i - \frac{L_i \cdot [F_1]}{12} \lambda$ along F coincides with $F^* L_i$, as $\lambda = 0$ on $\overline{M}_{0,g+n}$. In particular, this implies nefness. Hence, by [26, Theorem 0.3], each $L_i - \frac{L_i \cdot [F_1]}{12} \lambda$ is nef. Thus, the set T consists of line bundles that intersect $[F_1]$ trivially. Since T together with λ spans the Picard group, T generates a codimension-1 subspace. Consequently, $I([F_1]) = 1$, implying that $[F_1]$ is regular extremal. \square

Theorem 10.4. The type 2 F-curve on $\overline{M}_{g,n}$, for $g \geq 3$, is not extremal.

Proof. By [26, Theorem 2.1],

$$[F_2] = \frac{1}{2} ([F_3^1(\emptyset)] + [F_4^1(\emptyset)]).$$

Since $[F_2]$ is expressed as a positive linear combination of other effective curve classes, it is not extremal. \square

Theorem 10.5. Any type 4 F-curve on $\overline{M}_{g,n}$, $g \geq 2$, is regular extremal.

Proof. Let S be the set of all type 4 F-curves on $\overline{M}_{g,n}$. Choose any curve $F \in S$. Since κ is an ample divisor whose intersection with any F-curve is 1, it follows from [26, Theorem 2.1] that there exist perturbations of κ , denoted L_1, \dots, L_ρ , such that:

- (1) L_1, \dots, L_ρ form a basis of $\text{Pic}(\overline{M}_{g,n})$ consisting of nef line bundles.
- (2) For any $1 \leq i \leq \rho$, we have $L_i \cdot F = \min_{F' \in S} L_i \cdot F'$.

Consider the set of line bundles

$$T = \left\{ L_i - \frac{L_i \cdot F}{2} (12\lambda - \delta_{\text{irr}}) \mid 1 \leq i \leq \rho \right\}.$$

By the same reasoning as in Theorem 10.3, the elements of T are nef divisors that intersect F trivially and span a codimension-1 subspace of the Picard group. Consequently, $I(F) = 1$, which implies that F is regular extremal. \square

This completes the proof of Theorem 1.3 (1).

10.2. **Type 3.** Here, we prove Theorem 10.1 (2).

Theorem 10.6. Except for the three cases described in Theorem 10.1 (2), any type 3 F-curve does not form an extremal ray.

Proof. If $F_3^i(I) \neq F_3^{g-2}([n])$, then by Theorem 2.2,

$$[F_3^i(I)] = \frac{1}{2} \left([F_5^{1,i}(\emptyset, I)] + [F_5^{1,g-i-1}(\emptyset, I^c)] \right),$$

and since $[F_5^{1,i}(\emptyset, I)] \neq [F_5^{1,g-i-1}(\emptyset, I^c)]$, it follows that $[F_3^i(I)]$ is not an extremal ray.

If $F_3^i(I) = F_3^{g-2}([n])$ with $g \geq 4$ and $(g, n) \neq (4, 0)$, then

$$[F_3^{g-2}([n])] = \frac{1}{2} \left([F_5^{1,2}(\emptyset, \emptyset)] + [F_5^{g-3,2}([n], \emptyset)] \right),$$

and since $[F_5^{1,2}(\emptyset, \emptyset)] \neq [F_5^{g-3,2}([n], \emptyset)]$, it follows that $[F_3^{g-2}([n])]$ is not extremal. \square

Now, we need to establish the regular extremality of type 3 F-curves for the three cases in Theorem 10.1 (2). The last case follows from Example 5.10, provided that the characteristic is not equal to 2. In general characteristic, this can be seen explicitly, since λ , $12\lambda - \delta_{\text{irr}}$, and $\kappa + \delta_{2,\emptyset}$ all contract the curve $F_3^2(\emptyset)$ and are nef by Theorem 7.5.

Theorem 10.7 (All characteristics). $F_3^0([n])$ is a regular extremal on $\overline{NE}_1(\overline{M}_{2,n})$ for $n \geq 1$

Proof. We will prove that $I(F_3^0([n])) = 1$, which is equivalent to the given statement. For $n = 1$, this holds since both λ and $12\lambda - \delta_{\text{irr}}$ contract $F_3^0([1])$. The case $n = 2$ follows from Example 5.1 and Proposition 7.16.

We will first verify that $F_3^0([n])$ satisfies the conditions of Lemma 9.2. Let N_n be the set of nef divisors that intersect $F_3^0([n])$ trivially. Note that the image of $F_3^0([n])$ under any projection is $F_3^0([n-1])$. Hence, it suffices to show that the image of $f : N_n \rightarrow V_{2,n}$ spans $V_{2,n}$ for each n .

Define the divisor

$$D_n = \kappa + \delta_{1,\emptyset} + \delta_{2,\emptyset}$$

on $\overline{M}_{2,n}$ for $n \geq 2$. This divisor is nef by Theorem 7.5, see also Example 7.7 (3). Furthermore, by [26, Theorem 2.1], we have $D_n \cdot F_3^0([n]) = 0$. For $n = 2$, we compute

$$f(D_2) = 2 \cdot F_5, \quad f(\psi_1) = F_5 + F_6.$$

Thus, f satisfies the assertion. For $n > 2$ and n odd,

$$f(\psi_1) = -e_0.$$

For $n > 2$ and n even,

$$f(D_n) = -2e_0 - e_1, \quad f(\psi_1) = -e_0.$$

Hence, in all cases, f satisfies the assertion. Therefore, by Lemma 9.2, for any $n \geq 2$, the image of $N_n \rightarrow \mathbb{R}^{F_{g,n}^{\text{Knu}}}$ generates $\mathbb{R}^{F_{g,n}^{\text{Knu}}}$.

Now, we apply Lemma 9.1 with $C_n = [F_3^0([n])]$. We have already verified condition (1) for $n = 1, 2$. Moreover, condition (2) holds for every n by the preceding paragraph. Hence, by induction using Lemma 9.1, we conclude that $I(F_3^0([n])) = 1$ for every $n \geq 1$. \square

Theorem 10.8 (All characteristics). $F_3^1([n])$ is regular extremal on $\overline{\text{NE}}_1(\overline{M}_{3,n})$ for $n \geq 0$

Proof. The proof is almost identical to Theorem 10.7, except that we are using a different divisor here. We will prove that $I(F_3^1([n])) = 1$. For $n = 0$, this holds since $\lambda, 12\lambda - \delta_{\text{irr}}$ contract $F_3^1(\emptyset)$. The case of $n = 1$ follows from Example 5.5 and Proposition 7.16.

We will first verify that $F_3^1([n])$ satisfies the conditions of Lemma 9.2. Let N_n be the set of nef divisors that intersect $F_3^1([n])$ trivially. Since the image of $F_3^1([n])$ under any projection is $F_3^1([n])$, it suffices to show that the image of $f : N_n \rightarrow V_{3,n}$ spans $V_{3,n}$ for each n .

Define the divisor

$$D_n = \kappa + \delta_{2,\emptyset}$$

on $\overline{M}_{3,n}$ for $n \geq 2$. This divisor is nef by Theorem 7.5, see also Example 7.7 (1). Furthermore, by [26, Theorem 2.1], we have $D_n \cdot F_3^1([n]) = 0$. For $n = 2$, we compute

$$f(D_2) = F_5, \quad f(\psi_1) = F_5 + F_6.$$

Thus, f satisfies the assertion. For $n > 2$,

$$f(D_n) = -e_0 + (-1)^{n+1}e_1, \quad f(\psi_1) = -e_0.$$

Hence, in all cases, f satisfies the assertion. Therefore, by Lemma 9.2, for any $n \geq 2$, the image of $N_n \rightarrow \mathbb{R}^{F_{g,n}^{\text{Knu}}}$ generates $\mathbb{R}^{F_{g,n}^{\text{Knu}}}$.

Now, we apply Lemma 9.1 with $C_n = [F_3^1([n])]$. We have already verified condition (1) for $n = 0$. Moreover, condition (2) holds for every n by the preceding paragraph. Hence, by induction using Lemma 9.1, we conclude that $I(F_3^1([n])) = 1$ for every $n \geq 1$. \square

This completes the proof of Theorem 1.3 (2). To prove Theorem 1.1, we need the following theorem:

Theorem 10.9. For $g \geq 3$ and $n \geq 1$, $I(F_3^1([n])) = \lfloor \frac{g}{2} \rfloor$ on $\overline{M}_{g,n}$.

Proof. Define

$$C_n := \left\{ F_3^1([n]), F_5^{1,i}([n], \emptyset) \mid 1 \leq i \leq g-2 \right\}.$$

Step 1. If a nef divisor intersects $F_3^1([n])$ trivially, then it also intersects any element of C_n trivially. Let

$$D = a\lambda - b_{\text{irr}}\delta_{\text{irr}} - \sum b_{i,I}\delta_{i,I}$$

be such a nef divisor, where we use the convention $\delta_{0,\{i\}} = -\psi_i$. Then

$$D \cdot [F_3^1([n])] = b_{1,[n]} = 0.$$

Since D is nef, we obtain the following inequalities:

$$\begin{aligned} D \cdot F_5^{1,i}([n], \emptyset) &= b_{i,\emptyset} + b_{1,[n]} - b_{i+1,[n]} = b_{i,\emptyset} - b_{g-i-1,\emptyset} \geq 0, \\ D \cdot F_5^{1,g-i-1}([n], \emptyset) &= b_{g-i-1,\emptyset} + b_{1,[n]} - b_{g-i,[n]} = b_{g-i-1,\emptyset} - b_{i,\emptyset} \geq 0. \end{aligned}$$

Thus, $D \cdot F_5^{1,i}([n], \emptyset) = D \cdot F_5^{1,g-i-1}([n], \emptyset) = 0$, so D satisfies the assertion.

Therefore, it suffices to prove that $I(C_n) = \lfloor \frac{g}{2} \rfloor$.

Step 2. The dimension of the subspace generated by C_n is $\lfloor \frac{g}{2} \rfloor$.

By the explicit intersection formula shown in Step 1, identifying A_1 as the dual space of Pic , the mappings

$$\begin{aligned} D &\mapsto b_{1,[n]}, \\ D &\mapsto b_{i,\emptyset} - b_{g-i-1,\emptyset} \quad \text{for } 1 \leq i \leq \lfloor \frac{g}{2} \rfloor - 1 \end{aligned}$$

form a basis of the subspace. Hence, the dimension is $\lfloor \frac{g}{2} \rfloor$.

Therefore, $I(F_3^1([n])) = \lfloor \frac{g}{2} \rfloor$ implies the last condition of Lemma 9.1 (1).

Step 3. $I(F_3^1([n])) = \lfloor \frac{g}{2} \rfloor$ for $n = 0, 1$.

Let N_n be the set of nef divisors that intersect each element of C_n trivially. To prove the case of $n = 0$, by Lemma 8.5, it is enough to show that N_0 contains two linearly independent divisors. Since $\lambda, 12\lambda - \delta_{\text{irr}} \in N_0$, this holds.

Consider the case of $n = 1$. Again, by Lemma 8.5, we need to produce $g - \lfloor \frac{g}{2} \rfloor + 2$ linearly independent divisors in N_1 . The divisors

$$\lambda, 12\lambda - \delta_{\text{irr}}, \kappa + \delta_{g-1,\emptyset}, \kappa + \delta_{1,\{1\}} + \frac{1}{4}(\delta_{i,\emptyset} + \delta_{g-1-i,\emptyset}) \quad \text{for } 1 \leq i \leq \lfloor \frac{g-1}{2} \rfloor$$

are nef by Theorem 7.5, Proposition 7.11 and Proposition 7.12, and they intersect $F_3^1([n])$ trivially by Theorem 2.2. This forms a linearly independent set of

$$3 + \lfloor \frac{g-1}{2} \rfloor = g - \lfloor \frac{g}{2} \rfloor + 2$$

divisors in N_1 . This establishes the case of $n = 1$.

Step 4. Using Lemma 9.2.

Note that $\pi_{i*}C_{n+1} = C_n$ for any $1 \leq i \leq n+1$. Let $f : N_n \rightarrow V_{g,n}$ be the composition of E and the intersection pairing. We need to show that the image of f spans $V_{g,n}$. Define

$$D_1 := \kappa + \delta_{1,[n]}, \quad \text{and} \quad D_i := \kappa + \delta_{1,[n]} + \delta_{i,\{1\}} \quad \text{for } 2 \leq i \leq \lfloor \frac{g}{2} \rfloor.$$

These are nef by Theorem 7.5, and they intersect trivially with $F_3^1([n])$, hence with any element of C_n . If $n = 2$,

$$\begin{aligned} f(\psi_1) &= \sum_{j=1}^{\lfloor \frac{g}{2} \rfloor} F_6^{0,0,j,g-j}, & f(D_1) &= \sum_{j=2}^{\lfloor \frac{g}{2} \rfloor} F_6^{0,0,j,g-j}, \\ f(D_i) &= \sum_{j=2}^{\lfloor \frac{g}{2} \rfloor} F_6^{0,0,j,g-j} - F_6^{0,0,i,g-i} & \text{for } 2 \leq i \leq \lfloor \frac{g}{2} \rfloor. \end{aligned}$$

Hence, the image of f spans $V_{g,n}$. Here, $F_6^{0,0,j,g-j}$ is an abbreviation for $F_6^{0,0,j,g-j}(\{1\}, \{2\}, \emptyset, \emptyset)$.

If $n > 2$ and g is odd or n is even,

$$f(\psi_1) = -e_0, \quad f(D_1) = -e_0 + (-1)^{n+1}e_1, \quad f(D_i) = -e_0 + (-1)^{n+1}e_1 + e_i \quad \text{for } 2 \leq i \leq \lfloor \frac{g}{2} \rfloor.$$

A similar result holds when g is even and n is odd. Hence, in all cases, f satisfies the assertion. By Lemma 9.2, the image of $N_n \rightarrow \mathbb{R}^{F_{g,n}^{\text{Knu}}}$ spans $\mathbb{R}^{F_{g,n}^{\text{Knu}}}$.

By Steps 3 and 4, we can apply Lemma 9.1, and we obtain $I(F_3^1([n])) = \lfloor \frac{g}{2} \rfloor$ for every n . \square

10.3. Type 5 and 6. In this subsection, we prove Theorem 10.1 (3) and (4). The situation of type 5 and 6 F-curves is more subtle. Based on an explicit computation using [11], we propose the following conjecture:

Conjecture 10.10. Any type 5 F-curve on $\overline{M}_{g,n}$ spans a regular extremal ray, except for the following cases:

- (1) $F_5^{i,j}(I, \emptyset)$, where $i + 2j < g$ and $j \neq 0$.
- (2) $F_5^{i,j}([n], \emptyset)$, where $i + 2j = g$, $j \neq 0$, and $(g, n, i) \neq (2, n, 0), (3, n, 1), (4, 0, 2)$.

As reported in Theorem 1.4, we have verified Conjecture 10.10 for $g \leq 4$. This will be proved in Section 10.4. Note that the two exceptions in Conjecture 10.10 are clearly not extremal, since

$$[F_5^{i,j}(I, \emptyset)] = \frac{1}{2} \left([F_6^{j,i,g-i-2j}(\emptyset, \emptyset, I, I^c)] + [F_5^{i,2j}(I, \emptyset)] \right)$$

in the first case, and

$$[F_5^{i,j}([n], \emptyset)] = [F_3^i([n])]$$

in the second case (cf. Theorem 1.3).

Type 6 F-curves appear to be more complicated. For instance, Remark 5.11 presents an example of a non-extremal type 6 F-curve on \overline{M}_7 . However, this is not the minimal genus example. On $\overline{M}_{6,1}$, we have

$$[F_6^{0,1,2,3}(\{1\}, \emptyset, \emptyset, \emptyset)] = \frac{1}{2} \left([F_6^{0,2,2,2}(\{1\}, \emptyset, \emptyset, \emptyset)] + [F_6^{0,1,1,4}(\{1\}, \emptyset, \emptyset, \emptyset)] \right).$$

At present, the author does not have a clear conjecture regarding which type 6 F-curves are (regular) extremal.

Now, rather than attempting to classify all (regular) extremal rays, we instead focus on the following collection of type 5 and 6 curves.

Definition 10.11. An F-curve C on $\overline{M}_{g,n}$ is said to be of **Knudsen type** if there exists a projection map $\pi : \overline{M}_{g,n} \rightarrow \overline{M}_{g,m}$ such that $\pi_*[C] \in F_{g,m}^{\text{Knu}}$. Equivalently, such a curve is of the form $F_5^{0,0}(I, J)$ or $F_6^{0,0,i,g-i}(I, J, K, L)$.

The motivation behind this definition is as follows. Our method relies on establishing the regular extremality of certain curves for small values of n as a base case. However, in contrast to the situation for larger n , the space of semigroup κ divisors for small n is significantly more constrained, making it difficult to prove the base case directly. For F-curves of Knudsen type, however, we can invoke Corollary 8.6 as the base step.

The first main result of this section is the following:

Theorem 10.12. Any F-curve of Knudsen type forms a regular extremal ray.

The idea of the proof is similar to that of Section 10.2, but the argument is more involved due to two main reasons: (1) They are not stable under permutation of the marked points, and (2) a curve may become an element of $F_{g,n}^{\text{Knu}}$ after forgetting a marked point. Hence, extra care is needed when applying Lemma 9.1 and Lemma 9.2. Note that $F_{g,n}^{\text{Knu}}$ implicitly depends on the choice of two marked points i and j . We will use the original slightly ambiguous notation for simplicity.

Roughly speaking, the proof breaks into two steps. First, we consider the case of $F_5^{0,0}(p, \{q, r\})$ or $F_6^{0,0,i,g-i}(p, \{q, r\}, I, J)$ as in Proposition 10.15, which corresponds to the situation where there exists a point x such that $\pi_{x,*}C \in F_{g,n-1}^{\text{Knu}}$ for some choice of x and $F_{g,n-1}^{\text{Knu}}$. This part is more ad hoc, but manageable, since the setting is restricted. We will use induction on n , where Proposition 10.14 serves as the base step. After this, we may then assume that $\pi_{x,*}C \notin F_{g,n-1}^{\text{Knu}}$ for any x and $F_{g,n-1}^{\text{Knu}}$. This is the more general and simpler case.

We introduce the following notation. For any subset $S \subseteq \overline{NE}_1(\overline{M}_{g,n})$, define N_S to be the \mathbb{Q} -linear span of the set of nef divisors that intersect every element of S trivially.

Proposition 10.13. If C is a curve on $\overline{M}_{g,n}$ whose curve class is not contained in the linear span of $F_{g,n}^{\text{Knu}}$ and spans a regular extremal ray, then C satisfies the conclusion of Lemma 9.2, i.e.,

$$N_C \rightarrow \mathbb{Q}^{F_{g,n}^{\text{Knu}}}.$$

Proof. Since C is a regular extremal ray, N_C is a codimension 1 subspace of the \mathbb{Q} -Picard group consisting of divisors that intersect trivially with C . If the map $N_C \rightarrow \mathbb{Q}^{F_{g,n}^{\text{Knu}}}$ is not surjective, then there exists a class C' in $A_1(\overline{M}_{g,n})$, contained in the linear span of $F_{g,n}^{\text{Knu}}$, such that every

divisor in N_C intersects trivially with C' . However, this contradicts the codimension of N_C , since C and C' are linearly independent. \square

Proposition 10.14. If C is $F_5^{0,0}(p, \{q, r\})$ or $F_6^{0,0,i,g-i}(p, \{q, r\}, \emptyset, \emptyset)$ on $\overline{M}_{g,3}$, then

- (1) $N_C \rightarrow \mathbb{Q}^{F_{g,n}^{\text{Knu}}}$ for any choice of $F_{g,n}^{\text{Knu}}$, and
- (2) C spans a regular extremal ray.

Proof. Case 1. $F_{g,n}^{\text{Knu}}$ corresponds to $\{q, r\}$.

The space N_C contains $\pi_p^* \text{Pic}(\overline{M}_{g,2})$, so we apply Lemma 9.2. By Theorem 7.5, and in particular Example 7.7 (2), we have

$$D_j = \kappa + \delta_{0,\{q,r\}} + \delta_{j,p} + \delta_{g-j,\emptyset}$$

which are nef for $0 < j < \frac{g}{2}$, and contract both $F_5^{0,0}(p, \{q, r\})$ and $F_6^{0,0,i,g-i}(p, \{q, r\}, \emptyset, \emptyset)$.

For the map $f : N_3 \rightarrow V_{g,3}$ in Lemma 9.2, we have

$$f(\psi_r) = -e_0 \quad f(D_j) = -2e_0 - 2e_j$$

Hence, we obtain a surjection $N_C \rightarrow \mathbb{Q}^{F_{g,n}^{\text{Knu}}}$.

Case 2. $F_{g,n}^{\text{Knu}}$ corresponds to $\{p, q\}$ or $\{p, r\}$, and $C = F_5^{0,0}(p, \{q, r\})$.

Without loss of generality, assume that $F_{g,n}^{\text{Knu}}$ corresponds to $\{p, q\}$. We have $\pi_r^* N_{F_5^{0,0}(p,q)} \subseteq N_C$. Although we have Theorem 8.1, this is not enough, since (1) they cannot distinguish between $F_6^{0,0,j,g-j}(p, q, r, \emptyset)$ and $F_6^{0,0,j,g-j}(p, q, \emptyset, r)$, and (2) they always vanish on $F_5^{0,0}(p, q)$. Issue (1) is resolved by using exactly the same divisor as in Case 1. Issue (2) is resolved because $\kappa + \delta_{0,\{q,r\}}$ is a nef divisor that contracts $C = F_5^{0,0}(p, \{q, r\})$ but does not contract $C = F_5^{0,0}(p, q)$.

Case 3. $F_{g,n}^{\text{Knu}}$ corresponds to $\{p, q\}$ or $\{p, r\}$, and $C = F_6^{0,0,i,g-i}(p, \{q, r\}, \emptyset, \emptyset)$.

We follow the approach in Case 2. We also have $\pi_r^* N_{F_6^{0,0,i,g-i}(p,q,\emptyset,\emptyset)} \subseteq N_C$. This set of divisors (1) cannot distinguish between $F_6^{0,0,j,g-j}(p, q, r, \emptyset)$ and $F_6^{0,0,j,g-j}(p, q, \emptyset, r)$, and (2) always vanishes on $F_6^{0,0,i,g-i}(p, q, r, \emptyset)$ and $F_6^{0,0,i,g-i}(p, q, \emptyset, r)$. This issue is resolved because $\kappa + \delta_{i,r}$, $\kappa + \delta_{g-i,r}$, and the divisors in Step 1 are all contained in N_C .

(2) This follows from (1), Corollary 8.6, and Lemma 9.1 for $\{p, q\}$. \square

Proposition 10.15. If C is $F_5^{0,0}(p, \{q, r\})$ or $F_6^{0,0,i,g-i}(p, \{q, r\}, I, J)$ on $\overline{M}_{g,n}$, then

- (1) C spans a regular extremal ray, and
- (2) $N_C \rightarrow \mathbb{Q}^{F_{g,n}^{\text{Knu}}}$ for any choice of $F_{g,n}^{\text{Knu}}$ does not contain $[C]$.

Proof. We may exclude the cases $|I| = 1, i = 0$ and $|J| = 1, i = g$, since these cases follow from Corollary 8.6 and Proposition 10.13.

We will use induction on n and prove (1) and (2) at once. The $n = 3$ case is proved in Proposition 10.14. Consider the case $n \geq 4$. Fix a marked point $s \notin \{p, q, r\}$.

First, we will prove that $N_C \rightarrow \mathbb{Q}^{F_{g,n}^{\text{Knu}}}$, where this $F_{g,n}^{\text{Knu}}$ corresponds to $\{r, s\}$. By Corollary 8.6, Proposition 10.13, and the induction hypothesis for (2), (1) of Lemma 9.2 holds. Hence, it is enough to verify (2) of Lemma 9.2. Let $f : N_C \rightarrow V_{g,n}$ be the map in Lemma 9.2 (2). Note that

$$D_0 = \kappa + \delta_{0,\{q,r\}}, \quad D_j = \kappa + \delta_{0,\{q,r\}} + \delta_{j,p} + \delta_{g-j,\{p,q,r\}^c}$$

are nef line bundles by Theorem 7.5, and contained in N_C for $1 \leq j \leq \frac{g}{2}$. Then by Lemma 9.3,

$$f(D_0) = -2e_0, \quad f(D_j) = -2e_0 + 2e_j.$$

Thus, (2) of Lemma 9.2 is satisfied, so $N_C \rightarrow \mathbb{Q}^{F_{g,n}^{\text{Knu}}}$.

Now, (1) follows from Lemma 9.1 with $\{r, s\}$ and the induction hypothesis for (1). (2) follows from (1) and Proposition 10.13. \square

Proof of Theorem 10.12. We will use induction on n to prove the following statement: If C is of Knudsen type, i.e., of the form $F_5^{0,0}(I, J)$ or $F_6^{0,0,i,g-i}(I, J, K, L)$, then

- (1) C spans a regular extremal ray, and
- (2) $N_C \rightarrow \mathbb{Q}^{F_{g,n}^{\text{Knu}}}$ for any choice of $F_{g,n}^{\text{Knu}}$ such that $[C] \notin F_{g,n}^{\text{Knu}}$.

For simplicity of the argument, assume that if $C = F_6^{0,0,i,g-i}(I, J, K, L)$ and $i = 0$ (resp. $i = g$), then $|I|, |J| \leq |K|$ (resp. $|I|, |J| \leq |L|$), and $|I| \leq |J|$.

The case $n \leq 3$ is covered by Corollary 8.6 and Proposition 10.15. Hence, we may assume that $n \geq 4$. Since (2) follows from (1) and Proposition 10.13, it suffices to prove (1). If $|I| + |J| \leq 3$, then this is again proved in Corollary 8.6 and Proposition 10.15. Therefore, we may assume that $|I| + |J| \geq 4$.

Note that this condition implies $\pi_{x,*}C \notin F_{g,n-1}^{\text{Knu}}$ for any choice of $F_{g,n-1}^{\text{Knu}}$ and x . Hence, by the induction hypothesis for (2), condition (1) of Lemma 9.2 is satisfied. Choose $p \in I$ and $q, r \in J$ (this is possible since $|I| + |J| \geq 3$ and $|I| \leq |J|$), and let $F_{g,n}^{\text{Knu}}$ be the set of curves corresponding to $\{p, q\}$. Define

$$D_0 = \psi_r, \quad D_j = \kappa + \delta_{0,J} + \delta_{j,\{p,r\}}$$

for $1 \leq j \leq \frac{g}{2}$, which are nef line bundles contracting C by Theorem 7.5. Then by Lemma 9.3,

$$f(D_0) = -e_0, \quad f(D_j) = ((-1)^{|J|+1} - 1)e_0 - e_j,$$

Thus, condition (2) of Lemma 9.2 is satisfied. Therefore, $N_C \rightarrow \mathbb{Q}^{F_{g,n}^{\text{Knu}}}$

Note that condition (1) of Lemma 9.1 for $\{p, q\}$ also holds by the induction hypothesis. If $I = \{p\}$, then $I(\pi_{p,*}C) = I(\pi_{p,*}\pi_{q,*}C) = 0$. Otherwise, by the induction hypothesis, we have

$$I(\pi_{p,*}C) = I(\pi_{q,*}C) = I(\pi_{p,*}\pi_{q,*}C) = 1.$$

Hence, $I(C) = 1$, so C spans a regular extremal ray. This also implies condition (2). \square

Next, we prove the regular extremality of certain type 6 F-curves presented in Theorem 1.3 (4). The method used here is entirely different from that of Section 9. Instead of that approach, we employ Theorem 6.5 together with Hassett's moduli spaces of weighted pointed stable curves [28]. By Theorem 6.5, the pushforward of any regular extremal ray under the morphism $F : \bar{M}_{0,g+n}/S_g \rightarrow \bar{M}_{g,n}$ remains a regular extremal ray. By analyzing the F-curves contracted by a specific Hassett space, we deduce the regular extremality of certain F-curves on $\bar{M}_{0,g+n}/S_g$. To address the quotient by S_g , we require the following lemma.

Lemma 10.16. Let V, W be finite-dimensional \mathbb{Q} -vector spaces with an action of a finite group G , and $f : V \rightarrow W$ be a G -equivariant map. Let V^G (resp. V_G) be the set of G -invariants (resp. G -coinvariants), and note that $(V^*)^G$ can be naturally identified with V_G^* . Under this identification, the orthogonal complement of the image of $f^* : (W^*)^G \rightarrow (V^*)^G$ is $(\ker f)_G \subseteq V_G$.

Proof. By restriction, we have a natural map $(V^*)^G \rightarrow V_G^*$. It is straightforward to see that this is an isomorphism by breaking V into simple G -representations, since the base field is of characteristic 0.

Now we will consider the second assertion. We have an exact sequence

$$0 \rightarrow \ker f \rightarrow V \rightarrow W.$$

By taking dual and G -invariants, we have

$$(W^*)^G \rightarrow (V^*)^G \rightarrow (\ker f^*)^G \rightarrow 0.$$

By the identification above, the last map is equal to $(V_G^*)^* \rightarrow (\ker f)_G^*$ induced from the inclusion $(\ker f)_G \rightarrow V_G$. Hence the statement follows from taking duals. \square

Theorem 10.17. The following type 6 F-curves are regular extremal:

- (1) $F_6^{1,1,1,g-3}(\emptyset, \emptyset, \emptyset, [n])$,

- (2) $F_6^{1,1,2,g-4}(\emptyset, \emptyset, \emptyset, [n])$ for $n \neq 0$,
(3) $F_6^{0,1,1,g-2}(i, \emptyset, \emptyset, [n] \setminus i)$ for all $i \in [n]$.

Proof. We will denote the index set for points on $\overline{M}_{0,g+n}$ by $[g] \cup [n]$, where $[g]$ is the set of symmetric points, i.e. the points on which S_g acts, and $[n]$ is the set of asymmetric points. Moreover, the i th symmetric (resp. asymmetric) point will be denoted by i_s (resp. i_a).

Define a sequence $\mathcal{A}_1 := (a_j)_{j \in [g] \cup [n]}$ by

$$a_j = \begin{cases} \frac{1}{3} & \text{if } j \in [g], \\ 1 & \text{if } j \in [n]. \end{cases}$$

Consider the natural contraction $f : \overline{M}_{0,g+n} \rightarrow \overline{M}_{0,\mathcal{A}_1}$. By [28, Proposition 4.5], the exceptional locus of f is the union of $\overline{M}_{0,I+1} \times \overline{M}_{0,I^c+1}$ where $I \subseteq [g]$ and $|I| = 3$. Moreover, f contracts this boundary divisor into \overline{M}_{0,I^c+1} . Therefore, the set of F-curves contracted by f is exactly $F_6^{0,0,0,0}(p_s, q_s, r_s, I^c)$ for $I = \{p_s, q_s, r_s\}$. Since f is a smooth blowdown corresponding to images of F-curves (see also [17, Lemma 4.6, Proposition 4.7]), if we let U be the subspace of $A_1(\overline{M}_{0,g+n})$ generated by these F-curves, then we have a natural exact sequence

$$0 \rightarrow \text{Pic}(\overline{M}_{0,\mathcal{A}_1}) \rightarrow \text{Pic}(\overline{M}_{0,g+n}) \rightarrow U^*$$

and hence, by taking duals, we have

$$\ker f_* = \text{Im } U := \text{image of } U \text{ in } A_1(\overline{M}_{0,g+n})$$

where $f_* : A_1(\overline{M}_{0,g+n}) \rightarrow A_1(\overline{M}_{0,\mathcal{A}_1})$. Since \mathcal{A}_1 is S_g -invariant, f_* is also S_g -equivariant, so we can apply Lemma 10.16, which gives

$$(f^* \text{Pic}(\overline{M}_{0,\mathcal{A}_1})^{S_g})^\perp = \text{Im } U/S_g \subseteq A_1(\overline{M}_{0,g+n})/S_g = A_1(\overline{M}_{0,g+n}/S_g).$$

Note that $f^* \text{Pic}(\overline{M}_{0,\mathcal{A}_1})^{S_g}$ is generated by nef divisors, and $\text{Im } U/S_g$ is 1-dimensional since the set of $F_6^{0,0,0,0}(p, q, r, I^c)$ is transitive under the S_g -action. Hence, the image of $F_6^{0,0,0,0}(p_s, q_s, r_s, I^c)$ in $A_1(\overline{M}_{0,g+n}/S_g)$ is a regular extremal ray of $\overline{NE}_1(\overline{M}_{0,g+n}/S_g)$. By Theorem 6.5, applying the flag map yields that $F_6^{1,1,1,g-3}(\emptyset, \emptyset, \emptyset, [n])$ is a regular extremal ray.

For $F_6^{0,1,1,g-2}(i, \emptyset, \emptyset, [n] \setminus i)$, we use essentially the same argument, with $\mathcal{A}_2 := (a_j)_{j \in [g] \cup [n]}$ defined by

$$a_j = \begin{cases} \frac{1}{2} - \epsilon & \text{if } j \in [g], \\ \epsilon & \text{if } j = i_a, \\ 1 & \text{otherwise.} \end{cases}$$

for a sufficiently small positive number ϵ . Then the F-curves contracted by $f : \overline{M}_{0,g+n} \rightarrow \overline{M}_{0,\mathcal{A}_2}$ are $F_6^{0,0,0,0}(i_a, p_s, q_s, \{p_s, q_s, i_a\}^c)$ where $p_s, q_s \in [g]$. The proof is identical, so we omit the details.

For the last case, $F_6^{1,1,2,g-4}(\emptyset, \emptyset, \emptyset, [n])$, we use $\mathcal{A}_3 := (a_j)_{j \in [g] \cup [n]}$ defined by

$$a_j = \begin{cases} \frac{1}{4} & \text{if } j \in [g], \\ 1 & \text{if } j \in [n]. \end{cases}$$

By [28, Proposition 4.5], the exceptional locus of $f : \overline{M}_{0,g+n} \rightarrow \overline{M}_{0,\mathcal{A}_3}$ is the union of $\overline{M}_{0,I+1} \times \overline{M}_{0,I^c+1}$ where $I \subseteq [g]$ and $|I| = 3$ or 4, and f contracts this divisor to \overline{M}_{0,I^c+1} . Therefore, the set of F-curves contracted by f is $F_6^{0,0,0,0}(p_s, q_s, r_s, I^c)$ and $F_6^{0,0,0,0}(p_s, q_s, \{r_s, r'_s\}, I^c)$, where $I = \{p_s, q_s, r_s\}$ and $I = \{p_s, q_s, r_s, r'_s\}$, respectively. Note that the set of such F-curves has two S_g -orbits, corresponding to $|I| = 3$ and $|I| = 4$. We already know that the $|I| = 3$ case is regular extremal in $\overline{NE}_1(\overline{M}_{0,g+n}/S_g)$, and we will prove that the $|I| = 4$ case is also regular extremal. By applying the flag map and Theorem 6.5, this finishes the proof.

By the same argument as in the first case, we have

$$(f^* \text{Pic}(\overline{M}_{0,\mathcal{A}_3})^{S_g})^\perp = \text{Im } U/S_g \subseteq A_1(\overline{M}_{0,g+n})/S_g = A_1(\overline{M}_{0,g+n}/S_g).$$

However, in this case, U/S_g is 2-dimensional, generated by the images of $F_6^{0,0,0,0}(p_s, q_s, r_s, I^c)$ and $F_6^{0,0,0,0}(p_s, q_s, \{r_s, r'_s\}, I^c)$. Hence, it is enough to construct an S_g -invariant nef line bundle which contracts $F_6^{0,0,0,0}(p_s, q_s, \{r_s, r'_s\}, I^c)$ and does not contract $F_6^{0,0,0,0}(p_s, q_s, r_s, I^c)$.

For this, we use nef divisors on $\overline{M}_{0,g+n}$ coming from GIT quotients [2, 23]. We follow the notation of [23, Section 2]. Define a sequence $(x_i)_{i \in [g] \cup [n]}$ with $x_i = \frac{1}{2}$ for $i \in [g]$ and $x_i = 0$ or $\frac{1}{2}$ for $i \in [n]$ so that the sum of all x_i 's is an integer $d + 1$ for $d \geq 1$. Then, by [23, Theorem 2.1], we have a nef divisor $D := \varphi_{d,\bar{x}}^* \mathcal{O}(1)$ satisfying

$$D \cdot F_6^{0,0,0,0}(p_s, q_s, r_s, I^c) \neq 0, \quad D \cdot F_6^{0,0,0,0}(p_s, q_s, \{r_s, r'_s\}, I^c) = 0.$$

This proves the $|I| = 4$ case, and hence the theorem. \square

Note that, by Remark 5.11, the conclusion of Theorem 10.17 (2) does not hold when $n = 0$.

10.4. Small genus. In this subsection, we prove Theorem 1.4, which amounts to classifying regular extremal F-curves of types 5 and 6. Note that dealing with type 5 F-curves amounts to proving Conjecture 10.10 in this case. The non-extremal type 5 F-curves in Theorem 1.4 are exactly the exceptions listed in Conjecture 10.10, so it suffices to show the extremality of the remaining type 5 F-curves.

We begin with the case of $\overline{M}_{2,n}$. By Theorem 10.12, it suffices to show that the curves $F_5^{0,1}(I, J)$ are regular extremal.

Theorem 10.18. $F_5^{0,1}(I, J)$ are regular extremal on $\overline{M}_{2,n}$.

Proof. We proceed by induction on n . For $n \leq 5$, the F-conjecture is known to hold, so the F-cone coincides with the nef cone and is polyhedral. In this range, we can verify the extremality of these F-curves directly by brute-force computation. This verification is carried out using a Python script available at [11].

Now assume $n \geq 6$ and let $F = F_5^{0,1}(I, J)$ (in fact, $n \geq 4$ suffices). Choose three indices $p, q, r \in [n]$ such that $I \neq \{r\}$ and $I \not\subseteq \{p, q\}$. Then $\pi_{p,*}F$, $\pi_{q,*}F$, and $\pi_{q,*}\pi_{p,*}F$ are F-curves of the same type. By the induction hypothesis, Lemma 9.1 (1) holds with index of extremality 1. Therefore, to apply Lemma 9.1, it remains to verify that the conditions in Lemma 9.2 hold. Since each $\pi_{i,*}F$ is either an F-curve of the same type or zero, Lemma 9.2 (1) follows from the induction hypothesis and Proposition 10.13. Thus, it suffices to check Lemma 9.2 (2).

Case 1. $J \neq \emptyset$

By Theorem 7.5, the divisor $D = \kappa + \delta_{1,J}$ is nef and contracts F . Moreover, for the map $f : N_n \rightarrow V_{2,n}$, by Lemma 9.3, we have

$$f(\psi_r) = -e_0, \quad f(D) = -e_0 + (-1)^{|J|+1}e_1.$$

Hence, the condition in Lemma 9.2 (2) is satisfied.

Case 2. $J = \emptyset$

By Theorem 7.5, the divisor $D = \kappa + \delta_{1,\emptyset} + \delta_{2,\emptyset}$ is nef and contracts F . Moreover, for the map $f : N_n \rightarrow V_{2,n}$, by Lemma 9.3, we have

$$f(\psi_r) = -e_0, \quad f(D) = -2e_0 - e_1.$$

Therefore, the condition in Lemma 9.2 (2) is satisfied. \square

Now consider the case of $\overline{M}_{3,n}$. Apart from the cases covered by Theorem 10.12, we need to consider the following four cases:

- $F_5^{0,1}(I, J)$ with $J \neq \emptyset$,
- $F_5^{1,1}(I, J)$,
- $F_5^{0,2}(I, J)$,
- $F_6^{0,1,1,1}(I, J, K, L)$.

Theorem 10.19. All of the above four types of F-curves are regular extremal on $\overline{M}_{3,n}$.

Proof. The proof follows the strategy of Theorem 10.18. As before, we proceed by induction and verify the statement directly for $n \leq 4$, using [11]. Now assume $n \geq 4$. For each F-curve F of the above types, choose different indices p, q, r so that $\pi_{p,*}\pi_{q,*}F$ is nonzero and $\psi_r \cdot F = 0$. This is always possible since $n \geq 4$. Again, following the same reasoning as in Theorem 10.18, it suffices to check that the condition in Lemma 9.2 (2) holds with respect to p, q .

Case 1. $F = F_5^{0,1}(I, J)$ with $J \neq \emptyset$

By Theorem 7.5, the divisor $D = \kappa + \delta_{1,J}$ is nef and contracts F . Moreover, for the map $f : N_n \rightarrow V_{3,n}$, we have

$$f(\psi_r) = -e_0, \quad f(D) = -e_0 + (-1)^{|J|+1}e_1.$$

Hence, the condition in Lemma 9.2 (2) is satisfied.

Case 2. $F = F_5^{1,1}(I, J)$

Assume that not both I and J are empty. Without loss of generality, we may assume $J \neq \emptyset$. Then we can take $D = \kappa + \delta_{1,J}$, and the proof proceeds as in Case 1.

If $I = J = \emptyset$, let

$$D = \kappa + \frac{2}{3}\delta_{1,\emptyset} + \frac{1}{3}\delta_{2,\emptyset}.$$

Then D is nef by Proposition 7.13 and contracts F . Moreover, for the map $f : N_n \rightarrow V_{3,n}$, we have

$$f(\psi_1) = -e_0, \quad f(D) = -e_0 + \left(-\frac{1}{3} + (-1)^{n-1}\frac{2}{3}\right)e_1.$$

Hence, the condition in Lemma 9.2 (2) is satisfied.

Case 3. $F = F_5^{0,2}(I, J)$

By Theorem 7.5, the divisor $D = \kappa + \delta_{2,J}$ is nef and contracts F . Moreover, for the map $f : N_n \rightarrow V_{3,n}$, we have

$$f(\psi_r) = -e_0, \quad f(D) = -e_0 + (-1)^{|J^c|+1}e_1.$$

Hence, the condition in Lemma 9.2 (2) is satisfied.

Case 4. $F = F_6^{0,1,1,1}(I, J, K, L)$

This is a special case of Theorem 10.17 (1). □

Now we consider the case of $\overline{M}_{4,n}$. Except for the exceptions in Theorem 10.2 (3) and the cases covered by Theorem 10.12, we need to prove the regular extremality of the following F-curves:

- $F_6^{0,1,1,2}(I, J, K, L)$, $F_6^{1,1,1,1}(I, J, K, L)$, and $F_5^{0,3}(I, J)$.
- $F_5^{0,1}(I, J)$ for $J \neq \emptyset$.
- $F_5^{1,1}(I, J)$ for $I, J \neq \emptyset$.
- $F_5^{0,2}(I, J)$ and $F_5^{2,1}(I, J)$ for $(I, J) \neq ([n], \emptyset)$.

Theorem 10.20. All of the above F-curves are regular extremal on $\overline{M}_{4,n}$.

Proof. The overall strategy is the same. As before, if $n \leq 3$, the claim follows from the known cases of the F-conjecture and [11], so we assume $n \geq 4$. The following computation will be used multiple times (cf. Lemma 9.3):

$$\begin{aligned} f(\psi_r) &= -e_0, \quad f(\kappa + \delta_{1,J}) = -e_0 + (-1)^{|J|+1}e_1, \quad f(\kappa + \delta_{2,J}) = -e_0 + (-1)^{|J|+1}e_2, \\ f\left(\kappa + \sum_{i=1}^4 \delta_{i,\emptyset}\right) &= ((-1)^{n+1} - 1)e_0 + ((-1)^{n+1} - 1)e_1 - e_2, \\ f(\kappa + \delta_{2,\emptyset} + \delta_{4,\emptyset}) &= ((-1)^{n+1} - 1)e_0 - e_2. \end{aligned}$$

Note that all of these divisors are nef by Theorem 7.5 and Example 7.7. Moreover, in the proof, we need to verify that the listed divisors contract certain F-curves; however, we omit this step since the computation is evident from Theorem 2.2.

Case 1. $F_6^{0,1,1,2}(I, J, K, L)$.

Choose p, q, r so that $I \not\subseteq \{p, q\}$ and $I \neq \{r\}$, which is possible since $n \geq 4$. Then $\pi_{p,*}F$, $\pi_{q,*}F$, and $\pi_{q,*}\pi_{p,*}F$ are all F-curves of the same type. Hence, to apply Lemma 9.1, it remains to check that the condition in Lemma 9.2 holds, which reduces to verifying Lemma 9.2 (2) with respect to p and q .

Note that

$$\begin{aligned} f(\kappa + \delta_{0,I} + \delta_{1,L \cup i}) &= ((-1)^{|I|+1} - 1)e_0 + (-1)^{|L|}e_1, \\ f(\kappa + \delta_{2,J} + \delta_{1,I \cup j}) &= -e_0 + (-1)^{|I|}e_1 + (-1)^{|J|+1}e_2. \end{aligned}$$

These divisors are nef under certain condition, by Example 7.7 (4).

To conclude, we select in each situation three of the above divisors whose f -images generate $V_{4,n}$ (and, if n is odd, we may ignore e_2). When $J = K = L = \emptyset$, this is covered by Theorem 10.17 (2). If $L = \emptyset$ then at least one of J or K is nonempty (say J), so we can take

$$\psi_r, \kappa + \delta_{1,J}, \kappa + \sum_{i=1}^4 \delta_{i,\emptyset}.$$

If $L \neq \emptyset$ and at least one of J or K is nonempty (again say J), we may choose

$$\psi_r, \kappa + \delta_{1,J}, \kappa + \delta_{2,L}.$$

When $J = K = \emptyset$ and $L \neq \emptyset$, either $|I| > 1$ or $|L| > 1$, since $n \geq 4$. In the first case, for any $i \in I$ we can use

$$\psi_r, \kappa + \delta_{0,I} + \delta_{1,L \cup i}, \kappa + \delta_{2,L}.$$

In the second, for any $l \in L$ we can take

$$\psi_r, \kappa + \delta_{2,L} + \delta_{1,I \cup l}, \kappa + \delta_{2,L}.$$

In all subcases, these choices generate $V_{4,n}$, and thus $F_6^{0,1,1,2}(I, J, K, L)$ is extremal.

Case 2. $F_6^{1,1,1,1}(I, J, K, L)$

We proceed as in Case 1: choose any p, q, r and verify Lemma 9.2 (2) for p and q .

If at least three of I, J, K, L are empty, then the situation falls under Theorem 10.17 (1). When exactly two sets are nonempty, say I and J , a suitable choice is

$$\psi_r, \kappa + \delta_{1,I}, \kappa + \sum_{i=1}^4 \delta_{i,\emptyset}.$$

If at least three sets are nonempty, say I, J , and K , we may take

$$\psi_r, \kappa + \delta_{1,I}, \kappa + \delta_{1,I} + \delta_{1,J} + \delta_{2,I \cup J}.$$

In each case, the images of these divisors generate the required space, completing the argument.

Case 3. $F_5^{0,3}(I, J)$.

Choose p, q, r so that $I \not\subseteq \{p, q\}$ and $I \neq \{r\}$. We will check Lemma 9.2 (2) for p, q . Note that

$$f(\psi_r) = -e_0, \quad f(\kappa + \delta_{3,J}) = -e_0 + (-1)^{|J^c|+1}e_1$$

and that ψ_r and $\kappa + \delta_{3,J}$ are nef divisors. Since $I \neq \emptyset$, we can apply Example 7.7 (1). Hence, it suffices to find a nef divisor D_2 that contracts $F_5^{0,3}(I, J)$ and whose e_2 coefficient in $f(D_2)$ is nonzero when n is even.

If $J = \emptyset$, we can use $D_2 = \kappa + \sum_{i=1}^4 \delta_{i,\emptyset}$, for which

$$f(D_2) = -2e_0 - 2e_1 - e_2$$

when n is even.

If $|J| \geq 2$, choose $j \in J$ and use

$$D_2 = \kappa + \delta_{1,j} + \delta_{2,J \setminus j} + \delta_{3,J},$$

where

$$f(D_2) = -e_0 + (1 + (-1)^{|J^c|+1})e_1 + (-1)^{|J|}e_2.$$

Hence, from now on, assume $J = \{j\}$. If $|I| \geq 2$, choose $i \in I$ and use $D_2 = \kappa + \delta_{0,I} + \delta_{2,\{i,j\}}$, where

$$f(D_2) = \left((-1)^{|I|+1} - 1\right)e_0 - e_2.$$

Thus, we may assume $I = \{i\}$. Finally, let $D_2 = \kappa + \delta_{1,J^c} + \delta_{2,I^c}$, where

$$f(D_2) = -e_0 + (-1)^{|J^c|+1}e_1 + (-1)^{|I|+1}e_2.$$

Note that, since $n \geq 4$, this is a semigroup kappa divisor by Example 7.7 (4).

Case 4. $F_5^{0,1}(I, J)$ for $J \neq \emptyset$.

Choose p, q, r so that $I, J \not\subseteq \{p, q\}$ and $I \neq \{r\}$; this is possible since $n \geq 4$. Using ψ_r and $\kappa + \delta_{1,J}$, we can generate the space spanned by e_0 and e_1 , so it remains to find a nef divisor D contracting $F_5^{0,1}(I, J)$ whose e_2 -coefficient in $f(D)$ is nonzero. If $|J| \geq 2$, choose $j \in J$; then

$$\kappa + \delta_{1,J} + \delta_{2,j \cup I}$$

works by Example 7.7 (4). If $|I| \geq 2$, choose $i \in I$ and

$$\kappa + \delta_{0,I} + \delta_{2,i \cup J}$$

suffices. If $I = \{i\}$ and $J = \{j\}$, then

$$\kappa + \delta_{1,i} + \delta_{1,j} + \delta_{2,i \cup j}$$

is the desired divisor. Therefore, $F_5^{0,1}(I, J)$ is regular extremal.

Case 5. $F_5^{1,1}(I, J)$ for $I, J \neq \emptyset$.

Choose p, q, r so that $I, J \not\subseteq \{p, q\}$. This is possible since $n \geq 4$. Using ψ_r and $\kappa + \delta_{1,I}$, we can generate the space spanned by e_0 and e_1 , so it remains to find a nef divisor D contracting $F_5^{1,1}(I, J)$ whose e_2 -coefficient in $f(D)$ is nonzero. If $I \cup J \neq [n]$, then

$$\kappa + \delta_{1,I} + \delta_{1,J} + \delta_{2,I \cup J}$$

works. If $I \cup J = [n]$, then $|I| > 1$ or $|J| > 1$. Without loss of generality, assume $|J| > 1$ and choose $j \in J$. Then

$$\kappa + \delta_{1,J} + \delta_{2,I \cup j}$$

suffices by Example 7.7 (4). Therefore, $F_5^{1,1}(I, J)$ is regular extremal.

Case 6. $F_5^{0,2}(I, J)$ for $(I, J) \neq ([n], \emptyset)$.

Choose p, q, r so that $I \neq \{r\}$ and the image of $F_5^{0,2}(I, J)$ under $\pi_p, \pi_q, \pi_{\{p,q\}}$ is an F-curve of the same type. If $|I| \geq 3$, choose p, q , and r from I . If $|I| = 2$, choose $p, r \in I$ and q from the complement. If $|I| = 1$, choose all of the elements from I^c . This is possible since $n \geq 4$.

We divide this case into subcases, and for each subcase, we choose three divisors whose images under f generate $V_{4,n}$, spanned by e_0, e_1 , and e_2 . We can always take ψ_r for e_0 . If $J \neq \emptyset$, we may take $\kappa + \delta_{2,J}$ for e_2 , and we then need one more divisor for e_1 . If $|J| \geq 2$, choose $j \in J$ and take

$$\kappa + \delta_{2,J} + \delta_{1,j \cup I}.$$

This is nef by Example 7.7 (4), contracts $F_5^{0,2}(I, J)$, and

$$f(\kappa + \delta_{2,J} + \delta_{1,j \cup I}) = -e_0 + (-1)^{|I|}e_1 + (-1)^{|J|+1}e_2,$$

so it works. If $|I| \geq 2$, choose $i \in I$ and let

$$\kappa + \delta_{0,I} + \delta_{1,i \cup J}.$$

This is sufficient for the same reasoning. If $I = \{i\}$ and $J = \{j\}$, then, since $n \geq 3$, $(\delta_{2,j}, \delta_{3,i})$ forms a semigroup. Hence,

$$\kappa + \delta_{2,j} + \delta_{3,i}$$

is nef, contracts $F_5^{0,2}(I, J)$, and

$$f(\kappa + \delta_{2,j} + \delta_{3,i}) = -e_0 + (-1)^n e_1 + e_2.$$

Therefore, the case $J \neq \emptyset$ is settled.

Now consider the case $J = \emptyset$. In this case, $\kappa + \delta_{2,\emptyset} + \delta_{4,\emptyset}$ accounts for the e_2 term, so together with ψ_r , we still need one more divisor for the e_1 term. If $|I| \geq 2$, choose $i \in I$ and $k \in I^c$. Since $(I, J) \neq ([n], \emptyset)$, such k exists. Then

$$\kappa + \delta_{0,I} + \delta_{1,\{i,k\}}$$

works by a similar argument. If $I = \{i\}$, then by Proposition 7.15,

$$D := \kappa - \frac{1}{2}\psi_i + \frac{1}{2}\delta_{2,\emptyset} + \frac{1}{2}\delta_{4,\emptyset} + \frac{1}{2}\delta_{1,\{i\}}$$

is nef, contracts $F_5^{0,2}(I, J)$, and

$$f(D) = \left(\frac{1}{2}(-1)^{n+1} - \frac{1}{2}\right)e_0 + \frac{1}{2}e_1 - \frac{1}{2}e_2.$$

This completes the argument.

Case 7. $F_5^{2,1}(I, J)$ for $(I, J) \neq ([n], \emptyset)$.

Choose p, q, r such that the image under $\pi_p, \pi_q, \pi_{\{p,q\}}$ is an F-curve of the same type. The process is similar to Case 6, so we omit it.

Note that

$$f\left(\kappa + \frac{1}{2}\delta_{1,\emptyset} + \frac{1}{2}\delta_{2,\emptyset}\right) = -e_0 - \frac{1}{2}e_1 - \frac{1}{2}e_2.$$

This divisor is nef by Proposition 7.14.

To conclude, we select in each situation three of the above divisors whose f -images generate $V_{4,n}$. If $I, J \neq \emptyset$, we can take

$$\psi_r, \kappa + \delta_{1,J}, \kappa + \delta_{2,I}.$$

If $J \neq \emptyset$ and $I = \emptyset$, then

$$\psi_r, \kappa + \delta_{1,J}, \kappa + \delta_{2,\emptyset} + \delta_{4,\emptyset}$$

works. If $I = J = \emptyset$, then

$$\psi_r, \kappa + \frac{1}{2}\delta_{1,\emptyset} + \frac{1}{2}\delta_{2,\emptyset}, \kappa + \delta_{2,\emptyset} + \delta_{4,\emptyset}$$

suffices. Finally, if $J = \emptyset$ and $I \neq \emptyset$, since we already have ψ_r and $\kappa + \delta_{2,I}$, it remains to find one more nef divisor D contracting our F-curve whose e_1 -coefficient in $f(D)$ is nonzero. In this case, $I \neq \emptyset$ and $I^c \neq \emptyset$, so choose $i \in I$ and $j \in I^c$. Then $(\delta_{2,I}, \delta_{1,\{i,j\}})$ is a semigroup. Hence

$$D := \kappa + \delta_{2,I} + \delta_{1,\{i,j\}}$$

is nef, contracts $F_5^{2,1}(I, J)$, and

$$f(D) = -e_0 - e_1 + (-1)^{|I|+1}e_2,$$

so this divisor works. All in all, $F_5^{2,1}(I, J)$ with $(I, J) \neq ([n], \emptyset)$ spans a regular extremal ray. \square

11. PROOF OF THEOREM 1.1 AND COROLLARY 1.2

Proof of Theorem 1.1. We begin with the case $g = 2$. The case $n = 2$ is already covered in Section 5, so assume $n \geq 3$. Define

$$F = \{D \in \text{Nef}(\overline{\mathcal{M}}_{2,n}) \mid D \cdot F_3^0([n]) = 0\},$$

and let E be the linear subspace of F in which the coefficient of $-\delta_{0,[n-1]}$ is equal to the coefficient of ψ_n . By Theorem 10.7, F is a face of $\text{Nef}(\overline{\mathcal{M}}_{2,n})$ of codimension 1. Since $\psi_n \in F$, E is a proper subspace of F . Let $f : \overline{\mathcal{M}}_{2,2} \rightarrow \overline{\mathcal{M}}_{2,n}$ be the map attaching a rational stable curve with n marked points. Then, for any $D \in F \setminus E$, the divisor f^*D has the property that the coefficient of ψ_1 is not equal to the coefficient of ψ_2 . Moreover, since $D \cdot F_3^0([n]) = 0$, we have $f^*D \cdot F_3^0([2]) = b_{0,\{1,2\}} = 0$, so f^*D satisfies the condition of Theorem 3.1. Hence, f^*D is not semiample, and therefore D is also not semiample.

Now consider the case $g \geq 3$. The proof is almost the same as above. Define

$$F = \{D \in \text{Nef}(\overline{M}_{g,n}) \mid D \cdot F_3^1([n]) = 0\},$$

which is a face of codimension $\lfloor \frac{g}{2} \rfloor$ by Theorem 10.9, and let E be the linear subspace in which the coefficient of $-\delta_{1,[n-1]}$ is equal to the coefficient of ψ_n . Since $\psi_n \in F$, E is a proper subspace of F . Let $f : \overline{M}_{g-1,2} \rightarrow \overline{M}_{g,n}$ be the map attaching a genus 1 stable curve with n marked points. Then, by a similar argument, we can verify that if $D \in F \setminus E$, we may apply Theorem 3.1 to f^*D and deduce that D is not semiample. \square

Proof of Corollary 1.2. As above, let

$$F = \{D \in \text{Nef}(\overline{M}_{g,n}) \mid D \cdot F_3^1([n]) = 0\},$$

and let E be the linear subspace in which the coefficient of $-\delta_{1,[n-1]}$ is equal to the coefficient of ψ_n . Moreover, define

$$C_n := \left\{ F_3^1([n]), F_5^{1,i}([n], \emptyset) \mid 1 \leq i \leq g-2 \right\},$$

as in the proof of Theorem 10.9. Let C be the intersection of $\overline{NE}_1(\overline{M}_{g,n})$ with the subspace generated by C_n . By the proof of Theorem 10.9, since C_n generates a $\lfloor \frac{g}{2} \rfloor$ -dimensional subspace and $I(C_n) = \lfloor \frac{g}{2} \rfloor$, C is a $\lfloor \frac{g}{2} \rfloor$ -dimensional face of $\overline{NE}_1(\overline{M}_{g,n})$.

Assume there exists a projective contraction $f : \overline{M}_{g,n} \rightarrow X$ whose relative closed cone of curves is C . Let $D = f^*\mathcal{O}_X(1)$. Then $D \in F$ and D is semiample, so by the proof of Theorem 1.1, $D \in E$. Write

$$D = a\lambda - b_{\text{irr}}\delta_{\text{irr}} + \sum_{i=1}^n b_{0,i}\psi_i - \sum b_{i,I}\delta_{i,I}.$$

Since $D \in E$,

$$b_{1,[n]} = 0, \quad b_{0,n} = b_{1,[n-1]},$$

hence

$$D \cdot F_5^{g-1,0}(\emptyset, n) = b_{g-1,\emptyset} + b_{0,n} - b_{g-1,n} = b_{1,[n]} + b_{0,n} - b_{1,[n-1]} = 0.$$

Therefore, f also contracts $F_5^{g-1,0}(\emptyset, n)$. It is straightforward to see that $F_5^{g-1,0}(\emptyset, n) \notin C$, so such f does not exist. \square

Now we will prove that, in positive characteristic, $F_3^1([n])$ on $\overline{M}_{3,n}$ is contractible, i.e., there exists a projective contraction $f : \overline{M}_{3,n} \rightarrow X$ whose relative closed cone of curves is exactly the extremal ray spanned by $F_3^1([n])$.

Theorem 11.1. Assume that the base field has positive characteristic. Then there exists a divisorial contraction $f : \overline{M}_{3,n} \rightarrow X$ of relative Picard number 1, whose relative closed cone of curves is precisely the extremal ray spanned by $F_3^1([n])$. More precisely, f is an isomorphism outside $\Delta_{2,\emptyset}$ and restricts to the projection

$$\Delta_{2,\emptyset} \simeq \overline{M}_{2,1} \times \overline{M}_{1,n+1} \rightarrow \overline{M}_2 \times \overline{M}_{1,n+1}$$

on $\Delta_{2,\emptyset}$.

Proof. First, we prove all assertions except the last. We proceed by induction. Consider first the case $n = 1$. By the proof of Theorem 5.6, the face of the nef cone contracting $F_3^1([n])$ is generated by

$$\lambda, \quad 12\lambda - \delta_{\text{irr}}, \quad \psi_1, \quad 10\lambda - \delta_{\text{irr}} + 2\psi_1 - 2\delta_{1,\emptyset}.$$

The first three divisors are semiample by the proof of Proposition 7.16. Hence, to prove the existence of such a contraction, it is enough to show that there is another divisor in this cone that is also semiample. Let D be any divisor in the interior of the cone. Then D intersects positively with every F-curve except $F_3^1([1])$, so by a known case of the F-conjecture, $D - \epsilon\delta_{1,[1]}$ is ample for sufficiently small $\epsilon > 0$. In particular, the exceptional locus of D is contained in

$\Delta_{1,[1]}$, so by [32], it is enough to prove that $D|_{\Delta_{1,[1]}}$ is semiample. This follows from Lemma 7.4 and Proposition 7.16.

Now consider the induction step. We need to produce a codimension-1 subcone of the nef cone, intersecting trivially with $F_3^1([n])$, consisting entirely of semiample divisors. The proof is exactly the same as the proof of Theorem 10.8, using the fact that ψ_i and semigroup kappa classes are semiample (cf. Theorem 7.5).

Now we prove the last assertion, which implies that f is divisorial. It suffices to show that an integral subcurve $C \subseteq \overline{M}_{3,n}$ is contained in a fiber of

$$\overline{M}_{2,1} \times \overline{M}_{1,n+1} \longrightarrow \overline{M}_2 \times \overline{M}_{1,n+1}$$

if and only if its class in $A_1(\overline{M}_{3,n})$ is proportional to $F_3^1([n])$.

Let C be an integral curve contained in a fiber. Then $C = C_0 \times \{p\}$ for some integral subcurve $C_0 \subseteq \overline{M}_{2,1}$ contracted by $\pi : \overline{M}_{2,1} \rightarrow \overline{M}_2$. Since the Picard rank of $\overline{M}_{2,1}$ (resp. \overline{M}_2) is 3 (resp. 2), the classes of such curves are proportional. Noting that $F_3^0(\{1\})$ is contracted by π , we see that $[C_0]$ is proportional to $[F_3^0(\{1\})]$. As C is the image of C_0 under the clutching map attaching the curve corresponding to p , it follows that $[C]$ is proportional to $[F_3^1([n])]$.

Conversely, let C be an integral curve on $\overline{M}_{3,n}$ whose class is proportional to $[F_3^1([n])]$. Since

$$\delta_{2,\emptyset} \cdot F_3^1([n]) = -1,$$

we have

$$\delta_{2,\emptyset} \cdot C < 0,$$

so C is contained in $\Delta_{2,\emptyset} \simeq \overline{M}_{2,1} \times \overline{M}_{1,n+1}$.

Let π_1, π_2 be the projections from $\overline{M}_{2,1} \times \overline{M}_{1,n+1}$. We claim that $\pi_2(C)$ is a point, hence C is of the form $C_0 \times \{p\}$. This follows from

$$D \cdot \pi_2(C) = \pi_2^* D \cdot C = 0$$

for every $D \in \text{Pic}(\overline{M}_{1,n+1})$.

Let $i : \overline{M}_{2,1} \times \overline{M}_{1,n+1} \rightarrow \overline{M}_{3,n}$ be the clutching map. Define

$$S = \{\delta_{0,I} \mid |I| \geq 2, I \subseteq [n]\} \cup \{\delta_{1,I} \mid I \subsetneq [n], I \neq \emptyset\} \cup \{\psi_i \mid 1 \leq i \leq n\} \cup \{\lambda\}.$$

Note that each $D \in S$ can be regarded both as a divisor on $\overline{M}_{3,n}$ and on $\overline{M}_{1,n+1}$. To avoid confusion, view S as divisors on $\overline{M}_{3,n}$ and let S' be the corresponding set on $\overline{M}_{1,n+1}$; for $D \in S$ denote by $D' \in S'$ the corresponding divisor. For $D \in S$, we have $D \cdot F_3^1([n]) = 0$, hence $D \cdot C = 0$. Moreover, for $D \in S \setminus \{\lambda\}$,

$$i^* D = \pi_2^* D'$$

on $\overline{M}_{2,1} \times \overline{M}_{1,n+1}$. Therefore

$$D' \cdot \pi_2(C) = \pi_2^* D' \cdot C = i^* D \cdot C = D \cdot C = 0.$$

For λ ,

$$0 = \lambda \cdot C = i^* \lambda \cdot C = (\pi_1^* \lambda + \pi_2^* \lambda) \cdot C.$$

Since both $\pi_1^* \lambda$ and $\pi_2^* \lambda$ are nef, we obtain

$$\lambda \cdot \pi_2(C) = \pi_2^* \lambda \cdot C = 0.$$

Hence, for every $D' \in S'$,

$$\pi_2^* D' \cdot C = 0.$$

To finish the claim, it remains to show that S' spans $\text{Pic}(\overline{M}_{1,n+1})$. By [3, Theorem 4(c)], the boundary divisors span $\text{Pic}(\overline{M}_{1,n+1})$. Among the boundary divisors, all except $\delta_{0,[n+1]}$ lie in S' since $12\lambda = \delta_{\text{irr}}$. Moreover, $\psi_1 \in S'$, so [3, Theorem 4(c)] yields

$$\delta_{\text{irr}} + 12 \delta_{0,[n+1]} \in \text{Span } S'.$$

Thus S' spans $\text{Pic}(\overline{M}_{1,n+1})$, proving the claim.

Consequently, $C = C_0 \times \{p\}$ for some integral curve $C_0 \subseteq \overline{M}_{2,1}$. Since λ and $12\lambda - \delta_{\text{irr}}$ intersect trivially with $F_3^1([n])$, their pullbacks to $\overline{M}_{2,1}$ intersect trivially with C_0 . But these two classes span $\pi^*\text{Pic}(\overline{M}_2) \subseteq \text{Pic}(\overline{M}_{2,1})$. Therefore, C_0 is contained in a fiber of $\pi : \overline{M}_{2,1} \rightarrow \overline{M}_2$. \square

12. FURTHER QUESTIONS

In this section, we revisit some questions previously presented in the body of the paper, along with several new ones introduced here. They are organized into four themes.

12.1. Non-semiample nef divisors on $\overline{M}_{g,n}$. In this subsection, we assume that the characteristic of the base field is 0.

Theorem 1.1 shows that a large portion of the nef cone of $\overline{M}_{g,n}$ is non-semiample, and that if $g = 2$ or 3 , the subset of non-semiample nef divisors attains the smallest possible codimension. Hence, a natural question arising from Theorem 1.1 is Question 5.7, which asks whether the same is also true for higher genus. Unfortunately, the non-semiample criterion Theorem 3.1 does not appear to be sufficient for answering the question when $g \geq 4$, since it requires the divisor to contract a type 3 F-curve. Therefore, to address this question, one would need to find other examples of non-semiample nef divisors that can be utilized in this context, such as the divisor on $C \times C$ used in the proof of Theorem 3.1.

The following question about semiample divisors on $\overline{M}_{g,n}$ is natural, as the answer is known in other cases.

Question 12.1. Is every nef divisor on $\overline{M}_{0,n}$, $\overline{M}_{1,n}$, and \overline{M}_g semiample?

Question 12.1 for $\overline{M}_{0,n}$ is considered in [20, 38], and it is known to hold for $n \leq 6$, since $\overline{M}_{0,n}$ is log Fano for this range, as well as for symmetric divisors when $n \leq 19$ by [38]. The case of Question 12.1 for \overline{M}_g was posed in [18]. The author is not aware of other sources that have posed Question 12.1 for $\overline{M}_{1,n}$. The motivation for this question is to find a non-semiample divisor that is unrelated to ψ -classes. Note that Theorem 3.1 originated from the proof of the non-semiample of ψ -classes for higher genus in [32], whereas ψ -classes are semiample in genus 1 by Theorem 4.1.

12.2. Semiample divisors in positive characteristic. Thanks to [32], it is easier to prove the semiample of certain divisors in positive characteristic, and indeed, there are more semiample divisors in positive characteristic. This naturally leads to the following question.

Conjecture 12.2. Over a field of positive characteristic, every nef divisor on $\overline{M}_{g,n}$ is semiample.

If true, this conjecture would reveal a drastic difference in the nature of $\overline{M}_{g,n}$ between characteristic 0 and positive characteristic. We note the following fact regarding Conjecture 12.2.

Proposition 12.3 (char $k \neq 2$). Conjecture 12.2 for genus 0 implies Conjecture 12.2.

Proof. We proceed by induction on $\dim \overline{M}_{g,n}$. Let D be a nef divisor on $\overline{M}_{g,n}$. Since the conjecture is known for $g = 0$, we may assume $g \geq 1$. If $g \geq 2$, then by [26, Theorem 0.9], either $D = \pi_i^* D'$ for some projection map π_i , or D is big and the exceptional divisor is contained in the boundary of $\overline{M}_{g,n}$. In the first case, since D' is also nef, the induction hypothesis implies that D is semiample. In the second case, by [32, Theorem 0.2], it suffices to prove that the restrictions of D to boundary divisors are semiample. By Lemma 7.4, it is enough to show that the pullbacks of D along the attaching maps $\theta : \overline{M}_{g',n'} \rightarrow \overline{M}_{g,n}$ and $\xi : \overline{M}_{g-1,n+2} \rightarrow \overline{M}_{g,n}$ are semiample. This also follows from the induction hypothesis.

The case $g = 1$ is almost the same, except that in the first case we have $D = \pi_1^* D_1 + \pi_2^* D_2$, where $\pi_1 : \overline{M}_{1,n} \rightarrow \overline{M}_{1,S}$ and $\pi_2 : \overline{M}_{1,n} \rightarrow \overline{M}_{1,S^c}$, and D_1 (resp. D_2) is a nef divisor on $\overline{M}_{1,S}$ (resp. \overline{M}_{1,S^c}). \square

12.3. Modular interpretation of morphisms defined by semiample divisors. In Remark 6.7, we motivated Question 4.3 by relating it to contracting an F-curve on $\overline{M}_{1,n}$. There is another motivation for seeking a modular interpretation of other morphisms associated to semiample divisors. As mentioned, many divisors are semiample only in positive characteristic; for example, ψ_i is such a case by [32]. However, to the author’s knowledge, the modular interpretation of morphisms corresponding to ψ_i is not known, except in genus 0, where the corresponding map is given by Kapranov’s construction [29, 30]. If we could obtain a modular interpretation of such morphisms defined only in positive characteristic, we might uncover the origin of this difference.

A good analogue in the moduli of abelian varieties is the complete subvariety problem [27]. In that setting, there are more complete subvarieties of A_g in positive characteristic, given by the locus of abelian varieties with p -rank 0. Here, the difference arises from the p -rank, which is meaningful only in positive characteristic.

Regarding this, we ask for the modular interpretation of the morphism corresponding to semigroup kappa divisors (cf. Question 7.9). Moreover, it would be interesting to find a modular interpretation of the morphism on $\overline{M}_{3,n}$ that contracts only $F_3^1([n])$.

Finally, we note that in [45], the authors observed that the effectiveness of a divisor class depends on the characteristic of the base field.

12.4. Extremality of boundary strata of higher codimension. There is a body of literature devoted to extremal cycles of $\overline{M}_{g,n}$. However, relatively little is known in the case of higher codimension (see, e.g., [4, 40]). One of the obstacles is that, unlike in low codimension, there are many relations between higher-dimensional boundary strata, which makes the investigation significantly more complicated.

In this paper, we focus on the special case of 1-dimensional boundary strata, namely, F-curves. We study this case using the dual cone of nef divisors of $\overline{M}_{g,n}$. However, even in this setting, the case of type 5 F-curves (cf. Conjecture 10.10) remains unresolved, and for the type 6 case, we do not even have a conjectural description of the extremal curves. Hence, the following question is worth investigating.

Question 12.4. Which of the boundary strata of $\overline{M}_{g,n}$ are extremal?

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