ON THE FULLY NONLINEAR QUENCHING PROBLEM

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Abstract. We study the fully nonlinear quenching problem and establish sharp $C^{1,\alpha}_{\text{loc}}$-estimates and optimal growth at the free boundary in two distinct scenarios: the uniformly parabolic case and the degenerate elliptic case with varying singularities. For the former, we refine, in particular, the recent asymptotic results in [5].

1. Introduction

In this paper, we study local regularity properties of viscosity solutions to free boundary problems with singular absorption terms. The first model we consider is

$$\begin{cases} F(D^2u) - \partial_t u = \gamma u^{\gamma-1} & \text{in } \Omega_T \cap \{u > 0\}, \\ u = \varphi & \text{on } \partial_p \Omega_T, \end{cases} \tag{1.1}$$

governed by a fully nonlinear uniformly parabolic operator F. The second model is

$$\begin{cases} |Du|^{\kappa(x)} F(D^2u) = \gamma(x) u^{\gamma(x)-1} & \text{in } \Omega \cap \{u > 0\}, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

governed by a degenerate elliptic operator. Here, $\gamma, \gamma(x) \in (0,1)$ correspond to the singular absorption terms, while $\kappa(x) \geq 0$ in the second model represents the degeneracy associated with the gradient of the solution.

In the elliptic case and for $\kappa(x) = 0$, PDEs of this form arise as the Euler–Lagrange equations of the functional

$$\int \frac{1}{2}|Du|^2 + u^\gamma \, dx.$$

The cases $\gamma = 0$ and $\gamma = 1$ correspond to the cavitation problem and the obstacle problem, respectively. The intermediate case $\gamma \in (0,1)$ is the quenching problem we will address.

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The cavitation problem, also known as the Alt–Caffarelli problem, was treated in the variational setting by Alt and Caffarelli in [1]. Later, Ricarte and Teixeira in [18] studied the fully nonlinear case. The obstacle problem was studied by Caffarelli in [10] and later extended to the fully nonlinear setting by Lee and Shahgholian in [15]. In the case of the quenching problem, also known as the Alt–Phillips problem, Alt and Phillips studied it in [2], and Araújo and Teixeira extended it to the fully nonlinear uniformly elliptic case in [7].

The quenching problem refers to a phenomenon in which a process or reaction abruptly stops or vanishes, often encountered in combustion theory, heat transfer, and chemical reaction models. The quenching problem has been extensively studied over the years, and numerous results are available in the literature. For the variational setting involving the Laplacian operator, we refer readers to [2, 17, 16, 21]. In the nonvariational setting with fully nonlinear uniformly elliptic operators, the authors in [7] obtained optimal regularity along the free boundary by investigating the fine oscillation decay of limiting solutions. For degenerate elliptic operators, Teixeira in [19] established optimal regularity of solutions along the free boundary. In the uniformly parabolic case, a sharp regularity result was obtained in [5] by constructing proper barrier functions. We refer the reader to [4] and [6] for further related extensions.

In this paper, we first establish the existence of nonnegative viscosity solutions to (1.1) and (1.2) obtained as uniform limits of positive solutions to penalized problems. Subsequently, we derive sharp local regularity results by analyzing the regularity properties of the positive solutions to these penalized problems.

For the uniformly parabolic case (1.1), we improve upon the result in [5], which provides regularity for exponents strictly less than the optimal value. By applying Jensen–Ishii's lemma twice, we obtain more refined estimates, ultimately enabling us to achieve the optimal exponent. More precisely, we show that solutions are locally of class $C^{1,\alpha}$, for every

$$\alpha \in \left(0, \frac{\gamma}{2-\gamma}\right] \cap (0, \alpha_F) ,$$

while, at the free boundary, they are of class $C^{1,\frac{\gamma}{2-\gamma}}$. These results come with the corresponding quantitative estimates. Here, α_F denotes the optimal regularity exponent for F-caloric functions (cf. Remark 3.1).

Regarding the degenerate elliptic case (1.2), we address the more delicate scenario of varying singularities, under appropriate assumptions on $\kappa(x)$ and $\gamma(x)$. In this setting, we establish that, for each $x \in \Omega$, solutions belong to the class $C^{1,\alpha}$ at x, for every

$$\alpha \in \left(0, \frac{\gamma(x)}{\kappa(x)+2-\gamma(x)}\right] \cap (0, \alpha_F),$$

while, at the free boundary, they are of class $C^{1,\frac{\gamma(x)}{\kappa(x)+2-\gamma(x)}}$. In both cases, we derive the corresponding quantitative estimates. As in the parabolic setting, α_F corresponds to the optimal regularity exponent for F-harmonic functions (cf. Remark 4.1). The proof relies upon the use of both Jensen–Ishii's lemma and Hopf's lemma. Note that since the comparison principle does not hold in general for the operator $|Du|^{\kappa(x)}F(D^2u)$, we consider a suitable approximation to prove the existence of solutions to the penalized problem (see Proposition 4.1).

The paper is organized as follows. In Section 2, we introduce notation, the basic assumptions, and known results that will be used throughout the paper. In Section 3, we establish the sharp local regularity and the optimal growth at the free boundary for uniformly parabolic operators by repeatedly applying Jensen–Ishii's lemma. In Section 4, we treat the case of a degenerate elliptic operator with varying singularities.

2. Preliminaries

2.1. **Notation and definitions.** Let $\mathcal{S}^n$ denote the space of real $n\times n$ symmetric matrices. For parameters $0<\lambda\le\Lambda$, the Pucci extremal operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}:\mathcal{S}^n\to\mathbb{R}$ are defined as

$$\mathcal{M}^{+}_{\lambda,\Lambda}(M)=\Lambda\sum_{e_i>0}e_i+\lambda\sum_{e_i<0}e_i \quad \text{and} \quad \mathcal{M}^{-}_{\lambda,\Lambda}(M)=\lambda\sum_{e_i>0}e_i+\Lambda\sum_{e_i<0}e_i,$$

where $e_i=e_i(M)$ are the eigenvalues of $M\in\mathcal{S}^n$. We denote with $\mathcal{A}_{\lambda,\Lambda}$ the set of symmetric matrices M such that $\lambda I\le M \le \Lambda I$. Note that

$$\mathcal{M}^{+}_{\lambda,\Lambda}(M)=\sup_{A\in\mathcal{A}_{\lambda,\Lambda}}\operatorname{tr}(AM) \qquad \text{and} \qquad \mathcal{M}^{-}_{\lambda,\Lambda}(M)=\inf_{A\in\mathcal{A}_{\lambda,\Lambda}}\operatorname{tr}(AM).$$

For a bounded open domain $\Omega\subset\mathbb{R}^n$, with a smooth boundary, and $T>0$, let $\Omega_T=\Omega\times(-T,0]$. Denote by $\partial_p\Omega_T$ the parabolic boundary of Ω_T. For $(x_0,t_0)\in\Omega_T$ and $r>0$, we define the intrinsic parabolic cylinder

$$Q_r(x_0,t_0)=B_r(x_0)\times(t_0-r^2,t_0],$$

where $B_r(x_0)$ denotes an open ball in $\mathbb{R}^n$ centered at x_0 with radius r. For convenience, we denote $B_r=B_r(0)$ and $Q_r=Q_r(0,0)$.

Following [14], we introduce the definition of viscosity solution for the equation

$$F(D^2u)-\partial_t u=g(u,x,t) \quad \text{in } \Omega_T, \tag{2.1}$$

where $g \in C(\mathbb{R} \times \Omega_T)$. A similar definition applies to the elliptic case. We denote by $USC(\Omega_T)$, respectively $LSC(\Omega_T)$, the set of upper, respectively lower, semicontinuous functions on Ω_T.

Definition 2.1. *A function $u \in USC(\Omega_T)$ (resp., $u \in LSC(\Omega_T)$) is a viscosity subsolution (resp., supersolution) of (2.1) if, for every $(x_0, t_0) \in \Omega_T$ and $\phi \in C^{2,1}(\Omega_T)$ such that $u - \phi$ has a local maximum (resp. minimum) at (x_0, t_0), we have*

$$F(D^2\phi(x_0,t_0)) - \partial_t \phi(x_0,t_0) \geq \ (resp., \ \leq) \ g(u(x_0,t_0), x_0, t_0).$$

We say that $u \in C(\Omega_T)$ is a viscosity solution if u is both a viscosity supersolution and a subsolution.

We next recall the concept of parabolic superjet/subjet introduced in [12, Section 8].

Definition 2.2. *Let $v : \Omega_T \to \mathbb{R}$ be an upper semicontinuous function. For every $(x,t) \in \Omega_T$, the parabolic superjet of v at (x,t) is the set*

$$\begin{aligned}\mathcal{P}^+(v)(x,t) \ = \big\{ &(a,p,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mid \\ & v(y,s) \leqslant v(x,t) + a(s-t) + \langle p, y-x \rangle \\ & + \frac{1}{2}\langle X(y-x), y-x \rangle \\ & + o(|s-t| + |y-x|^2) \ \text{as} \ (y,s) \to (x,t) \big\}.\end{aligned}$$

The corresponding limiting superjet of v at (x,t) is

$$\begin{aligned}\overline{\mathcal{P}}^+(v)(x,t) \ = \big\{ &(a,p,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mid \\ & \exists (x_m, t_m, a_m, p_m, X_m) \ \text{such that} \\ & (a_m, p_m, X_m) \in \mathcal{P}^+(v)(x_m, t_m), \ \text{and} \\ & (x_m, t_m, v(x_m,t_m), a_m, p_m, X_m) \to (x, t, v(x,t), a, p, X) \\ & \text{as} \ m \to \infty \big\}.\end{aligned}$$

Subjets $\mathcal{P}^-$ and limiting subjets $\overline{\mathcal{P}}^-$ are defined analogously for lower semicontinuous functions, replacing $\leq$ with $\geq$ for the former and $\mathcal{P}^+$ with $\mathcal{P}^-$ for the latter. In the elliptic case, superjets and subjets are defined similarly, as described in [12, Section 2].

2.2. **Known results.** We recall here Jensen–Ishii's lemma (*cf.* [12]). We state it for the parabolic case, but a similar result also holds in the elliptic case.

Lemma 2.1 (Jensen–Ishii's lemma)**.** *Let $v \in C(Q_1)$ and suppose that*

$$\Phi(x,y,t) = v(x,t) - v(y,t) - L\phi(|x-y|) - K(|x|^2 + |y|^2 + (-t)^2)$$

has a local maximum at $(\overline{x}, \overline{y}, \overline{t}) \in B_1 \times B_1 \times (-1, 0]$ *with* $\overline{x} \neq \overline{y}$ *for* $L, K > 0$. *Then, for every sufficiently small* $\iota > 0$, *there exists* $\tau \in \mathbb{R}, p \in \mathbb{R}^n$ *and* $X, Y \in \mathcal{S}^n$ *such that*

$$(\tau + 2K\overline{t}, p + 2K\overline{x}, X) \in \overline{\mathcal{P}}^+(v)(\overline{x}, \overline{t}),$$

$$(\tau, p - 2K\overline{y}, Y) \in \overline{\mathcal{P}}^-(v)(\overline{y}, \overline{t}),$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq L \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix} + (2K + \iota) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{2.2}$$

where

$$|p| = L\phi'(|\overline{x} - \overline{y}|)$$

and

$$Z = \phi''(|\overline{x} - \overline{y}|) \frac{\overline{x} - \overline{y}}{|\overline{x} - \overline{y}|} \otimes \frac{\overline{x} - \overline{y}}{|\overline{x} - \overline{y}|} + \frac{\phi'(|\overline{x} - \overline{y}|)}{|\overline{x} - \overline{y}|} \left(I - \frac{\overline{x} - \overline{y}}{|\overline{x} - \overline{y}|} \otimes \frac{\overline{x} - \overline{y}}{|\overline{x} - \overline{y}|} \right).$$

Remark 2.1. *Note that applying the matrix inequality* (2.2) *to the vector* (ξ, ξ), *for* $\xi \in \mathbb{R}^n$ *arbitrary, implies that every eigenvalue of* $Y - X$ *is greater than or equal to* $-(4K + 2\iota)$. *Similarly, applying* (2.2) *to the vector* $\left(\frac{\overline{x} - \overline{y}}{|\overline{x} - \overline{y}|}, -\frac{\overline{x} - \overline{y}}{|\overline{x} - \overline{y}|} \right)$, *shows that at least one eigenvalue of* $Y - X$ *is greater than or equal to* $-4L\phi''(|\overline{x} - \overline{y}|) - (4K + 2\iota)$.

Next, we introduce the following modified version of Perron's method.

Lemma 2.2. [3, Theorem 2.1] *Let* $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ *be a degenerate elliptic operator, i.e.,* $F(x, r, p, X) \leq F(x, r, p, Y)$ *for all* $x \in \Omega$, $r \in \mathbb{R}$, $p \in \mathbb{R}^n$ *and* $X \leq Y$ *in* $\mathcal{S}^n$, *let* $g : \mathbb{R} \to \mathbb{R}$ *be a Lipschitz function and* $\varphi \in C(\partial\Omega)$. *Suppose that the comparison principle holds: if* u *is a viscosity subsolution and* v *is a viscosity supersolution of*

$$F(x, w, Dw, D^2 w) - \lambda w = f \quad \text{in } \Omega, \tag{2.3}$$

with $u \leq v$ *on* $\partial\Omega$, *where* $\lambda > 0$ *and* $f \in C(\Omega)$, *then* $u \leq v$ *in* Ω.
Assume that for each $\lambda > 0$ *and each* $f \in C(\Omega)$, *there exist a viscosity subsolution and a viscosity supersolution* (2.3) *with boundary condition* φ *on* $\partial\Omega$. *Assume that viscosity solutions of* $F(x, u, Du, D^2 u) = f(x) \in L^\infty(\Omega)$ *satisfy a priori interior* $C^{0,\alpha}$ *estimates and that there exist a continuous viscosity subsolution* $\underline{u}$ *and a continuous viscosity supersolution* $\overline{u}$ *to*

$$F(x, u, Du, D^2 u) = g(u) \quad \text{in } \Omega, \tag{2.4}$$

with $\underline{u} \leq \overline{u}$ *in* Ω *and* $\underline{u} = \overline{u} = \varphi$ *on* $\partial\Omega$. *Then the function defined by*

$$u(x) := \inf_{v \in \mathcal{S}} v(x) \quad \text{for } x \in \overline{\Omega},$$

where

$$\mathcal{S} = \left\{ v \in C(\overline{\Omega}) \;\middle|\; \begin{array}{l} v \text{ is a viscosity supersolution to (2.4)}, \\ \underline{u} \le v \le \overline{u} \text{ in } \Omega \end{array} \right\},$$

is a viscosity solution to (2.4) *satisfying* $u = \varphi$ *on* $\partial\Omega$.

The following results are used in establishing the existence of a solution to the degenerate elliptic problem stated in Proposition 4.1, via Lemma 2.2.

Lemma 2.3. [13, Lemma 2] *Let* Ω *be a bounded open domain satisfying a uniform exterior ball condition,* F *be a uniformly elliptic operator,* $f \in C(\Omega)$*, and* $\varphi \in C(\partial\Omega)$*. Then, there exist a viscosity subsolution* $\underline{w} \in C(\overline{\Omega})$ *and a viscosity supersolution* $\overline{w} \in C(\overline{\Omega})$*, independent of* $\kappa(\cdot)$ *and* δ*, to*

$$\begin{cases} (\delta + |Du|)^{\kappa(x)}(F(D^2u) - \delta u) = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

for every $\kappa \in C(\Omega)$*, with* $\kappa \ge 0$*, and every* $\delta \in (0,1)$*, with* $\overline{w} = \underline{w} = \varphi$ *on* $\partial\Omega$.

Remark 2.2. *By a simple modification of the proof of Lemma 2.3, for each* $\lambda > 0$*, we also obtain the existence of a continuous viscosity subsolution and a continuous viscosity supersolution to*

$$(\delta + |Du|)^{\kappa(x)}(F(D^2u) - \delta u) - \lambda u = f \quad \text{in } \Omega,$$

with boundary data φ *on* $\partial\Omega$.

Lemma 2.4. *Let* $\delta \in (0,1)$*,* $\lambda \ge 0$*,* $\kappa(\cdot) \ge 0$ *be a Lipschitz function in* Ω*,* F *be a uniformly elliptic operator and* $f \in C(\Omega)$*. Assume that* u *is a viscosity subsolution and* v *is a viscosity supersolution to*

$$(\delta + |Dw|)^{\kappa(x)}(F(D^2w) - \delta w) - \lambda w = f \quad \text{in } \Omega,$$

with $u \le v$ *on* $\partial\Omega$*. Then,* $u \le v$ *in* Ω.

Proof. The proof can be obtained by a straightforward modification of [13, Proposition 4], and is omitted here. □

Lemma 2.5. *Let* u *be a viscosity solution to*

$$(\delta + |Du|)^{\kappa(x)}(F(D^2u) - \delta u) = f \quad \text{in } B_1,$$

where $\delta \in (0,1)$*,* $\kappa \in C(B_1)$*, with* $\kappa \ge 0$*,* F *is a* (λ, Λ)*-uniformly elliptic operator and* $f \in C(B_1) \cap L^\infty(B_1)$*. Then, for each* $\mu \in (0,1)$*, there exists a constant* $C = C\left(n, \lambda, \Lambda, \mu, \|u\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)}\right)$*, independent of* δ*, such that*

$$|u(x) - u(y)| \le C|x-y|^\mu, \quad \text{for } x, y \in B_{1/2}.$$

Proof. Although a standard application of Lemma 2.1 yields the result, we include the proof here for the sake of completeness. See, for instance, [8] and [13]. Defining

$$\Phi(x,y) = u(x) - u(y) - L|x-y|^{\mu} - K(|x|^2 + |y|^2) \quad \text{in } B_1,$$

we will prove that

$$\max_{\overline{B_{\frac{1}{2}}\times B_{\frac{1}{2}}}} \Phi \leq 0,$$

for sufficiently large L and K. Then, the standard translation argument gives the desired result. Assume that Φ attains its positive maximum at $(\overline{x},\overline{y}) \in \overline{B_{\frac{1}{2}}} \times \overline{B_{\frac{1}{2}}}$. This implies that

$$\overline{x} \neq \overline{y}, \qquad u(\overline{x}) > u(\overline{y}) + L|\overline{x}-\overline{y}|^{\mu},$$

and

$$L|\overline{x}-\overline{y}|^{\mu} + K(|\overline{x}|^2 + |\overline{y}|^2) \leq 2\|u\|_{\infty}. \tag{2.5}$$

From (2.5), by choosing K sufficiently large, we ensure that

$$(\overline{x},\overline{y}) \in B_{\frac{1}{4}} \times B_{\frac{1}{4}}.$$

Note that

$$|\overline{x}-\overline{y}| \leq 1. \tag{2.6}$$

Now, we can obtain $p \in \mathbb{R}^n$ and $X, Y \in \mathcal{S}^n$, such that

$$(p_x, X) \in \overline{\mathcal{P}}^{+}(u)(\overline{x}), \tag{2.7}$$

$$(p_y, Y) \in \overline{\mathcal{P}}^{-}(u)(\overline{y}), \tag{2.8}$$

where $p_x = p + 2K\overline{x}$ and $p_y = p - 2K\overline{y}$, with the estimate given by Lemma 2.1. We can choose L sufficiently large so that

$$1 \leq \frac{1}{2}\mu L \leq \frac{1}{2}\mu L|\overline{x}-\overline{y}|^{\mu-1} \leq |p_x|, |p_y| \leq \frac{3}{2}\mu L|\overline{x}-\overline{y}|^{\mu-1}. \tag{2.9}$$

By applying (2.7) and (2.8) to the equation, we obtain the inequalities

$$(\delta + |p_x|)^{\kappa(\overline{x})}(F(X) - \delta u(\overline{x})) \geq f(\overline{x}),$$

and

$$(\delta + |p_y|)^{\kappa(\overline{y})}(F(Y) - \delta u(\overline{y})) \leq f(\overline{y}).$$

Then, we get

$$F(Y) - F(X) \leq \delta u(\overline{y}) + \frac{f(\overline{y})}{(\delta + |p_y|)^{\kappa(\overline{y})}} - \delta u(\overline{x}) - \frac{f(\overline{x})}{(\delta + |p_x|)^{\kappa(\overline{x})}} \tag{2.10}$$

By Lemma 2.1,

$$\begin{aligned} F(Y) - F(X) &\geq \lambda\Big(4\mu(1-\mu)L|\overline{x}-\overline{y}|^{\mu-2} - (4K+2\iota)\Big) - \Lambda(n-1)(4K+2\iota) \\ &\geq 4\lambda\mu(1-\mu)L|\overline{x}-\overline{y}|^{\mu-2} - (\lambda+\Lambda(n-1))(4K+2\iota) \\ &\geq 3\lambda\mu(1-\mu)L|\overline{x}-\overline{y}|^{\mu-2}, \end{aligned} \tag{2.11}$$

for sufficiently large L where we have used (2.6). On the other hand, using the assumptions $\delta \in (0,1)$ and $\kappa \geq 0$, together with (2.9), we obtain

$$\delta u(\overline{y}) + \frac{f(\overline{y})}{(\delta+|p_y|)^{\kappa(\overline{y})}} - \delta u(\overline{x}) - \frac{f(\overline{x})}{(\delta+|p_x|)^{\kappa(\overline{x})}} \leq 2\left\|u\right\|_{L^\infty} + 2\left\|f\right\|_{L^\infty}. \tag{2.12}$$

Combining (2.10), (2.11) and (2.12) and using (2.6), we deduce

$$2\left\|u\right\|_{L^\infty} + 2\left\|f\right\|_{L^\infty} \geq 3\lambda\mu(1-\mu)L|\overline{x}-\overline{y}|^{\mu-2} \geq 3\lambda\mu(1-\mu)L,$$

which leads to a contradiction for sufficiently large L. This completes the proof.

□

3. Optimal regularity in the uniformly parabolic case

In this section, we examine the fully nonlinear parabolic problem

$$\begin{cases} F(D^2u) - \partial_t u = \gamma u^{\gamma-1} & \text{in } \Omega_T \cap \{u>0\}, \\ u = \varphi & \text{on } \partial_p\Omega_T, \end{cases} \tag{3.1}$$

and establish the existence and optimal regularity of a solution under the following assumptions.

(A1): F is (λ,Λ)-uniformly elliptic, *i.e.*,

$$\mathcal{M}^-_{\lambda,\Lambda}(M-N) \leq F(M) - F(N) \leq \mathcal{M}^+_{\lambda,\Lambda}(M-N),$$

for every $M, N \in \mathcal{S}^n$.

(A2): F is 1-homogeneous, *i.e.*,

$$F(tM) = tF(M),$$

for every $t \geq 0$ and $M \in \mathcal{S}^n$.

The first main result of this paper is the following.

Theorem 3.1. *Let $\gamma \in (0,1)$, $\varphi \in C(\partial_p\Omega_T)$, with $\varphi \geq 0$, and assume (A1) and (A2). There exists a nonnegative bounded viscosity solution u to* (3.1) *in the sense of Definition 2.1, and u is locally of class $C^{1,\alpha}$, for every*

$$\alpha \in \left(0, \frac{\gamma}{2-\gamma}\right] \cap (0, \alpha_F),$$

with the estimate

$$\sup_{(y,s)\in Q_r(x,t)} |u(y,s) - u(x,t) - Du(x,t)\cdot(y-x)| \le Cr^{1+\alpha},$$

for $Q_r(x,t) \Subset \Omega_T$*, where* $C = C(n, \lambda, \Lambda, \gamma, \alpha, \|u\|_{L^\infty})$*. Moreover, for each free boundary point* (x,t)*,* u *is of class* $C^{1,\frac{\gamma}{2-\gamma}}$ *at* (x,t)*, with the estimate*

$$\sup_{Q_r(x,t)} u \le Cr^{1+\frac{\gamma}{2-\gamma}},$$

for $Q_r(x,t) \Subset \Omega_T$*, where* $C = C(n, \lambda, \Lambda, \gamma, \|u\|_{L^\infty})$*.*

Remark 3.1. *The constant* α_F *in the statement of the theorem denotes the optimal exponent associated with the* $C^{1+\mu,\frac{1+\mu}{2}}$*-regularity theory for solutions of* F*-caloric functions, i.e., solutions of the equation* $F(D^2h) - \partial_t h = 0$*. In the case of* $C^{1+\mu,\frac{1+\mu}{2}}$ *â€“regularity of solutions to* $F(D^2u) - \partial_t u = f$*, where* f *is a continuous function, we use an approximation lemma. Thus, we can establish regularity for every* $0 < \mu < \alpha_F$*, but not for* $\mu = \alpha_F$ *(see [20]).*

We construct our solution to (3.1) as the limit of solutions to singularly penalized approximating problems. Let $\rho \in C^\infty(\mathbb{R})$ be a nonnegative smooth function with compact support in [0, 1], such that $\int \rho = 1$. For each $\epsilon \in (0,1)$, define the real function

$$\beta_\epsilon(s) = \gamma \int_0^{\sigma(s)} \rho(\theta)\, d\theta,$$

where $\sigma(s) := s\epsilon^{\frac{2}{\gamma-2}} - \sigma_0$, for $\sigma_0 \in (0,1)$, which converges to $\gamma\chi_{\{s>0\}}$ as $\epsilon \to 0$. Here, χ_E denotes the characteristic function of a set E. We consider the penalized problem

$$\begin{cases} F(D^2u_\epsilon) - \partial_t u_\epsilon = \beta_\epsilon(u_\epsilon)u_\epsilon^{\gamma-1} & \text{in } \Omega_T, \\ u_\epsilon = \varphi_\epsilon & \text{on } \partial_p\Omega_T, \end{cases} \tag{3.2}$$

where $\epsilon \in (0,1)$ and $\varphi_\epsilon = \varphi + \epsilon^{\frac{2}{2-\gamma}}$. The following result concerns the existence of a positive solution to (3.2) and is taken from [5, Section 3].

Proposition 3.1. *There exists a viscosity solution* u_ϵ *to* (3.2)*. Moreover,* u_ϵ *satisfies*

$$0 < u_\epsilon \le \|\varphi\|_{L^\infty(\partial_p\Omega_T)} + 1 \quad \text{in } \Omega_T.$$

For ease of notation, hereafter in this section, we will denote u_ϵ by u. Following the approach of [7], we will examine the regularity of the auxiliary function

$$v := u^{\frac{2-\gamma}{2}}.$$

By direct calculation, we have

$$Dv = \frac{2-\gamma}{2} u^{-\frac{\gamma}{2}} Du \tag{3.3}$$

and

$$D^2 v = \frac{2-\gamma}{2} u^{-\frac{\gamma}{2}} D^2 u - \frac{2-\gamma}{2}\frac{\gamma}{2} u^{-\frac{\gamma}{2}-1} Du \otimes Du. \tag{3.4}$$

Using (3.3), (3.4) and (A2), we rewrite the equation in (3.2) as (see [5])

$$F(D^2 v + \delta v^{-1} Dv \otimes Dv) - \partial_t v = f(x,t) v^{-1} \quad \text{in } \Omega_T, \tag{3.5}$$

where

$$\delta = \frac{\gamma}{2-\gamma} \qquad \text{and} \qquad f(x,t) = \frac{2-\gamma}{2}\beta_\epsilon \left(v(x,t)^{\frac{2}{2-\gamma}} \right) \in [0,1).$$

The following Hölder regularity of v in the space variables has been obtained in [5, Theorem 1].

Proposition 3.2. *Let $v \in C(Q_1)$ be a positive viscosity solution to (3.5) in Q_1. Then, for each $\mu \in (0,1)$, there exists $C > 0$, depending only on $n, \lambda, \Lambda, \gamma, \mu$ and $\|v\|_{L^\infty(Q_1)}$, such that*

$$|v(x,t) - v(y,t)| \le C|x-y|^\mu,$$

for every $(x,t), (y,t) \in Q_{\frac{1}{2}}$.

Now, we build upon Proposition 3.2 to obtain the Lipschitz regularity of v in the space variables, thus unlocking our optimal regularity result.

Theorem 3.2. *Let $v \in C(Q_1)$ be a positive viscosity solution to (3.5) in Q_1. Then, there exists $C > 0$, depending only on $n, \lambda, \Lambda, \gamma$ and $\|v\|_{L^\infty(Q_1)}$, such that*

$$|v(x,t) - v(y,t)| \le C|x-y|, \tag{3.6}$$

for every $(x,t), (y,t) \in Q_{\frac{1}{2}}$.

Proof. Define in Q_1

$$\Phi(x,y,t) = v(x,t) - v(y,t) - Lw(|x-y|) - K(|x|^2 + |y|^2 + (-t)^2),$$

where, for a parameter $a \in (1,2)$ to be determined later,

$$w(t) = \begin{cases} t - \frac{1}{a}t^a & \text{if } 0 \le t < 1, \\ 1 - \frac{1}{a} & \text{if } t \ge 1. \end{cases}$$

Then, for $0 < t < 1$, we have $w'(t) = 1 - t^{a-1}$ and $w''(t) = -(a-1)t^{a-2}$. Note also that

$$w(t) \ge \frac{t}{2}, \tag{3.7}$$

for sufficiently small t. We will prove that

$$\max_{\overline{B_{\frac{1}{2}}}\times\overline{B_{\frac{1}{2}}}\times[-\frac{1}{4},0]} \Phi \leq 0, \tag{3.8}$$

for sufficiently large L and K. Then (3.6) follows from the standard translation argument.

To prove (3.8), assume, by contradiction, that Φ attains its positive maximum at $(\overline{x}, \overline{y}, \overline{t}) \in \overline{B_{\frac{1}{2}}} \times \overline{B_{\frac{1}{2}}} \times [-1/4, 0]$. Then, we have $\overline{x} \neq \overline{y}$,

$$v(\overline{x}, \overline{t}) - v(\overline{y}, \overline{t}) \geq Lw(|\overline{x} - \overline{y}|) > 0, \tag{3.9}$$

and

$$v(\overline{x}, \overline{t}) - v(\overline{y}, \overline{t}) > K\left(|\overline{x}|^2 + |\overline{y}|^2 + (-\overline{t})^2\right).$$

By Proposition 3.2, we have

$$|v(\overline{x}, \overline{t}) - v(\overline{y}, \overline{t})| \leq C|\overline{x} - \overline{y}|^{\mu},$$

where $\mu \in (0, 1)$ is to be determined later. From

$$\frac{1}{3}(|\overline{x}| + |\overline{y}| + |\overline{t}|)^2 \leq |\overline{x}|^2 + |\overline{y}|^2 + (-\overline{t})^2 \leq \frac{v(\overline{x}, \overline{t}) - v(\overline{y}, \overline{t})}{K} \leq \frac{C|\overline{x} - \overline{y}|^{\mu}}{K},$$

we get

$$|\overline{x}| + |\overline{y}| + |\overline{t}| \leq \left(\frac{3C}{K}\right)^{\frac{1}{2}} |\overline{x} - \overline{y}|^{\frac{\mu}{2}}. \tag{3.10}$$

From (3.10), by choosing K sufficiently large, we ensure that

$$(\overline{x}, \overline{y}, \overline{t}) \in B_{\frac{1}{10}} \times B_{\frac{1}{10}} \times (-\tfrac{1}{100}, 0].$$

Now, we can obtain $\tau \in \mathbb{R}, p \in \mathbb{R}^n$ and $X, Y \in \mathcal{S}^n$, such that

$$(\tau + 2K\overline{t}, p_x, X) \in \overline{\mathcal{P}}^{+}(v)(\overline{x}, \overline{t}), \tag{3.11}$$

$$(\tau, p_y, Y) \in \overline{\mathcal{P}}^{-}(v)(\overline{y}, \overline{t}), \tag{3.12}$$

where $p_x = p + 2K\overline{x}$ and $p_y = p - 2K\overline{y}$, with the estimate given by Lemma 2.1. We can choose L sufficiently large so that

$$1 \leq \frac{1}{2}|p| \leq |p_x|, |p_y| \leq \frac{3}{2}|p|. \tag{3.13}$$

Note that, by (3.10),

$$|p_y - p_x| = 2K|\overline{x} + \overline{y}| \leq 2(3CK)^{\frac{1}{2}}|\overline{x} - \overline{y}|^{\frac{\mu}{2}}. \tag{3.14}$$

By applying (3.11) and (3.12) to equation (3.5), we obtain the inequalities

$$F(X + \delta v(\overline{x}, \overline{t})^{-1} p_x \otimes p_x) - (\tau + 2K\overline{t}) \geq f(\overline{x}, \overline{t}) v(\overline{x}, \overline{t})^{-1},$$
$$F(Y + \delta v(\overline{y}, \overline{t})^{-1} p_y \otimes p_y) - \tau \leq f(\overline{y}, \overline{t}) v(\overline{y}, \overline{t})^{-1}.$$

Then, we get

$$F\big(Y+\delta v(\overline{y},\overline{t})^{-1}p_y\otimes p_y\big)-F\big(X+\delta v(\overline{x},\overline{t})^{-1}p_x\otimes p_x\big) \leqslant K+f(\overline{y},\overline{t})v(\overline{y},\overline{t})^{-1}-f(\overline{x},\overline{t})v(\overline{x},\overline{t})^{-1}. \tag{3.15}$$

On the other hand, for every $\eta>0$, there exists $M_\eta\in\mathcal{A}_{\lambda,\Lambda}$ such that

$$\begin{aligned} F\big(Y+\delta v(\overline{y},\overline{t})^{-1}p_y\otimes p_y\big)&-F\big(X+\delta v(\overline{x},\overline{t})^{-1}p_x\otimes p_x\big)\\ &\geqslant \operatorname{tr}\Big(M_\eta\Big(Y+\delta v(\overline{y},\overline{t})^{-1}p_y\otimes p_y \\ &\qquad -X-\delta v(\overline{x},\overline{t})^{-1}p_x\otimes p_x\Big)\Big)-\eta. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we obtain

$$\begin{aligned} K+\eta\geqslant{}& \operatorname{tr}\big(M_\eta(Y-X)\big)\\ &+v(\overline{y},\overline{t})^{-1}\big(\delta\operatorname{tr}(M_\eta p_y\otimes p_y)-f(\overline{y},\overline{t})\big)\\ &-v(\overline{x},\overline{t})^{-1}\big(\delta\operatorname{tr}(M_\eta p_x\otimes p_x)-f(\overline{x},\overline{t})\big). \end{aligned} \tag{3.17}$$

From Lemma 2.1,

$$\begin{aligned} \operatorname{tr}(M_\eta(Y-X)) &\geq \lambda(-4Lw''(|\overline{x}-\overline{y}|)-(4K+2\iota))-\Lambda(n-1)(4K+2\iota)\\ &\geq -4\lambda Lw''(|\overline{x}-\overline{y}|)-(\lambda+\Lambda(n-1))(4K+2\iota)\\ &\geq -3\lambda Lw''(|\overline{x}-\overline{y}|), \end{aligned} \tag{3.18}$$

for sufficiently large L. From (3.13), we know

$$\delta\operatorname{tr}(M_\eta p_y\otimes p_y)-f(\overline{y},\overline{t})\geq \delta\lambda|p_y|^2-\|f\|_{L^\infty}\geq\frac{\delta\lambda}{4}|p|^2-1>0, \tag{3.19}$$

for sufficiently large L. Using (3.9), (3.18) and (3.19), from (3.17), we obtain

$$\begin{aligned} K+\eta\geqslant{}& \operatorname{tr}\big(M_\eta(Y-X)\big)\\ &+v(\overline{x},\overline{t})^{-1}\big(\delta\operatorname{tr}(M_\eta p_y\otimes p_y)-f(\overline{y},\overline{t})\big)\\ &-v(\overline{x},\overline{t})^{-1}\big(\delta\operatorname{tr}(M_\eta p_x\otimes p_x)-f(\overline{x},\overline{t})\big)\\ \geqslant{}& -3\lambda Lw''(|\overline{x}-\overline{y}|)\\ &-v(\overline{x},\overline{t})^{-1}\big(2\delta n\Lambda|p_y||p_y-p_x|+\delta\Lambda|p_y-p_x|^2\\ &-f(\overline{x},\overline{t})+f(\overline{y},\overline{t})\big). \end{aligned} \tag{3.20}$$

Note that

$$f(\overline{x},\overline{t})-f(\overline{y},\overline{t})=\frac{2-\gamma}{2}(\beta_\epsilon(v(\overline{x},\overline{t})^{\frac{2}{2-\gamma}})-\beta_\epsilon(v(\overline{y},\overline{t})^{\frac{2}{2-\gamma}}))\geq 0, \tag{3.21}$$

which follows from $v(\overline{x},\overline{t})>v(\overline{y},\overline{t})$, and β_ϵ being a nondecreasing function. Denote $\Delta=|\overline{x}-\overline{y}|$. Using (3.7), (3.9), (3.13), (3.14) and (3.21), from (3.20), we have

$$\begin{aligned} K+\eta &\geq -3\lambda L w''(\Delta) - L^{-1}w(\Delta)^{-1}\Big(6(3C)^{\frac{1}{2}}\delta n\Lambda K^{\frac{1}{2}}\Delta^{\frac{\mu}{2}}Lw'(\Delta) \\ &\qquad + 12\delta CK\Lambda\Delta^{\mu}\Big) \\ &\geq L\Delta^{a-2}\Big(3(a-1)\lambda - 12(3C)^{\frac{1}{2}}\delta n\Lambda K^{\frac{1}{2}}L^{-1}\Delta^{1-a+\frac{\mu}{2}} \\ &\qquad - 24\delta CK\Lambda L^{-2}\Delta^{1-a+\mu}\Big) \\ &\geq \frac{3(a-1)\lambda}{2}L, \end{aligned}$$

for sufficiently large L, provided

$$1-a+\frac{\mu}{2}\geq 0 \qquad \text{and} \qquad 1-a+\mu\geq 0.$$

Note that we used $\Delta \leq 1$. Now, choose $a = \frac{5}{4}$ and $\mu = \frac{3}{4}$, and take the limit as $\eta \to 0$ and L sufficiently large, to obtain a contradiction. □

The next result is an improvement of [5, Theorem 2].

Theorem 3.3. *Let $v \in C(Q_1)$ be a positive viscosity solution to (3.5) in Q_1. Then, there exist r_0, $C > 0$, depending only on $n, \lambda, \Lambda, \gamma$ and $\|v\|_{L^\infty(Q_1)}$, such that*

$$|v(x,t)-v(x,s)| \leq C|t-s|^{\frac{1}{2}},$$

for every $x \in B_{\frac{1}{2}}$ and $t, s \in (-r_0, 0]$.

Proof. Upon construction of a proper barrier function and application of the comparison principle, the conclusion follows from Theorem 3.2 and [5, Lemma 2]. □

Proof of Theorem 3.1. Once Theorem 3.2 and Theorem 3.3 are proven, the remaining parts of the proof are similar to those in [5], where v was shown to exhibit $C^{1-,1/2-}$-regularity. Here, we have established $C^{1,1/2}$-regularity, an improvement enabling us to achieve the optimal regularity. Since only minor modifications are required, we omit further details.

Remark 3.2. *If we consider a variable coefficient operator $F = F(M, x, t)$, we can obtain the same result under the assumption that, for some $\mu > 0$, there exists a μ-Hölder modulus of continuity $\tilde{\omega}$ such that*

$$|F(M,x,t)-F(M,y,t)| \leq \tilde{\omega}(|x-y|)\|M\|.$$

4. Degenerate elliptic case with varying singularities

In this section, we examine the degenerate elliptic problem

$$\begin{cases} |Du|^{\kappa(x)}F(D^2u) = \gamma(x)u^{\gamma(x)-1} & \text{in } \Omega\cap\{u>0\}, \\ u=\varphi & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

for varying exponents $\kappa(\cdot)$ and $\gamma(\cdot)$, respectively, of degeneracy and singularity. In addition to (A1) and (A2), we also assume the following extra hypothesis.

(A3): The functions $\kappa,\gamma:\Omega\to\mathbb{R}$ are continuous and there exist constants κ_1, γ_0 and γ_1 such that, for all $x\in\Omega$,

$$0\le\kappa(x)\le\kappa_1 \qquad \text{and} \qquad 0<\gamma_0\le\gamma(x)\le\gamma_1<1.$$

Moreover, there exists a modulus of continuity $\overline{\omega}$ such that

$$\limsup_{t\to 0+}\overline{\omega}(t)\log\left(\frac{1}{t}\right)\le C,$$

for a constant $C>0$, and

$$|\kappa(x)-\kappa(y)|\le\overline{\omega}(|x-y|),$$
$$|\gamma(x)-\gamma(y)|\le\overline{\omega}(|x-y|),$$

for all $x,y\in\Omega$.

The second main result of this paper is the following.

Theorem 4.1. *Let $\varphi\in C(\partial\Omega)$, with $\varphi\ge0$, and assume (A1)–(A3). There exists a nonnegative bounded viscosity solution u to (4.1), and for each $x\in\Omega$, u is of class $C^{1,\alpha}$ at x for every*

$$\alpha\in\left(0,\frac{\gamma(x)}{\kappa(x)+2-\gamma(x)}\right]\cap(0,\alpha_F),$$

with the estimate

$$\sup_{y\in B_r(x)}|u(y)-u(x)-Du(x)\cdot(y-x)|\le Cr^{1+\alpha}, \tag{4.2}$$

for $B_r(x)\Subset\Omega$, where $C=C\left(n,\lambda,\Lambda,\gamma,\kappa,\alpha,\|u\|_{L^\infty}\right)$.

Moreover, for each free boundary point x, u is of class $C^{1,\frac{\gamma(x)}{\kappa(x)+2-\gamma(x)}}$ at x, with the estimate

$$\sup_{B_r(x)}u\le Cr^{1+\frac{\gamma(x)}{\kappa(x)+2-\gamma(x)}}, \tag{4.3}$$

for $B_r(x)\Subset\Omega$, where $C=C\left(n,\lambda,\Lambda,\gamma,\kappa,\|u\|_{L^\infty}\right)$.

Remark 4.1. *The constant α_F in the statement of the theorem denotes the optimal exponent associated with the $C^{1+\mu}$-regularity theory for solutions of F-harmonic functions, i.e., solutions of the equation $F(D^2h) = 0$. In the case of $C^{1+\mu}$â€“regularity of solutions to $F(D^2u) = f$, where f is a continuous function, we use an approximation lemma. Thus, we can establish regularity for every $0 < \mu < \alpha_F$, but not for $\mu = \alpha_F$ (see [11]).*

We will start by establishing the existence of a solution to (4.1). Similarly to Section 3, we first define, for $\epsilon \in (0, 1)$,

$$\beta_\epsilon(s) = \int_0^{\frac{s}{\epsilon^{1+\alpha}} - \sigma_0} \rho(\theta)\, d\theta,$$

where $\rho \in C^\infty(\mathbb{R})$ is a nonnegative smooth function with compact support in $[0, 1]$, satisfying $\int \rho = 1$, $\sigma_0 \in (0, 1)$ and

$$\alpha = \frac{\gamma_0}{\kappa_1 + 2 - \gamma_0}.$$

Note that $\beta_\epsilon(s) \to \chi_{\{s>0\}}$ as $\epsilon \to 0$.

Now, we analyze the penalized equation

$$\begin{cases} |Du_\epsilon|^{\kappa(x)} F(D^2 u_\epsilon) = \gamma(x)\beta_\epsilon(u_\epsilon) u_\epsilon^{\gamma(x)-1} & \text{in } \Omega, \\ u_\epsilon = \varphi_\epsilon & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

where $\varphi_\epsilon = \varphi + \epsilon^{1+\alpha}$.

Proposition 4.1. *For each $\epsilon \in (0, 1)$, there exists a viscosity solution u_ϵ to (4.4). Moreover, u_ϵ satisfies*

$$0 < u_\epsilon \leq \|\varphi\|_{L^\infty(\partial\Omega)} + 1 \quad \text{in } \Omega. \tag{4.5}$$

Proof. Let $(\eta_{1/n})_{n\in\mathbb{N}}$ be a standard mollifier, and define $\kappa_n := \kappa * \eta_{1/n}$, which is a smooth approximation of κ. We first prove that for each $\delta \in (0, 1)$ and $n \in \mathbb{N}$, there exists a viscosity solution $u_{\epsilon,n,\delta}$ to

$$\begin{cases} (\delta + |Du|)^{\kappa_n(x)} (F(D^2u) - \delta u) = \gamma(x)\beta_\epsilon(u) u^{\gamma(x)-1} & \text{in } \Omega, \\ u = \varphi_\epsilon & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

By Lemma 2.3, there exist a viscosity supersolution $\overline{u}$ to

$$(\delta + |Du|)^{\kappa_n(x)} (F(D^2u) - \delta u) = 0 \quad \text{in } \Omega$$

and a viscosity subsolution $\underline{u}$ to

$$(\delta + |Du|)^{\kappa_n(x)} (F(D^2u) - \delta u) = \gamma_1 (\sigma_0 \epsilon^{1+\alpha})^{\gamma_0 - 1} \quad \text{in } \Omega$$

with $\overline{u} = \underline{u} = \varphi_\epsilon$ on $\partial\Omega$. Since κ_n is Lipschitz continuous, Lemma 2.4 applies. Therefore, we have $\underline{u} \leq \overline{u}$ in Ω. Since

$$0 \leq \gamma(x)\beta_\epsilon(u) u^{\gamma(x)-1} \leq \gamma_1 (\sigma_0 \epsilon^{1+\alpha})^{\gamma_0 - 1} \quad \text{in } \Omega,$$

it follows that $\overline{u}$ is a viscosity supersolution and $\underline{u}$ is a viscosity subsolution to (4.6), respectively. Then, by applying Lemma 2.2 with Remark 2.2, Lemma 2.4 and Lemma 2.5, there exists a viscosity solution $u_{\epsilon,n,\delta}$ to (4.6) such that

$$\underline{u} \leq u_{\epsilon,n,\delta} \leq \overline{u} \quad \text{in } \Omega. \tag{4.7}$$

Also, note that the constant function $\|\varphi_\epsilon\|_{L^\infty(\partial\Omega)}$ is a viscosity supersolution to (4.6). Then, by Lemma 2.4, we obtain

$$u_{\epsilon,n,\delta} \leq \|\varphi_\epsilon\|_{L^\infty(\partial\Omega)} \leq \|\varphi\|_{L^\infty(\partial\Omega)} + 1 \quad \text{in } \Omega. \tag{4.8}$$

By (4.7) and Lemma 2.5, $(u_{\epsilon,n,\delta})_{n\in\mathbb{N},\,\delta\in(0,1)}$ is equibounded and equicontinuous. Hence, we apply the Arzelà–Ascoli Theorem to conclude that there exists $u_\epsilon \in C(\overline{\Omega})$ with $u_\epsilon = \varphi_\epsilon$ on $\partial\Omega$ such that $u_{\epsilon,n,\delta}$ converges to u_ϵ locally uniformly in Ω as $n \to \infty$ and $\delta \to 0$. Then, by the stability of viscosity solutions, u_ϵ is a viscosity solution to (4.4). Now, we prove (4.5). First, we claim that

$$u_\epsilon \geq \frac{\sigma_0}{2}\epsilon^{1+\alpha}.$$

Assume, for the sake of contradiction, the set

$$\mathcal{A} := \{x \in \Omega \mid u_\epsilon(x) < \frac{\sigma_0}{2}\epsilon^{1+\alpha}\},$$

is nonempty. Since

$$u_\epsilon = \varphi_\epsilon \geq \epsilon^{1+\alpha} > \frac{\sigma_0}{2}\epsilon^{1+\alpha} \quad \text{on } \partial\Omega,$$

we have

$$u_\epsilon \geq \frac{\sigma_0}{2}\epsilon^{1+\alpha} \quad \text{on } \partial\mathcal{A}.$$

From the definitions of β_ϵ and $\mathcal{A}$, u_ϵ is a viscosity solution to

$$|Du|^{\kappa(x)}F(D^2u) = 0 \quad \text{in } \mathcal{A}. \tag{4.9}$$

Then, by the cutting lemma([9, Lemma 5.1]), it follows that u_ϵ is also a viscosity solution to

$$F(D^2u) = 0 \quad \text{in } \mathcal{A}.$$

As a consequence, the maximum principle yields $u_\epsilon \geq \dfrac{\sigma_0}{2}\epsilon^{1+\alpha}$ in $\mathcal{A}$ which contradicts the definition of $\mathcal{A}$. On the other hand, from (4.8), we obtain $u_\epsilon \leq \|\varphi\|_{L^\infty(\partial\Omega)} + 1$ in Ω and (4.5) follows. □

As before, for simplicity of notation, we omit the subscript ϵ in u_ϵ from now on. We now consider the equation

$$|Du|^{\kappa(x)}F(D^2u) = h(x)\beta_\epsilon(u)u^{\gamma(x)-1} \quad \text{in } \Omega, \tag{4.10}$$

where h is a nonnegative function which is uniformly bounded by the universal constant $\overline{C}$.

Remark 4.2. *We will examine the scaling invariance of solutions to* (4.10)*. Let u be a positive viscosity solution to* (4.10) *in $B_R(x_0) \Subset \Omega$. Then, for parameters $R > 0$ and $A > 0$, the rescaled function*

$$\tilde{u}(x) = \frac{u(x_0 + Rx)}{A}$$

satisfies

$$|D\tilde{u}|^{\tilde{\kappa}(x)} F(D^2\tilde{u}) = \tilde{h}(x)\beta_{\tilde{\epsilon}}(\tilde{u})\tilde{u}^{\tilde{\gamma}(x)-1} \quad \text{in } B_1,$$

in the viscosity sense, where

$$\begin{aligned}
\tilde{\epsilon} &= \epsilon A^{-\frac{1}{1+\alpha}};\\
\tilde{\kappa}(x) &= \kappa(x_0 + Rx);\\
\tilde{\gamma}(x) &= \gamma(x_0 + Rx);\\
\tilde{h}(x) &= \frac{R^{\kappa(x_0+Rx)+2}}{A^{\kappa(x_0+Rx)+2-\gamma(x_0+Rx)}} h(x_0 + Rx).
\end{aligned}$$

Note that, as a consequence of (A3), $\tilde{h}$ is a nonnegative function, uniformly bounded by a constant that depends only on the universal constants, A and R.

Let us now denote

$$\alpha(x) := \frac{\gamma(x)}{\kappa(x) + 2 - \gamma(x)}. \tag{4.11}$$

To obtain the optimal growth of the solution, we first prove the following estimate.

Theorem 4.2. *Let α be as in* (4.11)*. Let $u \in C(B_1)$ be a positive viscosity solution to* (4.10) *in B_1. Then, for each $\mu \in (0, 1)$, there exists $C > 0$, depending only on $n, \lambda, \Lambda, \gamma, \kappa, \mu$ and $\|u\|_{L^\infty(B_1)}$, such that*

$$\left| u(x)^{\frac{1}{1+\inf\limits_{B_1}\alpha}} - u(y)^{\frac{1}{1+\inf\limits_{B_1}\alpha}} \right| \leq C|x - y|^{\mu}, \tag{4.12}$$

for every $x, y \in B_{\frac{1}{2}}$.

Proof. Denote

$$\alpha_0 := \inf_{B_1} \alpha \ \in \ \left[\tfrac{\gamma_0}{\kappa_1+2-\gamma_0}, \tfrac{\gamma_1}{2-\gamma_1}\right],$$

and let $v = u^{\frac{1}{1+\alpha_0}}$. Applying (A2), we rewrite (4.10) as

$$|Dv|^{\kappa(x)} F(D^2v + \alpha_0 v^{-1} Dv \otimes Dv) = \overline{h}(x)\beta_\epsilon(v^{1+\alpha_0})v^{\tilde{\alpha}(x)} \quad \text{in } B_1, \tag{4.13}$$

where

$$\tilde{\alpha}(x) = (1 + \alpha_0)(\gamma(x) - \kappa(x) - 2) + \kappa(x) + 1$$

and

$$\overline{h}(x) = \left(\frac{1}{1+\alpha_0}\right)^{\kappa(x)+1} h(x).$$

Note that

$$-1 \leq \tilde{\alpha}(x) \leq 0 \quad \text{in } B_1, \tag{4.14}$$

and $\overline{h}$ is a nonnegative function, uniformly bounded by $\overline{C}$. Denote also

$$f(x) := \overline{h}(x)\beta_\epsilon\left(v(x)^{1+\alpha_0}\right),$$

which is a nonnegative bounded function.

Defining

$$\Phi(x,y) = v(x) - v(y) - L|x-y|^\mu - K(|x|^2+|y|^2) \quad \text{in } B_1,$$

we will prove that

$$\max_{\overline{B_{\frac{1}{2}}}\times\overline{B_{\frac{1}{2}}}} \Phi \leq 0, \tag{4.15}$$

for sufficiently large L and K, thus obtaining (4.12). To obtain (4.15), assume that Φ attains its positive maximum at $(\overline{x},\overline{y}) \in \overline{B_{\frac{1}{2}}} \times \overline{B_{\frac{1}{2}}}$. This implies that

$$\overline{x} \neq \overline{y}, \qquad v(\overline{x}) > v(\overline{y}) + L|\overline{x}-\overline{y}|^\mu,$$

and

$$L|\overline{x}-\overline{y}|^\mu + K(|\overline{x}|^2+|\overline{y}|^2) \leq 2\|v\|_\infty. \tag{4.16}$$

From (4.16), by choosing K sufficiently large, we ensure that

$$(\overline{x},\overline{y}) \in B_{\frac{1}{4}} \times B_{\frac{1}{4}}.$$

Now, we can obtain $p \in \mathbb{R}^n$ and $X, Y \in \mathcal{S}^n$, such that

$$(p_x, X) \in \overline{\mathcal{P}}^+(v)(\overline{x}), \tag{4.17}$$

$$(p_y, Y) \in \overline{\mathcal{P}}^-(v)(\overline{y}), \tag{4.18}$$

where $p_x = p + 2K\overline{x}$ and $p_y = p - 2K\overline{y}$, with the estimate given by Lemma 2.1. We can choose L sufficiently large so that

$$1 \leq \frac{1}{2}\mu L \leq \frac{1}{2}\mu L|\overline{x}-\overline{y}|^{\mu-1} \leq |p_x|, |p_y| \leq \frac{3}{2}\mu L|\overline{x}-\overline{y}|^{\mu-1}. \tag{4.19}$$

By applying (4.17) and (4.18) to the equation (4.13), we obtain the inequalities

$$|p_x|^{\kappa(\overline{x})} F\big(X + \alpha_0 v(\overline{x})^{-1} p_x \otimes p_x\big) \geq f(\overline{x}) v(\overline{x})^{\tilde{\alpha}(\overline{x})},$$

and

$$|p_y|^{\kappa(\overline{y})} F\big(Y + \alpha_0 v(\overline{y})^{-1} p_y \otimes p_y\big) \leq f(\overline{y}) v(\overline{y})^{\tilde{\alpha}(\overline{y})}.$$

Then, we get

$$\begin{aligned}&F\big(Y+\alpha_0 v(\overline{y})^{-1}p_y\otimes p_y\big)-F\big(X+\alpha_0 v(\overline{x})^{-1}p_x\otimes p_x\big)\\&\le |p_y|^{-\kappa(\overline{y})}f(\overline{y})v(\overline{y})^{\tilde{\alpha}(\overline{y})}-|p_x|^{-\kappa(\overline{x})}f(\overline{x})v(\overline{x})^{\tilde{\alpha}(\overline{x})}.\end{aligned}\tag{4.20}$$

On the other hand, for every $\eta>0$, there exists $M_\eta\in\mathcal{A}_{\lambda,\Lambda}$ such that

$$\begin{aligned}&F(Y+\alpha_0 v(\overline{y})^{-1}p_y\otimes p_y)-F(X+\alpha_0 v(\overline{x})^{-1}p_x\otimes p_x)\\&\ge \operatorname{tr}(M_\eta(Y+\alpha_0 v(\overline{y})^{-1}p_y\otimes p_y-X-\alpha_0 v(\overline{x})^{-1}p_x\otimes p_x))-\eta.\end{aligned}\tag{4.21}$$

From (4.20) and (4.21), we obtain

$$\begin{aligned}\eta\ge{}&\operatorname{tr}(M_\eta(Y-X))\\&+v(\overline{y})^{-1}\Big(\alpha_0\operatorname{tr}(M_\eta p_y\otimes p_y)-|p_y|^{-\kappa(\overline{y})}f(\overline{y})v(\overline{y})^{\tilde{\alpha}(\overline{y})+1}\Big)\\&-v(\overline{x})^{-1}\Big(\alpha_0\operatorname{tr}(M_\eta p_x\otimes p_x)-|p_x|^{-\kappa(\overline{x})}f(\overline{x})v(\overline{x})^{\tilde{\alpha}(\overline{x})+1}\Big),\end{aligned}\tag{4.22}$$

and, from Lemma 2.1,

$$\begin{aligned}\operatorname{tr}(M_\eta(Y-X))&\ge\lambda\Big(4\mu(1-\mu)L|\overline{x}-\overline{y}|^{\mu-2}-(4K+2\iota)\Big)\\&\quad-\Lambda(n-1)(4K+2\iota)\\&\ge 4\lambda\mu(1-\mu)L|\overline{x}-\overline{y}|^{\mu-2}\\&\quad-(\lambda+\Lambda(n-1))(4K+2\iota)\\&\ge 3\lambda\mu(1-\mu)L|\overline{x}-\overline{y}|^{\mu-2},\end{aligned}\tag{4.23}$$

for sufficiently large L. From (4.14) and (4.19), we get

$$\begin{aligned}&\alpha_0\operatorname{tr}(M_\eta p_y\otimes p_y)-|p_y|^{-\kappa(\overline{y})}f(\overline{y})v(\overline{y})^{\tilde{\alpha}(\overline{y})+1}\\&\ge\frac{\gamma_0}{\kappa_1+2-\gamma_0}\lambda|p_y|^2-\overline{C}\max\left(1,\|v\|_{L^\infty}\right)\\&\ge\frac{\gamma_0}{\kappa_1+2-\gamma_0}\lambda\left(\frac{1}{2}\mu L\right)^2-\overline{C}\max\left(1,\|v\|_{L^\infty}\right)\\&>0,\end{aligned}\tag{4.24}$$

for sufficiently large L. Using that $v(\overline{y})^{-1} > v(\overline{x})^{-1}$, from (4.19), (4.22), (4.23) and (4.24), we obtain

$$\begin{aligned}
\eta &\geq \operatorname{tr}(M_\eta(Y-X)) \\
&\quad + v(\overline{x})^{-1}\Big(\alpha_0 \operatorname{tr}(M_\eta p_y \otimes p_y) - |p_y|^{-\kappa(\overline{y})} f(\overline{y}) v(\overline{y})^{\tilde{\alpha}(\overline{y})+1}\Big) \\
&\quad - v(\overline{x})^{-1}\Big(\alpha_0 \operatorname{tr}(M_\eta p_x \otimes p_x) - |p_x|^{-\kappa(\overline{x})} f(\overline{x}) v(\overline{x})^{\tilde{\alpha}(\overline{x})+1}\Big) \\
&\geq 3\lambda\mu(1-\mu)L|\overline{x}-\overline{y}|^{\mu-2} \\
&\quad - v(\overline{x})^{-1}\Big\{2\alpha_0 n\Lambda|p_y||p_y-p_x| + \alpha_0\Lambda|p_y-p_x|^2 + |p_y|^{-\kappa(\overline{y})} f(\overline{y}) \\
&\quad v(\overline{y})^{\tilde{\alpha}(\overline{y})+1}\Big\} \\
&\geq 3\lambda\mu(1-\mu)L|\overline{x}-\overline{y}|^{\mu-2} \\
&\quad - v(\overline{x})^{-1}\Big\{3\alpha_0 n\Lambda K\mu L|\overline{x}-\overline{y}|^{\mu-1} + \alpha_0\Lambda K^2 + \overline{C}\max(1, \|v\|_{L^\infty})\Big\}.
\end{aligned}$$

Finally, since $v(\overline{x})^{-1} < L^{-1}|\overline{x}-\overline{y}|^{-\mu}$ and $|\overline{x}-\overline{y}| \leq 1$, we get

$$\begin{aligned}
\eta &\geq 3\lambda\mu(1-\mu)L|\overline{x}-\overline{y}|^{\mu-2} \\
&\quad - L^{-1}|\overline{x}-\overline{y}|^{-\mu}\Big(3\alpha_0 n\Lambda K\mu L|\overline{x}-\overline{y}|^{\mu-1} + \alpha_0\Lambda K^2 + C\Big) \\
&\geq L|\overline{x}-\overline{y}|^{\mu-2}\Big\{3\lambda\mu(1-\mu) - 3\alpha_0 n\Lambda K\mu L^{-1}|\overline{x}-\overline{y}|^{1-\mu} \\
&\quad - (\alpha_0\Lambda K^2 + C)L^{-2}|\overline{x}-\overline{y}|^{2-2\mu}\Big\} \\
&\geq \lambda\mu(1-\mu)L,
\end{aligned}$$

for sufficiently large L. By taking the limit as $\eta \to 0$, we obtain a contradiction and conclude the proof. □

Remark 4.3. *Let $R \leq 1$ and $u \in C(B_R(x_0))$ be a viscosity solution to*

$$|Du|^{\kappa(x)} F(D^2u) = h(x)\beta_\epsilon(u)u^{\gamma(x)-1} \quad \text{in } B_R(x_0).$$

Then, for $\mu \in (0,1)$, it follows from Theorem 4.2 that

$$\left| u(x)^{\frac{1}{1+\inf_{B_R(x_0)} \alpha}} - u(y)^{\frac{1}{1+\inf_{B_R(x_0)} \alpha}} \right| \leq C|x-y|^\mu, \tag{4.25}$$

for every $x, y \in B_{\frac{R}{2}}(x_0)$. Indeed, denote $\overline{\alpha} := \inf_{B_R(x_0)} \alpha$ and let

$$\tilde{u}(x) = \frac{u(x_0+Rx)}{R^{1+\overline{\alpha}}} \quad \text{in } B_1.$$

By Remark 4.2, $\tilde{u}$ satisfies

$$|D\tilde{u}|^{\tilde{\kappa}(x)} F(D^2\tilde{u}) = \tilde{h}(x)\beta_{\tilde{\epsilon}}(\tilde{u})\tilde{u}^{\tilde{\gamma}(x)-1} \quad \text{in } B_1,$$

in the viscosity sense, where

$$\begin{aligned}\tilde{\epsilon} &= \epsilon R^{-\frac{1+\overline{\alpha}}{1+\alpha}};\\ \tilde{\kappa}(x) &= \kappa(x_0+Rx);\\ \tilde{\gamma}(x) &= \gamma(x_0+Rx);\\ \tilde{h}(x) &= R^{\gamma(x_0+Rx)-\overline{\alpha}(\kappa(x_0+Rx)+2-\gamma(x_0+Rx))}h(x_0+Rx),\end{aligned}$$

for $x\in B_1$. *For* $0\leq\beta\leq 1+\alpha(x_0)$, *defining in* B_1

$$h_\beta(x) = R^{\kappa(x_0+Rx)+2-\beta(\kappa(x_0+Rx)+2-\gamma(x_0+Rx))}h(x_0+Rx),$$

we have that h_β *is uniformly bounded. Indeed, from*

$$\gamma(x_0)-\alpha(x_0)(\kappa(x_0)+2-\gamma(x_0))=0$$

and (A3), we have

$$\begin{aligned}&R^{\kappa(x_0+Rx)+2-\beta(\kappa(x_0+Rx)+2-\gamma(x_0+Rx))}\\ &\leq R^{\gamma(x_0+Rx)-\alpha(x_0)(\kappa(x_0+Rx)+2-\gamma(x_0+Rx))}\\ &= R^{\gamma(x_0+Rx)-\gamma(x_0)+\alpha(x_0)(\kappa(x_0)-\kappa(x_0+Rx)+\gamma(x_0+Rx)-\gamma(x_0))}\\ &\leq R^{-3\overline{\omega}(R)}\\ &\leq C,\end{aligned}$$

in B_1. *The function* $\tilde{h}$ *corresponds to the case* $\beta=1+\overline{\alpha}$. *Then, applying Theorem 4.2 to* $\tilde{u}$, *we obtain*

$$\left|\tilde{u}(x)^{\frac{1}{1+\inf_{B_1}\tilde{\alpha}}}-\tilde{u}(y)^{\frac{1}{1+\inf_{B_1}\tilde{\alpha}}}\right|\leq C|x-y|^\mu, \tag{4.26}$$

for every $x,y\in B_{\frac{1}{2}}$, *where*

$$\tilde{\alpha}(x)=\frac{\tilde{\gamma}(x)}{\tilde{\kappa}(x)+2-\tilde{\gamma}(x)}=\alpha(x_0+Rx).$$

Note that $\inf_{B_1}\tilde{\alpha}=\overline{\alpha}$. *Hence,* (4.26) *implies* (4.25).

We can now prove the optimal growth of the solution using Theorem 4.2.

Theorem 4.3. *Let* α *be as in* (4.11). *Let* u *be a positive viscosity solution to* (4.10) *in* B_1. *There exists a universal constant* $C>0$, *depending only on* n, λ, Λ, γ, κ *and* $\|u\|_{L^\infty(B_1)}$, *but not depending on* ϵ, *such that*

$$\sup_{B_r} u \leq C\left(u(0)+r^{1+\alpha(0)}\right), \tag{4.27}$$

for every $r\leq\dfrac{1}{2}$.

Proof. Let us first observe that to establish (4.27), it suffices to show

$$\sup_{B_r} u \le C\left(u(0) + r^{1+\inf\limits_{B_{2r}} \alpha}\right), \tag{4.28}$$

for every $r \le 1/2$. Indeed, by the Mean Value Theorem, together with (A3), we obtain, for $x, y \in B_1$,

$$\begin{aligned}
|\alpha(x) - \alpha(y)| &\le \left|\tfrac{\gamma(x)}{\kappa(x)+2-\gamma(x)} - \tfrac{\gamma(x)}{\kappa(y)+2-\gamma(x)}\right| + \left|\tfrac{\gamma(x)}{\kappa(y)+2-\gamma(x)} - \tfrac{\gamma(y)}{\kappa(y)+2-\gamma(y)}\right| \\
&\le \sup_{t\ge 0} \tfrac{\gamma(x)}{(t+2-\gamma(x))^2} |\kappa(x) - \kappa(y)| + \sup_{0\le s\le 1} \tfrac{\kappa(y)+2}{(\kappa(y)+2-s)^2} |\gamma(x) - \gamma(y)| \\
&\le |\kappa(x) - \kappa(y)| + 2|\gamma(x) - \gamma(y)| \\
&\le 3\overline{\omega}(|x-y|).
\end{aligned} \tag{4.29}$$

Using (4.29) and again (A3), we conclude

$$\begin{aligned}
r^{1+\inf\limits_{B_{2r}} \alpha} &\le r^{1+\alpha(0)} r^{-3\overline{\omega}(2r)} \\
&\le C r^{1+\alpha(0)}.
\end{aligned}$$

To prove (4.28), we assume, by contradiction, that for every integer l, there exist F_l, u_l, κ_l, γ_l, h_l, ϵ_l and $r_l \le 1/2$ such that

$$|Du_l|^{\kappa_l(x)} F_l(D^2 u_l) = h_l(x) \beta_{\epsilon_l}(u_l) u_l^{\gamma_l(x)-1} \quad \text{in } B_1,$$

in the viscosity sense,

$$0 \le h_l \le \overline{C} \quad \text{in } B_1$$

but

$$s_l := \sup_{B_{r_l}} u_l > l\left(u_l(0) + r_l^{1+\inf\limits_{B_{2r_l}} \alpha_l}\right). \tag{4.30}$$

Denote $\overline{\alpha_l} = \inf\limits_{B_{2r_l}} \alpha_l$ and define w_l by

$$w_l(x) := \frac{u_l(r_l x)}{s_l}.$$

Then, by Remark 4.2, w_l satisfies

$$|Dw_l|^{\tilde{\kappa}_l(x)} F_l(D^2 w_l) = \tilde{h}_l(x) \beta_{\tilde{\epsilon}_l}(w_l) w_l^{\tilde{\gamma}_l(x)-1} \quad \text{in } B_1, \tag{4.31}$$

in the viscosity sense, where, for $x \in B_1$,

$$\begin{aligned} \tilde{\epsilon}_l &= \epsilon_l s_l^{-\frac{1}{1+\alpha}};\\ \tilde{\kappa}_l(x) &= \kappa_l(r_l x);\\ \tilde{\gamma}_l(x) &= \gamma_l(r_l x);\\ \tilde{h}_l(x) &= \frac{r_l^{\kappa_l(r_l x)+2}}{s_l^{\kappa_l(r_l x)+2-\gamma_l(r_l x)}} h_l(r_l x). \end{aligned}$$

Also, from (4.29) and (4.30), we have

$$w_l(0) = o(1), \qquad \sup_{B_1} w_l = 1, \tag{4.32}$$

and

$$\frac{r_l^{1+\alpha_l(r_l x)}}{s_l} \le \frac{r_l^{1+\overline{\alpha_l}}}{s_l} \le \frac{1}{l} \to 0 \quad \text{as} \quad l \to \infty, \tag{4.33}$$

in B_1. Using (4.33) and

$$\kappa_l(r_l x) + 2 - \gamma_l(r_l x) \ge 1 \quad \text{in } B_1,$$

we obtain

$$\frac{r_l^{\kappa_l(r_l x)+2}}{s_l^{\kappa_l(r_l x)+2-\gamma_l(r_l x)}} = \left(\frac{r_l^{1+\alpha_l(r_l x)}}{s_l}\right)^{\kappa_l(r_l x)+2-\gamma_l(r_l x)} \le \frac{1}{l} \to 0 \quad \text{as} \quad l \to \infty, \tag{4.34}$$

in B_1. Note that, from (4.34),

$$\tilde{h}_l \to 0 \quad \text{as} \quad l \to \infty. \tag{4.35}$$

By Theorem 4.2 and Remark 4.3, along with (4.32) and (4.35), for $0 < \mu < 1$, we have that $\{w_l\}_l$ is equicontinuous and

$$\begin{aligned} \sup_{B_{\frac{R}{2}}(x_0)} w_l &\le \left(w_l(x_0)^{\frac{1}{1+\inf_{B_R(x_0)} \tilde{\alpha}_l}} + CR^\mu \right)^{1+\inf_{B_R(x_0)} \tilde{\alpha}_l} \\ &\le C\left(w_l(x_0)^{\frac{2-\gamma_1}{2}} + R^\mu \right)^{1+\frac{\gamma_0}{\kappa_1+2-\gamma_0}}, \end{aligned} \tag{4.36}$$

for $B_R(x_0) \subset B_1$. Note that we used

$$\frac{\gamma_0}{\kappa_1 + 2 - \gamma_0} \le \tilde{\alpha}_l \le \frac{\gamma_1}{2-\gamma_1}.$$

From

$$\begin{aligned} |\tilde{\kappa}_l(x) - \tilde{\kappa}_l(y)| &= |\kappa_l(r_l x) - \kappa_l(r_l y)| \\ &\le \overline{\omega}(|r_l(x-y)|) \\ &\le \overline{\omega}(|x-y|), \end{aligned}$$

and

$$\begin{aligned}|\tilde{\gamma}_l(x)-\tilde{\gamma}_l(y)| &= |\gamma_l(r_l x)-\gamma_l(r_l y)|\\ &\le \overline{\omega}(|r_l(x-y)|)\\ &\le \overline{\omega}(|x-y|),\end{aligned}$$

for $x, y \in B_1$, we know that also $\{\tilde{\kappa}_l\}_l$, $\{\tilde{\gamma}_l\}_l$ are equicontinuous. Then, by the Arzelà–Ascoli Theorem, the equicontinuity of $\{\tilde{\kappa}_l\}_l$, $\{\tilde{\gamma}_l\}_l$ and $\{w_l\}_l$, combined with (A3) and (4.32), implies the existence of $\tilde{\kappa}_0, \tilde{\gamma}_0, w_0 \in C(B_1)$ such that, up to a subsequence,

$$\tilde{\kappa}_l \to \tilde{\kappa}_0, \qquad \tilde{\gamma}_l \to \tilde{\gamma}_0, \qquad w_l \to w_0,$$

locally uniformly in B_1. Furthermore, by (A1), possibly after passing to a subsequence, F_l converges locally uniformly to F_0, which satisfies (A1). We can rewrite (4.31) as

$$w_l^{1-\tilde{\gamma}_l(x)}|Dw_l|^{\tilde{\kappa}_l(x)}F_l(D^2 w_l) = \tilde{h}_l(x)\beta_{\tilde{\epsilon}_l}(w_l) \quad \text{in } B_1,$$

in the viscosity sense. From stability of viscosity solutions and (4.35), w_0 satisfies

$$w_0^{1-\tilde{\gamma}_0(x)}|Dw_0|^{\tilde{\kappa}_0(x)}F_0(D^2 w_0) = 0 \quad \text{in } B_1.$$

Note that, by the cutting lemma ([9, Lemma 5.1]), we have

$$F_0(D^2 w_0) = 0 \quad \text{in } \{w_0 > 0\} \cap B_1, \tag{4.37}$$

in the viscosity sense. By (4.36), we get

$$\sup_{B_{\frac{R}{2}}(x_0)} w_0 \le C\left(w_0(x_0)^{\frac{2-\gamma_1}{2}} + R^{\mu}\right)^{1+\frac{\gamma_0}{\kappa_1+2-\gamma_0}}, \tag{4.38}$$

for $0 < \mu < 1$ and $B_R(x_0) \subset B_1$. Furthermore, by (4.32), we obtain

$$w_0 \ge 0 \quad \text{in } B_1, \quad w_0(0) = 0 \quad \text{and} \quad \sup_{B_1} w_0 = 1. \tag{4.39}$$

From (4.39), there exist $z_+ \in \{w_0 > 0\} \cap B_1$ and $z_0 \in \{w_0 = 0\} \cap B_1$ such that

$$\operatorname{dist}(z_+, \{w_0 = 0\}) = |z_+ - z_0|.$$

By Hopf's lemma with (4.37), we obtain

$$\liminf_{h \to 0+} \frac{w_0(z_0 + h(z_+ - z_0)) - w_0(z_0)}{h} > 0. \tag{4.40}$$

On the other hand, by applying (4.38) with $x_0 = z_0$, we obtain

$$\sup_{B_{\frac{R}{2}}(z_0)} w_0 \le C R^{\mu\left(1+\frac{\gamma_0}{\kappa_1+2-\gamma_0}\right)},$$

for $0 < \mu < 1$ and sufficiently small $R > 0$. Then, choosing μ satisfying

$$\mu \left(1 + \tfrac{\gamma_0}{\kappa_1 + 2 - \gamma_0}\right) > 1,$$

we get

$$\begin{aligned}\limsup_{h \to 0+} \frac{w_0(z_0 + h(z_+ - z_0)) - w_0(z_0)}{h} &= \limsup_{h \to 0+} \frac{w_0(z_0 + h(z_+ - z_0))}{h} \\ &\leq \limsup_{h \to 0+} C h^{\mu\left(1 + \frac{\gamma_0}{\kappa_1 + 2 - \gamma_0}\right) - 1} \\ &= 0,\end{aligned}$$

which contradicts (4.40). □

Now, we will prove the Lipschitz continuity of the solution.

Proposition 4.2. *Let u be a positive viscosity solution to (4.10) in B_1. There exists a universal constant $C > 0$, depending only on $n, \lambda, \Lambda, \gamma, \kappa$ and $\|u\|_{L^\infty(B_1)}$, but not depending on ϵ, such that*

$$\sup_{B_r} |u(x) - u(0)| \leq Cr,$$

for every $r \leq \dfrac{1}{2}$.

Proof. Let $r \leq 1/2$. We will consider two cases based on the range of r in terms of $u(0) =: \theta$. By Remark 4.2, we may assume $\theta \leq 1/2$.

Case 1) $r \geq \theta$: by Theorem 4.3, we obtain

$$\sup_{B_r} u \leq C(\theta + r^{1+\alpha(0)}) \leq C(\theta + r) \leq 2Cr,$$

so we have

$$\sup_{B_r} |u(x) - u(0)| \leq \sup_{B_r} u + u(0) \leq 2Cr + \theta \leq (2C+1)r. \tag{4.41}$$

Case 2) $0 < r < \theta$: define $w(x) = \dfrac{u(\theta x)}{\theta}$ in B_1. Then, by Remark 4.2, w satisfies

$$|Dw|^{\tilde{\kappa}(x)} F(D^2 w) = \tilde{h}(x) \beta_{\tilde{\epsilon}}(w) w^{\tilde{\gamma}(x) - 1} \quad \text{in } B_1, \tag{4.42}$$

in the viscosity sense, where, for $x \in B_1$,

$$\begin{aligned}\tilde{\epsilon} &= \epsilon \theta^{-\frac{1}{1+\alpha}}; \\ \tilde{\kappa}(x) &= \kappa(\theta x); \\ \tilde{\gamma}(x) &= \gamma(\theta x); \\ \tilde{h}(x) &= \theta^{\gamma(\theta x)} h(\theta x).\end{aligned}$$

Note that

$$w(x)=\frac{u(\theta x)}{\theta}\le\frac{\sup\limits_{B_\theta} u}{\theta}\le\frac{C(u(0)+\theta)}{\theta}=2C\quad\text{in } B_1, \tag{4.43}$$

which follows from Theorem 4.3. We also have

$$\frac{\gamma_1}{2-\gamma_1}\ge\inf_{B_1}\tilde{\alpha}=\inf_{B_\theta}\alpha=:\alpha_\theta,$$

where

$$\tilde{\alpha}(x)=\frac{\tilde{\gamma}(x)}{\tilde{\kappa}(x)+2-\tilde{\gamma}(x)}.$$

Applying Theorem 4.2 to w, with $\mu=\frac{1}{2}$, we obtain

$$|w(x)^{\frac{1}{1+\alpha_\theta}}-w(0)^{\frac{1}{1+\alpha_\theta}}|=|w(x)^{\frac{1}{1+\alpha_\theta}}-1|\le C|x|^{\frac{1}{2}}\quad\text{in } B_{\frac{1}{2}},$$

which in turn implies

$$w(x)\ge\left(1-C|x|^{\frac{1}{2}}\right)^{1+\alpha_\theta}\ge\left(1-C|x|^{\frac{1}{2}}\right)^{\frac{2}{2-\gamma_1}}\quad\text{in } B_{\frac{1}{2}}.$$

Then,

$$w\ge\frac{1}{2}\quad\text{in}\quad B_{\delta_0}, \tag{4.44}$$

for a universal constant $\delta_0>0$. By (4.43) and (4.44), the right-hand side of (4.42) is bounded by

$$\overline{C}\left(\frac{1}{2}\right)^{\gamma_0-1}\quad\text{in } B_{\delta_0}.$$

The regularity result from [9] implies that

$$\sup_{B_r}|w(x)-w(0)|\le Cr,$$

for $0<r\le\frac{\delta_0}{2}$. By scaling back, we obtain

$$\sup_{B_r}|u(x)-u(0)|\le Cr,$$

for $0<r\le\frac{\theta\delta_0}{2}$.

For $\frac{\theta\delta_0}{2}<r<\theta$, using (4.41), we get

$$\begin{aligned}\sup_{B_r}|u(x)-u(0)| &\le \sup_{B_\theta}|u(x)-u(0)|\\ &\le (2C+1)\theta\\ &\le \frac{2}{\delta_0}(2C+1)r.\end{aligned}$$

□

As a consequence of Proposition 4.2, we derive a gradient bound for the solution, which is instrumental in establishing its sharp local regularity.

Proposition 4.3. *Let u be a positive viscosity solution to (4.10) in B_1. There exists a universal constant $C > 0$, depending only on $n, \lambda, \Lambda, \gamma, \kappa$ and $\|u\|_{L^\infty(B_1)}$, but not depending on ϵ, such that*

$$|Du(x)| \leq Cu(x)^{\frac{\gamma(x)}{\kappa(x)+2}} \quad \textit{in } B_{\frac{1}{2}}.$$

Proof. Let $x_0 \in B_{\frac{1}{2}}$ and

$$r_0 = \left(\frac{u(x_0)}{M}\right)^{\frac{1}{1+\alpha(x_0)}},$$

where M is a constant chosen such that $r_0 \leq \dfrac{1}{4}$. Define

$$w(x) := \frac{u(x_0 + r_0 x)}{r_0^{1+\alpha(x_0)}} \quad \text{in } B_1.$$

Then, by Remark 4.2, w satisfies

$$|Dw|^{\tilde{\kappa}(x)} F(D^2 w) = \tilde{h}(x)\beta_{\tilde{\epsilon}}(w) w^{\tilde{\gamma}(x)-1} \quad \text{in } B_1,$$

in the viscosity sense, where, for $x \in B_1$,

$$\begin{aligned}
\tilde{\epsilon} &= \epsilon r_0^{-\frac{1+\alpha(x_0)}{1+\alpha}};\\
\tilde{\kappa}(x) &= \kappa(x_0 + r_0 x);\\
\tilde{\gamma}(x) &= \gamma(x_0 + r_0 x);\\
\tilde{h}(x) &= r_0^{\gamma(x_0+r_0x)-\alpha(x_0)(\kappa(x_0+r_0x)+2-\gamma(x_0+r_0x))} h(x_0 + r_0 x).
\end{aligned}$$

Recall that, by Remark 4.3, $\tilde{h}$ is uniformly bounded.
Using Theorem 4.3, we get

$$\sup_{B_1} w = \sup_{B_1} \frac{u(x_0 + r_0 x)}{r_0^{1+\alpha(x_0)}} \leq \frac{C(u(x_0) + r_0^{1+\alpha(x_0)})}{r_0^{1+\alpha(x_0)}} = C(M+1).$$

Therefore, applying Proposition 4.2, we conclude

$$\begin{aligned}
|Du(x_0)| &= r_0^{\alpha(x_0)} |Dw(0)|\\
&\leq C r_0^{\alpha(x_0)}\\
&= C\left(\frac{u(x_0)}{M}\right)^{\frac{\alpha(x_0)}{1+\alpha(x_0)}}\\
&= \tilde{C} u(x_0)^{\frac{\gamma(x_0)}{\kappa(x_0)+2}}.
\end{aligned}$$

□

Using the optimal growth and gradient bound for the solutions, we obtain the following sharp local estimates, uniform in ϵ.

Theorem 4.4. *Let α be as in (4.11). Let u be a positive viscosity solution to (4.10) in B_1. For $x_0 \in B_{\frac{1}{2}}$ and $\beta \in (0, \alpha(x_0)] \cap (0, \alpha_F)$, there exists a universal constant $C > 0$, depending only on $n, \lambda, \Lambda, \gamma, \kappa, \beta$ and $\|u\|_{L^\infty(B_1)}$, and independent of ϵ, such that*

$$\sup_{x \in B_r(x_0)} |u(x) - u(x_0) - Du(x_0) \cdot (x - x_0)| \leq C r^{1+\beta}, \tag{4.45}$$

for every $r \leq \dfrac{1}{4}$.

Proof. Fix $x_0 \in B_{\frac{1}{2}}$, $\beta \in (0, \alpha(x_0)] \cap (0, \alpha_F)$ and $r \leq \dfrac{1}{4}$. Let

$$r_0 = \left(\frac{u(x_0)}{M}\right)^{\frac{1}{1+\beta}},$$

where $M > 1$ is a constant chosen in such a way that $r_0 \leq \dfrac{1}{4}$. We will consider two cases based on the range of r in terms of r_0.

Case 1) $r \geq r_0$: by the definition of r_0, $u(x_0) = M r_0^{1+\beta} \leq M r^{1+\beta}$; then, applying Theorem 4.3 and Proposition 4.3, we obtain

$$\begin{aligned}
&\sup_{x \in B_r(x_0)} \big|u(x) - u(x_0) - Du(x_0) \cdot (x - x_0)\big| \\
&\qquad \leq C\big(u(x_0) + r^{1+\alpha(x_0)}\big) + u(x_0) + C r u(x_0)^{\frac{\gamma(x_0)}{\kappa(x_0)+2}} \\
&\qquad \leq C\Big\{M r^{1+\beta} + r^{1+\alpha(x_0)} + r (M r^{1+\beta})^{\frac{\gamma(x_0)}{\kappa(x_0)+2}}\Big\} \\
&\qquad \leq C r^{1+\beta}.
\end{aligned} \tag{4.46}$$

Note that we used that

$$\begin{aligned}
1 + (1+\beta)\frac{\gamma(x_0)}{\kappa(x_0)+2} - (1+\beta) &= 1 - (1+\beta)\frac{\kappa(x_0) + 2 - \gamma(x_0)}{\kappa(x_0)+2} \\
&= 1 - (1+\beta)\frac{1}{1+\alpha(x_0)} \\
&\geq 0.
\end{aligned}$$

Case 2) $0 < r < r_0$: define

$$w(x) := \frac{u(x_0 + r_0 x)}{r_0^{1+\beta}} \quad \text{in } B_1.$$

Then, by Remark 4.2, w satisfies

$$|Dw|^{\tilde{\kappa}(x)} F(D^2 w) = \tilde{h}(x) \beta_{\tilde{\epsilon}}(w) w^{\tilde{\gamma}(x)-1} \quad \text{in } B_1, \tag{4.47}$$

in the viscosity sense, where, for $x \in B_1$,

$$\begin{aligned} \tilde{\epsilon} &= \epsilon r_0^{-\frac{1+\beta}{1+\alpha}}; \\ \tilde{\kappa}(x) &= \kappa(x_0 + r_0 x); \\ \tilde{\gamma}(x) &= \gamma(x_0 + r_0 x); \\ \tilde{h}(x) &= r_0^{\gamma(x_0+r_0x)-\beta(\kappa(x_0+r_0x)+2-\gamma(x_0+r_0x))} h(x_0 + r_0 x). \end{aligned}$$

Recall that, by Remark 4.3, $\tilde{h}$ is uniformly bounded.
Applying Theorem 4.3, we get

$$\begin{aligned} \sup_{B_1} w = \sup_{B_1} \frac{u(x_0 + r_0 x)}{r_0^{1+\beta}} &\leq \frac{C(u(x_0) + r_0^{1+\alpha(x_0)})}{r_0^{1+\beta}} \\ &\leq C(M + r_0^{\alpha(x_0)-\beta}) \\ &\leq C(M + 1). \end{aligned} \tag{4.48}$$

The definition of r_0 implies that $w(0) = M > 1$, thus, using Proposition 4.2, we obtain

$$w \geq \frac{1}{2} \text{ in } B_{\delta_0}, \tag{4.49}$$

for a universal constant $\delta_0 > 0$. By (4.48) and (4.49), the right-hand side of (4.47) is universally bounded in B_{δ_0}. Since

$$\beta \leq \alpha(x_0) = \frac{\gamma(x_0)}{\kappa(x_0) + 2 - \gamma(x_0)} = \frac{\tilde{\gamma}(0)}{\tilde{\kappa}(0) + 2 - \tilde{\gamma}(0)} < \frac{1}{\tilde{\kappa}(0) + 1},$$

the regularity result from [9] implies that there exists a universal constant $C > 0$ such that

$$\sup_{B_r} |w(x) - w(0) - Dw(0) \cdot x| \leq C r^{1+\beta},$$

for $0 < r \leq \dfrac{\delta_0}{2}$. By scaling back, we have

$$\sup_{x \in B_r(x_0)} |u(x) - u(x_0) - Du(x_0) \cdot (x - x_0)| \leq C r^{1+\beta},$$

for $0 < r \leq \dfrac{r_0 \delta_0}{2}$.

For $\frac{r_0\delta_0}{2} < r < r_0$, from (4.46), we obtain

$$\begin{aligned}
\sup_{x\in B_r(x_0)} & \left|u(x) - u(x_0) - Du(x_0)\cdot(x-x_0)\right| \\
&\le \sup_{x\in B_{r_0}(x_0)} \left|u(x) - u(x_0) - Du(x_0)\cdot(x-x_0)\right| \\
&\le C r_0^{1+\beta} \\
&\le C\left(\frac{2}{\delta_0}\right)^{1+\beta} r^{1+\beta}.
\end{aligned}$$

□

Combining the previous results through the limiting process, we finally establish the existence and sharp local regularity of viscosity solutions to equation (1.2).

Proof of Theorem 4.1. By Proposition 4.1, the sequence $\{u_\epsilon\}_\epsilon$ is uniformly bounded, and by Proposition 4.2, it is equicontinuous. Therefore, by the Arzelà–Ascoli Theorem, there exists a continuous function u such that, up to a subsequence, u_ϵ converges locally uniformly to u. By the properties of u_ϵ, the limit function u is nonnegative and bounded. Now, we show that u is a viscosity solution to (1.2). For $x \in \{u > 0\} \cap \Omega$, the continuity of u implies that $u > u(x)/2$ in $B_\delta(x)$, for some $\delta > 0$. Then by the uniform convergence of u_ϵ to u, we obtain

$$u_\epsilon > u(x)/4 > (\sigma_0 + 1)\epsilon^{1+\alpha} \quad \text{in } B_\delta(x),$$

for sufficiently small ϵ. By the definition of β_ϵ, we know that u_ϵ satisfies

$$|Du_\epsilon|^{\kappa(x)} F(D^2 u_\epsilon) = \gamma(x) u_\epsilon^{\gamma(x)-1} \text{ in } B_\delta(x),$$

in the viscosity sense. Taking the limit as $\epsilon \to 0$ and using the stability of viscosity solutions, we conclude that u is a viscosity solution to (1.2). The regularity along the free boundary with the estimate (4.3) follows from (4.27) and the limiting process. Similarly, the local regularity result with estimate (4.2) follows from (4.45) and the limiting process.

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