Minimization of the first eigenvalue for the Lamé system

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Abstract

We study the minimization, under a volume constraint, of the first eigenvalue of the Lamé system in $\mathbb{R}^N$. We first prove the existence of an optimal domain in the class of quasi-open sets and show that, in the physically relevant dimensions $N = 2$ and $N = 3$, every optimal domain is open. We then derive first and second-order optimality conditions. In dimension two, these conditions imply that the disk is not optimal when the Poisson ratio is below a certain threshold, whereas it is a local minimizer above this threshold. We further prove that the disk is not optimal for all Poisson ratios ν satisfying $\nu \leq 0.4$.

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Contents

1 Introduction

The Faber–Krahn inequality is one of the most fundamental results in spectral geometry. It states that, among all sets of a given volume (in any dimension), the ball uniquely minimizes the first eigenvalue of the Dirichlet–Laplacian, see [24], [34]. Similar results are known for other boundary conditions. For instance:

- the ball maximizes the first non-trivial eigenvalue of the Neumann–Laplacian, by the Szegö–Weinberger inequality [43, 44];
- the ball minimizes the first eigenvalue of the Robin–Laplacian when the boundary parameter is positive, by the Bossel–Daners inequality [2, 17];
- the ball maximizes the first non-trivial eigenvalue of the Steklov–Laplacian, by Brock's inequality [5].

In all of these problems, a volume constraint is imposed. For further discussion on eigenvalue optimization problems, see [29], [30].

For systems, the corresponding minimization and maximization problems are much less understood. In contrast with the scalar cases recalled above, the ball need not be an extremal domain. This issue has recently been studied for the Stokes operator. In three dimensions, the ball is known not to minimize the first eigenvalue among sets of prescribed volume. In two dimensions, however, the disk is a local minimizer and is conjectured to be a global minimizer. A numerical investigation of this problem is given in [38].

Another operator that has received considerable attention is the curl operator; see [12, 13, 26, 23, 22]. These works study several properties of possible optimal domains and show, in particular, that the ball does not minimize the first eigenvalue. In [23], the authors prove, under suitable regularity assumptions, that an optimal domain cannot be axially symmetric. It is conjectured in [12] that the optimal domain is a *spheromak*, namely a torus in $\mathbb{R}^3$ whose central hole is shrunk to form an almost spherical shape, often compared to a cored apple.

In the recent paper [35], the authors investigate the Maxwell operator (or vectorial Laplacian) with the boundary condition $u \times \nu = 0$. They demonstrate that, in three dimensions, the ball is neither a minimizer nor a maximizer for the first eigenvalue under volume or perimeter constraints. Specifically, the authors show that the infimum of the first eigenvalue is zero, while the supremum is $+\infty$ under both constraints.

In this article, we focus on the Lamé system with Dirichlet boundary conditions, a fundamental model in linear elasticity. Let Ω be a bounded open set in $\mathbb{R}^N$. We denote by $H_0^1(\Omega)^N$ the space of vector fields $u = (u_1, \dots, u_N)$ such that each component u_i belongs to the Sobolev space $H_0^1(\Omega)$.

The first eigenvalue of Ω for the Lamé system is defined by

$$\Lambda(\Omega) := \min_{u\in H^1_0(\Omega)^N\setminus\{0\}} \frac{\mu\int_\Omega |\nabla u|^2\,dx + (\lambda+\mu)\int_\Omega (\mathrm{div}(u))^2\,dx}{\int_\Omega |u|^2\,dx}, \tag{1.1}$$

where λ, μ are the Lamé coefficients that satisfy $\mu > 0, \lambda + \mu > 0$. In the previous expression, $|\nabla u|^2$ denotes $|\nabla u_1|^2 + \ldots |\nabla u_N|^2$ and $|u|^2$ denotes $u_1^2 + \ldots u_N^2$. The associated PDE solved by the minimizer u is

$$\begin{cases} -\mu\Delta u - (\lambda+\mu)\nabla(\mathrm{div}(u)) = \Lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We will explain below that a natural motivation for introducing the first eigenvalue Λ arises from the famous Korn inequality. In that context, let us mention the paper [37] where optimal constants for the Korn inequality are also computed but under tangential boundary conditions.

It is convenient to introduce the Poisson ratio ν, which is related to the Lamé parameters by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{1.3}$$

where E is the Young modulus and $\nu \in (-1, 0.5)$ (for many materials $\nu \in [0.2, 0.4]$). Dividing Λ by μ leads to the ratio

$$\frac{\lambda+\mu}{\mu} = \frac{1}{1-2\nu},$$

so that the minimization problem essentially depends on the Poisson ratio ν. In some works, such as [33], the eigenvalue is written as

$$\Lambda(\Omega, a) := \min_{u\in H^1_0(\Omega)^N\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx + a\int_\Omega (\mathrm{div}(u))^2\,dx}{\int_\Omega |u|^2\,dx}, \tag{1.4}$$

where a stands for $1/(1-2\nu)$. To make the underlying physics more apparent, we will explicitly retain the Lamé parameters and the Poisson coefficient in our paper.

Thus, this paper is dedicated to the study of the following shape optimization problem:

$$\boxed{\inf\{\Lambda(\Omega), \Omega \subset \mathbb{R}^N \text{ bounded }, |\Omega| = V_0\}} \tag{1.5}$$

or equivalently (since $\Lambda(t\Omega) = \Lambda(\Omega)/t^2$) to the unconstrained optimization problem

$$\inf\{|\Omega|^{2/N}\Lambda(\Omega), \Omega \subset \mathbb{R}^N\}. \tag{1.6}$$

Here $|\Omega|$ denotes the Lebesgue measure of the open set (or quasi-open set) Ω. The precise definition of quasi-open set will be given at the beginning of Section 3.

In Section 3, we first prove an existence result in the class of quasi-open sets, in any dimension. The proof follows a concentration-compactness strategy, classical in that context, adapted here to the vectorial setting. We then obtain a regularity result: in dimensions 2 and 3, every optimal domain is open. This is proved by relating the constrained minimization problem to a penalized one and by introducing Lamé quasi-minimizers, for which we establish global Hölder continuity. Note that this regularity result has been recently generalized by R. Frank, see Remark 3.1. Our first main result is therefore the following.

Theorem 1.1. *There exists a quasi-open set* Ω^* *solution of* (1.5) *or* (1.6)*. Moreover in dimension* $N = 2$ *and* $N = 3$ *this set is open and any eigenfunction associated with the eigenvalue* $\Lambda(\Omega^*)$ *belongs to* $\mathscr{C}^{0,\alpha}(\mathbb{R}^N)$ *for all* $\alpha < 1$ *if* $N = 2$*, and for all* $\alpha < \frac{1}{2}$ *if* $N = 3$*.*

In Section 4, we derive first and second order optimality conditions by calculating the first and second shape derivatives of the eigenvalue. These computations prove to be particularly useful in the subsequent Section 5, where we examine the potential optimality of the disk in two dimensions. In this context, we are able to prove:

Theorem 1.2. *If the Poisson coefficient* ν *is less than* 0.4*, the disk* **is not** *the minimizer of* Λ *(among sets of given area).*

The proof involves several steps. First, we explicitly compute the first eigenvalue of the disk, which is a non-trivial task, and show that if $\nu \leq 0.349\ldots$, the first eigenvalue is double. This finding allows us, through a straightforward variational argument, to conclude that the disk cannot be optimal in this case. It is worth noting that the value $0.349\ldots$ is explicitly related to the first zero of the Bessel function J_1 and its derivative.

Next, in Section 6.1, we identify explicit rhombi that yield a better first eigenvalue than the disk for the range $0.349\ldots \leq \nu \leq 0.3878\ldots$. We further extend our analysis by considering suitable rectangles in Section 6.2.

For these rectangles, the first eigenvalue cannot be computed explicitly, but suitable test functions provide sharp enough upper bounds to rule out the disk as a minimizer for $\nu \leq 0.4$. For $0.4 < \nu < 0.5$, our analytical arguments do not settle the question of global minimality. However, in Section 5.3, we prove that, once the first eigenvalue is simple, namely for $\nu > 0.349\ldots$, the disk is a local minimizer. This follows from the non-negativity of the second shape derivative, together with an estimate of the associated quadratic form in terms of the H^1-norm of the perturbation.

Finally, we present heuristic arguments suggesting that there exists a threshold ν^* such that the disk could be a minimizer when $\nu^* \leq \nu < 0.5$. This conclusion is based on the property that the Lamé eigenvalue Γ-converges to the Stokes eigenvalue as $\nu \to 1/2$, in conjunction with the previously established local minimality of the disk.

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2 Motivation and elementary comparisons

2.1 Reminders on the Korn inequalities

Let Ω denote any bounded open set in $\mathbb{R}^N$. For $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$ we denote by $e(u)$ the symmetric gradient defined by

$$e(u) := \frac{\nabla u + \nabla u^T}{2}.$$

Let us recall two standard Korn inequalities.

Theorem 2.1. *For all $u \in H_0^1(\Omega)^N$, one has*

$$\|\nabla u\|_{L^2(\Omega)} \leq 2\|e(u)\|_{L^2(\Omega)}, \tag{Korn}$$

and

$$\|u\|_{L^2(\Omega)} \leq C(\Omega)\|e(u)\|_{L^2(\Omega)}. \tag{Poincaré-Korn}$$

Moreover we can take $C(\Omega) = 2/\lambda_1^D(\Omega)$, where $\lambda_1^D(\Omega)$ is the usual scalar first eigenvalue for the Dirichlet Laplacian.

Proof. A straightforward integration by parts gives, for every $u \in C_c^\infty(\Omega, \mathbb{R}^N)$,

$$\int_\Omega |e(u)|^2 dx = \frac{1}{2}\left(\int_\Omega |\nabla u|^2 + (\mathrm{div}(u))^2\right). \tag{2.1}$$

Inequality (Korn) follows immediately. Notice that here the constant does not depend on Ω.

To prove (Poincaré-Korn), we combine (Korn) with the Poincaré inequality, applied to a scalar function $v \in H_0^1(\Omega)$:

$$\lambda_1(\Omega)\int_\Omega v^2\, dx \leq \int_\Omega |\nabla v|^2\, dx.$$

We deduce that, for $u = (u_1, u_2, \ldots u_N)$:

$$\begin{aligned}\frac{\int_\Omega |e(u)|^2\, dx}{\int_\Omega u^2\, dx} &= \frac{\frac{1}{2}\left(\int_\Omega |\nabla u|^2 + (\mathrm{div}(u))^2\right)}{\int_\Omega \sum_{i=1}^N (u_i)^2} \geq \frac{\frac{1}{2}\left(\int_\Omega \sum_{i=1}^N |\nabla u_i|^2\right)}{\int_\Omega \sum_{i=1}^N (u_i)^2}\\ &\geq \min_{i=1,2,\ldots N}\left(\frac{\frac{1}{2}\int_\Omega |\nabla u_i|^2}{\int_\Omega (u_i)^2}\right) \geq \frac{1}{2}\lambda_1(\Omega).\end{aligned}$$

□

Therefore, looking at the best constant in the (Poincaré-Korn) leads us to compute the eigenvalue Λ defined in (1.1) for the particular choice $\mu = 1/2$ and $\lambda = 0$.

2.2 Link with other eigenvalues

2.2.1 Link with the eigenvalues of the Stokes operator

For Ω, a bounded open set, let us introduce the so-called *Dirichlet Stokes first eigenvalue* $\lambda_1^{\text{Stokes}}(\Omega)$ by

$$\lambda_1^{\text{Stokes}}(\Omega) := \inf_{\substack{u \in (H_0^1(\Omega))^N \setminus \{0\} \\ \text{div}(u)=0 \text{ in } \Omega}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}.$$

We immediately deduce:

$$\mu \lambda_1^{\text{Stokes}}(\Omega) \geq \Lambda(\Omega). \tag{2.2}$$

Indeed, the minimization defining $\lambda_1^{\text{Stokes}}(\Omega)$ is performed over a subspace of $H_0^1(\Omega)^N$ on which the divergence term vanishes.

In some sense, the divergence term may appear as a penalization term, in particular when the Poisson coefficient goes to 1/2 (or equivalently when the Lamé coefficients are such that $(\lambda+\mu)/\mu \to +\infty$). We will make this more precise in Section 7 by proving the strong convergence of the Lamé operator to the Stokes operator when $\nu \to 1/2$. This will be done by means of Γ-convergence.

2.2.2 Comparison with Dirichlet Eigenvalues

We recall here some results already obtained, for instance, in [33]. Throughout this subsection, $\lambda_1^D(\Omega)$ denotes the first eigenvalue of the Dirichlet–Laplacian.

Proposition 2.1. *For any bounded domain Ω, one has*

$$\mu \lambda_1^D(\Omega) < \Lambda(\Omega) \leq \frac{(\lambda + (N+1)\mu)}{N} \lambda_1^D(\Omega). \tag{2.3}$$

Moreover,

$$\inf_\Omega \frac{\Lambda(\Omega)}{\lambda_1^D(\Omega)} = \mu,$$

and is achieved by a sequence of thin cuboids shrinking to a line.

Remark 2.1. *From the left inequality in* (2.3) *and the famous Faber-Krahn inequality, we observe that*

$$|\Omega|^{2/N} \Lambda(\Omega) \geq \mu\, |\Omega|^{2/N} \lambda_1^D(\Omega) \geq \mu j_{N/2-1,1}^2$$

where $j_{N/2-1,1}$ is the first zero of the Bessel function $J_{N/2-1}$. Thus we see that the infimum in (1.6) *is strictly positive.*

Proof. The inequality

$$\mu \lambda_1^D(\Omega) \leq \Lambda(\Omega)$$

follows the chain of inequalities that appear in the proof of Theorem 2.1 (multiplied by μ instead of 1/2).

Now we demonstrate that the inequality must be strict. Indeed, assuming that $\mu\lambda_1^D(\Omega) = \Lambda(\Omega)$ and applying the previously established chain of inequalities, we observe that each u_i must be a Dirichlet eigenfunction associated with $\lambda_1^D(\Omega)$. Since $\text{div}(u) = 0$ as well, Lemma 2.1 below gives a contradiction. This proves the strict inequality.

We now prove the upper bound. For that purpose we consider u_1 being the (normalized) Dirichlet eigenfunction associated to $\lambda_1(\Omega)$ and we consider the vector test functions $(0, 0, \dots, u_1, 0, \dots)$ composed of null functions except u_1 in i-th position. Then

$$\Lambda(\Omega) \leq \frac{\mu \int_\Omega |\nabla u_1|^2 \, dx + (\lambda + \mu) \int_\Omega (\partial_i u_1)^2 \, dx}{\int_\Omega u_1^2 \, dx}.$$

Summing these N inequalities yields

$$N\Lambda(\Omega) \leq N\mu \int_\Omega |\nabla u_1|^2 \, dx + (\lambda + \mu) \int_\Omega |\nabla u_1|^2 \, dx = (\lambda + (N+1)\mu)\lambda_1^D,$$

which finishes the proof of (2.3).

Let us now prove the last assertion. For that purpose, we consider the cuboid $\Omega_L = (0, L) \times \prod_{i=2}^N (0, 1)$ and take a first Dirichlet eigenfunction of Ω_L namely

$$u_1(X) = \sin\left(\pi \frac{x_1}{L}\right) \prod_{i=2}^N \sin(\pi x_i).$$

We will use the fact that

$$\lambda_1^D(\Omega_L) = \pi^2 (N - 1 + \frac{1}{L^2}).$$

Now, we plug in the Rayleigh quotient defining $\Lambda(\Omega_L)$ the vector $u = (u_1, 0, \dots, 0)$. Since

$$\int_{\Omega_L} |u|^2 \, dx = \frac{L}{2^N}$$

$$\int_{\Omega_L} |\nabla u|^2 \, dx = \int_{\Omega_L} |\nabla u_1|^2 \, dx = \frac{L\pi^2}{2^N} (N - 1 + \frac{1}{L^2}) = \frac{L}{2^N} \lambda_1^D(\Omega_L)$$

$$\int_{\Omega_L} (\operatorname{div}(u))^2 \, dx = \frac{\pi^2}{L^2} \frac{L}{2^N}$$

we deduce that

$$\Lambda(\Omega_L) \leq \frac{\mu \frac{L}{2^N} \lambda_1^D(\Omega_L) + (\lambda + \mu) \frac{L}{2^N} \frac{\pi^2}{L^2}}{\frac{L}{2^N}}$$

or

$$\Lambda(\Omega_L) \leq \mu \lambda_1^D(\Omega_L) + (\lambda + \mu) \frac{L}{2^N} \frac{\pi^2}{L^2}$$

and finally letting $L \to +\infty$ we conclude that

$$\inf_\Omega \frac{\Lambda(\Omega)}{\lambda_1^D(\Omega)} = \mu,$$

as claimed in the proposition. □

Lemma 2.1. *Let $u = (u_1, u_2, \dots, u_N)$ be a N-uple of functions in $H_0^1(\Omega)$. Then, if $\operatorname{div}(u) = 0$ and for all i, $u_i = \alpha_i u_1$ for some $\alpha_i \in \mathbb{R}$, it follows that u_1 and then all the u_i are identically zero. In particular, if Ω is connected and all the u_i are eigenfunctions associated to the first eigenvalue $\lambda_1^D(\Omega)$, it is not possible that $\operatorname{div}(u) = 0$.*

Proof. From the assumptions $\mathrm{div}(u) = 0$ and $u_i = \alpha_i u_1$ we deduce that u_1 satisfies

$$\frac{\partial u_1}{\partial x_1} + \sum_{i=2}^{N} \alpha_i \frac{\partial u_1}{\partial x_i} = 0$$

which means that u_1 must be constant on all affine lines directed by $(1, \alpha_2, \ldots, \alpha_N)$. Since all those lines touch the boundary of Ω, from the Dirichlet condition on u_1 we deduce that u_1 must be identically 0.
The last assertion comes from the fact that the first Dirichlet eigenvalue (for the Laplacian) of a connected domain is simple. □

3 Existence and regularity

3.1 Existence of an optimal quasi-open set

In this section, we fix the Lamé coefficients to be $\mu = 1/2$ and $\lambda = 0$, which corresponds to Korn's inequality. This choice does not affect the existence proof, since the general case only changes the energy by positive multiplicative constants, but it simplifies the notation.

We prove the existence of an optimal shape in the class of quasi-open sets, using a concentration-compactness strategy introduced by Lions [39]. This method has been used for shape optimization problems involving the Laplace operator, first by Bucur and later by several other authors (see [6, 8, 7, 20]). Recently, this strategy has also been applied to the Stokes operator [31]. We denote by $\mathrm{Cap}(A)$ the H^1-capacity of A (for instance the Bessel capacity $\mathrm{Cap}_{1,2}$). A set $A \subset \mathbb{R}^N$ is said to be quasi-open if, for every $\varepsilon > 0$ there exists an open set Ω_ε such that $A \subset \Omega_\varepsilon$ and $\mathrm{Cap}(\Omega_\varepsilon \setminus A) \leq \varepsilon$. We first introduce the class

$$\mathcal{O} := \{\Omega \subset \mathbb{R}^N \text{ quasi-open such that } 0 < |\Omega| < +\infty\}.$$

In the sequel the term *capacity* will always refer to the $\mathrm{Cap}_{1,2}$-capacity. Moreover, a property that holds outside a set of zero Capacity will be said to hold quasi-everywhere, or q.e. in short.

The space $H_0^1(\Omega)$ is defined as functions $u \in H^1(\mathbb{R}^N)$ such that $u = 0$ quasi-everywhere on Ω^c. Notice that a domain $\Omega \in \mathcal{O}$ is not necessarily bounded. However, the space $H_0^1(\Omega)$ is known to be a closed subspace of $H^1(\mathbb{R}^N)$ which is compactly embedded into $L^2(\mathbb{R}^N)$, when $|\Omega| < \infty$ (because by definition of being quasi-open there exists an open set E with $|E| < +\infty$ such that $\Omega \subset E$ thus $H_0^1(\Omega) \subset H_0^1(E)$ and the standard compact embedding of $H_0^1(E)$ into L^2 applies).

Notice also that thanks to Proposition 3.1 below, the space of all $u \in L^2(\Omega)^N$ such that $e(u) \in L^2(\Omega)$ and $u = 0$ q.e. in Ω^c coincides with the space $H_0^1(\Omega)^N$.

Then we can relax the definition of $\Lambda(\Omega)$ for $\Omega \in \mathcal{O}$ by considering

$$\Lambda(\Omega) := \min_{u \in H_0^1(\Omega)^N} \frac{\int_{\mathbb{R}^N} |e(u)|^2 \, dx}{\int_{\mathbb{R}^N} |u|^2 \, dx}.$$

Notice here that Ω is merely quasi-open and not necessarily open, but the definition coincides with the standard one when Ω is open.

Also, it is easy to check that the minimum in the definition of $\Lambda(\Omega)$ is achieved by an $H_0^1(\Omega)$ function, thanks to the compact embedding of $H_0^1(\Omega)$ into $L^2(\mathbb{R}^N)$ and the semicontinuity behavior of the convex functional $u \mapsto \int_{\mathbb{R}^N} |e(u)|^2 \, dx$ for the weak topology of H^1.

In the sequel we will need the famous Korn inequality but now in the whole $\mathbb{R}^N$, in particular valid for $u \in H^1_0(\Omega)^N$ with $\Omega \in \mathcal{O}$ quasi-open, as stated in the following proposition.

Proposition 3.1 (Korn inequality in $\mathbb{R}^N$)**.** *If $u \in L^2(\mathbb{R}^N)^N$ is such that $e(u) \in L^2(\mathbb{R}^N, \mathbb{R}^{N\times N})$, then $u \in H^1(\mathbb{R}^N)^N$ and*

$$2\int_{\mathbb{R}^N} |e(u)|^2 \, dx = \int_{\mathbb{R}^N} |\nabla u|^2 + (\mathrm{div}(u))^2 \, dx. \tag{3.1}$$

Proof. Let $R > 0$ be given and let $\varphi_R \in [0,1]$ be a cut-off function such that $\varphi_R = 1$ on $B(0,R)$, $\varphi_R = 0$ on $B(0,2R)^c$, and

$$|\nabla\varphi_R| \leq C\frac{1}{R}.$$

Then the function $u_R := \varphi_R u$ clearly belongs to $H^1_0(B(0,2R))^N$ and applying (2.1) we deduce that the equality in (3.1) holds true for the function $\varphi_R u$. For simplicity we will by now denote by φ the function φ_R. Notice that for a smooth function,

$$e(\varphi u)_{ij} = \frac{u^i\partial_j\varphi + u^j\partial_i\varphi}{2} + \varphi e(u)_{i,j},$$

so pointwisely in $\mathbb{R}^N$ it holds the following estimate

$$|e(u\varphi)| \leq |u||\nabla\varphi| + |\varphi||e(u)| \leq \frac{C}{R}|u| + |e(u)|,$$

$$|\mathrm{div}(u\varphi)| \leq \frac{C}{R}|u| + |\mathrm{div}(u)|,$$

$$|\nabla(u\varphi)| \leq \frac{C}{R}|u| + |\nabla(u)|.$$

The above remains true in H^1 by approximation by smooth functions. Recall also that $u\varphi = u$ in $B(0,R)$. Now applying (2.1) in $B(0,2R)$ to the function φu we obtain in particular the inequality

$$\int_{B(0,2R)} |\nabla(u\varphi)|^2 \, dx \leq \int_{B(0,2R)} |e(u\varphi)|^2 \, dx \leq C\int_{B(0,2R)} |e(u)|^2 + \frac{1}{R^2}|u|^2 \, dx,$$

which leads to

$$\int_{B(0,R)} |\nabla u|^2 \, dx \leq C\int_{B(0,2R)} |e(u)|^2 + \frac{1}{R^2}|u|^2 \, dx. \tag{3.2}$$

Thus passing (3.2) to the limit, and using that $e(u) \in L^2(\mathbb{R}^N)$ we can use Fatou lemma to get first $\nabla u \in L^2(\mathbb{R}^N)$, thus $u \in H^1(\mathbb{R}^N)^N$. Once this is known, we compute again more precisely to derive identity on $\mathbb{R}^N$. Indeed, we know that

$$2\int_{\mathbb{R}^N\cap B(0,R)} |e(u)|^2 \, dx = \int_{\mathbb{R}^N\cap B(0,R)} |\nabla u|^2 + (\mathrm{div}(u))^2 \, dx + E(R), \tag{3.3}$$

with

$$|E(R)| \leq \frac{C}{R^2} \int_{B(0,2R)\setminus B(0,R)} |u|^2 \, dx + C \int_{B(0,2R)\setminus B(0,R)} |Du|^2 \, dx.$$

We now let $R \to +\infty$ which yields,

$$|E(R)| \underset{R\to+\infty}{\longrightarrow} 0,$$

because we know that u and $\nabla u \in L^2(\mathbb{R}^N)$. The monotone convergence theorem allows to conclude that

$$2\int_{\mathbb{R}^N} |e(u)|^2 \, dx = \int_{\mathbb{R}^N} |\nabla u|^2 + (\mathrm{div}(u))^2 \, dx,$$

and the proposition follows. □

The purpose of this section is to prove the following result.

Theorem 3.1. *For all $V > 0$ there exists a solution for the problem*

$$\min_{\Omega\in\mathcal{O} \text{ such that } |\Omega|\leq V} \Lambda(\Omega).$$

Proof. The proof follows the same approach as in [31] reasoning on the scalar function $|u|$ and using the concentration-compactness strategy of Lions [39]. More precisely, we let Ω_k be a minimizing sequence with $|\Omega_k| \leq V$ and we consider $w_k := |u_k|$ where u_k is a chosen normalized eigenvector for $\Lambda(\Omega_k)$. In other words, $\|w_k\|_{L^2(\mathbb{R}^N)} = 1$ and by Proposition 3.1,

$$\int_{\mathbb{R}^N} |\nabla w_k|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla u_k|^2 \, dx \leq 2\int_{\mathbb{R}^N} |e(u_k)|^2 \, dx = 2\Lambda(\Omega_k) \leq C_0, \tag{3.4}$$

so that w_k is uniformly bounded in $H^1(\mathbb{R}^N)$. Let $Q_k : \mathbb{R}^+ \to \mathbb{R}^+$ be the sequence of concentration functions[1] defined by

$$Q_k(R) := \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |u_k|^2 \, dx.$$

Then Q_k is a sequence of nondecreasing functions on $\mathbb{R}$ which are uniformly bounded by 1. By Dini's theorem, up to extract a subsequence (not relabelled), $(Q_k)_k$ admits a pointwize limit function which is nondecreasing, bounded by 1, and that we denote $Q : \mathbb{R}^+ \to \mathbb{R}^+$. Then we let

$$\alpha := \lim_{R\to+\infty} Q(R) \in [0,1].$$

The value of α is usually referred to the "maximal concentration". Depending on the value of α, we know that one of the following occurs by the concentration-compactness principle of Lions [39, Lemma I.1].

- **If $\alpha = 1$: Compactness:** There exists a sequence $(y_k)_{k\in\mathbb{N}}$ such that $|w_k|^2(\cdot - y_k)$ is tight:

$$\forall \varepsilon > 0, \exists R < +\infty, \forall k \quad \int_{B(y_k,R)} w_k^2 \, dx \geq 1 - \varepsilon.$$

[1] According to Lions [39] this notion was first introduced by Lévy.

- **If $\alpha \in (0,1)$: Dichotomy:** There exist $(y_k)_{k\in\mathbb{N}}$ and two sequences of positive radii $(R_k)_k$, $(R'_k)_k$ satisfying
$$R'_k - R_k \to +\infty \text{ and } R_k, R'_k \to +\infty,$$
and such that
$$\int_{B(y_k,R_k)} w_k^2 \to \alpha, \;\; \int_{B(y_k,R'_k)^c} w_k^2 \to 1-\alpha. \tag{3.5}$$

- **If $\alpha = 0$: Vanishing.** For every $R > 0$,
$$\lim_{k\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} w_k^2 = 0.$$

As usual, our aim is to prove that only the compactness case can occur, by ruling out the two other cases. Let us first prove that the compactness situation implies the desired existence.

Step 1. *Compactness implies existence.* We consider the sequence of translated functions $u_k(\cdot - y_k)$ that we still denote by u_k. We know by assumption that this sequence is uniformly bounded in $H^1(\mathbb{R}^N)^N$ thus admits a weakly converging subsequence. Since $H^1(\mathbb{R}^N)^N$ is compactly embedded in $L^2_{loc}(\mathbb{R}^N)^N$, using a diagonal argument we can extract a subsequence (not relabelled) and a function $u \in L^2_{loc}(\mathbb{R}^N)^N$ such that $u_k \to u$ strongly in $L^2_{loc}(\mathbb{R}^N)^N$ and weakly in $H^1(\mathbb{R}^N)^N$. Now we use that $(w_k)_k$ is in the situation of compactness, and in particular for every $\varepsilon > 0$ there exists $R > 0$ such that
$$\forall k, \quad \int_{B_R} |u_k|^2 \, dx \geq 1 - \varepsilon.$$
Passing to the limit and using the convergence of u_k in $L^2(B_R)$ we deduce that $\int_{B_R} |u|^2 \, dx \geq 1 - \varepsilon$, which means in particular that
$$\int_{\mathbb{R}^N} |u|^2 \, dx \geq \; 1 - \varepsilon,$$
and since ε is arbitrary, we finally get $\int_{\mathbb{R}^N} |u|^2 \, dx \geq \; 1$. But of course the reverse is also true so in conclusion $\|u\|_{L^2(\mathbb{R}^N)} = 1$. But we already knew that u_k was converging weakly in $L^2(\mathbb{R}^N)$ to u. We just have proved that the sequence of norms are also converging so finally u_k converges strongly in $L^2(\mathbb{R}^N)$ to u. Passing to the limit in the Rayleigh quotient, strongly in L^2 for u_k and weakly in L^2 for $e(u_k)$ we deduce that
$$\inf_{\Omega\in\mathcal{O} \text{ such that } |\Omega|\leq V} \Lambda(\Omega) = \frac{\int_{\mathbb{R}^N} |e(u)|^2 \, dx}{\int_{\mathbb{R}^N} |u|^2 \, dx}. \tag{3.6}$$
Let us denote $\Omega = \{|u| > 0\}$, which is a quasi-open set, and from the equality in (3.6) we know that u must be an eigenfunction associated to $\Lambda(\Omega)$. Furthermore, since $\int_{\mathbb{R}^N} |u|^2 \, dx = 1$ we know that $|\Omega| > 0$. We can also assume that $|u_k|$ converges a.e. in $\mathbb{R}^N$ to $|u|$. This implies, for a.e. $x \in \mathbb{R}^N$,
$$\mathbb{1}_{\{|u|>0\}}(x) \leq \liminf_k \mathbb{1}_{\{|u_k|>0\}}(x),$$

and since $|\{|u_k| > 0\}| \le V$, we deduce by Fatou Lemma that $|\Omega| \le V$ and finally Ω is a solution.

Step 2. *Vanishing does not occur.* This case is easy to exclude by standard arguments. Indeed, Lemma 3.3 in [10] says that up to a subsequence, $w_k(\cdot - y_k)$ does not weakly converge in $H^1(\mathbb{R}^N)^N$. But this is a contradiction with the uniform bound in (3.4). Indeed, together with the fact that $\|w_k\|_2 = 1$, we obtain that that w_k is uniformly bounded in $H^1(\mathbb{R}^N)^N$ thus would admit a weakly converging subsequence, which is not possible.

Step 3. *Dichotomy cannot occur.* Assume that $(w_k)_{k\in\mathbb{N}}$ is in the dichotomy situation. Then the idea is to split the minimizing sequence in two disjoint pieces. For that purpose we define $\eta_k := (R'_k - R_k)/4$ and then we construct two cut-off functions: the first one $\varphi_{k,1}$ supported in $B(y_k, R_k + 2\eta_k)$ is such that $\varphi_{k,1} = 1$ in $B(y_k, R_k)$, and the second one $\varphi_{k,2}$ equal to 1 in $B(y_k, R'_k - \eta_n)^c$ and 0 on $B(y_k, R'_k - 2\eta_k)$ satisfying

$$|\nabla \varphi_{k,1}| + |\nabla \varphi_{k,2}| \le C/\eta_k \to 0.$$

Next, we define

$$v_{k,1} = \varphi_{k,1} u_k \quad \text{and} \quad v_{k,2} = \varphi_{k,2} u_k.$$

We want to prove that the sum $v_{k,1} + v_{k,2}$ has almost the same L^2 norm as the original function u_k because w_k is in a dichotomy situation. Let us define the annulus $A_k := B(y_k, R'_k) \backslash B(y_k, R_k)$. Because of (3.5) and the fact that $\|w_k\|_2 = 1$ for all k, we directly get

$$\int_{A_k} |w_k|^2 \, dx \to 0,$$

and since $|u_k| = |w_k|$ we also have for $i = 1, 2$,

$$\int_{A_k} |u_k \varphi_{k,i}|^2 \, dx \le \int_{A_k} |u_k|^2 \, dx = \int_{A_k} |w_k|^2 \, dx \to 0.$$

We deduce that

$$\int_{\mathbb{R}^N} |v_{k,1}|^2 \, dx = \int_{B(y_k,R_k)} |w_k|^2 \, dx + \int_{A_k} |u_k \varphi_{k,1}|^2 \to \alpha \tag{3.7}$$

$$\int_{\mathbb{R}^N} |v_{k,2}|^2 \, dx = \int_{B(y_k,R'_k)} |w_k|^2 \, dx + \int_{A_k} |u_k \varphi_{k,2}|^2 \to 1 - \alpha. \tag{3.8}$$

Then we want to estimate the difference of the symmetrized gradients. Recall that, as already used before, pointwisely in $\mathbb{R}^N$ it holds the following estimate

$$|e(u_k \varphi_{k,1})| \le |u||\nabla \varphi_{k,1}| + |\varphi_{k,1}||e(u_k)| \le \frac{1}{\eta_k}|u| + |e(u_k)|$$

and the same holds true for $v_{k,2}$,

$$|e(u_k \varphi_{k,2})| \le \frac{1}{\eta_k}|u| + |e(u_k)|.$$

Taking the square we get, for $i = 1, 2$,

$$|e(v_{k,i})|^2 \le \frac{1}{\eta_k^2}|u_k|^2 + 2\frac{1}{\eta_k}|u_k||e(u_k)| + |e(u_k)|^2. \tag{3.9}$$

Now remember that $v_{k,1}$ and $v_{k,2}$ have disjoint support, and that their sum coincide with u_k outside A_k, in which we can use (3.9) to estimate

$$\begin{aligned}\int_{\mathbb{R}^N} |e(u_k)|^2\,dx - \int_{\mathbb{R}^N} |e(v_{k,1})|^2\,dx &- \int_{\mathbb{R}^N} |e(v_{k,2})|^2\,dx \\ &\geq -2\int_{A_k} \frac{1}{\eta_k^2}|u_k|^2 + 2\frac{1}{\eta_k}|u_k||e(u_k)|\,dx. \qquad (3.10)\end{aligned}$$

Since $\frac{1}{\eta_k} \to 0$ and both u_k and $e(u_k)$ are uniformly bounded in L^2, we deduce that the term on the right-hand side converges to zero thus

$$\liminf_{k\to+\infty} \int_{\mathbb{R}^N} |e(u_k)|^2\,dx - \int_{\mathbb{R}^N} |e(v_{k,1})|^2\,dx - \int_{\mathbb{R}^N} |e(v_{k,2})|^2\,dx \geq 0. \qquad (3.11)$$

This allows to compare the Rayleigh quotient of u_k with the one of $v_{k,1} + v_{k,2}$. More precisely, using (3.11), the standard inequality on real nonnegative numbers a, b and positive numbers c, d,

$$\frac{a+b}{c+d} \geq \min\left\{\frac{a}{c}, \frac{b}{d}\right\},$$

and also (3.7) and (3.8), we obtain

$$\begin{aligned}\lambda^* := \inf_{|\Omega|\leq V} \Lambda(\Omega) &= \lim_{k\to+\infty} \int_{\mathbb{R}^N} |e(u_k)|^2\,dx \\ &\geq \liminf_{k\to+\infty} \int_{\mathbb{R}^N} |e(v_{k,1})|^2\,dx + \int_{\mathbb{R}^N} |e(v_{k,2})|^2\,dx \\ &= \liminf_{k\to+\infty} \frac{\int_{\mathbb{R}^N} |e(v_{k,1})|^2\,dx + \int_{\mathbb{R}^N} |e(v_{k,2})|^2\,dx}{\int_{\mathbb{R}^N} |v_{k,1}|^2\,dx + \int_{\mathbb{R}^N} |v_{k,2}|^2\,dx} \qquad (3.12) \\ &\geq \min\left\{\liminf_{k\to+\infty} \frac{\int_{\mathbb{R}^N} |e(v_{k,1})|^2\,dx}{\int_{\mathbb{R}^N} |v_{k,1}|^2\,dx}\,,\ \liminf_{k\to+\infty} \frac{\int_{\mathbb{R}^N} |e(v_{k,2})|^2\,dx}{\int_{\mathbb{R}^N} |v_{k,2}|^2\,dx}\right\}. \qquad (3.13)\end{aligned}$$

Notice that applying the concentration principle on the sequence v_k^1, we obtain that v_k^1 is in the compactness situation, with concentration value α. In particular, arguing as in the compactness case, we can assume that v_k^1 converges strongly in L^2 (and weakly in H^1) to a function $v \in H^1(\mathbb{R}^N)$. Then, if the minimum above is achieved for $v_{k,1}$, we deduce that the quasi-open set $\Omega^* = \{|v| > 0\}$ is an optimal domain, and the proof is concluded from the compactness situation. So we have to consider that it is not the case.

But then it means that v, being the L^2 limit of v_k^1, satisfies

$$\frac{\int_{\mathbb{R}^N} |e(v)|^2\,dx}{\int_{\mathbb{R}^N} |v|^2\,dx} > \lambda^*,$$

or differently,

$$\int_{\mathbb{R}^N} |e(v)|^2\,dx > \alpha\lambda^*.$$

We also know that

$$\liminf_{k\to+\infty} \frac{\int_{\mathbb{R}^N} |e(v_{k,2})|^2\,dx}{\int_{\mathbb{R}^N} |v_{k,2}|^2\,dx} = \lambda^*,$$

because by assumption the minimum in (3.13) is achieved with the sequence $v_{k,2}$, and since $\lim_{k\to+\infty}\int_{\mathbb{R}^N}|v_{k,2}|^2\,dx = 1-\alpha$ we deduce that

$$\liminf_{k\to+\infty}\int_{\mathbb{R}^N}|e(v_{k,2})|^2\,dx = (1-\alpha)\lambda^*.$$

Now returning back to (3.12), we have obtained

$$\lambda^* \geq \liminf_{k\to+\infty}\frac{\int_{\mathbb{R}^N}|e(v_{k,1})|^2\,dx + \int_{\mathbb{R}^N}|e(v_{k,2})|^2\,dx}{\int_{\mathbb{R}^N}|v_{k,1}|^2\,dx + \int_{\mathbb{R}^N}|v_{k,2}|^2\,dx} = \frac{\int_{\mathbb{R}^N}|e(v)|^2\,dx + \liminf_{k\to+\infty}\int_{\mathbb{R}^N}|e(v_{k,2})|^2\,dx}{\alpha+(1-\alpha)} > \lambda^*,$$

a contradiction. This achieves the proof of the Theorem. □

3.2 Regularity

The purpose of this section is to prove that any quasi-open solution of our shape optimization problem is in fact open. We obtain this conclusion for $N = 2$ and $N = 3$. The restriction comes from the need for an a priori L^p bound on an eigenfunction, which, to the best of our knowledge, is not available in arbitrary dimension; see also Remark 3.3 below.

Remark 3.1. *After completing the first version of this paper, Rupert Frank pointed out that the restriction $N = 2, 3$ can be removed. His approach differs from ours in two key respects. First, he incorporates the right-hand side f directly into the energy functional defining quasi-minimality, whereas in the present paper we treat f as an error term. Second, exploiting this quasi-minimality framework, he applies an elegant bootstrap argument, allowing f to be incorporated into the estimates and iterated to obtain higher regularity. A detailed proof for systems has recently appeared in [25].*

Remark 3.2. *From this section and the next ones, in order to lighten the notation, the superscript N on the functional spaces such as $H^1(\mathbb{R}^N)^N$, $C^{0,\alpha}(\mathbb{R}^N)^N$, $L^2(\mathbb{R}^N)^N$, etc., could be omitted for vector valued functions.*

Here is a general regularity result valid in any dimension.

Theorem 3.2. *Let $\Omega^* \subset \mathbb{R}^N$ be a quasi-open solution to the problem*

$$\min_{\Omega\in\mathcal{O}\ \textit{such that}\ |\Omega|\leq V}\Lambda(\Omega).$$

Assume moreover that an associated eigenfunction u belongs to $L^p(\mathbb{R}^N)$ with $p > N$. Then $u \in \mathscr{C}^{0,\alpha}(\mathbb{R}^N)$, for all $\alpha < 1-\frac{N}{p}$ and Ω^* is the positivity set of the modulus $|u|$. As a consequence, Ω^* is an open set.

In particular in dimension $N = 2$ and $N = 3$ we obtain the following.

Corollary 3.1. *Assume that $N = 2$ or $N = 3$. Then for all $V > 0$ there exists an open solution Ω^* for the problem*

$$\min\left\{\Lambda(\Omega)\ ,\ \Omega\subset\mathbb{R}^N\ \textit{open set such that}\ \ |\Omega|\leq V\right\}.$$

Moreover, any eigenfunction associated with $\Lambda(\Omega^*)$ belongs to $\mathscr{C}^{0,\alpha}(\mathbb{R}^N)$ for all $\alpha < 1$ if $N = 2$ and for all $\alpha < \frac{1}{2}$ if $N = 3$.

Proof. Let u be an eigenfunction associated with $\Lambda(\Omega^*)$. Since $u \in H^1(\mathbb{R}^N)$, by the Sobolev embedding we know that $u \in L^p(\mathbb{R}^N)$ with p arbitrary large for $N = 2$ or $p = 2^* = \frac{2N}{N-2}$ for $N > 2$. Then by Proposition 3.3 below we know that u is a Lamé quasi-minimizer with exponent $\gamma = N - \frac{2N}{p}$. To conclude that $u \in \mathscr{C}^{0,\alpha}$ we would need that $p > N$. This is true for $N = 2$ or $N = 3$. For $N = 2$ we deduce from Proposition 3.4 that $u \in \mathscr{C}^{0,\alpha}$ for all $\alpha < 1$. If $N = 3$ we deduce, from Proposition 3.4, that $u \in \mathscr{C}^{0,\alpha}$ for all $\alpha < \frac{1}{2}$. □

Remark 3.3. *Let us stress that the conclusion of Theorem 3.2 does not imply that $u \in L^\infty(\mathbb{R}^N)$. In other words by $u \in \mathscr{C}^{0,\alpha}(\mathbb{R}^N)$ we mean, that for a representative of u it holds $|u(x) - u(y)| \leq C|x - y|^\alpha$ which is enough to prove that u is continuous. Since Ω^* may not be a bounded set, we do not conclude that u is bounded. Let us mention that in the scalar case it is well known that any eigenfunction associated to the first eigenvalue of the Dirichlet Laplacian belongs to $L^\infty(\mathbb{R}^N)$ together with the following nice bound, for which one usually refers to [19, Example 2.1.8]:*

$$\|u\|_{L^\infty} \leq e^{\frac{1}{8\pi}} \lambda_1^D(\Omega)^{\frac{N}{4}} \|u\|_2.$$

It would be very interesting to know whether a similar bound is true for the Lamé eigenfunctions. This would directly imply the existence of an open solution in any dimension.

The proof of Theorem 3.2 is inspired by the seminal paper of Briançon, Hayouni and Pierre [3], and by later developments in related directions, see, for instance, [4, 9]. The general strategy is to show that a solution of the original problem also solves a penalized problem, and then to use regularity theory for free-boundary-type problems to obtain global Hölder continuity of the eigenfunction. In the present vectorial setting, this strategy requires a non-trivial adaptation because the energy involves the symmetrized gradient.

For instance, in [3], the first step relies on a truncated test function and the co-area formula. These tools are not available for the symmetric gradient, and we therefore need a different argument. As a result, we obtain Hölder continuity rather than Lipschitz regularity, which is nevertheless sufficient to prove that the optimal set is open.

3.2.1 Equivalence with a penalized problem

Proposition 3.2. *Let $V > 0$ be given and u be a solution for the problem*

$$\lambda_V := \min\left\{\int_{\mathbb{R}^N} |e(u)|^2 dx \quad s.t. \quad u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^2 = 1, \quad and \quad |\{|u| > 0\}| \leq V\right\}. \tag{3.14}$$

Then for all $\lambda \geq \frac{\lambda_V}{V}$, and all $v \in H^1(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} |e(u)|^2\, dx \leq \int_{\mathbb{R}^N} |e(v)|^2 dx + \lambda_V \left(1 - \int_{\mathbb{R}^N} |v|^2\right)^+ + \lambda\Big(|\{|v| > 0\}| - V\Big)^+. \tag{3.15}$$

Proof. For $v \in H^1(\mathbb{R}^N)$ and $\lambda > 0$ we introduce

$$F_\lambda(v) := \int_{\mathbb{R}^N} |e(v)|^2 dx + \lambda_V \left(1 - \int_{\mathbb{R}^N} |v|^2\right)^+ + \lambda\Big(|\{|v| > 0\}| - V\Big)^+.$$

Let u be a solution of the constrained problem (3.14), which exists thanks to Theorem 3.1. Since $|\{|u|>0\}|\le V$ and $\int_{\mathbb{R}^N}|u|^2=1$, we have

$$F_\lambda(u)=\int_{\mathbb{R}^N}|e(u)|^2=\lambda_V.$$

We will prove that, for $\lambda\ge\lambda_V/V$, one has

$$F_\lambda(u)\le F_\lambda(v)\qquad\text{for all } v\in H^1(\mathbb{R}^N),$$

which yields (3.15).

Step 1: the case $|\{|v|>0\}|\le V$. Let $a:=\int_{\mathbb{R}^N}|v|^2$. If $a=0$ then $F_\lambda(v)=\lambda_V$. Otherwise, setting $\tilde v:=v/\sqrt{a}$, we have $\int|\tilde v|^2=1$ and $|\{|\tilde v|>0\}|\le V$, hence $\tilde v$ is admissible for (3.14) and

$$\lambda_V\le\int_{\mathbb{R}^N}|e(\tilde v)|^2=\frac{1}{a}\int_{\mathbb{R}^N}|e(v)|^2.$$

Therefore $\int_{\mathbb{R}^N}|e(v)|^2\ge\lambda_V a$. If $a\ge1$ this implies $F_\lambda(v)\ge\lambda_V$, while if $a\le1$,

$$F_\lambda(v)\ge\int_{\mathbb{R}^N}|e(v)|^2+\lambda_V(1-a)\ge\lambda_V a+\lambda_V(1-a)=\lambda_V.$$

Step 2: the case $|\{|v|>0\}|>V$. Set $\Omega:=\{|v|>0\}$ and $m:=|\Omega|>V$. Let

$$t:=\left(\frac{m}{V}\right)^{1/N}>1\quad\text{and}\quad v_t(x):=v(tx).$$

Since $v\in H^1_0(\Omega)$, it follows that $v_t\in H^1_0(\frac{1}{t}\Omega)$. Moreover,

$$|\{|v_t|>0\}|=|\frac{1}{t}\Omega|=\frac{1}{t^N}|\Omega|=V.$$

We also have,

$$\int_{\mathbb{R}^N}|v_t|^2=t^{-N}\int_{\mathbb{R}^N}|v|^2,\qquad\int_{\mathbb{R}^N}|e(v_t)|^2=t^{2-N}\int_{\mathbb{R}^N}|e(v)|^2.$$

Let $w:=v_t/\|v_t\|_{L^2}$. Then $\int_{\mathbb{R}^N}|w|^2=1$ and $|\{|w|>0\}|=V$, so w is admissible in (3.14) and

$$\lambda_V\le\int_{\mathbb{R}^N}|e(w)|^2=\frac{\int_{\mathbb{R}^N}|e(v_t)|^2}{\int_{\mathbb{R}^N}|v_t|^2}=t^2\,\frac{\int_{\mathbb{R}^N}|e(v)|^2}{\int_{\mathbb{R}^N}|v|^2}.$$

Denoting again $a:=\int_{\mathbb{R}^N}|v|^2$, we obtain

$$\int_{\mathbb{R}^N}|e(v)|^2\ge\lambda_V\,\frac{a}{t^2}.$$

Hence

$$F_\lambda(v)\ge\lambda_V\frac{a}{t^2}+\lambda_V(1-a)_++\lambda(m-V).$$

Since $\lambda \geq \lambda_V/V$, we have

$$\lambda(m-V) \geq \frac{\lambda_V}{V}(m-V) = \lambda_V\Big(\frac{m}{V}-1\Big).$$

Setting $x := m/V > 1$, we note that $t^2 = x^{2/N}$ and, since $N \geq 2$, we know that $x^{-2/N} \geq x^{-1}$. Therefore $x + x^{-2/N} \geq x + \frac{1}{x}$. Moreover,

$$x + \frac{1}{x} - 2 = \frac{(x-1)^2}{x} \geq 0,$$

hence $x + \frac{1}{x} \geq 2$ and consequently $x + x^{-2/N} \geq 2$. This implies

$$x - 1 \geq 1 - x^{-2/N} = 1 - \frac{1}{t^2},$$

and thus

$$\lambda(m-V) \geq \lambda_V\Big(1 - \frac{1}{t^2}\Big).$$

Combining the above inequalities and using that

$$\frac{a}{t^2} + (1-a)_+ \geq \frac{1}{t^2},$$

we conclude that $F_\lambda(v) \geq \lambda_V$.

The proof is complete. □

3.2.2 Lamé quasi-minimizers

To study the regularity of optimal domains, we introduce the following notion.

Definition 3.1 (Quasi-minimizer)**.** *We say that $u \in H^1(\mathbb{R}^N)^N$ is a quasi-minimizer for the Lamé energy with exponent $\gamma > 0$ if and only if u satisfies the following minimality property: there exists $C > 0$ such that for all ball $B_r \subset \mathbb{R}^N$ of radius $r \in (0,1)$ and for all $v \in H^1(\mathbb{R}^N)^N$ such that $u = v$ on $\mathbb{R}^N \setminus B_r$ we have*

$$\int_{B_r} |\nabla u|^2 + (\mathrm{div}(u))^2\, dx \leq \int_{B_r} |\nabla v|^2 + (\mathrm{div}(v))^2 dx + Cr^\gamma. \tag{3.16}$$

The definition is motivated by the following observation.

Proposition 3.3. *Let u be a solution for the problem*

$$\min\left\{\int_{\mathbb{R}^N} |e(u)|^2 dx \quad s.t. \quad u \in H^1(\mathbb{R}^N)^N,\ \int_{\mathbb{R}^N} |u|^2 = 1, \quad and \quad |\{|u|>0\}| \leq V\right\}. \tag{3.17}$$

Assume moreover that $u \in L^p(\mathbb{R}^N)^N$ with $p \geq 2$. Then u is a quasi-minimizer for the Lamé energy in the sense of Definition 3.1 with $\gamma = N - \frac{2N}{p}$.

Proof. By Proposition 3.2 we already know that u satisfies the following minimality property: for all $v \in H^1(\mathbb{R}^N)^N$ we have

$$\int_{\mathbb{R}^N} |e(u)|^2 \, dx \leq \int_{\mathbb{R}^N} |e(v)|^2 dx + \lambda_V \left(1 - \int_{\mathbb{R}^N} |v|^2\right)^+ + \lambda\Big(|\{|v| > 0\}| - V\Big)^+, \tag{3.18}$$

or equivalently, using (3.1),

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + (\mathrm{div}(u))^2 \, dx \quad &\leq \quad \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + (\mathrm{div}(v))^2 \, dx \\ &\quad + \lambda_V \left(1 - \int_{\mathbb{R}^N} |v|^2\right)^+ + \lambda\Big(|\{|v| > 0\}| - V\Big)^+. \end{aligned} \tag{3.19}$$

Then let v be equal to u outside B_r, so that the volume $|\{|v| > 0\}|$ has at most increased by $\omega_N r^N$, and since $\int_{\mathbb{R}^N} |u|^2 \, dx = 1$, we can compute

$$1 - \int_{\mathbb{R}^N} |v|^2 = 1 - \int_{\mathbb{R}^N} |u|^2 + \int_{\mathbb{R}^N} |u|^2 - \int_{\mathbb{R}^N} |v|^2 = \int_{B_r} |u|^2 - \int_{B_r} |v|^2.$$

In other words from the minimality of u we obtain

$$\int_{B_r} |\nabla u|^2 + (\mathrm{div}(u))^2 \, dx \leq \int_{B_r} |\nabla v|^2 + (\mathrm{div}(v))^2 dx + C\left(\int_{B_r} |u|^2 - \int_{B_r} |v|^2\right)^+ + Cr^N.$$

If moreover $u \in L^p(\mathbb{R}^N)^N$ with $p \geq 2$, then denoting by q the exponent satisfying $2q = p$ we can estimate

$$\left(\int_{B_r} |u|^2 - \int_{B_r} |v|^2\right)^+ \leq \int_{B_r} |u|^2 \leq |B_r|^{\frac{1}{q'}} \left(\int_{B_r} |u|^{2q}\right)^{\frac{1}{q}} = Cr^{\frac{N}{q'}} \|u\|_p^{\frac{p}{q}} = Cr^\gamma$$

with $\gamma = \frac{N}{q'} = N - \frac{2N}{p}$. Since $\gamma < N$ and $r \leq 1$, it follows that $Cr^N \leq Cr^\gamma$ and finally u is a quasi-minimizer for the Lamé energy in the sense of Definition 3.1 with $\gamma = N - \frac{2N}{p}$. □

Theorem 3.2 follows by combining Proposition 3.3 with the next result, and by observing that the condition $\gamma > N - 2$, with $\gamma = N - \frac{2N}{p}$, is equivalent to $p > N$.

Proposition 3.4. *Let $u \in H^1(\mathbb{R}^N)^N$ be a quasi-minimizer for the Lamé energy with exponent $\gamma \in (N-2, N]$. Then $u \in C^{0,\alpha}(\mathbb{R}^N)^N$, for all $\alpha < \frac{\gamma-(N-2)}{2}$.*

Proof. The proof is based on standard arguments from free boundary theory, as in [18, Theorem 2.1]. The main difference is that the Lamé energy replaces the standard Dirichlet energy. Since the Lamé system is elliptic in the sense of systems, namely it satisfies the strong Legendre–Hadamard ellipticity condition, similar arguments from the regularity theory of elliptic systems apply.

In this proof we will keep denoting by $C > 0$ a universal constant that could change from line to line. Let $B_r \subset \mathbb{R}^N$ be a given ball of radius $r \in (0,1)$ and let v be the solution for the problem

$$\min_{v \in u + H_0^1(B_r)} \Big\{ \int_{B_r} |\nabla(v)|^2 + (\mathrm{div}(v))^2 \, dx \Big\}.$$

In other words v is the replacement of u in B_r, by a function satisfying $v = u$ on ∂B_r and solution for the homogeneous Lamé system

$$-\Delta v - \nabla(\mathrm{div} v) = 0 \text{ in } B_r. \tag{3.20}$$

Since the Lamé system satisfies the strong Legendre-Hadamard ellipticity condition, then v enjoys some nice decaying properties. Indeed by applying standard regularity theory for elliptic systems (see for instance [27, Theorem 4.11]) we know that

$$\sup_{B_{r/2}} |\nabla v|^2 \leq C \frac{1}{|B_r|} \int_{B_r} |\nabla v|^2 \, dx. \tag{3.21}$$

Let now $Q_s(v)$ be the quadratic form defined by

$$Q_s(v) := \int_{B_s} |\nabla v|^2 + (\mathrm{div}(v))^2 \, dx.$$

Then for $s \leq r/2$, using (3.21) we get

$$\begin{aligned} Q_s(v) = \int_{B_s} |\nabla v|^2 + (\mathrm{div}(v))^2 \, dx &\leq (1+N) \sup_{B_{r/2}} |\nabla v|^2 |B_s| \\ &\leq C \left(\frac{s}{r}\right)^N \int_{B_r} |\nabla v|^2 \, dx, \\ &\leq C \left(\frac{s}{r}\right)^N Q_r(v), \end{aligned} \tag{3.22}$$

where C depends only on dimension N.

Moreover, the weak formulation of (3.20) says that for all $\varphi \in H^1_0(B_r)^N$,

$$\int_{B_r} \nabla v : \nabla \varphi + (\mathrm{div} v)(\mathrm{div} \varphi) \, dx = 0.$$

In other words, if $A_r(u, v)$ denotes the bilinear form associated with Q_r and defined by

$$A_r(u, v) := \int_{B_r} \nabla u : \nabla v + \mathrm{div}(u)\mathrm{div}(v) \, dx,$$

we have $A_r(v, \varphi) = 0$ for all $\varphi \in H^1_0(B_r)^N$. This applies in particular to $\varphi = u - v \in H^1_0(B_r)^N$ and we deduce from Pythagoras equality that $Q_r(u - v) + Q_r(v) = Q_r(u)$ or differently,

$$Q_r(u - v) = Q_r(u) - Q_r(v). \tag{3.23}$$

We will use this property later.

Notice also that Q_s is a nonnegative quadratic form for any $s > 0$ and using that $Q_s(b-a) \geq 0$ and $Q_s(b + a) \geq 0$ we obtain, for arbitrary a, b,

$$2|A_s(a, b)| \leq Q_s(a) + Q_s(b),$$

so that, for all $s < r/2$,

$$Q_s(u) = Q_s(u - v + v) \leq 2Q_s(u - v) + 2Q_s(v).$$

Then using (3.22) we arrive at

$$\begin{aligned} Q_s(u) &\leq C\left(\frac{s}{r}\right)^N Q_r(v) + 2Q_r(u-v) \\ &\leq C\left(\frac{s}{r}\right)^N Q_r(u) + 2Q_r(u-v), \end{aligned}$$

where for the last line we have used that v is a minimizer of Q_r and u is a competitor. Now we recall (3.23) and we use that u is a quasi-minimizer with exponent γ, so that

$$Q_r(u-v) = Q_r(u) - Q_r(v) \leq Cr^\gamma.$$

All in all, we have proved that for all $s \leq r/2$ we have

$$Q_s(u) \leq C\left(\frac{s}{r}\right)^N Q_r(u) + Cr^\gamma. \tag{3.24}$$

Of course we can assume $C \geq 2$. The decaying in (3.24) looks promising but we would prefer s^γ on the last term instead of r^γ. We can obtain this up to decrease a bit the power γ and use a technical dyadic argument. This is standard (see for e.g. [41, Lemma 5.6.]) but let us write the full details for the reader's convenience.

Indeed, to lighten the notation we denote by $f(s)$ the non-decreasing function $f : s \mapsto Q_s(u)$. Let $r_0 \in (0,1)$ be fixed and $a \in (0, 1/2)$ that will be chosen later, and let $r_k := a^k r_0$. Let us prove by induction that for all $k \in \mathbb{N}$ it holds

$$f(a^k r_0) \leq C^k a^{Nk} f(r_0) + Ca^{(k-1)\gamma} r_0^\gamma \frac{C^k - 1}{C-1}. \tag{3.25}$$

For $k = 0$ the inequality is obvious. Now let us assume that it holds true for some k. Then from the decaying property (3.24) we infer that (using in particular that $\gamma \leq N$ and that $a \leq 1$ in (3.26)),

$$\begin{aligned} f(a^{k+1}r_0) &\leq Ca^N f(a^k r_0) + C(a^k r_0)^\gamma \\ &\leq Ca^N\left(C^k a^{Nk} f(r_0) + Ca^{(k-1)\gamma} r_0^\gamma \frac{C^k-1}{C-1}\right) + C(a^k r_0)^\gamma \\ &\leq C^{k+1} a^{N(k+1)} f(r_0) + Ca^{\gamma k} r_0^\gamma \frac{C^{k+1}-C}{C-1} + C(a^k r_0)^\gamma \qquad (3.26) \\ &= C^{k+1} a^{N(k+1)} f(r_0) + Ca^{\gamma k} r_0^\gamma \left(\frac{C^{k+1}-1}{C-1}\right), \end{aligned}$$

which proves (3.25). To simplify a bit we can write it differently, taking into account that $C \geq 2$,

$$f(r_k) \leq C^k \left(\frac{r_k}{r_0}\right)^N f(r_0) + C^{k+1} a^{-\gamma} r_k^\gamma. \tag{3.27}$$

The nice thing with (3.27) is that we have now r_k^γ on the last term (compare with (3.24)), but the price to pay is the C^k in factor. We will beat this factor by choosing well the constant a, and decreasing a bit the powers N and γ.

Indeed, let $\alpha \in (0,1)$ be given and let us fix

$$a := \frac{1}{C^{\frac{1}{\alpha}}},$$

so that

$$r_k^\alpha C^k = a^{\alpha k} r_0^\alpha C^k = r_0^\alpha.$$

Then from (3.27) we deduce that

$$\begin{aligned} f(r_k) &\leq C^k \left(\frac{r_k}{r_0}\right)^{N-\alpha} \left(\frac{r_k}{r_0}\right)^{\alpha} f(r_0) + C^{k+1} a^{-\gamma} r_k^{\gamma-\alpha} r_k^\alpha \\ &\leq \left(\frac{r_k}{r_0}\right)^{N-\alpha} f(r_0) + C' r_k^{\gamma-\alpha}, \end{aligned} \tag{3.28}$$

where

$$C' = C a^{-\gamma} r_0^\alpha.$$

Now let $s \in (0, 1/2)$ be given. There exists k such that $r_{k+1} \leq s \leq r_k$. In particular, $r_k \leq \frac{1}{a} s$.

Moreover, $s \mapsto f(s)$ is non-decreasing so

$$\begin{aligned} f(s) \leq f(r_k) &\leq \left(\frac{r_k}{r_0}\right)^{N-\alpha} f(r_0) + C' r_k^{\gamma-\alpha} \\ &\leq a^{-(N-\alpha)} \left(\frac{s}{r_0}\right)^{N-\alpha} f(r_0) + a^{-(\gamma-\alpha)} C' s^{\gamma-\alpha}. \end{aligned}$$

In conclusion we have proved that there exists a constant $C > 0$ (depending on N, α, γ, r_0) such that for all $s \leq 1/2$ we have

$$f(s) \leq C s^{N-\alpha} f(r_0) + C s^{\gamma-\alpha} \leq C s^{\gamma-\alpha} f(r_0) + C s^{\gamma-\alpha},$$

where for the last inequality we have used $N \geq \gamma$. Returning back to u, and estimating $f(r_0)$ by $\int_{\mathbb{R}^N} |e(u)|^2 \, dx$, we conclude, using also the Poincaré inequality, that for all $r \leq 1/2$,

$$\int_{B_r} |u - m_u|^2 \, dx \leq C r^2 \int_{B_r} |\nabla u|^2 \, dx \leq C r^{2+\gamma-\alpha} = C r^{N+(2+\gamma-N-\alpha)},$$

where m_u denotes the (vectorial) average of u. Remember that here α is arbitrary close to 0. Then by standard results about Campanato spaces (see for e.g. [41, Theorem 5.4]), provided that

$$2 + \gamma - N - \alpha > 0,$$

then $u \in \mathscr{C}^{0,\beta}(\mathbb{R}^N)$ with $\beta = \frac{2+\gamma-N-\alpha}{2}$. Since α is arbitrarily, this means that u belongs to $\mathscr{C}^{0,\beta}(\mathbb{R}^N)$ for all $\beta < \frac{\gamma-(N-2)}{2}$, as soon as $\gamma > N-2$, and the Proposition follows. □

4 Characterization of minimizers

4.1 Optimality conditions: first and second orders

We first give general formulae for the first and second order shape derivative of the Lamé eigenvalue. We will then apply them to get optimality conditions (assuming that the minimizer is smooth enough to justify our computations). As kindly mentioned by D. Buoso, these computations (and the criticality of the ball) already appeared in the review paper [11]. Moreover, they use weaker regularity assumptions on the domain Ω than those made in this paper. For completeness, we recall the main results that will be used in Section 5. Let Ω be a bounded domain with $\mathscr{C}^3$ boundary. This regularity assumption ensures that all boundary quantities appearing below belong to $L^2(\partial\Omega)$. Let $V \in W^{4,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ and introduce, for any $t \in (-1, 1)$ small enough,

$$\Omega_{tV} := (\mathrm{Id} + tV)\Omega.$$

In this section, shape functionals are considered on the class

$$\mathcal{O}^3 := \{\Omega \subset \mathbb{R}^N \text{ bounded open, } \partial\Omega \in \mathscr{C}^3\}.$$

Since $W^{4,\infty}(\mathbb{R}^N) \hookrightarrow \mathscr{C}^{3,1}(\mathbb{R}^N)$, for any t small enough the map $T_t := \mathrm{Id} + tV$ is a $\mathscr{C}^{3,1}$-diffeomorphism, hence $\Omega_{tV} = T_t(\Omega) \in \mathcal{O}^3$.

First order optimality conditions. Let $F : \mathcal{O}^3 \to \mathbb{R}$ be a shape functional and let $\Omega \in \mathcal{O}^3$. We say that F is differentiable at Ω if, for any $V \in W^{4,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ compactly supported, the limit

$$\langle dF(\Omega), V\rangle := \lim_{t\to 0} \frac{F(\Omega_{tV}) - F(\Omega)}{t}$$

exists and if it is a linear form in V. In this case, this limit is called the first-order shape derivative of F at Ω in the direction V.

We consider the case of the general eigenvalue

$$\Lambda(\Omega) = \inf_{u\in(H^1_0(\Omega))^N} \frac{\mu \int_\Omega |\nabla u|^2\, dx + (\lambda + \mu) \int_\Omega (\mathrm{div}(u))^2\, dx}{\int_\Omega |u|^2\, dx}.$$

Assume moreover that $\Lambda(\Omega)$ is simple. Then any minimizer realizing $\Lambda(\Omega)$ is unique up to multiplication by ± 1, once the $L^2(\Omega)$-normalization is fixed. We denote by u_Ω such a minimizer, normalized by $\|u_\Omega\|_{L^2(\Omega)} = 1$. The associated Euler–Lagrange equations read:

$$\begin{cases} -\mu\Delta u_\Omega - (\lambda + \mu)\nabla(\mathrm{div}(u_\Omega)) = \Lambda(\Omega)u_\Omega & \text{in } \Omega \\ u_\Omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

It is standard that the material state $t \mapsto u_{\Omega_{tV}} \circ T_t \in [H^1(\Omega)]^N$ is Fréchet differentiable at $t = 0$; see, e.g., [11, 32]. We denote its derivative by the *material derivative*

$$u'_V := \left.\frac{d}{dt}\right|_{t=0} \left(u_{\Omega_{tV}} \circ T_t\right) \in [H^1(\Omega)]^N.$$

The associated *Eulerian shape derivative* of the state is defined on the fixed domain Ω by

$$\dot{u}_V := \left.\frac{d}{dt}\right|_{t=0} u_{\Omega_{tV}} \in \mathcal{D}'(\Omega)^N,$$

and the two notions are related by the usual identity

$$\dot{u}_V = u'_V - \nabla u_\Omega \, V \qquad \text{in } \Omega,$$

whenever the terms are well-defined. In what follows we work with the Eulerian derivative $\dot{u}_V$ on the fixed domain Ω. Let us also define

$$\dot{\Lambda}_V := \frac{d}{dt}\bigg|_{t=0} \Lambda(\Omega_{tV}).$$

Differentiating at $t = 0$ the weak formulation of the eigenvalue problem on Ω_{tV} after pull-back to Ω yields the bulk equation in (4.2). The boundary condition follows from the Dirichlet constraint on the moving boundary: for $x \in \partial\Omega$, $u_{\Omega_{tV}}(T_t(x)) = 0$ for all small t, hence $\dot{u}_V + \nabla u_\Omega \, V = 0$ on $\partial\Omega$, i.e.

$$\dot{u}_V = -(\nabla u_\Omega \, n)\,(V \cdot n) \qquad \text{on } \partial\Omega.$$

Finally, the condition $\int_\Omega u_\Omega \cdot \dot{u}_V \, dx = 0$ is obtained by differentiating the $L^2(\Omega)$-normalization after pull-back. Equivalently, since $\Lambda(\Omega)$ is simple, it is the Fredholm compatibility condition for solvability of the linearized equation, the kernel being $\operatorname{span}\{u_\Omega\}$. We refer to [29, Chapter 5] (see also [11]) for these standard facts. Finally, $\dot{u}_V$ solves the system:

$$\begin{cases} -\mu\Delta\dot{u}_V - (\lambda+\mu)\nabla \operatorname{div} \dot{u}_V = \dot{\Lambda} u_\Omega + \Lambda(\Omega)\dot{u}_V & \text{in } \Omega \\ \dot{u}_V = -\nabla u_\Omega n (V \cdot n) & \text{on } \partial\Omega\,, \\ \int_\Omega u_\Omega \cdot \dot{u}_V = 0. \end{cases} \tag{4.2}$$

Remark 4.1. *The $[H^1(\Omega)]^N$-valued differentiability of the material state is a first-order property and only requires that T_t is bi-Lipschitz for $|t|$ small (hence $V \in W^{1,\infty}$ would be sufficient for this point). We impose the stronger assumption $V \in W^{4,\infty}$ in order to keep Ω_{tV} in the class $\mathcal{O}^3$ and to justify the second-order shape calculus used later. In particular, the second-order formula* (4.6) *involves boundary quantities such as the mean curvature and second normal derivatives.*

Let us multiply the main equation by u_Ω and then integrate by parts. We get

$$\mu \int_\Omega \nabla u_\Omega : \nabla \dot{u}_V + (\lambda+\mu) \int_\Omega \operatorname{div} \dot{u}_V \operatorname{div} u_\Omega = \Lambda(\Omega) \int_\Omega u_\Omega \cdot \dot{u}_V + \dot{\Lambda} \int_\Omega |u_\Omega|^2.$$

Similarly, let us multiply by $\dot{u}_V$ the main equation (4.1) solved by u_Ω and then integrate by parts. Using the boundary conditions, we get

$$\begin{aligned} \mu \int_\Omega \nabla u_\Omega : \nabla \dot{u}_V + (\lambda+\mu) \int_\Omega \operatorname{div} \dot{u}_V \operatorname{div} u_V \quad &= \quad \Lambda(\Omega) \int_\Omega u_\Omega \cdot \dot{u}_V - \mu \int_{\partial\Omega} |(\nabla u_\Omega) n|^2 (V \cdot n) \\ &\quad -(\lambda+\mu) \int_\Omega (\operatorname{div} u_\Omega)^2 V \cdot n, \end{aligned}$$

by using that $\nabla u_\Omega n \cdot n = \operatorname{div} u_\Omega$ on $\partial\Omega$. Finally, combining the two identities above yields

$$\dot{\Lambda} = -\mu \int_{\partial\Omega} |(\nabla u_\Omega) n|^2 (V \cdot n) - (\lambda+\mu) \int_\Omega (\operatorname{div} u_\Omega)^2 V \cdot n.$$

We have then obtained the following result.

Proposition 4.1. *Let Ω denote a $\mathscr{C}^3$ domain such that $\Lambda(\Omega)$ is simple. Let u_Ω be its associated (normalized) first eigenfunction. For any $V \in W^{4,\infty}(\mathbb{R}^N, \mathbb{R}^N)$, the mapping $W^{4,\infty}(\mathbb{R}^N, \mathbb{R}^N) \ni V \mapsto \Lambda(\Omega_V)$ is differentiable. Denoting by $\dot{\Lambda}$ its differential, the first order derivative of Λ is*

$$\dot{\Lambda} = \langle d\Lambda(\Omega), V\rangle = -\mu \int_{\partial\Omega} |(\nabla u_\Omega)n|^2 (V \cdot n) - (\lambda + \mu) \int_{\partial\Omega} (\operatorname{div} u_\Omega)^2 V \cdot n. \tag{4.3}$$

Remark 4.2 (Shape gradient)**.** *Observe that $\nabla u_\Omega \nu \cdot n = \operatorname{div} u_\Omega$ on $\partial\Omega$ and denote by $[\nabla u_{i,\Omega}]_\tau := \nabla u_{i,\Omega} - \frac{\partial u_{i,\Omega}}{\partial n} n$ the tangential part of the gradient $\nabla u_{i,\Omega}$. According to the result above, the shape gradient $\nabla\Lambda(\Omega)$ reads*

$$\nabla\Lambda(\Omega) = -\mu|(\nabla u_\Omega)n|^2 - (\lambda + \mu)(\operatorname{div} u_\Omega)^2 \tag{4.4}$$

and can be decomposed as:

$$\nabla\Lambda(\Omega) = -\mu \sum_i |[\nabla u_{i,\Omega}]_\tau|^2 - (\lambda + 2\mu)(\operatorname{div} u_\Omega)^2.$$

Corollary 4.1. *Let Ω^* be a solution, with $\mathscr{C}^3$ boundary, of the extremal eigenvalue problem*

$$\min_{|\Omega|=V_0} \Lambda(\Omega)$$

and assume that $\Lambda(\Omega^*)$ is simple. Then, denoting by u_{Ω^*} any associated eigenfunction,

$$\mu|(\nabla u^*_\Omega)n|^2 + (\lambda + \mu)(\operatorname{div} u^*_\Omega)^2$$

is constant on $\partial\Omega^*$.

Proof. This is a consequence of Proposition 4.1. Indeed, since we work with a volume constraint, there exists a Lagrange multiplier such that the shape gradient of the eigenvalue is proportional to the derivative of the volume, namely $\int_{\partial\Omega^*} V \cdot n$ whence the result. In particular, when $\mu > 0$ and $\lambda = 0$, we obtain that $|e(u_{\Omega^*})|$ is constant on the boundary. □

Second order optimality conditions. We rely on the standard pull-back/material-derivative approach in shape calculus (differentiation of weak formulations on a fixed reference domain). Although some references are stated for scalar problems, the same arguments apply to linear strongly elliptic systems such as the Lamé operator with Dirichlet boundary conditions; see, e.g., [29, Ch. 5],[11, 32] and the general monographs on shape calculus cited therein, for instance [21] or the older [40]. Hence, for any $\mathscr{C}^3$ domain Ω such that $\Lambda(\Omega)$ is simple, the mapping $\Omega \mapsto \Lambda(\Omega)$ is twice shape differentiable at Ω in the following sense: for any compactly supported vector field $V \in W^{4,\infty}(\mathbb{R}^N, \mathbb{R}^N)$, the map $f : t \mapsto \Lambda((\mathrm{Id} + t\Phi)\Omega)$ is twice differentiable at $t = 0$. We will use the notations

$$\langle d\Lambda(\Omega), V\rangle := f'(0), \qquad \langle d^2\Lambda(\Omega)V, V\rangle := f''(0).$$

Similarly, the mapping $\Omega \mapsto u_\Omega$ is twice shape-differentiable at Ω, where u_Ω is the first normalized eigenfunction of System (4.1) on Ω, in the sense that there exists $\varepsilon > 0$ such that

$$g : (-\varepsilon, \varepsilon) \ni t \longmapsto u_{(\mathrm{Id}+tV)\Omega} \in [H^2(\Omega)]^N$$

is twice differentiable at $t = 0$. Moreover, under the standing assumption $\partial\Omega \in \mathscr{C}^3$ (and $V \in W^{4,\infty}$), elliptic regularity ensures that u_Ω is regular enough so that the boundary quantities appearing in the second-order shape derivative are well-defined.

We let $\dot{u}_V$ be its first order derivative at $t = 0$.

Proposition 4.2. *For any $\mathscr{C}^3$ domain Ω such that $\Lambda(\Omega)$ is simple, let u_Ω be its associated first eigenfunction. For any $V \in W^{4,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ compactly supported, the shape derivative $\dot{u}_V$ solves the PDE*

$$\begin{cases} -\mu\Delta\dot{u}_V - (\lambda+\mu)\nabla \operatorname{div} \dot{u}_V = \Lambda(\Omega)\dot{u}_V + \langle d\Lambda(\Omega), V\rangle u_\Omega & \text{in } \Omega \\ \dot{u}_V = -\nabla u_\Omega n(V\cdot n) & \text{on } \partial\Omega\,, \\ \int_\Omega u_\Omega \cdot \dot{u}_V = 0. \end{cases} \tag{4.5}$$

If, in addition, the vector field V is normal to $\partial\Omega$, meaning that $V = (V\cdot n)n$ on $\partial\Omega$, the second-order shape derivative of Λ at Ω is given by

$$\begin{aligned} \langle d^2\Lambda(\Omega)V, V\rangle &= -\mu\int_{\partial\Omega}\left(H\left|\frac{\partial u_\Omega}{\partial n}\right|^2 + 2\frac{\partial^2 u_\Omega}{\partial n^2}\cdot\frac{\partial u_\Omega}{\partial n}\right)(V\cdot n)^2 \\ &\quad -(\lambda+\mu)\int_{\partial\Omega}\left(H(\operatorname{div} u_\Omega)^2 + \frac{\partial(\operatorname{div} u_\Omega)^2}{\partial n}\right)(V\cdot n)^2 \\ &\quad -2\Lambda\int_\Omega|\dot{u}_V|^2 + 2\mu\int_\Omega|\nabla\dot{u}_V|^2 + 2(\lambda+\mu)\int_\Omega(\operatorname{div}\dot{u}_V)^2 \end{aligned} \tag{4.6}$$

where H is the mean curvature of $\partial\Omega$.

Furthermore, when $N = 2$, one has

$$\frac{\partial(\operatorname{div} u_\Omega)^2}{\partial n} = -\frac{2\mu}{\lambda+\mu}\operatorname{div} u_\Omega\left(H\frac{\partial u_\Omega}{\partial n}\cdot n + \frac{\partial^2 u_\Omega}{\partial n^2}\cdot n\right) \qquad \text{on } \partial\Omega.$$

Remark 4.3. *The assumptions $\partial\Omega \in \mathscr{C}^3$ and $V \in W^{4,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ ensure that the boundary quantities appearing in* (4.6)*, in particular the mean curvature H and second normal derivatives are well-defined and that the perturbed domains Ω_{tV} remain of class $\mathscr{C}^3$.*

Proof. Let us denote by $u_\Omega = (u_1,\dots,u_N)^\top$ the solution of (4.1). General formulae for the shape differentiation of Dirichlet boundary value problem yield that $\dot{u}_V$ solves (4.5), we refer to [32, Chapter 5, §5.6, 5.7, 5.9] for the detailed computations.

Let us apply the Hadamard formula for integrals on variable boundaries [32, Proposition 5.4.18] to (4.3). This yields

$$\begin{aligned} \langle d^2\Lambda(\Omega)V, V\rangle &= -2\mu\int_{\partial\Omega}(\nabla\dot{u}_V)n\cdot(\nabla u_\Omega)n(V\cdot n) - \mu\int_{\partial\Omega}\left(H|(\nabla u_\Omega)n|^2 + \frac{\partial|(\nabla u_\Omega)n|^2}{\partial n}\right)(V\cdot n)^2 \\ &\quad -2(\lambda+\mu)\int_\Omega \operatorname{div} u_\Omega \operatorname{div}\dot{u}_V(V\cdot n) \\ &\quad -(\lambda+\mu)\int_{\partial\Omega}\left(H(\operatorname{div} u_\Omega)^2 + \frac{\partial(\operatorname{div} u_\Omega)^2}{\partial n}\right)(V\cdot n)^2 + R, \end{aligned}$$

where H denotes the mean curvature of $\partial\Omega$ and

$$\begin{aligned} R &= -2\mu\int_{\partial\Omega}\langle(\partial_n u_\Omega)n', (\partial_n u_\Omega)n\rangle(V\cdot n) - \mu\int_{\partial\Omega}\|(\partial_n u_\Omega)n\|^2(V\cdot n)' \\ &\quad -(\lambda+\mu)\int_{\partial\Omega}(\operatorname{div} u_\Omega)^2(V\cdot n)' \end{aligned}$$

where n' is the Eulerian derivative of n. The expression of $d^2\Lambda(\Omega)$ is independent of the specific extension chosen for n, which allows us to consider a regular extension of n, that is unitary in a neighborhood of $\partial\Omega$, without loss of generality. As a consequence, $n' = -\nabla_\Gamma(V\cdot n)$, the notation ∇_Γ standing for the tangential gradient. First, note that $(V\cdot n)' = 0$ since we are dealing with vector fields V that are normal to $\partial\Omega$. Furthermore, because $u_\Omega = 0$ on $\partial\Omega$ and n' is orthogonal to n, it follows that $\nabla u_i \cdot n' = 0$ for $i \in \{1, 2, \dots N\}$, which implies $(\nabla u_\Omega)n' = 0$. From this, we deduce that $R = 0$.

Let us multiply the main equation of (4.6) by $\dot{u}_V$ and then integrate by parts. We obtain

$$\mu\int_\Omega |\nabla \dot{u}_V|^2 + \mu\int_{\partial\Omega} (\nabla\dot{u}_V)n\cdot(\nabla u_\Omega)n(V\cdot n) + (\lambda+\mu)\int_\Omega (\operatorname{div}\dot{u}_V)^2$$
$$+(\lambda+\mu)\int_{\partial\Omega} \operatorname{div}\dot{u}_V(\nabla u_\Omega)n\cdot n(V\cdot n) = \dot{\Lambda}\int_\Omega \dot{u}_V\cdot u_\Omega + \Lambda\int_\Omega |\dot{u}_V|^2.$$

Using that $\int_\Omega \dot{u}_V\cdot u_\Omega = 0$ and that

$$(\nabla u_\Omega)n\cdot n = \sum_{i,j}\frac{\partial u_i}{\partial x_j}n_in_j = \sum_{i,j}\frac{\partial u_i}{\partial n}n_in_j^2 = \sum_i \frac{\partial u_i}{\partial n}n_i = \sum_i\frac{\partial u_i}{\partial x_i} = \operatorname{div} u_\Omega$$

on $\partial\Omega$, the equality above simplifies into

$$\mu\int_{\partial\Omega}(\nabla\dot{u}_V)n\cdot(\nabla u_\Omega)n(V\cdot n) + (\lambda+\mu)\int_{\partial\Omega}\operatorname{div}\dot{u}_V\operatorname{div}u_\Omega(V\cdot n)$$
$$= \Lambda\int_\Omega|\dot{u}_V|^2 - \mu\int_\Omega|\nabla\dot{u}_V|^2 - (\lambda+\mu)\int_\Omega(\operatorname{div}\dot{u}_V)^2.$$

As a result, the second order derivative of Λ rewrites

$$\begin{aligned}\langle d^2\Lambda(\Omega)V, V\rangle &= -\mu\int_{\partial\Omega}\left(H|(\nabla u_\Omega)n|^2 + \frac{\partial|(\nabla u_\Omega)n|^2}{\partial n}\right)(V\cdot n)^2\\ &\quad -2\Lambda\int_\Omega|\dot{u}_V|^2 + 2\mu\int_\Omega|\nabla\dot{u}_V|^2 + 2(\lambda+\mu)\int_\Omega(\operatorname{div}\dot{u}_V)^2\\ &\quad -(\lambda+\mu)\int_{\partial\Omega}\left(H(\operatorname{div}u_\Omega)^2 + \frac{\partial(\operatorname{div}u_\Omega)^2}{\partial n}\right)(V\cdot n)^2.\end{aligned}$$

Let us simplify the term

$$A := \frac{\partial|(\nabla u_\Omega)n|^2}{\partial n}.$$

By expanding $|(\nabla u_\Omega)n|^2$, we get

$$|(\nabla u_\Omega)n|^2 = \sum_i\left(\frac{\partial u_i}{\partial n}\right)^2.$$

Now since $u_i = 0$ on $\partial\Omega$, one has

$$\frac{\partial}{\partial n}\left(\frac{\partial u_i}{\partial n}\right)^2 = 2\left(\frac{\partial u_i}{\partial n}\right)\left(\frac{\partial^2 u_i}{\partial n^2}\right)$$

and therefore $A = 2\frac{\partial u_\Omega}{\partial n}\cdot\frac{\partial^2 u_\Omega}{\partial n^2}$.

Now we look at the term

$$B := \frac{\partial(\operatorname{div} u_\Omega)^2}{\partial n}.$$

To simplify this term, we will use the main equation in (4.1). We have

$$\frac{\partial \operatorname{div} u_\Omega}{\partial n} = \sum_i \frac{\partial \operatorname{div} u_\Omega}{\partial x_i} n_i.$$

According to (4.1) and the decomposition of the Laplacian

$$\Delta u_i = \Delta_\tau u_i + H\frac{\partial u_i}{\partial n} + \frac{\partial^2 u_i}{\partial n^2} \qquad \text{on } \partial\Omega,$$

we get

$$(\lambda+\mu)\frac{\partial \operatorname{div} u_\Omega}{\partial n} = -\mu\left(H\frac{\partial u_\Omega}{\partial n}\cdot n + \frac{\partial^2 u_\Omega}{\partial n^2}\cdot n\right) \qquad \text{on } \partial\Omega,$$

and thus

$$\frac{\partial(\operatorname{div} u_\Omega)^2}{\partial n} = -\frac{2\mu}{\lambda+\mu}\operatorname{div} u_\Omega\left(H\frac{\partial u_\Omega}{\partial n}\cdot n + \frac{\partial^2 u_\Omega}{\partial n^2}\cdot n\right) \qquad \text{on } \partial\Omega,$$

whence the last claim of the theorem. □

4.2 Multiplicity of minimal eigenvalues

Lemma 4.1. *Assume that $N = 2$ and let Ω^* be a minimizing domain with $\mathscr{C}^3$ boundary for the problem*

$$\min_{|\Omega|=V_0} \Lambda(\Omega).$$

Then, $\Lambda(\Omega^*)$ is at most of multiplicity 2.

Proof. In what follows, let us denote by $[y(x)]_\tau$ the tangential part of a vector field $y \in L^2(\partial\Omega, \mathbb{R}^2)$ at $x \in \partial\Omega$, in other words

$$[y(x)]_\tau = y(x) - (y(x)\cdot n(x))n(x).$$

Let us assume that $\Lambda(\Omega)$ has multiplicity $m \geq 3$. According to classical results for the derivative of multiple eigenvalues (see e.g. [29], [11], [15]), the first order optimality conditions read: let V denote a smooth vector field, then the directional derivative of $|\Omega|\Lambda(\Omega)$ exists and it is the smallest eigenvalue of the $m\times m$ matrix $\mathcal{M}$ whose entries are

$$V_0\mathcal{M} - \Lambda\int_{\partial\Omega}(V\cdot n)I_2, \quad \text{where } \mathcal{M}_{i,j} = -\int_{\partial\Omega}\left[\mu[\nabla u^i : \nabla u^j] + (\lambda+\mu)\operatorname{div} u^i \operatorname{div} u^j\right] V\cdot n,$$

where $(u^1,\dots,u^m)$ is an orthonormal basis of associated eigenfunctions. By minimality, this directional derivative has to be nonnegative. Since we can take both V and $-V$, this shows that $\mathcal{M} = \frac{\Lambda}{V_0}\int_{\partial\Omega}(V\cdot n)I_2$. In particular,

$$\mu[\nabla u^i : \nabla u^j] + (\lambda+\mu)\operatorname{div} u^i \operatorname{div} u^j = 0 \quad \text{on } \partial\Omega, \text{ for } i \neq j,$$

which rewrites

$$\mu\frac{\partial u^i}{\partial n}\cdot\frac{\partial u^j}{\partial n} + (\lambda+\mu)\operatorname{div} u^i \operatorname{div} u^j = 0 \quad \text{on } \partial\Omega \text{ for } i \neq j,$$

by using the Dirichlet boundary condition. Observe moreover that $\partial u^i/\partial n \cdot n = \sum_k \partial u^i_k/\partial n n_k = \operatorname{div} u^i$ and therefore, the condition above rewrites

$$\mu \left[\frac{\partial u^i}{\partial n}\right]_\tau \cdot \left[\frac{\partial u^j}{\partial n}\right]_\tau + (\lambda + 2\mu) \operatorname{div} u^i \operatorname{div} u^j = 0 \quad \text{on } \partial\Omega \text{ for } i \neq j,$$

or equivalently

$$\left(\sqrt{\mu}\left[\frac{\partial u^i}{\partial n}\right]_\tau + \sqrt{\lambda+2\mu}\operatorname{div}(u^i)n\right)\cdot\left(\sqrt{\mu}\left[\frac{\partial u^j}{\partial n}\right]_\tau + \sqrt{\lambda+2\mu}\operatorname{div}(u^j)n\right) = 0 \quad \text{on } \partial\Omega \text{ for } i \neq j.$$

We have obtained a family of (at least) three orthogonal vectors in $\mathbb{R}^2$, and thus a contradiction. □

5 The case of the disk

Our first aim is to compute the first eigenvalue of the unit disk in $\mathbb{R}^2$. We recall that the Lamé coefficients λ, μ are such that $\mu > 0, \lambda + \mu > 0$. The first eigenvalue is then defined by

$$\Lambda := \min_{u=(u_1,u_2)\in H^1_0(\Omega)^2} \frac{\mu\left(\int_\Omega |\nabla u_1|^2\,dx + \int_\Omega |\nabla u_2|^2\,dx\right) + (\lambda+\mu)\int_\Omega (\operatorname{div}(u))^2\,dx}{\int_\Omega (u_1)^2\,dx + \int_\Omega (u_2)^2\,dx} \tag{5.1}$$

and the PDE solved by the minimizer $u = (u_1, u_2)$ is

$$\begin{cases} -\mu\Delta u - (\lambda+\mu)\nabla(\operatorname{div}(u)) = \Lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

5.1 Eigenvalues and eigenvectors of the unit disk

We follow the strategy proposed in Capoferri et al, [14]. We will need some Helmholtz decomposition of the vector u. Let us state a more general Lemma that will be also useful for the derivative later.

Convention. For a scalar function ψ we set $\operatorname{curl}\psi := (\partial_y\psi, -\partial_x\psi)$, and for a vector field $w = (w_1, w_2)$ we set $\operatorname{curl} w := \partial_x w_2 - \partial_y w_1$. In particular, $\operatorname{curl}\operatorname{curl}\psi = -\Delta\psi$.

Lemma 5.1. *Let $v = (v_1, v_2)$ be a smooth function satisfying the equation*

$$-\mu\Delta v - (\lambda+\mu)\nabla(\operatorname{div}(v)) = \Lambda v + f \tag{5.3}$$

in a smooth domain Ω with a given function f. There exist two functions ψ_1 and ψ_2 in $C^\infty(\Omega)$ such that

$$v + \frac{1}{\Lambda} f = \nabla\psi_1 + \operatorname{curl}\psi_2 \quad \text{in } \Omega. \tag{5.4}$$

Furthermore ψ_1 and ψ_2 respectively satisfy the PDE

$$-(\lambda+2\mu)\Delta\psi_1 = \Lambda\psi_1 - \frac{\lambda+2\mu}{\Lambda}\operatorname{div} f \quad \text{in } \Omega,$$

and

$$-\mu\Delta\psi_2 = \Lambda\psi_2 - \frac{\mu}{\Lambda}\operatorname{curl} f \quad \text{in } \Omega.$$

Proof. The following argument was suggested by M. Levitin. Let us first set

$$\psi_1 = -\frac{\lambda+2\mu}{\Lambda}\,\mathrm{div}\ v \quad \text{and} \quad \psi_2 = -\frac{\mu}{\Lambda}\,\mathrm{curl}\,v = -\frac{\mu}{\Lambda}(\partial_x v_2 - \partial_y v_1).$$

According to (5.3), one has

$$\begin{aligned} v + \frac{f}{\Lambda} &= \frac{1}{\Lambda}\left(-\mu\Delta v - (\lambda+\mu)\nabla\,\mathrm{div}(v)\right) \\ &= \frac{1}{\Lambda}\left(-\mu\left(\Delta v - \nabla\,\mathrm{div}(v)\right) - (\lambda+2\mu)\nabla\,\mathrm{div}(v)\right). \end{aligned}$$

Note that

$$\Delta v - \nabla\,\mathrm{div}(v) = \begin{pmatrix} \partial_{yy}\ v_1 - \partial_{xy}v_2 \\ \partial_{xx}v_2 - \partial_{xy}v_1 \end{pmatrix} = \begin{pmatrix} -\partial_y \\ \partial_x \end{pmatrix}(\partial_x v_2 - \partial_y v_1).$$

Combining the identities above, we thus infer that v satisfies (5.4).

Now, observe that we can write the equation (5.3) as

$$\mu\,\mathrm{curl}\,\mathrm{curl}(v) - (\lambda+2\mu)\,\mathrm{grad}\,\mathrm{div}(v) = \lambda v + f$$

Now, passing to the divergence in this equation and using $\mathrm{div}\,\mathrm{curl} = 0$ and $\mathrm{div}\,\mathrm{grad}\,\mathrm{div} = \Delta\,\mathrm{div}$ yields

$$-(\lambda+2\mu)\Delta(\mathrm{div}\,v) = \Lambda\ \mathrm{div}\,v + \mathrm{div}\,f \quad \text{in } \Omega.$$

Recalling the definition $\psi_1 := -\frac{\lambda+2\mu}{\Lambda}\,\mathrm{div}\,v$, we obtain

$$-(\lambda+2\mu)\Delta\psi_1 = \Lambda\psi_1 - \frac{\lambda+2\mu}{\Lambda}\,\mathrm{div}\,f \quad \text{in } \Omega.$$

In the same way, taking the curl in this equation and using $\mathrm{curl}\,\mathrm{grad} = 0$ and $\mathrm{curl}\,\mathrm{curl} = -\Delta$ yields

$$-\mu\Delta\psi_2 = \Lambda\psi_2 - \frac{\mu}{\Lambda}\,\mathrm{curl}\,f, \quad \text{in } \Omega.$$

The conclusion follows. □

Now, to compute the eigenvalues and eigenvectors of the unit disk, we use the decomposition provided by Lemma 5.1 (with $v = u$ and $f = 0$),

$$u = \nabla\psi_1 + \mathrm{curl}\,\psi_2 = \begin{pmatrix} \frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y} \\ \frac{\partial\psi_1}{\partial y} - \frac{\partial\psi_2}{\partial x} \end{pmatrix} \tag{5.5}$$

and we use the fact that the scalar potentials ψ_i, $i = 1, 2$, satisfy the Helmholtz equation

$$-\Delta\psi_i = \omega_i^2\psi_i \quad \text{in } \Omega \tag{5.6}$$

where

$$\omega_1^2 = \frac{\Lambda}{\lambda+2\mu}, \quad \omega_2^2 = \frac{\Lambda}{\mu}.$$

We introduce

$$\omega = \sqrt{\Lambda},\ a_1 = \frac{1}{\sqrt{\lambda+2\mu}},\ a_2 = \frac{1}{\sqrt{\mu}}$$

therefore, $\omega_1 = a_1\omega$, $\omega_2 = a_2\omega$.

In polar coordinates, the general solution of (5.6) is given, for $i = 1, 2$ by

$$\psi_i(r,\theta) = a_{i,0}J_0(\omega_i r) + \sum_{k=1}^{\infty} J_k(\omega_i r)[a_{i,k}\cos k\theta + b_{i,k}\sin k\theta]. \tag{5.7}$$

It remains to express the Dirichlet boundary conditions $u_1 = u_2 = 0$ for $r = 1$. Using the expression of the derivatives in polar coordinates, this leads to the system

$$\begin{cases} \cos\theta[\frac{\partial\psi_1}{\partial r} + \frac{\partial\psi_2}{\partial\theta}] - \sin\theta[\frac{\partial\psi_1}{\partial\theta} - \frac{\partial\psi_2}{\partial r}] = 0 \\ \sin\theta[\frac{\partial\psi_1}{\partial r} + \frac{\partial\psi_2}{\partial\theta}] + \cos\theta[\frac{\partial\psi_1}{\partial\theta} - \frac{\partial\psi_2}{\partial r}] = 0 \end{cases}$$

for which we infer

$$\frac{\partial\psi_1}{\partial r} + \frac{\partial\psi_2}{\partial\theta} = 0, \qquad \frac{\partial\psi_1}{\partial\theta} - \frac{\partial\psi_2}{\partial r} = 0 \tag{5.8}$$

these equalities being true for $r = 1$. Using the expression of ψ_1, ψ_2 given in (5.7), we get by identification for the constant term (and using the fact that $J_0' = -J_1$):

$$a_{1,0}J_1(a_1\omega) = 0, \quad a_{2,0}J_1(a_2\omega) = 0.$$

This provides the sequence of eigenvalues $(\lambda + 2\mu)j_{1,k}^2$ and $\mu j_{1,k}^2$ where $j_{1,k}$ is the sequence of zeros of the Bessel function J_1. Among all these values, the smallest one is $\mu j_{1,1}^2$ since $\lambda + 2\mu > \mu$ by assumption. Therefore,

$$\text{a candidate to be the first eigenvalue } \Lambda \text{ is } \mu j_{1,1}^2. \tag{5.9}$$

We now consider the coefficients of $\cos k\theta$ and $\sin k\theta$ in (5.8). This gives the two systems

$$\begin{cases} \omega_1 J_k'(\omega_1)a_{k,1} + kJ_k(\omega_2)b_{k,2} = 0 \\ kJ_k(\omega_1)a_{k,1} + \omega_2 J_k'(\omega_2)b_{k,2} = 0 \end{cases} \tag{5.10}$$

and

$$\begin{cases} \omega_1 J_k'(\omega_1)b_{k,1} - kJ_k(\omega_2)a_{k,2} = 0 \\ kJ_k(\omega_1)b_{k,1} - \omega_2 J_k'(\omega_2)a_{k,2} = 0. \end{cases} \tag{5.11}$$

The determinant of these two systems is the same and it must vanish if we look for a non-trivial solution. This leads to the following transcendental equation, which determines the remaining eigenvalues:

$$a_1 a_2 \omega^2 J_k'(a_1\omega)J_k'(a_2\omega) - k^2 J_k(a_1\omega)J_k(a_2\omega) = 0. \tag{5.12}$$

Using the classical relations for the derivative of Bessel functions, we can rewrite (5.12)

$$\frac{k}{a_1\omega}J_k(a_1\omega)J_{k-1}(a_2\omega) + \frac{k}{a_2\omega}J_{k-1}(a_1\omega)J_k(a_2\omega) - J_{k-1}(a_1\omega)J_{k-1}(a_2\omega) = 0 \tag{5.13}$$

or

$$\frac{k}{a_1\omega}J_k(a_1\omega)J_{k+1}(a_2\omega) + \frac{k}{a_2\omega}J_{k+1}(a_1\omega)J_k(a_2\omega) - J_{k+1}(a_1\omega)J_{k+1}(a_2\omega) = 0. \tag{5.14}$$

Now to determine the first eigenvalue of the elasticity operator, we need to know whether the smallest solution of the previous transcendental equations can be smaller than the value $\mu j_{1,1}^2$ already obtained. In that case, the first eigenvalue would be double, systems (5.10) and (5.11) providing two independent solutions associated to the same eigenvalue.

Let us state the following characterization where we see that the first eigenvalue actually depends on the Poisson coefficient ν:

Theorem 5.1. *Let ν^* be the number*

$$\nu^* := \frac{j_{1,1}^2 - 2{j'_{1,1}}^2}{2j_{1,1}^2 - 2{j'_{1,1}}^2} \simeq 0.349895 \tag{5.15}$$

where $j_{1,1}$ is the first zero of the Bessel function J_1 and $j'_{1,1}$ is the first zero of its derivative J'_1. Assume that the Poisson coefficient ν satisfies

$$\nu \le \nu^*, \tag{5.16}$$

then the first eigenvalue is given as a solution of the transcendental equation (5.13) *for some k and then it is at least double. Assume that the Poisson coefficient ν satisfies*

$$\nu \ge \nu^* \tag{5.17}$$

then the first eigenvalue is $\Lambda = \mu j_{1,1}^2$. Moreover, it is a simple eigenvalue as soon as $\nu > \nu^*$.

Remark 5.1. *Note that when $\nu = \nu^*$ the first eigenvalue is (at least) triple and equal to $\mu j_{1,1}^2$.*

Proof. Let us introduce the function F_k defined by

$$F_k(\omega) = \frac{k}{a_1\omega} J_k(a_1\omega)J_{k-1}(a_2\omega) + \frac{k}{a_2\omega} J_{k-1}(a_1\omega)J_k(a_2\omega) - J_{k-1}(a_1\omega)J_{k-1}(a_2\omega).$$

When ω is small, using the Taylor expansion of the Bessel function near 0, we obtain

$$F_k(\omega) = \frac{a_1^{k-1}a_2^{k-1}(a_1^2 + a_2^2)}{[(k-1)!]^2 2^{2k+1}k(k+1)}\,\omega^2 + o(\omega^2)$$

that shows in particular that $F_k(\omega) > 0$ for $\omega > 0$ small.

Now let us look at F_1 and evaluate it at $\omega^* = \sqrt{\mu j_{1,1}^2} = j_{1,1}/a_2$. Since $J_1(a_2\omega^*) = 0$ we get

$$F_1(\omega^*) = \frac{J_0(a_2\omega^*)}{a_1\omega^*}\left(J_1(a_1\omega^*) - a_1\omega^* J_0(a_1\omega^*)\right).$$

If we can prove that $F_1(\omega^*) \le 0$, then F_1 changes its sign between 0 and ω^* that implies the fact that the first eigenvalue is a zero of the transcendental equation.

Now $J_1(x) - xJ_0(x) = -xJ'_1(x)$ and this is negative between 0 and $j'_{1,1}$ and positive between $j'_{1,1}$ and $j'_{1,2}$. On the other hand, the term $J_0(a_2\omega^*) = J_0(j_{1,1}) < 0$ therefore, we want to find the case where $x = a_1\omega^*$ belongs to the interval $[j'_{1,1}, j'_{1,2}]$. Now

$$a_1\omega^* = \sqrt{\frac{\mu}{\lambda + 2\mu}} j_{1,1} \in [j'_{1,1}, j'_{1,2}] \Leftrightarrow \frac{{j'_{1,1}}^2}{j_{1,1}^2} \le \frac{1-2\nu}{2-2\nu} \le \frac{{j'_{1,2}}^2}{j_{1,1}^2}$$

where we use the expression of $\mu/(\lambda + 2\mu)$ in term of ν. Solving the previous inequality in ν provides the desired result from the left inequality. The right inequality is automatically satisfied since $-1 \le \nu < 0.5$.

Now, it remains to prove that, when $\nu \ge \nu^*$ the first eigenvalue is $\mu j_{1,1}^2$ (and is simple when $\nu > \nu^*$). Let us introduce $\psi_k(x) := xJ'_k(x)/J_k(x)$. It is known that the function ψ_k is decreasing on all interval in $\mathbb{R}_+$ where it is defined, and in particular on $[0, j_{k,1})$. We refer to [42] or [36] for that assertion. Moreover $\psi_k(0) = k$. This implies that $\psi_k(x) < k$ for $x \in (0, j_{k,1})$ and for any k. Now, let us assume that ω is such that $\omega < \sqrt{\mu j_{1,1}^2}$. Since $\lambda + 2\mu > \mu$, we have $a_1 < a_2$ and therefore

$$a_1\omega < a_2\omega < a_2\sqrt{\mu j_{1,1}^2} = j_{1,1} \leq j_{k,1} \quad \text{for all } k \geq 1.$$

Therefore $J_k(a_1\omega) > 0$ and $J_k(a_2\omega) > 0$ for all $\omega < \sqrt{\mu j_{1,1}^2}$. Now, let us rewrite the transcendental equation (5.12) as (we can divide by $J_k(a_1\omega)J_k(a_2\omega)$ that is positive)

$$\psi_k(a_1\omega)\psi_k(a_2\omega) - k^2 = 0. \tag{5.18}$$

Now, the properties we recalled on ψ_k and the fact that $a_i\omega < j_{k,1}$ show that the first member of (5.18) is strictly negative when $\omega < \sqrt{\mu j_{1,1}^2}$. This proves the claim. □

5.2 Optimality of the disk: first order arguments

We investigate whether a Faber–Krahn type inequality holds for the elasticity operator. The answer depends on the Poisson ratio. Roughly speaking, when the first eigenvalue Λ is double, the disk is not a minimizer, whereas when Λ is simple, the disk is at least a local minimizer. We begin with the first case.

Theorem 5.2. *Assume that the Poisson coefficient ν satisfies* (5.16) *with a strict inequality. Then the disk does not minimize Λ among open sets of given volume.*

Proof. We will use a first order optimality argument for which we need the expression of the eigenvectors. As we have seen in Theorem 5.1, when ν satisfies (5.16), the eigenvalue is (at least) double and the two eigenvectors can be obtained through the systems (5.10) and (5.11) with ω defined as the smallest solution of all the equations (5.12) (or (5.13), (5.14)). The value of the integer k will not be really important here.

Let us choose for example

$$a_{1,k} = kJ_k(\omega_2), \qquad b_{2,k} = -\omega_1 J_k'(\omega_1)$$

that satisfy system (5.10). (we recall that $\omega_1 = a_1\omega$ and $\omega_2 = a_2\omega$). Then

$$\psi_1(r,\theta) = a_{1,k}J_k(\omega_1 r)\cos k\theta, \qquad \psi_2(r,\theta) = b_{2,k}J_k(\omega_2 r)\sin k\theta.$$

Using (5.5), we obtain $u = (u_1, u_2)$ with

$$u_1 = a_{1,k}\left(\omega_1\cos\theta\cos k\theta J_k'(\omega_1 r) + \frac{k\sin\theta}{r}\sin k\theta J_k(\omega_1 r)\right) + \\ b_{2,k}\left(\omega_2\sin\theta\sin k\theta J_k'(\omega_2 r) + \frac{k\cos\theta}{r}\cos k\theta J_k(\omega_2 r)\right)$$

$$u_2 = a_{1,k}\left(\omega_1\sin\theta\cos k\theta J_k'(\omega_1 r) - \frac{k\cos\theta}{r}\sin k\theta J_k(\omega_1 r)\right) + \\ b_{2,k}\left(-\omega_2\cos\theta\sin k\theta J_k'(\omega_2 r) + \frac{k\sin\theta}{r}\cos k\theta J_k(\omega_2 r)\right).$$

In principle, we must multiply the previous expressions by a normalization factor in order to satisfy $\int_\Omega u_1^2 + u_2^2 = 1$, but it turns out that this factor has no importance in the computation we present now.

The shape derivative of a multiple eigenvalue is now a classical topic: in the case of the elasticity operator, we refer for example to the recent paper [15]. To sum up, let us assume that the eigenvalue has multiplicity m and denote by $u^1, u^2, \ldots u^m$ a set of orthonormal eigenvectors. Then if we perturb the boundary of Ω by a vector field V, the first eigenvalue has a semi-derivative (or directional derivative) that is given as the smallest eigenvalue of the $m \times m$ matrix $\mathcal{M}$ whose entries are

$$\mathcal{M}_{i,j} = -\int_{\partial\Omega} \left[\mu[\nabla u^i : \nabla u^j] + (\lambda+\mu)\operatorname{div} u^i \operatorname{div} u^j\right] V \cdot n$$

(where n is here the exterior normal vector). It is therefore enough to prove that this matrix has a negative eigenvalue for a vector field preserving the area, i.e. a vector field V such that $\int_{\partial\Omega} V \cdot n = 0$. For that purpose, it is sufficient to look at the first term $\mathcal{M}_{1,1}$ and prove that it can be chosen negative (that will imply that the symmetric matrix $\mathcal{M}$ is not positive and therefore has a negative eigenvalue). This term being given by

$$\mathcal{M}_{1,1} = -\int_{\partial\mathbb{D}} \left[\mu(|\nabla u_1|^2 + |\nabla u_2|^2) + (\lambda+\mu)(\operatorname{div} u)^2\right] V \cdot n$$

we have to compute on the unit circle $|\nabla u_1|^2, |\nabla u_2|^2$ and $(\operatorname{div} u)^2$.

From the Helmholtz decomposition (5.5), it comes

$$\operatorname{div} u = \Delta\psi_1 = -\omega_1^2\psi_1 = -a_{1,k}\omega_1^2 J_k(\omega_1 r)\cos k\theta$$

so, on the unit circle

$$(\operatorname{div} u)^2 = a_{1,k}^2\omega_1^4 J_k(\omega_1)^2\cos^2 k\theta. \tag{5.19}$$

Now, u_1 and u_2 being constant on the unit circle, we have $|\nabla u_i|^2 = \left(\frac{\partial u_i}{\partial r}\right)^2$ with $r = 1$. Using the formula of u_1, u_2, we can write

$$\frac{\partial u_1}{\partial r} = A_1 \cos\theta\cos k\theta + B_1 \sin\theta\sin k\theta \tag{5.20}$$

with

$$A_1 = a_{1,k}\omega_1^2 J_k''(\omega_1) - kb_{2,k}J_k(\omega_2) + b_{2,k}k\omega_2 J_k'(\omega_2)$$
$$B_1 = -ka_{1,k}J_k(\omega_1) + a_{1,k}k\omega_1 J_k'(\omega_1) + b_{2,k}\omega_2^2 J_k''(\omega_2).$$

Using the Bessel differential equation to replace $J_k''(\omega_i), i = 1,2$ by a combination of $J_k'(\omega_i)$ and $J_k(\omega_i)$, together with the choice we have done for $a_{1,k}$ and $b_{2,k}$ and the transcendental equation (5.12), we can simplify the previous expressions as

$$A_1 = -k\omega_1^2 J_k(\omega_1)J_k(\omega_2), \qquad B_1 = \omega_1\omega_2^2 J_k'(\omega_1)J_k(\omega_2). \tag{5.21}$$

In the same way, we obtain

$$\frac{\partial u_2}{\partial r} = A_2 \sin\theta\cos k\theta + B_2 \cos\theta\sin k\theta \tag{5.22}$$

with

$$A_2 = -k\omega_1^2 J_k(\omega_1)J_k(\omega_2) = A_1, \qquad B_2 = -\omega_1\omega_2^2 J_k'(\omega_1)J_k(\omega_2) = -B_1. \tag{5.23}$$

Therefore

$$|\nabla u_1|^2 + |\nabla u_2|^2 = A_1^2\cos^2 k\theta + B_1^2 \sin^2 k\theta =$$
$$\omega_1^2 J_k^2(\omega_2)\left(\omega_1^2 k^2 J_k^2(\omega_1)\cos^2 k\theta + \omega_2^4 {J_k'}^2(\omega_1)\sin^2 k\theta\right).$$

With $(\operatorname{div} u)^2$ given by (5.19) we finally get

$$\mathcal{M}_{1,1} = \omega_1^2 J_k^2(\omega_2)\int_0^{2\pi}\left((\lambda+2\mu)k^2\omega_1^2 J_k^2(\omega_1)\cos^2 k\theta + \mu\omega_2^4 {J_k'}^2(\omega_1)\sin^2 k\theta\right) V\cdot n. \tag{5.24}$$

As explained before, in order to conclude the proof, it suffices to find a deformation field V such that $\int_0^{2\pi} V\cdot n = 0$ and $\mathcal{M}_{1,1} < 0$. Let us choose V such that $V(1,\theta) = \alpha\cos(2k\theta)$. Plugging this value in $\mathcal{M}_{1,1}$ yields

$$\mathcal{M}_{1,1} = \omega_1^2 J_k^2(\omega_2)\frac{\pi\alpha}{2}\left((\lambda+2\mu)k^2\omega_1^2 J_k^2(\omega_1) - \mu\omega_2^4 {J_k'}^2(\omega_1)\right). \tag{5.25}$$

The quantity $\mathcal{M}_{1,1}$ being linear in α, in order to conclude we just need to prove that the right-hand side of (5.25) cannot be zero. According to Theorem 5.1 we know that the eigenvalue satisfies $\Lambda < \mu j_{1,1}^2$, therefore $\omega_2 = \sqrt{\frac{\Lambda}{\mu}} < j_{1,1}$ and then $J_k(\omega_2) > 0$ (for $k \geq 1$, the first zero of J_k is always greater or equal to $j_{1,1}$). It remains to consider the quantity $(\lambda+2\mu)k^2\omega_1^2 J_k^2(\omega_1) - \mu\omega_2^4 {J_k'}^2(\omega_1)$. Using the expression of ω_1,ω_2 and up to the factor ω^2, it is equal to

$$Q = k^2 J_k^2(a_1\omega) - \frac{\omega^2}{\mu}{J_k'}^2(a_1\omega).$$

Since $a_1 < a_2$, we know that $J_k(a_1\omega) > 0$. Therefore, $Q = 0$ means

$$kJ_k(a_1\omega) - a_2\omega J_k'(a_1\omega) = 0 \quad \text{or} \quad kJ_k(a_1\omega) + a_2\omega J_k'(a_1\omega) = 0. \tag{5.26}$$

Let us analyze the first case. From the transcendental equation, we see that

$$kJ_k(a_1\omega) = a_2\omega J_k'(a_1\omega) \Rightarrow kJ_k(a_2\omega) = a_1\omega J_k'(a_2\omega).$$

Rewriting this in term of the function ψ_k already introduced, this means

$$\psi_k(a_2\omega) = \frac{ka_2}{a_1}$$

but since $ka_2/a_1 > k$ and $\psi_k(x) \leq k$ in this range we see that it is impossible.

Now in the other case, in the same way thanks to the transcendental equation, we get

$$\psi_k(a_2\omega) = -\frac{ka_2}{a_1}. \tag{5.27}$$

When $k \geq 3$ this is impossible since then $J_k(a_2\omega)$ and $J_k'(a_2\omega)$ are both positive (we recall that we are in the case where $\omega \leq \sqrt{\mu} j_{1,1} \Rightarrow a_2\omega \leq j_{1,1}$). It remains the case $k = 2$. In that case, due to the fact that ψ_2 is decreasing, we infer $\psi_2(a_2\omega) \geq \psi_2(j_{1,1})$. But since $j_{1,1}J_2'(j_{1,1}) = -2J_2(j_{1,1})$ this would imply with (5.27)

$$-\frac{2a_2}{a_1} \geq -2 \Rightarrow a_2 \leq a_1$$

a contradiction since we know that $a_2 > a_1$.

This finishes the proof of non optimality of the disk in that case. □

5.3 Optimality of the disk: second order arguments

Let us assume that Ω is the unit disk $\Omega = \mathbb{B}_2$ and assume that Λ is simple. We know, according to Theorem 5.1 that it is the case as soon as $\nu > \nu^*$ and moreover $\Lambda(\Omega) = \mu j_{1,1}^2$. We also know, from the proof of Theorem 5.1 that the associated eigenspace is spanned by the normalized vector $U = [u_1, u_2]^\top$, reading in polar coordinates (r, θ) as

$$u_1 = -\alpha \sin\theta J_1(j_{1,1}r) \quad \text{and} \quad u_2 = \alpha \cos\theta J_1(j_{1,1}r)$$

where $\alpha = \frac{1}{\sqrt{\pi}|J_0(j_{1,1})|}$. Our aim is to prove that in that case, the first order shape derivative of the functional $\mathcal{F}(\Omega) := |\Omega|\Lambda(\Omega)$ is zero (for any vector field V) while the second order shape derivative is a positive quadratic form.

A consequence of the general formulae for the second shape derivative given in Proposition 4.2 is:

Proposition 5.1. *Assume that $\Omega = \mathbb{B}_2$ is the unit disk in $\mathbb{R}^2$. Assume that the Poisson coefficient ν satisfies $\nu > \nu^*$. Then, the second order derivative of Λ at $\Omega = \mathbb{B}_2$ reads*

$$\begin{aligned} \langle d^2\Lambda(\Omega)V, V\rangle &= -\mu \int_{\partial\Omega} \frac{\partial^2 u_\Omega}{\partial n^2} \cdot \frac{\partial u_\Omega}{\partial n}(V\cdot n)^2 \\ &\quad -2\Lambda \int_\Omega |\dot{u}_V|^2 + 2\mu \int_\Omega |\nabla \dot{u}_V|^2 + 2(\lambda+\mu)\int_\Omega (\operatorname{div} \dot{u}_V)^2. \end{aligned} \tag{5.28}$$

Proof. This follows by observing that, in such a case,

- one has $\operatorname{div} u_\Omega = 0$ in Ω ;
- furthermore, using the standard decomposition of the Laplacian on $\partial\Omega$ yields

$$0 = -\Lambda(\Omega)u_\Omega - (\lambda+\mu)\nabla \operatorname{div} u_\Omega = \mu\Delta u_\Omega = \frac{\partial^2 u_\Omega}{\partial n^2} + H\frac{\partial u_\Omega}{\partial n} + \Delta_{\partial\Omega} u_\Omega,$$

  where $\Delta_{\partial\Omega}$ stands for the Laplace-Beltrami tangential operator on $\partial\Omega$. It follows that $\Delta_{\partial\Omega} u_\Omega = 0$ on $\partial\Omega$, and thus,

$$H\left|\frac{\partial u_\Omega}{\partial n}\right|^2 + 2\frac{\partial^2 u_\Omega}{\partial n^2}\cdot\frac{\partial u_\Omega}{\partial n} = \frac{\partial^2 u_\Omega}{\partial n^2}\cdot\frac{\partial u_\Omega}{\partial n} \quad \text{on } \partial\Omega.$$

□

Our first task is to compute explicitly the second derivative of Λ. According to Proposition 5.1, one has

$$\langle d^2\Lambda(\Omega)V, V\rangle = -\mu A_1 - 2\Lambda \int_\Omega |\dot{u}_V|^2 + A_2 + 2(\lambda+\mu)A_3,$$

where

$$\begin{aligned} A_1 &= \int_{\partial\Omega} \frac{\partial^2 u_\Omega}{\partial n^2}\cdot\frac{\partial u_\Omega}{\partial n}(V\cdot n)^2 \\ A_2 &= 2\mu \int_\Omega |\nabla \dot{u}_V|^2 \\ A_3 &= \int_\Omega (\operatorname{div} \dot{u}_V)^2. \end{aligned}$$

Let us compute each term separately. To this aim, it is convenient to denote by φ the function $V\cdot n$, defined on the boundary of $\mathbb{B}_2$, expanding in the θ-coordinate as

$$\varphi(\theta)=\sum_{k=0}^{+\infty}\alpha_k\cos(k\theta)+\beta_k\sin(k\theta). \tag{5.29}$$

A simplified expression of $\langle d^2\Lambda(\Omega)V,V\rangle$. According to the Green formula and Proposition 4.5, one has

$$\begin{aligned}A_2 &= 2\mu\int_{\partial\Omega}\dot u_V\cdot\frac{\partial\dot u_V}{\partial n}-2\mu\int_{\Omega}\dot u_V\cdot\Delta\dot u_V\\ &= 2\mu\int_{\partial\Omega}\dot u_V\cdot\frac{\partial\dot u_V}{\partial n}+2\Lambda(\Omega)\int_{\Omega}|\dot u_V|^2+2(\lambda+\mu)\int_{\Omega}\dot u_V\cdot\nabla\operatorname{div}\dot u_V.\end{aligned}$$

Let us now use the equation satisfied by $\dot u_V$. We obtain

$$A_2 = 2\mu\int_{\partial\Omega}\dot u_V\cdot\frac{\partial\dot u_V}{\partial n}+\Lambda(\Omega)\int_{\Omega}|\dot u_V|^2-2(\lambda+\mu)\int_{\Omega}(\operatorname{div}\dot u_V)^2+2(\lambda+\mu)\int_{\partial\Omega}\operatorname{div}\dot u_V(\dot u_V\cdot n).$$

Note that

$$\dot u_V\cdot n = -\varphi\nabla u_\Omega n\cdot n=-\varphi\sum_{i,j}\frac{\partial u_{\Omega,i}}{\partial x_j}n_jn_i=-\varphi\sum_i\frac{\partial u_{\Omega,i}}{\partial n}n_i=-\varphi\operatorname{div}u_\Omega=0\quad\text{on }\partial\Omega.$$

We get

$$A_2 = 2\mu\int_{\partial\Omega}\dot u_V\cdot\frac{\partial\dot u_V}{\partial n}+2\Lambda(\Omega)\int_{\Omega}|\dot u_V|^2-2(\lambda+\mu)\int_{\Omega}(\operatorname{div}\dot u_V)^2. \tag{5.30}$$

As a consequence,

$$\langle d^2\Lambda(\Omega)V,V\rangle=-\mu A_1+2\mu\int_{\partial\Omega}\dot u_V\cdot\frac{\partial\dot u_V}{\partial n}.$$

Let us now expand this expression into a sum of squares.

Computation of A_1. One has

$$\frac{\partial^2u_\Omega}{\partial n^2}\cdot\frac{\partial u_\Omega}{\partial n}=\alpha^2j_{1,1}^3J_1'(j_{1,1})J_1''(j_{1,1})=-\alpha^2j_{1,1}^2J_0(j_{1,1})^2,$$

by noting that $J_1'(j_{1,1})=J_0(j_{1,1})$ and $j_{1,1}^2J_1''(j_{1,1})=-j_{1,1}J_1'(j_{1,1})$. It follows that

$$A_1=\mu\alpha^2j_{1,1}^2J_0(j_{1,1})^2\int_{\partial\Omega}(V\cdot n)^2=\Lambda\left(2\alpha_0^2+\sum_{k=1}^{+\infty}(\alpha_k^2+\beta_k^2)\right).$$

Computation of $I := \int_{\partial\Omega} \dot{u}_V \cdot \frac{\partial \dot{u}_V}{\partial n}$. Recall that $\dot{u}_V$ satisfies

$$\dot{u}_V = \alpha j_{1,1} J_0(j_{1,1})\varphi(\theta)[\sin\theta, -\cos\theta]^\top \quad \text{on } \partial\Omega. \tag{5.31}$$

To compute $\dot{u}_V$ inside the domain Ω, we will use Lemma 5.1 with $v = \dot{u}_V$ and $f = d\Lambda(\Omega, V)u_\Omega$. According to formulae (4.1) and since $\text{div} u_\Omega = 0$, we finally obtain for the unit disk

$$d\Lambda(\Omega, V) = -2\mu j_{1,1}^2 \alpha_0 = -2\Lambda(\Omega)\alpha_0. \tag{5.32}$$

Since the first derivative of the area is $dA(\Omega, V) = \int_{\partial\Omega} V \cdot n = 2\pi\alpha_0$, we recover the fact that the first derivative of the functional $\mathcal{F}$ is zero at the disk (in other terms, the disk is a critical point).

We now use the decomposition of Lemma 5.1 on the unit circle taking profit that u_Ω vanishes on the boundary. Recall that $\dot{u}_V$ denotes the Eulerian (shape) derivative and satisfies the boundary condition (5.31) on $\partial\Omega$. Therefore, by writing ψ_1 and ψ_2 in the polar coordinates (r, θ), one has on the boundary

$$\begin{cases} \dot{u}_{V,1} = \partial_x\psi_1 + \partial_y\psi_2 = \cos\theta \left(\partial_r\psi_1 + \frac{1}{r}\partial_\theta\psi_2\right) - \sin\theta\left(\frac{1}{r}\partial_\theta\psi_1 - \partial_r\psi_2\right), \\ \dot{u}_{V,2} = \partial_y\psi_1 - \partial_x\psi_2 = \sin\theta \left(\partial_r\psi_1 + \frac{1}{r}\partial_\theta\psi_2\right) + \cos\theta\left(\frac{1}{r}\partial_\theta\psi_1 - \partial_r\psi_2\right). \end{cases}$$

By using (5.31), we infer that

$$\partial_r\psi_1 + \partial_\theta\psi_2 = 0 \quad \text{and} \quad \partial_\theta\psi_1 - \partial_r\psi_2 = -\frac{j_{1,1}}{\sqrt{\pi}}\varphi(\theta) \qquad \text{at } r = 1. \tag{5.33}$$

For the sake of notational clarity, let us introduce

$$\omega = \sqrt{\frac{\mu}{\lambda + 2\mu}}.$$

According to Lemma 5.1, the functions ψ_1 and ψ_2 solve the PDEs

$$-\Delta\psi_1 = \omega^2 j_{1,1}^2 \psi_1 \quad \text{and} \quad -\Delta\psi_2 = j_{1,1}^2\psi_2 \quad \text{in } \Omega.$$

We infer that ψ_1 and ψ_2 expand as

$$\begin{aligned} \psi_1 &= a_{1,0}J_0(\omega j_{1,1}r) + \sum_{k=1}^{+\infty} \left(a_{1,k}\cos(k\theta) + b_{1,k}\sin(k\theta)\right) J_k(\omega j_{1,1}r) \\ \psi_2 &= a_{2,0}J_0(j_{1,1}r) + \sum_{k=1}^{+\infty} \left(a_{2,k}\cos(k\theta) + b_{2,k}\sin(k\theta)\right) J_k(j_{1,1}r). \end{aligned}$$

Plugging these expressions into (5.33) allows us to compute the Fourier coefficients characterizing ψ_1 and ψ_2:

$$\begin{cases} a_{1,0} = 0, \quad b_{1,0} \text{ is arbitrary} \\ a_{1,k}j_{1,1}\omega J_k'(j_{1,1}\omega) - jJ_1(j_{1,1})b_{2,k} = 0, \qquad k \geq 1 \\ -ka_{1,k}J_k(j_{1,1}\omega) + j_{1,1}J_k'(j_{1,1})b_{2,k} = -\frac{j_{1,1}}{\sqrt{\pi}}\beta_k \\ kb_{1,k}J_j(j_{1,1}\omega) + j_{1,1}J_k'(j_{1,1})a_{2,k} = -\frac{j_{1,1}}{\sqrt{\pi}}\alpha_k \\ b_{1,k}j_{1,1}\omega J_k'(j_{1,1}\omega) + kJ_k(j_{1,1})a_{2,k} = 0 \end{cases}$$

which, after easy computations, reduces into

$$\begin{aligned}
a_{1,k} &= -\frac{k j_{1,1} J_k(j_{1,1})}{\sqrt{\pi}(j_{1,1}^2 \omega J_k'(\omega j_{1,1}) J_k'(j_{1,1}) - k^2 J_k(j_{1,1}) J_k(\omega j_{1,1}))}\beta_k \\
a_{2,k} &= \frac{k j_{1,1} J_k'(j_{1,1})}{\sqrt{\pi}(j_{1,1}^2 \omega J_k'(\omega j_{1,1}) J_k'(j_{1,1}) - k^2 J_k(j_{1,1}) J_k(\omega j_{1,1}))}\alpha_k \\
b_{1,k} &= -\frac{\omega j_{1,1}^2 J_k(j_{1,1}\omega)}{\sqrt{\pi}(j_{1,1}^2 \omega J_k'(\omega j_{1,1}) J_k'(j_{1,1}) - k^2 J_k(j_{1,1}) J_k(\omega j_{1,1}))}\alpha_k \\
b_{2,k} &= -\frac{\omega j_{1,1}^2 J_k'(j_{1,1}\omega)}{\sqrt{\pi}(j_{1,1}^2 \omega J_k'(\omega j_{1,1}) J_k'(j_{1,1}) - k^2 J_k(j_{1,1}) J_k(\omega j_{1,1}))}\beta_k.
\end{aligned}$$

We know the explicit expression of ψ_1 and ψ_2. We are now in position to compute I. One has

$$\begin{aligned}
I &= \int_0^{2\pi} \left(\dot{u}_{V,1} \frac{\partial \dot{u}_{V,1}}{\partial r} + \dot{u}_{V,2} \frac{\partial \dot{u}_{V,2}}{\partial r} \right) d\theta \\
&= \frac{j_{1,1}}{\sqrt{\pi}} \int_0^{2\pi} \left(\frac{\partial \psi_1}{\partial \theta} - \left(\frac{\partial^2 \psi_1}{\partial r \partial \theta} + \frac{\partial^2 \psi_2}{\partial r^2} \right) \right) \varphi(\theta)\, d\theta + (2\alpha_0)^2 \int_0^{2\pi} u_\Omega . \frac{\partial u_\Omega}{\partial r}\, d\theta.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{\partial^2 \psi_1}{\partial r \partial \theta} &= \omega j_{1,1} \sum_{k=1}^{+\infty} k J_k'(\omega j_{1,1}) \left(b_{1,k} \cos(k\theta) - a_{1,k} \sin(k\theta) \right) \\
\frac{\partial^2 \psi_2}{\partial r^2} &= j_{1,1}^2 \sum_{k=1}^{+\infty} J_k''(j_{1,1}) \left(a_{2,k} \cos(k\theta) + b_{2,k} \sin(k\theta) \right) + j_{1,1}^2 a_{2,0} J_1''(j_{1,1})
\end{aligned}$$

on $\partial\Omega$. Now, using

$$j_{1,1}^2 J_k''(j_{1,1}) = -j_{1,1} J_k'(j_{1,1}) + (k^2 - j_{1,1}^2) J_k(j_{1,1}),$$

it follows from easy, but lengthly computations that

$$\begin{aligned}
-\frac{j_{1,1}}{\sqrt{\pi}} \int_0^{2\pi} \left(\frac{\partial^2 \psi_1}{\partial r \partial \theta} + \frac{\partial^2 \psi_2}{\partial r^2} \right) \varphi(\theta)\, d\theta &= -\omega j_{1,1}^3 \sum_{k=1}^{+\infty} \frac{k^2 J_k(j_{1,1}) J_k'(\omega j_{1,1}) (\alpha_k^2 + \beta_k^2)}{j_{1,1}^2 \omega J_k'(\omega j_{1,1}) J_k'(j_{1,1}) - k^2 J_k(j_{1,1}) J_k(\omega j_{1,1})} \\
&\quad + \omega j_{1,1}^3 \sum_{k=1}^{+\infty} \frac{(-j_{1,1} J_k'(j_{1,1}) + (k^2 - j_{1,1}^2) J_k(j_{1,1})) J_k'(\omega j_{1,1}) (\alpha_k^2 + \beta_k^2)}{j_{1,1}^2 \omega J_k'(\omega j_{1,1}) J_k'(j_{1,1}) - k^2 J_k(j_{1,1}) J_k(\omega j_{1,1})} \\
&= -\omega j_{1,1}^4 \sum_{k=1}^{+\infty} \frac{(J_k'(j_{1,1}) + j_{1,1} J_k(j_{1,1})) J_k'(\omega j_{1,1}) (\alpha_k^2 + \beta_k^2)}{j_{1,1}^2 \omega J_k'(\omega j_{1,1}) J_k'(j_{1,1}) - k^2 J_k(j_{1,1}) J_k(\omega j_{1,1})}.
\end{aligned}$$

Similarly, since

$$\frac{\partial \psi_1}{\partial \theta} = \sum_{k=1}^{+\infty} k J_k(\omega j_{1,1}) \left(b_{1,k} \cos(k\theta) - a_{1,k} \sin(k\theta) \right) \qquad \text{on } \partial\Omega,$$

it follows that

$$\frac{j_{1,1}}{\sqrt{\pi}} \int_0^{2\pi} \frac{\partial \psi_1}{\partial \theta}\, d\theta = -j_{1,1}^2 \sum_{k=1}^{+\infty} \frac{k^2 J_k(\omega j_{1,1}) J_k(j_{1,1}) (\alpha_k^2 + \beta_k^2)}{j_{1,1}^2 \omega J_k'(\omega j_{1,1}) J_k'(j_{1,1}) - k^2 J_k(j_{1,1}) J_k(\omega j_{1,1})}.$$

Conclusion. Finally, we obtain the following expression of $\langle d^2\Lambda(\Omega)V, V\rangle$ by combining all the results above:

$$\langle d^2\Lambda(\Omega)V, V\rangle = \mu j_{1,1}^2 \left(6\alpha_0^2 + \sum_{k=1}^{+\infty} c_k(\alpha_k^2 + \beta_k^2)\right),$$

where

$$c_k = \frac{k^2 J_k(\omega j_{1,1}) J_k(j_{1,1}) - \omega j_{1,1}^2 J_k'(j_{1,1}) J_k'(\omega j_{1,1}) - 2k\omega j_{1,1}^2 J_k(j_{1,1}) J_k'(\omega j_{1,1})}{j_{1,1}^2 \omega J_k'(\omega j_{1,1}) J_k'(j_{1,1}) - k^2 J_k(j_{1,1}) J_k(\omega j_{1,1})}.$$

These computations allow us to state:

Theorem 5.3. *Let $\mathcal{F}$ be the shape functional defined by $\mathcal{F}(\Omega) = |\Omega|\Lambda(\Omega)$, and let Ω be the unit disk. Then $d\mathcal{F}(\Omega, V) = 0$ and*

$$\langle d^2\mathcal{F}(\Omega), V, V\rangle \geq A_0 \|\hat{V}\|_{H^1(\partial\Omega)}^2, \tag{5.34}$$

where $\hat{V}$ denotes the projection of $V \cdot n$ onto the orthogonal complement of $\operatorname{span}\{1, \cos\theta, \sin\theta\}$. *Therefore the unit disk is a local minimum in a weak sense.*

Proof. The fact that the first derivative of $\mathcal{F}$ vanishes at the disk has already been proved. Let us compute the second shape derivative. Denoting by A the area, we have

$$d^2\mathcal{F} = \Lambda d^2 A + 2d\Lambda dA + A d^2\Lambda.$$

Using $dA = \int_{\partial\Omega} \varphi$, $d^2 A = \int_{\partial\Omega} H\varphi^2$ where the mean curvature H equals 1 and $d\Lambda = -2\Lambda \int_{\partial\Omega} \varphi$, we finally get with the above expression of $d^2\Lambda$ the following expansion for the second derivative

$$\langle d^2\mathcal{F}(\Omega)V, V\rangle = \pi\Lambda \sum_{k=1}^{+\infty} C_k(\alpha_k^2 + \beta_k^2)$$

where α_k and β_k are the coefficients in the expansion (5.29) of φ and

$$C_k = 2j_{1,1}^2 \omega \frac{k J_k'(\omega j_{1,1}) J_k(j_{1,1})}{k^2 J_k(\omega j_{1,1}) J_k(j_{1,1}) - j_{1,1}^2 \omega J_k'(\omega j_{1,1}) J_k'(j_{1,1})}$$

with $\omega = \sqrt{\mu/(\lambda + 2\mu)} < 1$. We remark that no terms come from $k = 0$ and $k = 1$ ($C_1 = 0$). This is due to the invariance of the functional $\mathcal{F}$ under dilation and rotation. We claim that each C_k is positive for $k \geq 2$. Indeed, we have already seen in the proof of Theorem 5.1 that the denominator of C_k is positive. The first term in the numerator is also positive since, for $k \geq 2$, $J_k(j_{1,1}) > 0$. For the second term we need to be more precise. Since we are in the case where the first eigenvalue of the disk is $\mu j_{1,1}^2$ we know that $\nu > \nu^*$. Now,

$$\omega = \sqrt{\frac{\mu}{\lambda + 2\mu}} = \sqrt{\frac{1 - 2\nu}{2 - 2\nu}} < \sqrt{\frac{1 - 2\nu^*}{2 - 2\nu^*}} = \frac{j_{1,1}'}{j_{1,1}} < \frac{1}{2}.$$

Therefore $\omega j_{1,1} < j_{1,1}/2 < j_{k,1}'$ for any $k \geq 2$ and $J_k'(\omega j_{1,1}) > 0$.

To conclude the proof, we look at the asymptotic behaviour of C_k for k large. When x is fixed, we have for k large

$$J_k(x) \sim \frac{x^k}{2^k k!} - \frac{x^{k+2}}{2^{k+2}(k+1)!} \quad \text{and} \quad J_k'(x) \sim \frac{x^{k-1}}{2^k(k-1)!} - \frac{(k+2)x^{k+1}}{2^{k+2}(k+1)!}.$$

Then the numerator N_k of C_k satisfies

$$N_k \sim \frac{j_{1,1}^{2k+1}\omega^k}{2^{2k-1}[(k-1)!]^2}$$

while the denominator D_k of C_k satisfies

$$D_k \sim \frac{j_{1,1}^{2k+2}\omega^k}{2^{2k+2}(k-1)!(k+1)!}\left(2(1+\omega^2) - \frac{\omega^2 j_{1,1}^2}{k}\right) \quad \text{as } k \to +\infty.$$

Finally

$$C_k \sim \frac{4k(k+1)}{j_{1,1}\left(2(1+\omega^2) - \frac{\omega^2 j_{1,1}^2}{k}\right)} \geq k(k+1) \quad \text{as } k \to +\infty.$$

The conclusion follows since the H^1 norm of the projection of φ on the orthogonal space to $\operatorname{span}\{1, \cos\theta, \sin\theta\}$ is

$$\|\varphi\|_{H^1}^2 = \sum_{k+2}^{+\infty}(k^2+1)(\alpha_k^2 + \beta_k^2).$$

□

6 Some particular domains

The aim of this section is to exhibit (simple) domains whose first eigenvalue is smaller than that of the disk, at least for some values $\nu \geq \nu^*$. We first give explicit examples for which the value of Λ can be computed exactly. These examples are closely related to those found by Kawohl and Sweers in [33]. We then consider rectangles, for which we cannot compute Λ explicitly but can obtain sufficiently accurate upper bounds.

6.1 Rhombi

In this section, we ask whether there are planar domains for which an eigenvector has the form $U(x,y) = (u(x,y), u(x,y))$. We show that this is indeed possible and, in this case, construct both an explicit eigenvector and an explicit eigenvalue. More precisely, we identify parallelograms, in fact rhombi depending on the Lamé coefficients λ, μ, for which this condition holds. The associated eigenvalue has a simple expression depending only on the area of the parallelogram.

We proceed by analysis and synthesis.

Analysis. Let us assume that the domain $\Omega \subset \mathbb{R}^2$ has the property that its eigenvector is given by $U = (u(x,y), u(x,y))$. Thus $\text{div}(U) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$. We replace in the eigenvector equation (4.1) and we make the difference of the two equations to obtain

$$\frac{\partial}{\partial x}\text{div}U - \frac{\partial}{\partial y}\text{div}U = 0. \tag{6.1}$$

Therefore (locally, but then globally by analyticity), we have

$$\text{div}U = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = f(x+y) \tag{6.2}$$

for some analytic function f. Solving this transport equation (6.2) provides the existence of two analytic functions φ and ψ such that finally

$$u(x,y) = \varphi(x-y) + \psi(x+y). \tag{6.3}$$

Now we come back to the system (4.1): we have $\Delta u = 2(\varphi''(x-y) + \psi''(x+y))$ and $\text{div}U = 2\psi'(x+y)$. Therefore, using the change of variable $v = x-y, w = x+y$ we see that φ and ψ must satisfy

$$-2\mu(\varphi''(v) + \psi''(w)) - 2(\lambda+\mu)\psi''(w) = \Lambda(\varphi(v) + \psi(w)).$$

In this equation, we can separate variables to get the existence of some constant C such that

$$-2(\lambda+2\mu)\psi''(w) - \Lambda\psi(w) = C = 2\mu(\varphi''(v) + \Lambda\varphi(v)).$$

Solving this equation separately in ψ and φ yields

$$\psi(w) = A_1 \cos\omega_1 w + B_1 \sin\omega_1 w - \frac{C}{\Lambda} \quad \text{with } \omega_1^2 = \frac{\Lambda}{2\lambda+4\mu} \tag{6.4}$$

and

$$\varphi(v) = -A_2 \cos\omega_2 v - B_2 \sin\omega_2 v + \frac{C}{\Lambda} \quad \text{with } \omega_2^2 = \frac{\Lambda}{2\mu}. \tag{6.5}$$

Adding (6.5) and (6.4), we get by (6.3)

$$u(x,y) = u(v,w) = A_1 \cos\omega_1 w + B_1 \sin\omega_1 w - A_2 \cos\omega_2 v - B_2 \sin\omega_2 v$$

that can also be rewritten as

$$u(v,w) = C_1 \sin(\omega_1 w - \theta_1) - C_2 \sin(\omega_2 v - \theta_2). \tag{6.6}$$

With this expression of u we have completely taken into account the eigen-equation. It just remain to express the Dirichlet boundary condition. In other words, domains Ω that will satisfy the property (that the eigenvector is of the kind (u,u)) are those domains on which a function $u(v,w)$ given by (6.6) vanishes on the boundary of Ω.

Synthesis. We will prove below that necessarily $C_1 = C_2$ in the expression (6.6). So let us assume that $C_1 = C_2$ and let us investigate the set of points where u vanishes. In that case we have to solve $\sin(\omega_1 w - \theta_1) = \sin(\omega_2 v - \theta_2)$, therefore, coming back to the variables x, y:

$$u = 0 \Leftrightarrow \begin{cases} \omega_1(x+y) - \omega_2(x-y) = \theta_1 - \theta_2 + 2k\pi, \ k \in \mathbb{Z} \\ \omega_1(x+y) + \omega_2(x-y) = \theta_1 + \theta_2 + (2k'+1)\pi, \ k' \in \mathbb{Z} \end{cases}$$

or, it can also be written using the definition of ω_1, ω_2 and introducing the real numbers $a_1 = \theta_1 - \theta_2$ and $a_2 = \theta_1 + \theta_2$

$$\begin{cases} \left(\frac{1}{\sqrt{\lambda+2\mu}} - \frac{1}{\sqrt{\mu}}\right) x + \left(\frac{1}{\sqrt{\lambda+2\mu}} + \frac{1}{\sqrt{\mu}}\right) y = \sqrt{\frac{2}{\Lambda}}(a_1 + 2k\pi), \ k \in \mathbb{Z} \\ \left(\frac{1}{\sqrt{\lambda+2\mu}} + \frac{1}{\sqrt{\mu}}\right) x + \left(\frac{1}{\sqrt{\lambda+2\mu}} - \frac{1}{\sqrt{\mu}}\right) y = \sqrt{\frac{2}{\Lambda}}(a_2 + +(2k'+1)\pi), \ k' \in \mathbb{Z}. \end{cases} \tag{6.7}$$

This corresponds to equations of line segments with two specific normal vectors. Therefore, the domain Ω should be a parallelogram delimited by such parallel line segments. But we have to make more precise what line segments. To simplify the notations, let us introduce

$$\alpha = \frac{1}{\sqrt{\lambda+2\mu}} - \frac{1}{\sqrt{\mu}}, \quad \beta = \frac{1}{\sqrt{\lambda+2\mu}} + \frac{1}{\sqrt{\mu}}$$

and the normal vectors

$$\mathbf{e_1} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \mathbf{e_2} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}.$$

Let us assume that the parallelogram is defined by the four equations

$$\begin{cases} \mathbf{e_1} \cdot X = \xi_1 \\ \mathbf{e_1} \cdot X = \hat{\xi}_1 \end{cases} \qquad \begin{cases} \mathbf{e_2} \cdot X = \xi_2 \\ \mathbf{e_2} \cdot X = \hat{\xi}_2. \end{cases}$$

According to (6.7), we must have

$$\xi_1 = \sqrt{\frac{2}{\Lambda}}(a_1 + 2k\pi) \quad \hat{\xi}_1 = \sqrt{\frac{2}{\Lambda}}(a_1 + 2\hat{k}\pi)$$

therefore $\hat{\xi}_1 - \xi_1 = \sqrt{\frac{2}{\Lambda}} 2m\pi$ for some integer m that cannot be zero. Let us take the smallest possible value $m = 1$ (or $m = -1$). This shows that

$$\hat{\xi}_1 - \xi_1 = \sqrt{\frac{2}{\Lambda}} 2\pi. \tag{6.8}$$

Exactly in the same way, we get

$$\hat{\xi}_2 - \xi_2 = \sqrt{\frac{2}{\Lambda}} 2\pi. \tag{6.9}$$

In particular we see that the parallelogram must satisfy $\hat{\xi}_1 - \xi_1 = \hat{\xi}_2 - \xi_2$ and therefore, it is a rhombus. We are going to give a simple relation between the area of the parallelogram and the eigenvalue Λ. Assume that the parallelogram has vertices A, B, C, D with $B = A + \rho_1 \mathbf{e_1}^\perp$ and $D = A + \rho_2 \mathbf{e_2}^\perp$ where $\mathbf{e_1}^\perp$ and $\mathbf{e_2}^\perp$ are the vectors respectively orthogonal to $\mathbf{e_1}$ and $\mathbf{e_2}$ with the same norm. The line (AB) corresponds to ξ_1 and the line (AD) to $\hat{\xi}_2$ in the previous notations. Then the length of the basis AB is $AB = \rho_1 \|\mathbf{e_1}^{\perp}\|$. On the other hand, the height h of the parallelogram is given by the distance between B and its orthogonal projection B_1 on the line (CD). In other words the height is given by

$$h = \frac{1}{\|\mathbf{e_1}\|} BB_1 \cdot \mathbf{e}_1.$$

Now $BB_1 \cdot \mathbf{e}_1 = AB_1 \cdot \mathbf{e}_1 = \hat{\xi}_1 - \xi_1$ by definition of the two lines. Finally the area of the parallelogram Ω, that is $AB \times h$ is given by

$$|\Omega| = \rho_1 \|\mathbf{e_1}^{\perp}\| \frac{1}{\|\mathbf{e_1}\|}(\hat{\xi}_1 - \xi_1) = \rho_1 \sqrt{\frac{2}{\Lambda}} 2\pi.$$

It remains to express ρ_1 taken into account the relation (6.9). Let B_2 be the orthogonal projection of B on the line (AD). By definition we have $BB_2 \cdot \mathbf{e_2} = -\hat{\xi}_2 + \xi_2$. Now

$$\rho_1 \mathbf{e_1}^{\perp} \cdot \mathbf{e_2} = AB \cdot \mathbf{e_2} = B_2 B \cdot \mathbf{e_2} = \hat{\xi}_2 - \xi_2.$$

Thus

$$\rho_1 = \frac{(\hat{\xi}_2 - \xi_2)}{\mathbf{e_1}^{\perp} \cdot \mathbf{e_2}} = \sqrt{\frac{2}{\Lambda}} 2\pi \frac{\sqrt{\mu(\lambda + 2\mu)}}{4}.$$

Therefore we have proved that the area of the parallelogram is given by

$$|\Omega| = \frac{2\pi^2 \sqrt{\mu(\lambda + 2\mu)}}{\Lambda}. \tag{6.10}$$

Let us rephrase this formula in stating the following theorem:

Theorem 6.1. *Let Ω be a parallelogram defined by the four lines*

$$\begin{cases} \mathbf{e_1} \cdot X = \xi_1 \\ \mathbf{e_1} \cdot X = \hat{\xi}_1 \end{cases} \qquad \begin{cases} \mathbf{e_2} \cdot X = \xi_2 \\ \mathbf{e_2} \cdot X = \hat{\xi}_2 \end{cases}$$

where

$$\mathbf{e_1} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \mathbf{e_2} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

and

$$\alpha = \frac{1}{\sqrt{\lambda + 2\mu}} - \frac{1}{\sqrt{\mu}} \quad \beta = \frac{1}{\sqrt{\lambda + 2\mu}} + \frac{1}{\sqrt{\mu}}.$$

Assume that $\hat{\xi}_1 - \xi_1 = \hat{\xi}_2 - \xi_2$. Then an eigenvalue of the parallelogram is given by

$$\Lambda = \frac{2\pi^2 \sqrt{\mu(\lambda + 2\mu)}}{|\Omega|} \tag{6.11}$$

with an eigenvector U of the form $U = (u, u)$ where

$$u(x, y) = \sin(\omega_1(x + y) - \theta_1) - \sin(\omega_2(x - y) - \theta_2) \tag{6.12}$$

with

$$\omega_1^2 = \frac{\Lambda}{2\lambda + 4\mu} \quad \omega_2^2 = \frac{\Lambda}{2\mu}.$$

Remark 6.1. *In the above synthesis, we have studied the case $C_1 = C_2$. We claim that in the case $C_1 \neq C_2$ there are no (bounded) domain Ω in the plane such that*

$$u(v,w) = C_1 \sin(\omega_1 w - \theta_1) - C_2 \sin(\omega_2 v - \theta_2) = 0 \quad \text{on the boundary } \partial\Omega.$$

Indeed if we would have two level lines of the function $u(v,w)$ crossing at some point A, necessarily the gradient of u must vanish at A. That would provide the three relations

$$\begin{cases} C_1 \sin(\omega_1 w - \theta_1) - C_2 \sin(\omega_2 v - \theta_2) = 0 \\ C_1 \cos(\omega_1 w - \theta_1) = 0 \\ C_2 \cos(\omega_2 v - \theta_2) = 0 \end{cases}$$

that are clearly incompatible since we can assume $C_1 \neq 0$ and $C_2 \neq 0$ for a bounded domain.

Remark 6.2. *If we look for domains for which the eigenvector is $U = (u(x,y), Au(x,y))$ for some real number A, following the same approach we get other parallelograms but their eigenvalue is still given by the formula* (6.11).

Let us come back to the possible minimality of the disk. We have seen in Theorem 5.2 that the disk cannot be a minimizer if the Poisson coefficient is less than $\nu^* \simeq 0.349...$ but we were not able to conclude for larger values of the Poisson coefficient (between ν^* and 0.5) since we know that in this case the first eigenvalue of the disk is simple and the disk is a local minimizer (at least in a weak sense). Now our previous analysis allows us to increase the interval of values of the Poisson coefficient for which the disk is not optimal:

Corollary 6.1. *Assume that the Poisson coefficient ν satisfies*

$$\nu < \frac{j_{1,1}^4 - 8\pi^2}{2(j_{1,1}^4 - 4\pi^2)} \simeq 0.3879$$

then the disk is not a minimizer of Λ (among sets of given volume).

Proof. According to Theorem 5.2, it suffices to compare our previous parallelogram of area π with the first eigenvalue of the disk that is $\mu j_{1,1}^2$. Thus we get the thesis as soon as $2\pi\sqrt{\mu(\lambda + 2\mu)} < \mu j_{1,1}^2$. This is equivalent to $\frac{\lambda}{\mu} + 2 < \frac{j_{1,1}^4}{4\pi^2}$. Now using the relation between the Lamé coefficents and the Poisson coefficient, we know that $\lambda/\mu = 2\nu/(1-2\nu)$. Therefore

$$\frac{\lambda}{\mu} + 2 < \frac{j_{1,1}^4}{4\pi^2} \Leftrightarrow 8\pi^2(1-\nu) < j_{1,1}^4(1-2\nu) \Leftrightarrow \nu < \frac{j_{1,1}^4 - 8\pi^2}{2(j_{1,1}^4 - 4\pi^2)}.$$

□

6.2 Rectangles

We now consider the range $\frac{3}{8} \leq \nu \leq \frac{2}{5}$, which corresponds to $a = 1/(1-2\nu) \in [4,5]$, and use suitable rectangles as competitors. Since $3/8 < 0.38$, this will allow us, together with the previous arguments, to cover the whole range $\nu \in (-1, 0.4]$ and prove that the disk is not optimal there.

We consider a rectangle $\Omega_L = (0, L) \times (0, \ell)$ of area π. It will be useful to write the length and the width of the rectangle as

$$L = \sqrt{\frac{\pi}{t}} \quad \text{and} \quad \ell = \sqrt{t\pi}, \qquad t \in (0, 1].$$

Let us denote by φ_1 the first (normalized) eigenfunction for the Dirichlet-Laplacian of Ω_L, defined by

$$\varphi_1(x, y) = \frac{2}{\sqrt{\pi}} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{\ell}\right),$$

and another eigenfunction

$$\varphi_2(x, y) = \frac{2}{\sqrt{\pi}} \sin\left(\frac{2\pi x}{L}\right) \sin\left(\frac{2\pi y}{\ell}\right).$$

This other eigenfunction could be the fourth one (for a rectangle not too far from the square), but can also have a larger index. We will explain below why we do this particular choice.

Now the idea is to plug in the variational formulation defining $\Lambda(\Omega_L)$ a family of vectors, for $X = (\alpha_1, \alpha_2, \beta_1, \beta_2)$:

$$U_X = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\varphi_1 + \alpha_2\varphi_2 \\ \beta_1\varphi_1 + \beta_2\varphi_2 \end{pmatrix}.$$

Since the eigenfunctions of the Laplace operator define an orthonormal basis, we have

$$\int_{\Omega_L} |\nabla u_1|^2 + |\nabla u_2|^2 = \left(\frac{\pi^2}{L^2} + \frac{\pi^2}{\ell^2}\right)(\alpha_1^2 + 4\alpha_2^2 + \beta_1^2 + 4\beta_2^2)$$

and

$$\int_{\Omega_L} u_1^2 + u_2^2 = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2.$$

It remains to compute $\int_{\Omega_L} (\text{div}(U_X))^2$. We obtain

$$\int_{\Omega_L} (\text{div}(U_X))^2 = \frac{\pi^2}{L^2}\left(\alpha_1^2 + 4\alpha_2^2\right) + \frac{\pi^2}{\ell^2}\left(\beta_1^2 + 4\beta_2^2\right) - \frac{128}{9\pi}\left(\alpha_1\beta_2 + \alpha_2\beta_1\right). \tag{6.13}$$

Using $a = 1/(1 - 2\nu)$, this implies using this admissible test function, that

$$\frac{\Lambda(\Omega_L)}{\mu} \leq \frac{Q(X)}{\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2} \tag{6.14}$$

where Q is the quadratic form defined by

$$Q(X) = \alpha_1^2\left((1 + a)\frac{\pi^2}{L^2} + \frac{\pi^2}{\ell^2}\right) + \alpha_2^2\left(4(1 + a)\frac{\pi^2}{L^2} + \frac{4\pi^2}{\ell^2}\right) + \beta_1^2\left(\frac{\pi^2}{L^2} + (1 + a)\frac{\pi^2}{\ell^2}\right) + \beta_2^2\left(\frac{4\pi^2}{L^2} + 4(1 + a)\frac{\pi^2}{\ell^2}\right) - \frac{128a}{9\pi}\left(\alpha_1\beta_2 + \alpha_2\beta_1\right).$$

Now we have to choose $X = (\alpha_1, \alpha_2, \beta_1, \beta_2)$ that give the lowest possible value for the ratio in (6.14). This lowest value exactly corresponds to the smallest eigenvalue of the 4×4 matrix of the quadratic form Q. This matrix $\mathcal{M}$ has the simple structure

$$\mathcal{M} = \begin{pmatrix} a_1 & 0 & 0 & b \\ 0 & a_2 & b & 0 \\ 0 & b & a_3 & 0 \\ b & 0 & 0 & a_4 \end{pmatrix}.$$

Its characteristic polynomial factorizes as $P(x) = [(a_2 - x)(a_3 - x) - b^2][(a_1 - x)(a_4 - x) - b^2]$ with $b = -64a/9\pi$ and

$$\begin{aligned} a_1 &= \pi(1+a)t + \tfrac{\pi}{t} \\ a_2 &= 4\pi(1+a)t + \tfrac{4\pi}{t} \\ a_3 &= \pi t + \tfrac{(1+a)\pi}{t} \\ a_4 &= 4\pi t + \tfrac{4(1+a)\pi}{t}. \end{aligned}$$

We observe that $a_2 a_3 = a_1 a_4$ and $a_1 + a_4 \geq a_2 + a_3$ because $t \leq 1$. Therefore the trinome $(a_1 - x)(a_4 - x) - b^2$ is always less than $(a_2 - x)(a_3 - x) - b^2$ and the smallest root of $P(x)$ is the smallest root of $q_1(a, t, x) := (a_1 - x)(a_4 - x) - b^2$. More precisely, the question is to know whether the smallest root of q_1 is smaller than $j_{1,1}^2$ because our aim is to compare the rectangle Ω_L with the unit disk. Since $q_1(a, t, 0) = a_1 a_4 - b^2$, we see that

$$q_1(a, t, 0) \geq q_1(a, \frac{1}{2}, 0) = 4\pi^2 \left(a^2 + 4a + 4 - \frac{1024a^2}{81\pi^4} \right) > 0.$$

Therefore, we get the thesis as soon as we can find some $t^* \in (0, 1]$ such that $q_1(a, t^*, j_{1,1}^2) < 0$ for all $a \in [4, 5]$. It turns out that the particular choice $t^* = 2/5 = 0.4$ achieves this aim. This is an elementary analysis to prove that the polynomial expression

$$q_1(a, \frac{2}{5}, j_{1,1}^2) = j_{1,1}^4 - \pi j_{1,1}^2 \left(\frac{29}{2} + \frac{52a}{5} \right) + 4\pi^2 \left(a^2 + \frac{169}{36}(a+1) \right) - \frac{4096a^2}{81\pi^2}$$

remains negative for all $a \in [4, 5]$. Thus we have proved

Theorem 6.2. *Let Ω_L be the rectangle of length $L = \sqrt{5\pi/2}$ and width $\ell = \sqrt{2\pi/5}$. Then its first eigenvalue satisfies*

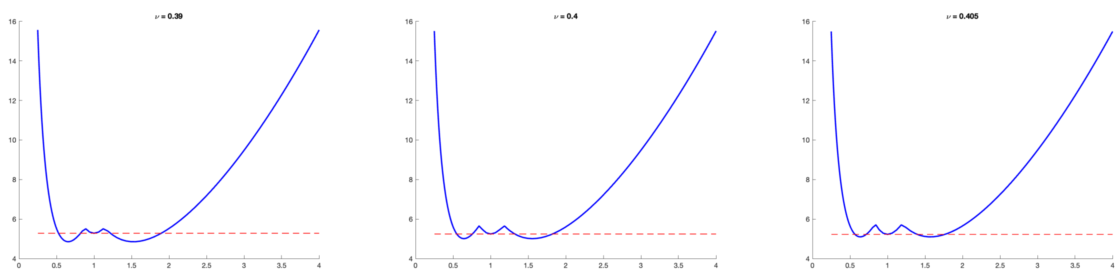
$$\Lambda(\Omega_L) < \mu j_{1,1}^2$$

for all values of the Poisson coefficient $\nu \in [\frac{3}{8}, \frac{2}{5}]$. Therefore the disk is not a minimizer in this range of values of ν.

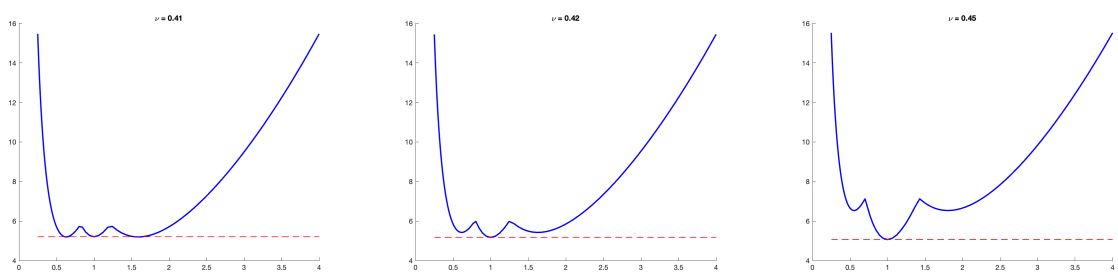
Remark 6.3. *Let us explain why we use the pair φ_1, φ_2 as test functions. The goal is to obtain a cross term from the divergence contribution that is strong enough to lower the first eigenvalue of the matrix $\mathcal{M}$. The first two eigenfunctions of the rectangle do not provide such a term, and a direct comparison shows that the above choice is more effective.*

6.3 The case of ellipses

When Ω is an ellipse, we do not have explicit formulae for the eigenvalues and eigenfunctions, nor a sufficiently sharp upper bound, as in Sections 6.1 and 6.2. We therefore complement the previous analysis with numerical simulations. We compute an approximation of $\Lambda(\Omega_a)$, where Ω_a denotes the ellipse with semi-axes a and $1/a$. For $a = 1$, this gives the disk, whose first eigenvalue is $\mu j_{1,1}^2$ for $\nu \geq 0.35$, according to Theorem 5.1. The numerical results, obtained with Matlab and summarized in Figure 1, suggest that the disk is not optimal for $\nu \leq \bar{\nu}$, while it appears to be optimal among ellipses of area π for $\nu > \bar{\nu}$, where $\bar{\nu}$ is close to 0.41.



(a) Case $\nu = 0.39$ (b) Case $\nu = 0.40$ (c) Case $\nu = 0.405$



(d) Case $\nu = 0.41$ (e) Case $\nu = 0.42$ (f) Case $\nu = 0.45$

Figure 1: Graph of $\Lambda(\Omega_a)$ with respect to a. The dotted line corresponds to the first eigenvalue of the disk.

7 Conclusion

7.1 A conjecture

Our numerical simulations do not produce a competitor with a first eigenvalue lower than that of the disk when $\nu \geq 0.41$. For instance, the best rectangles improve on the disk only below a value close to 0.41, and the same behavior is observed for ellipses. This suggests the existence of a threshold $\hat{\nu}$ such that the disk is a minimizer whenever $\nu \geq \hat{\nu}$. The following heuristic argument supports this conjecture.

1. First we prove in Section 7.2 below that the eigenvalues of the Lamé system converge to the eigenvalues of the Stokes system when $\nu \to 1/2$.

2. If we assume that the disk minimizes the first Stokes eigenvalue in the plane (this is another conjecture as explained in [31]),
3. if we could then use the local minimality of the disk for our problem in a strong sense (for example for the Hausdorff convergence),

then the conjecture would follow. Indeed, the Γ-convergence result stated below implies that minimizers of the Lamé problem converge to minimizers of the Stokes problem. Hence, for ν close enough to $1/2$, a minimizer of the Lamé problem should lie in a neighborhood of the disk where the disk is locally minimizing.

7.2 Γ-convergence as $\nu \to 1/2$

As explained just above, it is interesting to prove that when $\nu \to 1/2$, the eigenvalues of the Lamé system converge to those of the Stokes system.

For that purpose we will renormalize the eigenvalue $\Lambda(\Omega)$ and work with the parameter $a := \frac{\lambda+\mu}{\mu} = \frac{1}{1-2\nu}$ that we consider satisfying $a \to +\infty$. In other words the right quantity to study becomes

$$\Lambda^a(\Omega) := \frac{1}{\mu}\Lambda(\Omega) = \min_{u\in H_0^1(\Omega)^N\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx + a\int_\Omega (\mathrm{div}(u))^2\,dx}{\int_\Omega |u|^2\,dx}.$$

In this section we would like to investigate the limiting behavior as $a \to +\infty$ (or equivalently $\nu \to 1/2$) of $\Lambda^a(\Omega)$. In particular we will show that for Ω fixed, $\Lambda^a(\Omega)$ converges to the Stokes eigenvalue, and moreover under some standard geometrical restrictions on the admissible sets Ω, the shape functional $\Omega \mapsto \Lambda^a(\Omega)$ Γ-converges to $\Omega \mapsto \lambda_1^{\text{Stokes}}(\Omega)$. Here, $\lambda_1^{\text{Stokes}}(\Omega)$ is the Stokes eigenvalue already defined in Section 2.2.1 as

$$\lambda_1^{\text{Stokes}}(\Omega) := \min_{\substack{u\in H_0^1(\Omega)^N\setminus\{0\}\\ \text{s.t. } \mathrm{div}(u)=0}} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega |u|^2\,dx}.$$

We first establish the convergence of $\Lambda^a(\Omega)$ for Ω, fixed. For this purpose we define the following two quadratic forms on $H_0^1(\Omega)^N$:

$$Q_a(u) := \int_\Omega |\nabla u|^2\,dx + a\int_\Omega (\mathrm{div}(u))^2\,dx.$$

$$Q_\infty(u) := \begin{cases} \int_\Omega |\nabla u|^2\,dx & \text{if } \mathrm{div}(u)=0 \\ +\infty & \text{otherwise.}\end{cases}$$

Proposition 7.1. *Let Ω be a bounded open set. Then Q_a Γ-converges to Q_∞ for the L^2 topology when $a \to +\infty$. As a consequence, the associated Lamé operator converges in the strong resolvent sense to the Stokes operator and in particular*

$$\lim_{a\to\infty} \Lambda^a(\Omega) = \lambda_1^{\text{Stokes}}(\Omega).$$

Proof. The proof is standard, but we include the details for completeness.
Step 1. Γ*-limsup.* Let $u \in H_0^1(\Omega)^N$ be such that $Q_\infty(u) < +\infty$ (otherwise there is nothing to

prove). Then we take as a recovery sequence the constant sequence $u_\lambda = u$ and we use that $\operatorname{div}(u) = 0$, together with Korn inequality (see (3.1)) to deduce that $Q_a(u) = Q_\infty(u)$ and a fortiori,

$$\limsup_{a\to+\infty} Q_a(u) = Q_\infty(u),$$

which directly proves the limsup inequality.

Step 2. Γ*-liminf.* Assume that $u_a \to u$ in $L^2(\Omega)$. We can assume that

$$\sup_a Q_a(u_a) \le C,$$

otherwise there is nothing to prove. But this means thanks to Korn inequality, that u_a is uniformly bounded in $H^1(\Omega)^N$ thus converges weakly in $(H^1)^N$ and strongly in $(L^2)^N$, up to a subsequence, to some function $u \in H^1_0(\Omega)^N$ and

$$\int_\Omega |\nabla u|^2 \le \liminf_{a\to+\infty} \int_\Omega |\nabla u_a|^2\, dx.$$

Passing to the liminf in the inequality

$$\int_\Omega (\operatorname{div}(u_a))^2\, dx \le \frac{C}{a},$$

we deduce that $\operatorname{div}(u) = 0$ thus

$$Q_\infty(u) = \int_\Omega |\nabla u|^2 dx$$

which finishes the liminf inequality, and the proof of Γ-convergence.

Then the end of the statement of the Proposition follows from the standard theory of Γ-convergence that asserts that Γ-convergence of quadratic forms implies the convergence in the strong resolvent sense of the associated operators (see [16, Chapter 12]). A review of these properties can also be found in [1, Section 1.1]. In particular, the convergence of the eigenvalues follows from the fact that the associated operators have compact resolvent. More precisely, we first notice that the quadratic forms Q_a and Q_∞ are equi-coercive thanks to Poincaré-Korn inequality. They are also semi-continuous with respect to the $(L^2)^N$ topology. The associated operators are thus self-adjoint, invertible and thanks to the compact embedding of $H^1_0(\Omega)^N$ into $L^2(\Omega)^N$, their inverse are compact operators. We then apply Proposition 7 in [1] with $X = H^1_0(\Omega)^N$ and $H = L^2(\Omega)^N$ which establishes the convergence of the spectrum for the inverse operators, from the Γ-convergence of Q_a to Q_∞. The spectrum of the operators itself then follows immediately. □

We now consider the Γ-convergence of $\Lambda^a(\Omega)$ with respect to the domain variable Ω. For simplicity, we restrict ourselves to a class of uniformly Lipschitz domains, more precisely domains satisfying a uniform ε-cone property; see [32, Definition 2.4.1]. This class is endowed with the complementary Hausdorff distance; see [32, Definition 2.2.8]. We shall also use the stability of the Dirichlet problem along convergent sequences in this class; see [32, Theorem 3.2.13].

Proposition 7.2. *Let $D \subset \mathbb{R}^N$, $\varepsilon_0 > 0$ and $V > 0$ be fixed, and let $\mathcal{A}_0$ be the class of domains $\Omega \subset D$ satisfying the ε_0-cone property and the volume constraint $|\Omega| = V$. Then, as $a \to +\infty$, the family of functionals $\Lambda^a : \mathcal{A}_0 \to \mathbb{R}$ Γ-converges to $\lambda_1^{\text{Stokes}}$ with respect to the complementary Hausdorff distance.*

Proof. Step 1. Γ*-limsup.* For $\Omega \in \mathcal{A}$ being given we take the constant sequence $\Omega_a = \Omega$ as a recovery sequence. Thanks to Proposition 7.1 we know that

$$\lim_{a \to +\infty} \Lambda^a(\Omega) = \lambda_1^{\text{Stokes}}(\Omega),$$

which proves the Γ-limsup property.

Step 2. Γ*-liminf.* Let Ω_a converging to Ω for the complementary Hausdorff distance. Since $\mathcal{A}$ is closed for the complementary Hausdorff convergence (see [32, Theorem 2.4.10]), it follows that $\Omega \in \mathcal{A}$, and it is easily seen that $|\Omega| = V$.

Let u_a be a sequence of normalized eigenfunctions, associated to $\Lambda^a(\Omega_a)$. In particular, $u_a \in H_0^1(\Omega_a)^N$ and

$$\Lambda^a(\Omega_a) = \int_{\Omega_a} |\nabla u_a|^2 \, dx + a \int_{\Omega_a} (\text{div}(u_a))^2 \, dx.$$

We may assume without loss of generality that $(\Lambda^a(\Omega_a))_a$ is a bounded sequence. Therefore, the sequence u_a is uniformly bounded in $H_0^1(D)^N$ and converges up to a subsequence (not relabelled) to a function $u \in H_0^1(D)^N$, weakly in $H^1(D)^N$ and strongly in $L^2(D)^N$. In particular $\|u\|_{L^2(D)^N} = 1$. Moreover by the Mosco convergence of Ω_a (see [28, Theorem 4]) we know that $u \in H_0^1(\Omega)^N$. Finally, the bound on $\Lambda^a(\Omega_a)$ tells us that

$$\int_{\Omega_a} (\text{div}(u_a))^2 \, dx \leq \frac{C}{a} \xrightarrow[a \to +\infty]{} 0,$$

and we deduce from the weak convergence of u_a to u in $H^1(D)^N$ that $\text{div}(u) = 0$ thus u is an admissible competitor for the Rayleigh quotient that defines $\lambda_1^{\text{Stokes}}(\Omega)$.

Therefore, the following sequence of inequalities holds:

$$\begin{aligned} \lambda_1^{\text{Stokes}}(\Omega) &\leq \int_{\Omega} |\nabla u|^2 \, dx \leq \liminf \int_D |\nabla u_a|^2 \, dx \\ &\leq \liminf \int_D |\nabla u_a|^2 \, dx + a \int_D (\text{div}(u_a))^2 \, dx \\ &= \liminf \Lambda^a(\Omega_a), \end{aligned}$$

which finishes the proof of the liminf inequality, and so follows the proof of Γ-convergence. $\square$

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