EXISTENCE OF COMPLEMENTS FOR FOLIATIONS

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Abstract. This paper demonstrates the existence of $\mathbb{Q}$-complements for algebraically integrable log-Fano foliations on klt varieties. Additionally, we investigate properties of algebraically integrable Fano foliations such as a form of inversion of adjunction, which allows to infer information on the non-log canonical locus of a foliated pair from the behavior of its restriction to a normal prime divisor, as well as a foliated analog of the classical connectedness principle by Kollár on the properties of the exceptional locus.

1. Introduction

The notion of complements, a distinguished class of global sections of the canonical sheaf of a variety, was introduced by Shokurov in [30]. Subsequently, the theory of complements was substantially developed and has come to play a central role in birational geometry (see, for example, [31, 28, 29, 5, 14, 6, 19, 12, 16, 13, 21]). In [5], Birkar established the existence of monotone complements for varieties of Fano type in fixed dimension. More precisely, he shows that there exists an integer n only depending on the dimension of X and the coefficient set Γ of B such that if (X, B) is an lc pair of Fano type, then there is a divisor $B^+ \geq B$ satisfying (X, B^+) is lc and $n(K_X + B^+) \sim 0$. This result is commonly referred to as the existence of bounded complements. The existence of bounded complements — or more precisely, the existence of a uniform bound on the index n under suitable assumptions on the pair (X, B) — was a key ingredient in Birkar's proof [6] to verify a long standing conjecture due to Borisov-Alexeev-Borisov (BAB) which states that the set of mild singular Fano varieties of a fixed dimension forms a bounded family.

In parallel, significant progress has been achieved in recent years in the minimal model program (MMP) for foliated varieties. In this setting, the canonical divisor K_X of a variety X is replaced by the canonical class $K_{\mathcal{F}}$ associated to the foliation $\mathcal{F}$, and the resulting foliated MMP provides a framework for studying the geometry of both the ambient variety and the foliation. This constitutes a natural generalization of the classical MMP, as $K_{\mathcal{F}} = K_X$ when $\mathcal{F} = \mathcal{T}_X$, the tangent sheaf. It is therefore expected that many results from the classical theory admit analogues in the foliated setting.

Using the classification of log canonical foliation singularities on surfaces (cf. [18]), we can show the existence of 1-complements for log canonical Fano foliations on klt surfaces:

Proposition 1.1 (= Proposition 4.5)**.** *Let $\mathcal{F}$ be an lc Fano foliation of rank one on a klt surface X. Then $\mathcal{F}$ has a 1-complement, that is, there is an effective divisor B such that the foliated pair $(\mathcal{F}, B)$ is log canonical and $K_{\mathcal{F}} + B \sim 0$.*

In higher dimensions, recent advances in the development of the Minimal Model Program (MMP) for foliated varieties, in particular on algebraically integrable foliations (see [10, 15, 24]) allow to investigate the existence of (bounded) complements in the foliated setting. As opposed to complements for varieties, even the existence of $\mathbb{Q}$-complements due to the general failure of Bertini-type results (cf. [9, Section 3.5.2]) is by no means trivial and the main content of the present publication. By a $\mathbb{Q}$-complement for a foliated triple $(X, \mathcal{F}, B)$ (cf. Definition 3.5) we mean a $\mathbb{Q}$-divisor

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$B^+ \geq B$ such that $K_{\mathcal{F}} + B^+ \sim_{\mathbb{Q}} 0$ and $(\mathcal{F}, B^+)$ is log canonical. So far we are unfortunately unable to demonstrate the existence of bounded complements in full generality. On the one hand because we fall back on classical vanishing theorems to compensate for the lack of their analogues for $K_{\mathcal{F}}$ in the process of lifting sections from invariant subvarieties to X, on the other hand because we face technical difficulties when trying to settle the base case on the closure of general fibres. However, we are confident to overcome these difficulties in ongoing work, in which we moreover address boundedness questions for Fano foliations. We see our paper as a partial resolution of a problem set out by Cascini and Spicer at a conference [27]. Our main theorem on the existence of $\mathbb{Q}$-complements is the following:

Theorem 1.2 (= Theorem 5.6, Main Theorem). *Let $(X, \mathcal{F}, B)$ be an lc Fano foliated triple with $\mathcal{F}$ algebraically integrable. Then there exists a $\mathbb{Q}$-complement. That is, there exists an effective divisor $\Delta \sim_{\mathbb{Q}} -(K_{\mathcal{F}} + B)$ such that $(X, \mathcal{F}, B + \Delta)$ is log canonical.*

As mentioned in the abstract, we furthermore provide proofs of a number of side results on Fano and log Calabi-Yau foliations that may be of independent interest. To be quoted first, a weak form of a foliated inversion of adjunction to foliation invariant divisors:

Theorem 1.3 (= Theorem 5.3, Inversion of adjunction). *Let $(X, \mathcal{F}, B)$ be a foliated triple with $\mathcal{F}$ being algebraically integrable. Let $S \subseteq X$ be a prime invariant divisor with normalisation $\nu\colon S^\nu \to S$ such that $(S^\nu, \mathcal{F}_{S^\nu}, \mathrm{Diff}_{S^\nu}(\mathcal{F}, B))$ is log canonical. Then*

$$\mathrm{Nlc}(X, \mathcal{F}, B) \cap S = \varnothing.$$

It is to be pointed out that we exclusively deduce log canonicity of a pair from log canonicity of its restriction to an invariant divisor, however do not consider the behavior of any other type of singularities with respect to such restriction. In the same spirit, we have:

Proposition 1.4 (= Proposition 5.4). *Let $(X, \mathcal{F}, B)$ be a foliated triple where $\mathcal{F}$ is an algebraically integrable foliation with $K_{\mathcal{F}} + B \sim_{\mathbb{Q}} 0$. If $\mathrm{Nlc}(X, \mathcal{F}, B) \neq \varnothing$, then it intersects every closure of leaf of $\mathcal{F}$.*

This can be seen as a weak inversion of subadjunction for foliated log Calabi-Yau triples. The "weak" is to be understood in the same way as above. We further notice that even though we phrase the statement here only for restrictions to fibres, it equally applies to general members of covering families of invariant subvarieties of any dimension.

Lastly, we include the following foliated version of a connectedness principle for log canonical centres of algebraically integrable log Fano foliations:

Lemma 1.5 (= Lemma 5.8). *Let $(\mathcal{F}, B)$ be a foliated pair on a normal variety X such that*

(1) $\mathcal{F}$ is induced by a proper morphism $g\colon X \to Z$ with connected fibres;
(2) $-(K_{\mathcal{F}} + B)$ is g-big and g-nef;

Then $\lfloor B \rfloor$ is connected.

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2. Sketch of the proof of main theorem

Analogous to the classical case, we aim to give an inductive proof; that is, we seek to invoke the existence of foliated complements under the assumption that such complements exist on invariant subvarieties.

Let $(X, \mathcal{F}, B)$ be an lc Fano foliated triple with $\mathcal{F}$ algebraically integrable. Let S be the closure of a general leaf of $\mathcal{F}$. In general, S need not be normal, but for simplicity we assume that S is normal. By adjunction formula (cf. Proposition 3.7 and 5.1), we obtain $(K_{\mathcal{F}} + B)|_S = K_{\mathcal{F}_S} + B_S$ for some effective divisor B_S and $(\mathcal{F}_S, B_S)$ is an lc Fano foliated triple with

$$\operatorname{corank}(\mathcal{F}_S) = \dim S - \operatorname{rank} \mathcal{F}_S = \dim X - s - \operatorname{rank} \mathcal{F} = \operatorname{corank} \mathcal{F} - 1.$$

By ampleness of $-(K_{\mathcal{F}} + B)$, Serre vanishing gives $\mathrm{H}^1(X, -m(K_{\mathcal{F}} + B) - S) = 0$ for all sufficiently large m. Assume further that $m(K_{\mathcal{F}_S} + B_S^+) \sim 0$. Then the restriction map

$$\mathrm{H}^0(X, -m(K_{\mathcal{F}} + B)) \twoheadrightarrow \mathrm{H}^0(S, -m(K_{\mathcal{F}_S} + B_S))$$

is surjective. Since $m(B_S^+ - B_S) \sim -m(K_{\mathcal{F}_S} + B_S)$, there exists an effective divisor $\Delta \sim -m(K_{\mathcal{F}} + B)$ on X such that $\Delta|_S = m(B_S^+ - B_S)$. Set $B^+ = B + \frac{1}{m}\Delta$. Then we have $K_{\mathcal{F}} + B^+ \sim_{\mathbb{Q}} 0$. By inversion of adjunction (Theorem 5.3), the non-lc locus $\operatorname{Nlc}(X, \mathcal{F}, B)$ of $(X, \mathcal{F}, B^+)$ does not meet S. However, Proposition 5.4 implies that, if $\operatorname{Nlc}(X, \mathcal{F}, B^+) \neq \varnothing$, then it intersects every closure of leaf of $\mathcal{F}$. Thus, we conclude that $(X, \mathcal{F}, B^+)$ is an lc foliated triple.

Last but not least, to avoid any misunderstandings from the start, the authors wish to explicitly point out that since we do not have a foliated version of Kawamata-Viehweg vanishing to replace Serre vanishing in our proofs, the methods applied in this work are not suitable to demonstrate the existence of n-complements for bounded n which is of greater technical relevance.

3. Preliminaries

We work over an algebraically closed field of characteristic zero.

Definition 3.1. Let X be a normal variety. A *foliation* $\mathcal{F}$ on X is a coherent subsheaf of the tangent sheaf $\mathcal{T}_X$ such that

- $\mathcal{F}$ is saturated, that is, $\mathcal{T}_X/\mathcal{F}$ is torsion free and
- $\mathcal{F}$ is closed under Lie bracket.

The *rank* of a foliation $\mathcal{F}$, denoted by $\operatorname{rank} \mathcal{F}$, is the rank of $\mathcal{F}$ as a sheaf. The *corank* of $\mathcal{F}$ is $\dim X - \operatorname{rank} \mathcal{F}$. If $\mathcal{F} = 0$, then we say $\mathcal{F}$ is the foliation by points.

Definition 3.2. Suppose $\mathcal{F}$ is a rank r foliation on a normal variety X. Notice that there exists an open embedding $j \colon X_0 \hookrightarrow X$ such that X_0 is smooth, $\operatorname{codim}(X \setminus X_0) \geq 2$ and $\mathcal{F}$ is locally free on X_0, so $\wedge^r \mathcal{F}$ is an invertible sheaf on X_0. We define the *canonical divisor* of $\mathcal{F}$ to be any divisor $K_{\mathcal{F}}$ on X such that $\mathcal{O}_X(-K_{\mathcal{F}}) \cong j_*(\wedge^r \mathcal{F})$.

Definition 3.3. Let X be a normal variety and $\mathcal{F}$ a rank r foliation on X. A subvariety $S \subseteq X$ is called $\mathcal{F}$*-invariant* if for any open subset $U \subseteq X$ and any section $\partial \in \mathrm{H}^0(U, \mathcal{F})$ we have:

$$\partial(I_{S\cap U}) \subseteq I_{S\cap U}$$

where $I_{S\cap U}$ denotes the ideal sheaf of $S \cap U$ in U. If $\Delta \subseteq X$ is a prime divisor, one defines $\epsilon(\Delta) = 1$ if Δ is invariant in the above sense and $\epsilon(\Delta) = 0$ otherwise.

Let $f\colon Y \dashrightarrow X$ be a dominant map between normal varieties and $\mathcal{F}$ be a foliation on X, then there is a foliation on Y as constructed in [20, Section 3.2], which is denoted by $f^{-1}\mathcal{F}$. For any birational map $g\colon X \to X'$, we denote by $g_*\mathcal{F} := (g^{-1})^{-1}\mathcal{F}$ the *pushforward* of $\mathcal{F}$ on X'. A foliation $\mathcal{F}$ on X is *algebraically integrable* if there exists a dominant map $f\colon X \dashrightarrow Z$ such that $\mathcal{F} = f^{-1}\mathcal{F}_Z$ where $\mathcal{F}_Z$ is the foliation by points on Z, and we say $\mathcal{F}$ is induced by f.

Definition 3.4. Let $\mathcal{F}$ be a foliation on a normal variety X. We say that $\mathcal{F}$ is *algebraically integrable* if it is induced by some dominant map $f\colon X \dashrightarrow Y$. In this case we also call a divisor D *vertical* if it is $\mathcal{F}$-invariant, and *horizontal* if it is not $\mathcal{F}$-invariant.

Definition 3.5. Let X be a normal variety, B be an effective $\mathbb{Q}$-divisor on X, and $\mathcal{F}$ be a foliation on X. Then we call $(\mathcal{F}, B)$ a *foliated pair* (or $(X, \mathcal{F}, B)$ a *foliated triple*) if $K_{\mathcal{F}} + B$ is $\mathbb{Q}$-Cartier. Let $f\colon Y \to X$ be any birational morphism from some normal variety Y and $\mathcal{G} := f^{-1}\mathcal{F}$ be the pullback foliation of $\mathcal{F}$ on Y. We have

$$K_{\mathcal{G}} + B_Y = f^*(K_{\mathcal{F}} + B) + \sum a(E_i, \mathcal{F}, B)E_i$$

where B_Y is the strict transform of B on Y. We say that the triple $(X, \mathcal{F}, B)$ is *log canonical* if for any such model Y over X, we have $a(E_i, \mathcal{F}, B) \geq -\epsilon(E_i)$. The triple $(X, \mathcal{F}, B)$ is called *strictly log canonical* if there is an exceptional divisor E over X such that $a(E, \mathcal{F}, B) = -\epsilon(E) = -1$.

Let $V \subseteq X$ be a subvariety. We say that V is an *lc* (resp. *strictly lc* resp. *non-lc*) *centre* of $(X, \mathcal{F}, B)$ if there exists a divisor E over X such that

- $a(E, \mathcal{F}, B) = -\epsilon(E)$ (resp. $a(E, \mathcal{F}, B) = -\epsilon(E) = -1$ resp. $a(E, \mathcal{F}, B) < -\epsilon(E)$) and
- $\mathrm{centre}_X(E) = V$.

We denote by $\mathrm{Nlc}(\mathcal{F}, B)$ the union of all the non-lc centres of $(\mathcal{F}, B)$.

We would like to mention that for algebraically integrable foliations, the best resolution we have are toroidal models (see [1, Theorem 2.1 resp. Proposition 4.4]). Thus, we cannot guarantee the existence of F-dlt modification. Instead, we use Property $(*)$-modifications, which are derivates of the above toroidalisations in *op.cit.* that made their first appearance in [2].

Definition 3.6 (cf. [2, Definition 3.8], Property $(*)$ modification)**.** Let X be a normal variety and $(\mathcal{F}, B)$ be a foliated pair on X where $\mathcal{F}$ is algebraically integrable and none of components of B is $\mathcal{F}$-invariant. A *Property* $(*)$ *modification* of $(\mathcal{F}, B)$ is a birational morphism $f\colon Y \to X$ such that

(1) Y is klt,
(2) $\mathcal{G} := f^{-1}\mathcal{F}$ is induced by an equidimensional morphism $Y \to Z$,
(3) $(\mathcal{G}, B_Y := f_*^{-1}B + \sum \epsilon(E)E)$ is log canonical where the summation is over all π-exceptional prime divisors, and
(4) $K_{\mathcal{G}} + B_Y + F = f^*(K_{\mathcal{F}} + B)$ where F is an effective divisor.

Proposition 3.7 (Restricted foliation, cf. [9, Proposition-Definition 3.7])**.** *Let $(\mathcal{F}, B)$ be a foliated pair on a normal variety X and S be an $\mathcal{F}$-invariant prime divisor with normalisation $n\colon S^\nu \to S$. Then there exists a* restricted foliation $\mathcal{F}_{S^\nu}$, *and an effective* $\mathbb{Q}$*-divisor* $\mathrm{Diff}_{S^\nu}(\mathcal{F}, B)$, *called the* different, *such that*

$$n^*(K_{\mathcal{F}} + B) \sim_{\mathbb{Q}} K_{\mathcal{F}_{S^\nu}} + \mathrm{Diff}_{S^\nu}(\mathcal{F}, B).$$

Definition 3.8. Let X be a normal variety and $(\mathcal{F}, B)$ be a foliated pair on X. We say that $(\mathcal{F}, B)$ is a *Fano foliated pair* (or $(X, \mathcal{F}, B)$ a *Fano foliated triple*) if the following conditions are satisfied:

(1) $-(K_{\mathcal{F}} + B)$ is ample and
(2) $(\mathcal{F}, B)$ is log canonical.

We call $(X, \mathcal{F}, B)$ *weak Fano* if it satisfies (1) and (3) above, and $-(K_{\mathcal{F}} + B)$ is big and nef.

We recall the definition of complements for classical pairs.

Definition 3.9. Let (X, B) be a pair. Then an n-complement is of the form $K_X + B^+$ such that

- (X, B^+) is log canonical,
- $n(K_X + B^+) \sim 0$, and
- $nB^+ \geq n\lfloor B \rfloor + \lfloor (n+1)\{B\} \rfloor$.

For the existence of n-complements for Fano pairs with bounded n, we refer to [5, Theorem 1.10]. Similarly, we define n-complements for foliations.

Definition 3.10. Let $(X, \mathcal{F}, B)$ be a foliated triple. Then an *n-complement* is of the form $K_{\mathcal{F}} + B^+$ such that

- $(X, \mathcal{F}, B^+)$ is log canonical,
- $n(K_{\mathcal{F}} + B^+) \sim 0$, and
- $nB^+ \geq n\lfloor B \rfloor + \lfloor (n+1)\{B\} \rfloor$.

Example 3.11 (Unbounded log canonical Fano foliations)**.** Let $\mathbb{F}_n$ be the Hirzebruch surface and E be the negative section. Let $\mathcal{G}_n$ be the foliation induced by the canonical fibration on $\mathbb{F}_n$ and $\pi\colon \mathbb{F}_n \to S_n$ be the contraction of E. Let $\mathcal{F}_n = \pi_*\mathcal{G}_n$ be the pushforward of $\mathcal{G}_n$. Then we have the following:

(1) $(S_n, \mathcal{F}_n)$ is strictly log canonical,
(2) $-K_{\mathcal{F}_n}$ is ample, and
(3) S_n has only one singularity, which is ϵ-lc if and only if $\epsilon \geq \frac{2}{n}$.

Similarly, we can consider ruled surfaces $\mathbb{P}(\mathcal{E})$ over curves of arbitrary genus and $\pi\colon \mathbb{P}(\mathcal{E}) \to S$ be the contraction of the negative section. Let $\mathcal{G}$ be the foliation induced by the fibration and $\mathcal{F}$ be the pushforward of $\mathcal{G}$ by π. Then $\mathcal{F}$ is strictly log canonical and S has only one singularity which is possibly non-rational.

4. Fano foliations on klt surfaces

In this section, we construct 1-complements for Fano foliations of rank one on klt surfaces.

Lemma 4.1 (Precise adjunction)**.** *Let $\mathcal{F}$ be a foliation of rank one with canonical singularities on a normal projective surface X. Let C be an $\mathcal{F}$-invariant curve with $\nu\colon C^\nu \to C$ the normalization map. Then there is an effective divisor $\operatorname{Diff}_C(\mathcal{F})$ on C^ν such that $K_{\mathcal{F}}|_{C^\nu} = K_{C^\nu} + \operatorname{Diff}_C(\mathcal{F})$, where $\operatorname{mult}_P \operatorname{Diff}_C(\mathcal{F})$ has the following form:*

(1) $\frac{r-1}{r}$ *if $\nu(P)$ is a terminal foliation singularity and a cyclic quotient singularity of index r;*
(2) 1 *if $\nu(P) \in \operatorname{Sing}(X)$ and is a canonical non-terminal foliation singularity;*
(3) $\operatorname{Z}(\mathcal{F}, C, P)$ *(cf.* [18, Definition 2.13]*) if $\nu(P) \in X$ is a non-singular point.*

Proof. Let $\pi\colon X' \to X$ be the minimal resolution of $\mathcal{F}$ with $\mathcal{F}' = \pi^{-1}\mathcal{F}$ and $C' = \pi_*^{-1}C$. By [18, Theorem 1.1], $\mathcal{F}'$ has at worst reduced singularities and C' is non-singular. Then by [7, Proposition 2.3] and its proof, we have $K_{\mathcal{F}'}|_{C'} \sim K_{C'} + \sum_{P' \in C'} \operatorname{Z}(\mathcal{F}', C', P')P'$.

Write $K_{\mathcal{F}'} = \pi^*K_{\mathcal{F}} + E$, where E is effective as $\mathcal{F}$ has at worst canonical singularities. By [18, Theorem 1.1], it is a direct computation that the support of E is the union of $\mathcal{F}'$-chains and $E|_{C'} = \sum_{P' \in E \cap C'} \frac{1}{r_{P'}} P'$ where $r_{P'}$ is the index of $K_{\mathcal{F}}$ at $\nu(P')$.

Let $\pi_C\colon C' \to C^\nu$ be morphism induced from π. By [9, Remark 3.10], we have $\operatorname{Diff}_C(\mathcal{F}) = (\pi_C)_* \operatorname{Diff}_{C'}(\mathcal{F}', -E)$ and hence $\operatorname{mult}_P \operatorname{Diff}_C(\mathcal{F})$ satisfies (1)-(3). $\square$

Lemma 4.2. *Let $f\colon W \to C$ be a fibration from a projective surface W onto a curve with relative Picard number 1. For any fibre F of f, if $|\operatorname{Sing}(W) \cap F| = 1$, then the point $\operatorname{Sing}(W) \cap F$ on W is not a cyclic quotient singularity.*

Proof. Suppose there is exactly one singularity of W, say p, on a fibre F of f which is a cyclic quotient singularity. Let $g\colon Y \to X$ be the minimal resolution of p with $E_1, \ldots, E_\ell$ all prime g-exceptional divisors on Y such that $E_i \cdot E_{i+1} = 1$ for $i = 1, \ldots, \ell - 1$. Note that the intersection matrix $(E_i \cdot E_j)$ is negative definite.

Let $\widetilde{F} := (g_*)^{-1}F$ be the strict transform of F on Y. We write $\widetilde{F} = g^*F + \sum_{i=1}^{\ell} a_i E_i$ where $a_i \in \mathbb{Q}$ and possibly after re-indexing, we can assume $\widetilde{F} \cdot E_1 = 1$ and $\widetilde{F} \cdot E_j = 0$ for all $j \geq 2$. Note that $0 \leq \widetilde{F} \cdot E_j \leq (-\sum_{i=1}^{\ell} E_i) \cdot E_j$ for all j and both inequalities are strict when $j = 1$. Thus, by [23, Corollary 4.2], we have $-1 < a_i < 0$ for all i. Moreover, $\widetilde{F}^2$ is an integer as Y is smooth. Hence, $0 = F^2 = g^*F \cdot \widetilde{F} = (\widetilde{F} - \sum_{i=1}^{\ell} a_i E_i) \cdot \widetilde{F} = \widetilde{F}^2 - a_1$, and therefore $a_1 = \widetilde{F}^2$ is an integer, which is impossible. □

Lemma 4.3. *Let $f\colon W \to \mathbb{P}^1$ be a fibration from a projective klt surface W of Picard number 2 and $\mathcal{F}$ be the foliation induced by f. If there is a section C of f such that $(\mathcal{F}, C)$ is log canonical and $(K_{\mathcal{F}} + C) \cdot F < 0$ for all fibres F of f, then $\mathcal{F}$ has only terminal singularities and thus W has only cyclic quotient singularities. Moreover, on each fibre, there is at most one cyclic quotient singularity of W not on C.*

Proof. By [17, Lemma 2.1], $\mathcal{F}$ is terminal at any point p on C. Suppose there is a point $q \in W$ which is not a terminal foliation singularity. Let F_q be the fibre passing through q. Then $q \notin C$ and by Lemma 4.1, $0 > (K_{\mathcal{F}} + C) \cdot F_q \geq -2 + \frac{r-1}{r} + 1 + \frac{1}{r} = 0$, which is impossible where r is the index at the point $C \cap F_q$. Hence $\mathcal{F}$ has only terminal singularities and therefore W has only cyclic quotient singularities by [18, Theorem 1.1].

For the moreover part, suppose there are two distinct cyclic quotient singularities q_1 and q_2 of W of index r_1 and r_2 on the same fibre F but not on C. Then

$$0 > (K_{\mathcal{F}} + C) \cdot F \geq -2 + \frac{r_0 - 1}{r_0} + \frac{r_1 - 1}{r_1} + \frac{r_2 - 1}{r_2} + \frac{1}{r_0} \geq 0,$$

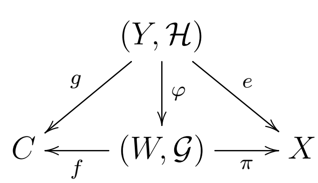
which is impossible where r_0 is the index at the point $C \cap F$. □

Proposition 4.4. *Let $\mathcal{F}$ be an lc Fano foliation of rank one on a klt surface X. Then X is of Picard number 1 and $\mathcal{F}$ is an algebraically integrable foliation with exactly one strictly lc centre. Moreover, there exists a birational morphism $\pi\colon W \to X$ such that*

(1) $E_0 := \mathrm{Exc}(\pi)$ is a non-invariant irreducible rational curve over the strictly lc centre,
(2) $\mathcal{G} := \pi^*\mathcal{F}$ is induced by the fibration $f\colon W \to \mathbb{P}^1$,
(3) W is a projective surface of Picard number 2 with only cyclic quotient singularities,
(4) there are at most three singular fibres of f, on which there are exactly two cyclic quotient singularities of W and one of them is on E_0,
(5) all singularities of W are on the singular fibres of f.

Proof. As $\mathcal{F}$ is Fano, by bend and break, $\mathcal{F}$ is algebraically integrable. By [3, Lemma 3.2], there is a fibration $g\colon Y \to C$ parametrizing the general leaves of $\mathcal{F}$ with a birational morphism $e\colon Y \to X$. By [1, Theorem 2.1], we can assume that Y and C are smooth and $\mathrm{Exc}(e)$ is a simple normal crossing divisor. We may further assume Y is the minimal resolution of $\mathcal{F}$. Note that $\mathcal{H} := e^{-1}\mathcal{F}$ is induced by the fibration g.

Let $\varphi\colon Y \to W$ be the morphism contracting all irreducible e-exceptional $\mathcal{H}$-invariant curves. Note that this morphism is also over C. Put $\mathcal{G} = \varphi_*\mathcal{H}$, $\pi\colon W \to X$, and $f\colon W \to C$.



Then all π-exceptional divisors are not $\mathcal{G}$-invariant. Let Δ be the sum of all irreducible π-exceptional divisors. By [18, Theorem 1.1], there is exactly one irreducible π-exceptional divisor over each strictly lc centre. Thus we have $\pi^*K_{\mathcal{F}} = K_{\mathcal{G}} + \Delta$ by adjunction.

Let F be a general fibre of f. Then $\Delta \cdot F$ is the number of irreducible components of Δ, which is a positive integer. As $\mathcal{F}$ is Fano, $0 > K_{\mathcal{F}} \cdot \pi_*F = (K_{\mathcal{G}} + \Delta) \cdot F = -2 + \Delta \cdot F$ and thus $\Delta \cdot F = 1$. Hence, $\Delta = E_0$ is irreducible. Since X is klt, E_0 is rational. Therefore, this proves (1).

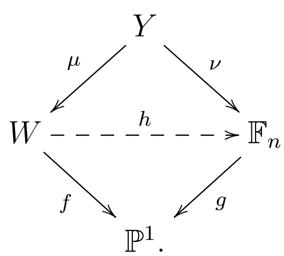
Since E_0 is rational and dominates C, we have $C \cong \mathbb{P}^1$. This proves (2).

Suppose there is a reducible fibre of f. There is an irreducible component of this fibre intersecting E_0, say F_0. Since the fibres are connected, there is another irreducible component intersecting F_0, say F_1. As $\mathcal{F}$ is Fano and by Lemma 4.1, $0 > K_{\mathcal{F}} \cdot \pi_*F_0 = (K_{\mathcal{G}} + E_0) \cdot F_0 = -2 + \frac{r_0-1}{r_0} + 1 + \frac{1}{r_0} = 0$, which is impossible where r_0 is the index at the point $E_0 \cap F_0$. Thus all fibres of f are irreducible and hence W is of Picard number 2. For any fibre F of f, we have $0 > K_{\mathcal{F}} \cdot \pi_*F = (K_{\mathcal{G}} + E_0) \cdot F$. By Lemma 4.3, W has only cyclic quotient singularities. This proves (3).

By [18, Theorem 1.1] and the classification of klt surface singularities, there are at most three cyclic quotient singularities of W on E_0. Then (4) and (5) follow from Lemma 4.2 and 4.3. □

Proposition 4.5. *Let $\mathcal{F}$ be an lc Fano foliation of rank one on a klt surface X. Then $\mathcal{F}$ has a 1-complement, that is, there is an effective divisor B such that the foliated pair $(\mathcal{F}, B)$ is log canonical and $K_{\mathcal{F}} + B \sim 0$.*

Proof. By Proposition 4.4, we have a birational morphism $\pi\colon W \to X$ such that the pullback foliation $\mathcal{G} := \pi^{-1}\mathcal{F}$ is induced by a fibration $f\colon W \to \mathbb{P}^1$, and $\pi^*K_{\mathcal{F}} = K_{\mathcal{G}} + E_0$ where E_0 is the irreducible π-exceptional divisor. Then there are birational (possibly identity) morphisms $\mu\colon Y \to W$ and $\nu\colon Y \to \mathbb{F}_n$ where $\mathbb{F}_n$ is a Hirzebruch surface. Moreover, μ and ν are sequences of at most three weighted blowups.



More precisely, let C_0 be the negative section for $\mathbb{F}_n \to \mathbb{P}^1$ and L_i be a fibre for $\mathbb{F}_n \to \mathbb{P}^1$ such that $h_*^{-1}L_i$ is a singular fibre for f. By Proposition, there are at most three L_i. At each $L_i \cap C_0$, ν is a weighted blowup. And μ contracts the strict transform of L_i on Y.

Let $\mathcal{H}$ be the foliation on $\mathbb{F}_n$ induced by g. Then $h_*(K_{\mathcal{G}} + E_0) = K_{\mathcal{G}} + C_0$ where C_0 is the negative section for g and $h = \nu \circ \mu^{-1}$. It is clear that there is a 1-complement for $(\mathcal{G}, C_0)$, say $(\mathcal{G}, C_0 + C)$ where C is a section for g with $C \cap C_0 = \varnothing$. Thus, pulling back to W and then pushing forward to X, we get a 1-complement $(\mathcal{F}, B := \pi_*h_*^{-1}C)$ for $\mathcal{F}$. □

5. Algebraically integrable Fano foliations

In this section, we show the existence of $\mathbb{Q}$-complements for Fano foliated triples $(X, \mathcal{F}, B)$ with $\mathcal{F}$ algebraically integrable.

First, we prove adjunction and inversion of adjunction for invariant divisors in this case:

Proposition 5.1 (cf. [15, Theorem 2.4.1], Adjunction)**.** *Let $(X, \mathcal{F}, B)$ be a foliated triple. Suppose $\mathcal{F}$ is algebraically integrable and $S \subseteq X$ a prime invariant divisor with normalisation $S^\nu \to S$. Assume that $(X, \mathcal{F}, B)$ is log canonical, then the restriction $(S^\nu, \mathcal{F}_{S^\nu}, \mathrm{Diff}_{S^\nu}(\mathcal{F}, B))$ is log canonical.*

Proof. We give a direct proof here, which is essentially the same as the one given in [15].

Let $f\colon Y \to X$ be a Property $(*)$ modification of $(X, \mathcal{F}, B)$ by [2, Theorem 3.10]. Then since $(X, \mathcal{F}, B)$ is log canonical, we have

$$K_{\mathcal{G}} + B_Y = f^*(K_{\mathcal{F}} + B)$$

where $\mathcal{G} := f^{-1}\mathcal{F}$ and $B_Y := f_*^{-1}B + \sum \epsilon(E)E$. Let T be the strict transform of S with normalisation $T^\nu \to T$. By taking the restriction to T^ν, we have

$$K_{\mathcal{G}_{T^\nu}} + \mathrm{Diff}_{T^\nu}(\mathcal{G}, B_Y) = (f|_{T^\nu})^*(K_{\mathcal{F}_{S^\nu}} + \mathrm{Diff}_{S^\nu}(\mathcal{F}, B)).$$

Since the left hand side is log canonical by [2, Proposition 3.2], so is the right hand side. □

Next, we prove the following inversion of adjunction theorem for foliated pairs. We will need the following lemma:

Lemma 5.2. *Let $\pi\colon Y \to Z$ be an equi-dimensional fibration between normal varieties with $\mathcal{F}$ the induced foliation on Y such that $\dim Y = n$ and $\mathcal{F}$ is Gorenstein and has rank r. Let $H_Z \subseteq Z$ be a prime divisor and assume that*

- *S and T are two irreducible components of $\pi^{-1}(H_Z)$,*
- *$G \subseteq S \cap T$ is a component of $S \cap T$ and $\mathrm{codim}(G; S) = 1$ such that S and T are both Cartier around η_G, and*
- *$\pi(G) = H_Z$.*

Then $G \subseteq \mathrm{Sing}(\mathcal{F})$.

Proof. Since the statement is local around the generic point η_G, we can assume that:

- Both Y and Z are affine.
- Z and H_Z are both smooth. In particular, we may assume that Ω_Z is locally free and generated by $\mathrm{d}z_1$, $\mathrm{d}z_2$, …, $\mathrm{d}z_{n-r}$ where $H_Z := \{z_1 = 0\}$.
- $\pi^\sharp(z_1) = x^{n_1}y^{n_2}f$ where x, y are the local generators of S and T respectively, n_1, $n_2 \geq 1$ and $f \neq 0$ a holomorphic function. Here $\pi^\sharp\colon \mathcal{O}_Z \to \pi_*\mathcal{O}_Y$ is the morphism induced by $\pi\colon Y \to Z$.

Now we consider the first fundamental exact sequence [26, Theorem 25.1]:

$$\pi^*\Omega_Z \to \Omega_Y \to \Omega_{Y/Z} \to 0.$$

Notice that $\Omega_{Y/Z} \cong \Omega_Y/\langle \mathrm{d}(\pi^\sharp z_1), \mathrm{d}(\pi^\sharp z_2), \ldots, \mathrm{d}(\pi^\sharp z_{n-r})\rangle$ and that

$$\mathrm{d}(\pi^\sharp z_1) = x^{n_1}\mathrm{d}(y^{n_2}f) + y^{n_2}\mathrm{d}(x^{n_1}f) \in \mathfrak{m}_{\eta_G}\Omega_{Y/\operatorname{Spec}\mathbb{C}}.$$

Thus, we have $\dim_{k(\eta_G)} \Omega_{Y/Z} \otimes k(\eta_G) \geq r+1$ and hence, $\dim_{k(\eta_G)} \Omega^r_{Y/Z} \otimes k(\eta_G) > 1$.

Now we consider the induced morphism $\varphi_r\colon \Omega^r_{Y/Z} \to \Omega^{[r]}_{Y/Z} = \mathcal{O}_Y(K_{\mathcal{F}})$. Since $\mathcal{O}_Y(K_{\mathcal{F}})$ is locally free by assumption, φ_r factors by some inclusion $\psi_r\colon \Omega^r_{Y/Z}/(\Omega^r_{Y/Z})^{\mathrm{tor}} \hookrightarrow \mathcal{O}_Y(K_{\mathcal{F}})$:

$$\begin{array}{ccc} \Omega^r_{Y/Z} & \twoheadrightarrow & \Omega^r_{Y/Z}/(\Omega^r_{Y/Z})^{\mathrm{tor}} \\ & \searrow^{\varphi_r} & \big\downarrow \psi_r \\ & & \mathcal{O}_Y(K_{\mathcal{F}}). \end{array}$$

Since $\dim_{k(\eta_G)}(\Omega^r_{Y/Z}/(\Omega^r_{Y/Z})^{\mathrm{tor}}) \otimes k(\eta_G) = \dim_{k(\eta_G)} \Omega^r_{Y/Z} \otimes k(\eta_G) > 1$, we have φ_r is not surjective at η_G. Then $\mathrm{Im}(\Omega^r_Y \to \mathcal{O}_Y(K_{\mathcal{F}})) = \mathrm{Im}(\Omega^r_{Y/Z} \xrightarrow{\varphi_r} \mathcal{O}_Y(K_{\mathcal{F}})) = \mathrm{Im}\,\psi_r$ has cosupport containing G. Thus, $\mathrm{Im}(\Omega^r_Y(-K_{\mathcal{F}}) \to \mathcal{O}_Y) \subseteq \mathcal{I}_G$ and therefore, $G \subseteq \mathrm{Sing}(\mathcal{F})$. □

Theorem 5.3 (Inversion of adjunction)**.** *Let $(X, \mathcal{F}, B)$ be a foliated triple with $\mathcal{F}$ being algebraically integrable. Let $S \subseteq X$ be a prime invariant divisor with normalisation $\nu\colon S^\nu \to S$ such that $(S^\nu, \mathcal{F}_{S^\nu}, \mathrm{Diff}_{S^\nu}(\mathcal{F}, B))$ is log canonical. Then*

$$\mathrm{Nlc}(X, \mathcal{F}, B) \cap S = \varnothing.$$

Proof. Suppose $\mathrm{Nlc}(X, \mathcal{F}, B) \cap S \neq \varnothing$ for the sake of contradiction.

By [2, Theorem 3.10], there is a Property $(*)$ modification $f\colon Y \to X$ such that $(Y, \mathcal{G}, B_Y)$ is an lc foliated pair with

$$K_{\mathcal{G}} + B_Y + F = f^*(K_{\mathcal{F}} + B) \tag{1}$$

where $\mathcal{G} := f^{-1}\mathcal{F}$, $B_Y = f_*^{-1}B + \sum \epsilon(E_i)E_i$ and F is effective and f-exceptional. Let $\pi\colon Y \to Z$ be the morphism inducing $\mathcal{G}$ on Y. Let $T := f_*^{-1}(S)$ be the strict transform of S. By the assumption that $\mathrm{Nlc}(X, \mathcal{F}, B) \cap S \neq \varnothing$, we have $\mathrm{Supp}(F) \cap T \neq \varnothing$. Let $\rho\colon T^\rho \to T$ be the normalisation of T. We have the following commutative diagram:

$$\begin{array}{ccccccc}
T^\rho & \xrightarrow{\rho} & T & \hookrightarrow & (Y, \mathcal{G}, B_Y + F) & \xrightarrow{\pi} & Z \\
\downarrow & & \downarrow & & \downarrow f & \nearrow & \\
S^\nu & \xrightarrow{\nu} & S & \hookrightarrow & (X, \mathcal{F}, B) & &
\end{array}$$

Now we consider the adjunction of $(K_{\mathcal{G}} + B_Y + F)$ and $(K_{\mathcal{F}} + B)$ to S and T separately. Then we may use equation (1) to get

$$K_{\mathcal{G}_{T^\rho}} + \mathrm{Diff}_{T^\rho}(\mathcal{G}, B_Y) + F|_{T^\rho} = f^*(K_{\mathcal{F}_{S^\nu}} + \mathrm{Diff}_{S^\nu}(\mathcal{F}, B)). \tag{2}$$

Since we have the hypothesis that the restriction $(S^\nu, \mathcal{F}_{S^\nu}, \mathrm{Diff}(\mathcal{F}, B))$ is log canonical, so is

$$(T^\rho, \mathcal{G}_{T^\rho}, \mathrm{Diff}_{T^\rho}(\mathcal{G}, B_Y) + F|_{T^\rho}).$$

In particular, we have therefore

$$\mathrm{mult}_G(\mathrm{Diff}_{T^\rho}(\mathcal{G}, B_Y) + F|_{T^\rho}) \leq \epsilon(G) \tag{3}$$

for any prime divisor $G \subseteq T^\rho$. Since $\mathrm{Supp}(F) \cap T \neq \varnothing$ and Y is $\mathbb{Q}$-factorial, we may take G to be the strict transform of a component of $\mathrm{Supp}(F) \cap T$ such that $\mathrm{codim}_{T^\rho}(G) = 1$ and $G = \mathrm{Supp}(F_1|_{T^\rho})$ where $F_1 \subseteq \mathrm{Supp}(F)$. Let $a = \mathrm{mult}_{F_1} F$. Note that $a > 0$, $\mathrm{mult}_G(F|_{T^\rho}) = a$, and

$$\begin{aligned}
\epsilon(G) &\geq \mathrm{mult}_G(\mathrm{Diff}_{T^\rho}(\mathcal{G}, B_Y) + F|_{T^\rho}) \\
&= \mathrm{mult}_G \mathrm{Diff}_{T^\rho}(\mathcal{G}) + \mathrm{mult}_G((B_Y + F)|_{T^\rho}) \\
&\geq \mathrm{mult}_G(F|_{T^\rho}) = a > 0
\end{aligned}$$

where

- the first inequality is (3);
- the equality $\mathrm{mult}_G(\mathrm{Diff}_{T^\rho}(\mathcal{G}, B_Y) + F|_{T^\rho}) = \mathrm{mult}_G \mathrm{Diff}_{T^\rho}(\mathcal{G}) + \mathrm{mult}_G((B_Y + F)|_{T^\rho})$ comes from [9, Remark 3.10];
- by definition we have $B_Y \geq 0$, and according to [9, Proposition-Definition 3.7] we have $\mathrm{mult}_G \mathrm{Diff}_{T^\rho}(\mathcal{G}) \geq 0$. Therefore, the second inequality holds.

Thus, $\epsilon(G) = 1$, $\epsilon(F_1) = 0$, and

$$\mathrm{mult}_G \mathrm{Diff}_{T^\rho}(\mathcal{G}) \leq 1 - a < 1. \tag{4}$$

Therefore, $G \subseteq \mathrm{Sing}(\mathcal{G})$ by Lemma 5.2.

Let $p\colon (Y',\mathcal{G}') \to (Y,\mathcal{G})$ be the quasi-étale index one cover of $K_\mathcal{G}$, T and, F around η_G, with T' the normalisation of $p_*^{-1}T$, $G' := p_*^{-1}G$. Let e be the ramification index of p along G'. By [20, Lemma 3.4] (see also [8, Proposition 2.2]), we have

$$\operatorname{mult}_{G'} \operatorname{Diff}_{T'}(\mathcal{G}') = e \operatorname{mult}_G(\operatorname{Diff}_T \mathcal{G}) - (e-1).$$

As $G \subseteq \operatorname{Sing}(\mathcal{G})$ and $\mathcal{G}'$ is Cartier around $\eta_{G'}$, we have $\operatorname{mult}_{G'} \operatorname{Diff}_{T'}(\mathcal{G}') \geq 1$ and thus $\operatorname{mult}_G \operatorname{Diff}_T(\mathcal{G}) \geq 1$, which contradicts the inequality (4). □

Next, we prove that log canonical on a general fibre implies globally log canonical:

Proposition 5.4. *Let $(X,\mathcal{F},B)$ be a foliated triple where $\mathcal{F}$ is an algebraically integrable foliation with $K_\mathcal{F}+B \sim_\mathbb{Q} 0$. If $\operatorname{Nlc}(X,\mathcal{F},B) \neq \varnothing$, then it intersects every closure of leaf of $\mathcal{F}$.*

Proof. Let $f\colon Y \to X$ be a Property $(*)$ modification of $(X,\mathcal{F},B)$. Then we have

$$K_\mathcal{G} + B_Y + G = f^*(K_\mathcal{F}+B)$$

where $\mathcal{G} := f^{-1}\mathcal{F}$ is induced by a projective contraction $Y \to Z$, $B_Y = f_*^{-1}\tilde{B}$, and G is an effective divisor. Note that $\operatorname{Nlc}(Y,\mathcal{G},B_Y+G)$ is supported in G and by construction, $K_\mathcal{G}+B_Y+G = f^*(K_\mathcal{F}+B) \sim_\mathbb{Q} 0$. As $K_\mathcal{G}+B_Y \sim_\mathbb{Q} -G$ is not pseudo-effective over Z, by [24, Thereom 1.4], we can run a $(K_\mathcal{G}+B_Y)$-MMP$/Z$ with scaling of an ample$/Z$ $\mathbb{R}$-divisor which terminates with $(W,\mathcal{H},B_W)/T$ a Mori fibre space$/Z$ of $(Y,\mathcal{G},B_Y)$ where B_W and $\mathcal{H}$ denote the pushforward of the divisor B on X and the foliation $\mathcal{F}$, respectively. Then $-(K_\mathcal{H}+B_W)$ is ample over T. Let $\phi\colon Y \dashrightarrow W$ be the birational map induced by the MMP and $\psi\colon W \to T$ be the contraction introduced by the Mori fibre space structure.

Let G_W be the pushforward of the divisor G on W. Since $K_\mathcal{H}+B_W+G_W = \phi_*(K_\mathcal{F}+B+G) \sim_\mathbb{Q} 0$, $-(K_\mathcal{H}+B_W) \sim_\mathbb{Q} G_W$ is ample over T as well. Therefore, G_W is horizontal over T. We label some morphisms in the following commutative diagram:

$$\begin{array}{ccc} Y & \overset{\phi}{\dashrightarrow} & W \\ \downarrow & \swarrow_{\nu} & \downarrow{\scriptstyle\psi} \\ Z & \underset{\eta}{\longleftarrow} & T \end{array}$$

As ν is surjective, so is η, which implies that G_W dominates Z. Hence, G also dominates Z and therefore, G intersects all leaves of $\mathcal{G}$. The Proposition follows upon observing that

$$f_*(\operatorname{Nlc}(Y,\mathcal{G},B_Y+G)) \subseteq \operatorname{Nlc}(X,\mathcal{F},B).$$

□

Remark 5.5. Notice that this Proposition together with Lemma 5.3 proves that lifts of complements from vertical divisors S to the ambient space remain log canonical. We will give details of this process in the next Theorem.

Now we are able to show the existence of $\mathbb{Q}$-complements for algebraically integrable Fano foliations.

Theorem 5.6. *Let $(X,\mathcal{F},B)$ be an lc Fano foliated triple with $\mathcal{F}$ algebraically integrable. Then there exists a $\mathbb{Q}$-complement. That is, there exists an effective divisor $\Delta \sim_\mathbb{Q} -(K_\mathcal{F}+B)$ such that $(X,\mathcal{F},B+\Delta)$ is log canonical.*

Proof. We will proceed by induction on the corank c of $\mathcal{F}$. When $c=0$, the existence of $\mathbb{Q}$-complement is given by [5, Theorem 1.7].

Now we assume the theorem holds when the corank is $c-1$. Let $f\colon Y\to X$ be a Property $(*)$ modification of $(X,\mathcal{F},B)$ with $\mathcal{G}:=f^{-1}\mathcal{F}$ induced by $\pi\colon Y\to Z$. Note that

$$K_{\mathcal{G}}+B_Y+E=f^*(K_{\mathcal{F}}+B)$$

where $B_Y=f_*^{-1}B$ is the strict transform of B on Y and $E=\sum\epsilon(E)E$ is the sum of all f-exceptional prime divisors. We fix a general prime divisor $D_Z\subseteq Z$ such that $F:=\pi^{-1}(D_Z)$ is normal and irreducible. Let $G:=f(F)$ be the image. Notice that G is not necessarily normal. Therefore, we denote the normalisation by $\nu\colon G^\nu\to G$ and the normality of F naturally gives us an induced morphism $g^\nu\colon F\to G^\nu$.

$$\begin{array}{ccccccc} & & F & \xhookrightarrow{\iota_F} & (Y,\mathcal{G},B_Y+E) & \xrightarrow{\pi} & Z \\ & \swarrow^{g^\nu} & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & \nearrow & \\ G^\nu & \xrightarrow{\nu} & G & \xhookrightarrow{\iota_G} & (X,\mathcal{F},B) & & \end{array}$$

By adjunction (Proposition 5.1), $(\mathcal{F}_{G^\nu},\mathrm{Diff}_{G^\nu}(\mathcal{F},B))$ is an lc Fano foliated pair of corank $c-1$. Thus by induction hypothesis, there exists an effective divisor Δ_{G^ν} on G^ν such that

$$K_{\mathcal{F}_{G^\nu}}+\mathrm{Diff}_{G^\nu}(\mathcal{F},B)+\Delta_{G^\nu}\sim_{\mathbb{Q}}0$$

and $(\mathcal{F}_{G^\nu},\mathrm{Diff}_{G^\nu}(\mathcal{F},B)+\Delta_{G^\nu})$ is lc. Let $m_0\in\mathbb{N}$ such that $m_0\Delta_{G^\nu}\sim -m_0(K_{\mathcal{F}_{G^\nu}}+\mathrm{Diff}_{G^\nu}(\mathcal{F},B))$ and $L:=-m(K_{\mathcal{F}}+B)$ where m is sufficiently divisible by m_0. Then

$$L|_{G^\nu}=(\iota_G\circ\nu)^*L=-m(K_{\mathcal{F}_{G^\nu}}+\mathrm{Diff}_{G^\nu}(\mathcal{F},B))\sim m\Delta_{G^\nu}$$

and thus, $m\Delta_{G^\nu}\in|L|_{G^\nu}|$. Therefore, there is a non-zero section $\alpha_{G^\nu}\in\mathrm{H}^0(G^\nu,\mathcal{O}_{G^\nu}(L|_{G^\nu}))$ such that $m\Delta_{G^\nu}=\mathrm{Div}(\alpha_{G^\nu})+L|_{G^\nu}$.

Now we consider the following short exact sequence induced by the normalisation ν and twisted by the sheaf $\mathcal{O}_G(L|_G)$:

$$0\to\mathcal{O}_G(L|_G)\to\nu_*\mathcal{O}_{G^\nu}(L|_G)\to(\nu_*\mathcal{O}_{G^\nu}/\mathcal{O}_G)(L|_G)\to 0$$

and thus, we have the following exact sequence:

$$0\to\mathrm{H}^0(G,\mathcal{O}_G(L|_G))\xrightarrow{j}\mathrm{H}^0(G,\nu_*\mathcal{O}_{G^\nu}(L|_G))\xrightarrow{\beta}\mathrm{H}^0(G,(\nu_*\mathcal{O}_{G^\nu}/\mathcal{O}_G)(L|_G)).$$

Note that $\alpha_{G^\nu}\in\mathrm{H}^0(G,\nu_*\mathcal{O}_{G^\nu}(L|_G))$ and $E|_F$ is a union of some lc centres of $(\mathcal{G}_F,\mathrm{Diff}_F(\mathcal{G},B_Y+E))$ and thus, $g^\nu(E|_F)$ is a union of some lc centres of $(\mathcal{F}_{G^\nu},\mathrm{Diff}_{G^\nu}(\mathcal{F},B))$. Thus, no component of Δ_{G^ν} is contained in $g^\nu(E|_F)$. Hence, the image $\beta(\alpha_{G^\nu})$ is 0 and therefore, there is an $\alpha_G\in\mathrm{H}^0(G,\mathcal{O}_G(L|_G))$ such that $j(\alpha_G)=\alpha_{G^\nu}$.

We then consider the following short exact sequence on X:

$$0\to\mathcal{O}_X(L-G)\to\mathcal{O}_X(L)\to\mathcal{O}_G(L)\to 0.$$

We recall that $L=-m(K_{\mathcal{F}}+B)$ is an ample divisor. Then by Serre vanishing theorem, we have $\mathrm{H}^1(X,\mathcal{O}_X(L-G))=0$ for m sufficiently large and thus, the following morphism is surjective:

$$\mathrm{H}^0(X,\mathcal{O}_X(L))\twoheadrightarrow\mathrm{H}^0(G,\mathcal{O}_G(L)).$$

Let $\alpha\in\mathrm{H}^0(X,\mathcal{O}_X(L))$ be a lifting of α_G. Then we have an effective divisor $D:=\mathrm{Div}(\alpha)+L$ and thus $K_{\mathcal{F}}+B+\Delta\sim_{\mathbb{Q}}0$ where $\Delta=\frac{1}{m}D$.

Note that $K_{\mathcal{G}}+B_Y+E+f^*\Delta=f^*(K_{\mathcal{F}}+B+\Delta)$. Taking adjunction to F, we have

$$\begin{aligned}K_{\mathcal{G}_F}+\mathrm{Diff}_F(\mathcal{G},B_Y+E+f^*\Delta)&=(g^\nu)^*(K_{\mathcal{F}_{G^\nu}}+\mathrm{Diff}_{G^\nu}(\mathcal{F},B+\Delta))\\&=(g^\nu)^*(K_{\mathcal{F}_{G^\nu}}+\mathrm{Diff}_{G^\nu}(\mathcal{F},B)+\Delta_{G^\nu}).\end{aligned}$$

Since $(\mathcal{F}_{G^\nu},\mathrm{Diff}_{G^\nu}(\mathcal{F},B)+\Delta_{G^\nu})$ is log canonical, so is $(\mathcal{G}_F,\mathrm{Diff}_F(\mathcal{G},B_Y+E+f^*\Delta))$.

According to the inversion of adjunction (Theorem 5.3), we have

$$\mathrm{Nlc}(\mathcal{G}, B_Y + E + f^*\Delta) \cap F = \varnothing$$

and thus $\mathrm{Nlc}(\mathcal{G}, B_Y + E + f^*\Delta)$ cannot be horizontal as $F := \pi^{-1}(D_Z)$ is irreducible. Hence, by Proposition 5.4, $\mathrm{Nlc}(\mathcal{G}, B_Y + E + f^*\Delta) = \varnothing$. Therefore, $(Y, \mathcal{G}, B_Y + E + f^*\Delta)$ is log canonical and so is $(X, \mathcal{F}, B + \Delta)$. □

We now make note of a couple of immediate corollaries of the proof of the above proposition, which are not made further use of in the present paper, however appear to be of independent interest. The first can be seen as log-extension of a classification result obtained in [4].

We notice that Proposition 5.4 provides us with an alternative algebraic proof of [4, Proposition 3.14] (see also [25, Proposition 10.3] for a proof using generalised foliated quadruples):

Proposition 5.7. *Let $(X, \mathcal{F}, B)$ be an lc Fano foliated triple, then $\mathcal{F}$ is not induced by a morphism.*

Proof. Suppose, for the sake of contradiction, that $\mathcal{F}$ is induced by a morphism $\pi\colon X \to Z$. We will then proceed the proof by induction on the corank c. When $c = 1$, we have $\dim Z = 1$. We fix a general point $z_0 \in Z$ and $F_0 := \pi^{-1}(z_0)$.

Let $z_1 \in Z \setminus \{z_0\}$ be another general point and $F_1 := \pi^{-1}(z_1)$ such that F_1 is irreducible and normal. Note that $(F_1, \mathrm{Diff}_{F_1}(\mathcal{F}, B))$ is a Fano pair, hence there exists a $\mathbb{Q}$-complement D_{z_1} such that $K_{F_1} + \mathrm{Diff}_{F_1}(\mathcal{F}, B) + D_{z_1} \sim_{\mathbb{Q}} 0$. Since $-(K_{\mathcal{F}} + B)$ is ample, there exists an $m \gg 0$ such that $L := -m(K_{\mathcal{F}} + B) - F_0$ is very ample.

Note that $mD_{z_1} \sim L|_{F_1}$. We consider the following short exact sequence on X:

$$0 \to \mathcal{O}_X(L - F_1) \to \mathcal{O}_X(L) \to \mathcal{O}_{F_1}(L) \to 0.$$

By Serre vanishing theorem, we have the induced surjective morphism:

$$\mathrm{H}^0(X, \mathcal{O}_X(L)) \twoheadrightarrow \mathrm{H}^0(F_z, \mathcal{O}_{F_z}(L)).$$

Let $mD \in \mathrm{H}^0(X, \mathcal{O}_X(L))$ be a lifting of mD_{z_1}. Then the triple $(X, \mathcal{F}, B + D + \frac{1}{m}\Delta)$ has the following properties:

- $K_{\mathcal{F}} + B + D + \frac{1}{m}F_0 \sim_{\mathbb{Q}} 0$.
- $(F_1, \mathrm{Diff}_{F_1}(\mathcal{F}, B + D + \frac{1}{m}F_0)) = (F_1, \mathrm{Diff}_{F_1}(\mathcal{F}, B) + D_{z_1})$ is log canonical since $z_0 \neq z_1$.
- $(X, \mathcal{F}, B + D + \frac{1}{m}F_0)$ is obviously not log canonical since $F_0 > 0$ is a vertical divisor.

Thus, we get a contradiction with Proposition 5.4.

Now we suppose the theorem holds for the cases when corank $\leq c - 1$. We fix a general Cartier divisor $H_Z > 0$ on Z and $S := \pi^* H_Z$. Then $(S, \mathcal{F}_S, \mathrm{Diff}_S(\mathcal{F}, B))$ is an lc Fano triple with corank $c - 1$ and $\mathcal{F}_S$ is a foliation induced by the morphism $\pi|_S\colon S \to H_Z$, which contradicts the induction hypothesis. □

Last, as an addendum, we show the following version of connectedness principle for foliations induced by morphisms:

Lemma 5.8. *Let $(\mathcal{F}, B)$ be a foliated pair on a normal variety X such that*

(1) $\mathcal{F}$ is induced by a proper morphism $g\colon X \to Z$ with connected fibres;
(2) $-(K_{\mathcal{F}} + B)$ is g-big and g-nef;

Then $\lfloor B \rfloor$ is connected.

Proof. We prove the result by induction on $\dim Z$. For the case when $\dim Z = 0$, we have Z is a point, $\mathcal{F} = \mathcal{T}_X$ the tangent sheaf of X, and $-(K_{\mathcal{F}} + B) = -(K_X + B)$ is big and nef. By [22, Theorem 17.4], we have $\lfloor B \rfloor$ is connected.

Now suppose $\dim Z \geq 1$. Let H be a general vertical divisor with respect to g, that is, $H = g^*A$ for some general ample divisor A. Let $m\colon X' \to X$ be a log resolution of $(\mathcal{F}, B)$ such that the strict transform $H' := m_*^{-1}H$ is also smooth. Let $\mathcal{F}'$ be the induced foliation on X'. We consider the adjunction with respect to H':

$$(K_{\mathcal{F}} + B)|_{H'} = K_{\mathcal{F}_{H'}} + \Theta = K_{\mathcal{F}_{H'}} + B|_{H'} + \Theta_0.$$

According to [2, Proposition 3.2], we have $\mathrm{Supp}(\lfloor\Theta_0\rfloor) \subseteq \mathrm{Sing}(\mathcal{F}') \cap H'$. However since $H = g^*A$ for some general A and $\mathrm{codim}(\mathrm{Sing}(\mathcal{F})) \geq 2$, by moving H in an infinitesimal neighbourhood in $\mathbb{P}(\mathrm{H}^0(Z, \mathcal{O}_Z(A)))$ if necessary, we can assume that we have $\mathrm{Supp}(\lfloor\Theta_0\rfloor) = 0$. Then by the induction assumption, we get that $\lfloor\Theta\rfloor = \lfloor B|_H\rfloor$ is connected. Therefore, $\lfloor B\rfloor$ is connected. $\square$

The following example demonstrates that the assumption in Lemma 5.8 cannot be weakened to require only relative nefness.

Example 5.9. Let $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ be the lattice of rank two. Let $v_1 = v_5 = e_1$, $v_2 = e_2$, $v_3 = -e_1$, and $v_4 = -e_2$. Let $\sigma_i = \mathbb{R}_{\geq 0}v_i + \mathbb{R}_{\geq 0}v_{i+1}$ for $1 \leq i \leq 4$. Note that $\Sigma = \{\{0\}, v_i, \sigma_i \mid 1 \leq i \leq 4\}$ is a fan in $N_{\mathbb{R}}$ and gives the toric variety $X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$. Let $W = \mathbb{C}(e_1 + e_2) \subseteq N_{\mathbb{C}}$ be a complex vector subspace. Then W introduces a toric foliation on X_Σ. (For a reference, see [11].) We have the following observations:

(1) $K_{\mathcal{F}} \sim 0$.
(2) $\mathcal{F}$ is algebraically integrable and log canonical.
(3) $\mathcal{F}$ has four singular points, two of which are dicritical and another two are non-dicritical.
(4) Let $g\colon Y \to X_\Sigma$ be blowup of two dicritical points with exceptional curves E_1 and E_2. Then $\mathcal{G} := g^{-1}\mathcal{F}$ is induced by a morphism $f\colon Y \to Z$, $(\mathcal{G}, E_1 + E_2)$ is log canonical, and $K_{\mathcal{G}} + E_1 + E_2 = g^*K_{\mathcal{F}}$.

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