INFINITESIMAL DEFORMATIONS OF LIE ALGEBROID PAIRS

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ABSTRACT. We study infinitesimal deformations of Lie algebroid pairs in the category of smooth manifolds enriched with a local Artinian $\mathbb{K}$-algebra. Given a Lie algebroid pair (L, A), i.e. a Lie algebroid L together with a Lie subalgebroid A, we investigate isomorphism classes of infinitesimal deformations of (L, A) modulo automorphisms from exponentials of derivations of L and those from the exponentials of inner derivations of L, respectively. For the associated two deformation functors, we find the associated governing L_∞-algebras in the sense of extended deformation theory. Furthermore, when (L, A) is a matched Lie pair, i.e. the quotient L/A is also a Lie subalgebroid of L, we investigate isomorphism classes of infinitesimal deformations modulo automorphisms from exponentials of derivations along the normal direction L/A. The extended deformation theory of the associated deformation functor recovers the formal deformation theory of complex structures and that of transversely holomorphic foliations.

Keywords: Infinitesimal deformation, Lie algebroid pair, L_∞-algebra, Maurer-Cartan element, Gauge equivalence class.

AMS subject classification: 13D10 , 17B70, 16E45, 53C05, 53C12

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Introduction

The motivation.

The "Deligne principle" in deformation theory posits a fundamental correspondence: every formal deformation problem is governed by a differential graded (dg) Lie algebra (or, more generally, an L_∞-algebra). Specifically, the dg Lie algebra controls the deformations via solutions of the Maurer-Cartan equation modulo gauge actions [11, 23, 24].

This paper aims to initiate the study of such a formal deformation theory for **Lie algebroid pairs**, which we refer to as Lie pairs for brevity. To contextualize this, recall that a Lie algebroid over $\mathbb{K}$ (where $\mathbb{K}$ represents either the field of real numbers, $\mathbb{R}$, or complex numbers, $\mathbb{C}$) is a $\mathbb{K}$-vector bundle L over a base manifold M. This bundle is endowed with a Lie bracket $[-,-]$ defined on its sections, along with a bundle map $\rho\colon L \to T_M \otimes_{\mathbb{R}} \mathbb{K}$, called the anchor. The anchor map ρ must satisfy two conditions: it acts as a morphism of Lie algebras on the section spaces, and it adheres to the Leibniz identity:

$$[X, fY] = f[X, Y] + (\rho(X)f)Y,$$

for all $X, Y \in \Gamma(L)$ and $f \in C^\infty(M, \mathbb{K})$. A *Lie (algebroid) pair* is then defined as an inclusion $A \hookrightarrow L$ of Lie algebroids sharing a common base space, denoted by (L, A). These Lie pairs arise naturally in diverse mathematical domains, including Lie theory, complex geometry, foliation theory, and Poisson geometry. For instance, a complex manifold X gives rise to the Lie pair $(T_X \otimes \mathbb{C}, T_X^{0,1})$ over $\mathbb{C}$. Similarly, a regular foliation $\mathcal{F}$ on M defines a Lie pair (T_M, F) over $\mathbb{R}$, where $F \subset T_M$ represents the integrable distribution tangent to the foliation $\mathcal{F}$. The Molino class of a foliation $\mathcal{F} \subset M$ and the Atiyah class of a complex manifold X can be interpreted as the Atiyah classes of their corresponding Lie pairs [4]. Furthermore, the Atiyah class of any Lie pair (L, A) induces an L_∞-algebra structure on the shifted tangent complex $\Gamma(\Lambda^\bullet A^* \otimes L/A[1])$ [22]. In the specific case where the Lie pair originates from a compact Kähler manifold, this induced L_∞-algebra structure recovers the fundamental construction in Kapranov's formulation of Rozansky-Witten theory [19].

Inspired by the success of formal deformation theory for complex structures [23] and regular foliations [7, 15, 25] in differential geometry, we initiate a study of infinitesimal deformations of Lie pairs. Specifically, our aim is to classify infinitesimal deformations of a Lie pair (L, A) up to a suitably defined notion of isomorphism. We anticipate that this classification is equivalent to identifying gauge relations between Maurer-Cartan elements within certain L_∞-algebras.

The main results.

We outline the contents and main results of this paper. First in Section 1, we introduce the concept of infinitesimal thickenings of Lie algebroids, a tool specifically designed to describe infinitesimal deformations of Lie pairs. The notion of an $\mathscr{A}$-ringed manifold $M_{\mathscr{A}}$ refers to a smooth manifold M enriched with a local Artinian $\mathbb{K}$-algebra $\mathscr{A}$. More precisely, it is a locally ringed space over M whose structure sheaf is the sheaf of smooth functions thickened by $\mathscr{A}$ (see Definition 1.1).

Within the category of $\mathscr{A}$-ringed manifolds, we identify vector bundle objects, which we call $\mathscr{A}$-ringed vector bundles (see Definition 1.6). An $\mathscr{A}$-ringed vector bundle can be interpreted as an infinitesimal thickening of a conventional vector bundle, achieved through the application of the local Artinian $\mathbb{K}$-algebra $\mathscr{A}$. Analogously, we define $\mathscr{A}$-ringed Lie algebroids (see Definition 1.12) as Lie algebroid objects within the category of $\mathscr{A}$-ringed manifolds, representing infinitesimal thickenings of standard Lie algebroids. Given an $\mathscr{A}$-ringed Lie algebroid $L_{\mathscr{A}}$, evaluating it at the unique maximal ideal of $\mathscr{A}$ yields a Lie algebroid L, which we designate as the center Lie algebroid of $L_{\mathscr{A}}$.

Second, in Section 2, we define an infinitesimal deformation of a Lie pair (L, A) as a Lie pair $(L^0_{\mathscr{A}}, A_{\mathscr{A}})$ within the category of $\mathscr{A}$-ringed manifolds, centered around the original Lie pair (L, A). Here, $L^0_{\mathscr{A}}$ denotes the $\mathscr{A}$-linear extension of the Lie algebroid L, specifically what we term the $\mathscr{A}$-Cartesian extension of L (refer to Definition 2.1). The fundamental concept underlying this definition is that the Lie subalgebroid $A_{\mathscr{A}}$ is subject to variation, parameterized by $\mathscr{A}$, while the extended Lie algebroid $L^0_{\mathscr{A}}$ remains stable with respect to $\mathscr{A}$.

We also need to define isomorphisms between infinitesimal deformations. To this end, we introduce a specific type of automorphism of $\mathscr{A}$-ringed Lie algebroids called a "small automorphism." It is characterized by inducing the identity map on the associated center Lie algebroid, obtained via evaluation at the maximal ideal of $\mathscr{A}$. An interesting fact is that any small automorphism of the $\mathscr{A}$-Cartesian extension $L^0_{\mathscr{A}}$ can be represented as the exponential $\exp(\delta)$ of a nilpotent derivation δ of $L^0_{\mathscr{A}}$ (see Proposition 1.19). Consequently, the small automorphism group $\mathrm{sAut}(L^0_{\mathscr{A}})$ possesses a subgroup, denoted $\mathrm{sIAut}(L^0_{\mathscr{A}})$, comprising exponential elements derived from nilpotent inner derivations of $L^0_{\mathscr{A}}$. Using these concepts, we define weak and semistrict isomorphisms. Two infinitesimal deformations, $(L^0_{\mathscr{A}}, A_{\mathscr{A}})$ and $(L^0_{\mathscr{A}}, A'_{\mathscr{A}})$, of (L, A) are defined as weak (respectively, semistrict) isomorphic if there exists a small morphism $\Pi_A \colon A'_{\mathscr{A}} \to A_{\mathscr{A}}$ of $\mathscr{A}$-ringed Lie algebroids and a small automorphism $\exp(\delta)$, belonging to $\mathrm{sAut}(L^0_{\mathscr{A}})$ (respectively, $\mathrm{sIAut}(L^0_{\mathscr{A}})$), that relates the two deformations in a proper manner. For a rigorous definition, refer to Definition 2.17.

Given a Lie pair (L, A), assigning its weak or semistrict isomorphic infinitesimal deformations to each local Artinian $\mathbb{K}$-algebra determines two infinitesimal deformation functors, denoted by $\mathrm{wDef}_{(L,A)}$ and $\mathrm{sDef}_{(L,A)}$, respectively, from the category of local Artinian $\mathbb{K}$-algebras to the category of sets. Consequently, it is pertinent to inquire about the L_∞-algebras, denoted as $\mathfrak{h}$ and $\mathfrak{h}_0$, that govern these deformation functors. Specifically, we seek $\mathfrak{h}$ and $\mathfrak{h}_0$ such that the associated algebraic deformation functors $\mathrm{Def}_{\mathfrak{h}}$ and $\mathrm{Def}_{\mathfrak{h}_0}$ are isomorphic to $\mathrm{wDef}_{(L,A)}$ and $\mathrm{sDef}_{(L,A)}$, respectively. Here, $\mathrm{Def}_{\mathfrak{h}}$ (resp. $\mathrm{Def}_{\mathfrak{h}_0}$) maps each local Artinian $\mathbb{K}$-algebra $\mathscr{A}$ to the set of gauge equivalent classes of Maurer-Cartan elements of the nilpotent L_∞-algebra $\mathfrak{h} \otimes \mathfrak{m}_{\mathscr{A}}$ (resp. $\mathfrak{h}_0 \otimes \mathfrak{m}_{\mathscr{A}}$) in the sense of Getzler [9] (see also [14]).

In fact, associated with a Lie pair (L, A), there exists a cubic L_∞-algebra

$$\mathfrak{c} := \Gamma(\Lambda^\bullet A^* \otimes L/A),$$

as demonstrated in [1]. This L_∞-algebra, which we call the basic cubic L_∞-algebra of (L, A), differs from the construction presented in [22]. A concrete illustration of this arises when considering the Lie pair $(T_X \otimes \mathbb{C}, T^{0,1}_X)$ derived from a compact complex manifold X. In this specific case, the corresponding basic cubic L_∞-algebra $\mathfrak{c}$ is isomorphic to the Kodaira-Spencer algebra $\Omega^{0,\bullet}_X(T^{1,0}_X)$, which governs the infinitesimal deformations of complex structures on X.

The significance of $\mathfrak{c}$ lies in the fact that the set of Maurer-Cartan elements within the cubic (and nilpotent) L_∞-algebra $\mathfrak{c} \otimes \mathfrak{m}_{\mathscr{A}}$ is isomorphic to the set of standard deformations of (L, A) (see Proposition 2.28), which indeed controls deformations of the Dirac structure $D = A \oplus A^\perp$ of the Courant algebroid $L \oplus L^*$ [1, 13, 20, 32, 33]. However, when considering infinitesimal deformations modulo small automorphisms, the degree 0 component of $\mathfrak{c}$, specifically $\Gamma(L/A)$, is insufficient to generate the small automorphism group $\mathrm{sAut}(L^0_{\mathscr{A}})$ via the L_∞ exponential map. Furthermore, there are numerous examples involving general Lie algebra pairs that show the algebraic deformation functor associated with $\mathfrak{c}$ is neither isomorphic to the infinitesimal deformation functor $\mathrm{wDef}_{(L,A)}$ nor to $\mathrm{sDef}_{(L,A)}$.

To address this problem, we couple the Lie algebra of derivations of the Lie algebroid L, denoted by $\mathrm{Der}(L)$, with the cubic L_∞-algebra $\mathfrak{c}$. As demonstrated in [27], there exists a natural action of $\mathrm{Der}(L)$ on $\mathfrak{c}$. Consequently, the direct sum

$$\mathfrak{h} := \mathrm{Der}(L) \oplus \mathfrak{c},$$

inherits a cubic L_∞-algebra structure that extends the canonical structure on $\mathfrak{c}$. We call $\mathfrak{h}$ the extended cubic L_∞-algebra of the Lie pair (L, A). Furthermore, $\mathfrak{h}$ contains an L_∞-subalgebra given by

$$\mathfrak{h}_0 = \mathrm{IDer}(L) \oplus \mathfrak{c},$$

where $\mathrm{IDer}(L) \subset \mathrm{Der}(L)$ represents the Lie subalgebra of inner derivations of L. These two cubic L_∞-algebras, $\mathfrak{h}$ and $\mathfrak{h}_0$, are central to our investigation. Our main result, detailed in Theorems 2.30 and 2.36, can be summarized as follows:

Theorem A. *The infinitesimal deformation functor* $\mathrm{wDef}_{(L,A)}$ *of weak isomorphisms classes is controlled by the extended cubic* L_∞*-algebra* $\mathfrak{h}$*, while* $\mathrm{sDef}_{(L,A)}$ *of semistrict isomorphism classes is controlled by the* L_∞*-subalgebra* $\mathfrak{h}_0$ *of* $\mathfrak{h}$*.*

To illustrate the application of this theorem, let us consider deformations of a Lie algebra pair $\mathfrak{a} \subset \mathfrak{l}$. In this specific case, the base manifold M reduces to a single point. Consequently, the tangent space of the functor $\mathrm{sDef}_{(\mathfrak{l},\mathfrak{a})}$ is isomorphic to the deformation space of the Lie subalgebra $\mathfrak{a}$ within $\mathfrak{l}$, as defined by Crainic, Schätz, and Struchiner in [6].

In the third part of this paper, Section 3, we investigate the specific scenario where $L = A \bowtie (L/A)$ constitutes a matched Lie pair, implying that L/A can be embedded into L as a Lie subalgebroid. Consequently, the set $\{\mathrm{L}_b \mid b \in \Gamma(L/A)\}$ of inner derivations along the "normal direction" L/A forms a Lie subalgebra within the inner derivations of L. The exponential of these derivations, defined as

$$\mathrm{hAut}(L^0_{\mathscr{A}}) = \{\exp(\mathrm{L}_b) \mid b \in \Gamma(L/A) \otimes \mathfrak{m}_{\mathscr{A}}\},$$

forms a subgroup of the small automorphism group $\mathrm{sAut}(L^0_{\mathscr{A}})$. In this context, we employ $\mathrm{hAut}(L^0_{\mathscr{A}})$ to define isomorphism classes of infinitesimal deformations, as detailed in Definition 3.1. The functor corresponding to this relation is denoted by $\mathrm{hDef}_{A\bowtie B}$. Furthermore, the basic cubic L_∞-algebra $\mathfrak{c}$ associated with the matched Lie pair simplifies to a dg Lie algebra. This dg Lie algebra governs the aforementioned isomorphism classes of infinitesimal deformations, as formalized by the following statement (see Theorem 3.2 for more details):

Theorem B. *For a matched Lie pair* $L = A \bowtie (L/A)$*, the infinitesimal deformation functor* $\mathrm{hDef}_{A\bowtie B}$ *is isomorphic to the algebraic deformation functor associated with the dg Lie algebra* $\mathfrak{c}$*.*

As an application, we consider the matched Lie pair $T^{0,1}_X \bowtie T^{1,0}_X$ on a complex manifold X. The associated infinitesimal deformation functor $\mathrm{hDef}_{T^{0,1}_X \bowtie T^{1,0}_X}$ is isomorphic to the infinitesimal deformation functor of complex structures on X. Furthermore, the basic dg Lie algebra $\mathfrak{c}$ corresponds to the Kodaira-Spencer algebra of X. Consequently, this recovers the established result regarding the isomorphism between the functor of infinitesimal deformations of complex structures and the algebraic deformation functor associated with the Kodaira-Spencer algebra.

We also extend our investigation to transversely holomorphic foliations $\mathcal{F}$ on compact smooth manifolds M. Let F denote the tangent bundle of $\mathcal{F}$, and let $B = T_M/F$ represent the normal bundle. This normal bundle B possesses a natural complex structure, inducing a splitting $B^{\mathbb{C}} = B^{1,0} \oplus B^{0,1}$ of its complexified bundle $B^{\mathbb{C}}$. We then construct a matched Lie pair $(F^{\mathbb{C}} \oplus B^{0,1}) \bowtie B^{1,0}$. The deformation functor associated with this matched Lie pair is isomorphic to the deformation functor of the transversely holomorphic foliation $\mathcal{F}$ itself [8, 12, 34, 35].

Related works.

Several works in the literature relate to the deformation theory presented in this paper. The study of Lie algebra deformations originates with the work of Nijenhuis and Richardson [29, 31]. For a more recent overview of Lie algebra pair deformations, see Crainic [6]. Ji [17] investigated the simultaneous deformations of Lie algebroids and their Lie subalgebroids, deriving an L_∞-algebra via higher derived brackets. However, this approach differs significantly from the Lie pair deformation framework presented here. Specifically, our Lie pair setting (L, A) considers a Lie algebroid L and its Lie subalgebroid A defined over the "same" base manifold. This choice is motivated by examples arising from foliations and complex manifolds. In contrast, Ji's framework typically involves a Lie algebroid E over a smooth manifold M, where the base manifold of its Lie subalgebroid is a submanifold of M. This is motivated by examples from Poisson geometry involving coisotropic submanifolds of Poisson manifolds. In subsequent work, Ji [18] further explored the relationship between deformations of Lie subalgebroids and deformations of coisotropic submanifolds (see also [2, 30] on deformations of coisotropic submanifolds).

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List of commonly used notations.

(1) $\mathbb{K}$ — the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers;
(2) $\mathfrak{m}_{\mathscr{A}}$ — the maximal ideal of a local Artinian $\mathbb{K}$-algebra $\mathscr{A}$;
(3) $M_{\mathscr{A}} = (M, \mathcal{O}_{M_{\mathscr{A}}})$ — an $\mathscr{A}$-ringed manifold centered on a smooth manifold M;
(4) $\mathrm{Mfld}_{\mathscr{A}}$ — the category of $\mathscr{A}$-ringed manifolds;
(5) $\mathrm{sAut}(M_{\mathscr{A}})$ — the group of small automorphisms of an $\mathscr{A}$-ringed manifold $M_{\mathscr{A}}$;
(6) $\Gamma(T_M^{\mathbb{K}})$ — the space of $\mathbb{K}$-valued vector fields on M, i.e., $\mathbb{K} \otimes_{\mathbb{R}} \Gamma(T_M)$;
(7) $T_{\mathscr{A}} := (T_M^{\mathbb{K}})_{\mathscr{A}}$ — the $\mathscr{A}$-ringed vector bundle with center $T_M^{\mathbb{K}} \to M$;
(8) $\Gamma(T_{\mathscr{A}}) := \Gamma(T_M) \otimes_{\mathbb{R}} \mathscr{A}$ — the space of global sections of $T_{\mathscr{A}}$;
(9) $\mathbb{K}_2[t]$ — the $\mathbb{K}$-algebra of dual numbers ;
(10) $(E_{\mathscr{A}}, [-,-]_{E_{\mathscr{A}}}, \rho_{E_{\mathscr{A}}})$ — an $\mathscr{A}$-ringed Lie algebroid;
(11) $E^0_{\mathscr{A}}$ — the $\mathscr{A}$-Cartesian extension of a Lie algebroid E;
(12) $\mathrm{sAut}(E_{\mathscr{A}})$ — the group of small automorphisms of an $\mathscr{A}$-ringed Lie algebroid $E_{\mathscr{A}}$;
(13) $\mathrm{Der}(E)$ — the space of derivations of a Lie algebroid E;
(14) $\mathrm{IDer}(E)$ — the space of inner derivations of a Lie algebroid E;
(15) $\mathrm{sIAut}(E^0_{\mathscr{A}})$ — the group of small inner automorphisms of $E^0_{\mathscr{A}}$ (see (11));
(16) $\mathrm{Sd}(L, A, \mathscr{A})$ — the set of solutions $\xi \in \Gamma(A^* \otimes B) \otimes \mathfrak{m}_{\mathscr{A}}$ to Equation (2.8);
(17) $\mathrm{MC}(\mathfrak{g})$ — the set of Maurer-Cartan elements of a nilpotent L_∞-algebra $\mathfrak{g}$;
(18) $\Omega^\bullet_A(B) = \Gamma(\Lambda^\bullet A^* \otimes B)$ — the basic cubic L_∞-algebra arising from the Lie pair (L, A);
(19) $\mathfrak{h} = (\mathrm{Der}(L) \oplus \Gamma(B)) \bigoplus \left(\bigoplus_{n \geqslant 1} \Omega^n_A(B)\right)$ — the extended cubic L_∞-algebra arising from the Lie pair (L, A);
(20) $\mathrm{Def}_{\mathfrak{g}}$ — the algebraic deformation functor associated to an L_∞-algebra $\mathfrak{g}$;
(21) $\mathrm{wDef}_{(L,A)}$ — the weak infinitesimal deformation functor of the Lie pair (L, A);
(22) **Art** — the category of local Artinian $\mathbb{K}$-algebras;
(23) **Set** — the category of sets;
(24) $\mathrm{sDef}_{(L,A)}$ — the semistrict infinitesimal deformation functor of the Lie pair (L, A);
(25) $\mathrm{hDef}_{A \bowtie B}$ — the deformation functor of the matched Lie pair $L = A \bowtie B$;
(26) $\mathbf{H}^i(\mathfrak{g}, [-]_1)$ — the i-th cohomology of an L_∞-algebra $\mathfrak{g}$ with respect to the first bracket $[-]_1$.

1. Lie algebroids enriched with local Artinian algebras

We begin by defining smooth manifolds enriched with a local Artinian $\mathbb{K}$-algebra $\mathscr{A}$.

1.1. **Local Artinian ringed manifolds and vector bundles.** Given a smooth manifold M, we denote by $\mathcal{O}_M$ the sheaf of $\mathbb{K}$-valued smooth functions on M and by $C^\infty(M, \mathbb{K})$ the $\mathbb{K}$-algebra of smooth functions on M.

Definition 1.1. *Let M be a smooth manifold, and $\mathscr{A}$ a local Artinian $\mathbb{K}$-algebra. Define a sheaf $\mathcal{O}_{M_{\mathscr{A}}}$ of $\mathscr{A}$-algebras over M by assigning to every open subset $U \subset M$ the $\mathscr{A}$-algebra*

$$\mathcal{O}_{M_{\mathscr{A}}}(U) := C^\infty(U, \mathbb{K}) \otimes_{\mathbb{K}} \mathscr{A}.$$

We call $M_{\mathscr{A}} := (M, \mathcal{O}_{M_{\mathscr{A}}})$ the $\mathscr{A}$-ringed manifold (centered on M). The space of global sections of $\mathcal{O}_{M_{\mathscr{A}}}$ is denoted by $C^\infty(M_{\mathscr{A}})(= C^\infty(M, \mathbb{K}) \otimes_{\mathbb{K}} \mathscr{A})$; its elements are called smooth functions on $M_{\mathscr{A}}$.

Let $\mathcal{E}$ be a sheaf of $\mathbb{K}$-modules over M. The *evaluation map* $\mathrm{ev}\colon \mathscr{A} \to \mathscr{A}/\mathfrak{m}_{\mathscr{A}} = \mathbb{K}$ at the maximal ideal $\mathfrak{m}_{\mathscr{A}}$ of $\mathscr{A}$ induces a map $\mathcal{E} \otimes_{\mathbb{K}} \mathscr{A} \to \mathcal{E}$, also denoted by ev, sending $\xi \otimes a$ to $\xi \cdot \mathrm{ev}(a)$ for all local sections ξ and all $a \in \mathscr{A}$. In particular, when $\mathcal{E}$ is the structural sheaf $\mathcal{O}_{M_{\mathscr{A}}}$, the evaluation map induces an embedding from the $\mathbb{K}$-ringed manifold $(M, \mathcal{O}_M)$ to the $\mathscr{A}$-ringed manifold $(M, \mathcal{O}_{M_{\mathscr{A}}})$.

Definition 1.2. *A morphism of $\mathscr{A}$-ringed manifolds $M_{\mathscr{A}} \to N_{\mathscr{A}}$ consists of a pair $(\varphi, \lambda^\sharp)$ where*

(1) $\varphi\colon M \to N$ is a smooth map of smooth manifolds;

(2) $\lambda^\sharp\colon \mathcal{O}_{N_{\mathscr{A}}} \to \varphi_* \mathcal{O}_{M_{\mathscr{A}}}$ is a morphism of sheaves of $\mathscr{A}$-algebras over N extending the pullback map $\varphi^*\colon \mathcal{O}_N \to \varphi_* \mathcal{O}_M$ in the following sense — for each local section $f \otimes a$ of $\mathcal{O}_{N_{\mathscr{A}}}(U) = C^\infty(U, \mathbb{K}) \otimes_{\mathbb{K}} \mathscr{A}$, where $U \subset N$ is open, one has

$$\mathrm{ev} \circ \lambda^\sharp(f \otimes a) = \mathrm{ev} \circ (\lambda^\sharp(f) \otimes a) = \varphi^*(f) \cdot \mathrm{ev}(a).$$

Since the restriction map to each stalk at $x \in M$, $\lambda^\sharp_x\colon \mathcal{O}_{N_{\mathscr{A}}, \varphi(x)} \to \mathcal{O}_{M_{\mathscr{A}}, x}$ is local, the morphism $(\varphi, \lambda^\sharp)$ of $\mathscr{A}$-ringed manifolds is indeed a morphism of locally ringed spaces. Meanwhile, the morphism $\lambda^\sharp\colon \mathcal{O}_{N_{\mathscr{A}}} \to \varphi_* \mathcal{O}_{M_{\mathscr{A}}}$ of sheaves is completely determined by the corresponding morphism on the level of global sections,

$$\lambda^*\colon C^\infty(N_{\mathscr{A}}) \to C^\infty(M_{\mathscr{A}}).$$

So we can regard a morphism of $\mathscr{A}$-ringed manifolds from $M_{\mathscr{A}}$ to $N_{\mathscr{A}}$ as a morphism of $\mathscr{A}$-algebras $\lambda^*\colon C^\infty(N_{\mathscr{A}}) \to C^\infty(M_{\mathscr{A}})$ covering a smooth map $\varphi\colon M \to N$ in the sense that

$$\mathrm{ev} \circ \lambda^* = \varphi^*\colon C^\infty(N, \mathbb{K}) \to C^\infty(M, \mathbb{K}).$$

In this situation, we say that λ^* is a **lifting** of φ^* and that φ^* is the **center** of λ^*.

It is clear that the collection of $\mathscr{A}$-ringed manifolds and their morphisms forms a category, denoted by $\mathrm{Mfld}_{\mathscr{A}}$. We need a particular type of automorphisms of $\mathscr{A}$-ringed manifolds.

Definition 1.3. *A **small automorphism** of an $\mathscr{A}$-ringed manifold $M_{\mathscr{A}}$ is a lifting of the identity map Id on $C^\infty(M, \mathbb{K})$, i.e., a morphism $\lambda^*\colon C^\infty(M_{\mathscr{A}}) \to C^\infty(M_{\mathscr{A}})$ satisfying $\mathrm{ev} \circ \lambda^* = \mathrm{Id}$.*

We denote by $\mathrm{sAut}(M_{\mathscr{A}})$ the group of small automorphisms of $M_{\mathscr{A}}$.

Example 1.4. *Let $\mathscr{A} = \mathbb{K}_2[t] := \mathbb{K}[t]/(t^2)(\cong \mathbb{K} \oplus \mathbb{K}t)$ be the algebra of dual numbers. Then for any $X \in \Gamma(T_M^{\mathbb{K}}) := \mathbb{K} \otimes_{\mathbb{R}} \Gamma(T_M)$, the exponential map*

$$\exp(X \otimes t) = \mathrm{Id} + X \otimes t\colon C^\infty(M, \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}_2[t] \to C^\infty(M, \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}_2[t],$$

which sends $f \in C^\infty(M, \mathbb{K})$ to $f + X(f) \otimes t$, is an element in $\mathrm{sAut}(M_{\mathbb{K}_2[t]})$. It is clear that the inverse of $\exp(X \otimes t)$ is $\exp(-X \otimes t)$.

Remark 1.5. *In fact, any small automorphism of an $\mathscr{A}$-ringed manifold $M_{\mathscr{A}}$ is of the form $\exp(D)$ for some $D \in \Gamma(T_M^{\mathbb{K}}) \otimes_{\mathbb{K}} \mathfrak{m}_{\mathscr{A}}$. Here $\exp$ denotes the exponential map:*

$$\exp\colon \Gamma(T_M^{\mathbb{K}}) \otimes_{\mathbb{K}} \mathfrak{m}_{\mathscr{A}} \to \mathrm{sAut}(M_{\mathscr{A}}), \quad D \mapsto \exp(D),$$

where $\exp(D)$ is defined by

$$\exp(D)(f) := \sum_{n \geq 0} \frac{1}{n!} \langle D^n, f \rangle,$$

for all $f \in C^\infty(M_{\mathscr{A}})$. The bracket $\langle D^n, f \rangle$ means the $\mathscr{A}$-linear action of the differential operator D^n on $f \in C^\infty(M_{\mathscr{A}})$. The above sum is indeed finite as $D \in \Gamma(T_M^{\mathbb{K}}) \otimes_{\mathbb{K}} \mathfrak{m}_{\mathscr{A}}$ is nilpotent. (See Proposition 1.19.)

We now turn to vector bundle objects in the category of $\mathscr{A}$-ringed manifolds.

Definition 1.6. *An* ***$\mathscr{A}$-ringed vector bundle*** *over an $\mathscr{A}$-ringed manifold $M_{\mathscr{A}}$ is an $\mathscr{A}$-ringed manifold $E_{\mathscr{A}}$ together with a morphism $E_{\mathscr{A}} \to M_{\mathscr{A}}$, which covers a smooth $\mathbb{K}$-vector bundle $\pi\colon E \to M$.*

The underlying vector bundle $E \to M$ is called the ***center*** *of $E_{\mathscr{A}} \to M_{\mathscr{A}}$. The space of global sections of $E_{\mathscr{A}}$ is defined by*

$$\Gamma(E_{\mathscr{A}}) := \Gamma(E) \otimes_{\mathbb{K}} \mathscr{A},$$

which is a $C^\infty(M_{\mathscr{A}})$-module.

Definition 1.7. *A morphism of $\mathscr{A}$-ringed vector bundles from $F_{\mathscr{A}}$ to $E_{\mathscr{A}}$ (over the same base $M_{\mathscr{A}}$) is specified by an $\mathscr{A}$-linear map $\Pi\colon \Gamma(F_{\mathscr{A}}) \to \Gamma(E_{\mathscr{A}})$ satisfying the following two conditions:*

(1) The map Π covers a small automorphism λ^* of $M_{\mathscr{A}}$ in the sense that

$$\Pi(f\nu) = \lambda^*(f)\Pi(\nu),$$

for all $f \in C^\infty(M_{\mathscr{A}}), \nu \in \Gamma(F_{\mathscr{A}})$.

(2) The center of Π:

$$\Pi_0 := \mathrm{ev} \circ \Pi\colon \Gamma(F) \to \Gamma(E)$$

defines a morphism of vector bundles over M.

Such a morphism will be denoted by (Π, λ^*). In particular, when the small automorphism λ^* of $C^\infty(M_{\mathscr{A}})$ is the identity Id, we call (Π, Id) a ***strict morphism*** *and denote it by Π for simplicity.*

Compositions of morphisms of $\mathscr{A}$-ringed vector bundles are naturally defined.

Remark 1.8. *In general, morphisms between $\mathscr{A}$-ringed vector bundles over distinct $\mathscr{A}$-ringed base manifolds are more complicated than what we have discussed here.*

Proposition 1.9. *Suppose that $(\Pi, \lambda^*)\colon F_{\mathscr{A}} \to E_{\mathscr{A}}$ is a morphism of $\mathscr{A}$-ringed vector bundles over $M_{\mathscr{A}}$. Then the map $\Pi\colon \Gamma(F_{\mathscr{A}}) \to \Gamma(E_{\mathscr{A}})$ is injective (resp. surjective) if and only if the center $\Pi_0\colon \Gamma(F) \to \Gamma(E)$ of Π is injective (resp. surjective). Consequently, Π is an isomorphism of $\mathscr{A}$-modules if and only if its center Π_0 is an isomorphism of vector bundles.*

Proof. Assume that $\mathfrak{m}_{\mathscr{A}}^n \neq 0$ while $\mathfrak{m}_{\mathscr{A}}^{n+1} = 0$ for some $n \geqslant 1$. We now show that Π is injective if and only if $\Pi_0\colon \Gamma(F) \to \Gamma(E)$ is injective.

Assume first that Π is injective. If $\Pi_0(e) = 0$ for some $e \in \Gamma(F)$, it then follows from $\mathrm{ev}(\Pi(e)) = \Pi_0(e) = 0$ that $\Pi(e) \in \Gamma(E) \otimes_{\mathbb{K}} \mathfrak{m}_{\mathscr{A}}$. Thus, for any nonzero element $a \in \mathfrak{m}_{\mathscr{A}}^n$, we have $\Pi(e \otimes a) = \Pi(e)a = 0$, which implies that $e \otimes a = 0$ since Π is injective. Hence, we have $e = 0$, and thus Π_0 is injective.

Conversely, assume that $\Pi_0\colon \Gamma(F) \to \Gamma(E)$ is injective. We next show that Π is injective as well. Consider the following family of $\mathscr{A}$-linear maps induced by Π,

$$\Pi_k\colon \Gamma(F_{\mathscr{A}/\mathfrak{m}_{\mathscr{A}}^{k+1}}) \to \Gamma(E_{\mathscr{A}/\mathfrak{m}_{\mathscr{A}}^{k+1}}),$$

for all $0 \leqslant k \leqslant n$, where $\Pi_n = \Pi$. By an induction argument on k, it suffices to prove that Π_k is injective provided that Π_{k-1} is injective for all k.

By choosing a splitting of the short exact sequence of vector spaces

$$0 \to \mathfrak{m}_{\mathscr{A}}^k/\mathfrak{m}_{\mathscr{A}}^{k+1} \to \mathscr{A}/\mathfrak{m}_{\mathscr{A}}^{k+1} \to \mathscr{A}/\mathfrak{m}_{\mathscr{A}}^k \to 0,$$

we may decompose the map

$$\Pi_k\colon \Gamma(F_{\mathscr{A}/\mathfrak{m}_{\mathscr{A}}^{k+1}}) \cong \Gamma(F_{\mathscr{A}/\mathfrak{m}_{\mathscr{A}}^{k}}) \oplus \Gamma(F_{\mathfrak{m}_{\mathscr{A}}^k/\mathfrak{m}_{\mathscr{A}}^{k+1}}) \to \Gamma(E_{\mathscr{A}/\mathfrak{m}_{\mathscr{A}}^{k+1}}) \cong \Gamma(E_{\mathscr{A}/\mathfrak{m}_{\mathscr{A}}^{k}}) \oplus \Gamma(E_{\mathfrak{m}_{\mathscr{A}}^k/\mathfrak{m}_{\mathscr{A}}^{k+1}})$$

as the sum

$$\Pi_k = \Pi_k^{(1)} + \Pi_k^{(2)} := \mathrm{pr}_1 \circ \Pi_k + \mathrm{pr}_2 \circ \Pi_k,$$

where $\mathrm{pr}_1\colon \Gamma(E_{\mathscr{A}/\mathfrak{m}_{\mathscr{A}}^{k+1}}) \to \Gamma(E_{\mathscr{A}/\mathfrak{m}_{\mathscr{A}}^{k}})$ and $\mathrm{pr}_2\colon \Gamma(E_{\mathscr{A}/\mathfrak{m}_{\mathscr{A}}^{k+1}}) \to \Gamma(E_{\mathfrak{m}_{\mathscr{A}}^{k}/\mathfrak{m}_{\mathscr{A}}^{k+1}})$ are projections. Note that the restriction of $\Pi_k^{(1)}$ onto the subspace $\Gamma(F_{\mathfrak{m}_{\mathscr{A}}^{k}/\mathfrak{m}_{\mathscr{A}}^{k+1}})$ vanishes. It follows that $\Pi_k^{(1)}$ coincides with Π_{k-1} up to an extension by zero. For any

$$\sum_i s_i \otimes a_i + \sum_j t_j \otimes b_j \in \Gamma(F_{\mathscr{A}/\mathfrak{m}_{\mathscr{A}}^{k}}) \oplus \Gamma(F_{\mathfrak{m}_{\mathscr{A}}^{k}/\mathfrak{m}_{\mathscr{A}}^{k+1}}) \cong \Gamma(F_{\mathscr{A}/\mathfrak{m}_{\mathscr{A}}^{k+1}}),$$

where $s_i, t_j \in \Gamma(F)$ and $a_i \in \mathscr{A}/\mathfrak{m}_{\mathscr{A}}^{k}, b_j \in \mathfrak{m}_{\mathscr{A}}^{k}/\mathfrak{m}_{\mathscr{A}}^{k+1}$, we have

$$\begin{aligned}\Pi_k\left(\sum_i s_i \otimes a_i + \sum_j t_j \otimes b_j\right) &= \Pi_k\left(\sum_i s_i \otimes a_i\right) + \Pi_k\left(\sum_j t_j \otimes b_j\right)\\ &= \Pi_{k-1}\left(\sum_i s_i \otimes a_i\right) + \Pi_k^{(2)}\left(\sum_i s_i \otimes a_i\right) + \sum_j \Pi_0(t_j) \otimes b_j.\end{aligned}$$

If the above expression vanishes, then both $\Pi_{k-1}\left(\sum_i s_i \otimes a_i\right) \in \Gamma(F_{\mathscr{A}/\mathfrak{m}^k})$ and $\Pi_k^{(2)}(\sum_i s_i \otimes a_i) + \sum_j \Pi_0(t_j) \otimes b_j = 0 \in \Gamma(F_{\mathfrak{m}^k/\mathfrak{m}^{k+1}})$ vanishes. Note that Π_{k-1} is injective by inductive assumption. It follows that $\sum_i s_i \otimes a_i = 0$, which implies that $\sum_j t_j \otimes b_j = 0$ since Π_0 is also injective. Hence, Π_k is injective.

To prove that Π is surjective if and only if Π_0 is surjective, it suffices to adopt an analogous approach, which we omit. □

Example 1.10 (Tangent bundle of an $\mathscr{A}$-ringed manifold)**.** *Consider the associated $\mathscr{A}$-ringed tangent bundle $T_{\mathscr{A}} := (T_M^{\mathbb{K}})_{\mathscr{A}}$ over the $\mathscr{A}$-ringed manifold $M_{\mathscr{A}}$. Each small automorphism λ^* of $M_{\mathscr{A}}$ induces an automorphism $(\Pi_{\lambda^*}, \lambda^*)$ of $T_{\mathscr{A}}$ by conjugation, i.e.,*

$$\Pi_{\lambda^*}(D) = \lambda^* \circ D \circ (\lambda^*)^{-1},$$

for all $D \in \Gamma(T_{\mathscr{A}}) = \Gamma(T_M) \otimes_{\mathbb{R}} \mathscr{A}$.

In particular, when $\mathscr{A} = \mathbb{K}_2[t]$ is the algebra of dual numbers, by Example 1.4, each vector field $D_0 \in \Gamma(T_M^{\mathbb{K}})$ induces a small automorphism $\exp(D_0 \otimes t) = \mathrm{Id} + D_0 \otimes t$ of $M_{\mathscr{A}}$. In this case, the associated morphism is

$$\begin{aligned}\Pi_{\exp(D_0\otimes t)}(D) &= \exp(D_0 \otimes t) \circ D \circ \exp(-D_0 \otimes t) = (\mathrm{Id} + D_0 \otimes t) \circ D \circ (\mathrm{Id} - D_0 \otimes t)\\ &= D + [D_0, D] \otimes t, \end{aligned} \tag{1.11}$$

for all $D \in \Gamma(T_{\mathscr{A}})$.

1.2. Local Artinian ringed Lie algebroids.

We now study Lie algebroid objects in the category of $\mathscr{A}$-ringed manifolds.

Definition 1.12. *An **$\mathscr{A}$-ringed Lie algebroid** consists of a triple $(E_{\mathscr{A}}, [-,-]_{E_{\mathscr{A}}}, \rho_{E_{\mathscr{A}}})$, where*

(1) $E_{\mathscr{A}}$ is an $\mathscr{A}$-ringed vector bundle over $M_{\mathscr{A}}$;

(2) $\rho_{E_{\mathscr{A}}}$, called the anchor, is a morphism of $\mathscr{A}$-ringed vector bundles from $E_{\mathscr{A}}$ to the tangent bundle $T_{\mathscr{A}}$ of $M_{\mathscr{A}}$, covering the identity $\mathrm{Id}\colon M_{\mathscr{A}} \to M_{\mathscr{A}}$;

(3) $[-,-]_{E_{\mathscr{A}}}\colon \Gamma(E_{\mathscr{A}}) \times \Gamma(E_{\mathscr{A}}) \to \Gamma(E_{\mathscr{A}})$ is an $\mathscr{A}$-bilinear Lie bracket on the space $\Gamma(E_{\mathscr{A}})$ of global sections, satisfying the Leibniz rule

$$[u, fv]_{E_{\mathscr{A}}} = \langle \rho_{E_{\mathscr{A}}}(u), f\rangle v + f[u,v]_{E_{\mathscr{A}}},$$

for all $f \in C^\infty(M_{\mathscr{A}})$ and $u, v \in \Gamma(E_{\mathscr{A}})$.

It follows from the above Leibniz rule that the anchor map $\rho_{E_{\mathscr{A}}}$ is indeed a morphism of Lie algebras on the section spaces.

Given an $\mathscr{A}$-ringed Lie algebroid $(E_{\mathscr{A}}, [-,-]_{E_{\mathscr{A}}}, \rho_{E_{\mathscr{A}}})$, the evaluation map $\mathrm{ev}\colon E_{\mathscr{A}} \to E$ determines a Lie algebroid $(E, [-,-]_E, \rho_E)$. Conversely, given a Lie algebroid $(E, [-,-]_E, \rho_E)$ over a smooth manifold M, there exists an $\mathscr{A}$-ringed Lie algebroid, denote by $E^0_{\mathscr{A}}$, whose anchor $\rho_{E^0_{\mathscr{A}}}$ and bracket $[-,-]_{E^0_{\mathscr{A}}}$ are $\mathscr{A}$-linear extensions of the anchor ρ_E and the bracket $[-,-]_E$, respectively. We call $E^0_{\mathscr{A}}$ the **$\mathscr{A}$-Cartesian extension** of the Lie algebroid E.

Definition 1.13. *Let $(E_{\mathscr{A}}, [-,-]_{E_{\mathscr{A}}}, \rho_{E_{\mathscr{A}}})$ and $(F_{\mathscr{A}}, [-,-]_{F_{\mathscr{A}}}, \rho_{F_{\mathscr{A}}})$ be two $\mathscr{A}$-ringed Lie algebroids over $M_{\mathscr{A}}$. A morphism of $\mathscr{A}$-ringed Lie algebroids from $F_{\mathscr{A}}$ to $E_{\mathscr{A}}$ is a morphism $(\Pi, \lambda^*)\colon F_{\mathscr{A}} \to E_{\mathscr{A}}$ of the underlying $\mathscr{A}$-ringed vector bundles satisfying the following two conditions:*

(1) The $\mathscr{A}$-linear map $\Pi\colon \Gamma(F_{\mathscr{A}}) \to \Gamma(E_{\mathscr{A}})$ preserves the brackets, i.e.,

$$\Pi([s_1, s_2]_{F_{\mathscr{A}}}) = [\Pi(s_1), \Pi(s_2)]_{E_{\mathscr{A}}},$$

for all $s_1, s_2 \in \Gamma(F_{\mathscr{A}})$.

(2) The pair (Π, λ^*) is compatible with the two anchors $\rho_{E_{\mathscr{A}}}$ and $\rho_{F_{\mathscr{A}}}$ in the following sense:

$$\lambda^*\langle \rho_{F_{\mathscr{A}}}(s), f\rangle = \langle \rho_{E_{\mathscr{A}}}(\Pi(s)), \lambda^* f\rangle, \tag{1.14}$$

for all $f \in C^\infty(M_{\mathscr{A}})$ and $s \in \Gamma(F_{\mathscr{A}})$.

It is straightforward to verify that the center $\Pi_0\colon F \to E$ of a morphism Π of $\mathscr{A}$-ringed Lie algebroids is itself a morphism of Lie algebroids over M. In particular, when λ^* is the identity Id of $C^\infty(M_{\mathscr{A}})$, the morphism (Π, Id) will be denoted by Π for simplicity and will be referred to as a **strict** morphism.

Remark 1.15. *Given two Lie algebroids over different base manifolds, Liu and Chen have studied various characterizations of morphisms and comorphisms between them in [3]. The definition of morphisms and comorphisms of ordinary Lie algebroids can be generalized to the setting of $\mathscr{A}$-ringed Lie algebroids. In fact, our definition of morphisms of $\mathscr{A}$-ringed Lie algebroids over the same $\mathscr{A}$-ringed manifold can be viewed as a special case of* comorphisms *of $\mathscr{A}$-ringed Lie algebroids in the spirit of [3].*

In what follows, we focus on a special kind of automorphisms of an $\mathscr{A}$-ringed Lie algebroid.

Definition 1.16. *A **small automorphism** of an $\mathscr{A}$-ringed Lie algebroid $(E_{\mathscr{A}}, [-,-]_{E_{\mathscr{A}}}, \rho_{E_{\mathscr{A}}})$ over $M_{\mathscr{A}}$ is a morphism (Π, λ^*) of $\mathscr{A}$-ringed Lie algebroids from $E_{\mathscr{A}}$ to itself whose center Π_0 is the identity of the center Lie algebroid E over M.*

Denote by $\mathrm{sAut}(E_{\mathscr{A}})$ the group of small automorphisms of the $\mathscr{A}$-ringed Lie algebroid $E_{\mathscr{A}}$.

For any $\mathscr{A}$-Cartesian extension $E^0_{\mathscr{A}}$ of a Lie algebroid E, we now establish that every element of $\mathrm{sAut}(E^0_{\mathscr{A}})$ can be expressed as the exponential of a nilpotent derivation of $E^0_{\mathscr{A}}$.

Definition 1.17. *A derivation of the Lie algebroid $(E, [-,-]_E, \rho_E)$ over M is a linear operator $\delta\colon \Gamma(E) \to \Gamma(E)$ equipped with a vector field $\sigma(\delta) \in \Gamma(T^{\mathbb{K}}_M)$, called the symbol of δ, satisfying*

$$\begin{aligned} \delta(fu) &= \sigma(\delta)(f)u + f\delta(u), \\ [\sigma(\delta), \rho_E(u)] &= \rho_E(\delta(u)), \\ \delta[u,v]_E &= [\delta(u), v]_E + [u, \delta(v)]_E, \end{aligned}$$

for all $f \in C^\infty(M, \mathbb{K}), u, v \in \Gamma(E)$.

For a Lie algebroid E, the space $\mathrm{Der}(E)$ of derivations together with the standard commutator is a Lie algebra. The subspace $\mathrm{IDer}(E) := \{\mathrm{ad}_u \,|\, u \in \Gamma(E)\}$ of inner derivations is a Lie subalgebra of $\mathrm{Der}(E)$.

Note that the $\mathscr{A}$-linear extension of derivations of a Lie algebroid E yields derivations of the $\mathscr{A}$-Cartesian extension $E^0_{\mathscr{A}}$ of E. In particular, for any element $\delta \in \mathrm{Der}(E) \otimes_{\mathbb{K}} \mathfrak{m}_{\mathscr{A}}$, the operator δ is a nilpotent derivation of $E^0_{\mathscr{A}}$. Its exponential

$$\exp(\delta) := \mathrm{Id} + \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \cdots$$

defines a small automorphism of $E^0_{\mathscr{A}}$. Indeed, every small automorphism of $E^0_{\mathscr{A}}$ arises in this manner. To illustrate this fact, consider the special case where $\mathscr{A} = \mathbb{K}_2[t]$ is the algebra of dual numbers over $\mathbb{K}$.

Example 1.18. *Consider the $\mathbb{K}_2[t]$-Cartesian extension $E^0_{\mathbb{K}_2[t]}$ of a Lie algebroid $(E, [\cdot,\cdot]_E, \rho_E)$. Suppose that $\Pi \in \mathrm{sAut}(E_{\mathbb{K}_2[t]})$ is a small automorphism covering a small automorphism $\exp(D \otimes t) = \mathrm{Id} + D \otimes t$ of $M_{\mathscr{A}}$ for some $D \in \Gamma(T_M^{\mathbb{K}})$ (see Example 1.4). Then*

$$\Pi = \mathrm{Id} + \Pi_1 \otimes t \colon \Gamma(E) \to \Gamma(E) \otimes_{\mathbb{K}} \mathbb{K}_2[t],$$

where $\Pi_1 \colon \Gamma(E) \to \Gamma(E)$ is $\mathbb{K}$-linear, satisfying

$$\Pi_1(fe) = D(f)e + f\Pi_1(e).$$

Since $\Pi([u,v]_E) = [\Pi(u), \Pi(v)]_E$ for all $u, v \in \Gamma(E)$, it follows that

$$\Pi_1[u,v]_E = [\Pi_1(u), v]_E + [u, \Pi_1(v)]_E.$$

By Equations (1.11) and (1.14), one obtains

$$\rho_E(\Pi_1(e)) = [D, \rho_E(e)],$$

and Π_1 is a derivation of E with symbol D. Hence, we have $\Pi = \exp(\Pi_1 \otimes t)$.

Proposition 1.19. *Any small automorphism of the $\mathscr{A}$-Cartesian extension $E^0_{\mathscr{A}}$ can be uniquely expressed as $\exp(\delta)$ for some $\delta \in \mathrm{Der}(E) \otimes_{\mathbb{K}} \mathfrak{m}_{\mathscr{A}}$ of $E^0_{\mathscr{A}}$. Moreover, $\exp(\delta)$ covers the small automorphism $\exp(\sigma(\delta))$ of the $\mathscr{A}$-ringed manifold $M_{\mathscr{A}}$ generated by the symbol $\sigma(\delta) \in \Gamma(T_{\mathscr{A}})$ of δ.*

Proof. We prove by induction on the dimension $\dim_{\mathbb{K}} \mathscr{A} := n \geqslant 2$ of the local Artinian $\mathbb{K}$-algebra $\mathscr{A}$. When $n = 2$, we have $\mathscr{A} \cong \mathbb{K}_2[t]$. By Example 1.18, any small automorphism $\Pi \colon E_{\mathscr{A}} \to E_{\mathscr{A}}$ is of the form $\Pi = \mathrm{Id} + \Pi_1 \otimes t = \exp(\Pi_1 \otimes t)$ for some $\Pi_1 \in \mathrm{Der}(E)$.

Suppose that the proposition holds for all local Artinian $\mathbb{K}$-algebras $\mathscr{A}'$ with $\dim_{\mathbb{K}} \mathscr{A}' \leqslant n-1$ for some integer $n \geqslant 3$. Given an n-dimensional local Artinian $\mathbb{K}$-algebra $\mathscr{A}$, for each nonzero element $t \in \mathfrak{m}_{\mathscr{A}}$ such that $t\mathfrak{m}_{\mathscr{A}} = 0$, one has a short exact sequence of $\mathbb{K}$-vector spaces

$$0 \to \mathbb{K}t \to \mathscr{A} \xrightarrow{\mathrm{pr}} \mathscr{A}' := \mathscr{A}/\mathbb{K}t \to 0,$$

where $\mathscr{A}'$ is a local Artinian $\mathbb{K}$-algebra of dimension $(n-1)$, and pr is a morphism of Artinian algebras.

Given any $\Pi \in \mathrm{sAut}(E^0_{\mathscr{A}})$, the composition

$$\widetilde{\Pi} = (\mathrm{Id} \otimes \mathrm{pr}) \circ \Pi \colon \Gamma(E) \to \Gamma(E) \otimes \mathscr{A}'$$

determines an element $\Pi' := \widetilde{\Pi} \otimes \mathrm{Id} \in \mathrm{sAut}(E^0_{\mathscr{A}'})$. By the induction assumption, there exists an element $\delta' \in \mathrm{Der}(E) \otimes_{\mathbb{K}} \mathfrak{m}_{\mathscr{A}'}$ such that $\Pi' = \exp(\delta')$.

Now we choose an element $\delta_0 \in \mathrm{Der}(E) \otimes_{\mathbb{K}} \mathfrak{m}_{\mathscr{A}}$ satisfying $(\mathrm{Id} \otimes \mathrm{pr}) \circ \delta_0 = \delta'$. Consider the small automorphism $\Pi \circ \exp(-\delta_0)$ of $E_{\mathscr{A}}$, which is subject to the relation

$$\begin{aligned}(\mathrm{Id} \otimes \mathrm{pr}) \circ (\Pi \circ \exp(-\delta_0)) &= \big((\mathrm{Id} \otimes \mathrm{pr}) \circ \Pi\big) \circ \big((\mathrm{Id} \otimes \mathrm{pr}) \circ \exp(-\delta_0)\big) \\ &= \exp(\delta')\exp(-\delta') = \mathrm{Id}.\end{aligned}$$

It follows that $\Pi \circ \exp(-\delta_0)$ sends $\Gamma(E) \otimes_{\mathbb{K}} \mathscr{A}$ to $\Gamma(E) \otimes_{\mathbb{K}} (\mathbb{K} \oplus \mathbb{K}t)$. Since $\mathbb{K} \oplus \mathbb{K}t \cong \mathbb{K}_2[t]$, by Example 1.18, we can find an element $\delta_1 \in \mathrm{Der}(E) \otimes_{\mathbb{K}} \mathbb{K}t$ such that $\Pi \circ \exp(-\delta_0) = \exp(\delta_1)$. Hence, we have

$$\Pi = \exp(\delta_1) \circ \exp(\delta_0) = \exp(\delta_1 + \delta_0).$$

To see the uniqueness of δ, we observe that if $\exp(\delta) = \mathrm{Id} \otimes \mathrm{Id}_{\mathscr{A}}$, then we have

$$\delta + \frac{1}{2}\delta^2 + \frac{1}{6}\delta^3 + \cdots = 0 \colon \Gamma(E) \to \Gamma(E) \otimes_{\mathbb{K}} \mathfrak{m}_{\mathscr{A}}.$$

Since $\delta^n(\Gamma(E)) \subset \Gamma(E) \otimes_{\mathbb{K}} \mathfrak{m}^n_{\mathscr{A}}$ for all $n > 0$, solving the above equation degreewisely, we obtain that $\delta(\Gamma(E)) \subset \Gamma(E) \otimes_{\mathbb{K}} \mathfrak{m}^n_{\mathscr{A}}$ for all $n > 0$, which implies that $\delta = 0$. □

A small automorphism φ of $E^0_{\mathscr{A}}$ is called **inner** if it is of the form $\exp(\delta)$, where $\delta \in \mathrm{IDer}(E) \otimes_{\mathbb{K}} \mathfrak{m}_{\mathscr{A}}$ is an inner derivation. The set $\mathrm{sIAut}(E^0_{\mathscr{A}})$ consisting of all small inner automorphisms forms a subgroup of $\mathrm{sAut}(E^0_{\mathscr{A}})$.

2. Infinitesimal deformations of Lie pairs

Let (L, A) denote an inclusion $i\colon A \hookrightarrow L$ of Lie algebroids over a common base manifold M. This section examines infinitesimal deformations of such Lie pairs. The definition of these deformations necessarily involves a local Artinian $\mathbb{K}$-algebra, denoted by $\mathscr{A}$, which serves as the parameter space for the deformation.

2.1. Infinitesimal deformations and their standard realizations.

Definition 2.1. *An* ***infinitesimal deformation*** *of a Lie pair (L, A) parameterized by a local Artinian $\mathbb{K}$-algebra $\mathscr{A}$ consists of the following data:*

(1) an $\mathscr{A}$-ringed Lie algebroid $(A_{\mathscr{A}}, [-,-]_{A_{\mathscr{A}}}, \rho_{A_{\mathscr{A}}})$ whose center Lie algebroid is the given Lie subalgebroid A of L;

(2) a morphism of $\mathscr{A}$-ringed Lie algebroids from $(A_{\mathscr{A}}, [-,-]_{A_{\mathscr{A}}}, \rho_{A_{\mathscr{A}}})$ to the $\mathscr{A}$-Cartesian extension $L^0_{\mathscr{A}}$ of L,
$$(I, \lambda^*)\colon (A_{\mathscr{A}}, [-,-]_{A_{\mathscr{A}}}, \rho_{A_{\mathscr{A}}}) \to L^0_{\mathscr{A}},$$
whose center I_0 coincides with the given inclusion $i\colon A \hookrightarrow L$ of Lie algebroids.

Conceptually, an infinitesimal deformation of (L, A) depicts an $\mathscr{A}$-parameterized family of Lie algebroid structures $(A_{\mathscr{A}}, [-,-]_{A_{\mathscr{A}}}, \rho_{A_{\mathscr{A}}})$ on the vector bundle A, while the 'bigger' Lie algebroid L (which contains $A_{\mathscr{A}}$) remains unchanged with respect to $\mathscr{A}$. We shall denote an infinitesimal deformation of (L, A) by the quadruple $([-,-]_{A_{\mathscr{A}}}, \rho_{A_{\mathscr{A}}}; I, \lambda^*)$ or by (I, λ^*) for simplicity.

Definition 2.2. *Let $([-,-]_{A_{\mathscr{A}}}, \rho_{A_{\mathscr{A}}}, I, \lambda^*)$ and $([-,-]'_{A_{\mathscr{A}}}, \rho'_{A_{\mathscr{A}}}, I', \lambda'^*)$ be infinitesimal deformations of a Lie pair (L, A). They are said to be* ***strictly isomorphic*** *if there exists an $\mathscr{A}$-ringed Lie algebroid morphism*
$$(\Pi_A, \lambda^*_A)\colon (A_{\mathscr{A}}, \rho'_{A_{\mathscr{A}}}, [-,-]'_{A_{\mathscr{A}}}) \to (A_{\mathscr{A}}, \rho_{A_{\mathscr{A}}}, [-,-]_{A_{\mathscr{A}}})$$
such that the following diagram of morphisms of $\mathscr{A}$-ringed Lie algebroids
$$\begin{array}{ccc}
L^0_{\mathscr{A}} & \xleftarrow{\quad\mathrm{Id}\quad} & L^0_{\mathscr{A}} \\
\uparrow{\scriptstyle (I,\lambda^*)} & & \uparrow{\scriptstyle (I',\lambda'^*)} \\
(A_{\mathscr{A}}, \rho_{A_{\mathscr{A}}}, [-,-]_{A_{\mathscr{A}}}) & \xleftarrow{(\Pi_A,\lambda^*_A)} & (A_{\mathscr{A}}, \rho'_{A_{\mathscr{A}}}, [-,-]'_{A_{\mathscr{A}}})
\end{array}$$
commutes.

In this definition, the center $(\Pi_A)_0$ of Π_A is necessarily the identity map on the given vector bundle A.

Given a Lie pair (L, A), the $\mathscr{A}$-linear extension of i defines a strict morphism $I\colon A^0_{\mathscr{A}} \to L^0_{\mathscr{A}}$ between the $\mathscr{A}$-Cartesian extensions of L and A. This is indeed the *trivial* infinitesimal deformation of (L, A). Next, we present a class of 'nontrivial' infinitesimal deformations. Note that, each Lie pair (L, A) determines a short exact sequence of vector bundles over M:
$$0 \to A \xrightarrow{i} L \xrightarrow{\mathrm{Pr}_B} B := L/A \to 0. \tag{2.3}$$
Choose a splitting of the exact sequence above, given by an injective vector bundle map $j\colon B \to L$ and a surjective vector bundle map $\mathrm{Pr}_A\colon L \to A$, which induces an isomorphism $L \cong A \oplus B$. For each element $\xi \in \Gamma(A^* \otimes B) \otimes \mathfrak{m}_{\mathscr{A}}$, consider the following strict morphism of $\mathscr{A}$-ringed vector bundles
$$\begin{aligned} I_\xi\colon \Gamma(A_{\mathscr{A}}) &\to \Gamma(L_{\mathscr{A}}) \cong \Gamma(A_{\mathscr{A}}) \oplus \Gamma(B_{\mathscr{A}}), \\ a &\mapsto I_\xi(a) := i(a) + j(\xi(a)), \end{aligned} \tag{2.4}$$
which in fact determines a strict bundle map
$$\rho^\xi_{A_{\mathscr{A}}}\colon \Gamma(A_{\mathscr{A}}) \to \Gamma(T_{\mathscr{A}}), \qquad \text{(see Example 1.10)}$$

$$a \mapsto \rho^{\xi}_{A_{\mathscr{A}}}(a) := \rho_{L^0_{\mathscr{A}}}(I_\xi(a)), \tag{2.5}$$

and a bracket

$$\begin{aligned}[\cdot,\cdot]^{\xi}_{A_{\mathscr{A}}} \colon \Gamma(A_{\mathscr{A}}) \times \Gamma(A_{\mathscr{A}}) &\to \Gamma(A_{\mathscr{A}}) \\ (a_1, a_2) &\mapsto [a_1, a_2]^{\xi}_{A_{\mathscr{A}}} := \mathrm{Pr}_A\big([I_\xi(a_1), I_\xi(a_2)]_{L^0_{\mathscr{A}}}\big).\end{aligned} \tag{2.6}$$

Lemma 2.7. *Suppose that the map I_ξ satisfies*

$$I_\xi\left(\mathrm{Pr}_A\big([I_\xi(a_1), I_\xi(a_2)]_{L^0_{\mathscr{A}}}\big)\right) = [I_\xi(a_1), I_\xi(a_2)]_{L^0_{\mathscr{A}}}, \tag{2.8}$$

for all $a_1, a_2 \in \Gamma(A)$. Then the anchor map $\rho^{\xi}_{A_{\mathscr{A}}}$ (2.5) and the bracket $[-,-]^{\xi}_{A_{\mathscr{A}}}$ (2.6) together define an $\mathscr{A}$-ringed Lie algebroid structure on $A_{\mathscr{A}}$. Moreover, the strict morphism I_ξ in (2.4) is an infinitesimal deformation of (L, A).

Proof. We first check that $(A_{\mathscr{A}}, [-,-]^{\xi}_{A_{\mathscr{A}}}, \rho^{\xi}_{A_{\mathscr{A}}})$ is an $\mathscr{A}$-ringed Lie algebroid. It suffices to verify the Leibniz rule and the Jacobi identity. In fact, for all $a_1, a_2 \in \Gamma(A)$ and $f \in C^\infty(M, \mathbb{K})$, we have

$$\begin{aligned}[a_1, f a_2]^{\xi}_{A_{\mathscr{A}}} &= \mathrm{Pr}_A([I_\xi(a_1), I_\xi(f a_2)]_{L^0_{\mathscr{A}}}) = \mathrm{Pr}_A([I_\xi(a_1), f I_\xi(a_2)]_{L^0_{\mathscr{A}}}) \\ &= \mathrm{Pr}_A(\rho_{L^0_{\mathscr{A}}}(I_\xi(a_1))(f) \cdot I_\xi(a_2) + f[I_\xi(a_1), I_\xi(a_2)]_{L^0_{\mathscr{A}}}) \\ &= \rho^{\xi}_{A_{\mathscr{A}}}(a_1) f a_2 + f[a_1, a_2]^{\xi}_{A_{\mathscr{A}}}.\end{aligned}$$

Meanwhile, for all $a_1, a_2, a_3 \in \Gamma(A_{\mathscr{A}})$, using Equation (2.8), we have

$$\begin{aligned}[a_1, [a_2, a_3]^{\xi}_{A_{\mathscr{A}}}]^{\xi}_{A_{\mathscr{A}}} &= \mathrm{Pr}_A[I_\xi(a_1), I_\xi(\mathrm{Pr}_A([I_\xi(a_2), I_\xi(a_3)]_{L^0_{\mathscr{A}}}))]_{L^0_{\mathscr{A}}} \\ &= \mathrm{Pr}_A[I_\xi(a_1), [I_\xi(a_2), I_\xi(a_3)]_{L^0_{\mathscr{A}}}]_{L^0_{\mathscr{A}}}.\end{aligned}$$

Thus, the Jacobi identity for $[-,-]_{L^0_{\mathscr{A}}}$ implies the Jacobi identity for $[-,-]^{\xi}_{A_{\mathscr{A}}}$. Now using Equation (2.8) again, we see that $I_\xi \colon A_{\mathscr{A}} \to L^0_{\mathscr{A}}$ is an inclusion of $\mathscr{A}$-ringed Lie algebroids whose center is the given inclusion $i \colon A \to L$. This proves that I_ξ is an infinitesimal deformation of (L, A). □

We also need the converse fact of Lemma 2.7 which is easily seen.

Lemma 2.9. *If $A_{\mathscr{A}}$ is endowed with an $\mathscr{A}$-ringed Lie algebroid structure such that $(I_\xi, \mathrm{Id}) \colon A_{\mathscr{A}} \to L^0_{\mathscr{A}}$ is a morphism of $\mathscr{A}$-ringed Lie algebroids, then*

(1) The $\mathscr{A}$-ringed Lie algebroid structure on $A_{\mathscr{A}}$ is the one determined by Equations (2.5) and (2.6);
(2) The strict morphism I_ξ is an infinitesimal deformation of (L, A);
(3) The map I_ξ is subject to Equation (2.8).

Infinitesimal deformations of the form I_ξ (subject to Equation (2.8)) will be referred to as **standard deformations**.

Indeed, any infinitesimal deformation of Lie pairs is strictly isomorphic to a standard one: Given any infinitesimal deformation (I, λ^*) of (L, A), we can use the chosen splitting $j \colon B \to L$ of the short exact sequence (2.3) and the associated decomposition $L = A \oplus B$ to express I explicitly. When restricted onto $\Gamma(A)$, the inclusion I decomposes as follows:

$$\begin{aligned}I \colon \Gamma(A) &\to \Gamma(L_{\mathscr{A}}) = \Gamma(A_{\mathscr{A}}) \oplus \Gamma(B_{\mathscr{A}}), \\ a &\mapsto I(a) = \mathcal{I}_A(a) + j(\mathcal{I}_B(a)),\end{aligned} \tag{2.10}$$

where $\mathcal{I}_B \colon \Gamma(A) \to \Gamma(B_{\mathscr{A}})$ and $\mathcal{I}_A \colon \Gamma(A) \to \Gamma(A_{\mathscr{A}})$ are both $\mathbb{K}$-linear. Moreover, they satisfy the following conditions.

Lemma 2.11. *(1) The pair $(\mathcal{I}_A, \lambda^*)$ defines a small automorphism of the $\mathscr{A}$-ringed vector bundle $A_{\mathscr{A}}$;*

(2) The map

$$\mathcal{I}_B \circ \mathcal{I}_A^{-1} \colon \Gamma(A) \to \Gamma(B) \otimes \mathfrak{m}_{\mathscr{A}}$$

is $C^\infty(M, \mathbb{K})$-linear.

(3) The corresponding element $\xi := \mathcal{I}_B \circ \mathcal{I}_A^{-1}$ in $\Gamma(A^* \otimes B) \otimes \mathfrak{m}_{\mathscr{A}}$ satisfies Equation (2.8), thus the associated map $I_\xi \colon A^\xi_{\mathscr{A}} \to L^0_{\mathscr{A}}$ defines a standard deformation of (L, A).

Proof. (1) Since $I \colon A_{\mathscr{A}} \to L_{\mathscr{A}}$ is an $\mathscr{A}$-ringed vector bundle morphism covering a small automorphism λ^* of $M_{\mathscr{A}}$, it follows that $I(fa) = \lambda^*(f)I(a)$ for all $a \in \Gamma(A_{\mathscr{A}})$ and $f \in C^\infty(M_{\mathscr{A}})$. Thus, we have

$$\mathcal{I}_A(fa) = \lambda^*(f)\mathcal{I}_A(a), \tag{2.12}$$

and

$$\mathcal{I}_B(fa) = \lambda^*(f)\mathcal{I}_B(a). \tag{2.13}$$

The center of $\mathcal{I}_A$ is the identity Id of A. This (according to Proposition 1.9) implies that $\mathcal{I}_A$ is an automorphism of $A_{\mathscr{A}}$. Thus, by Equation (2.12), $\mathcal{I}_A$ covers the small automorphism λ^* of $M_{\mathscr{A}}$. This proves Statement (1).

(2) Note that, the inverse $\mathcal{I}_A^{-1}$ of $\mathcal{I}_A$ is a small automorphism of $A_{\mathscr{A}}$ covering $(\lambda^*)^{-1}$, i.e.,

$$\mathcal{I}_A^{-1}(fa) = (\lambda^*)^{-1}(f)\mathcal{I}_A^{-1}(a).$$

Combining this equation with Equation (2.13), we see that $\mathcal{I}_B \circ \mathcal{I}_A^{-1}$ is $C^\infty(M, \mathbb{K})$-linear.

(3) Obviously, $\mathcal{I}_B \circ \mathcal{I}_A^{-1}$ determines an element $\xi \in \Gamma(A^* \otimes B) \otimes \mathfrak{m}_{\mathscr{A}}$ satisfying

$$I_\xi(a) = a + j(\xi(a)) = a + j(\mathcal{I}_B(\mathcal{I}_A^{-1}(a))) = I(\mathcal{I}_A^{-1}(a)).$$

Since $I = \mathcal{I}_A + j \circ \mathcal{I}_B$ is an infinitesimal deformation, we have

$$I([a_1, a_2]_{A_{\mathscr{A}}}) = [I(a_1), I(a_2)]_{L^0_{\mathscr{A}}}, \tag{2.14}$$

for all $a_1, a_2 \in \Gamma(A_{\mathscr{A}})$. Thus,

$$\begin{aligned}
& (I_\xi \circ \mathrm{Pr}_A)\big([I_\xi(a_1), I_\xi(a_2)]_{L^0_{\mathscr{A}}}\big) \\
= \;& (I_\xi \circ \mathrm{Pr}_A)\big([I(\mathcal{I}_A^{-1}(a_1)), I(\mathcal{I}_A^{-1}(a_2))]_{L^0_{\mathscr{A}}}\big) && \text{by Equation (2.14)} \\
= \;& (I_\xi \circ \mathcal{I}_A)\big([\mathcal{I}_A^{-1}(a_1), \mathcal{I}_A^{-1}(a_2)]_{A_{\mathscr{A}}}\big) \\
= \;& I\big([\mathcal{I}_A^{-1}(a_1), \mathcal{I}_A^{-1}(a_2)]_{A_{\mathscr{A}}}\big) && \text{by Equation (2.14)} \\
= \;& [I \circ \mathcal{I}_A^{-1}(a_1), I \circ \mathcal{I}_A^{-1}(a_2)]_{L^0_{\mathscr{A}}} \\
= \;& [I_\xi(a_1), I_\xi(a_2)]_{L^0_{\mathscr{A}}}.
\end{aligned}$$

□

In the sequel, $([-,-]^\xi_{A_{\mathscr{A}}}, \rho^\xi_{A_{\mathscr{A}}}; I_\xi, \mathrm{Id})$, induced from (I, λ^*) and the splitting $j \colon B \to L$, will be called the **standard realization** of (I, λ^*).

Proposition 2.15. *The standard realization $([-,-]^\xi_{A_{\mathscr{A}}}, \rho^\xi_{A_{\mathscr{A}}}; I_\xi, \mathrm{Id})$ is strictly isomorphic to (I, λ^*).*

Proof. The goal is to establish the following commutative diagram of morphisms of $\mathscr{A}$-ringed Lie algebroids:

$$\begin{array}{ccc}
L^0_{\mathscr{A}} & \xleftarrow{\;\mathrm{Id}\;} & L^0_{\mathscr{A}} \\
\uparrow{\scriptstyle (I_\xi, \mathrm{Id})} & & \uparrow{\scriptstyle (I, \lambda^*)} \\
(A_{\mathscr{A}}, \rho^\xi_{A_{\mathscr{A}}}, [-,-]^\xi_{A_{\mathscr{A}}}) & \xleftarrow{(\mathcal{I}_A, \lambda^*)} & (A_{\mathscr{A}}, \rho_{A_{\mathscr{A}}}, [-,-]_{A_{\mathscr{A}}}).
\end{array}$$

This diagram naturally commutes by definitions of these arrows. It suffices to show that $\mathcal{I}_A$ is a morphism of $\mathscr{A}$-ringed Lie algebroids covering λ^*. In fact, since the map $I = \mathcal{I}_A + j \circ \mathcal{I}_B$ is a morphism of $\mathscr{A}$-ringed Lie algebroids, we have

$$\mathcal{I}_A([a_1, a_2]_{A_{\mathscr{A}}}) = \mathrm{Pr}_A\big([I(a_1), I(a_2)]_{L^0_{\mathscr{A}}}\big) = [\mathcal{I}_A(a_1), \mathcal{I}_A(a_2)]^\xi_{A_{\mathscr{A}}},$$

and
$$\rho^{\xi}_{A_{\mathscr{A}}}(\mathcal{I}_A(a)) = \rho_{L^0_{\mathscr{A}}}(\mathcal{I}_A(a) + j(\xi(\mathcal{I}_A(a)))) = \rho_{L^0_{\mathscr{A}}}(I(a)),$$
for all $a, a_1, a_2 \in \Gamma(A)$. Thus, $\mathcal{I}_A$ intertwines the relative $\mathscr{A}$-ringed Lie algebroid structures. □

Example 2.16. *Consider a codimension k foliation $\mathcal{F}$ in an n-dimensional real smooth manifold M. Let $F \subset T_M$ be the involutive subbundle tangent to the leaves of $\mathcal{F}$. Then (T_M, F) is a Lie pair with $B = T_M/F$ the normal bundle of $\mathcal{F}$. It follows that the set of infinitesimal deformations of (T_M, F) parameterized by the algebra $\mathbb{R}_2[t]$ of dual numbers coincides with that of infinitesimal deformations of the foliation $\mathcal{F}$ [15], the latter of which is defined by a smooth family of involutive distributions $\{F_t\}_{t\in\mathbb{R}}$ in T_M such that $F_0 = F$.*

More precisely, given an infinitesimal deformation F_t of F, each Riemannian metric on T_M determines a family of splittings $\mathrm{pr}_{F_t}\colon T_M \to F_t$ of the short exact sequences of vector bundles (over M)
$$0 \to F_t \to T_M \xrightarrow{\mathrm{pr}_{B_t}} B_t = T_M/F_t \to 0.$$
According to [15], there exists an element $\sigma \in \Gamma(F^* \otimes B)$ defined for all $X \in \Gamma(F)$ by
$$\sigma(X) = \mathrm{pr}_B\left(\frac{d}{dt}|_{t=0}\, \mathrm{pr}_{F_t}(X)\right).$$
Let
$$\xi = \sigma \cdot t \in \Gamma(F^* \otimes B) \otimes \mathfrak{m}_{\mathbb{R}_2[t]}.$$
Then the associated map $I_\xi\colon F_{\mathbb{R}_2[t]} \to (T_M)_{\mathbb{R}_2[t]}$ satisfies Equation (2.8). This condition is indeed equivalent to the vanishing of the integrability tensor Λ_t of the deformation F_t (see [15, Proposition 2.10]). Thus, I_ξ defines a standard deformation of (T_M, F).

To obtain a local moduli space for the smooth foliation $\mathcal{F}$, it is necessary to consider infinitesimal deformations of the foliation $\mathcal{F}$ up to certain diffeomorphisms of M [25], whose infinitesimal counterparts indeed corresponds to infinitesimal deformations of the associated Lie pair (F, T_M) up to certain automorphisms of the tangent Lie algebroid T_M instead of the identity of T_M. It is thus reasonable to classify such infinitesimal deformations up to small automorphisms of the Lie algebroid $L^0_{\mathscr{A}}$, instead of the identity as in Definition 2.2 of strict isomorphisms.

2.2. Weak deformation functors and their standard realizations.

We now introduce isomorphism classes of infinitesimal deformations of a Lie pair (L, A) defined up to *small automorphisms* of the Cartesian extension $L^0_{\mathscr{A}}$ of L.

Definition 2.17. *Two infinitesimal deformations of a Lie pair (L, A),*
$$([-,-]_{A_{\mathscr{A}}}, \rho_{A_{\mathscr{A}}}; I, \lambda^*) \quad \text{and} \quad ([-,-]'_{A_{\mathscr{A}}}, \rho'_{A_{\mathscr{A}}}; I', \lambda'^*),$$
are said to be weak isomorphic (resp. semistrict isomorphic) if there exist

1. *an element $(\exp(\Delta), \exp(\sigma(\Delta)))$ in the small automorphism group $\mathrm{sAut}(L^0_{\mathscr{A}})$ (resp. small inner automorphism group $\mathrm{sIAut}(L^0_{\mathscr{A}})$) of the $\mathscr{A}$-Cartesian extension $L^0_{\mathscr{A}}$ of L, and*
2. *an $\mathscr{A}$-ringed Lie algebroid morphism*
$$(\Pi_A, \lambda^*_A)\colon\ (A_{\mathscr{A}}, [-,-]'_{A_{\mathscr{A}}}, \rho'_{A_{\mathscr{A}}}) \to (A_{\mathscr{A}}, [-,-]_{A_{\mathscr{A}}}, \rho_{A_{\mathscr{A}}}),$$

such that the following diagram
$$\begin{array}{ccc} L^0_{\mathscr{A}} & \xleftarrow{(\exp(\Delta),\exp(\sigma(\Delta)))} & L^0_{\mathscr{A}} \\ {\scriptstyle (I,\lambda^*)}\uparrow & & \uparrow{\scriptstyle (I',\lambda'^*)} \\ (A_{\mathscr{A}}, \rho_{A_{\mathscr{A}}}, [-,-]_{A_{\mathscr{A}}}) & \xleftarrow{(\Pi_A,\lambda^*_A)} & (A_{\mathscr{A}}, \rho'_{A_{\mathscr{A}}}, [-,-]'_{A_{\mathscr{A}}}), \end{array}$$
commutes.

Note that analogous to the definition of strict isomorphisms, the center $(\Pi_A)_0$ of Π_A in the above definition is necessarily the identity map on the vector bundle A as well.

Definition 2.18. *The weak deformation functor* $\mathrm{wDef}_{(L,A)}$*, associated with the Lie pair* (L, A)*, is a functor from the category* ***Art*** *of local Artinian* $\mathbb{K}$*-algebras to the category* ***Set*** *of sets. Specifically, for each* $\mathscr{A} \in$ ***Art****, the resulting set* $\mathrm{wDef}_{(L,A)}(\mathscr{A})$ *is defined as:*

$$\mathrm{wDef}_{(L,A)}(\mathscr{A}) := \left\{ \begin{array}{c} \textit{weak isomorphism classes of} \\ \textit{infinitesimal deformations of} \\ (L, A) \ \ \textit{parameterized by } \mathscr{A} \end{array} \right\}.$$

For a morphism ϑ in the category **Art**, the resulting map of sets $\mathrm{wDef}_{(L,A)}(\vartheta)$ is naturally defined. This convention is adopted for all types of deformation functors throughout the paper.

Similarly, we denote by $\mathrm{sDef}_{(L,A)}$ the **semistrict deformation functor**, which sends each $\mathscr{A}$ to the set of semistrict isomorphism classes of $\mathscr{A}$-parameterized infinitesimal deformations of (L, A).

Next, we represent the weak and the semistrict infinitesimal deformation functors by standard realizations. Let us fix a splitting j of the short exact sequence (2.3) and denote by $\mathrm{Sd}(L, A, \mathscr{A})$ the set of solutions $\xi \in \Gamma(A^* \otimes B) \otimes \mathfrak{m}_{\mathscr{A}}$ to Equation (2.8), which can be identified (via the chosen splitting j) with the set of standard deformations of (L, A).

The small automorphism group $\mathrm{sAut}(L^0_{\mathscr{A}})$ of the $\mathscr{A}$-Cartesian Lie algebroid $L^0_{\mathscr{A}}$ acts on the set of infinitesimal deformations, and on the set $\mathrm{Sd}(L, A, \mathscr{A})$ accordingly. Let us explain this fact.

(1) Given $\Pi \in \mathrm{sAut}(L^0_{\mathscr{A}})$ and $\xi \in \Gamma(A^* \otimes B) \otimes \mathfrak{m}_{\mathscr{A}}$, we first establish the following commutative diagram in the category of $\mathscr{A}$-ringed vector bundles:

$$\begin{array}{ccc} L^0_{\mathscr{A}} & \xleftarrow{\ \Pi\ } & L^0_{\mathscr{A}} \\ {\scriptstyle I_{\Pi \triangleright \xi}}\uparrow & & \uparrow{\scriptstyle I_\xi} \\ A_{\mathscr{A}} & \xleftarrow{\ \Pi_\xi\ } & A_{\mathscr{A}}. \end{array} \tag{2.19}$$

In this diagram, we define

$$\Pi_\xi := \mathrm{Pr}_A \circ \Pi \circ I_\xi \colon \quad A_{\mathscr{A}} \to A_{\mathscr{A}},$$

as a morphism of $\mathscr{A}$-ringed vector bundles. Note that, the center of Π_ξ is the identity map $\mathrm{Id}\colon A \to A$. By Proposition 1.9, Π_ξ is a small automorphism of the $\mathscr{A}$-ringed vector bundle $A_{\mathscr{A}}$. So we can consider the map

$$\mathrm{Pr}_B \circ \Pi \circ I_\xi \circ \Pi_\xi^{-1} : \quad A_{\mathscr{A}} \to B_{\mathscr{A}}.$$

It can be easily seen that it corresponds to an element $\Pi \triangleright \xi \in \Gamma(A^* \otimes B) \otimes \mathfrak{m}_{\mathscr{A}}$. Moreover, the morphism $I_{\Pi \triangleright \xi} = \mathrm{Id} + j \circ \Pi \triangleright \xi \colon A_{\mathscr{A}} \to A_{\mathscr{A}} \oplus B_{\mathscr{A}} = L_{\mathscr{A}}$ (of $\mathscr{A}$-ringed vector bundles) satisfies

$$\begin{aligned} I_{\Pi \triangleright \xi}(\Pi_\xi(a)) &= \Pi_\xi(a) + j(\Pi \triangleright \xi(\Pi_\xi(a))) \\ &= \mathrm{Pr}_A(\Pi(I_\xi(a))) + (j \circ \mathrm{Pr}_B)(\Pi(I_\xi(a))) \\ &= \Pi(I_\xi(a)), \end{aligned} \tag{2.20}$$

for all $a \in \Gamma(A)$. Therefore, we see that Diagram (2.19) is indeed commutative.

(2) If ξ is in the smaller subset $\mathrm{Sd}(L, A, \mathscr{A}) (\subset \Gamma(A^* \otimes B) \otimes \mathfrak{m}_{\mathscr{A}})$, then I_ξ is an infinitesimal deformation of the Lie pair (L, A). In this situation, we can equip $A_{\mathscr{A}}$ with a new $\mathscr{A}$-ringed Lie algebroid structure $([\cdot,\cdot]'_{A_{\mathscr{A}}}, \rho'_{A_{\mathscr{A}}})$ by pulling back the original one $([\cdot,\cdot]^\xi_{A_{\mathscr{A}}}, \rho^\xi_{A_{\mathscr{A}}})$ through the small automorphism Π_ξ. In doing so, we obtain the following commutative diagram in the category of $\mathscr{A}$-ringed Lie algebroids:

$$\begin{array}{ccc} L^0_{\mathscr{A}} & \xleftarrow{\qquad\quad \Pi \quad\qquad} & L^0_{\mathscr{A}} \\ {\scriptstyle I_{\Pi \triangleright \xi}}\uparrow & & \uparrow{\scriptstyle I_\xi} \\ (A_{\mathscr{A}}, [\cdot,\cdot]'_{A_{\mathscr{A}}}, \rho'_{A_{\mathscr{A}}}) & \xleftarrow{\ \Pi_\xi\ } & (A_{\mathscr{A}}, [\cdot,\cdot]^\xi_{A_{\mathscr{A}}}, \rho^\xi_{A_{\mathscr{A}}}). \end{array}$$

(3) According to Lemma 2.9, the $\mathscr{A}$-ringed Lie algebroid $(A_{\mathscr{A}}, [\cdot,\cdot]'_{A_{\mathscr{A}}}, \rho'_{A_{\mathscr{A}}})$ is exactly $(A_{\mathscr{A}}, [\cdot,\cdot]^{\Pi\triangleright\xi}_{A_{\mathscr{A}}}, \rho^{\Pi\triangleright\xi}_{A_{\mathscr{A}}})$, and the element $\Pi \triangleright \xi$ belongs to $\mathrm{Sd}(L, A, \mathscr{A})$.

As a summary of the above construction, we have the desired action

$$- \triangleright - \colon \mathrm{sAut}(L^0_{\mathscr{A}}) \times \mathrm{Sd}(L, A, \mathscr{A}) \to \mathrm{Sd}(L, A, \mathscr{A}),$$
$$(\Pi, \xi) \mapsto \Pi \triangleright \xi := \mathrm{Pr}_B \circ \Pi \circ I_\xi \circ \Pi_\xi^{-1}. \tag{2.21}$$

In a similar fashion, the group $\mathrm{sIAut}(L^0_{\mathscr{A}})$ of small inner automorphisms of $L^0_{\mathscr{A}}$ also acts on the set $\mathrm{Sd}(L, A, \mathscr{A})$. So the following proposition is now obvious.

Proposition 2.22. *For any $\xi, \eta \in \mathrm{Sd}(L, A, \mathscr{A})$, the standard deformations I_ξ and I_η are weak isomorphic (resp. semistrict isomorphic) if and only if ξ and η are in the same orbit, i.e., $\eta = \Pi \triangleright \xi$ for some $\Pi \in \mathrm{sAut}(L^0_{\mathscr{A}})$ (resp. $\mathrm{sIAut}(L^0_{\mathscr{A}})$).*

As a consequence, there exists a one-to-one correspondence between the set $\mathrm{wDef}_{(L,A)}(\mathscr{A})$ of weak isomorphism classes of infinitesimal deformations (resp. $\mathrm{sDef}_{(L,A)}(\mathscr{A})$ of semistrict isomorphism classes) and the orbit space $\mathrm{Sd}(L, A, \mathscr{A})/\,\mathrm{sAut}(L^0_{\mathscr{A}})$ (resp. $\mathrm{Sd}(L, A, \mathscr{A})/\,\mathrm{sIAut}(L^0_{\mathscr{A}})$).

Determining whether two elements, ξ and η, in $\mathrm{Sd}(L, A, \mathscr{A})$ belong to the same orbit under the action of either $\mathrm{sAut}(L^0_{\mathscr{A}})$ or $\mathrm{sIAut}(L^0_{\mathscr{A}})$ is a significant challenge. The primary obstacle lies in the need to explicitly compute the inverse Π_ξ^{-1} within the group action map defined in Equation (2.21), a task that is often analytically difficult. To circumvent this difficulty, we will introduce an alternative approach in the subsequent section.

2.3. Gauge equivalence of Maurer-Cartan elements.

In deformation theory, every formal deformation problem should be governed by an L_∞-algebra. Specifically, isomorphic deformations correspond to gauge-equivalent Maurer-Cartan elements within this L_∞-algebra. In this section, we investigate the weak (resp. semistrict) deformation functor of Lie pairs by means of the associated L_∞-algebras. To begin, we recall some fundamental concepts related to L_∞-algebras, as presented in [9]. Note that our sign convention differs slightly from that used in [21].

Definition 2.23. *An L_∞-algebra is a graded vector space $\mathfrak{g}$ equipped with a collection of skew-symmetric maps $[\cdots]_k \colon \wedge^k \mathfrak{g} \to \mathfrak{g}$ of degree $(2-k)$ for all $k \geqslant 1$, called the k-bracket, satisfying the n-Jacobi identity*

$$\sum_{i=1}^{n} (-1)^i \sum_{\sigma \in \mathrm{Sh}(i,n-i)} \chi(\sigma)[[x_{\sigma(1)}, \cdots, x_{\sigma(i)}]_i, x_{\sigma(i+1)}, \cdots, x_{\sigma(n)}]_{n-i+1} = 0,$$

for all $n \geqslant 1$ and all homogeneous elements $x_1, \cdots, x_n \in \mathfrak{g}$. Here $\mathrm{Sh}(i, n-i)$ is the set of $(i, n-i)$-shuffles, and $\chi(\sigma)$ is the Koszul sign of the $(i, n-i)$-shuffle σ of the n-input $(x_1, \cdots, x_n)$.

In particular, the 1-bracket $[-]_1 \colon \mathfrak{g} \to \mathfrak{g}$ is of degree 1 and square zero, thus defines a cochain complex. We denote by $\mathbf{H}^i(\mathfrak{g}, [-]_1)$ the i-th cohomology of $\mathfrak{g}$ with respect to $[-]_1$.

An L_∞-algebra with $[\cdots]_k = 0$ for all $k \geqslant 3$ is a **dg Lie algebra**. An L_∞-algebra with $[\cdots]_k = 0$ for all $k \geqslant 4$ is called a **cubic L_∞-algebra** [13] (also known as an $L_{\leqslant 3}$-algebra).

The *lower center filtration* $F^i\mathfrak{g}$ on an L_∞-algebra $\mathfrak{g}$ is the decreasing filtration defined by $F^1\mathfrak{g} = \mathfrak{g}$ and, for $i \geqslant 2$, is defined inductively by

$$F^i\mathfrak{g} = \sum_{i_1 + \cdots + i_k = i} [F^{i_1}\mathfrak{g}, \cdots, F^{i_k}\mathfrak{g}]_k.$$

An L_∞-algebra $\mathfrak{g}$ is called **nilpotent** if the lower center series terminates, that is, $F^i\mathfrak{g} = 0$ for $i \gg 0$.

Example 2.24. *If $\mathfrak{g}$ is an L_∞-algebra and $\mathscr{A}$ is a local Artinian $\mathbb{K}$-algebra with maximal ideal $\mathfrak{m}_{\mathscr{A}}$, then the $\mathscr{A}$-extensions of all k-brackets on $\mathfrak{g} \otimes \mathfrak{m}_{\mathscr{A}}$, i.e.,*

$$[x_1 \otimes v_1, \cdots, x_k \otimes v_k]_k = [x_1, \cdots, x_k]_k \otimes v_1 \cdots v_k,$$

for all $x_i \in \mathfrak{g}, v_i \in \mathfrak{m}_{\mathscr{A}}$, together make $\mathfrak{g} \otimes \mathfrak{m}_{\mathscr{A}}$ into a nilpotent L_∞-algebra.

A **Maurer-Cartan element** of a nilpotent L_∞-algebra $\mathfrak{g}$ is a degree 1 element $\xi \in \mathfrak{g}^1$ satisfying the following Maurer-Cartan equation

$$\sum_{k=1}^{\infty} \frac{1}{k!}[\xi, \cdots, \xi]_k = 0.$$

Denote by $\mathrm{MC}(\mathfrak{g})$ the set of Maurer-Cartan elements of a nilpotent L_∞-algebra $\mathfrak{g}$. Any Maurer-Cartan element $\xi \in \mathrm{MC}(\mathfrak{g})$ determines a new sequence of brackets

$$[x_1, \cdots, x_i]_i^\xi = \sum_{k=0}^{\infty} \frac{1}{k!}[\xi^{\wedge k}, x_1, \cdots, x_i]_{k+i},$$

where $[\xi^{\wedge k}, x_1, \cdots, x_i]_{k+i}$ is an abbreviation for $[\xi, \cdots, \xi, x_1, \cdots, x_i]_{k+i}$, in which ξ occurs k times. We call $[\cdots]_i^\xi$ the i-th ξ-bracket. These ξ-brackets $\{[\cdots]_i^\xi\}_{i\geqslant 1}$ defines a new nilpotent L_∞-algebra structure on $\mathfrak{g}$ (see [9, Proposition 4.4]).

Two Maurer-Cartan elements $\xi, \eta \in \mathrm{MC}(\mathfrak{g})$ are said to be **gauge equivalent** if they are connected by **the L_∞-exponential** of an element $b \in \mathfrak{g}^0$ in the following sense:

$$\eta = e^b * \xi := \xi - \sum_{k=1}^{\infty} \frac{1}{k!} e_\xi^k(b) \in \mathfrak{g}^1,$$

where $e_\xi^1(b) = [b]_1^\xi$, and the components $e_\xi^{k+1}(b) \in \mathfrak{g}^1$ for $k \geqslant 1$ are inductively determined by

$$e_\xi^{k+1}(b) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{k_1+\cdots+k_n=k \\ k_i \geqslant 1}} \frac{k!}{k_1! \cdots k_n!}[b, e_\xi^{k_1}(b), \cdots, e_\xi^{k_n}(b)]_{n+1}^\xi.$$

Remark 2.25. *The definition of L_∞-exponentials arises from Getzler's formula for the generalized Campbell-Hausdorff series, which is expressed as a sum of terms indexed by rooted trees. Specifically, Proposition 5.9 in Getzler's work provides a detailed exposition of this formula. Notably, when applied to a dg Lie algebra $\mathfrak{g}$, the L_∞-exponential map reduces to the classical exponential map for dg Lie algebras.*

Definition 2.26. *The algebraic deformation functor (associated to an L_∞-algebra $\mathfrak{g}$)*

$$\mathrm{Def}_\mathfrak{g} \colon \boldsymbol{Art} \to \boldsymbol{Set},$$

sends each local Artinian $\mathbb{K}$-algebra $\mathscr{A} \in \boldsymbol{Art}$ to the set $\mathrm{Def}_\mathfrak{g}(\mathscr{A})$ of gauge equivalent classes in $\mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}_\mathscr{A})$, and each morphism $\vartheta \colon \mathscr{A} \to \mathscr{A}'$ to the map

$$\mathrm{Def}_\mathfrak{g}(\vartheta) \colon \mathrm{Def}_\mathfrak{g}(\mathscr{A}) \to \mathrm{Def}_\mathfrak{g}(\mathscr{A}'),$$

which maps the gauge equivalent class of $\xi \in \mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}_\mathscr{A})$ to that of $(\mathrm{Id}_\mathfrak{g} \otimes \vartheta)\xi$.

Example 2.27. *Consider the algebra $\mathscr{A} = \mathbb{K}_2[t] = \mathbb{K}[t]/(t^2)$ of dual numbers. Note that,*

$$\mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}_{\mathbb{K}_2[t]}) \cong \{\xi \otimes t | \xi \in \mathfrak{g}^1, [\xi]_1 = 0\},$$

*and that $b \in \mathfrak{g}^0 \otimes \mathfrak{m}_{\mathbb{K}_2[t]}$ acts on ξ by $e^b * \xi = \xi - [b]_1$. Therefore, the set $\mathrm{Def}_\mathfrak{g}(\mathbb{K}_2[t])$ is isomorphic to the first cohomology $\boldsymbol{H}^1(\mathfrak{g}, [-]_1)$ of the L_∞-algebra $\mathfrak{g}$, thus a $\mathbb{K}$-vector space, known as **the tangent space** of the functor $\mathrm{Def}_\mathfrak{g}$.*

We now describe the governing L_∞-algebra of infinitesimal deformations of (L, A). In fact, according to [1, Proposition 4.1], each splitting $j \colon B \to L$ of the short exact sequence (2.3) of vector bundles induces a cubic L_∞-algebra structure $\{[\cdots]_k\}_{k=1}^3$ on the graded vector space

$$\Omega_A^\bullet(B) := \Gamma(\wedge^\bullet A^* \otimes B),$$

whose unary bracket $[-]_1$ is the Chevalley-Eilenberg differential

$$d_{\mathrm{CE}} \colon \Omega_A^\bullet(B) \to \Omega_A^{\bullet+1}(B)$$

of the A-module structure on B from the flat Bott A-connection ∇ on B defined by

$$\nabla_a b = \mathrm{pr}_B([a, j(b)]_L), \quad \forall a \in \Gamma(A), b \in \Gamma(B).$$

For each local Artinian $\mathbb{K}$-algebra $\mathscr{A}$ with maximal ideal $\mathfrak{m}_{\mathscr{A}}$, by Example 2.24, the $\mathscr{A}$-linear extension $\Omega_A^\bullet(B) \otimes \mathfrak{m}_{\mathscr{A}}$ of $\Omega_A^\bullet(B)$, when equipped with the $\mathscr{A}$-linear extension of the structure maps $\{[\cdots]_k\}_{k=1}^3$, is a nilpotent cubic L_∞-algebra. By abuse of notations, we also denote these extended structure maps on $\Omega_A^\bullet(B) \otimes \mathfrak{m}_{\mathscr{A}}$ by $\{[\cdots]_k\}_{k=1}^3$.

Proposition 2.28. *Given a splitting $j\colon B \to L$ of the short exact sequence (2.3), for each $\xi \in \Omega_A^1(B) \otimes \mathfrak{m}_{\mathscr{A}}$, the map $I_\xi\colon A_{\mathscr{A}} \to L_{\mathscr{A}}$ defined in (2.4) is a standard deformation of (L, A) if and only if $\xi \in \mathrm{MC}(\Omega_A^\bullet(B) \otimes \mathfrak{m}_{\mathscr{A}})$.*

Proof. For each $\xi \in \Omega_A^1(B) \otimes \mathfrak{m}_{\mathscr{A}}$, according to the formulas in [1, Proposition 4.3], the elements $[\xi]_1, [\xi,\xi]_2, [\xi,\xi,\xi]_3 \in \Omega_A^2(B) \otimes \mathfrak{m}_{\mathscr{A}}$ are defined by for all $a_1, a_2 \in \Gamma(A)$,

$$\begin{aligned}
[\xi]_1(a_1,a_2) &= \mathrm{Pr}_B([a_1, j(\xi(a_2))]_{L^0_{\mathscr{A}}}) - \mathrm{Pr}_B([a_2, j(\xi(a_1))]_{L_{\mathscr{A}^0}}) - \xi([a_1,a_2]_A),\\
\frac{1}{2!}[\xi,\xi]_2(a_1,a_2) &= \mathrm{Pr}_B([j(\xi(a_1)), j(\xi(a_2))]_{L^0_{\mathscr{A}}}) - \xi(\mathrm{Pr}_A([j(\xi(a_1)), a_2]_{L^0_{\mathscr{A}}}))\\
&\qquad - \xi(\mathrm{Pr}_A([a_1, j(\xi(a_2))]_{L^0_{\mathscr{A}}})),\\
\frac{1}{3!}[\xi,\xi,\xi]_3(a_1,a_2) &= -\xi(\mathrm{Pr}_A[j(\xi(a_1)), j(\xi(a_2))]_{L^0_{\mathscr{A}}}).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&(d_{\mathrm{CE}}\xi + \frac{1}{2}[\xi,\xi]_2 + \frac{1}{6}[\xi,\xi,\xi]_3)(a_1,a_2)\\
&= \mathrm{Pr}_B([a_1, j(\xi(a_2))]_{L^0_{\mathscr{A}}}) - \mathrm{Pr}_B([a_2, j(\xi(a_1))]_{L^0_{\mathscr{A}}})\\
&\quad - \xi(\mathrm{Pr}_A[I_\xi(a_1), I_\xi(a_2)]_{L^0_{\mathscr{A}}}) + \mathrm{Pr}_B([j(\xi(a_1)), j(\xi(a_2))]_{L^0_{\mathscr{A}}})\\
&= [I_\xi(a_1), I_\xi(a_2)]_{L^0_{\mathscr{A}}} - I_\xi(\mathrm{Pr}_A[I_\xi(a_1), I_\xi(a_2)]_{L^0_{\mathscr{A}}}).
\end{aligned}$$

By Lemma 2.7, Equation (2.8) in particular, the inclusion I_ξ defines a standard deformation of (L, A) if and only if ξ solves the Maurer-Cartan equation of the cubic L_∞-algebra $\Omega_A^\bullet(B) \otimes \mathfrak{m}_{\mathscr{A}}$. $\square$

It is natural to expect that the basic cubic L_∞-algebra $\Omega_A^\bullet(B)$ controls the infinitesimal deformations of (L, A). That is, the associated algebraic deformation functor is isomorphic to the weak (or semistrict) deformation functor. However, the degree 0 component $\Gamma(B)$ of $\Omega_A^\bullet(B)$ cannot generate (by the L_∞ exponentials) the weak symmetry group $\mathrm{Aut}(L)$ of the Lie algebroid L that acts on infinitesimal deformations of the Lie pair (L, A). To remedy this issue, we introduce another L_∞-algebra $\mathfrak{h}$, which is the extension of the basic cubic L_∞-algebra $\Omega_A^\bullet(B)$ by the Lie algebra $\mathrm{Der}(L)$ of derivations of the Lie algebroid L.

Proposition 2.29 ([27])**.** *The graded vector space*

$$\mathfrak{h} := (\mathrm{Der}(L) \oplus \Gamma(B)) \bigoplus \left(\bigoplus_{n \geqslant 1} \Omega_A^n(B) \right)$$

is a cubic L_∞-algebra extending the structure maps of the basic L_∞-algebra $\Omega_A^\bullet(B)$ as follows:

(1) The 1-bracket $[-]_1\colon \mathrm{Der}(L) \to \Omega_A^1(B)$ is given by

$$[\delta]_1(a) = -\mathrm{Pr}_B(\delta(a)),$$

for all $\delta \in \mathrm{Der}(L)$, $a \in \Gamma(A)$.

(2) The extended 2-bracket $[-]_2$ is determined by the canonical commutator

$$[\delta_1, \delta_2]_2 := \delta_1\delta_2 - \delta_2\delta_1,$$

for all $\delta_1, \delta_2 \in \mathrm{Der}(L)$, *and the Lie algebra* $\mathrm{Der}(L)$ *action on* $\Omega_A^\bullet(B)$ *defined by*

$$[\delta, X]_2(a_1, \cdots, a_k) := (\mathrm{Pr}_B \circ \delta \circ j)(X(a_1, \cdots, a_k)) - \sum_{j=1}^{k} X(a_1, \cdots, \mathrm{Pr}_A \delta(a_j), \cdots, a_k),$$

for all $\delta \in \mathrm{Der}(L), X \in \Omega_A^k(B)$, *and all* $a_1, \cdots, a_k \in \Gamma(A)$.

(3) The 3*-bracket is the operation*

$$[-]_3 \colon \mathrm{Der}(L) \otimes \Omega_A^p(B) \otimes \Omega_A^q(B) \to \Omega_A^{p+q-1}(B)$$

defined by

$$\begin{aligned} &[\delta, X, Y]_3(a_1, \cdots, a_{p+q-1}) \\ &= (-1)^{p+1} \sum_{\sigma \in \mathrm{sh}(p,q-1)} \mathrm{sgn}(\sigma) Y((\mathrm{Pr}_A \circ \delta \circ j) X(a_{\sigma(1)}, \cdots, a_{\sigma(p)}), a_{\sigma(p+1)}, \cdots, a_{\sigma(p+q-1)}) \\ &\quad + \sum_{\tau \in \mathrm{sh}(p-1,q)} \mathrm{sgn}(\tau) X((\mathrm{Pr}_A \circ \delta \circ j) Y(a_{\tau(p+1)}, \cdots, a_{\tau(p+q-1)}), a_{\tau(1)}, \cdots, a_{\tau(p)}), \end{aligned}$$

for all $\delta \in \mathrm{Der}(L), X \in \Omega_A^p(B), Y \in \Omega_A^q(B)$ *and* $a_1, \cdots, a_{i+j-1} \in \Gamma(A)$.

We call $\mathfrak{h}$ the **extended cubic L_∞-algebra** of (L, A). Denote by $\mathrm{Def}_{\mathfrak{h}}$ the algebraic deformation functor associated to the L_∞-algebra $\mathfrak{h}$.

Our first main theorem of this paper states that the cubic L_∞-algebra $\mathfrak{h}$ controls weak deformations of the Lie pair (L, A).

Theorem 2.30. *The algebraic deformation functor* $\mathrm{Def}_{\mathfrak{h}}$ *associated to the extended cubic* L_∞*-algebra* $\mathfrak{h}$ *of* (L, A) *is naturally isomorphic to the weak infinitesimal deformation functor* $\mathrm{wDef}_{(L,A)}$.

In order to prove Theorem 2.30, we need to build a natural transformation of functors

$$\gamma \colon \mathrm{Def}_{\mathfrak{h}} \Rightarrow \mathrm{wDef}_{(L,A)}.$$

Let us first define, for any local Artinian $\mathbb{K}$-algebra $\mathscr{A}$, a map of sets

$$\gamma_{\mathscr{A}} \colon \mathrm{Def}_{\mathfrak{h}}(\mathscr{A}) \to \mathrm{wDef}_{(L,A)}(\mathscr{A}).$$

For every gauge equivalent class $[\xi] \in \mathrm{Def}_{\mathfrak{h}}(\mathscr{A})$, where $\xi \in \mathrm{MC}(\mathfrak{h} \otimes \mathfrak{m}_{\mathscr{A}})$, we define $\gamma_{\mathscr{A}}([\xi])$ to be the equivalence class of the standard deformation $([-,-]^{\xi}_{A_{\mathscr{A}}}, \rho^{\xi}_{A_{\mathscr{A}}}; I_\xi, \mathrm{Id})$, or the orbit in $\mathrm{Sd}(L, A, \mathscr{A}) / \mathrm{sAut}(L^0_{\mathscr{A}})$ passing through ξ by Proposition 2.22. To see that $\gamma_{\mathscr{A}}$ is well-defined, we prove the following proposition.

Proposition 2.31. *Two Maurer-Cartan elements* $\xi, \eta \in \mathrm{MC}(\mathfrak{h} \otimes \mathfrak{m}_{\mathscr{A}})$ *are gauge equivalent if and only if the associated standard deformations* I_ξ *and* I_η *of the Lie pair* (L, A) *are weak isomorphic.*

We note that the two Maurer-Cartan elements ξ and η are gauge equivalent means that there exists a nilpotent derivation $\delta \in \mathrm{Der}(L) \otimes \mathfrak{m}_{\mathscr{A}}$ of $L^0_{\mathscr{A}}$ such that

$$\eta = e^{\delta} * \xi := -\sum_{k=0}^{\infty} \frac{1}{k!} e^k_\xi(\delta),$$

where $e^k_\xi(\delta) \in \Omega^1_A(B) \otimes \mathfrak{m}_{\mathscr{A}}, k \geqslant 0$ are inductively defined by $e^0_\xi(\delta) = -\xi$,

$$e^1_\xi(\delta) = d_1(\delta) - [\delta, \xi]_2 + \frac{1}{2}[\delta, \xi, \xi]_3 \tag{2.32}$$

and

$$e^{k+1}_\xi(\delta) = [\delta, e^k_\xi(\delta)]_2 - [\delta, \xi, e^k_\xi(\delta)]_3 + \frac{1}{2} \sum_{\substack{k_1+k_2=k \\ k_1 \geqslant 1, k_2 \geqslant 1}} \frac{k!}{k_1! k_2!} [\delta, e^{k_1}_\xi(\delta), e^{k_2}_\xi(\delta)]_3. \tag{2.33}$$

Then we introduce two family of maps

$$
\begin{aligned}
x^k &\colon \Gamma(A_{\mathscr{A}}) \to \Gamma(A_{\mathscr{A}}), & x^k(a) &:= (\mathrm{Pr}_A \circ \delta^k \circ I_\xi)(a),\\
y^k &\colon \Gamma(A_{\mathscr{A}}) \to \Gamma(B_{\mathscr{A}}), & y^k(a) &:= (\mathrm{Pr}_B \circ \delta^k \circ I_\xi)(a),
\end{aligned}
$$

for all $k \geqslant 0$ and all $a \in \Gamma(A_{\mathscr{A}})$. These maps satisfy the following key lemma, whose proof will be given in Appendix A.

Lemma 2.34. *The above maps x^k and y^k (for all $k \geqslant 0$) are related by the following relation*

$$
y^k = -\sum_{p=0}^{k} \binom{k}{p} e_\xi^p(\delta) \circ x^{k-p}, \qquad \text{as a map } \Gamma(A_{\mathscr{A}}) \to \Gamma(B_{\mathscr{A}}).
$$

With the help of this lemma, we now prove Proposition 2.31.

Proof of Proposition 2.31. To see the necessity, assume that two Maurer-Cartan elements ξ and η are gauge equivalent and let δ, x^k, and y^k be as earlier. Then we define a small automorphism of the $\mathscr{A}$-Cartesian extension $L^0_{\mathscr{A}}$ of the Lie algebroid L by

$$
\Pi := \exp(\delta) = \sum_{k=0}^{\infty} \frac{\delta^k}{k!} \colon L^0_{\mathscr{A}} \to L^0_{\mathscr{A}},
$$

and a small automorphism of the vector bundle $A_{\mathscr{A}}$ by

$$
\Pi_A := \sum_{k=0}^{\infty} \frac{x^k}{k!} \colon A_{\mathscr{A}} \to A_{\mathscr{A}}.
$$

We claim that (Π, Π_A) gives an isomorphism from I_ξ to I_η, i.e., the following commutative diagram

$$
\begin{array}{ccc}
L^0_{\mathscr{A}} & \xleftarrow{\quad\Pi\quad} & L^0_{\mathscr{A}} \\
{\scriptstyle I_\eta}\uparrow & & \uparrow{\scriptstyle I_\xi} \\
(A_{\mathscr{A}}, [\cdot,\cdot]^\eta_{A_{\mathscr{A}}}, \rho^\eta_{A_{\mathscr{A}}}) & \xleftarrow[\Pi_A]{} & (A_{\mathscr{A}}, [\cdot,\cdot]^\xi_{A_{\mathscr{A}}}, \rho^\xi_{A_{\mathscr{A}}})
\end{array}
\tag{2.35}
$$

commutes in the category of $\mathscr{A}$-ringed Lie algebroids. In fact, using Lemma 2.34, we have

$$
\begin{aligned}
\Pi \circ I_\xi &= \sum_{k=0}^{\infty} \frac{1}{k!} x^k + j \circ \sum_{k=0}^{\infty} \frac{1}{k!} y^k \\
&= \Pi_A - j \circ \sum_{k=0}^{\infty} \frac{1}{k!} \Big(\sum_{q=0}^{k} \binom{k}{q} e_\xi^q(\delta) \circ x^{k-q} \Big) \\
&= \Pi_A - j \circ \Big(\sum_{p=0}^{\infty} \frac{1}{p!} e_\xi^p(\delta) \Big) \circ \Big(\sum_{k=0}^{\infty} \frac{1}{k!} x^k \Big) \\
&= \Pi_A + j \circ \eta \circ \Pi_A = I_\eta \circ \Pi_A,
\end{aligned}
$$

which implies that Diagram (2.35) commutes in the category of $\mathscr{A}$-ringed vector bundles. Meanwhile, since we have

$$
\begin{aligned}
\Pi_A([a_1, a_2]^\xi_A) &= (\mathrm{Pr}_A \circ \Pi \circ I_\xi)([a_1, a_2]^\xi_A) \\
&= \mathrm{Pr}_A \Big([(\Pi \circ I_\xi)(a_1), (\Pi \circ I_\xi)(a_2)]_{L^0_{\mathscr{A}}} \Big) \\
&= \mathrm{Pr}_A \Big([(I_\eta \circ \Pi_A)(a_1), (I_\eta \circ \Pi_A)(a_2)]_{L^0_{\mathscr{A}}} \Big) \\
&= (\mathrm{Pr}_A \circ I_\eta) \Big([\Pi_A(a_1), \Pi_A(a_2)]^\eta_A \Big) \\
&= [\Pi_A(a_1), \Pi_A(a_2)]^\eta_A,
\end{aligned}
$$

and

$$(\rho_A^\eta \circ \Pi_A)(a) = (\rho_{L^0_{\mathscr{A}}} \circ I_\eta \circ \Pi_A)(a) = (\rho_{L^0_{\mathscr{A}}} \circ \Pi \circ I_\xi)(a) = (\rho_{L^0_{\mathscr{A}}} \circ I_\xi)(a) = \rho_A^\xi(a),$$

for any $a_1, a_2, a \in \Gamma(A_{\mathscr{A}})$, it follows that Diagram (2.35) indeed commutes in the category of $\mathscr{A}$-ringed Lie algebroids. Hence, I_ξ and I_η are isomorphic.

Conversely, to prove sufficiency, assume that the standard deformations I_ξ and I_η are isomorphic. By Proposition 2.22, there exists a small automorphism $\Pi \in \mathrm{sAut}(L^0_{\mathscr{A}})$ of the $\mathscr{A}$-Cartesian extension $L^0_{\mathscr{A}}$ of L such that

$$\eta = \Pi \triangleright \xi = \mathrm{Pr}_B \circ \Pi \circ I_\xi \circ \Pi_A^{-1},$$

where Π_A^{-1} is the inverse of $\Pi_A := \mathrm{Pr}_A \circ \Pi \circ I_\xi$.

By Proposition 1.19, we may assume that $\Pi = \exp(\delta)$ for some $\delta \in \mathrm{Der}(L) \otimes \mathfrak{m}_{\mathscr{A}}$. Using Lemma 2.34, we have

$$\begin{aligned}
\eta \circ \Pi_A &= \mathrm{Pr}_B \circ \Pi \circ I_\xi = \mathrm{Pr}_B \circ \exp(\delta) \circ I_\xi = \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{Pr}_B \circ \delta^k \circ I_\xi = \sum_{k=0}^{\infty} \frac{1}{k!} y^k \\
&= -\sum_{k=0}^{\infty} \frac{1}{k!} \Big(\sum_{q=0}^{k} \tbinom{k}{q} e_\xi^q(\delta) \circ x^{k-q} \Big) = -\Big(\sum_{p=0}^{\infty} \frac{1}{p!} e_\xi^p(\delta) \Big) \circ \Big(\sum_{k=0}^{\infty} \frac{1}{k!} x^k \Big) \\
&= -\Big(\sum_{p=0}^{\infty} \frac{1}{p!} e_\xi^p(\delta) \Big) \circ \Pi_A.
\end{aligned}$$

Since Π_A is a small automorphism of the vector bundle $A_{\mathscr{A}}$, we obtain

$$\eta = -\sum_{p=0}^{\infty} \frac{1}{p!} e_\xi^p(\delta) = e^\delta * \xi,$$

which implies that η and ξ are gauge equivalent. □

We are now in a position to prove the main theorem.

Proof of Theorem 2.30. By Proposition 2.31, the assignment

$$\gamma_{\mathscr{A}} \colon \mathrm{Def}_{\mathfrak{h}}(\mathscr{A}) \to \mathrm{wDef}_{(L,A)}(\mathscr{A})$$

is well-defined. Given any morphism $\vartheta \colon \mathscr{A} \to \mathscr{A}'$ of local Artinian $\mathbb{K}$-algebras, it is easy to see that the following diagram

$$\begin{array}{ccc}
\mathrm{Def}_{\mathfrak{h}}(\mathscr{A}) & \xrightarrow{\gamma_{\mathscr{A}}} & \mathrm{wDef}_{(L,A)}(\mathscr{A}) \\
\big\downarrow {\scriptstyle \mathrm{Def}_{\mathfrak{h}}(\vartheta)} & & \big\downarrow {\scriptstyle \mathrm{wDef}_{(L,A)}(\vartheta)} \\
\mathrm{Def}_{\mathfrak{h}}(\mathscr{A}') & \xrightarrow{\gamma_{\mathscr{A}'}} & \mathrm{wDef}_{(L,A)}(\mathscr{A}')
\end{array}$$

commutes. Therefore, γ is indeed a natural transformation. By Proposition 2.31, the natural transformation $\gamma_{\mathscr{A}}$ is injective for any Artinian algebra $\mathscr{A}$. It follows from Proposition 2.15 that any infinitesimal deformation is realized by some Maurer-Cartan element. So the natural isomorphism $\gamma_{\mathscr{A}}$ is also surjective. □

Note that, the extension of the basic L_∞-algebra $\Omega_A^\bullet(B)$ by the Lie algebra $\mathrm{IDer}(L)$ of inner derivations of L, denoted by

$$\mathfrak{h}_0 := (\mathrm{IDer}(L) \oplus \Gamma(B)) \bigoplus \Big(\bigoplus_{n \geqslant 1} \Omega_A^n(B) \Big)$$

is an L_∞-subalgebra of $\mathfrak{h}$. By a similar argument as in the proof of Theorem 2.30, we obtain the following

Theorem 2.36. *The algebraic deformation functor* $\mathrm{Def}_{\mathfrak{h}_0}$ *associated to the cubic* L_∞*-algebra* $\mathfrak{h}_0$ *is naturally isomorphic to the semistrict deformation functor* $\mathrm{sDef}_{(L,A)}$ *of the Lie pair* (L, A).

As a consequence, we have the following corollary:

Corollary 2.37. *The tangent space* $\mathrm{wDef}_{(L,A)}(\mathbb{K}_2[t])$ *(resp.* $\mathrm{sDef}_{(L,A)}(\mathbb{K}_2[t])$*) is isomorphic to the tangent cohomology* $\boldsymbol{H}^1(\mathfrak{h},[-]_1)$ *(resp.* $\boldsymbol{H}^1(\mathfrak{h}_0,[-]_1)$*).*

Remark 2.38. *A natural relation between infinitesimal deformations of a Lie pair* (L,A) *and infinitesimal deformations of the Lie algebroid* A *studied in* [5] *can be inferred. In fact, there exists a morphism* $\phi=\{\phi_1,\phi_2\}$ *of* L_∞*-algebras from the cubic* L_∞*-algebra* $\mathfrak{h}$ *(or its* L_∞*-subalgebra* $\mathfrak{h}_0$*) to the dg Lie algebra* $C^\bullet_{\mathrm{def}}(A)$*, called the deformation complex of* A*, which controls the infinitesimal deformations of* A*.*

2.4. Examples.

2.4.1. *Lie algebra pairs.* Here we compare two types of deformations of a Lie algebra pair — the one defined in the present paper and the one introduced by Crainic-Schätz-Struchiner in [6].

Let $\mathfrak{l}$ be a Lie algebra and $\mathfrak{a}\subset\mathfrak{l}$ a Lie subalgebra. Suppose that $\mathfrak{a}$ is of dimension k. Denote by $\mathrm{Gr}_k(\mathfrak{l})$ the Grassmannian manifold of k-dimensional subspaces of $\mathfrak{l}$. Following [6], a *deformation of the Lie subalgebra* $\mathfrak{a}$ inside $\mathfrak{l}$ is a smooth curve $\mathfrak{a}_t\in C^\infty([0,1],\mathrm{Gr}_k(\mathfrak{l}))$ such that $\mathfrak{a}_0=\mathfrak{a}$ and $\mathfrak{a}_t$ are Lie subalgebras of $\mathfrak{l}$ for all $t\in[0,1]$. Two deformations $\mathfrak{a}_t$ and $\tilde{\mathfrak{a}}_t$ of $\mathfrak{a}$ are said to be *isomorphic* if there exists a smooth curve $g_t, t\in[0,1]$ in the connected and simply connected Lie group G integrating $\mathfrak{l}$, such that g_0 is the identity of G and $\tilde{\mathfrak{a}}_t=\mathrm{Ad}_{g_t}\mathfrak{a}_t$. It can be verified that the set of isomorphism classes of deformations of $\mathfrak{a}$ is isomorphic to the tangent space $\mathrm{sDef}_{(\mathfrak{l},\mathfrak{a})}(\mathbb{K}_2[t])$ of the semistrict deformation functor of the Lie pair $(\mathfrak{l},\mathfrak{a})$, and also isomorphic to the first Chevalley-Eilenberg cohomology $\mathbf{H}^1_{\mathrm{CE}}(\mathfrak{a},\mathfrak{l}/\mathfrak{a})$ of the $\mathfrak{a}$-module $\mathfrak{l}/\mathfrak{a}$.

On the other hand, by Theorem 2.30, weak isomorphism classes of infinitesimal deformations of the Lie pair $(\mathfrak{l},\mathfrak{a})$ is controlled by the cubic L_∞-algebra $\mathfrak{h}=\mathrm{Der}(\mathfrak{l})\oplus\mathrm{Hom}(\Lambda^\bullet\mathfrak{a},\mathfrak{l}/\mathfrak{a}))$. In particular, by Corollary 2.37, we have the following identifications

$$\mathrm{wDef}_{(\mathfrak{l},\mathfrak{a})}(\mathbb{K}_2[t])\cong\mathrm{Def}_{\mathfrak{h}}(\mathbb{K}_2[t])\cong\mathbf{H}^1(\mathfrak{h},[-]_1).$$

In general, the two cohomology spaces $\mathbf{H}^1_{\mathrm{CE}}(\mathfrak{a},\mathfrak{l}/\mathfrak{a})$ and $\mathbf{H}^1(\mathfrak{h},[-]_1)$ are different. For example, let $\mathfrak{l}=\mathfrak{b}(3,\mathbb{K})$ be the 6-dimensional Lie algebra consisting of 3×3 upper triangular matrices. Consider a 3-dimensional Lie subalgebra $\mathfrak{a}$ of $\mathfrak{l}$ generated by e_{11},e_{12} and e_{13}. Here e_{ij} represents the 3×3 matrix with 1 in the (i,j)-entry and zeros elsewhere. We have

$$[e_{11},e_{12}]=e_{12},\quad[e_{11},e_{13}]=e_{13},\quad[e_{12},e_{13}]=0.$$

By direct computations, one obtains $\mathbf{H}^1_{\mathrm{CE}}(\mathfrak{a},\mathfrak{l}/\mathfrak{a})\cong\mathbb{K}^3$ and $\mathbf{H}^1(\mathfrak{h},[-]_1)\cong\mathbb{K}^2$.

However, if $\mathfrak{l}$ is semisimple, then all derivations of $\mathfrak{l}$ are inner. In this case, the two deformation functors $\mathrm{wDef}_{(\mathfrak{l},\mathfrak{a})}$ and $\mathrm{sDef}_{(\mathfrak{l},\mathfrak{a})}$ are isomorphic.

2.4.2. *Extensions of Lie algebroids.* We now consider a particular example from a construction of extensions of Lie algebroids in [26].

Let (M,π) be a Poisson manifold. Then the cotangent bundle T^*M is a Lie algebroid with the anchor $\pi^\sharp\colon T^*M\to TM$ and the Lie bracket $[-,-]_{T^*M}$ defined by

$$[\alpha,\beta]_{T^*M}=\mathcal{L}_{\pi^\sharp(\alpha)}\beta-\mathcal{L}_{\pi^\sharp(\beta)}\alpha-d\pi(\alpha,\beta),$$

for all $\alpha,\beta\in\Omega^1(M)$. Given a Poisson vector field V (i.e. $\mathcal{L}_V\pi=0$), there is an extension of Lie algebroids on $T^*M\oplus(M\times\mathbb{R})$, where the anchor map is defined by $\rho_V(\alpha+f)=\pi^\sharp(\alpha)+fV$ for all $\alpha\in\Omega^\bullet(M)$ and $f\in C^\infty(M,\mathbb{R})$, and the Lie bracket is defined by

$$[\alpha+f,\beta+g]_V:=[\alpha,\beta]_{T^*M}+\pi^\sharp(\alpha)g-\pi^\sharp(\beta)f+f\mathcal{L}_V\beta-g\mathcal{L}_V\alpha+fV(g)-gV(f),$$

for all $\alpha,\beta\in\Omega^1(M)$ and $f,g\in C^\infty(M)$. It follows that $(L=T^*M\oplus(M\times\mathbb{R}),A=T^*M)$ is a Lie pair with $B=L/A\cong M\times\mathbb{R}$.

Suppose that V is the Hamiltonian vector field of a smooth function $\phi\in C^\infty(M)$, i.e. $V=\pi^\sharp(d\phi)$. Let us analyze the tangent space $\mathrm{wDef}_{(L,A)}(\mathbb{K}_2[t])=\mathbf{H}^1(\mathfrak{h},[-]_1)$ of the weak deformation functor $\mathrm{wDef}_{(L,A)}$ of this Lie pair. First of all, note that the set $\mathrm{MC}(\mathfrak{h})$ of Maurer-Cartan element of the associated cubic L_∞-algebra $\mathfrak{h}$ is indeed the set of Poisson vector fields on M. In fact, any Poisson vector field Y induces a derivation $\delta^Y \in \mathrm{Der}(T^*M \oplus (M \times \mathbb{R}))$ with zero symbol defined by

$$\delta^Y(\alpha + f) := -\langle Y, \alpha \rangle d\phi + \langle Y, \alpha \rangle - fY(\phi) d\phi,$$

for all $\alpha + f \in \Gamma(T^*M \oplus (M \times \mathbb{R}))$. One can directly examine that $[\delta^Y]_1 = -Y$, which implies that $\mathbf{H}^1(\mathfrak{h}, [-]_1) = 0$. Hence, every infinitesimal deformation (parameterized by $\mathbb{K}_2[t]$) of this particular Lie pair $(T^*M \oplus (M \times \mathbb{R}), T^*M)$ arising from a Hamiltonian vector field is *trivial.*

3. Infinitesimal deformations of matched Lie pairs

Let (L, A) be a matched Lie pair, i.e., the short exact sequence (2.3) admits a canonical splitting such that B is also Lie subalgebroid of L. We denote such a matched Lie pair by $A \bowtie B$. In this section, we study infinitesimal deformations of this particular type of Lie pairs.

3.1. **The deformation functor.** Consider the subgroup $\mathrm{hAut}(L^0_{\mathscr{A}})$ of the group $\mathrm{sAut}(L^0_{\mathscr{A}})$ of small automorphisms of $L^0_{\mathscr{A}}$ defined by

$$\mathrm{hAut}(L^0_{\mathscr{A}}) := \{\exp(\mathrm{L}_b) \mid b \in \Gamma(B \otimes \mathfrak{m}_{\mathscr{A}})\}.$$

We elect the notation "hAut" because it arises from 'half' of the collection of inner derivations $\mathrm{L}_l := [l, -]_L$ for $l \in \Gamma(L) \otimes \mathfrak{m}_{\mathscr{A}} = \Gamma(A \oplus B) \otimes \mathfrak{m}_{\mathscr{A}}$.

Definition 3.1. *Two infinitesimal deformations of a matched Lie pair $L = A \bowtie B$*

$$([-,-]_{A_{\mathscr{A}}}, \rho_{A_{\mathscr{A}}}; I, \lambda^*) \quad \text{and} \quad ([-,-]'_{A_{\mathscr{A}}}, \rho'_{A_{\mathscr{A}}}; I', \lambda'^*)$$

are said to be isomorphic if there exists an element $\exp(\mathrm{L}_b) \in \mathrm{hAut}(L^0_{\mathscr{A}})$ and an $\mathscr{A}$-ringed Lie algebroid morphism (Π_A, λ^*_A) from $(A_{\mathscr{A}}, [-,-]'_{A_{\mathscr{A}}}, \rho'_{A_{\mathscr{A}}})$ to $(A_{\mathscr{A}}, [-,-]_{A_{\mathscr{A}}}, \rho_{A_{\mathscr{A}}})$ whose center is the identity of A such that the following diagram

$$\begin{array}{ccc}
L^0_{\mathscr{A}} & \xleftarrow{\exp(\mathrm{L}_b)} & L^0_{\mathscr{A}} \\
{\scriptstyle (I,\lambda^*)}\uparrow & & \uparrow{\scriptstyle (I',\lambda'^*)} \\
(A_{\mathscr{A}}, \rho_{A_{\mathscr{A}}}, [-,-]_{A_{\mathscr{A}}}) & \xleftarrow{(\Pi_A, \lambda^*_A)} & (A_{\mathscr{A}}, \rho'_{A_{\mathscr{A}}}, [-,-]'_{A_{\mathscr{A}}})
\end{array}$$

commutes.

The assignment for each local Artinian $\mathbb{K}$-algebra $\mathscr{A}$ the set $\mathrm{hDef}_{A \bowtie B}(\mathscr{A})$ of isomorphism classes of infinitesimal deformations of the matched Lie pair $L = A \bowtie B$ parameterized by $\mathscr{A}$ determines a functor

$$\mathrm{hDef}_{A \bowtie B} \colon \boldsymbol{Art} \to \boldsymbol{Set},$$

called the deformation functor of the matched Lie pair $L = A \bowtie B$.

Note that, the basic L_∞-algebra $\Omega^\bullet_A(B)$ of the matched Lie pair $L = A \bowtie B$ degenerates to a canonical dg Lie algebra, since its third bracket $[-,-,-]_3$ vanishes in this case. Here is the main theorem.

Theorem 3.2. *The deformation functor $\mathrm{hDef}_{A \bowtie B}$ of the matched Lie pair $L = A \bowtie B$ is isomorphic to the algebraic deformation functor associated to the dg Lie algebra $\Omega^\bullet_A(B)$.*

Proof. Since $B \subset L$ is also a Lie subalgebroid in this case, by Proposition 2.15, each infinitesimal deformation I admits a canonical standard realization I_ξ for some $\xi \in \Gamma(A \otimes B^*) \otimes \mathfrak{m}_{\mathscr{A}}$ satisfying (2.8). On the other hand, by Proposition 2.22, the restriction of the map (2.21) defines an action of $\mathrm{hAut}(L^0_{\mathscr{A}})$ on the set $\mathrm{Sd}(L, A, \mathscr{A})$ of standard realizations, such that the standard deformations I_ξ and I_η are isomorphic if and only if $\eta = \Pi \triangleright \xi$ for some $\Pi \in \mathrm{hAut}(L^0_{\mathscr{A}})$. As a consequence, the set $\mathrm{wDef}_{(L,A)}(\mathscr{A})$ of weak isomorphism classes of infinitesimal deformations is isomorphic to the set of orbits $\mathrm{Sd}(L, A, \mathscr{A}) / \mathrm{hAut}(L^0_{\mathscr{A}})$ of this subgroup action.

On the other hand, by Proposition 2.28, the set $\mathrm{Sd}(L, A, \mathscr{A})$ of standard deformations can be identified with the set $\mathrm{MC}(\Omega^\bullet_A(B) \otimes \mathfrak{m}_{\mathscr{A}})$ of Maurer-Cartan elements of the dg Lie algebra $\Omega^\bullet_A(B) \otimes$

$\mathfrak{m}_{\mathscr{A}}$. Meanwhile, by Proposition 2.31, two Maurer-Cartan elements ξ and η are gauge equivalent via an inner derivation L_b of $L^0_{\mathscr{A}}$ for some $b \in \Gamma(B) \otimes \mathfrak{m}_{\mathscr{A}}$, if and only if standard deformations I_ξ and I_η are isomorphic via $\exp(\mathrm{L}_b) \in \mathrm{hAut}(L^0_{\mathscr{A}})$. The conclusion is thus immediate. □

3.2. **Examples.**

3.2.1. *Deformation of complex structures.* Recall from [23] that an infinitesimal deformation of a complex manifold X over $\mathrm{Spec}(\mathscr{A})$ is a morphism of sheaves of $\mathbb{C}$-algebras $\mathcal{F} \to \mathcal{O}_X$ over X. It is required that $\mathcal{F}$ is flat over $\mathscr{A}$ and $\mathcal{F} \otimes_{\mathscr{A}} \mathbb{C} \to \mathcal{O}_X$ is an isomorphism. Two infinitesimal deformations $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be isomorphic if there exists an isomorphism $\phi\colon \mathcal{F}_1 \to \mathcal{F}_2$ of $\mathscr{A}$-algebras such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{\ \phi\ } & \mathcal{F}_2 \\ & \searrow \quad \swarrow & \\ & \mathcal{O}_X. & \end{array}$$

The assignment for each local Artinian $\mathbb{C}$-algebra $\mathscr{A}$ the set $\mathrm{Def}_X(\mathscr{A})$ of isomorphism classes of infinitesimal deformations of X over $\mathrm{Spec}(\mathscr{A})$ defines a functor

$$\mathrm{Def}_X\colon \mathbf{Art} \to \mathbf{Set}.$$

Note that, the complexified tangent bundle $T^{\mathbb{C}}_X = T^{0,1}_X \bowtie T^{1,0}_X$ is a matched Lie pair. We claim that the deformation functor Def_X of complex structures on X is isomorphic to the deformation functor $\mathrm{hDef}_{T^{0,1}_X \bowtie T^{1,0}_X}$ of the matched Lie pair $(T^{\mathbb{C}}_X, T^{0,1}_X)$.

In fact, the dg Lie algebra $\Omega^\bullet_A(B)$ under this circumstance is exactly the Kodaira-Spencer algebra

$$\mathrm{KS}_X := \Omega^{0,\bullet}_X(T^{1,0}_X) = \Gamma(\Lambda^\bullet(T^{0,1}_X)^* \otimes T^{1,0}_X).$$

Any infinitesimal deformation of the matched Lie pair $(T^{\mathbb{C}}_X, T^{0,1}_X)$ is realized by some Maurer-Cartan element $\xi \in \Omega^{0,1}_X(T^{1,0}_X) \otimes_{\mathbb{C}} \mathfrak{m}_{\mathscr{A}}$ such that it is of the form:

$$I_\xi = i + \xi\colon T^{0,1}_X \otimes \mathscr{A} \hookrightarrow T^{\mathbb{C}}_X \otimes \mathscr{A}.$$

Consider the composition

$$I^*_\xi \circ d\colon \Omega^{0,0}_X \otimes \mathscr{A} \to \Omega^{0,1}_X \otimes \mathscr{A}$$

of the de Rham differential d and the linear dual I^*_ξ of the bundle map I_ξ. Here $\Omega^{0,0}_X$ coincides with the space of smooth $\mathbb{C}$-valued functions on X. It can be seen that $I^*_\xi \circ d = \bar{\partial} + \xi \lrcorner \partial$, and hence we define a sheaf as the kernel

$$\mathcal{F}_\xi := \ker(\Omega^{0,0}_X \otimes_{\mathbb{C}} \mathscr{A} \xrightarrow{\bar{\partial} + \xi \lrcorner \partial} \Omega^{0,1}_X \otimes \mathscr{A}).$$

It is clear that $\mathcal{F}_\xi$, together with the evaluation map $\mathrm{ev}\colon \mathscr{A} \to \mathbb{C}$, defines an infinitesimal deformation $\mathcal{F}_\xi \to \mathcal{O}_X$ of X.

On the other hand, if two Maurer-Cartan elements ξ and η are gauge equivalent via some $b \in \Gamma(T^{1,0}_X) \otimes \mathfrak{m}_{\mathscr{A}}$, i.e., $\eta = e^b * \xi$. Then by Proposition 2.31, the standard deformations I_ξ and I_η are isomorphic via the small automorphism $\exp(\mathrm{L}_b)$, i.e., $I_\eta = \exp(\mathrm{L}_b) \circ I_\xi$. Here $\mathrm{L}_b\colon \Omega^{0,\bullet}_X \otimes \mathscr{A} \to \Omega^{0,\bullet}_X \otimes \mathscr{A}$ is the inner Lie derivative of $T^{\mathbb{C}}_X \otimes \mathscr{A}$ along b. It follows that the following diagram is commutative:

$$\begin{array}{ccc} \Omega^{0,0}_X \otimes \mathscr{A} & \xrightarrow{I^*_\xi \circ d} & \Omega^{0,1}_X \otimes \mathscr{A} \\ \downarrow e^b & & \downarrow e^b \\ \Omega^{0,0}_X \otimes \mathscr{A} & \xrightarrow{I^*_\eta \circ d} & \Omega^{0,1}_X \otimes \mathscr{A}. \end{array}$$

Further, it can be verified that $\mathcal{F}_\xi$ and $\mathcal{F}_\eta$ are isomorphic via the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}_\xi & \longrightarrow & \Omega_X^{0,0}\otimes\mathscr{A} & \xrightarrow{\bar\partial+\xi\lrcorner\partial} & \Omega_X^{0,1}\otimes\mathscr{A} \\ & & \downarrow\cong & & \downarrow e^b & & \downarrow e^b \\ 0 & \longrightarrow & \mathcal{F}_\eta & \longrightarrow & \Omega_X^{0,0}\otimes\mathscr{A} & \xrightarrow{\bar\partial+\eta\lrcorner\partial} & \Omega_X^{0,1}\otimes\mathscr{A}. \end{array}$$

So in this setting, what Theorem 3.2 states is exactly the well-known fact of isomorphism of functors

$$\gamma\colon \operatorname{Def}_{\mathrm{KS}_X}\cong\operatorname{Def}_X.$$

The reader may wish to see [11,16,23] for more in-depth discussions on this subject.

3.2.2. *Transversely holomorphic foliations.* Assume that $\mathcal{F}$ is a transversely holomorphic foliation defined on a compact manifold M, with real dimension p and complex codimension q. Such a foliation structure leads to a connection between the geometry of the said manifold and the algebraic properties of the associated bundles [8].

Let F denote the tangent bundle of the foliation $\mathcal{F}$. A key feature of a transversely holomorphic foliation is that its transverse complex structure induces a well-defined complex structure on the normal bundle $B := T_M/F$. This normal bundle captures the behavior of the manifold M in directions transverse to the leaves of $\mathcal{F}$. Upon complexifying the normal bundle, we obtain a decomposition

$$B^{\mathbb{C}} = B^{1,0}\oplus B^{0,1},$$

where $B^{1,0}$ and $B^{0,1}$ are the holomorphic and anti-holomorphic components respectively.

Let $F^{\mathbb{C}}$ be the complexified tangent bundle of the foliation $\mathcal{F}$. Then the direct sum

$$A := F^{\mathbb{C}}\oplus B^{0,1}$$

is a Lie subalgebroid of the complexified tangent bundle $T_M^{\mathbb{C}}$ of M. Thus, $(T_M^{\mathbb{C}}, A)$ is a Lie pair. Moreover, the quotient bundle $T_M^{\mathbb{C}}/A$ is naturally identified with $B^{1,0}$. Hence, we obtain a matched Lie pair

$$T_M^{\mathbb{C}} = A\bowtie B^{1,0},$$

which plays a crucial role in the deformation theory of the initial foliation $\mathcal{F}$. In fact, the set of isomorphism classes of infinitesimal deformations of this matched Lie pair $A\bowtie B^{1,0}$ is isomorphic to the set of infinitesimal deformations of the transversely holomorphic foliation $\mathcal{F}$ [8,10,12,28,34,35]. This correspondence provides a useful tool for studying the deformation theory of $\mathcal{F}$ from the perspective of Lie algebroids and their deformations. Thus, by Theorem 3.2, we recover the following fundamental result:

Theorem 3.3 ([8,12])**.** *The infinitesimal deformations of the transversely holomorphic foliation $\mathcal{F}$ are controlled by the dg Lie algebra $\Gamma\big(\wedge^\bullet(F^{\mathbb{C}}\oplus B^{0,1})^*\otimes B^{1,0}\big)$.*

To the best of our knowledge, comprehensive results on the existence of local moduli spaces for smooth foliations are currently lacking (see [25]). Consequently, this problem warrants further investigation and is a direction for future research.

Appendix A. Proof of Lemma 2.34

We proceed by induction on k. When $k=0$, since $x^0(a)=a, y^0(a)=\xi(a)$, Lemma 2.34 holds trivially, as

$$y^0(a)=\xi(a)=-e^0_\xi(\delta)(a)=-e^0_\xi(\delta)\circ x^0(a),$$

for all $a\in\Gamma(A)$.

Suppose that Lemma 2.34 holds for $k\leqslant n$ for some $n\geqslant 1$. For the inductive step $k=n+1$, we note that

$$x^{n+1}(a)=(\mathrm{Pr}_A\circ\delta^{n+1}\circ I_\xi)(a)=\mathrm{Pr}_A\circ\delta\circ\big(\delta^n(I_\xi(a))\big)$$

$$= \mathrm{Pr}_A \circ \delta\big(x^n(a) + j(y^n(a))\big), \tag{A.1}$$

and that

$$y^{n+1}(a) = (\mathrm{Pr}_B \circ \delta)(x^n(a) + (j \circ y^n)(a)). \tag{A.2}$$

for all $a \in \Gamma(A)$. We also need to compute $e_\xi^{n+1-m}(\delta) \circ x^m$ for all $0 \leqslant m \leqslant n$. We divide the computation into the following three cases:

(1) For $m = 0$, by Equation (2.33), one has

$$e_\xi^{n+1}(\delta) = [\delta, e_\xi^n(\delta)]_2 - [\delta, \xi, e_\xi^n(\delta)]_3 + \frac{1}{2} \sum_{\substack{p+q=n \\ p\geqslant 1, q\geqslant 1}} \frac{n!}{p!q!} [\delta, e_\xi^p(\delta), e_\xi^q(\delta)]_3$$
$$= (\mathrm{Pr}_B \circ \delta \circ j) \circ e_\xi^n(\delta) - e_\xi^n(\delta) \circ \mathrm{Pr}_A \circ \delta - e_\xi^n(\delta) \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ \xi - \xi \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ e_\xi^n(\delta)$$
$$+ \sum_{p=1}^{n-1} \tbinom{n}{p} e_\xi^{n-p}(\delta) \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ e_\xi^p(\delta)$$
$$= (\mathrm{Pr}_B \circ \delta \circ j) \circ e_\xi^n(\delta) - e_\xi^n(\delta) \circ \mathrm{Pr}_A \circ \delta \circ (x^0 + j \circ y^0) + \sum_{p=1}^{n} \tbinom{n}{p} e_\xi^{n-p}(\delta) \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ e_\xi^p(\delta)$$
$$= (\mathrm{Pr}_B \circ \delta \circ j) \circ e_\xi^n(\delta) - e_\xi^n(\delta) \circ x^1 + \sum_{p=1}^{n} \tbinom{n}{p} e_\xi^{n-p}(\delta) \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ e_\xi^p(\delta).$$

(2) For all $m = 1, \cdots, n-1$, by Equation (2.33) and the inductive assumption, we have

$$e_\xi^{n-m+1}(\delta) \circ x^m = (\mathrm{Pr}_B \circ \delta \circ j) \circ e_\xi^{n-m}(\delta) \circ x^m - e_\xi^{n-m}(\delta) \circ (\mathrm{Pr}_A \circ \delta) \circ x^m$$
$$+ e_\xi^{n-m}(\delta) \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ e_\xi^0(\delta) \circ x^m$$
$$+ \sum_{p=1}^{n-m} \tbinom{n-m}{p} e_\xi^{n-m-p}(\delta) \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ e_\xi^p(\delta) \circ x^m$$
$$= (\mathrm{Pr}_B \circ \delta \circ j) \circ e_\xi^{n-m}(\delta) \circ x^m - e_\xi^{n-m}(\delta) \circ \mathrm{Pr}_A \circ \delta \circ (x^m + j \circ y^m)$$
$$- e_\xi^{n-m}(\delta) \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ \Big(\sum_{i=1}^{m} \tbinom{m}{i} e_\xi^i(\delta) \circ x^{m-i} \Big)$$
$$+ \sum_{p=1}^{n-m} \tbinom{n-m}{p} e_\xi^{n-m-p}(\delta) \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ e_\xi^p(\delta) \circ x^m$$
$$= (\mathrm{Pr}_B \circ \delta \circ j) \circ e_\xi^{n-m}(\delta) \circ x^m - e_\xi^{n-m}(\delta) \circ x^{m+1}$$
$$- e_\xi^{n-m}(\delta) \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ \Big(\sum_{i=1}^{m} \tbinom{m}{i} e_\xi^i(\delta) \circ x^{m-i} \Big)$$
$$+ \sum_{p=1}^{n-m} \tbinom{n-m}{p} e_\xi^{n-m-p}(\delta) \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ e_\xi^p(\delta) \circ x^m,$$

where we have used Equation (A.1) in the last step.

(3) For $m = n$, using Equation (2.32) and the inductive assumption for $k = n$, we have

$$e_\xi^1(\delta) \circ x^n$$
$$= -\mathrm{Pr}_B \circ \delta \circ x^n - (\mathrm{Pr}_B \circ \delta \circ j) \circ \xi \circ x^n + \xi \circ (\mathrm{Pr}_A \circ \delta) \circ x^n + \xi \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ \xi \circ x^n$$
$$= -\mathrm{Pr}_B \circ \delta \circ x^n - (\mathrm{Pr}_B \circ \delta \circ j) \circ \xi \circ x^n - e_\xi^0(\delta) \circ \mathrm{Pr}_A \circ \delta \circ (x^n + j \circ y^n)$$
$$- e_\xi^0(\delta) \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ \Big(\sum_{i=1}^{n} \tbinom{n}{i} e_\xi^i(\delta) \circ x^{n-i} \Big)$$

$$
\begin{aligned}
&= -\mathrm{Pr}_B \circ \delta \circ x^n - (\mathrm{Pr}_B \circ \delta \circ j) \circ \xi \circ x^n - e_\xi^0(\delta) \circ x^{n+1} \\
&\quad - e_\xi^0(\delta) \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ \Big(\sum_{i=1}^{n} \tbinom{n}{i} e_\xi^i(\delta) \circ x^{n-i} \Big).
\end{aligned}
$$

Summing up the above three equalities, we obtain

$$
\begin{aligned}
\sum_{m=0}^{n} \tbinom{n}{m} e_\xi^{n+1-m}(\delta) \circ x^m &= e_\xi^{n+1}(\delta) + \sum_{m=1}^{n-1} \tbinom{n}{m} e_\xi^{n+1-m}(\delta) \circ x^m + e_\xi^1(\delta) \circ x^n \\
&= (\mathrm{Pr}_B \circ \delta \circ j) \circ \Big(\sum_{m=0}^{n} \tbinom{n}{m} e_\xi^{n-m}(\delta) \circ x^m \Big) - \sum_{m=0}^{n} \tbinom{n}{m} e_\xi^{n-m}(\delta) \circ x^{m+1} - \mathrm{Pr}_B \circ \delta \circ x^n \\
&\qquad - \sum_{m=1}^{n} \sum_{i=1}^{m} \tbinom{n}{m} \tbinom{m}{i} e_\xi^{n-m}(\delta) \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ e_\xi^i(\delta) \circ x^{m-i} \\
&\qquad\quad + \sum_{m=0}^{n-1} \sum_{p=1}^{n-m} \tbinom{n}{m} \tbinom{n-m}{p} e_\xi^{n-m-p}(\delta) \circ (\mathrm{Pr}_A \circ \delta \circ j) \circ e_\xi^p(\delta) \circ x^m \\
&= -\mathrm{Pr}_B \circ \delta \circ j \circ y^n - \sum_{m=0}^{n} \tbinom{n}{m} e_\xi^{n-m}(\delta) \circ x^{m+1} - \mathrm{Pr}_B \circ \delta \circ x^n \quad \text{by Equation (A.2)} \\
&= -y^{n+1} - \sum_{m=0}^{n} \tbinom{n}{m} e_\xi^{n-m}(\delta) \circ x^{m+1}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
y^{n+1} &= - \sum_{m=0}^{n} \tbinom{n}{m} e_\xi^{n+1-m}(\delta) \circ x^m - \sum_{m=0}^{n} \tbinom{n}{m} e_\xi^{n-m}(\delta) \circ x^{m+1} \\
&= - \sum_{m=0}^{n} \tbinom{n}{m} e_\xi^{n+1-m}(\delta) \circ x^m - \sum_{m=1}^{n+1} \tbinom{n}{m-1} e_\xi^{n+1-m}(\delta) \circ x^m \\
&= - \sum_{m=0}^{n+1} \tbinom{n+1}{m} e_\xi^{n+1-m}(\delta) \circ x^m,
\end{aligned}
$$

which proves the case for $k = n+1$ as desired. The proof of Lemma 2.34 is complete.

References

[1] R. Bandiera, Z. Chen, M. Stiénon, and P. Xu, *Shifted derived Poisson manifolds associated with Lie pairs*, Comm. Math. Phys. **375** (2020), no. 3, 1717–1760.

[2] A. Cattaneo and G. Felder, *Coisotropic submanifolds in Poisson geometry and branes in the Poisson sigma model*, Lett. Math. Phys. **69** (2004), 157–175.

[3] Z. Chen and Z. Liu, *On (co-)morphisms of Lie pseudoalgebras and groupoids*, J. Algebra **316** (2007), no. 1, 1–31.

[4] Z. Chen, M. Stiénon, and P. Xu, *From Atiyah classes to homotopy Leibniz algebras*, Comm. Math. Phys. **341** (2016), no. 1, 309–349.

[5] M. Crainic and I. Moerdijk, *Deformations of Lie brackets: cohomological aspects*, J. Eur. Math. Soc. (JEMS) **10** (2008), no. 4, 1037–1059.

[6] M. Crainic, F. Schätz, and I. Struchiner, *A survey on stability and rigidity results for Lie algebras*, Indag. Math. (N.S.) **25** (2014), no. 5, 957–976.

[7] M. del Hoyo and R.L. Fernandes, *On deformations of compact foliations*, Proc. Amer. Math. Soc. **147** (2019), no. 10, 4555–4561.

[8] T. Duchamp and M. Kalka, *Deformation theory for holomorphic foliations*, J. Differential Geometry **14** (1979), no. 3, 317–337.

[9] E. Getzler, *Lie theory for nilpotent L_∞-algebras*, Ann. of Math. (2) **170** (2009), no. 1, 271–301.

[10] J. Girbau, A. Haefliger, and D. Sundararaman, *On deformations of transversely holomorphic foliations*, J. Reine Angew. Math. **345** (1983), 122–147.

[11] W. M. Goldman and J. J. Millson, *The homotopy invariance of the Kuranishi space*, Illinois J. Math. **34** (1990), no. 2, 337–367.
[12] X. Gómez-Mont, *Transversal holomorphic structures*, J. Differential Geometry **15** (1980), no. 2, 161–185.
[13] M. Gualtieri, M. Matviichuk, and G. Scott, *Deformation of Dirac structures via L_∞ algebras*, Int. Math. Res. Not. IMRN **14** (2020), 4295–4323.
[14] A. Guan, *Gauge equivalence for complete L_∞-algebras*, Homology Homotopy Appl. **23** (2021), no. 2, 283–297.
[15] J. Heitsch, *A cohomology for foliated manifolds*, Comment. Math. Helv. **50** (1975), 197–218.
[16] D. Iacono, *L_∞-algebras and deformations of holomorphic maps*, Int. Math. Res. Not. IMRN **8** (2008), Art. ID rnn013, 36.
[17] X. Ji, *Simultaneous deformations of a Lie algebroid and its Lie subalgebroid*, J. Geom. Phys. **84** (2014), 8–29.
[18] ———, *On equivalence of deforming Lie subalgebroids and deforming coisotropic submanifolds*, J. Geom. Phys. **116** (2017), 258–270.
[19] M. Kapranov, *Rozansky-Witten invariants via Atiyah classes*, Compositio Math. **115** (1999), no. 1, 71–113.
[20] F. Keller and S. Waldmann, *Formal deformations of Dirac structures*, J. Geom. Phys. **57** (2007), no. 3, 1015–1036.
[21] T. Lada and M. Markl, *Strongly homotopy Lie algebras*, Comm. Algebra **23** (1995), no. 6, 2147–2161.
[22] C. Laurent-Gengoux, M. Stiénon, and P. Xu, *Poincaré-Birkhoff-Witt isomorphisms and Kapranov dg-manifolds*, Adv. Math. **387** (2021), Paper No. 107792, 62.
[23] M. Manetti, *Lectures on deformations of complex manifolds (deformations from differential graded viewpoint)*, Rend. Mat. Appl. (7) **24** (2004), no. 1, 1–183.
[24] ———, *Deformation theory via differential graded Lie algebras*, Algebraic Geometry Seminars, 1998–1999 (Italian) (Pisa), Scuola Norm. Sup., Pisa, 1999, pp. 21–48.
[25] L. Meersseman, M. Nicolau, and J. Ribón, *On the automorphism group of foliations with geometric transverse structures*, Math. Z. **301** (2022), no. 2, 1603–1630.
[26] R. Mehta and M. Zambon, *L_∞-algebra actions*, Differential Geom. Appl. **30** (2012), no. 6, 576–587.
[27] D. Ni, J. Cheng, Z. Chen, and C. He, *Internal symmetry of the $L_{\leqslant 3}$ algebra arising from a Lie pair*, Pure Appl. Math. Q. **19** (2023), no. 4, 2195–2234.
[28] M. Nicolau, *Deformations of holomorphic and transversely holomorphic foliations*, Complex manifolds, foliations and uniformization, Panor. Synthèses, vol. 34/35, Soc. Math. France, Paris, 2011, pp. 259–297 (English, with English and French summaries).
[29] A. Nijenhuis and R. W. Richardson, *Deformations of homomorphisms of Lie groups and Lie algebras*, Bull. Amer. Math. Soc. **73** (1967), 175–179.
[30] Y. Oh and J. Park, *Deformations of coisotropic submanifolds and strong homotopy Lie algebroids*, Invent. Math. **161** (2005), no. 2, 287–360.
[31] R. W. Richardson, *Deformations of subalgebras of Lie algebras*, J. Differential Geometry **3** (1969), 289–308.
[32] D. Roytenberg, *Quasi-Lie bialgebroids and twisted Poisson manifolds*, Lett. Math. Phys. **61** (2002), no. 2, 123–137.
[33] P. Ševera, *Letters to Alan Weinstein about Courant algebroids* (2017), available at arXiv:1707.00265.
[34] D. C. Spencer, *Deformation of structures on manifolds defined by transitive, continuous pseudogroups. I. Infinitesimal deformations of structure*, Ann. of Math. (2) **76** (1962), 306–398.
[35] ———, *Deformation of structures on manifolds defined by transitive, continuous pseudogroups. II. Deformations of structure*, Ann. of Math. (2) **76** (1962), 399–445.

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