

GENERIC CONFIGURATIONS IN 2D STRONGLY COMPETING SYSTEMS

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ABSTRACT. We study a problem modelling segregation of an arbitrary number of competing species in planar domains. The solutions give rise to a well known free boundary problem with the domain partitioning itself into subdomains occupied by different species. Several subdomains can coexist in a neighborhood of any point. However, we show that *generically* (in the sense of Baire category) only triple junctions appear, meaning that at most three subdomains meet at their common free boundary. Our main tools are the use of the formalism of harmonic maps into singular spaces and the introduction of a complex structure via the Hopf differential.

MSC classification: 35Bxx, 35J47, 35R35.

1. INTRODUCTION

For any positive natural number $N \geq 2$, we consider the metric space

$$\Sigma_N := \{te_i : i = 1, \dots, N, t \geq 0\} \subset \mathbb{R}^N,$$

endowed with the induced geodesic distance (e_i running over the standard basis). Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain. In the spirit of the pioneering work of Gromov and Schoen [14], we consider the harmonic maps $U \in H^1(\Omega; \Sigma^N)$, which are the stationary points of the Dirichlet energy

$$(1.1) \quad E(U) := \frac{1}{2} \int_{\Omega} |\nabla U|^2 dx.$$

In this paper we study the geometry of the *generic* (in the sense of Baire category) harmonic maps with values into Σ_N . To this aim, for given N we introduce the set $\mathcal{U}_N(\Omega) \subset H^1(\Omega; \Sigma_N)$ of all the stationary points of (1.1) and we set

$$(1.2) \quad \mathcal{U}(\Omega) := \bigcup_{N \geq 2} \left\{ u_1 + \dots + u_N : (u_1, \dots, u_N) \in \mathcal{U}_N(\Omega) \right\}$$

and

$$\overline{\mathcal{U}(\Omega)} := H^1\text{-closure of } \mathcal{U}(\Omega).$$

It is well-known (see Corollary 3.1) that for every $U = (u_1, \dots, u_N) \in \mathcal{U}_N(\Omega)$ each component u_j is locally Lipschitz continuous and, by the very definition, it is harmonic in its domains of positivity

$$-\Delta u_j = 0 \quad \text{in} \quad \omega_j := \{u_j > 0\} \quad \forall j = 1, \dots, N.$$

The multiplicity at a point $z \in \Omega$ is defined as the positive integer

$$(1.3) \quad m_U(z) := \lim_{r \rightarrow 0^+} \#\{j : |\omega_j \cap \mathbb{D}_r(z)| > 0\},$$

where $\mathbb{D}_r(z)$ denotes the open disk centered in z and radius r and $\#$ the cardinality. When $m_U(z) = m$ we call z an m -point. The interesting points in our analysis are those with multiplicity at least 3. The 1-points are the interiors of the positivity regions ω_j , whereas the 2-points are the locally smooth interfaces between exactly two such regions (see §3):

$$m(z) = 2 \implies z \in \partial\omega_i \cap \partial\omega_j \quad i \neq j.$$

The set of m -points with $m \geq 3$ in Ω is discrete and can be regarded as the singular set of the harmonic maps $U : \Omega \rightarrow \Sigma_N$.

The aim of this paper is to classify the structures of the singular set of the *generic* harmonic maps $\mathcal{U}(\Omega)$. Leaving apart the cases $N = 2, 3$ which are not relevant to the present study, a classification has been completed for $N = 4$ in [16], where the authors prove that two configurations are possible, namely the case of a single 4-point, or configurations with two 3-points (see Figure 1).

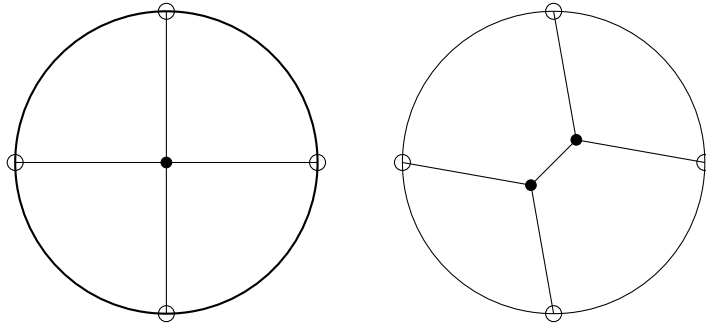


FIGURE 1. Four species: configurations with one 4-point (on the left) and two 3-points (on the right).

As explained in [16], when $N = 4$ the existence of 4-points requires an additional condition and, therefore, it is not a stable configuration. This suggests that the solutions with 4-points lie on a Banach manifold with finite co-dimension, and therefore are not generic in the space of solutions.

In this paper we show that the result obtained for $N = 4$ is paradigmatic of the general case $N > 3$. We prove that the configurations where the subdomains meet exclusively on 2-points or 3-points are residual in the sense of Baire category. A direct consequence is that the points with higher multiplicity are unstable and can be perturbed away by small variations.

To state our main result we introduce the set of functions in $\mathcal{U}(\Omega)$ corresponding to harmonic maps $U \in \mathcal{U}_N(\Omega)$ with at most triple junctions:

$$(1.4) \quad \mathcal{U}_N^\gamma(\Omega) := \left\{ U = (u_1, \dots, u_N) \in \mathcal{U}_N(\Omega), \ m_U(z) \leq 3, \ \forall z \in \Omega \right\}$$

and

$$(1.5) \quad \mathcal{U}^\gamma(\Omega) := \bigcup_{N \geq 2} \left\{ u_1 + \dots + u_N : (u_1, \dots, u_N) \in \mathcal{U}_N^\gamma(\Omega) \right\} \subset \mathcal{U}(\Omega).$$

Our main result is the following.

Theorem 1.1. *For $\Omega \subset \mathbb{C}$ a bounded simply connected domain, $\mathcal{U}^\gamma(\Omega) \subset \overline{\mathcal{U}(\Omega)}$ is residual in the sense of Baire category.*

1.1. A model for competing species. Problem (1.1) is linked to the following competition-diffusion system of N differential equations

$$(1.6) \quad \begin{cases} -\Delta u_j = -\mu u_j \sum_{k \neq j} u_k & \text{in } \Omega, \\ u_j \geq 0 & \text{in } \Omega, \\ u_j = g_j & \text{on } \partial\Omega, \end{cases} \quad j = 1, \dots, N,$$

This system can be viewed as a simplified model for the study of the segregation phenomena, when one interpretes u_j in (1.6) as the density of the j -th specie. From the PDEs point of view, this is an obstacle problem with unilateral constraints $u_j \geq 0$ and Dirichlet boundary values

$$g_j \geq 0, \quad g_j g_k = 0, \quad G := g_1 + \dots + g_N \neq 0 \quad \mathcal{H}^1\text{-a.e.}$$

The assumption $g_j \geq 0$ not identically zero is motivated by the model. The parameter $\mu > 0$ quantifies the interaction strength among the populations.

In [10, 21] the existence and uniqueness of the solutions of (1.6) are shown. Problem (1.1) is the limiting case of (1.6) when the parameter μ tends to $+\infty$, see [10]. In the last decade the qualitative properties of the solutions have been the object of an intensive study, see [2, 3, 4, 6, 5, 7, 8, 9, 10, 11, 12, 13, 16, 17, 19, 21] and the references therein.

The connection between (1.1) and (1.6) arises because, as it has been proved in [10], the one parameter solutions to the reaction-diffusion system (1.6) converge, as $\mu \rightarrow +\infty$, to the unique solution of the variational problem (1.1) with the same boundary conditions.

1.2. Geometry of nodal set. In this paper, we depart from the well-studied approach of fixing the boundary values. We introduce a novel method to analyze the nodal set of a typical map within the space of all solutions, regardless of the possible traces of $H^1(\Omega; \Sigma_N)$ -maps.

Our starting point is the work [16], where for $N = 4$ the possible 4-point configurations are characterized by means of the solution of a Dirichlet problem for the Laplace equation. The necessary and sufficient conditions on the datum which generates a 4-point suggest that in the most common configurations only 3-points appear. The geometry of the solutions $\mathcal{U}(\Omega)$

for any N was already studied in [17]. In particular, when the multiplicity of each singular point is even, it is proved that the solution is the absolute value of the harmonic function which assumes the value $\sum_{j=1}^N (-1)^j g_j$ on $\partial\Omega$.

The connection to the harmonic functions envisaged in [16, 17] suggests to introduce a global approach based on complex analytic functions. We consider the whole set $\mathcal{U}(\Omega)$ of harmonic maps for varying boundary values and varying N . Following a well-known idea long exploited in harmonic map theory (see, e.g., [18]), we associate to each solution $U \in \mathcal{U}$ its Hopf differential

$$f_U := U_z^2 = \frac{1}{4}(U_{x_1}^2 - U_{x_2}^2 - 2iU_{x_1}U_{x_2}), \quad z = x_1 + ix_2.$$

It follows from the inner variations of the Dirichlet energy that f_U is holomorphic for every $U \in \mathcal{U}$ (see Proposition 2.2). Moreover, the points z with multiplicity $m_U(z) \geq 3$ are zeros of the holomorphic function f_U and, specifically, the multiplicity 3-points are in one to one correspondence with the simple zeros of f_U :

$$m_U(z) = 3 \quad \Longleftrightarrow \quad \text{ord}(f_U; z) = 1.$$

The problem of the genericity of multiplicity at most 3-points is therefore translated to the genericity of Hopf differentials with simple zeros. Although holomorphic functions with zeros with order 1 are clearly residual (actually an open dense set), the set of Hopf differentials is itself very non-generic. It is therefore necessary to study in detail the range of the map $I : \mathcal{U} \rightarrow \mathcal{H}$ associating to each map $U \in \mathcal{U}(\Omega)$ its Hopf differential $I(U) := U_z^2$. The heuristic idea is to consider an appropriate primitive

$$U = 2 \int f^{1/2}.$$

This procedure may end up with functions which are not in $\mathcal{U}(\Omega)$. Actually, any f belonging to the image of the Hopf differential I is characterized by a system of equations (cf. (2.4) in Proposition 2.4) and, hence, the residuality needs to be proven for holomorphic functions solving suitable differential constraints. The example discussed in §5.1 shows the high degree of rigidity of such conditions, suggesting that there are only finitely many directions for perturbing a Hopf differential with higher order zeros into one with simple roots. However, a careful and detailed analysis shows that this can be achieved through an iterative argument.

We think that our approach is prototypical of a variety of other contexts where generically singularities with reduced complexity (such as triple junctions in the plane) are expected.

1.3. Structure of the paper. The paper is so organized. In Section 2 after recalling some basic facts and known results of the set $\mathcal{U}(\Omega)$, we give a characterization of the functions belonging to the Bergman space of integrable holomorphic functions which are Hopf differentials of stationary maps

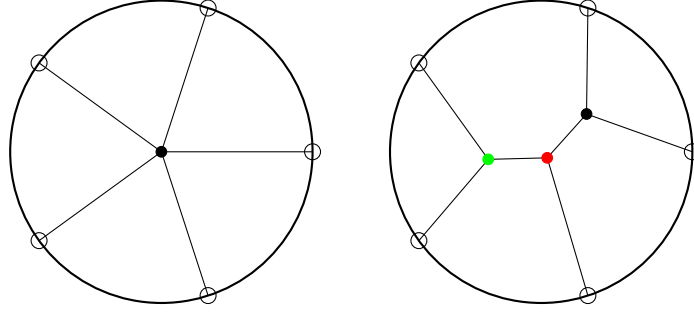


FIGURE 2. Five species: configuration with one 5-point (on the left); configuration with three 3-points (on the right).

$U : \Omega \rightarrow \Sigma_N$ for any N . Section 3 is then entirely devoted to the proofs of the results stated in Section 2. Next Section 4 contains some results on the structure of the free boundary of the functions in $\mathcal{U}(\Omega)$ and a generalization of the index formula proved in [17] on the number of critical points counted with multiplicity. Section 5 deals with the main analytical step of the genericity result, namely the desingularizing procedure of a higher order zero point. We need to take into account the linear algebra constraints coming from the above hinted rigidity, in addition to the global structure of holomorphic functions. By reducing the order of the zeros one by one, after a finite iteration, one can prove that Hopf differentials with simple zeros are dense, thus providing the principal ingredient for the main theorem (Figure 2 schematically shows the desingularization of a 5-point).

Finally, in Section 6 we prove the main genericity result for harmonic maps $\mathcal{U}(\Omega)$ with points of multiplicity at most 3.

2. HOLOMORPHIC FUNCTIONS AND SEGREGATED STATES

We recall some basic facts about solutions to (1.1). For $N \geq 2$ the set of nontrivial stationary maps $U : \Omega \rightarrow \Sigma_N$ which are stationary for (1.1) is characterized (see [10, 11, 21]) by the following analytical conditions:

$$\mathcal{U}_N(\Omega) = \left\{ \begin{array}{l} U = (u_1, \dots, u_N) : u_j \in H^1(\Omega), \\ \Delta u_j \geq 0, \quad \Delta(u_j - \sum_{k \neq j} u_k) \leq 0, \\ u_j \geq 0, \quad u_j \not\equiv 0, \quad u_j \cdot u_k = 0 \text{ a.e., } j \neq k \end{array} \right\}.$$

The set $\mathcal{U}_1(\Omega)$ is made of positive harmonic functions and plays no role in our analysis.

Remark 2.1. For $U \in \mathcal{U}_N(\Omega)$ defined in (1.2), it may happen that the nodal region $\omega_j = \{u_j > 0\}$ (which will be shown to be an open set) has an arbitrary number of connected components. In the case the number of connected components of $\{U > 0\}$ is finite, we assume that

$$\omega_j = \{u_j > 0\} \quad \text{is connected} \quad \forall j = 1, \dots, N.$$

Such condition is not restrictive, and can be achieved by a suitable relabeling.

Let $\mathcal{H}(\Omega)$ be the set of the holomorphic functions and

$$\mathcal{A}^1(\Omega) := \mathcal{H}(\Omega) \cap L^1(\Omega)$$

its Bergman space. For $f \in \mathcal{H}(\Omega)$ we set $\mathcal{Z}_f := \{z \in \Omega : f(z) = 0\}$. We use the Wirtinger derivatives

$$U_z := \frac{1}{2}(U_{x_1} - iU_{x_2}), \quad U_{\bar{z}} := \frac{1}{2}(U_{x_1} + iU_{x_2}) \in L^2(\Omega; \mathbb{C}).$$

2.1. The map I . A crucial role will be played by the Hopf differential. The main result is contained in the following proposition.

Proposition 2.2. *Let $\Omega \subset \mathbb{C}$ be an open set. If $U \in \mathcal{U}(\Omega)$ then $U_z^2 \in \mathcal{A}^1(\Omega)$.*

We denote $I : \mathcal{U}(\Omega) \rightarrow \mathcal{A}^1(\Omega)$ the map associating to each $U \in \mathcal{U}(\Omega)$ its Hopf differential $I(U) = U_z^2$. The proof is postponed to §3.1. A straightforward consequence of Proposition 2.2 is that every $U \in \mathcal{U}(\Omega)$ is a locally Lipschitz continuous in Ω , because its Wirtinger derivative is locally bounded (see Corollary 3.1).

The map I is not surjective, its range is actually a very small set of holomorphic functions. Characterizing $I(\mathcal{U}(\Omega))$ as a subset of $\mathcal{A}^1(\Omega)$ is one of the main points of our analysis. To this aim we recall some elementary facts about holomorphic functions. Since the problem is invariant under conformal change of variables, if Ω is a bounded simply connected domain, then we can assume, without loss of generality, that

$$\Omega \equiv \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Therefore, we omit the dependence on the domain in $\mathcal{U}_N, \mathcal{U}, \mathcal{H}, \mathcal{A}^1$. The zero set of a non-constant holomorphic function $f : \mathbb{D} \rightarrow \mathbb{C}$ is discrete. Therefore, the set $\mathcal{Z}_f^{\text{odd}}$ of the zeros of f with odd order is at most countable. We write

$$(2.1) \quad \mathcal{Z}_f^{\text{odd}} = \{z_1, z_2, \dots\}.$$

We fix a family of non-intersecting closed segments L_j with an endpoint in z_j and the other on $\partial\mathbb{D}$. Let $P \in \mathbb{N} \cup \{+\infty\}$ be the number of such segments. We set

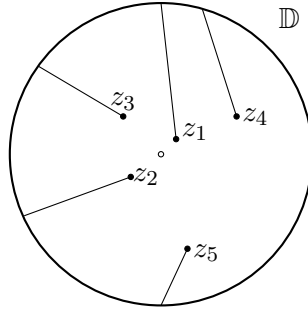
$$(2.2) \quad \Omega^- := \mathbb{D} \setminus \bigcup_{j=1}^P L_j.$$

with the convention that if $P = 0$ then $\Omega^- = \mathbb{D}$.

Since f has no odd zeros in the simply connected domain Ω^- , hence there exist exactly two holomorphic functions $\pm f^{1/2} : \Omega^- \rightarrow \mathbb{C}$ such that $(\pm f^{1/2})^2 = f$.

We fix a point $z_0 \in \Omega^-$ and we set

$$(2.3) \quad F_{z_0, f}(z) := 2 \int_{z_0}^z f^{1/2}(\zeta) d\zeta.$$

FIGURE 3. The set Ω^-

Remark 2.3. Both $f^{1/2}$ and $F_{z_0, f}$ are holomorphic in Ω and extend continuously to $\Omega^- \cup \mathcal{Z}_f^{\text{odd}}$. In particular, we may choose $z_0 \in \mathcal{Z}_f^{\text{odd}}$ in (2.3).

The characterization of the map $I : \mathcal{U} \longrightarrow \mathcal{A}^1$ is given in terms of $F_{z_0, f}$.

Proposition 2.4. For $f \in \mathcal{A}^1$ the following are equivalent:

- i. there exists $U \in \mathcal{U}$ such that $I(U) = f$;
- ii. there exists $z_0 \in \Omega^- \cup \mathcal{Z}_f^{\text{odd}}$ such that $F_{z_0, f}$ satisfies

$$(2.4) \quad \Re F_{z_0, f}(z) = 0 \quad \forall z \in \mathcal{Z}_f^{\text{odd}}.$$

Moreover, if ii. holds, then $|\Re F_{z_0, f}| \in \mathcal{U}$ and $I(|\Re F_{z_0, f}|) = f$.

The proof is postponed to the §3.4. If $\mathcal{Z}_f^{\text{odd}}$ is empty, then ii. is always verified for every z_0 and $|\Re F_{z_0, f}| \in \mathcal{U}$. In particular, the function U such that $f = I(U)$ is not unique. Uniqueness, on the contrary, holds in the case $\mathcal{Z}_f^{\text{odd}} \neq \emptyset$.

Proposition 2.5. Let $f \in I(\mathcal{U})$.

- i. If $\mathcal{Z}_f^{\text{odd}} \neq \emptyset$, then there exists a unique function $U \in \mathcal{U}$ such that $I(U) = f$.
- ii. If $\mathcal{Z}_f^{\text{odd}} = \emptyset$, then $\{|\Re F_{z_0, f}| : z_0 \in \mathbb{D}\} = I^{-1}(f)$.

The proof of the proposition is postponed to the §3.5.

Remark 2.6. Propositions 2.4 and 2.5 imply the commutativity of the following diagram:

$$(2.5) \quad \begin{array}{ccc} f_U \in \mathcal{A}^1 & \xrightarrow{\quad} & F_U \in \mathcal{H}(\Omega^-) \\ I \uparrow & & \swarrow \pi \\ U \in \mathcal{U} & & \end{array}$$

with $f_U = I(U)$, $F_U := F_{z_0, f_U}$ with $z_0 \in \Omega^- \cup \mathcal{Z}_f^{\text{odd}}$ such that $U(z_0) = 0$ and $\pi : \mathcal{H}(\Omega^-) \rightarrow \mathcal{U}$ the nonlinear projection

$$\pi(F)(z) := |\Re(F(z))|.$$

3. PROOFS OF PROPOSITIONS 2.2, 2.4 AND 2.5

3.1. Proof of Proposition 2.2. The result is a direct consequence of the fact that $\mathcal{U}_N$ are the stationary points of the Dirichlet energy E in (1.1) (see [21]). Fix any $U \in \mathcal{U}$ and compute inner variations of E at U : namely, for every smooth vector field $\lambda : \Omega \rightarrow \mathbb{C}$ with compact support in Ω , we consider the diffeomorphism (for $|\varepsilon|$ sufficiently small)

$$\Phi_\varepsilon : \Omega \rightarrow \Omega, \quad \Phi_\varepsilon(z) := z + \varepsilon \lambda(z),$$

and set $U^\varepsilon := U \circ \Phi_\varepsilon$. By construction $\text{supp}(U^\varepsilon - U) \subset\subset \Omega$ for every ε sufficiently small and by stationarity

$$(3.1) \quad \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} E(U^\varepsilon) = 0 \quad \forall \lambda \in C_c^1(\Omega).$$

The fact that f_U is holomorphic is a direct consequence of (3.1). In order to compute the inner variations, we write the Dirichlet energy in terms of the Wirtinger operators:

$$E(U^\varepsilon) = \frac{i}{2} \int_{\Omega} \left(|U_z^\varepsilon|^2 + |U_{\bar{z}}^\varepsilon|^2 \right) dz \wedge d\bar{z},$$

Using the chain rule for $V \in H^1(\mathbb{C}, \mathbb{R})$ and $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ smooth

$$\begin{aligned} (V \circ \Phi)_z &:= (V_z \circ \Phi) \Phi_z + (V_{\bar{z}} \circ \Phi) \bar{\Phi}_z, \\ (V \circ \Phi)_{\bar{z}} &:= (V_z \circ \Phi) \Phi_{\bar{z}} + (V_{\bar{z}} \circ \Phi) \bar{\Phi}_{\bar{z}}, \end{aligned}$$

and the formulas $\overline{\Phi_z} = \bar{\Phi}_{\bar{z}}$ and $\overline{\Phi_{\bar{z}}} = \bar{\Phi}_z$, we can compute that

$$\begin{aligned} |(V \circ \Phi)_z|^2 &:= |V_z \circ \Phi|^2 (\Phi_z \overline{\Phi_z} + \Phi_{\bar{z}} \overline{\Phi_{\bar{z}}}) + (V_z \circ \Phi)^2 \Phi_z \Phi_{\bar{z}} \\ &\quad + (\overline{V_z \circ \Phi})^2 \bar{\Phi}_z \bar{\Phi}_{\bar{z}}, \\ |(V \circ \Phi)_{\bar{z}}|^2 &:= |V_{\bar{z}} \circ \Phi|^2 (\Phi_z \overline{\Phi_z} + \Phi_{\bar{z}} \overline{\Phi_{\bar{z}}}) + (\overline{V_{\bar{z}} \circ \Phi})^2 \Phi_z \Phi_{\bar{z}} \\ &\quad + (V_{\bar{z}} \circ \Phi)^2 \bar{\Phi}_z \bar{\Phi}_{\bar{z}}, \end{aligned}$$

and summing up the two identities

$$\begin{aligned} |(V \circ \Phi)_z|^2 + |(V \circ \Phi)_{\bar{z}}|^2 &:= \left[|V_z \circ \Phi|^2 + |V_{\bar{z}} \circ \Phi|^2 \right] (\Phi_z \overline{\Phi_z} + \Phi_{\bar{z}} \overline{\Phi_{\bar{z}}}) \\ &\quad + 2(V_z \circ \Phi)^2 \Phi_z \Phi_{\bar{z}} + 2(V_{\bar{z}} \circ \Phi)^2 \bar{\Phi}_z \bar{\Phi}_{\bar{z}}. \end{aligned}$$

We use the previous identity with $V = U$ and $\Phi(z) = \Phi_\varepsilon(z) = z + \varepsilon \lambda(z)$,

$$\Phi_z = 1 + \varepsilon \lambda_z, \quad \Phi_{\bar{z}} = \varepsilon \lambda_{\bar{z}},$$

which implies

$$\begin{aligned} |U_z^\varepsilon|^2 + |U_{\bar{z}}^\varepsilon|^2 &:= \left[|U_z \circ \Phi_\varepsilon|^2 + |U_{\bar{z}} \circ \Phi_\varepsilon|^2 \right] (1 + \varepsilon(\lambda_z + \overline{\lambda_z}) + O(\varepsilon^2)) \\ &\quad + 2\varepsilon (U_z \circ \Phi_\varepsilon)^2 (\lambda_{\bar{z}} + O(\varepsilon^2)) + 2\varepsilon (U_{\bar{z}} \circ \Phi_\varepsilon)^2 (\overline{\lambda_z} + O(\varepsilon^2)). \end{aligned}$$

We can then write

$$\begin{aligned}
E(U^\varepsilon) &= \frac{i}{2} \int_{\Omega} \left(|U_z^\varepsilon|^2 + |U_{\bar{z}}^\varepsilon|^2 \right) dz \wedge d\bar{z} \\
&= \frac{i}{2} \int_{\Omega} \left[|U_z \circ \Phi_\varepsilon|^2 + |U_{\bar{z}} \circ \Phi_\varepsilon|^2 \right] \left(1 + \varepsilon(\lambda_z + \bar{\lambda}_{\bar{z}}) + O(\varepsilon^2) \right) dz \wedge d\bar{z} \\
&\quad + \varepsilon i \int_{\Omega} (U_z \circ \Phi_\varepsilon)^2 (\lambda_{\bar{z}} + O(\varepsilon^2)) dz \wedge d\bar{z} \\
&\quad + \varepsilon i \int_{\Omega} (U_{\bar{z}} \circ \Phi_\varepsilon)^2 (\bar{\lambda}_z + O(\varepsilon^2)) dz \wedge d\bar{z}.
\end{aligned}$$

We now make the change of variables $z = \Phi_\varepsilon^{-1}(\zeta)$, which in turn implies

$$\begin{aligned}
dz \wedge d\bar{z} &= \left[(\Phi_\varepsilon^{-1})_\zeta d\zeta + (\Phi_\varepsilon^{-1})_{\bar{\zeta}} d\bar{\zeta} \right] \wedge \left[(\overline{\Phi_\varepsilon^{-1}})_\zeta d\zeta + (\overline{\Phi_\varepsilon^{-1}})_{\bar{\zeta}} d\bar{\zeta} \right] \\
&= \left[(\Phi_\varepsilon^{-1})_\zeta (\overline{\Phi_\varepsilon^{-1}})_{\bar{\zeta}} - (\Phi_\varepsilon^{-1})_{\bar{\zeta}} (\overline{\Phi_\varepsilon^{-1}})_\zeta \right] d\zeta \wedge d\bar{\zeta}.
\end{aligned}$$

In particular, since $\Phi_\varepsilon^{-1}(\zeta) = \zeta - \varepsilon\lambda(\zeta) + O(\varepsilon^2)$, we get

$$\begin{aligned}
dz \wedge d\bar{z} &= (1 - \varepsilon\lambda_\zeta + O(\varepsilon^2)) (1 - \varepsilon\bar{\lambda}_{\bar{\zeta}} + O(\varepsilon^2)) d\zeta \wedge d\bar{\zeta} \\
&\quad - (-\varepsilon\lambda_{\bar{\zeta}} + O(\varepsilon^2)) (-\varepsilon\bar{\lambda}_\zeta + O(\varepsilon^2)) d\zeta \wedge d\bar{\zeta} \\
&= [1 - \varepsilon(\lambda_\zeta + \bar{\lambda}_{\bar{\zeta}}) + O(\varepsilon^2)] d\zeta \wedge d\bar{\zeta}.
\end{aligned}$$

We can then rewrite the Dirichlet energy with respect to the variable ζ :

$$\begin{aligned}
E(U^\varepsilon) &= \frac{i}{2} \int_{\Omega} \left[|U_\zeta|^2 + |U_{\bar{\zeta}}|^2 \right] \left(1 + \varepsilon(\lambda_\zeta \circ \Phi_\varepsilon^{-1} + \bar{\lambda}_{\bar{\zeta}} \circ \Phi_\varepsilon^{-1}) + O(\varepsilon^2) \right) \cdot \\
&\quad \cdot [1 - \varepsilon(\lambda_\zeta + \bar{\lambda}_{\bar{\zeta}}) + O(\varepsilon^2)] d\zeta \wedge d\bar{\zeta} \\
&\quad + \varepsilon i \int_{\Omega} (U_\zeta)^2 (\lambda_{\bar{\zeta}} \circ \Phi_\varepsilon^{-1} + O(\varepsilon^2)) [1 - \varepsilon(\lambda_\zeta + \bar{\lambda}_{\bar{\zeta}}) + O(\varepsilon^2)] d\zeta \wedge d\bar{\zeta} \\
&\quad + \varepsilon i \int_{\Omega} (U_{\bar{\zeta}})^2 (\bar{\lambda}_\zeta \circ \Phi_\varepsilon^{-1} + O(\varepsilon^2)) [1 - \varepsilon(\lambda_\zeta + \bar{\lambda}_{\bar{\zeta}}) + O(\varepsilon^2)] d\zeta \wedge d\bar{\zeta}.
\end{aligned}$$

Taking the derivative in $\varepsilon = 0$ of the energy $E(U^\varepsilon)$ is now straightforward to infer that

$$\begin{aligned}
\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} E(U^\varepsilon) &= \frac{i}{2} \int_{\Omega} \left[|U_\zeta|^2 + |U_{\bar{\zeta}}|^2 \right] (\lambda_\zeta + \bar{\lambda}_{\bar{\zeta}}) d\zeta \wedge d\bar{\zeta} \\
&\quad - \frac{i}{2} \int_{\Omega} \left[|U_\zeta|^2 + |U_{\bar{\zeta}}|^2 \right] (\lambda_\zeta + \bar{\lambda}_{\bar{\zeta}}) d\zeta \wedge d\bar{\zeta} \\
&\quad + i \int_{\Omega} U_\zeta^2 \lambda_{\bar{\zeta}} d\zeta \wedge d\bar{\zeta} + i \int_{\Omega} U_{\bar{\zeta}}^2 \bar{\lambda}_\zeta d\zeta \wedge d\bar{\zeta} \\
(3.2) \quad &= 2i \int_{\Omega} \Re(U_\zeta^2 \lambda_{\bar{\zeta}}) d\zeta \wedge d\bar{\zeta} = 0 \quad \forall \lambda \in C_c^\infty(\Omega; \mathbb{C}).
\end{aligned}$$

Recalling that λ takes values in $\mathbb{C}$, by its arbitrariness we conclude that $(U_z^2)_{\bar{z}} = 0$, i.e. U_z^2 satisfies the Cauchy-Riemann equations in a weak sense and therefore is holomorphic in Ω , since $U_z^2 \in L^1(\Omega)$. $\square$

A simple corollary is the Lipschitz continuity of the functions in $\mathcal{U}$.

Corollary 3.1. *Let $U \in \mathcal{U}$, with $U = u_1 + \cdots + u_N \in \mathcal{U}_N$ for some $N \geq 2$. Then, the functions U and u_j are locally Lipschitz continuous and the open sets $\omega_j := \{u_j > 0\} \subset \Omega$ have no connected component compactly contained in Ω .*

Proof. The Lipschitz continuity of U follows from the fact that $(U_z)^2 = f_U$ is holomorphic and, hence, locally bounded in Ω . As a consequence, also the functions u_j are locally Lipschitz (because they are the zero extension of the Lipschitz functions $U|_{\omega_j}$, which in turns satisfy $U|_{\partial\omega_j \cap \Omega} = 0$). Moreover, by the very definition of $\mathcal{U}$ it follows that u_j are harmonic in ω_j . Therefore, there cannot exist connected components $C \subset \omega_j$ with $\overline{C} \cap \partial\Omega = \emptyset$, because otherwise $u_j|_C$ would be a positive harmonic function with $u_j = 0$ on ∂C . $\square$

3.2. Definition and properties of $F_{z_0, f}$. Given any holomorphic function $f \in \mathcal{H}$, we use the notation introduced in §2.

The main step of the proof of Propositions 2.4 and 2.5 is contained in the following lemma.

Lemma 3.2. *Let $f \in \mathcal{A}^1$ and $F_{z_0, f}$ defined in (2.3) with $z_0 \in \Omega^- \cup \mathcal{Z}_f^{\text{odd}}$. If $\Re F_{z_0, f}(z) = 0$ for all $z \in \mathcal{Z}_f^{\text{odd}}$, then $|\Re F_{z_0, f}|$ extends by continuity to a function $U \in \overline{\mathcal{U}}$.*

Remark 3.3. *In the case $\mathcal{Z}_f^{\text{odd}} = \emptyset$, the hypotheses are automatically satisfied and the lemma asserts that $|\Re F_{z_0, f}| \in \mathcal{U}$ (see also [17, Proposition 3.9], where the case of solutions with singular points with even multiplicity is explicitly considered).*

Proof. Assume $\mathcal{Z}_f^{\text{odd}} \neq \emptyset$. Fix an index j and any point on a cut $\xi \in L_j \cap \mathbb{D}$, $\xi \neq z_j$. We consider the limits of $F_{z_0, f}$ as z tends to ξ from the two sides of the cut L_j , i.e.

$$z \rightarrow \xi^\pm \iff \begin{cases} z \rightarrow \xi, \\ \pm (\text{Arg}(z - z_j) - \text{Arg}(\xi - z_j)) > 0, \end{cases}$$

for any local determination of the argument of $(\xi - z_j)$ (see Figure 4).

Namely, we consider

$$\begin{aligned} F_{z_0, f}(\xi^\pm) &:= 2 \lim_{z \rightarrow \xi^\pm} \int_{z_0}^z f^{1/2}(\zeta) d\zeta \\ (3.3) \qquad &= F_{z_0, f}(z_j) + 2 \lim_{z \rightarrow \xi^\pm} \int_{z_j}^z f^{1/2}(\zeta) d\zeta. \end{aligned}$$

Observe that $f^{1/2}$ changes sign across L_j , i.e.

$$\lim_{z \rightarrow \xi^+} f^{1/2}(z) = - \lim_{z \rightarrow \xi^-} f^{1/2}(z),$$

because z_j is assumed to be a zero with odd order. Therefore, (3.3) reads as

$$F_{z_0, f}(\xi^+) - F_{z_0, f}(z_j) = -F_{z_0, f}(\xi^-) + F_{z_0, f}(z_j).$$

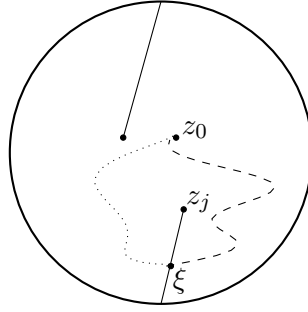


FIGURE 4. $z \rightarrow \xi^-$ along the dotted path, and $z \rightarrow \xi^+$ along the dashed path

Using the hypothesis $\Re F_{z_0, f}(z_j) = 0$, we infer that

$$\Re F_{z_0, f}(\xi^+) = -\Re F_{z_0, f}(\xi^-),$$

and, hence, the following limit is well-defined

$$|\Re F_{z_0, f}(\xi)| := \lim_{z \rightarrow \xi} |\Re F_{z_0, f}(z)|.$$

We then conclude that there exists a continuous extension V of $|\Re F_{z_0, f}|$ to the whole $\mathbb{D}$.

The extension V is independent from the choice of the cuts L_j . Indeed, consider a different simply connected domain $\tilde{\Omega}$ obtained by taking away a set of non-intersecting segments $\tilde{L}_j$ with $L_j = \tilde{L}_j$ for every j except one index j_0 (see Figure 5). Then, we can choose the determinations of the square root

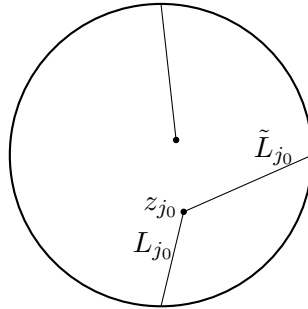


FIGURE 5. The domains Ω^- and $\tilde{\Omega}$.

in such a way that $\tilde{f}^{1/2} = f^{1/2}$ in one of the two connected components of $\Omega^- \cap \tilde{\Omega}$, and $\tilde{f}^{1/2} = -f^{1/2}$ in the other. In computing $F_{z_0, f}(z)$, we can consider paths $\gamma : [0, 1] \rightarrow \mathbb{D}$ with $\gamma(0) = z_0$, $\gamma(1/2) = z_{j_0}$ and $\gamma(1) = z$, with $\gamma((0, 1/2))$ and $\gamma((1/2, 1))$ contained inside a connected component of $\Omega^- \cap \tilde{\Omega}$. Therefore, it follows from (3.3) that $\Re F_{z_0, f} = \pm \Re \tilde{F}_{z_0, f}$, thus concluding that

$$|\Re F_{z_0, f}| = |\Re \tilde{F}_{z_0, f}|,$$

from which the uniqueness of the extension V follows.

We denote by $\{\omega_j\}_j$ the connected components of $\{V > 0\}$ and set

$$v_j := \begin{cases} V(x) & \text{if } x \in \omega_j, \\ 0 & \text{elsewhere.} \end{cases}$$

Note that v_j is subharmonic, because v_j is a continuous nonnegative function which is harmonic where it is positive. We remark that V is locally the modulus of a harmonic function in $\mathbb{D} \setminus \mathcal{Z}_f^{\text{odd}}$; therefore, the level set $\{V = 0\}$ is made by analytic curves with endpoints on $\mathcal{Z}_f \cup \partial\mathbb{D}$. This implies that for every j_0 and for every $z \in \partial\omega_{j_0} \cap \mathbb{D} \setminus \mathcal{Z}_f$ there exists exactly one i_0 such that $z \in \partial\omega_{j_0} \cap \partial\omega_{i_0}$. Therefore,

$$\tilde{V} := v_{j_0} - \sum_{j \neq j_0} v_j$$

equals the harmonic function $v_{j_0} - v_{i_0} = \pm \Re F_{z_0, f}$ in a neighborhood of z . Moreover, $\tilde{V}$ is superharmonic away from $\overline{\omega_{j_0}}$, because it is the sum of superharmonic functions $-v_j$, $j \neq j_0$. This implies that globally $\tilde{V}$ is superharmonic in $\mathbb{D} \setminus (\mathcal{Z}_f \cap \{V = 0\})$.

To complete the proof, we need to verify that the Euler-Lagrange equations are satisfied also in a neighborhood of the isolated points $\mathcal{Z}_f \cap \{V = 0\}$. Consider any test function $\phi \in C_c(\mathbb{D})$ with $\text{supp}(\phi) \cap \mathcal{Z}_f \cap \{V = 0\} = \{z_j\}$ and consider a smooth cut-off function χ_s such that $\chi_s(z) = 1$ if $|z - z_j| > 2s$ and $\chi_s(z) = 0$ if $|z - z_j| < s$, with

$$s \|\partial_z \chi_s\|_\infty + s^2 \|\partial_{z\bar{z}}^2 \chi_s\|_\infty \leq C.$$

A simple computation yields

$$\int_{\mathbb{D}} v_{j_0} \partial_{z\bar{z}}^2 \phi = \lim_{s \rightarrow 0^+} \int_{\mathbb{D}} v_{j_0} \partial_{z\bar{z}}^2 (\chi_s \phi) \quad \forall j_0 \text{ with } z_{j_0} \in \mathcal{Z}_f \cap \{V = 0\},$$

taking into account that $v_{j_0}(z_j) = 0$. This implies that v_{j_0} is subharmonic in the whole of $\mathbb{D}$, as well as $v_{j_0} - \sum_{j \neq j_0} v_j$ superharmonic in $\mathbb{D}$.

Finally, we verify that V belongs to $\overline{\mathcal{U}}$. To this aim, we notice that for every $r < 1$ there exists an integer N_r such that $V|_{\mathbb{D}_r} \in \mathcal{U}_{N_r}(\mathbb{D}_r)$ with $\|V\|_{H^1(\mathbb{D}_r)} \leq \|f\|_{L^1(\mathbb{D})}$. By a simple rescaling we deduce that $V(r \cdot) \in \mathcal{U}$ and $V(r \cdot) \rightarrow V$ in $H^1(\mathbb{D})$, thus $V \in \overline{\mathcal{U}}$. $\square$

3.3. Commutativity of the diagram (2.5). Next we show that for every $U \in \mathcal{U}$ the function $f_U := I(U)$ satisfies (2.4) of Proposition 2.4.

We need the following lemma.

Lemma 3.4. *Let $h(z)$ be holomorphic, $u = \Re h$ e $v = \Im h$. Then, $4(\partial_z |u|)^2 = (h')^2$ for a.e. z .*

Proof. We have,

$$h'(z) = h_{x_1} = u_{x_1} + iv_{x_1} = u_{x_1} - iu_{x_2} = 2u_z,$$

where we used the Cauchy-Riemann conditions $u_{x_1} = v_{x_2}$, $u_{x_2} = -v_{x_1}$. Next, for every z such that $u(z) \neq 0$ (recall that $\{u = 0\}$ is a negligible set because of the harmonicity of u), we can compute as follows

$$4|u|_z^2 = \left[\frac{u}{|u|} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) u \right]^2 = 4u_z^2 = (h'(z))^2.$$

□

We can then show the following.

Proposition 3.5. *Let $U \in \mathcal{U}$, $f_U := I(U)$ and F_{z_0, f_U} be as in (2.3) with $z_0 \in \{U = 0\}$ any fixed point. Then, $\Re F_{z_0, f_U}(z) = 0$ for all $z \in \mathcal{Z}_{f_U}^{\text{odd}}$ and $U = |\Re F_{z_0, f_U}|$.*

Proof. Let $\Omega^- \subseteq \mathbb{D}$ be the domain in (2.2) for the holomorphic function f_U . We set $V(z) := |\Re F_{z_0, f_U}(z)|$. Then, by Lemma 3.4 we have that

$$(3.4) \quad V_z^2 = f_U = U_z^2 \quad \text{and} \quad V(z_0) = U(z_0) = 0.$$

We claim that (3.4) implies that $U = V$. Condition (3.4) implies that $U_z = \pm V_z$ a.e. By the regularity of U , the sign is constant in any connected component of $\{U > 0\} \cap \{V > 0\}$. If $U_z = -V_z$, then $U + V = C > 0$ with $C \in \mathbb{R}$. If $U_z = V_z$, then $U - V = C$ with $C \in \mathbb{R}$. In particular, for any connected component $A \subseteq \{U > 0\} \cap \{V > 0\}$ it holds that

$$\partial A \subseteq \{U = 0, V = C\} \cup \{U = C, V = 0\}.$$

Since $\{V = C\}$ are locally the level sets of a harmonic function, there cannot be accumulation of the level sets $\{V = C\} \cap \{U = 0\}$ towards z_0 . This implies that z_0 is on the boundary of some connected components of $\{U > 0\} \cap \{V > 0\}$, and since $U(z_0) = V(z_0) = 0$ actually $U = V$ in a neighborhood of z_0 . By unique continuation, it then follows that $U = V$ is an open subset and, by connectedness, in the whole of Ω^- . Therefore, we conclude that V can be extended by continuity in $\mathbb{D}$ as $U = V$. To conclude the proof we need only to notice that

$$\mathcal{Z}_{f_U}^{\text{odd}} \subset \{U = 0\},$$

because in a neighborhood of any point of $\mathcal{Z}_{f_U} \cap \{U > 0\}$ the function U is harmonic and U_z^2 in such neighborhood can only have zeros of even order. This implies that $\Re F_{z_0, f_U}(z) = 0$ for all $z \in \mathcal{Z}_{f_U}^{\text{odd}}$. □

3.4. Proof of Proposition 2.4. We start showing that i. implies ii. in Proposition 2.4, i.e., we show that if $f = I(U)$ for some $U \in \mathcal{U}$, then there exists z_0 such that

$$\Re F_{z_0, f}(z) = 0 \quad \forall z \in \mathcal{Z}_f^{\text{odd}} \quad \text{and} \quad |\Re F_{z_0, f}| \in \mathcal{U}.$$

In fact, this is a direct consequence of Proposition 3.5, if z_0 is any point such that $U(z_0) = 0$.

Viceversa, ii. implies i.: indeed, by Lemma 3.2, if

$$\Re F_{z_0, f}(z) = 0 \quad \forall z \in \mathcal{Z}_f^{\text{odd}},$$

then $|\Re F_{z_0, f}| \in \mathcal{U}$ and by Lemma 3.4 $I(|\Re F_{z_0, f}|) = f$. $\square$

3.5. Proof of Proposition 2.5. i. Assume that $f = I(U)$ with $U \in \mathcal{U}$. Then, $\emptyset \neq \mathcal{Z}_f^{\text{odd}} \subset \{U = 0\}$ implies that there exists $z_0 \in \mathcal{Z}_f^{\text{odd}}$ such that $U = |\Re F_{z_0, f}|$ by Proposition 3.5. In particular, U depends uniquely on f and it is the only function with $I(U) = f$.

For ii. we notice that by Lemma 3.4 we have that any function of the form $V = |\Re F_{z_0, f}|$ with $z_0 \in \mathbb{D}$ is defined in the whole $\mathbb{D}$, belongs to $\mathcal{U}$ and satisfies $I(V) = f$. Viceversa, any V such that $I(V) = f$ can be recovered by Proposition 3.5 as $V = |\Re F_{z_0, f}|$, where z_0 is any point in $\{V = 0\}$, thus showing that

$$\{|\Re F_{z_0, f}| : z_0 \in \mathbb{D}\} = I^{-1}(f).$$

$\square$

4. NODAL SET OF THE FUNCTIONS IN $\mathcal{U}$

The main result of this section (Proposition 4.3) is the proof of a Euler-type formula relating the number of vertices, edges and faces of a planar graph (see [15, Theorem 1.5.2]), applied to the study of the nodal set of $U \in \mathcal{U}$. In order to do so, we need to show that the nodal set of U can be viewed as a graph with some very special properties.

Therefore we need to introduce the definition of nodal and critical set of $U \in \mathcal{U}$ and to relate to the order of the zeros of f_U . This is done in Proposition 4.1.

We start by reviewing results which are already known in the literature [8, 11].

4.1. Structure of $U \in \mathcal{U}$. We give direct proofs of several regularity properties of the functions $U \in \mathcal{U}$. Recall the definition of $m_U(z)$ in (1.3). If $m_U(z_0) = 1$, then in a neighborhood of z_0 there exists only one function, say u_{j_0} , not identically zero. By Harnack inequality, $u_{j_0} > 0$ in that neighborhood and $z_0 \in \omega_{j_0} \subset \{U > 0\}$. Therefore, the set of positivity of U is equivalently characterized as the set of multiplicity one points:

$$\{U > 0\} = \{z \in \mathbb{D} : m_U(z) = 1\}.$$

Moreover, there are no points with $m_U(z_0) = 0$, because otherwise $U \equiv 0$ in a neighborhood of z_0 with the consequence that both ∇U and f_U are identically zero. The 2-points characterize the interfaces between the regions ω_j . We denote $\mathcal{N}_U, \mathcal{C}_U$ the nodal and critical set of U , respectively:

$$\mathcal{N}_U := \{z \in \mathbb{D} : U(z) = 0\} \quad \text{and} \quad \mathcal{C}_U := \{z \in \mathbb{D} : U(z) = f_U(z) = 0\}.$$

The next result links $m_U(z_0)$ with the order of the zero z_0 of f_U .

Proposition 4.1. *Let $U \in \mathcal{U}$. Then,*

$$(4.1) \quad m_U(z_0) = 2 + \text{ord}(f_U; z_0) \quad \forall z_0 \in \mathcal{N}_U,$$

and the following holds:

i. $\mathcal{C}_U = \{z \in \mathbb{D} : m_U(z) \geq 3\}$.

ii. $\mathcal{N}_U$ is the union of non-intersecting analytic arcs with endpoints on $\mathcal{C}_U \cup \partial\mathbb{D}$, which meets in $\mathcal{C}_U$ at equal angles.

iii. If $U = u_1 \cdots u_N$, then $u_\ell - u_j$ is harmonic in $\text{Int}(\overline{\omega}_\ell \cup \overline{\omega}_j)$.

iv. For every $z_0 \in \mathbb{D}$ such that $m_U(z_0) = 2$ there exist $\ell \neq j$ such that $z_0 \in \overline{\omega}_\ell \cap \overline{\omega}_j$ and

$$(4.2) \quad \lim_{\substack{z \rightarrow z_0 \\ z \in \omega_\ell}} \nabla u_\ell(z) = - \lim_{\substack{z \rightarrow z_0 \\ z \in \omega_j}} \nabla u_j(z) \neq 0.$$

v. If $z_0 \in \mathbb{D}$ with $m_U(z_0) = h \geq 3$, then there exists $\theta_0 \in (-\pi, \pi]$ such that

$$(4.3) \quad U(z_0 + re^{i\theta}) = r^{h/2} \left| \cos\left(\frac{h}{2}(\theta + \theta_0)\right) \right| + o(r^{h/2}) \quad \text{as } r \rightarrow 0.$$

In particular, $\|\nabla U\|_{L^\infty(\mathbb{D}_r(z_0))} \rightarrow 0$ as $r \rightarrow 0$.

Proof. Let $z_0 \in \mathcal{N}_U$. If $f_U(z_0) \neq 0$, then we know that $U = |\Re F_{z_0, f_U}|$ is the absolute value of a harmonic function in a neighborhood of z_0 and therefore $\{U = 0\}$ is locally an analytic arc separating two connected components of $\{U > 0\}$. In this case, (4.1) is verified because

$$m_U(z_0) = 2 \quad \text{and} \quad \text{ord}(f_U; z_0) = 0.$$

Viceversa let $z_0 \in \mathcal{N}_U \cap \mathcal{Z}_{f_U}$ be a zero of f_U of order $n \geq 1$. Without loss of generality, assume that $z_0 = 0$ and in a neighborhood we can write

$$f_U(z) = z^n(a_n + a_{n+1}z + \dots), \quad a_n \neq 0.$$

If n is odd, in a sufficiently small neighborhood of $z_0 = 0$ we can select a single valued branch of the square root of the term inside the parentheses, say

$$(a_n + a_{n+1}z + \dots)^{1/2} = b_0 + b_1z + b_2z^2 + \dots$$

We then get for $f_U^{1/2}$, locally outside $z = 0$,

$$f_U^{1/2}(z) = z^{n/2}(b_0 + b_1z + b_2z^2 + \dots).$$

If n is a zero of even order, putting $n = 2s$ we get

$$f_U^{1/2}(z) = z^s(b_0 + b_1z + b_2z^2 + \dots).$$

In both cases, integrating term by term leads to

$$F_{0, f_U}(z) = 2 \int_0^z f_U^{1/2}(\zeta) d\zeta = z^{(n+2)/2}(c_0 + c_1z + c_2z^2 + \dots),$$

for suitable coefficients c_i with $c_0 \neq 0$. We use Proposition 3.5 to infer that

$$(4.4) \quad U(z) = |\Re F_{0, f_U}(z)| = |c_0| |z|^{(n+2)/2} |\cos(\text{Arg } z^{\frac{n+2}{2}})| + o(|z|^{(n+2)/2}).$$

The nodal set of U around 0 consists of $n + 2$ analytic curves having a common endpoint in the origin with the angle between two adjacent rays is

equal to $2\pi/(n+2)$. Hence we infer that the origin is a point of multiplicity $n+2$ for U and order n for f_U :

$$m_U(0) = n+2 \quad \text{and} \quad \text{ord}(f_U; 0) = n.$$

In particular, we get all the conclusions of the proposition. Indeed, i. is a direct consequence of (4.1), because in any zero of f_U the order is greater or equal 1 and, therefore, the multiplicity is bigger or equal 3. The structure of $\mathcal{N}_U$ in ii. and v. are a consequence of (4.4). Finally, the properties iii. and iv. follows from the fact that $U = |\Re F_{z_0, f_U}|$ is locally the modulus of a harmonic function away from the critical set, so that changing sign across the regular part of $\mathcal{N}_U$ gives back a harmonic function for which ii. and iii. trivially hold. $\square$

4.2. Index formula. We are now ready to state and prove the main result of the present section.

We consider functions $U \in \mathcal{U}$ such that U extends continuously on $\partial\mathbb{D}$ (with an abuse of notation denoted by U itself) and the number of connected components of $\{z \in \partial\mathbb{D} : U(z) > 0\}$ is an integer $M \in \mathbb{N}$:

$$(4.5) \quad M := \#\{\text{connected components of } \{z \in \partial\mathbb{D} : U(z) > 0\}\}.$$

Set, moreover,

$$(4.6) \quad T := \#\{\text{connected components of } \overline{\mathcal{N}_U}\},$$

with $\overline{\mathcal{N}_U}$ denoting the closure of the nodal set:

$$\overline{\mathcal{N}_U} = \{z \in \overline{\mathbb{D}} : U(z) = 0\}.$$

Recalling Remark 2.1, from now on we assume that any function U satisfying (4.5) is written as $U = (u_1, \dots, u_N) \in \mathcal{U}_N$ with the condition

$$(4.7) \quad \omega_j = \{u_j > 0\} \quad \text{is connected for } j \in \{1, \dots, N\}.$$

Set, moreover, for all $\ell, j \in \{1, \dots, N\}$ with $\ell \neq j$,

$$\Gamma_{\ell j} := \partial\omega_\ell \cap \partial\omega_j \cap \{z \in \mathbb{D} : m_U(z) = 2\},$$

and let $\{\gamma_j\}_{j=1, \dots, M}$ be the connected components of $\{U > 0\} \cap \partial\mathbb{D}$.

There is a simple connection between the integers N, M and T .

Lemma 4.2. *Let $N \geq 2$ and $U \in \mathcal{U}_N$ satisfying (4.5)-(4.7). Then,*

$$(4.8) \quad M = N + T - 1.$$

Proof. The proof is easily done by induction. The case $\overline{\mathcal{N}_U}$ connected, i.e. $T = 1$, follows from the observation that $\overline{\mathcal{N}_U}$ is connected if and only if $\overline{\omega_j} \cap \partial\mathbb{D}$ is connected, i.e. $M = N$.

If the formula is proven for any $T' < T$, then it follows for T . Indeed, if $\mathcal{N}_U$ is not connected, there exists $j_0 \in \{1, \dots, N\}$ such that $\overline{\omega_{j_0}} \cap \partial\mathbb{D}$ is not connected. One can then consider a curve γ joining two of the connected

components of $\bar{\omega}_{j_0} \cap \partial\mathbb{D}$ (see Figure 6) and create two domains Ω_1 and Ω_2 homeomorphic to $\mathbb{D}$ with

$$\#\{\text{connected components } \overline{\mathcal{N}}_U \cap \Omega_\ell\} =: T_\ell < T \quad \ell = 1, 2.$$

We can than use the inductive hypothesis and deduce that

$$M_\ell = N_\ell + T_\ell - 1 \quad \ell = 1, 2,$$

with M_ℓ and N_ℓ the number of connected components of $\{U > 0\} \cap \partial\Omega_\ell$ and the number of species in Ω_ℓ , respectively. We have that $M = M_1 + M_2$, while $N = N_1 + N_2 - 1$, because the set ω_{j_0} intersect both Ω_1 and Ω_2 . Summing the two equations

$$M = M_1 + M_2 = N_1 + N_2 + T_1 + T_2 - 2 = N + 1 + T - 2 = N + T - 1.$$

□

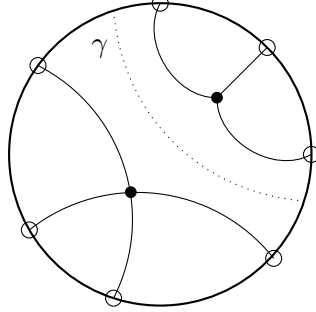


FIGURE 6. $M = 7, N = 6, T = 2$; $M_1 = 4, N_1 = 4, T_1 = 1$,
 $M_2 = 3, N_2 = 3, T_2 = 1$.

We consider the planar graph $(\mathcal{V}, \mathcal{E})$ where the set of vertices $\mathcal{V}$ are the points in $\mathcal{C}_U$ (the critical points belonging to the nodal set) and the zeros of U on $\partial\mathbb{D}$, the edges $\mathcal{E}$ are the arcs $\bar{\Gamma}_{ij}$ and $\bar{\gamma}_j$ with $i, j \in \{1, \dots, N\}$. For every $z \in \mathcal{V}$ we define the index

$$i(z) := m_U(z) - 2.$$

Note that for every $z \in \mathcal{C}_U$ we have that $i(z) = \text{ord}(f_U; z)$. The following result is a generalization of the index formula proved in [17].

Proposition 4.3. *Let $N \geq 2$ and $U \in \mathcal{U}_N$ satisfying (4.5) and (4.7). Then, the following index formula holds*

$$(4.9) \quad \sum_{z \in \mathcal{V}} i(z) = N - T - 1,$$

where T is the number of connected components of $\overline{\mathcal{N}}_U$.

Proof. For any $z \in \mathcal{C}_U$ the number $m_U(a) = i(a) + 2$ corresponds to the number of the arcs Γ_{ij} such that $p \in \bar{\Gamma}_{ij}$, whereas for the others vertices $z \in \partial\mathbb{D}$ it holds that $m_U(z) + 1$ is the number of the arcs Γ_{ij} such that $z \in \bar{\Gamma}_{ij}$. Then,

$$\begin{aligned}
\sum_{z \in \mathcal{V}} i(z) &= \sum_{z \in \mathcal{C}_U} [m_U(z) - 2] + \sum_{z \in \mathcal{V} \cap \partial\mathbb{D}} [m_U(z) - 2] \\
&= \sum_{z \in \mathcal{C}_U} [\#\{e \in \mathcal{E} : z \in e\} - 2] + \sum_{z \in \mathcal{V} \cap \partial\mathbb{D}} [\#\{e \in \mathcal{E} : z \in e\} - 3] \\
&= \sum_{p \in \mathcal{V}} [\#\{e \in \mathcal{E} : z \in e\} - 2] - M \\
&= 2\#\mathcal{E} - 2\#\mathcal{V} - M = 2\#\mathcal{F} - 4 - M \\
&= 2(N + 1) - 4 - M \stackrel{(4.8)}{=} 2N - 2 - N - T + 1 = N - T - 1,
\end{aligned}$$

where we used the Euler formula $\#\mathcal{E} - \#\mathcal{V} = \#\mathcal{F} - 2$, with $\mathcal{F}$ the number of faces which equals $N + 1$ (the nodal regions plus the complement of $\mathbb{D}$), and we used that each edge is counted twice in the last sum (once for each endpoint). $\square$

Remark 4.4. A consequence of Proposition 4.3 is that for every $U \in \mathcal{U}_N$ satisfying (4.5) and (4.7) one has $m_U(z) \leq N$ for all $z \in \mathcal{C}_U$.

5. UNTIE SINGULAR POINTS

In this section we provide the main step of the proof of genericity in $\mathcal{U}$ of functions with only triple junctions. Here we consider functions $U \in \mathcal{U}$ satisfying the following conditions:

- (H1) The associate holomorphic function $f_U := I(U)$ extends to a holomorphic function in a neighborhood of the unit disk $\mathbb{D}$ and therefore its zero set is finite in $\mathbb{D}$:

$$\mathcal{C}_U = \mathcal{Z}_{f_U} \cap \mathcal{N}_U = \{z_0, z_1, \dots, z_M\}, \quad M \in \mathbb{N}.$$

- (H2) The order of z_0 is bigger than 1:

$$\text{ord}(f_U; z_0) = m_0 + 1 \quad m_0 > 0,$$

- (H3) The zeros z_j are in *general position* with respect to z_0 , meaning that
- i. the distances $|z_j - z_0|$ are all different,
 - ii. the half lines $\Gamma_j = \{z_j + t(z_j - z_0) : t \geq 0\}$ does not intersect $\mathcal{Z}_{f_U}$.

Under assumptions (H1)-(H3) we show how to perturb f_U in such a way to find a new segregated state V close to U and a point ω_0 close to z_0 satisfying

$$\mathcal{C}_V = \{\omega_0, z_0, \dots, z_M\},$$

and

$$\begin{aligned} \text{ord}(f_V; \omega_0) &= 1, \quad \text{ord}(f_V; z_0) = m_0, \\ \text{ord}(f_V; z_j) &= \text{ord}(f_U; z_j) \quad \forall j = 1, \dots, M. \end{aligned}$$

Before entering the details of the proof of the main result of the present section, we give, in subsection 5.1, an example which shows the kind of “desingularization” one can observe in Figure 2, although possible, can only happen by choosing the directions connecting the newly generated critical points in a very rigid way.

5.1. An example of rigidity. The set of holomorphic functions with simple zeros is trivially generic. However, the subset of holomorphic functions belonging to $I(\mathcal{U})$ is highly non-generic. This fact can be appreciated by a simple example. Consider the harmonic function U and its holomorphic Hopf differential

$$U(z) = \left| \Re \left(\frac{2}{5} z^{5/2} \right) \right|, \quad I(U) = U_z^2 = f = \frac{1}{4} z^3,$$

In order to untangle the critical point $z = 0$, we have to consider the holomorphic function f_w , depending on a complex parameter written in polar form as $w = \varepsilon e^{i\phi} \in \mathbb{C}$,

$$f_w(z) = \frac{1}{4} z(z - w)^2.$$

Clearly, $\mathcal{Z}_{f_w} = \{0, w\}$. Consider the square root of f_w defined in the simply connected open set $\Omega^- := \{z = \rho e^{i\theta} \in \mathbb{C} : 0 < \rho < 1, \theta \in (-\pi, \pi)\}$

$$f_w^{1/2}(z) = \frac{1}{2} z^{1/2} (z - w),$$

where $z^{1/2} = \sqrt{\rho} e^{i\frac{\theta}{2}}$ for every $z = \rho e^{i\theta}$ with $\theta \in (-\pi, \pi)$. The primitive of $f_w^{1/2}$ which vanishes in 0 is given by

$$\begin{aligned} F_w(z) &= 2 \int_0^z \zeta^{1/2} (\zeta - w) d\zeta = \frac{2}{5} z^{5/2} - \frac{2}{3} z^{3/2} w \\ &= \frac{2}{5} \rho^{5/2} e^{i5\theta/2} - \frac{2}{3} \varepsilon \rho^{3/2} e^{i(3\theta/2 + \phi)}. \end{aligned}$$

The trace of $\Re F_w$ on the boundary of $\mathbb{D}$ is

$$G_\phi(\theta) = \frac{2}{5} \cos\left(\frac{5}{2}\theta\right) - \frac{2}{3} \varepsilon \cos\left(\frac{3}{2}\theta + \phi\right).$$

It is easy to verify that, for $\varepsilon > 0$ sufficiently small, G_ϕ vanishes exactly in five points on $\partial\mathbb{D}$. For $\varepsilon = 0$, $G_0(\theta) = 0$ if and only if $\theta = \pi/5 + 2k\pi/5$ for $k = 0, \dots, 4$.

Set now $U_w(z) = |\Re F_w(z)|$. If $w = 0$, then $U_0 \in \mathcal{U}_5$ and it is the solution with boundary value G_0 (see Figure 7, left). However, for $\varepsilon > 0$ this is almost never the case. Notice that

$$U_w(0) = 0 \quad \text{and} \quad U_w(w) = \frac{4}{15} |w|^{5/2} \left| \cos\left(\frac{5}{2}\phi\right) \right|.$$

Hence, $U_w(w) = 0$ if and only if $\phi = \frac{\pi}{5} + \frac{2k\pi}{5}$ for $k = 0, \dots, 4$. In this five instances, Proposition 2.4 implies that $U_w \in \mathcal{U}_5$ and $f_{U_w} = f_w$ (see Figure 7, right, to help intuition). In all the other cases, i.e. $\phi \neq \frac{\pi}{5} + \frac{2k\pi}{5}$, we have

that $w \in \mathcal{Z}_{f_{U_w}}$ but $w \notin \mathcal{N}_{U_w}$, and hence $\mathcal{C}_{U_w} = \mathcal{Z}_{f_{U_w}} \cap \mathcal{N}_{U_w} = \{0\}$. From the index formula (4.9) we infer that, if $U_w \in \mathcal{U}_5$, then the origin should be a point with multiplicity 5. This is however impossible because

$$U_w(z) = |\Re F_w(z)| = \left| \frac{2}{5} \rho^{5/2} \cos(\frac{5}{2}\theta) - \frac{2}{3} \rho^{3/2} \epsilon \cos(\frac{3}{2}\theta + \phi_w) \right| = O(\rho^{3/2})$$

contradicts (4.3) in Proposition 4.1.

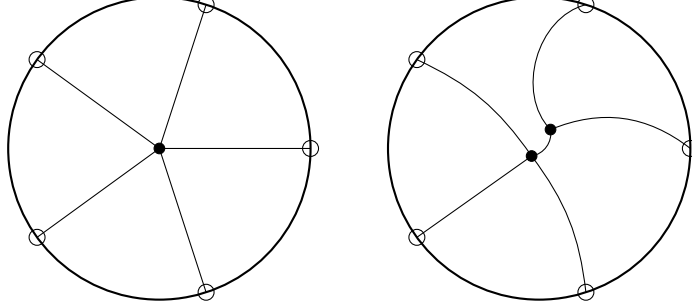


FIGURE 7. Five species: configuration with one 5-point (on the left); configuration with one 3-point and one 4-point (on the right).

As we will see in the next result, the desingularization is a global process which involves simultaneously all critical points.

5.2. The main lemma. The Main Theorem on the multiplicity of points $z \in \Omega^-$ for the generic solutions U to (1.1) stated in the Introduction requires the following lemma, which proves, roughly speaking, that there exists a suitable small perturbation of the function U , which decrease the order of one zero of f_U , leaving all the other unchanged.

For the readers' convenience, we illustrate the main steps of the proof:

- Step 1: given $U \in \mathcal{U}$, we consider the corresponding Hopf differential f_U and propose a modification of it which consists in multiplying by a polynomial of several complex variables w_ℓ with coefficients holomorphic functions $h_\ell(z)$.

The main idea of Step 1 is the introduction of new complex parameters w_ℓ which will be chosen in such a way to satisfy the necessary algebraic conditions (2.4) in order to produce a new Hopf differential.

- Step 2: we study the linear system in the new variables introduced in Step 1 corresponding to the solutions of (2.4). A convenient choice for the holomorphic coefficients $h_\ell(z)$ is a sequence of carefully chosen powers z^{k_ℓ} with $k_\ell := \ell R$ for a large constant R .
- Step 3 shows that the choices in the previous step allow to prove the invertibility of the linear system guaranteeing (2.4) in all points except one (corresponding to the new zero of the holomorphic perturbation). The solution of Step 3 depends on the single parameter

corresponding to the position of the new zero of the holomorphic perturbation.

- Step 4 contains the implicit function argument which ensures that the algebraic equation (2.4) can be satisfied also in the new zero, while preserving all the other solutions. This step takes care of the rigidity presented in §5.1, showing that only finitely many directions are admissible for this final perturbation.
- The final Step 5 shows that the perturbation we make is arbitrarily small in $H^1(\mathbb{D})$ and $L^\infty(\mathbb{D})$.

Lemma 5.1. *Let $U \in \mathcal{U}$ with f_U satisfying (H1)-(H3). Then, for every $\varepsilon > 0$ there exists $U_\varepsilon \in \mathcal{U}$ such that*

$$\|U - U_\varepsilon\|_{L^\infty(\mathbb{D})} + \|U - U_\varepsilon\|_{W^{1,2}(\mathbb{D})} \leq \varepsilon,$$

and

$$\begin{aligned} \mathcal{C}_{U_\varepsilon} &= \{\omega_0, z_0, \dots, z_M\}, \\ \text{ord}(f_{U_\varepsilon}; \omega_0) &= 1, \quad \text{ord}(f_{U_\varepsilon}; z_0) = m_0, \\ \text{ord}(f_{U_\varepsilon}; z_j) &= \text{ord}(f_U; z_j) \quad \forall j = 1, \dots, M. \end{aligned}$$

Proof. We consider the simply connected domain

$$\Omega^- := \mathbb{D} \setminus \bigcup_{j=0}^M \Gamma_j,$$

where $\Gamma_j = \{z_j + t(z_j - z_0) : t \geq 0\}$ for $j = 1, \dots, M$ and

$$\Gamma_0 := \{z_0 + t\gamma_0 : t \geq 0\}, \quad \gamma_0 \in \mathbb{C}, \quad |\gamma_0| = 1$$

is any half-line not intersecting $\bigcup_{j=1}^M \Gamma_j$. We translate our domain by $-z_0$:

$$\Omega_0 := \{z \in \mathbb{C} : z + z_0 \in \Omega^-\}.$$

The functions

$$U_0(z) := U(z - z_0), \quad f_0(z) := f_U(z - z_0)$$

are defined in $\mathbb{D}_0 := \{z + z_0 : |z| < 1\}$ and satisfy

$$\mathcal{C}_{U_0} = \{0, \omega_1, \dots, \omega_M\} \quad \text{with} \quad \omega_j := z_j - z_0 \quad \forall j = 1, \dots, M.$$

The set Ω_0 is star-shaped and we have that

$$\Omega_0 = \mathbb{D}_0 \setminus \left(\bigcup_{j=1}^M \{t\omega_j : t \geq 1\} \cup \{t\gamma_0 : t \geq 0\} \right).$$

Note that by assumption (H3) we have that

$$(5.1) \quad 0 < |\omega_1| < \dots < |\omega_M|,$$

and

$$\frac{\omega_j}{|\omega_j|} \neq \frac{\omega_\ell}{|\omega_\ell|} \quad \text{and} \quad \frac{\omega_j}{|\omega_j|} \neq \gamma_0 \quad \forall j \neq \ell \in \{1, \dots, M\}.$$

For the sake of readability we split the proof into different steps.

Step 1: modified holomorphic function. By (H1) - (H2) the function f_0 can be written in the form

$$(5.2) \quad f_0(z) = z^{m_0+1} h^2(z) \prod_{j=1}^M (z - \omega_j)^{q_j}, \quad q_j \in \mathbb{N}, \quad j = 1, \dots, M,$$

with $h : \mathbb{D}_0 \rightarrow \mathbb{C}$ a holomorphic function with $h(\omega_j) \neq 0$ for all j . We fix M holomorphic functions $h_1, \dots, h_M$ in $\mathbb{D}_0$ to be specified later and for every vector $W = (w_1, \dots, w_M) \in \mathbb{C}^M$ we set

$$(5.3) \quad q(z, W) := 1 + \sum_{\ell=1}^M w_\ell h_\ell(z) \quad z \in \mathbb{D}_0.$$

Let $\omega_0 \in \mathbb{C}$ with $0 < |\omega_0| < |\omega_1|$ and consider the function

$$(5.4) \quad f_{\omega_0, W}(z) := z^{m_0} (z - \omega_0) h^2(z) q^2(z, W) \prod_{j=1}^M (z - \omega_j)^{q_j}.$$

We consider a simply connected domain $\Omega_{\omega_0} := \Omega_0 \setminus L_{\omega_0}$, with $L_{\omega_0} := \{\omega_0 + t\gamma_0 : t \geq 0\}$. On the simply connected domain Ω_{ω_0} we consider a determination of the square root $f_{\omega_0, W}^{1/2} : \Omega_{\omega_0} \rightarrow \mathbb{C}$ written in the following form:

$$\begin{aligned} f_{\omega_0, W}^{1/2}(z) &:= z^{m_0/2} (z - \omega_0)^{1/2} H(z) q(z, W) \\ &= z^{m_0/2} (z - \omega_0)^{1/2} H(z) + z^{m_0/2} (z - \omega_0)^{1/2} H(z) \sum_{\ell=1}^M w_\ell h_\ell(z), \end{aligned}$$

with

$$H(z) := \prod_{j=1}^M (z - \omega_j)^{q_j/2} h(z),$$

where we fixed suitable determinations of the square roots $z^{m_0/2}$, $(z - \omega_0)^{1/2}$ and $(z - \omega_j)^{q_j/2}$ in Ω_{ω_0} . We set

$$\begin{aligned} F_{\omega_0, W}(z) &:= 2 \int_0^z \zeta^{m_0/2} (\zeta - \omega_0)^{1/2} H(\zeta) d\zeta \\ &\quad + 2 \sum_{\ell=1}^M w_\ell \int_0^z h_\ell(\zeta) \zeta^{m_0/2} (\zeta - \omega_0)^{1/2} H(\zeta) d\zeta, \end{aligned}$$

where the integration is taken along curves in Ω_{ω_0} . By construction, we have that $F_{\omega_0, W}(0) = 0$. Define the matrix $A = A(\omega_0) = \{A_{j, \ell}(\omega_0)\} \in \mathbb{C}^{M \times M}$ as follows

$$A_{j, \ell} = A_{j, \ell}(\omega_0) := 2 \int_0^{\omega_j} h_\ell(\zeta) \zeta^{m_0/2} (\zeta - \omega_0)^{1/2} H(\zeta) d\zeta \quad j, \ell = 1, \dots, M,$$

and the vector $B = B(\omega_0) \in \mathbb{C}^M$ with components

$$b_j = b_j(\omega_0) := 2 \int_0^{\omega_j} \zeta^{m_0/2} (\zeta - \omega_0)^{1/2} H(\zeta) d\zeta \quad j = 1, \dots, M.$$

With these notations

$$(5.5) \quad F_{\omega_0, W}(\omega_j) = b_j + \sum_{\ell=1}^M A_{j,\ell} w_\ell = (AW + B)_j \quad j = 1, \dots, M.$$

Step 2: invertibility of A . We show that one can choose $h_1, \dots, h_\ell \in \mathcal{H}(\mathbb{C})$ such that the matrix $A(0)$ is non-singular and, hence, by continuity the same holds for $A(\omega_0)$ if $|\omega_0|$ is sufficiently small.

Consider the case

$$h_\ell(z) = z^{k_\ell}, \quad \text{with } k_1 < k_2 < \dots < k_M, \quad k_\ell \in \mathbb{N}.$$

We integrate along the path $\zeta = t\omega_j$ with $t \in [0, 1]$,

$$\begin{aligned} A_{j,\ell}(0) &= 2 \int_0^{\omega_j} \zeta^{\frac{m_0}{2} + \frac{1}{2} + k_\ell} \prod_{s=1}^M (\zeta - \omega_s)^{q_s/2} h(\zeta) d\zeta \\ &= 2(\omega_j)^{\frac{m_0}{2} + \frac{3}{2} + k_\ell} \int_0^1 t^{\frac{m_0}{2} + \frac{1}{2} + k_\ell} \prod_{s=1}^M (t\omega_j - \omega_s)^{q_s/2} h(t\omega_j) dt \\ &= 2(\omega_j)^{\frac{m_0}{2} + \frac{3}{2} + Q + k_\ell} \int_0^1 t^{\frac{m_0}{2} + \frac{1}{2} + k_\ell} \prod_{s=1}^M \left(t - \frac{\omega_s}{\omega_j}\right)^{q_s/2} h(t\omega_j) dt, \end{aligned}$$

with $Q = (q_1 + \dots + q_M)/2$. Set

$$\begin{aligned} a_{j,\ell} &= \int_0^1 t^{\frac{m_0}{2} + \frac{1}{2} + k_\ell} \prod_{s=1}^M \left(t - \frac{\omega_s}{\omega_j}\right)^{q_s/2} h(t\omega_j) dt \\ &= i^{q_j} \int_0^1 t^{k_\ell} (1-t)^{q_j/2} t^{\frac{m_0}{2} + \frac{1}{2}} \prod_{s \neq j} \left(t - \frac{\omega_s}{\omega_j}\right)^{q_s/2} h(t\omega_j) dt \\ &= \int_0^1 t^{k_\ell} (1-t)^{q_j/2} g_j(t) dt, \end{aligned}$$

with

$$g_j(t) := i^{q_j} t^{\frac{m_0}{2} + \frac{1}{2}} \prod_{s \neq j} \left(t - \frac{\omega_s}{\omega_j}\right)^{q_s/2} h(t\omega_j).$$

Then,

$$A_{j,\ell}(0) = 2(\omega_j)^{\frac{m_0}{2} + \frac{3}{2} + Q + k_\ell} a_{j,\ell}, \quad j, \ell = 1, \dots, M.$$

The matrix A is non singular if and only if the following matrix is non-singular:

$$\mathcal{A} = \begin{pmatrix} \omega_1^{k_1} a_{11} & \omega_1^{k_2} a_{12} & \dots & \omega_1^{k_M} a_{1M} \\ \omega_2^{k_1} a_{21} & \omega_2^{k_2} a_{22} & \dots & \omega_2^{k_M} a_{2M} \\ \vdots & \vdots & & \vdots \\ \omega_M^{k_1} a_{M1} & \omega_M^{k_2} a_{M2} & \dots & \omega_M^{k_M} a_{MM} \end{pmatrix}.$$

We assume $k_\ell = \ell R$ with $R > 0$ and claim that:

$$\det \mathcal{A} = \begin{vmatrix} \omega_1^R a_{11} & \omega_1^{2R} a_{12} & \dots & \omega_1^{MR} a_{1M} \\ \omega_2^R a_{21} & \omega_2^{2R} a_{22} & \dots & \omega_2^{MR} a_{2M} \\ \vdots & \vdots & & \vdots \\ \omega_M^R a_{M1} & \omega_M^{2R} a_{M2} & \dots & \omega_M^{MR} a_{MM} \end{vmatrix} \neq 0$$

for R sufficiently large.

To this aim, we start looking at the functions

$$m_{j\ell}(t) := t^{\ell R} (1-t)^{q_j/2}.$$

By an explicit computation we have that

$$M_{j\ell} := \int_0^1 m_{j\ell}(t) dt = \frac{\Gamma(1 + \ell R) \Gamma(1 + q_j/2)}{\Gamma(2 + \ell R + q_j/2)},$$

where Γ is Euler's Gamma function. In particular, by the well-known relations $\Gamma(1+x) = x\Gamma(x)$ and $\Gamma(x)\Gamma(x + \frac{1}{2}) = 2^{1-2x}\sqrt{\pi}\Gamma(2x)$, we infer that

$$M_{j\ell} = (\ell R)^{-1-q_j/2} (1 + o(1)) \quad \text{as } R \rightarrow +\infty.$$

Moreover, we have that

$$(5.6) \quad \mu_{j\ell} := \frac{m_{j\ell}(t)}{M_{j\ell}} dt \xrightarrow{*} \delta_1.$$

Indeed, $\mu_{j\ell}$ are probability measures such that for every $\tau < 1$ we have

$$\begin{aligned} \mu_{j\ell}([0, \tau]) &= M_{j\ell}^{-1} \int_0^\tau t^{\ell R} (1-t)^{q_j/2} dt \leq M_{j\ell}^{-1} \int_0^\tau t^{\ell R} dt \\ &\leq \frac{\tau^{\ell R}}{(\ell R + 1) (\ell R)^{1+q_j/2} (1 + o(1))} \rightarrow 0 \quad \text{as } R \rightarrow +\infty \end{aligned}$$

This implies that any weak* limit μ of $\mu_{j\ell}$ is zero in $[0, \tau]$ for every $\tau < 1$. Since the space of probability measures on a compact set is weak* closed, we infer (5.6). In particular, we have that

$$\begin{aligned} \lim_{R \rightarrow +\infty} (\ell R)^{1+q_j/2} a_{j\ell} &= \lim_{R \rightarrow +\infty} M_{j\ell}^{-1} a_{j\ell} \\ (5.7) \quad &= \lim_{R \rightarrow +\infty} M_{j\ell}^{-1} \int_0^1 t^{k_\ell} (1-t)^{q_j/2} g_j(t) dt = g_j(1). \end{aligned}$$

We can now conclude the proof of the invertibility of $\mathcal{A}(0)$. We use the Leibnitz formula for the determinants

$$\begin{aligned} \det \mathcal{A} &= \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{j=1}^M \omega_j^{\sigma(j)R} a_{j\sigma(j)} \\ &= \omega_1^R \omega_2^{2R} \cdot \dots \cdot \omega_M^{MR} \left[a_{11} \dots a_{MM} \right. \\ &\quad \left. + \sum_{\sigma \neq (1,2,\dots,M)} \operatorname{sgn}(\sigma) \prod_{j=1}^M a_{j\sigma(j)} \frac{\omega_1^{\sigma(1)R} \omega_2^{\sigma(2)R} \dots \omega_M^{\sigma(M)R}}{\omega_1^R \omega_2^{2R} \dots \omega_M^{MR}} \right], \end{aligned}$$

where as usual $\sigma = (\sigma(1), \dots, \sigma(M))$ is a permutation of $(1, \dots, M)$ and $\operatorname{sgn}(\sigma)$ denotes its sign. Using (5.7) in the form

$$a_{j\ell} = (\ell R)^{-1-q_j/2} g_j(1)(1 + o(1)),$$

we get (recall that $Q = q_1/2 + \dots + q_M/2$)

$$\begin{aligned} \det \mathcal{A} &= \omega_1^R \omega_2^{2R} \dots \omega_M^{MR} R^{-M-Q} \prod_{j=1}^M g_j(1) \left[\prod_{j=1}^M j^{-1-q_j/2} \right. \\ &\quad \left. + \sum_{\sigma \neq (1,2,\dots,M)} \operatorname{sgn}(\sigma) \prod_{j=1}^M \sigma(j)^{-1-q_j/2} \frac{\omega_1^{\sigma(1)R} \omega_2^{\sigma(2)R} \dots \omega_M^{\sigma(M)R}}{\omega_1^R \omega_2^{2R} \dots \omega_M^{MR}} + o(1) \right]. \end{aligned}$$

By the assumption (H3) on the zeros of f_U in general position (i.e. (5.1)), we deduce that

$$\lim_{R \rightarrow \infty} \frac{\omega_1^{\sigma(1)R} \omega_2^{\sigma(2)R} \dots \omega_M^{\sigma(M)R}}{\omega_1^R \omega_2^{2R} \dots \omega_M^{MR}} = 0, \quad \forall \sigma \neq (1, 2, \dots, M).$$

Then, taking into account that $g_j(1) \neq 0$, it follows that $\det \mathcal{A} \neq 0$ for sufficiently large R (depending only on the exponents q_j 's), thus concluding the proof of the claim.

Step 3: $\Re F_{\omega_0, W}(\omega_j) = 0$ for $j = 1, \dots, M$. We prove that it is possible to find a column vector

$$W = W(\omega_0) = (w_1(\omega_0), \dots, w_M(\omega_0)),$$

such that the conditions (2.4) are satisfied. Keeping in mind (5.5), conditions (2.4) can be written in the single equation

$$(5.8) \quad A(\omega_0)W(\omega_0) + B(\omega_0) = i\Lambda(\omega_0),$$

with an arbitrary $\Lambda(\omega_0) \in \mathbb{R}^M$. Note that, since for $\omega_0 = 0$ we have that $f_{0,0} = f_0$, then $F_{0,0}$ satisfies (2.4), i.e.,

$$(F_{0,0}(\omega_j))_{j=1,\dots,m} =: B(0) \in i\mathbb{R}.$$

Since the matrix A is non-singular, the system (5.8) is then solved by

$$(5.9) \quad W(\omega_0) = (A(\omega_0))^{-1}(B(0) - B(\omega_0)).$$

In particular, the function $\omega_0 \mapsto W(\omega_0)$ is continuous and

$$\lim_{\omega_0 \rightarrow 0} W(\omega_0) = W(0) = 0.$$

Step 4: $\Re F_{\omega_0, W(\omega_0)}(\omega_0) = 0$. We show that it is possible to choose ω_0 such that also the real part of $F_{\omega_0, W(\omega_0)}(\omega_0)$ is zero.

In order to compute $F_{\omega_0, W(\omega_0)}$ we integrate along the path $\zeta = t\omega_0$, $0 \leq t \leq 1$ and we get

$$F_{\omega_0, W(\omega_0)}(\omega_0) = \int_0^1 (t\omega_0)^{m_0/2} ((t-1)\omega_0)^{1/2} H(t\omega_0) q(t\omega_0, W(\omega_0)) \omega_0 dt.$$

We consider $\omega_0 = \varepsilon e^{i\vartheta}$ and

$$\begin{aligned} G(\varepsilon, \vartheta) &:= \Re F_{\varepsilon e^{i\vartheta}, W(\varepsilon e^{i\vartheta})}(\varepsilon e^{i\vartheta}) \\ &= \varepsilon^{\frac{m_0+3}{2}} \Re \left(i e^{\frac{i(m_0+3)\vartheta}{2}} \int_0^1 t^{m_0/2} \sqrt{1-t} Z(t\varepsilon e^{i\vartheta}) dt \right), \end{aligned}$$

with

$$Z(t\varepsilon e^{i\vartheta}) := H(t\varepsilon e^{i\vartheta}) q(t\varepsilon e^{i\vartheta}, W(t\varepsilon e^{i\vartheta})).$$

Set

$$K(\varepsilon, \vartheta) := \varepsilon^{-\frac{m_0+3}{2}} G(\varepsilon, \vartheta) = \Re \left(i e^{\frac{i(m_0+3)\vartheta}{2}} \int_0^1 t^{m_0/2} \sqrt{1-t} Z(t\varepsilon e^{i\vartheta}) dt \right).$$

We show that the equation $K(0, \vartheta) = 0$ has a finite number of roots. Indeed, we write $Z(0) = |Z(0)|e^{-i\varphi}$ and

$$c_{m_0} = \int_0^1 t^{m_0/2} \sqrt{1-t} dt = \frac{\sqrt{\pi}}{2} \frac{\Gamma(1 + \frac{m_0}{2})}{\Gamma(\frac{5+m_0}{2})}.$$

Then

$$K(0, \vartheta) = |Z(0)|c_{m_0} \Re \left(i e^{i(\frac{(3+m_0)\vartheta}{2} - \varphi)} \right) = -|H(0)|c_{m_0} \sin \left(\frac{3+m_0}{2} \vartheta - \varphi \right).$$

Since $H(0) \neq 0$, we infer that

$$\begin{aligned} K(0, \vartheta) = 0 &\iff \sin \left(\frac{3+m_0}{2} \vartheta - \varphi \right) = 0 \\ &\iff \frac{3+m_0}{2} \vartheta - \varphi = k\pi, \quad k \in \mathbb{Z}, \end{aligned}$$

that is the zeros of $K(0, \vartheta)$ are

$$\vartheta_k = \frac{2\varphi + 2k\pi}{3+m_0}, \quad k = 0, \dots, 2+m_0.$$

Let us fix an index $k \in \{0, \dots, 2+m_0\}$. We have

$$\frac{\partial K(0, \vartheta)}{\partial \vartheta} \Big|_{(0, \vartheta_k)} = -\frac{3+m_0}{2} c_{m_0} |H(0)| \cos \left(\frac{3+m_0}{2} \vartheta_k - \varphi \right) \neq 0,$$

because $H(0) \neq 0$. Given the regularity of $K(\varepsilon, \vartheta)$, for the implicit function theorem there exists a function $\vartheta_k(\varepsilon)$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with sufficiently small $\varepsilon_0 > 0$, such that

$$\vartheta_k(0) = \vartheta_k, \quad K(\varepsilon, \vartheta_k(\varepsilon)) = 0.$$

It follows that there exist a finite number of directions $\vartheta_k(\varepsilon)$ such that, if we set $\omega_0(\varepsilon) = \varepsilon e^{i\vartheta_k(\varepsilon)}$, condition

$$\Re F_{\omega_0, W(\omega_0)}(\omega_0) = 0$$

is satisfied.

Step 5: conclusion of the proof. For $\varepsilon > 0$ sufficiently small, we consider a solution $\omega_0(\varepsilon)$. Then, $f_\varepsilon(z) := f_{\omega_0(\varepsilon), W(\varepsilon)}(z + z_0) \in \mathcal{H}$. By construction we have that $\{z_0, z_0 + \omega_0, z_1, \dots, z_M\} \subseteq \mathcal{Z}_{f_\varepsilon}$ and

$$\text{ord}(f_\varepsilon; z_j) = \text{ord}(f_U; z_j) \quad \forall j = 1, \dots, M,$$

and $\text{ord}(f_\varepsilon; z_0) = m_0$ and $\text{ord}(f_\varepsilon; z_0 + \omega_0) = 1$. Since $W(\omega_0(\varepsilon)) \rightarrow W(0) = 0$, we have also that

$$(5.10) \quad \lim_{\varepsilon \rightarrow 0^+} \|f_\varepsilon - f_U\|_{L^\infty(\mathbb{D})} = 0.$$

Moreover, from $\mathcal{Z}_{f_U}^{\text{odd}} \subset \mathcal{Z}_{f_U} \cap \mathcal{C}_U$, we have $\mathcal{Z}_{f_\varepsilon}^{\text{odd}} \subset \{\omega_0, z_0, \dots, z_M\}$. Set $F_\varepsilon := F_{\omega_0(\varepsilon), W(\omega_0(\varepsilon))}$. By Steps 3 and 4 $\Re F_\varepsilon(z) = 0$ for all $z \in \mathcal{Z}_{f_\varepsilon}^{\text{odd}}$. By Proposition 2.4 to infer that there exists a function $U_\varepsilon = |\Re F_\varepsilon| \in \mathcal{U}$ such that $f_\varepsilon = I(U_\varepsilon)$.

Notice that, since $U_\varepsilon(z_0) = 0$ and (5.10) holds, we have $U_\varepsilon \rightarrow U$ uniformly. Furthermore, $|\nabla U_\varepsilon|^2 \rightarrow |\nabla U|^2$ in L^1 , which follows from $f_\varepsilon \rightarrow f_U$ in L^1 . In particular, $\nabla U_\varepsilon \rightharpoonup \nabla U$ in L^2 and the Dirichlet energies converge, thus implying $U_\varepsilon \rightarrow U$ in $H^1(\mathbb{D})$. $\square$

6. GENERIC SEGREGATED STATES

In this section we prove Theorem 1.1 on the genericity of functions $U \in \mathcal{U}$ with only triple junctions. By (4.1), we need to prove that the subset of functions $U \in \mathcal{U}$ such that $I(U)$ has only simple zeros is the intersection of open dense subsets.

6.1. Finite number of critical points. First of all we show that the functions with only finitely many critical points constitute a dense set of $\mathcal{U}$.

Lemma 6.1. *The set of $U \in \mathcal{U}$ such that the corresponding holomorphic functions f_U have finitely many zeros is dense in $\mathcal{U}$.*

Proof. Given any function $U \in \mathcal{U}$, it is enough to consider the functions

$$U_\varepsilon(z) := U\left(\frac{z}{1 + \varepsilon}\right) \quad z \in \mathbb{D}.$$

Then, clearly $U_\varepsilon \in \mathcal{U}$ and its squared Wirtinger derivative satisfies

$$f_{U_\varepsilon}(z) := (\partial_z U_\varepsilon)^2 = (1 + \varepsilon)^{-2} f_U \left(\frac{z}{1 + \varepsilon} \right).$$

Therefore, since the set of its zeros of the holomorphic function f_U is locally finite on compact subsets of $\mathbb{D}$, we deduce in particular that

$$\#\mathcal{Z}_{f_{U_\varepsilon}} = \#(\mathcal{Z}_{f_U} \cap \mathbb{D}_{\frac{1}{1+\varepsilon}}) < +\infty.$$

Moreover, by the very definition we also have that

$$\|U_\varepsilon - U\|_{H^1(\mathbb{D})} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.$$

□

Remark 6.2. *Considering that $\mathcal{C}_{U_\varepsilon} \subseteq \mathcal{Z}_{f_{U_\varepsilon}}$, we conclude that the set of functions with finitely many critical points is dense in $\mathcal{U}$.*

6.2. Points in general position. In the perspective of proving a density result, we can then assume that the number of critical points of $U \in \mathcal{U}$ is finite, say

$$\mathcal{C}_U = \{z_0, z_1, \dots, z_M\}, \quad M \in \mathbb{N}.$$

We show in the next lemma that there exists a conformal diffeomorphism ϕ of the unit disk such that the image of the critical set is in general position with respect to $\phi(z_0)$ as defined in (H3) in Section 5.

We recall that the conformal diffeomorphisms of the unit disk are characterized in terms of two parameters

$$\phi_{\alpha, \theta}(z) = e^{i\theta} \frac{z + \alpha}{\bar{\alpha}z + 1} \quad \alpha \in \mathbb{D}, \quad \theta \in \mathbb{R}.$$

When $\theta = 0$ we write ϕ_α in place of $\phi_{\alpha, 0}$.

Lemma 6.3. *Let $\{p_0, \dots, p_M\} \subset \mathbb{D}$. Then, there exists $\alpha \in \mathbb{D}$ such that*

$$\{\phi_\alpha(p_0), \dots, \phi_\alpha(p_M)\}$$

are in general position with respect to $\phi_\alpha(p_0)$.

Proof. First of all we consider the conformal diffeomorphism ϕ_{-p_0} and notice that $\phi_{-p_0}(p_0) = 0$. Therefore, we can consider without loss of generality points

$$\{q_0, \dots, q_M\} \quad q_j := \phi_{-p_0}(p_j), \quad q_0 = 0.$$

We will show that the points $\{\phi_\alpha(q_j)\}$ are in general position with respect to $\alpha = \phi_\alpha(0)$, provided α is small enough and

$$\text{Arg}(\alpha) \neq \frac{\text{Arg}(q_j) + \text{Arg}(q_\ell)}{2} + k\pi, \quad \forall j, \ell = 1, \dots, M, \quad \forall k \in \mathbb{Z}.$$

We now verify the claim. By continuity, we have $\phi_\alpha(z) \rightarrow z$ for $\alpha \rightarrow 0$. Hence, there exists $\delta_0 > 0$ such that for $|\alpha| < \delta_0$ any triple of non-aligned points in $\{q_0, \dots, q_M\}$ remains non-aligned and, similarly,

$$|q_j - q_0| \neq |q_\ell - q_0| \implies |\phi_\alpha(q_j) - \phi_\alpha(q_0)| \neq |\phi_\alpha(q_\ell) - \phi_\alpha(q_0)|.$$

Furthermore, if there were two points aligned with 0, i.e. there exists a line r such that $q_j, q_\ell, 0 \in r$, then since $\text{Arg}(\alpha) \neq \text{Arg}(q_j) + k\pi$ it follows $\phi_\alpha(r)$ is a circle and therefore the points $\phi_\alpha(q_j)$ and $\phi_\alpha(q_\ell)$ are not any more aligned with $\phi_\alpha(0)$.

Finally, recall that the circles with center the origin are sent by ϕ_α to circles with the center on the open segment between 0 and α . Therefore, if two points q_j, q_ℓ belong to the same circle around the origin (i.e., have the same distance from 0), their images are on an other circle whose center is not α . The only possibility in order to have the same distance from α is that q_j and q_ℓ are symmetric with respect to the line $r = \{t\alpha : t \in \mathbb{R}\}$. But this instance is excluded by the fact that

$$\text{Arg}(q_j) \neq \frac{\text{Arg}(q_j) + \text{Arg}(q_\ell)}{2} + k\pi, \quad k \in \mathbb{Z}.$$

The claim is then proven and we find a conformal diffeomorphism of the disk $\phi_{-p_0+\alpha} = \phi_\alpha \circ \phi_{-p_0}$ which sends $\{p_0, \dots, p_M\}$ in general position with respect to $\phi_{-p_0+\alpha}(p_0) = \alpha$. $\square$

6.3. Proof of Theorem 1.1. We are now ready to give the proof of the main theorem. We show that the sets

$$\mathcal{U}_r^\gamma := \left\{ U \in \mathcal{U} : m_U(z) = 3 \quad \forall z \in \mathcal{C}_U \cap \overline{\mathbb{D}}_r \right\}, \quad r \in (0, 1),$$

are open dense set in $\overline{\mathcal{U}}$. Therefore, the set $\mathcal{O}$ of functions U with only triple junctions

$$\mathcal{U}^\gamma := \bigcap_{r \in (0, 1)} \mathcal{U}_r^\gamma$$

is residual in $\overline{\mathcal{U}}$.

Openness of $\mathcal{U}_r^\gamma$. Let $U \in \mathcal{U}_r^\gamma$ be fixed and assume that U has M critical points in $\overline{\mathbb{D}}_r$ which are triple points:

$$\mathcal{C}_U \cap \overline{\mathbb{D}}_r = \{z_1, \dots, z_M\}, \quad m_U(z_i) = 3 \quad \forall i = 1, \dots, M.$$

We show that there exists $\varepsilon > 0$ such that for every $V \in \mathcal{U}$ with $\|U - V\|_{H^1(\mathbb{D})} \leq \varepsilon$ we have that $V \in \mathcal{O}_r$, i.e. V has only triple points in $\overline{\mathbb{D}}_r$. We start observing that

$$\|f_U - f_V\|_{L^1(\mathbb{D})} = \|U_z^2 - V_z^2\|_{L^1(\mathbb{D})} \leq C\|U - V\|_{H^1(\mathbb{D})} \leq C\varepsilon,$$

for a dimensional constant $C > 0$. On the other hand, since the functions f_U, f_V are holomorphic in $\mathbb{D}$, from the Cauchy formula we deduce that for every $s < 1$ there exists a constant $C(s) > 0$ such that

$$(6.1) \quad \|f_U - f_V\|_{C^1(\mathbb{D}_s)} \leq C(s)\|f_U - f_V\|_{L^1(\mathbb{D})} \leq C(s)\varepsilon.$$

We consider a radius $r < s < 1$ such that $\mathcal{C}_U \cap \mathbb{D}_s \subset \mathcal{C}_U \cap \overline{\mathbb{D}}_r$. We show that, if $\varepsilon > 0$ is sufficiently small, then $\mathcal{C}_V \cap \mathbb{D}_{\frac{r+s}{2}}$ is all made of triple junctions too. Indeed, we can consider disjoint circles $C_j = \partial\mathbb{D}_{s_j}(z_j) \subset \mathbb{D}_{\frac{r+s}{2}}$ (oriented

counterclockwise) around every z_j for $j = 1, \dots, M$, for a suitable $s_j > 0$. Then, by the argument principle we have that

$$\frac{1}{2\pi i} \oint_{C_j} \frac{f'_U(x)}{f_U(z)} dz = 1.$$

By (6.1), if $\varepsilon > 0$ is small enough, then

$$\left| \frac{1}{2\pi i} \oint_{C_j} \frac{f'_V(x)}{f_V(z)} dz - \frac{1}{2\pi i} \oint_{C_j} \frac{f'_U(x)}{f_U(z)} dz \right| \leq C \varepsilon,$$

where the constant C depends on $s < 1$ and on f_U (through the choice of the circles C_j and the norm of $\|f_U\|_{C^1(\mathbb{D}_s)}$). Since $\frac{1}{2\pi i} \oint_{C_j} \frac{f'_V(x)}{f_V(z)} dz \in \mathbb{N}$, we conclude that

$$\frac{1}{2\pi i} \oint_{C_j} \frac{f'_V(x)}{f_V(z)} dz = 1 \quad \forall j = 1, \dots, M,$$

i.e., by the argument principle, f_V has a simple zero in $\mathbb{D}_{s_j}(z_j)$. Moreover, since $|U| \geq c > 0$ in $\mathbb{D}_{\frac{r+s}{2}} \setminus \bigcup_{j=1}^M \mathbb{D}_{s_j}(z_j)$ and $\|U - V\|_{L^\infty(\mathbb{D})} = o(1)$ for $\varepsilon \rightarrow 0^+$, we also infer that there are no other critical points of V in $\mathbb{D}_{\frac{r+s}{2}}$, thus concluding the proof.

Density of $\mathcal{U}_r^\gamma$. By definition $\mathcal{U} \subset \overline{\mathcal{U}}$ is dense. By Lemma 6.1 the set of functions $U \in \mathcal{U}$ with finitely many critical points is dense. Therefore, it is enough to show the density of $\mathcal{U}_r^\gamma$ in such set. We fix then $U \in \mathcal{U}$ with finitely many critical point and we define the index

$$\alpha_U := \sum_{z \in \mathcal{C}_U} (m_U(z) - 3).$$

Note that U has only triple junctions if and only if $\alpha_U = 0$. We show that, given any $U \in \mathcal{U}$ with finitely many critical points and $\alpha_U > 0$, for every $\varepsilon > 0$ there exists $U_\varepsilon \in \mathcal{U}$ with

$$(6.2) \quad \alpha_{U_\varepsilon} = \alpha_U - 1 \quad \text{and} \quad \|U - U_\varepsilon\|_{H^1(\mathbb{D})} \leq \varepsilon.$$

Iterating the argument α_U times, we find a segregated state $V \in \mathcal{U}$ with $\alpha_V = 0$, i.e. with only triple junctions, and $\|U - V\|_{H^1(\mathbb{D})} \leq \alpha_U \varepsilon$. By the arbitrariness of ε , we hence conclude.

In order to prove (6.2), let

$$\mathcal{C}_U = \{z_0, \dots, z_{\alpha_U-1}\},$$

and assume that $\text{ord}(f_U; z_0) = m_0 + 1$, with $m_0 > 0$. By Lemma 6.3 we can consider a conformal map $\phi : \mathbb{D} \rightarrow \mathbb{D}$ such that $\phi(\mathcal{C}_U)$ is in general position with respect to $\phi(z_0)$. Then, we can apply Lemma 5.1 to $U \circ \phi$ and, for every $\varepsilon_0 > 0$, we find $V \in \mathcal{U}$ with $\|U \circ \phi - V\|_{H^1(\mathbb{D})} \leq \varepsilon_0$ and

$$\mathcal{C}_V = \{\omega_0, \phi(z_0), \dots, \phi(z_{\alpha_U-1})\},$$

with

$$\text{ord}(f_V; \omega_0) = 1, \quad \text{ord}(f_V; \phi(z_0)) = m_0$$

and

$$\text{ord}(f_V; \phi(z_i)) = \text{ord}(f_U; \phi(z_i)) \quad \forall i = 1, \dots, \alpha_U - 1.$$

Then, we conclude that $U_\varepsilon := V \circ \phi^{-1} \in \mathcal{U}$ with

$$\|U - U_\varepsilon\|_{H^1(\mathbb{D})} \leq C\|U \circ \phi - V\|_{H^1(\mathbb{D})} \leq C\varepsilon_0,$$

for a constant $C > 0$ depending on conformal diffeomorphism ϕ , and hence on U . By the arbitrariness of ε_0 , the proof is concluded. $\square$

Remark 6.4. *The set $\mathcal{U}^\gamma$ is not open in $\mathcal{U}$. If $U \in \mathcal{U}^\gamma$ extends to a holomorphic function in a neighborhood of $\overline{\mathbb{D}}$ and has a critical point with multiplicity higher than 3 on the boundary of $\mathbb{D}$, then every neighborhood of U in $\mathcal{U}$ cannot be contained in $\mathcal{U}^\gamma$: e.g., any traslation moving the critical point from the boundary to the interior produces a function $V \notin \mathcal{U}^\gamma$.*

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