EQUI-DIMENSIONALIZATION VIA SUBDIVISION OF SIMPLICES

WATARU KAI

Abstract. We give an alternative proof of Suslin's equi-dimensionalization moving lemma using a different geometric construction. The new construction provides better control of the degrees of the polynomials describing the geometric procedure.

The new degree bound can be used to improve an earlier result of Hiroyasu Miyazaki and the present author on algebraic cycles *with modulus*: the isomorphism in question is now valid for any fixed divisor, without the need to take the limit over thickenings.

Contents

1. Introduction 1
2. Subdivision of simplices (fixing the notation) 4
3. Equi-dimensionalization 7
4. Proof of Suslin's quasi-isomorphism 13
5. The modulus variant 15
References 17

1. Introduction

Let X be an affine scheme of finite type over a field k. Suslin [14] showed how to render a given closed set $V \subset X \times \mathbb{A}^n$ by pullback along an appropriate X-morphism $X \times \mathbb{A}^n \to X \times \mathbb{A}^n$ (compatibly with a prescribed map $X \times Z \to X \times Z$ for a hypersurface $Z \subset \mathbb{A}^n$ if any such data is given) and make it equi-dimensional over $\mathbb{A}^n \setminus Z$.

By orchestrating this construction to form an endomorphism of the cosimplicial scheme $X \times \Delta^\bullet$ (where Δ^n is the algebraic n-simplex (1.3.1)), he showed that Bloch's higher Chow group is isomorphic to its equi-dimensional variant:

Theorem 1.1 (Suslin [14, Th. 2.1])**.** *Let X be an affine scheme of finite type over a field k and $t \geq 0$ be a non-negative integer. Then the inclusion*

$$z_t^{\mathsf{equi}}(X, \bullet) \to z_t(X, \bullet)$$

is a quasi-isomorphism. (See §1.3 for the definitions of these complexes.)

This result was used by Friedlander–Suslin [6] to prove that Bloch's higher Chow group is isomorphic to a version of motivic cohomology. Voevodsky [15] then established that this motivic cohomology is the same as the motivic cohomology they had defined earlier; see also the beautiful exposition in [11, Lect. 18A, 19].

Date: May 11, 2026.

2020 *Mathematics Subject Classification.* 14C25 (Primary), 14C15, 14F42, 19E15 (Secondary).

In these notes, we offer an alternative proof of Theorem 1.1 based on different constructions. The motivation for seeking new constructions is twofold. First, the identification of the higher Chow group and the motivic cohomology is such an important result that one could not have enough of different proofs thereof.

Second, Suslin's constructions do not give nice numerical bounds for the polynomials describing the endomorphisms on $X \times \Delta^\bullet$. If we write $X = \operatorname{Spec} A$, an X-endomorphism on $X \times \Delta^n \cong \operatorname{Spec} A[Y_1, \ldots, Y_n]$ corresponds to the choice of n elements $f_1^{(n)}, \ldots, f_n^{(n)}$ of $A[Y_1, \ldots, Y_n]$. If we demand that the endomorphisms form a cosimplicial map, then Suslin's construction produces $f_k^{(n)}$'s having degrees at least $n+1$ in the variables Y_j.

Also, when $X = \mathbb{A}^m = \operatorname{Spec} k[X_1, \ldots, X_m]$ is an affine space (to which case the proof is reduced), we can consider the degrees of $f_k^{(n)}$ in the variables X_i. These degrees heavily depend on the closed set V and his proof does not imply any uniform bound.

Our constructions are based on the classical idea known as *subdivision of simplices*. This involves $(n+1)!$ endomorphisms of $X \times \Delta^n$. (E.g. a triangle is divided into 6 small triangles by (barycentric) subdivision.) They do not form an endomorphism of the cosimplicial scheme $X \times \Delta^\bullet$ but with some simplicial maneuver, their signed sums do give rise to a map of cycle complexes. The counterparts to $f_k^{(n)}$'s above will have degree 1 in the variables Y_j because subdivision is written by affine-linear transformations of $\Delta^\bullet$.

When $X = \mathbb{A}^m = \operatorname{Spec} k[X_1, \ldots, X_m]$, it turns out that $f_k^{(n)}$ can be taken to have degrees as low as $n+1$ in the variables X_i. The advance from no uniform bound to this rather explicit one is in part due to an improvement of the proof rather than the construction. In Suslin's arguments, at the cost of letting $\deg_X(f_k^{(n)})$ soar, the proof of equi-dimensionality goes in a sense uniformly for all fibers. [1] In our treatment, more objects appearing in the discussion depend genuinely on the fiber. We have to respect their individual personalities and duly respond. The dedication is rewarded by the uniform bound of $\deg_X(f_k^{(n)})$.

We demonstrate that these new bounds of degrees can be useful by showing a *modulus* analog of Theorem 1.1:

Theorem 1.2 (Theorem 5.3)**.** *Let X be an affine scheme of finite type over a field k and $D \subset X$ be an effective Cartier divisor. Let $t \geq 0$ be a non-negative integer. Then the inclusion*

$$z_t^{\mathsf{equi}}(X|D, \bullet) \to z_t(X|D, \bullet)$$

is a quasi-isomorphism. (See §5 for the definitions of these complexes.)

Formerly, Hiroyasu Miyazaki and the present author [8] had proven a similar result (copying Suslin's very constructions) but we needed to consider the pro-objects on both sides associated with the infinitesimal thickenings mD ($m \geq 1$). [2]

---

[1] For the interested reader: in the proof of [14, Th. 1.1], he considers the fiber of $\Phi^{-1}(V) \subset \mathbb{A}^m \times \mathbb{A}^n$ over each point $y \in \mathbb{A}^n \setminus Z$ and eventually shows that the *infinite part* of the fiber $\Phi^{-1}(V)_y$ (the closure in $\mathbb{P}^m$, intersected with the hyperplane $\mathbb{P}^{m-1}$ at infinity) is contained in a closed subset W_0 (independent of y!) of dimension $\leq t-1$.

[2] In the earlier work, we treated the cubical version of the higher Chow group with modulus, but the same proof works for the simplicial version as well. The new result, without the need of thickenings,

1.3. **Recollection of the cycle complexes.** Let us briefly recall Bloch's higher Chow complex and its equi-dimensional variant from [2] [14]. Denote by

$$\Delta^n = \operatorname{Spec} k[t_0, \dots, t_n]/(t_0 + \cdots + t_n - 1) \tag{1.3.1}$$

the algebraic simplex over the base field k. For each injection $i\colon \{0, 1, \dots, n'\} \hookrightarrow \{0, 1, \dots, n\}$, there is a corresponding closed immersion $i\colon \Delta^{n'} \hookrightarrow \Delta^n$. An irreducible scheme V over Δ^n is said to satisfy the *face condition* if for each (ordered) inclusion $i\colon \Delta^{n'} \hookrightarrow \Delta^n$ as above, the fiber product $V \times_{\Delta^n} \Delta^{n'}$ has dimension $\leq \dim(V) - (n - n')$. When this is the case, the fiber product is either empty or purely of dimension $\dim(V) - (n - n')$.

Let t be an integer. For each $n \geq 0$, define $z_t(X, n)$ to be the free abelian group

$$z_t(X, n) := \mathbb{Z}\left[V \subset X \times \Delta^n \,\middle|\, \begin{matrix}\text{integral closed subscheme of dimension} \\ n + t \text{ satisfying the face condition}\end{matrix} \right].$$

Thanks to the face condition, the alternating sums of pullbacks along various ordered inclusions $\{0, \dots, n-1\} \hookrightarrow \{0, \dots, n\}$ give well-defined differential maps

$$\cdots \to z_t(X, n) \to z_t(X, n-1) \to \cdots .$$

This is Bloch's cycle complex $z_t(X, \bullet)$.

When $t \geq 0$, one can ask if V is equi-dimensional over Δ^n (i.e., dominant and the fibers have dimensions $\leq t$, or equivalently the non-empty fibers have pure dimension t). One thus obtains a subcomplex $z_t^{\mathsf{equi}}(X, \bullet) \subset z_t(X, \bullet)$.

1.4. **Organization of the notes.** In §2 we define the *subdivision maps* $\mathsf{sd}_n^\sigma\colon X \times \Delta^n \to X \times \Delta^n$ for each permutation $\sigma \in \mathfrak{S}_{\{0,1,\dots,n\}}$, and the *homotopy* maps $\mathsf{sd}_{n,k}^\sigma\colon X \times \Delta^{n+1} \to X \times \Delta^n \times \Delta^1$ for each choice of $0 \leq k \leq n$ and $\sigma \in \mathfrak{S}_{\{0,\dots,k\}}$. All these maps depend on the preliminary choice of *centers* $c^i\colon X \to \Delta^i$ for $i \leq n$.

Section 3 is the technical key of these notes, where we prove that if the choice of centers is generic, then pullback along subdivision renders a given cycle equi-dimensional.

In §4 we complete our proof of Theorem 1.1 using the subdivision construction.

In §5 we explain that the exact same proof yields the modulus analog (Theorem 1.2).

Acknowledgment. This work grew out of conversations with Ryomei Iwasa and Hiroyasu Miyazaki in 2019. I did not manage to write it up in a timely manner due to the addition of a new member to my household and a global sanitary crisis that followed. In the meantime I have been supported by JSPS KAKENHI grants (JP18K13382, JP22K13886 and JP26K06736) and the JSPS Overseas Research Fellowship (the class of 2020). I also thank the University of Milan for hospitality, where I was a long-term visitor in the years 2022–24 funded by the above-mentioned JSPS fellowship. Last but not least, I am deeply grateful to the referees for their careful reading and valuable comments, which led to significant improvements of the exposition.

---

is *only available for the simplicial* version because we failed to cook up subdivision of cubes and (importantly) homotopy by affine-linear maps.

2. Subdivision of simplices (fixing the notation)

Here we set up *subdivision maps*

$$\Delta^n \to \Delta^n$$

which are determined by the choice of the center and a permutation of $\{0, 1, \ldots, n\}$. We also have to introduce the homotopy maps

$$\Delta^{n+1} \to \Delta^n \times \Delta^1.$$

A similar (and more complicated) construction has been used in the proof of the localization property of the higher Chow group [3] [10].

2.1. **Simplices.** Endow the set $[n] = \{0, \ldots, n\}$ with the order $0 < \cdots < n$. Recall $\Delta^n := \mathrm{Spec}(k[t_0, \ldots, t_n]/(t_0 + \cdots + t_n - 1))$ is the algebraic n-simplex. Denote by $v_i\colon \mathrm{Spec}(k) = \Delta^0 \to \Delta^n$ the i^{th} vertex defined by the k-algebra map $t_j \mapsto 0$ for $j \in \{0, \ldots, n\} \setminus \{i\}$ and $t_i \mapsto 1$.

2.2. **The centers.** Let A be a k-algebra. Suppose that we are given a section $c^n \in \Delta^n(A)$ for each $n \geq 0$. They will serve as the center of subdivision. For a non-empty subset $S \subset [n]$ consisting of $s + 1$ elements, consider the corresponding embedding $i_S\colon \Delta^s \hookrightarrow \Delta^n$. Denote by

$$c^n_S \in \Delta^n(A)$$

the image of $c^s \in \Delta^s(A)$ under this embedding. If S is a singleton $\{i\}$, then $c^n_{\{i\}}$ equals the structure map $\mathrm{Spec}(A) \to \mathrm{Spec}(k)$ followed by the i^{th} vertex $v_i\colon \mathrm{Spec}(k) \to \Delta^n$.

2.3. **The subdivision maps.** For a finite set T, denote by $\mathfrak{S}_T$ the group of permutations on T. For each permutation $\sigma \in \mathfrak{S}_{[n]} = \mathfrak{S}_{\{0,\ldots,n\}}$, we consider the *subdivision* map (a morphism of A-schemes determined by the choice of centers $\{c^i\}_{i \leq n}$)

$$\mathsf{sd}^\sigma_n\colon \Delta^n_A \to \Delta^n_A,$$

defined to be the affine-A-linear map sending the vertex v_k $(0 \leq k \leq n)$ to $c^n_{\sigma(\{0,\ldots,k\})}$. Concretely, this is the restriction of the map between the ambient spaces

$$\mathbb{A}^{n+1}_A \to \mathbb{A}^{n+1}_A$$

corresponding to the A-linear map $A^{n+1} \to A^{n+1}$ sending $(t_0, \ldots, t_n)$ to $\sum_{k=0}^n t_k c^n_{\sigma(\{0,\ldots,k\})}$. See Figures 1, 2.

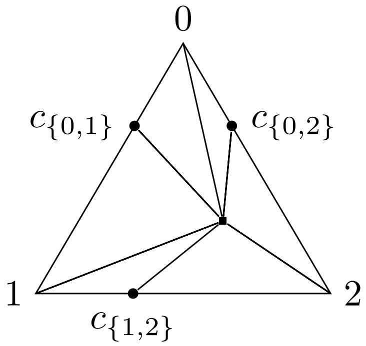

FIGURE 1. Subdivision $n = 2$

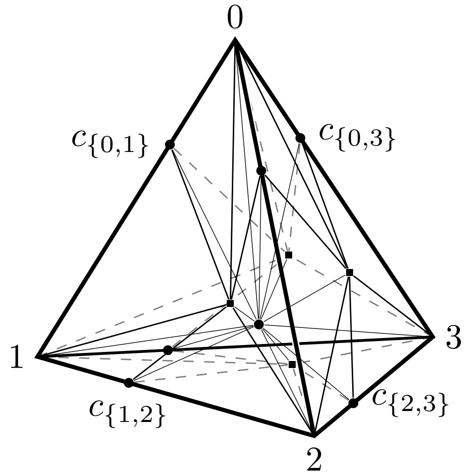

FIGURE 2. Subdivision $n = 3$. There are $4! = 24$ tetrahedra.

The following relations of maps $\Delta^{n-1} \rightrightarrows \Delta^n$ hold:

$$\mathsf{sd}_n^\sigma \circ \partial_i = \mathsf{sd}_n^{\sigma\circ(i,i+1)} \circ \partial_i \qquad (0 \le i \le n-1), \tag{2.3.1}$$
$$\mathsf{sd}_n^\sigma \circ \partial_n = \partial_{\sigma(n)} \circ \mathsf{sd}_{n-1}^{\partial^{-1}_{\sigma(n)}\circ\sigma},$$

where $\partial^{-1}_{\sigma(n)} \circ \sigma$ denotes the permutation on $[n-1]$ obtained as the composite $[n-1] \xrightarrow{\sigma} [n] \setminus \{\sigma(n)\} \xleftarrow[\cong]{\partial_{\sigma(n)}} [n-1]$ from the left to the right. For each n, σ, if we choose appropriate linear coordinates on the source and target (both of which are non-canonically isomorphic to $\mathbb{A}^n_A$), the map sd_n^σ can be written as a morphism

$$\begin{array}{ccc} \mathbb{A}^n_A = \operatorname{Spec}(A[Y_1, \dots, Y_n]) & \to & \mathbb{A}^n_A = \operatorname{Spec}(A[Z_1, \dots, Z_n]) \\ \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} & \mapsto & \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} := \begin{pmatrix} C_1^1 & \cdots & C_1^n \\ & \ddots & \vdots \\ & & C_n^n \end{pmatrix} \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \end{array} \tag{2.3.2}$$

where $C^i_j \in A$ (which also depend on the choice of centers) and the blank entries in the lower half are zero. For example, we may choose v_0 as the origin of the source and the vectors $v_i - v_0$ as the coordinate axes $(1 \le i \le n)$. On the target, we take $v_{\sigma(0)}$ as the origin and $v_{\sigma(i)} - v_{\sigma(0)}$ as the axes $(1 \le i \le n)$.

2.4. **The homotopy.** Given $0 \le k \le n$ and a bijection σ on $\{0, \dots, k\}$, we consider the subdivision map (Figure 3)

$$\mathsf{sd}^\sigma_{n,k} \colon \Delta^{n+1}_A \to \Delta^n_A \times_A \Delta^1_A$$

which is the affine-linear map sending the vertices

$$v_0, \; \dots \;, v_k \qquad \text{to} \qquad (c_{\{\sigma(0)\}}, v_0), \dots, (c_{\sigma(\{0,\dots,k\})}, v_0)$$

and the vertices

$$v_{k+1}, \; \dots \;, v_{n+1} \qquad \text{to} \qquad (v_k, v_1) \; \dots \;, (v_n, v_1).$$

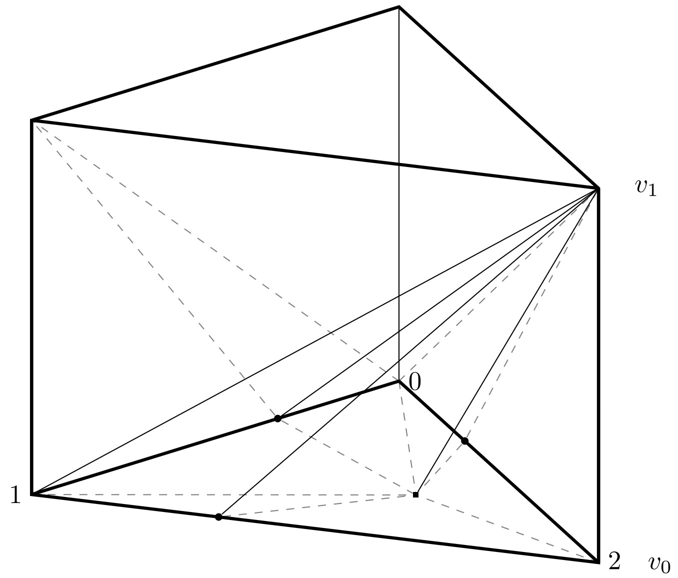

FIGURE 3. Homotopy for $n = 2$. There are $1! + 2! + 3! = 9$ tetrahedra.

Let $\mathrm{pr}_1 \colon \Delta^n \times \Delta^1 \to \Delta^n$ be the first projection. The collection of maps

$$\mathrm{pr}_1 \circ \mathsf{sd}^{\sigma}_{n,k} \colon \Delta^{n+1}_A \rightrightarrows \Delta^n_A$$

will serve as a homotopy later on. It takes some patience to verify the following relations of maps $\Delta^n \rightrightarrows \Delta^n \times \Delta^1$ (which, by the way, imply the relations (2.3.1)):

(2.4.3)

$$\begin{aligned}
\mathsf{sd}^{\mathrm{id}}_{n,0} \circ \partial_0 &= (\mathrm{id}, \mathrm{const}_{v_1}) && \text{(where } \mathrm{const}_{v_1} \text{ is the constant map } \Delta^n \to \Delta^1 \text{ to } v_1\text{);} \\
\mathsf{sd}^{\sigma}_{n,n} \circ \partial_{n+1} &= (\mathsf{sd}^{\sigma}_n, \mathrm{const}_{v_0}) && (\sigma \in \mathfrak{S}_{[n]}); \\
\mathsf{sd}^{\sigma}_{n,k} \circ \partial_i &= \mathsf{sd}^{\sigma \circ (i,i+1)}_{n,k} \circ \partial_i && (0 \le i < k,\ \sigma \in \mathfrak{S}_{[k]}); \\
\mathsf{sd}^{\sigma}_{n,k} \circ \partial_i &= (\partial_{i-1} \times \mathrm{id}_{\Delta^1}) \circ \mathsf{sd}^{\sigma}_{n-1,k} && (k+1 < i \le n, \sigma \in \mathfrak{S}_{[k]}); \\
\mathsf{sd}^{\sigma}_{n,k} \circ \partial_{k+1} &= \mathsf{sd}^{\sigma}_{n,k+1} \circ \partial_{k+1} && (0 \le k < n, \sigma \in \mathfrak{S}_{[k]}); \\
\mathsf{sd}^{\sigma}_{n,k} \circ \partial_k &= (\partial_{\sigma(k)} \times \mathrm{id}_{\Delta^1}) \circ \mathsf{sd}^{\partial^{-1}_{\sigma(k)} \circ \sigma}_{n-1,k-1} && (1 \le k \le n, \sigma \in \mathfrak{S}_{[k]}),
\end{aligned}$$

where $\partial^{-1}_{\sigma(k)} \circ \sigma \in \mathfrak{S}_{[k-1]}$ is the unique element τ satisfying $\partial_{\sigma(k)}(\tau(j)) = \sigma(j)$ for all $0 \le j \le k-1$. In the second to last formula, we wrote σ also for the permutation on $[k+1]$ acting on the subset $[k]$ by σ and fixing $k+1$.

For each n, k, σ, under appropriate choice of linear coordinates (uniform in the choices of the centers), the morphism $\mathrm{pr}_1 \circ \mathsf{sd}^{\sigma}_{n,k}$ can be written as

$$\mathbb{A}^{n+1}_A = \operatorname{Spec}(A[Y_1, \dots, Y_{n+1}]) \quad \to \quad \mathbb{A}^n_A = \operatorname{Spec}(A[Z_1, \dots, Z_n]) \tag{2.4.4}$$

$$\begin{pmatrix} Y_1 \\ \vdots \\ \vdots \\ \vdots \\ Y_{n+1} \end{pmatrix} \mapsto \begin{pmatrix} Z_1 \\ \vdots \\ \vdots \\ Z_n \end{pmatrix} := \left(\begin{array}{ccc|c|ccc} C^1_1 & \cdots & C^k_1 & & & & \\ & \ddots & \vdots & 1 & & & \\ & & C^k_k & & & & \\ \hline & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{array}\right) \begin{pmatrix} Y_1 \\ \vdots \\ \vdots \\ \vdots \\ Y_{n+1} \end{pmatrix}$$

where $C^i_j \in A$ also depend on the centers, the blank entries are zero and the "1" in the $(k+1)^{\mathrm{st}}$ column is placed in the $\sigma^{-1}(k)^{\mathrm{th}}$ row. An example of such a choice is to take v_0 as the origin of the source Δ^{n+1}_A and the vectors $v_i - v_0 \in \Delta^{n+1}(A)$ as its coordinate axes ($1 \le i \le n+1$); on the target Δ^n_A, take $v_{\sigma(0)}$ as the origin and the vectors $v_{\sigma(i)} - v_{\sigma(0)}$ ($1 \le i \le k$), $v_i - v_{\sigma(0)}$ ($k+1 \le i \le n$) as the axes.

3. Equi-dimensionalization

We work over an *infinite* field k. We set the base ring A to be the polynomial ring $k[X_1, \dots, X_m]$. Products "$\times$" of schemes will mean the fiber products over k. The main purpose of this section is to show that given a closed subset $V \subset \operatorname{Spec} k[X_1, \dots, X_m] \times \Delta^n$ of dimension $\le n+t$, then for a generic choice of the center the inverse image by the subdivision map $(\mathsf{sd}^{\sigma}_n)^{-1}(V) \subset \operatorname{Spec} k[X_1, \dots, X_m] \times \Delta^n$ has fiber dimensions $\le t$ over Δ^n (Corollary 3.7).

3.1. **Affine-linear maps of vector spaces.** Recall that an *affine-linear* map $V_1 \to V_2$ between vector spaces (over some field) is the composite of a linear map and translation on the target. An *affine-linear subset* of a vector space is the image of an affine-linear map. It can be written as $\boldsymbol{v} + W$ with $\boldsymbol{v}$ an element and W a (uniquely determined) linear subspace. The *dimension* and *codimension* of an affine-linear subset is defined to be those of W. (In our treatment, it might be more consistent to include the empty subset as an affine-linear subset, but this is of minor importance.) Suppose an affine-linear map as above has image with dimension $\ge c$. Then the inverse image of any element has codimension $\ge c$ in the source, because it is either empty or an affine-linear subset having codimension equaling the dimension of the image.

Lemma 3.2. *Let $W \subset \mathbb{A}^m \times \mathbb{A}^n = \operatorname{Spec}(k[X_1, \dots, X_m; Z_1, \dots, Z_n])$ be an irreducible closed subscheme of dimension $\ge n$. Consider the following affine-linear map of vector spaces, where the source denotes the vector space of polynomials of degree $\le N$:*

$$\begin{aligned} k[X_1, \dots, X_m]_{\le N} &\to \Gamma(W, \mathcal{O}_W) \\ C(\underline{X}) &\mapsto Z_1 - C(\underline{X}). \end{aligned}$$

Then the inverse image of $0 \in \Gamma(W, \mathcal{O}_W)$ has codimension $\ge N+1$ on the source.

Proof. If the image of W to $\mathbb{A}^m$ is a (necessarily closed) point x, then for a dimension reason we see that the support of W is $\mathbb{A}^n_{k(x)}$ and that the inverse image is empty.

Next, suppose that the image of $W \to \mathbb{A}^m$ has dimension ≥ 1. By §3.1, it suffices to show that the image of the restriction map $k[X_1, \dots, X_m]_{\leq N} \to \Gamma(W, \mathcal{O}_W)$ has dimension $\geq N+1$. Since the image of $W \to \mathbb{A}^m$ now has dimension ≥ 1 and since the base field is infinite, we can find a hyperplane in $\mathbb{A}^m$ not containing the image of W. It follows that there is a linear form $l(\underline{X}) \in k[X_1, \dots, X_m]_1$ mapping to an element θ in the ring $\Gamma(W, \mathcal{O}_W)$ which is transcendental over k. Then the elements $1, l(\underline{X}), \dots, l(\underline{X})^N \in k[\underline{X}]_{\leq N}$ map to $1, \theta, \dots, \theta^N \in \Gamma(W, \mathcal{O}_W)$ which are linearly independent over k. This completes the proof. □

3.3. **Subdivision with respect to the universal center.** When we later actuate the moving machinery, we will want to choose centers of subdivision $c^i(\underline{X}) \in \Delta^i(k[X_1, \dots, X_m])$ whose coordinates are represented by polynomials of degree $\leq N$ for some integer N. As is always the case with moving lemmas, provided N is large enough, any generic choice of c^i will serve our purposes. However, to discuss which closed subset our "generic" centers have to avoid (and how large N has to be), we must consider the universal situations first.

Let $\mathbb{A}_{\leq N}$ be the affine space parametrizing polynomials in variables $X_1, \dots, X_m$ of degree $\leq N$. It has dimension $\frac{(N+m)!}{N!m!}$.

Let $p \geq 0$ be an integer and $\Lambda \subset \{1, \dots, p\} \times \{1, \dots, n\}$ be a set of indices. For pairs (i, j) not in Λ, suppose we are given constant polynomials

$$C^i_j(\underline{X}) := C^i_j \in k \quad \text{for } (i, j) \notin \Lambda.$$

Given these data, we can consider the morphism

$$\begin{aligned} \mathsf{Sd}\colon (\mathbb{A}_{\leq N})^\Lambda \times \mathbb{A}^m \times \mathbb{A}^p &\to \mathbb{A}^m \times \mathbb{A}^n \\ (\{C^i_j(\underline{X})\}_{(i,j)\in\Lambda}, \underline{X}, \underline{Y}) &\mapsto (\underline{X}, \left(\sum_{i=1}^p C^i_j(\underline{X}) Y_i\right)_{1\leq j\leq n}). \end{aligned}$$

When a rational point $\{C^i_j(\underline{X})\}_{(i,j)\in\Lambda} \in (\mathbb{A}_{\leq N})^\Lambda(k)$ is given, write

$$\mathsf{sd}_{\{C^i_j\}} := \mathsf{Sd}(\{C^i_j(\underline{X})\}, -, -)\colon \mathbb{A}^m \times \mathbb{A}^p \to \mathbb{A}^m \times \mathbb{A}^n.$$

We saw in (2.3)(2.4) that under appropriate choice of linear coordinates on simplices $\Delta^\bullet$, the morphisms sd^σ_n and $\mathrm{pr}_1 \circ \mathsf{sd}^\sigma_{n,k}$ are equal to $\mathsf{sd}_{\{C^i_j\}}$ for appropriate choices of Λ and $C^i_j(\underline{X})$. (Recall that we now set $A = k[X_1, \dots, X_m]$, which amounts to $\mathbb{A}^m$.)

3.4. **Bad locus.** Now suppose we are given a closed subset $V \subset \mathbb{A}^m \times \mathbb{A}^n$ of dimension $\leq n + t$. We want to argue that $\mathsf{sd}^{-1}_{\{C^i_j\}}(V) \subset \mathbb{A}^m \times \mathbb{A}^p$ has fibers of dimension $\leq t$ over $\mathbb{A}^p$ for a generic choice of $\{C^i_j(\underline{X})\}$. For given constants $C^i_j \in k$ (for each $(i, j) \notin \Lambda$), define the "bad locus" on which this condition is not true:

(3.4.1)

$$B(V) := \left\{ \alpha \in (\mathbb{A}_{\leq N})^\Lambda \times \mathbb{A}^p \;\middle|\; \begin{array}{c} \text{the fiber} \\ \mathsf{Sd}^{-1}(V)_\alpha \subset \mathbb{A}^m_{k(\alpha)} \\ \text{has dimension} > t \end{array} \right\} \subset (\mathbb{A}_{\leq N})^\Lambda \times \mathbb{A}^p.$$

It is a constructible subset (a consequence of e.g. [4, IV$_1$, 1.8.4 and IV$_3$, 13.1.3]). Note that if the first component of α is $\{C^i_j(\underline{X})\}_{(i,j)\in\Lambda} \in (\mathbb{A}_{\le N})^\Lambda(k(\alpha))$, then $\mathsf{Sd}^{-1}(V)_\alpha = \mathsf{sd}^{-1}_{\{C^i_j\}}(V \otimes k(\alpha))$.

3.5. **A pointwise argument.** The following assertion is useful when estimating the dimension of a constructible subset (seen as a topological space by the subset topology).

Let $f\colon X \to Y$ be a map of finite-type k-schemes and $C \subset X$ be a constructible subset. If the fiber C_y has codimension $\ge c$ in X_y for all $y \in Y$, then C has codimension $\ge c$ in X.

This follows for example from the fact that the dimension of a constructible subset of a finite-type k-scheme is the maximum of the transcendental degrees of its points (which is true for locally open subsets hence for constructible subsets of such schemes). When we apply it, the map f is often a projection $X \times Z \to Z$.

Proposition 3.6. *Let $V \subset \mathbb{A}^m \times \mathbb{A}^n$ be a closed subset of dimension $\le n+t$ with some $t \ge 0$. Suppose a $\Lambda \subset \{1,\dots,p\}\times\{1,\dots,n\}$ and $C^i_j \in k$ for $(i,j) \notin \Lambda$ have been chosen and define the constructible subset $B(V) \subset (\mathbb{A}_{\le N})^\Lambda \times \mathbb{A}^p$ by* (3.4.1). *Let $y \in \mathbb{A}^p_k$ be a point and write $(y_1,\dots,y_p) \in \mathbb{A}^p(k(y))$ for its coordinates. Assume that either of the following is true:*

(i) *for all $1 \le j \le n$, there exists $1 \le i \le p$ such that $y_i \ne 0$ and $(i,j) \in \Lambda$; or*

(ii) *the projection $V \to \mathbb{A}^n$ has fiber dimensions $\le t$.*

Then the fiber $B(V)_y := B(V) \times_{\mathbb{A}^p} \{y\}$ has codimension $\ge N+1$ in $(\mathbb{A}_{\le N})^\Lambda_{k(y)}$.

Proof. Base-change to $k(y)$ does not affect Assumptions (i)(ii) and the conclusion. Thus, making this base-change, we may assume y is a k-rational point.

For a point $\alpha = (\{C^i_j(\underline{X})\}_{i,j}, (y_1,\dots,y_p)) \in (\mathbb{A}_{\le N})^\Lambda \times \mathbb{A}^p$, the fiber $\mathsf{Sd}^{-1}(V)_\alpha$ can be obtained by intersecting

$$V_{k(\alpha)} \quad \text{in} \quad \mathbb{A}^m_{k(\alpha)} \times_{k(\alpha)} \mathbb{A}^n_{k(\alpha)} = \operatorname{Spec}(k(\alpha)[X_1,\dots,X_m; Z_1,\dots,Z_n])$$

with hypersurfaces

$$H_j := \{Z_j = \sum_{i=1}^p C^i_j(\underline{X}) y_i\} \tag{3.6.2}$$

($j = 1,\dots,n$) and projecting it to $\mathbb{A}^m_{k(\alpha)}$ (the projection is then automatically a closed immersion). Thus our goal is to bound $\dim\left(V_{k(\alpha)} \cap \bigcap_{j=1}^n H_j\right)$ by t except when $\{C^i_j(\underline{X})\}_{(i,j)\in\Lambda}$ is in some codimension $\ge N+1$ subset of $(\mathbb{A}^{\le N})^\Lambda$.

Let J be the set of those j's such that the set of indices

$$\Lambda(y,j) := \Lambda \cap (\{1 \le i \le p \mid y_i \ne 0\} \times \{j\}) \tag{3.6.3}$$

is empty. For a given j, this is equivalent to the condition that the functions $C^i_j(\underline{X})y_i$ on $(\mathbb{A}_{\le N})^\Lambda$ are constants: $C^i_j y_i \in k$ for all $1 \le i \le p$. When this is the case, the hypersurface $H_j \subset \mathbb{A}^m_{k(\alpha)} \times_{k(\alpha)} \mathbb{A}^n_{k(\alpha)}$ from (3.6.2) is the pullback of a hyperplane $H_{j,0} \subset \mathbb{A}^n$.

Suppose J is non-empty, i.e., Assumption (i) fails, so that Assumption (ii) holds. Regardless of the choice of $C^i_j(\underline{X}) \in \mathbb{A}_{\le N}$ for $(i,j) \in \Lambda \cap (\{1,\dots,p\} \times J)$ (all defined over some overfield $k(y) \subset \Omega$), the following intersection of subsets from (3.6.2):

$$H_J := \bigcap_{j \in J} H_j$$

is the pull-back of the linear subscheme $H_{J,0} = \bigcap_{j \in J} H_{j,0} \subset \mathbb{A}^n$ of codimension $|J|$. So we may compute the intersection $V \cap H_J$ in $\mathbb{A}^m \times \mathbb{A}^n$ as

$$V \cap H_J = V \times_{\mathbb{A}^n} H_{J,0},$$

which has dimension $\leq \dim(H_{J,0}) + t$, because Assumption (ii) says V has fiber dimensions $\leq t$ over $\mathbb{A}^n$. Since the intersection we are interested in is

$$V_{k(\alpha)} \cap \bigcap_{j=1}^{n} H_j = \left(V_{k(\alpha)} \cap H_J\right) \cap \bigcap_{j \notin J} H_j,$$

we can make the following replacements

$$\begin{aligned} \{1, \dots, n\} &\rightsquigarrow \{1, \dots, n\} \setminus J \\ \mathbb{A}^m \times \mathbb{A}^n &\rightsquigarrow \mathbb{A}^m \times H_{J,0} (\cong \mathbb{A}^m \times \mathbb{A}^{n-|J|}) \\ V &\rightsquigarrow V \cap H_J \\ \Lambda &\rightsquigarrow \Lambda \setminus (\{1, \dots, p\} \times J) \end{aligned}$$

to reduce the problem to the case where J is empty, i.e., Assumption (i) is satisfied. Here we used the fact recalled in §3.5 that in order to bound the codimension of $B(V)_y$ in $(\mathbb{A}_{\leq N})^{\Lambda}_{k(y)}$ from below, it suffices to give a lower bound of the codimension of each fiber of

$$B(V)_y \to (\mathbb{A}_{\leq N})^{\Lambda \cap (\{1, \dots, p\} \times J)}_{k(y)}$$

in $(\mathbb{A}_{\leq N})^{\Lambda \setminus (\{1, \dots, p\} \times J)}_{k(y)}$.

The rest of this proof is an adaptation of a step in the Friedlander–Lawson moving lemma [5, proof of Prop. 2.2, especially p.107].

From now on we assume Assumption (i) holds. Let us examine the condition that a tuple $\{C^i_j(\underline{X})\}_{i,j} \in (\mathbb{A}_{\leq N})^{\Lambda}(\Omega)$, with Ω an overfield of k, belongs to the bad locus $B(V)_y$. Below, we shall often omit the base-change to Ω from notation.

The condition $\{C^i_j(\underline{X})\}_{i,j} \in B(V)_y$ implies (though not necessarily equivalent to) that there is $1 \leq j_0 \leq n$ such that the following two conditions hold:

(I-j_0) the intersection $V_{j_0 - 1} := V \cap \bigcap_{j=1}^{j_0 - 1} H_j$ in $\mathbb{A}^m \times \mathbb{A}^n$ has dimension $n + t - (j_0 - 1)$ (which is necessarily positive); and

(II-j_0) the intersection $V_{j_0 - 1} \cap H_{j_0}$ still has the same dimension, i.e., H_{j_0} contains an irreducible component of $V_{j_0 - 1}$.

If $V_{j_0 - 1} = \bigcup_{\omega} V^{\omega}_{j_0 - 1}$ is the decomposition into reduced irreducible components, Condition (II-j_0) is equivalent to that the function $Z_{j_0} - \sum_{i=1}^{p} C^i_j(\underline{X}) y_i$ restricts to the zero element on one of $V^{\omega}_{j_0 - 1}$.

Denote by $B(V)^{j_0}_y \subset (\mathbb{A}_{\leq N})^{\Lambda}$ the constructible set of tuples satisfying (I-j_0)(II-j_0). We have

$$B(V)_y \subset \bigsqcup_{j_0 = 1}^{n} B(V)^{j_0}_y.$$

Hence it suffices to prove that $B(V)^{j_0}_y$ has codimension $\geq N + 1$ in $(\mathbb{A}_{\leq N})^{\Lambda}$. Note that the conditions (I-j_0)(II-j_0) do not depend on the choice of $C^i_j(\underline{X})$ for indices $j > j_0$. By §3.5, it suffices to show that for each choice of $C^i_j(\underline{X})$ $(j < j_0)$ satisfying (I-j_0), the set of tuples $\{C^i_{j_0}(\underline{X})\}_{(i,j_0)\in\Lambda(y,j_0)}$ satisfying (II-j_0) form a constructible subset of $(\mathbb{A}_{\leq N})^{\Lambda(y,j_0)}$ having codimension $\geq N+1$, where the set of indices $\Lambda(y, j_0)$ is as defined in (3.6.3).

By Assumption (i), the set $\Lambda(y, j_0)$ is non-empty. In particular, the affine-linear map

$$(k[X_1, \dots, X_m]_{\leq N})^{\Lambda(y,j_0)} \to k[X_1, \dots, X_m]_{\leq N}$$
$$\{C^i_{j_0}(\underline{X})\}_{(i,j_0)\in\Lambda(y,j_0)} \mapsto \sum_{i=1}^{p} C^i_{j_0}(\underline{X})y_i$$

is surjective. (It is not necessarily a linear map because $C^i_{j_0}$ with $(i, j_0) \notin \Lambda$ are prescribed constants, so that the zero element may be mapped to a non-zero constant.) By this surjectivity and Lemma 3.2 for $W := V^\omega_{j_0-1}$, the inverse image of zero along the affine-linear map

$$(k[X_1, \dots, X_m]_{\leq N})^{\Lambda(y,j_0)} \to \Gamma(V^\omega_{j_0-1}, \mathcal{O}_{V^\omega_{j_0-1}})$$
$$\{C^i_{j_0}(\underline{X})\}_{(i,j_0)\in\Lambda(y,j_0)} \mapsto Z_{j_0} - \sum_{i=1}^{p} C^i_{j_0}(\underline{X})y_i$$

has codimension $\geq N+1$ in the source for all ω. This verifies that $B(V)^{j_0}_y$ has codimension $\geq N+1$ in $(\mathbb{A}_{\leq N})^{\Lambda(y,j_0)}$ and completes the proof of Proposition 3.6. $\square$

Corollary 3.7. *Let $V \subset \mathbb{A}^m \times \Delta^n$ be a closed subset of pure dimension $n + t$ for some $t \geq 0$, satisfying the face condition with respect to the faces of Δ^n. Let $N \geq n+1$. Then for a generic choice of centers $c^i \in \mathrm{Hom}_k(\mathbb{A}^m, \Delta^i)$ of degrees $\leq N$, $i = 1, \dots, n$, the subdivision maps in* (2.3)(2.4) *satisfy the following.*

(i) *For all $\sigma \in \mathfrak{S}_{[n]}$, the closed subset $(\mathsf{sd}^\sigma_n)^{-1}(V) \subset \mathbb{A}^m \times \Delta^n$ is equi-dimensional over Δ^n (of fiber dimension t).*

(ii) *Suppose V is equi-dimensional over Δ^n. Then for all $1 \leq k \leq n$ and $\sigma \in \mathfrak{S}_{[k]}$, the closed subset $(\mathrm{pr}_1 \circ \mathsf{sd}^\sigma_{n,k})^{-1}(V) \subset \mathbb{A}^m \times \Delta^{n+1}$ is equi-dimensional over Δ^{n+1}.*

Proof. It suffices to treat individual σ (and k) because the conjunction of finitely many conditions which are true for a generic choice is again true for a generic choice. Denote by $\mathcal{A}$ the (finite-dimensional) affine space parametrizing the choice of centers $c^i \in \mathrm{Hom}_k(\mathbb{A}^m, \Delta^i)$ of degrees $\leq N$ $(1 \leq i \leq n)$. It suffices to show that the bad locus $B \subset \mathcal{A} \times \Delta^p$ defined as in (3.4.1) (where $p = n$ or $n+1$) has codimension $\geq N+1$, because then its image into $\mathcal{A}$ will have codimension $\geq N+1-p \geq 1$.

This being said, (ii) is obviously a particular case of Proposition 3.6(ii).

Let us consider (i). Let $y \in \Delta^n$ be any point. By the pointwise argument in §3.5, it suffices to show $B(V)_y \subset \mathcal{A}_y$ has codimension $\geq N$. Take coordinate systems on the source and target as in (2.3.2). Let $y = (y_1, \dots, y_n) \in \mathbb{A}^n(k(y))$ (on the source) be the coordinate of y with respect to this coordinate system and let n_0 be the greatest index i with $y_i \neq 0$. We may regard y as the point $(y_1, \dots, y_{n_0})$ of $\mathbb{A}^{n_0}$

and our map may be regarded as an endomorphism on $\mathbb{A}^m \times \mathbb{A}^{n_0}$ given by

$$(\underline{X}, \begin{pmatrix} Y_1 \\ \vdots \\ Y_{n_0} \end{pmatrix}) \quad \mapsto \quad (\underline{X}, \begin{pmatrix} Z_1 \\ \vdots \\ Z_{n_0} \end{pmatrix} := \begin{pmatrix} C_1^1(\underline{X}) & \cdots & C_1^{n_0}(\underline{X}) \\ & \ddots & \vdots \\ & & C_{n_0}^{n_0}(\underline{X}) \end{pmatrix} \begin{pmatrix} Y_1 \\ \vdots \\ Y_{n_0} \end{pmatrix})$$

Since this $\mathbb{A}^{n_0}$ corresponds to a face $\Delta^{n_0} \hookrightarrow \Delta^n$, we may replace the problem with that for $V \times_{\Delta^n} \Delta^{n_0}$ which falls into Case (i) of Proposition 3.6 (with $i = n_0$ fulfilling the assumption for all $1 \le j \le n_0$). $\square$

In application, this Corollary 3.7 is supplemented by the following assertion:

Lemma 3.8. *Let $V \subset \mathbb{A}^m \times \Delta^n$ be a closed subset of pure dimension $n + t$ for some $t \ge 0$ satisfying the face condition. Let $N \ge 0$ be any integer. Then for a generic choice of centers $c^i \in \mathrm{Hom}_k(\mathbb{A}^m, \Delta^i)$ of degrees $\le N$, $1 \le i \le n$, the closed subset*

$$(\mathrm{pr}_1 \circ \mathsf{sd}^{\sigma}_{n,k})^{-1}(V) \subset \mathbb{A}^m \times \Delta^{n+1}$$

has dimension $n + t + 1$ and satisfies the face condition.

Proof. The following proof is routine but tedious to write down, reflecting the fact that the map $\mathsf{sd}^{\sigma}_{n,k}$ restricted to the faces can have complicated form. As in the proof of Corollary 3.7, it suffices to treat individual k and σ. Let $\mathcal{A}$ be the finite-dimensional affine space parametrizing the choice of centers. The choice of linear coordinates on Δ^n gives an isomorphism $\mathcal{A} \cong (\mathbb{A}_{\le N})^{\Lambda}$ with $\Lambda = \{(i, j) \mid 1 \le i \le j \le n\}$.

As in §3.3, the map $\mathsf{sd}^{\sigma}_{n,k}$ is the restriction to the point $\{c^i\}_{1 \le i \le n} \in \mathcal{A}$ of a map:

$$\mathsf{Sd}^{\sigma}_{n,k} \colon \mathcal{A} \times \mathbb{A}^m \times \Delta^{n+1} \to \mathbb{A}^m \times \Delta^n.$$

We know that $\mathsf{Sd}^{\sigma}_{n,k}$ can be written in the form (2.4.4). Faces of Δ^{n+1} and Δ^n correspond to equations $\{Y_i = 0\}$ and $\{Y_1 + \cdots + Y_{n+1} = 1\}$, and $\{Z_j = 0\}$ and $\{Z_1 + \cdots + Z_n = 1\}$ respectively. From this, restriction of $\mathsf{Sd}^{\sigma}_{n,k}$ to a face $\mathcal{A} \times \mathbb{A}^m \times \Delta^{p_0}$ of the source has the following form.

$$F \colon (\mathbb{A}_{\le N})^{\Lambda} \times \mathbb{A}^m \times \Delta^{p_0} \to \mathbb{A}^m \times \Delta^n$$

$$(\{C^i_j(\underline{X})\}_{i,j}, \underline{x}, \underline{y}) \mapsto (\underline{x}, \left(\begin{array}{cc|c} \bar{C}^1_1(\underline{x}) \cdots & \bar{C}^{k_0}_1(\underline{x}) & \\ \vdots & \vdots & * \\ \bar{C}^1_{\lambda(1)}(\underline{x}) \ddots & & \\ & \bar{C}^{k_0}_{\lambda(k_0)}(\underline{x}) & \\ \hline & & I \end{array}\right) \begin{pmatrix} y_1 \\ \vdots \\ y_{p_0} \end{pmatrix})$$

where:

- $1 \le k_0 \le k$ and $1 \le \lambda(1) \le \cdots \le \lambda(k_0) \le n$ are some integers;
- $\bar{C}^{\alpha}_{\beta}(\underline{X})$ are certain (linearly independent) k-linear combinations of $C^i_j(\underline{X})$'s;
- entries denoted by "$*$" are some functions on $(\mathbb{A}_{\le N})^{\Lambda}$;
- I is a matrix each of whose rows and columns has at most one non-zero entry, and whose non-zero entries are all 1.

Therefore it suffices to prove the following property, which we prove by induction on p_0:

(**P**(p_0, n, V)) Let $V \subset \mathbb{A}^m \times \Delta^n$ be a closed subset of dimension $n+t$ ($t \geq 0$) satisfying the face condition. Then the inverse image of V along any map F from $(\mathbb{A}_{\leq N})^\Lambda \times \mathbb{A}^m \times \Delta^{p_0}$ as above has codimension $\geq m-t$ on the source.

This condition implies that a generic fiber of the projection $F^{-1}(V) \to (\mathbb{A}_{\leq N})^\Lambda$ has codimension $\geq m-t$ in $\mathbb{A}^m \times \Delta^{p_0}$ (base-changed to the residue field of the chosen point of $(\mathbb{A}_{\leq N})^\Lambda$). In our original setting, this says that the subset

$$\Big((\mathrm{pr}_1 \circ \mathsf{sd}^\sigma_{n,k})^{-1}(V)\Big) \times_{\Delta^{n+1}} \Delta^{p_0} \quad \subset \quad \mathbb{A}^m \times \Delta^{p_0}$$

has codimension $\geq m-t$ for a generic $\{c^i\} \in \mathcal{A}$. This is precisely the face condition for this face.

Moving on to the proof of Property **P**(p_0, n, V), let J be the set of $j \in \{1, \dots, n\}$ such that the j^{th} row of the above matrix is zero. Set $n_0 := n - |J|$. Then the map F lands in the face $\bigcap_{j \in J}\{Z_j = 0\} \cong \mathbb{A}^m \times \Delta^{n_0}$. By the face condition on V, the closed set $V \cap \bigcap_{j\in J}\{Z_j = 0\}$ has dimension $n_0 + t$ (or is empty). Replacing Δ^n with this face Δ^{n_0} and V with $V \cap \bigcap_{j \in J}\{Z_j = 0\}$, we may assume that the matrix I has no trivial row. Under this assumption, the map F is smooth on the locus $\{Y_{k_0} \neq 0\}$. Therefore, the set $F^{-1}(V) \cap \{Y_{k_0} \neq 0\}$ has codimension $\geq m-t$. On the other hand, the inverse image $(F|_{\{Y_{k_0}=0\}})^{-1}(V)$ has codimension $\geq m-t$ in the locus $\{Y_{k_0} = 0\}$ by induction on p_0. This concludes that $F^{-1}(V)$ has codimension $\geq m-t$. $\square$

4. Proof of Suslin's quasi-isomorphism

We have made all the necessary geometric preparation. Now it is time to complete our alternative proof of Theorem 1.1. By the pullback/pushforward functoriality of the cycle complexes with respect to field extensions of finite degree, we may assume that the base field k is an *infinite* field. Let X be an affine k-scheme of finite type.

4.1. The complex obtained from the faces of a fixed $[n]$.

To assemble the subdivision maps into a map of cycle complexes, we borrow a construction from simpicial techniques.

For an integer $n \geq 0$ we consider the following complex $z_t(X, \bullet \hookrightarrow [n])$: it is a chain complex concentrated in degrees $[0,n]$ such that in homological degree $0 \leq s \leq n$, we have (recall that $[n] = \{0, \dots, n\}$ has $n+1$ elements)

$$z_t(X, s \hookrightarrow [n]) := \bigoplus_{S \subset [n],\ |S| = s+1} z_t(X, s). \tag{4.1.1}$$

Let us denote by $z_t(X,S)$ the summand corresponding to S. For each $S = \{v_0, \dots, v_s\}$ and $0 \leq k \leq s$, we give a differential map from $z_t(X,S)$ to $z_t(X, S \setminus \{v_k\})$ which is equal to $(-1)^k \partial_k^*\colon z_t(X,S) \to z_t(X, S\setminus\{v_k\})$. The following map obtained by taking the sum on each $z_t(X,s)$

$$z_t(X, s \hookrightarrow [n]) = \bigoplus_{S \subset [n],\ |S|=s+1} z_t(X,s) \to z_t(X,s)$$

defines a map of complexes $z_t(X, \bullet \hookrightarrow [n]) \to z_t(X, \bullet)$, which we call the canonical map $\mathsf{can}^{[n]}$.

By imposing the equi-dimensionality condition, we can define the subcomplex $z_t^{\mathsf{equi}}(X, \bullet \hookrightarrow [n])$, which is carried into $z_t^{\mathsf{equi}}(X, \bullet)$ by $\mathsf{can}^{[n]}$.

Lemma 4.2. *The maps*

$$\begin{aligned} &\mathsf{can}^{[n]} \colon z_t(X, \bullet \hookrightarrow [n]) \to z_t(X, \bullet) \\ \textit{and} \quad &\mathsf{can}^{[n]} \colon z_t^{\mathsf{equi}}(X, \bullet \hookrightarrow [n]) \to z_t^{\mathsf{equi}}(X, \bullet) \end{aligned}$$

induce isomorphisms on homology groups in degrees $0 \le s \le n - 1$.

Proof. This is a general fact on simplicial abelian groups. See for example [6, Cor. 1.3]. (Beware the difference of conventions here and there. The set $[n]$ has $n + 1$ elements.) □

Let us write $z_t(X, \bullet)/(\mathsf{equi})$ and $z_t(X, \bullet \hookrightarrow [n])/(\mathsf{equi})$ for the quotients by the subcomplexes of equi-dimensional cycles. In order to show Theorem 1.1 saying that $z_t(X, \bullet)/(\mathsf{equi})$ is acyclic, by Lemma 4.2 it suffices to show that the map

$$\mathsf{can}^{[n]} \colon z_t(X, \bullet \hookrightarrow [n])/(\mathsf{equi}) \to z_t(X, \bullet)/(\mathsf{equi}) \tag{4.2.2}$$

induces the zero maps on homology. This will follow if we show that the map (4.2.2) is weakly nullhomotopic. [3] To this end, fix an arbitrary finitely generated subcomplex

$$z_t^{\mathsf{f.g.}}(X, \bullet \hookrightarrow [n]) \subset z_t(X, \bullet \hookrightarrow [n])$$

and let $z_t^{\mathsf{f.g.}}(X, \bullet \hookrightarrow [n])/(\mathsf{equi})$ be its image to $z_t(X, \bullet \hookrightarrow [n])/(\mathsf{equi})$.

4.3. **The subdivision maps induce a map of complexes.** Keep $n \ge 0$ fixed. Choose a finitely generated subcomplex $z_t^{\mathsf{f.g.}}(X, \bullet \hookrightarrow [n])$. By enlarging it slightly, we may assume that its degree s part admits the decomposition

$$z_t^{\mathsf{f.g.}}(X, s \hookrightarrow [n]) = \bigoplus_{S \subset [n], |S| = s+1} z_t^{\mathsf{f.g.}}(X, S),$$

where $z_t^{\mathsf{f.g.}}(X, S) := z_t^{\mathsf{f.g.}}(X, s \hookrightarrow [n]) \cap z_t(X, S)$.

Choose a closed immersion $X \hookrightarrow \mathbb{A}^m$. All the results in §3 are valid also for closed subsets in $X \times \Delta^s$ via the immersion into $\mathbb{A}^m \times \Delta^s$ because all the maps we constructed are $\mathbb{A}^m$-maps. Below when we apply results from §3, we assume that an integer $N \ge n + 1$ has been fixed and the centers $c^s \in \Delta^s(\mathbb{A}^m)(\twoheadrightarrow \Delta^s(X))$ $(0 \le s \le n)$ of degree $\le N$ have been chosen appropriately generically, which is possible because we are working on a fixed $z_t^{\mathsf{f.g.}}(X, \bullet \hookrightarrow [n])$.

Lemma 3.8 implies that for each subset $S \subset [n]$ of order $s + 1$, each $0 \le k \le s$ and each permutation $\sigma \in \mathfrak{S}_{[k]}$, we have well-defined pullback maps along $\mathrm{pr}_1 \circ \mathsf{sd}_{s,k}^{\sigma}$ $(0 \le k \le s \le n)$:

$$(\mathrm{pr}_1 \circ \mathsf{sd}_{s,k}^{\sigma})^* \colon z_t^{\mathsf{f.g.}}(X, S) \to z_t(X, s + 1),$$

[3] Recall that a map of chain complexes is said to be *weakly nullhomotopic* if it is nullhomotopic on every finitely generated subcomplex (finitely many non-trivial terms, each finitely generated) of the source.

which carries the equi-dimensional subgroup into its counterpart by Corollary 3.7(ii). The relations (2.4.3) imply that the sum $\sum_{\sigma,k}\operatorname{sgn}(\sigma)(-1)^k(\mathrm{pr}_1\circ\mathsf{sd}^\sigma_{s,k})^*$ for various s and S defines a homotopy between $\mathsf{can}^{[n]}$ and the map $\mathsf{sd}^{[n]}$ defined by $\sum_{\sigma\in\mathfrak{S}_s}\operatorname{sgn}(\sigma)(\mathsf{sd}^\sigma_s)^*\colon z_t^{\mathsf{f.g.}}(X,S)/(\mathsf{equi})\to z_t(X,s)/(\mathsf{equi})$ for each $S\subset[n]$:

$$\mathsf{can}^{[n]}\simeq\mathsf{sd}^{[n]}\colon\quad z_t^{\mathsf{f.g.}}(X,\bullet\hookrightarrow[n])/(\mathsf{equi})\rightrightarrows z_t(X,\bullet)/(\mathsf{equi}).$$

Corollary 3.7(i) says that the map $\mathsf{sd}^{[n]}$ is actually the zero map (for a generic choice of centers). Thus we have constructed a nullhomotopy for $\mathsf{can}^{[n]}$ restricted to $z_t^{\mathsf{f.g.}}(X,\bullet\hookrightarrow[n])/(\mathsf{equi})$. As the finitely generated subcomplex $z_t^{\mathsf{f.g.}}(X,\bullet\hookrightarrow[n])$ can be taken arbitrarily large, we conclude that $\mathsf{can}^{[n]}$ itself is weakly nullhomotopic and by Lemma 4.2 the proof of Theorem 1.1 is now complete.

5. The modulus variant

We quickly describe what can be said about the *modulus* analog of Bloch's cycle complex. Suppose a finite-type k-scheme X is endowed with a Cartier divisor D. Let $\overline{\Delta^n}$ be the closure of Δ^n in $\mathbb{P}^{n+1}$ (recall by definition $\Delta^n\subset\mathbb{A}^{n+1}\subset\mathbb{P}^{n+1}$), which is isomorphic to $\mathbb{P}^n$. Let $\overline{\Delta^n}_\infty$ denote the hyperplane $(\overline{\Delta^n}\setminus\Delta^n)_{\mathsf{red}}$ in $\overline{\Delta^n}$.

A closed subset $V\subset(X\setminus|D|)\times\Delta^n$ is said to satisfy the *modulus condition* if, writing $\overline{V}^N$ for the normalization of (each irreducible component of) the closure $\overline{V}$ in $X\times\overline{\Delta^n}$, the following inequality of Cartier divisors on $\overline{V}^N$ holds:

$$\overline{\Delta^n}_\infty|_{\overline{V}^N}\geq D|_{\overline{V}^N}.$$

The advantage of the subdivision construction is the following fact.

Lemma 5.1. *Assume D is effective. If a closed subset $V\subset(X\setminus D)\times\Delta^n$ satisfies the modulus condition, then so do $(\mathrm{pr}_1\circ\mathsf{sd}^\sigma_{n,k})^{-1}(V)\subset(X\setminus D)\times\Delta^{n+1}$ and $(\mathsf{sd}^\sigma_n)^{-1}(V)\subset(X\setminus D)\times\Delta^n$ for every $0\leq k\leq n$ and $\sigma\in\mathfrak{S}_{[k]}$ regardless of the choice of centers.*

Proof. The proof is similar to [8, Prop. 3.6 or Rem. 3.10]. Just use the fact that the morphisms $\mathrm{pr}_1\circ\mathsf{sd}^\sigma_{n,k}$, sd^σ_n are written by polynomials whose degrees in the coordinates of $\Delta^\bullet$ are 1.

Alternatively, once one has checked the modulus condition for $(\mathrm{pr}_1\circ\mathsf{sd}^\sigma_{n,k})^{-1}(V)$, then the rest follows because the modulus condition is inherited by subsets thanks to the so-called *containment lemma* [9, Prop. 2.4]. □

Remark 5.2. With Suslin's construction $X\times\Delta^{n+1}\to X\times\Delta^n$, one can only say that if V satisfies the modulus condition with respect to the divisor $(n+1)D$ (a stronger condition), then the pullback satisfies the modulus condition with respect to D because Suslin's maps have degrees $n+1$ in the coordinates of Δ^n, Δ^{n+1}. This is why formerly [8] we only managed to show the pro-quasi-isomorphism over the thickenings mD $(m\geq 1)$.

Define the (simplicial) *cycle complexes with modulus*

$$z_t^{\mathsf{equi}}(X|D,\bullet)\subset z_t(X|D,\bullet)\quad(\subset z_t(X\setminus|D|,\bullet))$$

by adding the modulus condition (and the equi-dimensionality condition). The next theorem improves (the hypothetical simplicial analogue of) our former result [8, Th. 1.2] with Miyazaki in that we do not need the limit over the thickenings of D anymore.

Theorem 5.3. *Let X be an affine k-scheme equipped with an effective Cartier divisor D. Then for any $t \geq 0$, the inclusion*

$$z_t^{\mathsf{equi}}(X|D,\bullet) \hookrightarrow z_t(X|D,\bullet)$$

is a quasi-isomorphism.

Proof. Thanks to Lemma 5.1, the proof of Theorem 1.1 given in §4 works verbatim. □

Remark 5.4. As was mentioned in a footnote in §1, Theorem 5.3 for the cubically defined $z_t(X|D,\bullet)$ is not yet available because the author has not been able to (and has no plan of further trying to) write down similar subdivision and *homotopy* maps of cubes using only linear polynomials. For example, the homotopy in [13, §3] involves polynomials such as $\eta_c^1(y,z) = 1-(1-c)y(1-z)$ of degree 2. Unlike in the case of Bloch's cycle complex, the equivalence of the simplicial and cubical versions is not known due to the lack of an honest $(\mathbb{P}^1, -\{\infty\})$-invariance on the simplicial side at the moment.

Remark 5.5. Theorem 5.3 used the bound $\deg_Y(f_k^n) \leq 1$ (notation from §1). The other bound $\deg_X(f_k^{(n)}) \leq n+1$ (Corollary 3.7) can be used to obtain the following (rather incomplete) result. Let us explain the motivation behind its formulation first.

In light of the use of complexes $(C_*\mathbb{Z}^{\mathcal{Q},\mathbb{A}^q})(X) = \mathbb{Z}^{\mathcal{Q},\mathbb{A}^q}(X \times \Delta^\bullet)$ of algebraic cycles $V \subset \mathbb{A}^q \times X \times \Delta^\bullet$ quasi-finite and dominant over $X \times \Delta^\bullet$ in the Friedlander-Suslin theory [6, §12], it might be worthwhile to consider its modulus analog.

Because in Voevodsky's theory, quasi-isomorphisms of sheaves can be detected at spectra of function fields $X = \operatorname{Spec} F$, the only relevant case of the moving lemma (Theorem 1.1) of Suslin here is for the cycle complex $z_0(\mathbb{A}_F^q,\bullet)$ (so by base change, for $z_0(\mathbb{A}^q,\bullet)$).

Now, in more general motive theories, the $\mathbb{A}^1$-invariance should be replaced by the $(\mathbb{P}^q,\mathbb{P}^{q-1})$-invariance, where the specific meaning of the compactification $\mathbb{A}^q \hookrightarrow (\mathbb{P}^q,\mathbb{P}^{q-1})$ differs from theory to theory: it is either the log compactification, the avatar for the cofiber of the Gysin map for $\mathbb{P}^{q-1} \hookrightarrow \mathbb{P}^q$ on cohomology, or the scheme $\mathbb{P}^q$ with the divisor $\mathbb{P}^{q-1}$ encoding the fact that they are only allowing tame ramification. The correct way to see it in our setup should be as the pair $(\mathbb{P}^q, -\mathbb{P}^{q-1})$ where the divisor $-\mathbb{P}^{q-1}$ has a *minus sign* because it has been shown by Miyazaki [12] that the (cubical) cycle complex with modulus $z^*(X|D,\bullet)$ is $(\mathbb{P}^q, -\mathbb{P}^{q-1})$-invariant. Schemes equipped with not necessarily effective Cartier divisors have been studied also in [1] [7].

Since it is probably not true in whatever modulus category that quasi-isomorphisms between sheaves can be checked at function fields, it seems to the author that we should ask if the following inclusion is a quasi-isomorphism for affine X and effective D:

(5.5.1) $$z_t^{\mathsf{equi}}((X,D) \otimes (\mathbb{P}^q, -\mathbb{P}^{q-1}),\bullet) \quad \subset \quad z_t((X,D) \otimes (\mathbb{P}^q, -\mathbb{P}^{q-1}),\bullet).$$

Here, the tensor product of pairs denotes the scheme $X \times \mathbb{P}^q$ equipped with the divisor $D \times \mathbb{P}^q - X \times \mathbb{P}^{q-1}$. One might think that one could use the $(\mathbb{P}^q, -\mathbb{P}^{q-1})$-invariance on both sides, but if $t - q < 0$, the equi-dimensional cycle complex $z_{t-q}^{\mathsf{equi}}(X|D,\bullet)$ does not make sense. [4]

[4] In fact, the point of the presence of $\mathbb{A}^q$ in Friedlander-Suslin's $C_*\mathbb{Z}^{\mathcal{Q},\mathbb{A}^q}$ was to make sense of the non-existent "z_{-q}^{equi}".

Hoping to show that (5.5.1) is a quasi-isomophism, we deploy the subdivision machinery from §3. In doing so, we first choose an embedding $X \hookrightarrow \mathbb{A}^p$ and take the product with $\mathbb{P}^q \setminus \mathbb{P}^{q-1} \cong \mathbb{A}^q$ to get an embedding $X \times (\mathbb{P}^q \setminus \mathbb{P}^{q-1}) \hookrightarrow \mathbb{A}^p \times \mathbb{A}^q =: \mathbb{A}^m$. As the cycle complexes with modulus in this case are by definition certain complexes of cycles on $(X \setminus D) \times \mathbb{A}^q \times \Delta^\bullet$, the arguments concerning the equi-dimensionality goes just fine.

The exact analog of Lemma 5.1 that the subdivision maps preserve the modulus condition is *not necessarily true* because we are concerned with a negative Cartier divisor $-\mathbb{P}^{q-1}$ and because we took it away from our sight when we set up the subdivision machinery, i.e., the use of $\mathbb{P}^q \setminus \mathbb{P}^{q-1}$. However, by the fact (Corollary 3.7) that the polynomials describing $\mathsf{sd}^\sigma_{s,k}$ have bounded degrees $\leq N$ in the coordinates of $\mathbb{P}^q \setminus \mathbb{P}^{q-1} \cong \mathbb{A}^q$ (where we can take $N := n+1$ if we are interested in homology degrees $\leq n$), an argument analogous to [8, Prop. 3.6] shows that the homotopy maps $(\mathrm{pr}_1 \circ \mathsf{sd}^\sigma_{n,k})^*$ carry the group $z_t((X, ND) \otimes (\mathbb{P}^q, -\mathbb{P}^{q-1}), S)$, which has coefficient N in the modulus, into $z_t((X, D) \otimes (\mathbb{P}^q, -\mathbb{P}^{q-1}), s+1)$ when $|S| - 1 = s \leq n$.

This establishes isomorphisms of pro-homology groups in each degree for the inclusion

$$(5.5.2)\quad \left\{ z_t^{\mathsf{equi}}((X, D') \otimes (\mathbb{P}^q, -\mathbb{P}^{q-1}), \bullet) \right\}_{D \subset D' \text{ thickenings}} \to \left\{ z_t((X, D') \otimes (\mathbb{P}^q, -\mathbb{P}^{q-1}), \bullet) \right\}_{D \subset D' \text{ thickenings}}.$$

Desirably we should be able to show the quasi-isomorphism for any fixed D.

References

[1] Federico Binda, *A motivic homotopy theory without $\mathbb{A}^1$-invariance*, Math. Z., **295** (3), 2020, 1475–1519.

[2] Spencer Bloch, *Algebraic cycles and higher K-theory*, Advances in Mathematics, **61** (3), 1986, 267–304.

[3] Spencer Bloch, *The moving lemma for higher Chow groups*, J. Algebraic Geom., **3**, 1994, 537–568.

[4] Jean Dieudonné and Alexander Grothendieck, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas (Parts 1, 3)*, Publ. Math. Inst. Hautes Études Sci., **20** (1964), **28** (1966).

[5] Eric M. Friedlander and H. Blaine Lawson, *Moving algebraic cycles of bounded degree*, Invent. Math., **132** (1), 1998, 91–119.

[6] Eric M. Friedlander and Andrei Suslin, *The spectral sequence relating algebraic K-theory to motivic cohomology*, Ann. Scient. Éc. Norm. Sup. 4e série, **35**, 2002, 773–875.

[7] Bruno Kahn and Hiroyasu Miyazaki, *Modulus triples*, arXiv:2303.02453, 2023.

[8] Wataru Kai and Hiroyasu Miyazaki, *Suslin's moving lemma with modulus*, Annals of K-Theory, **3** (1), 2018, 55–70.

[9] Amalendu Krishna and Jinhyun Park, *Moving lemma for additive higher Chow groups*, Algebra & Number Theory, **6** (2), 2012, 293–326.

[10] Marc Levine, *Techniques of localization in the theory of algebraic cycles*, J. Algebraic Geom., **10**, 2001, 299–363.

[11] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel, "Lecture Notes on Motivic Cohomology", volume 2 of Clay Mathematics Monographs, American Mathematical Society, Providence, RI, 2006.

[12] Hiroyasu Miyazaki, *Cube invariance of higher Chow groups with modulus*, J. Algebraic Geom., **28**, 2019, 339–390.

[13] Jinhyun Park, *On localization for cubical higher Chow groups*, Tohoku Mathematical Journal, **75** (2), 2023, 251–281.

[14] Andrei Suslin, *Higher Chow groups and étale cohomology*, In: "Cycles, Transfers and Motivic Homology Theories", Vladimir Voevodsky, Andrei Suslin, and Eric Friedlander (eds.), volume 143 of Annals of Math. Studies, Princeton University Press, NJ, 2000, 237–252.

[15] Vladimir Voevodsky, *Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic*, Int. Math. Res. Not., **2002** (7), 2002, 351–355.

Mathematical Institute, Tohoku University, Aramaki Aoba 6-3, 980-8578 Sendai, Japan
Email address: `kaiw@tohoku.ac.jp`