KÄHLER DUALITY AND PROJECTIVE EMBEDDINGS

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ABSTRACT. Motivated by the duality theory between Hermitian symmetric spaces of noncompact and compact types, we introduce and examine the concept of Kähler duality between exact domains of $\mathbb{C}^n$ (cf. Definitions 2 and 3).

Specifically, we address the following question (cf. Definition 1).

Question. *Let (U, g) be an exact domain admitting a Kähler dual (U^*, g^*). Assume that there exists $\alpha > 0$ such that the following conditions hold:*

(a) *$(U, \alpha g)$ has a Fubini-Study completion;*

(b) *$(U^*, \alpha g^*)$ has a Fubini-Study compactification.*

Is it true that (U, g) is biholomorphically isometric to a bounded symmetric domain?

In Theorem 1.1, we affirmatively answer this question for a Cartan-Hartogs domain $M_{\Omega,\mu}$ equipped with an appropriate metric $g_{\Omega,\mu}$. In Theorem 1.2, we provide a counterexample by introducing another Kähler metric $\hat{g}_{\Omega,\mu}$ on $M_{\Omega,\mu}$, that satisfies both conditions (a) and (b). Furthermore, in Theorem 1.3, we characterize Hermitian symmetric spaces of compact type among classical flag manifolds based solely on Kähler duality. We also propose and thoroughly investigate a conjecture that our question has a positive answer if the metric g is Kähler-Einstein.

CONTENTS

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1. Introduction

A Hermitian symmetric space of noncompact type is a Kähler manifold (M, g) characterized by the property that, for all $p \in M$, the geodesic symmetry

$$s_p : \exp_p(v) \mapsto \exp_p(-v), \forall v \in T_pM$$

is a globally defined holomorphic isometry of (M, g).

Up to homotheties, (M, g) is biholomorphically isometric to a bounded symmetric domain $\Omega \subset \mathbb{C}^n$ centred at the origin $0 \in \mathbb{C}^n$ equipped with the Kähler metric g_Ω whose associated Kähler form is

$$\omega_\Omega = -\frac{i}{2\pi}\partial\bar{\partial}\log N_\Omega \tag{1}$$

where

$$N_\Omega(z, \overline{z}) = (V(\Omega)K_\Omega(z, \overline{z}))^{-\frac{1}{\gamma}} \tag{2}$$

is the *generic norm*, $V(\Omega)$ is the *Euclidean volume* of Ω, γ the *genus* of Ω and K_Ω its *Bergman kernel*.

Thus

$$g_\Omega = \frac{1}{\gamma}g_B \tag{3}$$

where g_B is the *Bergman metric* on Ω.

Notice that the genus γ of Ω is nothing but the Einstein constant of the Kähler-Einstein (KE in the sequel) metric g_Ω.

Every Hermitian symmetric space of noncompact type is a homogenous Kähler manifold and there exists a complete classification of the irreducible ones, known as *Cartan domains*, which include four classical series and two exceptional cases of complex dimensions 16 and 27, respectively.

A Cartan domain Ω is uniquely determined by a triple of integers (r, a, b), where r represents the rank of Ω, namely the maximal dimension of a complex totally geodesic submanifold of Ω, and a and b are positive integers such that

$$\gamma = (r-1)a + b + 2 \tag{4}$$

and

$$n = r + \frac{r(r-1)}{2}a + rb,$$

where n is the dimension of Ω . The *Wallach set* $W(\Omega) \subset \mathbb{R}$ of a Cartan domain $\Omega \subset \mathbb{C}^n$ is a subset of $\mathbb{R}$ which depends on a and r. More precisely we have

$$W(\Omega) = \left\{0, \, \frac{a}{2}, \, 2\frac{a}{2}, \, \ldots, \, (r-1)\frac{a}{2}\right\} \cup \left((r-1)\frac{a}{2}, \, +\infty\right). \tag{5}$$

The importance of the Wallach set for our aim is due to the following fact (see the beginning of Section 2 for the definition of infinitely projectively induced Kähler metric).

Theorem A.[34, Th.2] *Let (Ω, g_Ω) be a Cartan domain. The Kähler metric αg_Ω is infinitely projectively induced iff $\alpha \in W(\Omega) \setminus \{0\}$. Consequently, if (Ω, g_Ω) is a bounded symmetric domain, αg_Ω is infinitely projectively induced for all sufficiently large α.*

One of the remarkable aspects of the theory of Hermitian symmetric spaces is the concept of *duality*, which provides a way to transition from a Hermitian symmetric space of noncompact type (Ω, g_Ω) to a Hermitian symmetric space of compact type (Ω_c, g_{Ω_c}), known as the compact dual of (Ω, g_Ω) (see, e.g. [12] and references therein).

An interesting feature of this duality is reflected in the following chain of holomorphic embeddings, which hold for any bounded symmetric domain Ω:

$$0 \in \Omega \subset \mathbb{C}^n \xrightarrow{J} \Omega_c \xrightarrow{BW} \mathbb{C}P^d, \tag{6}$$

where J is a holomorphic embedding with dense image and BW is the *Borel-Weil* embedding satisfying $BW^*(g_{FS}) = g_{\Omega_c}$ (see [39] for more details on the value of d). Moreover, $\Omega_c = J(\mathbb{C}^n) \sqcup H$, where $H = BW^{-1}(\{Z_0 = 0\})$, and where $[Z_0, \dots, Z_d]$ denotes the homogeneous coordinates on $\mathbb{C}P^d$. The embeddings (6) have been extensively used to study the symplectic geometry of bounded symmetric domains and theirs duals in [12], [13], [28] (see also [10] for interesting rigidity properties of holomorphic isometries of the complex hyperbolic space into bounded symmetric domains).

If ω_{Ω_c} is the Kähler form associated to g_{Ω_c} it turns out that

$$J^*\omega_{\Omega_c} = \frac{i}{2\pi}\partial\bar\partial \log N^*_\Omega,$$

where we define

$$N^*_\Omega(z, \bar z) := N_\Omega(z, -\bar z) \tag{7}$$

and where $N_\Omega(z, \bar z)$ is the generic norm of Ω given by (2).

Example 1. The prototypical example of Hermitian symmetric space of noncompact type is the unit ball $\mathbb{C}H^n = \{z \in \mathbb{C}^n \mid |z|^2 < 1\}$ equipped with the hyperbolic metric $g_{\mathbb{C}H^n} = g_{hyp}$ such that

$$\omega_{hyp} = -\frac{i}{2\pi}\partial\bar\partial \log N_{\mathbb{C}H^n} = -\frac{i}{2\pi}\partial\bar\partial \log(1 - |z|^2)$$

In this case $a = 2, b = n-1$, $r = 1$ and $\gamma = n+1$. The compact dual of $(\mathbb{C}H^n, g_{hyp})$ is $(\mathbb{C}P^n, g_{FS})$ the complex projective space with the Fubini-Study metric g_{FS}. In the affine chart $U_0 = \{Z_0 \neq 0\} \subset \mathbb{C}P^n$, the Fubini-Study form, with respect the affine coordinates $z_j = \frac{Z_j}{Z_0}$, is given by $\omega_{FS} = \frac{i}{2\pi}\partial\bar\partial \log(1 + |z|^2)$. Notice that in this case the BW embedding is the identity map of $\mathbb{C}P^n$.

Using the fact that g_{Ω_c} is projectively induced by the Borel-Weil embedding and ω_{Ω_c} is such that $\omega_{\Omega_c}(B) = \pm\pi$, where B is a generator of $H_2(M, \mathbb{Z})$ (see e.g. [31]) we derive the following compact version of Theorem A.

Theorem B. *Let $(\Omega_c, \alpha g_{\Omega_c})$ be a Hermitian symmetric space of compact type, with $\alpha > 0$. Then αg_{Ω_c} is projectively induced iff α is a positive integer.*

Theorems A and B prompt the question of whether the concept of duality is specific to Hermitian symmetric spaces and, in particular, whether it characterizes them. To rigorously address this question (see Question A below), it is natural to begin by introducing some definitions.

Definition 1. *A compact (resp. complete) Kähler manifold $(\tilde M, \tilde g)$ is a* Fubini-Study compactification (resp. completion) *of a Kähler manifold (M, g) if there exists a holomorphic isometric* *embedding $J : (M, g) \to (\tilde{M}, \tilde{g})$ such that $J(M)$ is an open and dense subset of $\tilde{M}$ and $\tilde{g}$ is finitely (resp. infinitely) projectively induced.*

Remark 1. Although a compact manifold is (of course) complete, Calabi's rigidity theorem [8, Th.9] implies that a Kähler manifold admitting a Fubini-Study compactification does *not* admit a Fubini-Study completion, and viceversa.

Example 2. Being g_Ω complete it follows from Theorem A that a bounded symmetric domain $(\Omega, \alpha g_\Omega)$ admits a Fubini-Study completion iff $\alpha \in W(\Omega) \setminus \{0\}$, by trivially setting $(\tilde{\Omega}, \alpha \tilde{g}_\Omega) = (\Omega, \alpha g_\Omega)$ and $J = id_\Omega$. On the other hand, by Theorem B, $(\mathbb{C}^n, J^* \alpha g_{\Omega_c})$ with $\alpha \in \mathbb{Z}^+$ admits a Fubini-Study compactification where $J : \mathbb{C}^n \to \Omega_c$ is the embedding in (6).

It is not hard to see that the Kähler potentials $-\log N_\Omega$ and $\log N_\Omega^*$ (see (7)) are the *Calabi's diastasis functions* $D_0^{g_\Omega}$ and $D_0^{J^* g_{\Omega_c}}$ at the origin $0 \in \Omega \subset \mathbb{C}^n$ for the metrics g_Ω and $J^* g_{\Omega_c}$, respectively (see [8] or Section 4 below).

Taking inspiration from this fact and by the duality theory between hermitian symmetric spaces, we propose the following definitions (cf. also [14]).

Definition 2. A pair (U, g) is called an *exact domain* if U is a connected complex domain of $\mathbb{C}^n$ containing the origin and g is a real analytic Kähler metric whose associated Kähler form is given by $\omega = \frac{i}{2\pi} \partial\bar{\partial} D_0^g$, where D_0^g is Calabi's diastasis function for g at $0 \in \mathbb{C}^n$ and D_0^g cannot be extended to an open subset V strictly containing U.

Remark 2. The exploration of the maximal extension domain of Calabi's diastasis function is a complex and profound issue (see [8] and also the recent preprint [36]). All examples at our disposal of exact domains (U, g) with g infinitely projectively induced are complete. Thus, such domains admit a Fubini-Study completion by simply setting $(\tilde{U}, \tilde{g}) = (U, g)$ and $J = id_U$.

Definition 3. Let (U, g) and (U^*, g^*) be two exact domains of $\mathbb{C}^n$. We say that (U^*, g^*) is the *Kähler dual* of (U, g) and g^* is a Kähler metric dual to g if

$$D_0^{g^*}(z, \bar{z}) = -D_0^g(z, -\bar{z}), \forall z \in U \cap U^*. \tag{8}$$

One can easily construct explicit examples of exact domains not admitting a Kähler dual (cf. Section 4).

Example 3. Any bounded symmetric domain (Ω, g_Ω) is an exact domain and its Kähler dual is the exact domain given by $(\Omega^*, g_\Omega^*) = (\mathbb{C}^n, J^* g_{\Omega_c})$, where g_{Ω_c} is the Kähler metric on the compact dual Ω_c of Ω and $J : \mathbb{C}^n \to \Omega_c$ is the embedding in (6).

Inspired by Theorems A and Theorem B, Definitions 1, 2, 3 and Example 3 we address the following

Question A. *Let (U, g) be an exact domain admitting a Kähler dual (U^*, g^*). Suppose that there exists $\alpha > 0$ for which the following properties hold true:*

(a) *$(U, \alpha g)$ has a Fubini-Study completion;*
(b) *$(U^*, \alpha g^*)$ has a Fubini-Study compactification.*

Is it true that (U, g) is biholomorphically isometric to a bounded symmetric domain $(\Omega, \lambda g_\Omega)$, for some $\lambda \in \mathbb{R}^+$ (and hence $\lambda\alpha \in W(\Omega) \cap \mathbb{Z}^+$)?

In this paper we consider Question A in the context where U is a *Cartan-Hartogs domain* (CH domain in the sequel) equipped with two natural Kähler metrics $g_{\Omega,\mu}$ and $\hat{g}_{\Omega,\mu}$ (see below). CH domains are a 1-parameter family of noncompact domains of $\mathbb{C}^{n+1}$, given by:

$$M_{\Omega,\mu} := \left\{(z,w) \in \Omega \times \mathbb{C} \mid |w|^2 < N_\Omega^\mu(z,\bar{z})\right\}$$

where $\Omega \subset \mathbb{C}^n$ is a Cartan domain, known as the base of $M_{\Omega,\mu}$, $N_\Omega(z,\bar{z})$ is its generic norm, and $\mu > 0$ is a positive real parameter.

The first metric on $M_{\Omega,\mu}$ that we consider is the Kähler metric $g_{\Omega,\mu}$, introduced and studied by G. Roos, A. Wang, W. Yin, L. Zhang, W. Zhang in [44] and [45]. The associated Kähler form is given by

$$\omega_{\Omega,\mu} = -\frac{i}{2\pi}\partial\bar{\partial}\log\left(N_\Omega^\mu(z,\bar{z}) - |w|^2\right). \tag{9}$$

One of the important features of this metric is the following result which shows that for suitable values of the parameter μ it is KE.

Theorem C.([44] and [45]) *Let $M_{\Omega,\mu}$ be a CH domain. The metric $g_{\Omega,\mu}$ is KE iff $\mu = \frac{\gamma}{n+1}$, where n is the complex dimension of Ω and γ its genus.*

The following theorem represents the first result of this paper. It provides a positive answer to Question A for a CH domain $(M_{\Omega,\mu}, g_{\Omega,\mu})$ and also characterizes the complex hyperbolic space among CH domains.

Theorem 1.1. *The CH domain $(M_{\Omega,\mu}, g_{\Omega,\mu})$ is an exact domain that admits a Kähler dual $(M^*_{\Omega,\mu} = \mathbb{C}^{n+1}, g^*_{\Omega,\mu})$. Moreover, $(M_{\Omega,\mu}, \alpha g_{\Omega,\mu})$ admits a Fubini-Study completion for all sufficiently large α. Additionally, $(\mathbb{C}^{n+1}, \alpha g^*_{\Omega,\mu})$ admits a Fubini-Study compactification for some α iff $M_{\Omega,\mu}$ is homogeneous, i.e. $(M_{\Omega,\mu}, \alpha g_{\Omega,\mu}) = (\mathbb{C}H^{n+1}, g_{hyp})$.*

Remark 3. It is worth pointing out that the Kähler dual $(\mathbb{C}^{n+1}, \alpha g^*_{\Omega,\mu})$ of $(M_{\Omega,\mu}, g_{\Omega,\mu})$ has also been examined in [38] from a symplectic perspective, with the aim to extend the symplectic duality given in [12] for symmetric spaces to CH domains.

The second metric $\hat{g}_{\Omega,\mu}$ on a CH domain $M_{\Omega,\mu}$ that we consider in this paper is the Kähler metric whose associated Kähler form is given by (see Section 3 for details):

$$\hat{\omega}_{\Omega,\mu} = -\frac{i}{2\pi}\partial\bar{\partial}\log(N_\Omega^\mu(z,\bar{z}) - |w|^2) - \frac{i}{2\pi}\partial\bar{\partial}\log N_\Omega^\mu(z,\bar{z}) \tag{10}$$

In the following theorem, which represents our second main result, we show that the CH domain $(M_{\Omega,\mu}, \hat{g}_{\Omega,\mu})$ satisfies the assumptions (a) and (b) of Question A, for suitable values of α and μ. Therefore, Question A has a negative answer for these values.

Theorem 1.2. *The CH domain $(M_{\Omega,\mu}, \hat{g}_{\Omega,\mu})$ is an exact domain that admits a Kähler dual $(M^*_{\Omega,\mu} = \mathbb{C}^{n+1}, \hat{g}^*_{\Omega,\mu})$. For $\mu \in \mathbb{Z}^+$ and sufficiently large integer α, $(M_{\Omega,\mu}, \alpha\hat{g}_{\Omega,\mu})$ has a Fubini-Study completion and $(\mathbb{C}^{n+1}, \alpha\hat{g}^*_{\Omega,\mu})$ has a Fubini-Study compactification.*

This theorem draws us with a strong analogy with the duality theory between Hermitian symmetric spaces. Moreover, the existence of such a compactification offers a bridge between the complex geometry of a given CH domain and the algebraic geometry of the Fubini-Study compactification of its Kähler dual.

Remark 4. One natural metric to consider on a CH domain is, of course, the Bergman one (see [46]). We can prove that although the Bergman metric on CH domains admits a Kähler dual, the latter is not finitely projectively induced. This and other properties of the Bergman metric on CH domains and its Kähler dual will be explored in a forthcoming paper.

A natural extension of the concept of Hermitian symmetric space of compact type is that of a flag manifold (F, g_F), i.e., compact and simply-connected homogeneous complex manifold F equipped with a homogeneous Kähler metric g_F. Note that if the Kähler form ω_F associated to g_F is integral then g_F is finitely projectively induced (cf. [43] and [15]).

Moreover, by restricting to the case of a flag manifold of classical type, for each $p \in F$ there exists a holomorphic embedding with dense image $J_F : \mathbb{C}^n \to F$, $J_F(0) = p$. This embedding is analogous to the embedding J given by (6) for a Hermitian symmetric space of compact type (see Section 4 below for the definition of J_F and more details).

Thus a flag manifold of classical type (F, g_F) of complex dimension n and integral ω_F is a Fubini-Study compactification of $(\mathbb{C}^n, J_F^* g_F)$. Consequently, it is natural to investigate whether the latter admits a Kähler dual. In the following theorem, which represents our final main result, we prove that this is the case only when the flag is indeed a Hermitian symmetric space of compact type.

Theorem 1.3. *Let (F, g_F) be a flag manifold of classical type. If $(\mathbb{C}^n, J_F^* g_F)$ admits a Kähler dual, then (F, g_F) is biholomorphically isometric to a Hermitian symmetric space of compact type.*

It is worth pointing out that in Theorem 1.3 we are not assuming any of the conditions (a) and (b) of Question A. Therefore, this theorem provide a characterization of Hermitian symmetric spaces of compact type among classical flag manifolds solely in terms of Kähler duality.

The paper is organized as follows. In Sections 2 and 3, we describe the values of α and μ for which the metrics $\alpha g^*_{\Omega,\mu}$, $\alpha \hat{g}_{\Omega,\mu}$ and $\alpha \hat{g}^*_{\Omega,\mu}$ are projectively induced (see Propositions 2.1, 3.1 and 3.2, respectively). We also prove Theorems 1.1 and 1.2. In the final part of Section 3 we show that the homogeneity assumption in Theorem 1.3 cannot be weakened to cohomogeneity 1 by showing that the Fubini-Study compactification of $(\mathbb{C}^{n+1}, \hat{g}^*_{\Omega,\mu})$ given in Theorem 1.2, is a cohomogeneity 1 G-space.

In Section 4, after recalling the construction of the embedding $J_F : \mathbb{C}^n \to F$ for a flag manifold of classical type and the definition of Alekseevsky-Peremolov coordinates, we prove Theorem 1.3. In Section 5 we introduce a conjecture (Conjecture A) asserting that Question A has a positive answer if the metric g is Kähler-Einstein. We also analyze the necessity of conditions (a), (b) and the KE assumption in Conjecture A. Specifically, we show the necessity of assumption (b) in Example 4, examine the role of assumption (a) by comparing it with other long-standing conjectures within the context of flag manifolds, and show the necessity of the KE assumption by proving that the metric $\hat{g}_{\Omega,\mu}$ is never KE for any value of μ. In the last section (Section 6) we show how our results on CH domains could be extended to generalized CH domains.

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2. The Kähler dual of $(M_{\Omega,\mu}, g_{\Omega,\mu})$ and the proof of Theorem 1.1

Consider the Kähler dual of a CH domain $(M_{\Omega,\mu}, g_{\Omega,\mu})$ namely the Kähler manifold $(M^*_{\Omega,\mu}, g^*_{\Omega,\mu})$, with associated Kähler form given by

$$\omega^*_{\Omega,\mu} = \frac{i}{2\pi}\partial\bar\partial \log\left(N^\mu_\Omega(z,-\bar z) + |w|^2\right). \tag{11}$$

These domains were introduced in [38] where symplectic aspects were studied in analogy with those of [12]. In particular, one can prove (see [38, Section 3]) that $M^*_{\Omega,\mu} = \mathbb{C}^{n+1}$ and it can be easily verified that $(M^*_{\Omega,\mu}, g^*_{\Omega,\mu})$ is indeed the Kähler dual since $N^\mu_\Omega(z,-\bar z)$ is real valued.

In the proof of Theorem 1.1 we need the following Propositions 2.1 and 2.3 and Lemma 2.2.

First we recall some basic facts. Throughout this paper we say that a Kähler metric g on a complex manifold M is *projectively induced* if there exists a holomorphic map $f : M \to \mathbb{C}P^N$, $N \in \mathbb{N}\cup\{\infty\}$, such that $f^*g_{FS} = g$, where g_{FS} is the Fubini-Study metric on $\mathbb{C}P^N$. Such a map is called a *holomorphic isometry*. Notice that we are not assuming either M compact or complete. A Kähler metric g is said to be *finitely projectively induced* if N can be chosen finite, and *infinitely projectively induced* if $N = \infty$ and the immersion is full, meaning its image is not contained within any finite dimensional complex projective space. For more on projectively induced Kähler metrics, refer to [8] and [35].

Proposition 2.1. *The Kähler metric $\alpha g^*_{\Omega,\mu}$ on $\mathbb{C}^{n+1}$ is finitely projectively induced if and only if $\alpha,\mu \in \mathbb{Z}^+$.*

Proof. Let (Ω_c, g_{Ω_c}) be the compact dual of the Cartan domain (Ω, g_Ω). We know that the Kähler metric g_{Ω_c} is finitely projectively induced via the Borel–Weil embedding $BW : \Omega_c \to \mathbb{C}P^d$, $BW(p) = [s_0(p), \cdots, s_d(p)]$, i.e. $BW^*g_{FS} = g_{\Omega_c}$ for a suitable $d = \dim H^0(L) - 1$ depending on Ω_c (cf. (6)). Here $s_0, \ldots, s_d$ are global holomorphic sections of the holomorphic line bundle L over Ω_c such that $c_1(L) = [\omega_{\Omega_c}]$. If $\mu \in \mathbb{Z}^+$ then μg_{Ω_c} is finitely projectively induced, i.e. there exists a holomorphic embedding

$$F_\mu : \Omega_c \to \mathbb{C}P^{d_\mu},\ F_\mu(p) = \left[s_0^{(\mu)}(p), \cdots, s_{d_\mu}^{(\mu)}(p)\right], \tag{12}$$

such that $F^*_\mu g_{FS} = \mu g_{\Omega_c}$, where $s_0^{(\mu)}, \ldots, s_{d_\mu}^{(\mu)}$ is a basis of global holomorphic sections of $L^{\otimes\mu}$ and $d_\mu + 1 = \dim H^0(L^{\otimes\mu})$. Moreover, we can also assume $s_0^{(\mu)}(J(z)) \neq 0$, for all $z \in \mathbb{C}^n$ where $J : \mathbb{C}^n \to \Omega_c$ is given by (6). The existence of F_μ with these properties follows by the fact that $L^{\otimes\mu}$ defines a regular quantization of the homogeneous Kähler manifold (Ω_c, g_{Ω_c}) (the reader is referred to [5, Th. 5.1] for details).

One then easily deduces that the map

$$\tilde F_\mu : \mathbb{C}^{n+1} \to \mathbb{C}P^{d_\mu+1},\ (z,w) \mapsto \left[s_0^{(\mu)}(J(z)), \ldots, s_{d_\mu}^{(\mu)}(J(z)), w s_0^{(\mu)}(J(z))\right], \tag{13}$$

is a holomorphic isometric embedding of $\left(\mathbb{C}^{n+1}, g^*_{\Omega,\mu}\right)$ into $\left(\mathbb{C}P^{d_\mu+1}, g_{FS}\right)$. If also α is a positive integer we get that $\alpha g^*_{\Omega,\mu}$ is finitely projectively induced by the holomorphic map $V_\alpha \circ \tilde F_\mu$, where $V_\alpha : \mathbb{C}P^{d_\mu+1} \to \mathbb{C}P^{\binom{d_\mu+1+\alpha}{d_\mu+1}}$ is Calabi's map (see [8, Th.13]), namely a suitable normalization of the Veronese map.

Conversely, assume that $\alpha g^*_{\Omega,\mu}$ is projectively induced. Then the same will be true for the metric $\alpha g^*_{\mathbb{C}H^1,\mu}$ on $\mathbb{C}^2$. Indeed, by the hereditary property of the Calabi's diastasis function (see [8, Prop.6]), we get that

$$D_0^{g^*_{\Omega,\mu}} = \log\left(N^\mu_\Omega(z,-\bar z)+|w|^2\right)$$

on $\mathbb{C}^{n+1}$ restricts to $D_0^{g^*_{\mathbb{C}P^1,\mu}} = \log\left((1+|\xi|^2)^\mu+|w|^2\right)$ on $\mathbb{C}^2$, where we choose a $(\mathbb{C}P^1,\mu g_{FS})$ complex and totally geodesic embedded into Ω_c.

Take now $\{(\xi,w)\in\mathbb{C}^2 \mid \xi=0\}$ and $\{(\xi,w)\in\mathbb{C}^2 \mid w=0\}$, equipped with the Kähler metrics induced by $\alpha g^*_{\mathbb{C}H^1,\mu}$. It is immediate to see that they are nothing but $(\mathbb{C},\alpha g_{FS})$ and $(\mathbb{C},\alpha\mu g_{FS})$ respectively, where g_{FS} is the Fubini-Study metric of $\mathbb{C}P^1$ restricted to $\mathbb{C}=\{Z_0\neq 0\}\subset\mathbb{C}P^1$. Thus, the assumption that $\alpha g^*_{\Omega,\mu}$ (and hence $\alpha g^*_{\mathbb{C}H^1,\mu}$) is projectively induced implies that both αg_{FS} and $\alpha\mu g_{FS}$ are projectively induced. This forces $\alpha\in\mathbb{Z}^+$ and $\alpha\mu\in\mathbb{Z}^+$ and we can then write

$$\mu=\frac{a}{b} \quad\text{and}\quad \alpha=bk,$$

for some $a,b,k\in\mathbb{Z}^+$ and a,b coprime. Consider now $\{(\xi,w)\in\mathbb{C}^2 \mid w=1\}$ equipped with the Kähler metric g induced by $\alpha g^*_{\mathbb{C}H^1,\mu}$. The proof will be completed if we show that g is not projectively induced if $\frac{a}{b}\notin\mathbb{Z}^+$. Notice that the Kähler form ω associated to g is

$$\omega=\frac{i}{2\pi}\partial\bar\partial\Phi, \qquad \Phi(\xi)=bk\log\left(\left(1+|\xi|^2\right)^{\frac{a}{b}}+1\right).$$

It is immediate to verify that $D(\xi)=\Phi(\xi)-bk\log 2$, is the diastasis function for the metric g centred at the origin. Moreover, by using the radiality of D and Calabi's criterium (see [8] or [35]), in order to show that g is not projectively induced one needs to show that at least one the coefficients of the Taylor expansion in ξ and $\bar\xi$ around the origin of e^D-1 is negative. We can see that this is equivalent to show that the Taylor expansion of $\Psi(|\xi|^2)=\left((1+|\xi|^2)^{\frac{a}{b}}+1\right)^{bk}$ in $|\xi|^2$, around the origin, has negative coefficients when $\frac{a}{b}\notin\mathbb{Z}^+$.

This expansion reads as

$$\begin{aligned}\Psi(|\xi|^2) &= \sum_{p=0}^{kb}\binom{kb}{p}\left(1+|\xi|^2\right)^{\frac{pa}{b}}\\ &= \sum_{p=0}^{kb}\binom{kb}{p}\Big[1+\frac{pa}{b}|\xi|^2+\frac{pa}{2b}\left(\frac{pa}{b}-1\right)|\xi|^4+\frac{pa}{6b}\left(\frac{pa}{b}-1\right)\left(\frac{pa}{b}-2\right)|\xi|^6\\ &\quad+\cdots+\frac{pa}{h!b}\left(\frac{pa}{b}-1\right)\cdots\left(\frac{pa}{b}-(h-1)\right)|\xi|^{2h}+\ldots\Big]\\ &= \sum_{h=0}^{\infty}\frac{A_h}{h!}|\xi|^{2h},\end{aligned} \tag{14}$$

where $A_0=\sum_{p=0}^{kb}\binom{kb}{p}$ and

$$A_h=\sum_{p=0}^{kb}B_{h,p}, \quad B_{h,p}=\binom{kb}{p}\frac{pa}{b}\left(\frac{pa}{b}-1\right)\cdots\left(\frac{pa}{b}-(h-1)\right), \quad h>0. \tag{15}$$

Hence we have

$$\frac{B_{\ell,p}}{B_{\ell,1}} = \binom{kb}{p}\frac{p}{bk}\prod_{s=1}^{\ell-1}\frac{bs-ap}{bs-a} = \binom{kb}{p}\frac{p}{bk}\prod_{s=1}^{\ell-1}\left(1-\frac{a\,(p-1)}{bs-a}\right), \forall \ell.$$

Since $\sum_{s=1}^{\infty}\frac{a(p-1)}{bs-a} = +\infty$, we conclude that (see e.g. [24, Section 3.7])

$$\lim_{\ell\to\infty}\frac{B_{\ell,p}}{B_{\ell,1}} = 0, \qquad p = 2, 3, \dots, kb$$

and therefore that

$$\lim_{h\to\infty}\frac{A_h}{B_{h,1}} = 1. \tag{16}$$

Since we are assuming that $\frac{a}{b} \notin \mathbb{Z}^+$, from (15) we get

$$B_{h+1,1} = ka\left(\frac{a}{b}-1\right)\dots\left(\frac{a}{b}-h\right) \neq 0,\ \forall h \in \mathbb{Z}^+.$$

Clearly $\frac{a}{b} - h < 0$, for any $h > \frac{a}{b}$. Hence, if h_0 is sufficiently large, from (16), we see that $\{A_h\}_{h>h_0}$ is an alternate sequence. We have proved that the Taylor expansion (14) has infinite negative coefficients when $\frac{a}{b} \notin \mathbb{Z}$, as desired. The proof is complete. □

Remark 5. Theorem 1.1 and Proposition 2.1 show that condition (b) in Question A cannot be weakened to the condition that the metric αg^* is finitely projectively induced.

Lemma 2.2. *Let (M, g) be a Kähler manifold such that g is projectively induced. Assume that there exists $\alpha \in \mathbb{Z}^+$ such that $(M, \alpha g)$ admits a Fubini-Study compactification $(\tilde{M}, \alpha\tilde{g})$. Then $(\tilde{M}, \tilde{g})$ is a Fubini-Study compactification of (M, g).*

Proof. By assumption there exist holomorphic isometric embeddings $\varphi : M \to \mathbb{C}P^N$, $J : (M, \alpha g) \to (\tilde{M}, \alpha\tilde{g})$ and $\varphi_\alpha : \tilde{M} \to \mathbb{C}P^{s_\alpha}$ for some positive integers N and s_α. Let $V_\alpha : \mathbb{C}P^N \to \mathbb{C}P^{N_\alpha}$ be the Calabi's map (see (cf. [8, Th.13])), i.e. $V_\alpha^* g_{FS} = \alpha g_{FS}$. Since without loss of generality we can assume φ_α to be full we can also assume $s_\alpha \leq N_\alpha$. Then if $i : \mathbb{C}P^{s_\alpha} \to \mathbb{C}P^{N_\alpha}$ denotes the natural totally geodesic inclusion of $\mathbb{C}P^{s_\alpha}$ into $\mathbb{C}P^{N_\alpha}$ it follows that the maps $V_\alpha \circ \varphi : M \to \mathbb{C}P^{N_\alpha}$ and $i \circ \varphi_{\alpha|M} \circ J : M \to \mathbb{C}P^{N_\alpha}$ are two holomorphic isometric immersions inducing the same Kähler metric αg. By Calabi's rigidity theorem [8, Th.9] there exists a unitary transformation U of $\mathbb{C}P^{N_\alpha}$ such that $V_\alpha \circ \varphi = U \circ i \circ \varphi_{\alpha|M} \circ J$. Then the holomorphic map

$$\tilde{\varphi} := (V_{\alpha|V_\alpha(\mathbb{C}P^N)})^{-1} \circ U \circ i \circ \varphi_\alpha : \tilde{M} \to \mathbb{C}P^N$$

satisfies $\tilde{\varphi}^* g_{FS} = \tilde{g}$ and hence $(\tilde{M}, \tilde{g})$ turns out to be a Fubini-Study compactification of (M, g). □

The following proposition describes the projective inducibility of multiples of the metric $g_{\Omega,\mu}$.

Proposition 2.3. *(*[34, Th.2]*) Let $M_{\Omega,\mu}$ be a CH domain. For a real number $\alpha > 0$, the Kähler metric $\alpha g_{\Omega,\mu}$ is infinitely projectively induced iff $(\alpha + s)\mu$ belongs to $W(\Omega) \setminus \{0\}$ for all integer $s \geq 0$.*

Proof of Theorem 1.1. By combining Proposition 2.1, Proposition 2.3 and the fact that the metric $g_{\Omega,\mu}$ is complete we deduce that the assumptions that $\mu \in \mathbb{Z}^+$, and α is an integer sufficiently large are necessary and sufficient conditions for $(M_{\Omega,\mu}, \alpha g_{\Omega,\mu})$ to admit a Fubini-Study completion (cf. Remark 1), and the dual Kähler metric $\alpha g^*_{\Omega,\mu}$ to be finitely projectively induced.

Notice that if $M_{\Omega,\mu}$ is homogeneous then $\Omega = \mathbb{C}H^n$, $\mu = 1$ and hence $(M_{\Omega,\mu} = \mathbb{C}H^{n+1}, g_{\Omega,\mu} = g_{hyp})$ which admits a Fubini-Study compactification given by $(\mathbb{C}P^{n+1}, g_{FS})$. Hence, it remains to prove that if one assumes that $(M_{\Omega,\mu}, \alpha g^*_{\Omega,\mu})$ admits a Fubini-Study compactification then $M_{\Omega,\mu}$ is homogeneous, i.e. $\Omega_c = \mathbb{C}P^n$. By Lemma 2.2 we can assume $\alpha = 1$ and hence there exists a holomorphic isometry Ψ from $\mathbb{C}^{n+1}$ into a finite dimensional complex projective space inducing $g^*_{\Omega,\mu}$. Assume by contradiction that $\Omega_c \neq \mathbb{C}P^n$. By Calabi's rigidity, up to a unitary transformations of the ambient projective space we can assume that $\Psi = \tilde{F}_\mu \circ J : \mathbb{C}^{n+1} \to \mathbb{C}P^{d_\mu+1}$ is given by the embedding (13). Let now

$$P_j(X_0, \dots, X_{d_\mu}) = 0, \;\; j = 1, \dots, k, \; \deg P_j \geq 2, \tag{17}$$

be the homogeneous polynomial equations which define the image $F_\mu(\Omega_c)$ of the embedding $F_\mu : \Omega_c \to \mathbb{C}P^{d_\mu}$ given by (12) (the condition $\deg P_j \geq 2$ for all $j = 1, \dots, k$ comes from the fact that F_μ is a full embedding and $\Omega_c \neq \mathbb{C}P^n$).

We claim that

$$\overline{(\tilde{F}_\mu \circ J)(\mathbb{C}^{n+1})} = K, \tag{18}$$

where

$$K := \{[X_0, \dots, X_{d_\mu}, X_{d_\mu+1}] \mid P_j(X_0, \dots, X_{d_\mu}) = 0, \;\; j = 1, \dots, k \}.$$

The inclusion $\overline{(\tilde{F}_\mu \circ J)(\mathbb{C}^{n+1})} \subseteq K$ is immediate by construction. In order to prove that $K \subseteq \overline{(\tilde{F}_\mu \circ J)(\mathbb{C}^{n+1})}$ let $Q = \left[X_0, \dots, X_{d_\mu}, X_{d_\mu+1}\right] \in K$.

We have two cases: on the one hand, if $X_0 = \cdots = X_{d_\mu} = 0$ (i.e. $Q = [0, \dots, 0, 1]$), then $Q \in \overline{(\tilde{F}_\mu \circ J)(\mathbb{C}^{n+1})}$ since it is the limit of any sequence

$$(\tilde{F}_\mu \circ J)(z, w_j) = [s_0^{(\mu)}(J(z)), \dots, s_{d_\mu}^{(\mu)}(J(z)), w_j s_0^{(\mu)}(J(z))]$$

with $w_j \to +\infty$ and any (fixed) $z \in \mathbb{C}^{n+1}$.

On the other hand, let $(X_0, \dots, X_{d_\mu}) \neq (0, \dots, 0)$, say $X_k \neq 0$. Then $[X_0, \dots, X_{d_\mu}] \in F_\mu(\Omega_c) \subset \mathbb{C}P^{d_\mu}$, i.e. $\left[X_0, \dots, X_{d_\mu}\right] = \left[s_0^{(\mu)}(p), \cdots, s_{N_\mu}^{(\mu)}(p)\right]$ for some $p \in \Omega_c$. Since $J(\mathbb{C}^n)$ is dense in Ω_c, there exists a sequence $z_j \in \mathbb{C}^n$ such that $p_j := J(z_j) \to p$ and then the sequence

$$\tilde{F}_\mu\left(p_j, \frac{X_{d_\mu+1}}{s_k^{(\mu)}(p)}\right) = \left[s_0^{(\mu)}(p_j), \dots, s_{N_\mu}^{(\mu)}(p_j), s_k^{(\mu)}(p_j)\frac{X_{d_\mu+1}}{s_k^{(\mu)}(p)}\right] \to Q.$$

This shows that $Q = [X_0, \dots, X_{d_\mu}, X_{d_\mu+1}] \in \overline{(\tilde{F}_\mu \circ J)(\mathbb{C}^{n+1})}$ and proves the claim (18).

In light of the above, in order to end the proof it will suffice to show that K is not smooth unless $\Omega = \mathbb{C}H^n$ and $\mu = 1$.

In order to do that, notice that $[0, \dots, 0, 1] \in K$ and that, by the implicit function theorem, in order for the equations (17) to define a smooth submanifold of $\mathbb{C}P^{d_\mu+1}$ in a neighbourhood of $(X_0, \dots, X_{d_\mu+1}) = (0, \dots, 0, 1)$ at least one of the homogeneous polynomials P_j must be linear in constrast with $\deg P_j \geq 2$, for all $j = 1, \dots, k$. □

3. THE KÄHLER DUAL OF $(M_{\Omega,\mu}, \hat{g}_{\Omega,\mu})$ AND THE PROOF OF THEOREM 1.2

In the proof of Theorem 1.2 we need the following proposition interesting on its own sake.

Proposition 3.1. *Let $M_{\Omega,\mu}$ be a CH domain. Then the Kähler metric $\hat{g}_{\Omega,\mu}$ whose associated Kähler form is given by* (10) *is complete. Futhermore, for a real number $\alpha > 0$, the Kähler metric $\alpha\hat{g}_{\Omega,\mu}$ is infinitely projectively induced iff $(2\alpha + s)\mu$ belongs to $W(\Omega) \setminus \{0\}$ for all integer $s \geq 0$.*

The second part of the proposition can be considered the analogous of Proposition 2.3 above for the metric $g_{\Omega,\mu}$. Indeed its proof is an adaptation of the proof given in [34, Th.2].

Proof of Proposition 3.1. Let $\gamma(t) = (z_1(t), \dots, z_n(t), w(t))$ be a curve in $M_{\Omega,\mu}$. Then, it is easily seen that

$$\|\dot{\gamma}(t)\|^2_{\hat{g}_{\Omega,\mu}} = \|\dot{\gamma}(t)\|^2_{g_{\Omega,\mu}} + \|p(\dot{\gamma})(t)\|^2_{\mu g_\Omega} \tag{19}$$

where $p(\gamma)(t) = (z_1(t), \dots, z_n(t))$ denotes the projection of γ on the base Ω equipped with the metric μg_Ω.

We deduce that the length of γ with respect to $\hat{g}_{\Omega,\mu}$ is

$$\int \|\dot{\gamma}(t)\|_{\hat{g}_{\Omega,\mu}} dt = \int \sqrt{\|\dot{\gamma}(t)\|^2_{g_{\Omega,\mu}} + \|p(\dot{\gamma})(t)\|^2_{\mu g_\Omega}} dt \geq \int \|\dot{\gamma}(t)\|_{g_{\Omega,\mu}} dt \tag{20}$$

Now, we know (see e.g. [16]) that a Riemannian metric is complete if and only if the length of every *divergent curve* (this means that is gets out of every compact in $M_{\Omega,\mu}$) is not finite. Let then $\gamma(t)$ be a divergent curve in $M_{\Omega,\mu}$. Since the metric $g_{\Omega,\mu}$ is complete, then $\int \|\dot{\gamma}(t)\|_{g_{\Omega,\mu}}\, dt = \infty$. But then, from (20) we deduce that also $\int \|\dot{\gamma}(t)\|_{\hat{g}_{\Omega,\mu}}\, dt = \infty$ and by the above criterium we deduce that also $\hat{g}_{\Omega,\mu}$ is complete.

In order to prove the second part of the proposition, notice that a potential of the multiple $\alpha\hat{g}_{\Omega,\mu}$ is

$$D = \log[N_\Omega^{-\mu\alpha}(N_\Omega^{\mu} - |w|^2)^{-\alpha}]. \tag{21}$$

In fact, it is easily seen that (21) is indeed the diastasis centred at the origin for the metric $\alpha s\hat{g}_{\Omega,\mu}$, i.e. $D = D_0^{\alpha\hat{g}_{\Omega,\mu}}$. In order to apply Calabi's criterium (see [8, Th.9]) consider the expansion

$$e^D - 1 = \sum_{j,k=0}^{\infty} B_{jk} (zw)^{m_j} (\bar{z}\bar{w})^{m_k}, \tag{22}$$

where $m_j = (m_{j,1}, \dots, m_{j,n}, m_{j,n+1})$ is a multiindex, $(zw)^{m_j} = z_1^{m_{j,1}} \cdots z_n^{m_{j,n}} w^{m_{j,n+1}}$ and the multiindices m_j are ordered so that their norms $|m_j| = m_{j,1} + \cdots + m_{j,n} + m_{j,n+1}$ satisfy $|m_j| \leq |m_{j+1}|$ and by using the lexicographic order when $|m_j| = |m_k|$. The coefficient B_{jk} in (22) is clearly the partial derivative

$$B_{jk} = \frac{1}{m_j! m_k!} \frac{\partial^{|m_j|+|m_k|}}{\partial (zw)^{m_j} (\bar{z}\bar{w})^{m_k}} e^D$$

evaluated at the origin.

Now, by following the same outline of the proof in [34], we first notice that $B_{jk} = 0$ when the "truncated norms" $|m_j|_n = m_{j,1} + \cdots + m_{j,n}$ and $|m_k|_n = m_{k,1} + \cdots + m_{k,n}$ are different or when $m_{j,n+1} \neq m_{k,n+1}$.

Indeed, this follows from the fact that the map $z \mapsto e^{i\theta} z$ is an automorphism of the base domain Ω. Then, since the diastasis $\log N_\Omega(z, \bar{z})$ is invariant by automorphisms, we have $N_\Omega(e^{i\theta} z, e^{-i\theta}\bar{z}) =$

$N_\Omega(z,\bar z)$. Similarly, since the diastasis (21) depends radially on w, also the map $w \mapsto e^{i\phi}w$ is an isometry.

Then, if in the expansion (22) of $e^D - 1 = N_\Omega(z,\bar z)^{-\mu\alpha}[N_\Omega(z,\bar z)^\mu - |w|^2)]^{-\alpha}$ there is a monomial $(zw)^{m_j}(\bar z\bar w)^{m_k}$ (with $B_{jk} \neq 0$) we must have

$$(zw)^{m_j}(\bar z\bar w)^{m_k} = e^{i\theta(|m_j|_n - |m_k|_n)}e^{i\phi(m_{j,n+1}-m_{k,n+1})}(zw)^{m_j}(\bar z\bar w)^{m_k}$$

for every θ, ϕ, which implies the claim.

Since $B_{jk} \neq 0$ implies $m_{j,1} + \cdots + m_{j,n} = m_{k,1} + \cdots + m_{k,n}$ and $m_{j,n+1} = m_{k,n+1}$, it follows that $B_{jk} \neq 0$ implies $|m_j| = |m_k|$. Then, by the ordering chosen for the m_j's, we deduce that the matrix $B = (B_{jk})$ has the following diagonal block structure:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \\ 0 & E_1 & 0 & 0 & 0 & \\ 0 & 0 & E_2 & 0 & 0 & \\ 0 & 0 & 0 & E_3 & 0 & \vdots \\ 0 & 0 & 0 & 0 & E_4 & \\ & & \cdots & & & \end{pmatrix}$$

where E_i contains the entries corresponding to derivatives with respect to multiindices m_j having norm $|m_j| = i$.

In turn, up to rearranging the order of the m_j's having norm i, we can assume that every block E_i has the following diagonal block structure[1]

$$E_i = \begin{pmatrix} F_{z(i),w(0)} & 0 & 0 & 0 & 0 & & 0 \\ 0 & F_{z(0),w(i)} & 0 & 0 & 0 & & 0 \\ 0 & 0 & F_{z(i-1),w(1)} & 0 & 0 & & 0 \\ 0 & 0 & 0 & F_{z(i-2),w(2)} & 0 & \cdots & 0 \\ & & \cdots & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & & F_{z(1),w(i-1)} \end{pmatrix}$$

where we are denoting by $F_{z(i-s),w(s)}$ $(s = 0, \ldots, i)$ the block which contains the entries B_{jk} corresponding to the derivatives with respect to multiindices such that $|m_j|_n = |m_k|_n = i - s$ and $m_{j,n+1} = m_{k,n+1} = s$ i.e., in other words, such that there are $i - s$ derivatives with respect to z and $\bar z$ and s derivatives with respect to w and $\bar w$ (recall that in order for B_{jk} to be non zero we must have $|m_j|_n = |m_k|_n$ and $m_{j,n+1} = m_{k,n+1}$).

In order to prove that the matrix is positive definite, we just need to show that each of these blocks $F_{z(i-s),w(s)}$ (for any $i = 1, 2, \ldots$ and any $s = 0, \ldots, i$) is positive definite.

Let us begin from the block $F_{z(i),w(0)}$, By definition, its entries are obtained by deriving the function $N_\Omega^{-\mu\alpha}(N_\Omega^\mu - |w|^2)^{-\alpha}$ i times with respect to the z's and i times with respect to the $\bar z$'s (with no derivatives with respect to w or $\bar w$). Since the derivatives are evaluated at the origin, one gets the same by applying the same derivatives to the function

$$N_\Omega^{-\mu\alpha}(N_\Omega^\mu)^{-\alpha} = N_\Omega^{-2\alpha\mu} = e^{2\alpha\mu D_0^{g_\Omega}}$$

[1]Rearranging the order has the effect to apply to E_i a permutation both to the rows and the columns, which in turn can be obtained by replacing E_i with a congruent matrix ${}^TPE_iP$ (where P is a permutation matrix), which is permitted since E_i is positive definite if and only if ${}^TPE_iP$ is.

where $D_0^{g_\Omega}$ denotes the diastasis function of g_Ω (cf. (1) above). By Calabi's criterium and Theorem A one gets that the blocks under consideration are positive definite if and only if

$$2\alpha\mu \in W(\Omega) \setminus \{0\}. \tag{23}$$

Let us now consider the block $F_{z(0),w(i)}$. by definition, its entries are obtained by deriving the function $N_\Omega^{-\mu\alpha}(N_\Omega^\mu - |w|^2)^{-\alpha}$ i times with respect to w and i times with respect to $\bar{w}$ (with no derivatives with respect to the z's or the $\bar{z}$'s): since the derivatives are evaluated at the origin, by using the fact that $N_\Omega(0,0) = 1$ one gets the same by applying the same derivatives to the function

$$(1 - |w|^2)^{-\alpha} = \left(\sum_{j=0}^{\infty} |w_j|^2\right)^{\alpha},$$

which shows that the block $F_{z(0),w(i)}$ is positive definite.

Finally, let us consider the block $F_{z(i-s),w(s)}$, which by definition consists of the entries obtained by deriving the function $N_\Omega^{-\alpha\mu}(N_\Omega^\mu - |w|^2)^{-\alpha}$ $i-s$ times with respect to the z's, $i-s$ times with respect to the $\bar{z}$'s, s times with respect to w and s times with respect to $\bar{w}$.

We first notice that

$$\frac{\partial^{2s}}{\partial w^s \partial \bar{w}^s}|_{w=0} N_\Omega^{-\alpha\mu}(N_\Omega^\mu - |w|^2)^{-\alpha} = N_\Omega^{-\alpha\mu} s!\alpha(\alpha+1)\cdots(\alpha+s-1)(N_\Omega^\mu - |w|^2)^{-\alpha-s} \tag{24}$$

Indeed, this immediately follows from the general formula[2] for radial functions $\frac{\partial^{2s}}{\partial w^s \partial \bar{w}^s}|_{w=0} f(|w|^2) = s! f^{(s)}(0)$ and the fact that

$$\frac{d^s}{dx^s}(A - x)^{-\alpha} = \alpha(\alpha+1)\cdots(\alpha+s-1)(A-x)^{-\alpha-s}.$$

Then, since the derivatives are evaluated at the origin, we deduce that the entries of $F_{z(i-s),w(s)}$ can be equivalently be obtained by deriving the function

$$N_\Omega^{-\alpha\mu} s!\alpha(\alpha+1)\cdots(\alpha+s-1)(N_\Omega^\mu)^{-\alpha-s} = s!\alpha(\alpha+1)\cdots(\alpha+s-1)(N_\Omega^\mu)^{-2\alpha-s}$$

$i-s$ times with respect to the z's and $i-s$ times with respect to the $\bar{z}$'s.

Since

$$(N_\Omega^\mu)^{-2\alpha-s} = e^{(2\alpha+s)\mu D_0^{g_\Omega}}.$$

by applying again Calabi's criterium, by Theorem A and taking into account also (23) we finally deduce that $\alpha\hat{g}_{\Omega,\mu}$ is projectively induced if and only if $(2\alpha+s)\mu \in W(\Omega)\setminus\{0\}$ for every integer $s \geq 0$. This concludes the proof of the proposition. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. It is easily seen that Calabi's diastasis function $D_0^{\hat{g}_{\Omega,\mu}}$ for the Kähler metric $\hat{g}_{\Omega,\mu}$ (cf. (10)) on the CH domain $M_{\Omega,\mu} \subset \mathbb{C}^{n+1}$ is given by

$$D_0^{\hat{g}_{\Omega,\mu}}(z,\bar{z}) = -\log(N_\Omega(z,\bar{z})^\mu - |w|^2) - \log N_\Omega^\mu(z,\bar{z})$$

[2]One immediately sees this formula by comparing the Taylor expansions $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)(0)}}{k!} x^k$ (with $x = |w|^2$) and $f(w,\bar{w}) = \sum_{i,j=0}^{\infty} \frac{1}{i!j!} \frac{\partial^{i+j}}{\partial w^i \partial \bar{w}^j}|_{w=0} w^i \bar{w}^j$.

and that it cannot be extended to an open subset of $\mathbb{C}^{n+1}$ strictly containing $M_{\Omega,\mu}$. Thus, the CH domain $(M_{\Omega,\mu}, \hat{g}_{\Omega,\mu})$ is an exact domain (cf. Definition 2). Moreover, the dual $(1,1)$-form

$$\hat{\omega}^*_{\Omega,\mu} = \frac{i}{2\pi}\partial\bar{\partial}\log(N_\Omega(z,-\bar{z})^\mu + |w|^2) + \frac{i}{2\pi}\partial\bar{\partial}\log N^\mu_\Omega(z,-\bar{z}) \tag{25}$$

obtained by (10) with the "dual trick" (8) in Definition 3 turns out to be a Kähler form on the whole $\mathbb{C}^{n+1}$ and if $\hat{g}^*_{\Omega,\mu}$ denotes the associated Kähler metric and $D_0^{\hat{g}^*_{\Omega,\mu}}$ its Calabi's diastasis functions, one has

$$D_0^{\hat{g}^*_{\Omega,\mu}}(z,\bar{z}) = \log(N_\Omega(z,-\bar{z})^\mu + |w|^2) + \log N^\mu_\Omega(z,-\bar{z}). \tag{26}$$

Thus $(\mathbb{C}^{n+1}, \hat{g}^*_{\Omega,\mu})$ is the Kähler dual of $(M_{\Omega,\mu}, \hat{g}_{\Omega,\mu})$. We are going to show that for all $\alpha, \mu \in \mathbb{Z}^+$, the pair $(\mathbb{C}^{n+1}, \alpha\hat{g}^*_{\Omega,\mu})$ has a Fubini-Study compactification. Specifically, for $\mu \in \mathbb{Z}^+$ we will construct a compact Kähler manifold $(P_{\Omega_c,\mu}, g_{P,\mu})$, where $g_{P,\mu}$ is (finitely) projectively induced, along with a holomorphic embedding with dense image $\hat{J} : \mathbb{C}^{n+1} \to P_{\Omega_c,\mu}$ such that $\hat{J}^* g_{P,\mu} = \hat{g}^*_{\Omega,\mu}$. This will complete the proof of Theorem 1.2. Indeed by choosing $\mu \in \mathbb{Z}^+$ and $\alpha \in \mathbb{Z}^+$ sufficiently large, Proposition 3.1 and the structure of $W(\Omega)$ ensure that $(M_{\Omega,\mu}, \alpha\hat{g}_{\Omega,\mu})$ has a Fubini-Study completion (cf. Remark 1). Furthermore, $\alpha g_{P,\mu}$ will be projectively induced using Calabi's map (see [8, Th.13]), and $(P_{\Omega_c,\mu}, \alpha g_{P,\mu})$ will serve as the Fubini-Study compactification of $(\mathbb{C}^{n+1}, \alpha\hat{g}^*_{\Omega,\mu})$.

In order to do that we simplify the notation as follows. Fix $\mu > 0$ and set $F = F_\mu$ for the map given by (12). We also set $N = d_\mu$, and we denote by F_k, $k = 0, \ldots, N$, the sections $s_k^{(\mu)}$ appearing as the components of the map F_μ.

Ultimately, we have an embedding

$$F : \Omega_c \to \mathbb{C}P^N, p \mapsto [F_0(p), \ldots, F_N(p)] \tag{27}$$

such that $F^* g_{FS} = \mu g_{\Omega_c}$ and $F_0(J(z)) \neq 0$, for all $z \in \mathbb{C}^n$, where $J : \mathbb{C}^n \to \Omega_c$ is the embedding given by (6). Moreover, if $U_k = \{[X_0, \ldots, X_N] \in \mathbb{C}P^N \mid X_k \neq 0\}$ are the open affine subsets we have

$$\Omega_c = J(\mathbb{C}^n) \sqcup F^{-1}(\{X_0 = 0\}). \tag{28}$$

Notice also that by

$$\mu\omega_{\Omega_c|F^{-1}(U_0)} = F^*\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log(1 + \sum_{k=1}^N |\frac{F_k(p)}{F_0(p)}|^2) = \frac{i}{2\pi}\partial\bar{\partial}\log\frac{\|F(p)\|^2}{|F_0(p)|^2}$$

and

$$J^*(\mu\omega_{\Omega_c}) = \frac{i}{2\pi}\partial\bar{\partial}\log N^\mu_\Omega(z,-\bar{z})$$

we deduce that

$$N^\mu_\Omega(z,-\bar{z}) = \frac{\|F(J(z))\|^2}{|F_0(J(z))|^2}. \tag{29}$$

Let now $P(\mathbb{C} \oplus O(1))$ be the $\mathbb{C}P^1$-bundle over $\mathbb{C}P^N \times \mathbb{C}P^N$, where $O(1) \to \mathbb{C}P^N$ is the hyperplane bundle. Denote by $Q_{\mathbb{C}P^N} = \Delta^*_{\mathbb{C}P^N} P(\mathbb{C} \oplus O(1))$ the pull-back bundle over $\mathbb{C}P^N$, where $\Delta_{\mathbb{C}P^N}$ is the diagonal map of $\mathbb{C}P^N \to \mathbb{C}P^N \times \mathbb{C}P^N$. Analogously let

$$P_{\Omega_c,\mu} = \Delta^*_{\Omega_c} P(\mathbb{C} \oplus L^{\otimes\mu})$$

be the $\mathbb{C}P^1$-bundle over Ω_c where $P(\mathbb{C} \oplus L^{\otimes\mu})$ is the $\mathbb{C}P^1$-bundle over $\Omega_c \times \Omega_c$, $L^{\otimes\mu} = F^*(O(1))$ and Δ_{Ω_c} is the diagonal map of $\Omega_c \to \Omega_c \times \Omega_c$.

Then we have a natural holomorphic embedding with dense image of $\mathbb{C}^{n+1}$ into $P_{\Omega_c,\mu}$ given by

$$\hat{J}: \mathbb{C}^{n+1} \to F^{-1}(U_0) \times \mathbb{C}P^1 \subseteq P_{\Omega_c,\mu}, \;\; (z,w) \mapsto (J(z), [1,w]). \tag{30}$$

Consider the biholomorphic map Ψ from $Q_{\mathbb{C}P^N}$ into the blowup $\mathrm{Bl}_{[0,\dots,0,1]}(\mathbb{C}P^{N+1})$ of $\mathbb{C}P^{N+1}$ at the point $[0,\dots,0,1]$ given locally on the trivializing open subset $U_k \times \mathbb{C}P^1 \subseteq Q_{\mathbb{C}P^N}$, $k = 0,\dots,N$, by

$$\begin{gathered}\Psi_{|U_k\times\mathbb{C}P^1}: U_k \times \mathbb{C}P^1 \to \mathrm{Bl}_{[0,\dots,0,1]}(\mathbb{C}P^{N+1}) \subset \mathbb{C}P^{N+1}\times\mathbb{C}P^N,\\ ([Z_0,\dots,Z_N],[W_0,W_1]) \mapsto ([Z_0W_0,\dots,Z_NW_0,Z_kW_1],[Z_0,\dots,Z_N]).\end{gathered} \tag{31}$$

It is easily seen (see e.g. [20, Chapter V]) that the maps $\Psi_{|U_k\times\mathbb{C}P^1}$ glue to a well-defined biholomorphism $\Psi: Q_{\mathbb{C}P^N} \to \mathrm{Bl}_{[0,\dots,0,1]}(\mathbb{C}P^{N+1})$). Recall that

$$\mathrm{Bl}_{[0,\dots,s0,1]}(\mathbb{C}P^{N+1}) = \{([X_0,\dots,X_N,X_{N+1}],[Y_0,\dots,Y_N]) \mid X_iY_j = X_jY_i,\ i,j=0,\dots,N\,\}. \tag{32}$$

Since the latter is a submanifold of the product $\mathbb{C}P^{N+1}\times\mathbb{C}P^N$, it can be embedded into a complex projective space by using the restriction of the Segre embeddings:

$$\mathrm{Bl}_{[0,\dots,0,1]}(\mathbb{C}P^{N+1}) \subset \mathbb{C}P^{N+1}\times\mathbb{C}P^N \overset{Segre}{\to} \mathbb{C}P^K,\ K = \frac{(N+1)(N+4)}{2}$$

Thus, one can consider the holomorphic embedding

$$\varphi = Segre \circ \Psi \circ \varphi_{P_{\Omega_c,\mu}} : P_{\Omega_c,\mu} \to \mathbb{C}P^K,$$

where $\varphi_{P_{\Omega_c,\mu}} : P_{\Omega_c,\mu} \to Q_{\mathbb{C}P^N}$ is given locally by

$$F^{-1}(U_k)\times\mathbb{C}P^1 \to U_k\times\mathbb{C}P^1,\ (p,[W_0,W_1]) \mapsto (F(p),[W_0,W_1]). \tag{33}$$

Thus, for $\mu \in \mathbb{Z}^+$ we define the desired projectively induced metric $g_{P,\mu}$ on $P_{\Omega_c,\mu}$ by

$$g_{P,\mu} := \varphi^* g_{FS}. \tag{34}$$

It remains then to show that $\hat{J}^* g_{P,\mu} = \hat{g}^*_{\Omega,\mu}$ (and hence $(P_{\Omega_c,\mu}, g_{P,\mu})$ will be the desired Fubini-Study compactification of $(\mathbb{C}^{n+1}, \hat{g}^*_{\Omega,\mu})$, with $\hat{J}: \mathbb{C}^{n+1} \to (P_{\Omega_c,\mu}, g_{P,\mu})$). In order to do this, notice that by (31) the embedding

$$\Psi \circ \varphi_{P_{\Omega_c,\mu}} : P_{\Omega_c,\mu} \to \mathrm{Bl}_{[0,\dots 0,1]}(\mathbb{C}P^{N+1}) \subset \mathbb{C}P^{N+1}\times\mathbb{C}P^N$$

reads on $F^{-1}(U_k)\times\mathbb{C}P^1$ as

$$\Psi \circ \varphi_{P_{\Omega_c,\mu}|F^{-1}(U_k)\times\mathbb{C}P^1} : (p,[W_0,W_1]) \mapsto ([F(p)W_0, F_k(p)W_1],[F(p)]). \tag{35}$$

Since the pull-back of the Fubini-Study metric on $\mathbb{C}P^K$ via the Segre embedding is given by the Kähler product of the Fubini-Study metrics on $\mathbb{C}P^{N+1}$ and $\mathbb{C}P^N$, one gets that the Kähler form $\omega_{P,\mu}$ associated to $g_{P,\mu}$ is given on $F^{-1}(U_k)\times\mathbb{C}P^1$ by:

$$\omega_{P,\mu|F^{-1}(U_k)\times\mathbb{C}P^1} = \frac{i}{2\pi}\partial\bar\partial\log\left(\frac{\|F(p)\|^2}{|F_k(p)|^2}|W_0|^2 + |W_1|^2\right) + \frac{i}{2\pi}\partial\bar\partial\log\frac{\|F(p)\|^2}{|F_k(p)|^2}. \tag{36}$$

Notice that

$$\hat{J}(\mathbb{C}^{n+1}) \subset F^{-1}(U_0)\times\{W_0\neq 0\},\ \varphi_{P_{\Omega_c,\mu}}(F^{-1}(U_0)\times\{W_0\neq 0\}) \subset U_0\times\{W_0\neq 0\}$$

$$\Psi(U_0\times\{W_0\neq 0\}) \subset \{X_0\neq 0\}\times\{Y_0\neq 0\},\ Segre(\{X_0\neq 0\}\times\{Y_0\neq 0\}) \subset \{T_0\neq 0\}.$$

where $[T_0, \dots, T_K]$ denote the homogeneous coordinates of $\mathbb{C}P^K$. Hence, by passing to affine coordinates $(\frac{X_1}{X_0}, \dots, \frac{X_{N+1}}{X_0})$, $(\frac{Y_1}{Y_0}, \dots, \frac{Y_N}{Y_0})$, $(\frac{T_1}{T_0}, \dots, \frac{T_K}{T_0})$ and by using (36) (with $k = 0$) one gets

$$\omega_{P,\mu|F^{-1}(U_0)\times\{W_0\neq 0\}} = \frac{i}{2\pi}\partial\bar{\partial}\log\left(\frac{\|F(p)\|^2}{|F_0(p)|^2} + |w|^2\right) + \frac{i}{2\pi}\partial\bar{\partial}\log\frac{\|F(p)\|^2}{|F_0(p)|^2},$$

Then (29) and (30) yield

$$\hat{J}^*\omega_{P,\mu} = \frac{i}{2\pi}\partial\bar{\partial}\log\left(\frac{\|F(J(z))\|^2}{|F_0(J(z))|^2} + |w|^2\right) + \frac{i}{2\pi}\partial\bar{\partial}\log\left(\frac{\|F(J(z))\|^2}{|F_0(J(z))|^2}\right)$$

$$= \frac{i}{2\pi}\partial\bar{\partial}\log\left(N_\Omega(z, -\bar{z})^\mu + |w|^2\right) + \frac{i}{2\pi}\partial\bar{\partial}\log\left(N_\Omega(z, -\bar{z})^\mu\right) = \hat{\omega}^*_{\Omega,\mu}.$$

This concludes the proof of the theorem. □

Remark 6. With the notation introduced in the proof of Theorem 1.2 one has

$$P_{\Omega_c,\mu} = \hat{J}(\mathbb{C}^{n+1}) \sqcup \varphi^{-1}(\{T_0 = 0\})$$

in analogy with (28) for Ω_c.

One can wonder if there exist any non integer values of α e μ such that $\alpha\hat{g}^*_{\Omega,\mu}$ is projectively induced (in analogy with Proposition 2.1 for the metric $\alpha g^*_{\Omega,\mu}$). This cannot happen.

Proposition 3.2. *$\alpha\hat{g}^*_{\Omega,\mu}$ is projectively induced iff $\alpha, \mu \in \mathbb{Z}^+$.*

Proof. If $\alpha, \mu \in \mathbb{Z}^+$ then the proof of Theorem 1.2 shows that $\alpha g_{P,\mu}$ and hence $\alpha\hat{g}^*_{\Omega,\mu}$ is finitely projectively induced. On the other hand, if $\alpha\hat{g}^*_{\Omega,\mu}$ is projectively induced then by taking $z = 0$ in (26) we see that

$$\alpha\frac{i}{2\pi}\partial\bar{\partial}\log(1 + |w|^2) = \alpha g_{FS|\mathbb{C}},$$

where $\mathbb{C} = U_0 = \{W_0 \neq 0\}$ with affine coordinate $w = \frac{W_1}{W_0}$, is projectively induced, forcing $\alpha \in \mathbb{Z}^+$. It remains to prove that if $\alpha\hat{g}^*_{\Omega,\mu}$ is projectively induced then $\alpha, \mu \in \mathbb{Z}^+$. As in the proof of Proposition 2.1, namely by first restricting to $\{(\xi, w) \in \mathbb{C}^2 \mid \xi = 0\}$ and $\{(\xi, w) \in \mathbb{C}^2 \mid w = 0\}$, and then to $\{(\xi, w) \in \mathbb{C}^2 \mid w = 1\}$, one deduces that if $\alpha\hat{g}^*_{\Omega,\mu}$ is projectively induced then the same holds true for the Kähler metric g whose Calabi's diastasis function at the origin is given by

$$D(\xi) = bk\log\left(\left(\left(1 + |\xi|^2\right)^{\frac{a}{2b}} + 1\right)\left(1 + |\xi|^2\right)^{\frac{a}{2b}}\right) - bk\log 2$$

for some $a, b, k \in \mathbb{Z}^+$ and a, b coprime. Following the exact same line of reasoning as in the proof of Proposition 2.1, one can deduce that the condition $\frac{a}{2b} \notin \mathbb{Z}$ implies that at least one of the coefficients of the Taylor expansion in ξ and $\bar{\xi}$ around the origin of $e^D - 1$ is negative. Therefore, the conclusion follows by Calabi's criterium. □

By combining Proposition 3.2 and Theorem 1.2 we immediately get

Corollary 3.3. *$(\mathbb{C}^{n+1}, \alpha\hat{g}^*_{\Omega,\mu})$ has a Fubini-Study compactification iff $\alpha, \mu \in \mathbb{Z}^+$.*

3.1. $(P_{\Omega_c,\mu}, g_{P,\mu})$ **as** $C1$ G**-space.** In this subsection we show that $(P_{\Omega_c,\mu}, g_{P,\mu})$ is a *cohomogeneity* 1 *space*, for all $\mu \in \mathbb{Z}^+$ (see Proposition 3.5 below). Since $(P_{\Omega_c,\mu}, g_{P,\mu})$ is a Fubini-Study compactification of the exact domain $(\mathbb{C}^{n+1}, \hat{g}^*_{\Omega,\mu})$, which admits a Kähler dual given by $(M_{\Omega,\mu}, \hat{g}_{\Omega,\mu})$, this proposition shows that the condition of homogeneity in Theorem 1.3 cannot be relaxed to cohomogeneity 1. Before proving this fact, we briefly review some basic facts about cohomogeneity 1 G-spaces, referring the reader to [40], [2] and [3] for more detailed information on the subject.

Given a semisimple Lie group G, a Kähler manifold (P, ω_P) is called a *cohomogeneity* 1 G-*space* (from now on, $C1$ G-space) if there exists an action by holomorphic isometries of G on P which has an orbit of (real) codimension 1 in P. Such an orbit is called a *regular orbit*, and the union of regular orbits is called the *regular set* P_{reg}; the orbits of higher codimension are called *singular orbits*.

A compact C1 G-space P has regular orbits of the kind G/S (where S is the stabilizer of a regular point) and two singular orbits $G/H_0, G/H_1$. We say that the action of G on P is *ordinary* if

$$\dim N_G(S) = \dim S + 1, \tag{37}$$

where $N_G(S)$ is the normalizer of S in G. This assumption allows us to associate to the regular orbits G/S a flag manifold $G/N_G(S)$. This flag manifold is called *the flag associated to the regular orbits*.

Let now Ω_c be an irreducible Hermitian symmetric space of compact type (i.e. the dual of a Cartan domain Ω). Write $\Omega_c = G/K$, where G is a simple Lie group and K is a maximal compact subgroup of G.

Notice that the $\mathbb{C}P^1$-bundle $P_{\Omega_c,\mu} \to \Omega_c$ can be trivialized on the open subsets $F^{-1}(U_j) = \{p \in M \mid F_j(p) \neq 0\}$ with the following identifications given by the transition functions

$$F^{-1}(U_j) \times \mathbb{C}P^1 \ni (p, [W_0, W_1]) \simeq \left(p, \left[W_0, \frac{F_j(p)}{F_k(p)} W_1\right]\right) \in F^{-1}(U_k) \times \mathbb{C}P^1,$$

where $F : \Omega_c \to \mathbb{C}P^N$ is given by (27).

Define an action of G on $P_{\Omega_c,\mu}$ locally in the following way. Given

$$(p, [W_0, W_1]) \in F^{-1}(U_j) \times \mathbb{C}P^1 \subset P_{\Omega_c,\mu},$$

and $gp \in F^{-1}(U_k)$, then

$$g \cdot (p, [W_0, W_1]) = \left(gp, \left[W_0, \frac{F_j(p)}{\lambda_g^{-1}(F_k(gp))} W_1\right]\right) \in F^{-1}(U_k) \times \mathbb{C}P^1, \tag{38}$$

where $\lambda_g : L_p^{\otimes\mu} \to L_{gp}^{\otimes\mu}$ denotes a lifting to the fibers of the action map $\Omega_c \to \Omega_c, p \mapsto gp$. Checking that this local definition extends to a well-defined action on $P_{\Omega_c,\mu}$ and that it is in fact a group action are straightforward by using the transition functions, and we omit that.

Let us see more in detail the structure of the orbits in order to see that this is a C1 G-action. This is done more easily by using the fact that, as we have seen in the proof of Theorem 1.2 (see (35)), the $\mathbb{C}P^1$-bundle $P_{\Omega_c,\mu}$ can be seen as a submanifold of the blowup $Bl_{[0,\ldots,0,1]}(\mathbb{C}P^{N+1})$ via the embedding whose local expression on $F^{-1}(U_k) \times \mathbb{C}P^1$ is

$$(p, [W_0, W_1]) \mapsto ([F(p)W_0, F_k(p)W_1], [F(p)]).$$

Then, by (38), one can easily see that the action can be written as

$$g\cdot\left(\left[\frac{F(p)}{F_j(p)}W_0,W_1\right],[F(p)]\right)=\left(\left[\frac{\lambda_g^{-1}\left(F(gp)\right)}{F_j(p)}W_0,W_1\right],[F(gp)]\right), \tag{39}$$

for any $p\in\Omega_c$ with $F_j(p)\neq 0$. s

Now, the singular orbits of this action (39) can be easily described as follows:

- if $W_0=0$, we have $g\cdot([0,\dots,0,1],[F(p)])=([0,\dots,0,1],[F(gp)])$ and then we have a singular orbit isomorphic to $F(\Omega_c)$;
- if $W_1=0$, we have $g\cdot([F(p),0],[F(p)])=([F(gp),0],[F(gp)])$ and we get again a singular orbit isomorphic to $F(\Omega_c)$.

Hence, in this case $G/H_0\cong G/H_1\cong G/K$.

The following lemma provides a description of the orbits when $W_0,W_1\neq 0$.

Lemma 3.4. *The orbit of a point* $\left(\left[\frac{W_0}{W_1}\frac{F(p)}{F_k(p)},1\right],[F(p)]\right)\in P_{\Omega_c,\mu}$ *with* $W_0,W_1\neq 0$ *can be identified with the subset of the sphere* $S^{2N+1}(r)\subseteq\mathbb{C}^{N+1}$ *of radius* $r=\|\frac{W_0}{W_1}\frac{F(p)}{F_k(p)}\|$ *such that the image of this subset in* $S^{2N+1}(r)/\{e^{i\phi}\}=\mathbb{C}P^N$ *is exactly* $F(\Omega_c)$.

Proof. Note that by (39) we can write

$$g\cdot\left(\left[\frac{W_0}{W_1}\frac{F(p)}{F_k(p)},1\right],[F(p)]\right)=\left(\left[\frac{W_0}{W_1}\frac{\lambda_g^{-1}(F(gp))}{F_k(p)},1\right],[F(gp)]\right). \tag{40}$$

Note also that

$$\|\frac{F(p)}{F_k(p)}\|=\|\frac{\lambda_g^{-1}\left(F(gp)\right)}{F_k(p)}\|, \tag{41}$$

for any $g\in G$ and any $p\in M$ with $F_k(p)\neq 0$. This can be proved, for example, by using Calabi's rigidity theorem.

From (40) and (41) one can define the following map from the orbit $\mathcal{O}$ of the point $\left(\left[\frac{W_0}{W_1}\frac{F(p)}{F_k(p)},1\right],[F(p)]\right)$ to the sphere $S^{2N+1}(r)\subset\mathbb{C}^{N+1}$ of radius $r=\|\frac{W_0}{W_1}\frac{F(p)}{F_k(p)}\|$:

$$f:\mathcal{O}\to S^{2N+1}(r),\quad \left(\left[\frac{W_0}{W_1}\frac{\lambda_g^{-1}(F(gp))}{F_k(p)},1\right],[F(gp)]\right)\mapsto\frac{W_0}{W_1}\frac{\lambda_g^{-1}(F(gp))}{F_k(p)} \tag{42}$$

One immediately sees that f is injective and is in fact a diffeomorphism onto its image, so that the orbit identifies with the set

$$X:=\left\{\frac{W_0}{W_1}\frac{\lambda_g^{-1}(F(gp))}{F_k(p)}\mid g\in G\right\}\subset S^{2N+1}(r)$$

We have to prove that $X=p^{-1}(F(\Omega_c))$, where

$$p:S^{2N+1}(r)\to S^{2N+1}(r)/\{e^{i\phi}\}=\mathbb{C}P^N$$

denotes the canonical projection to the quotient.

The inclusion $X\subset p^{-1}(F(\Omega_c))$ is straightforward. Hence, it remains to prove that for any $e^{i\phi}\in\mathbb{C}$ there exists $g\in G$ such that

$$\frac{\lambda_g^{-1}(F(gp))}{F_k(p)}\frac{W_1}{W_0}=e^{i\phi}\frac{F(p)}{F_k(p)}\frac{W_1}{W_0}. \tag{43}$$

In order to show this, notice that, since $\Omega_c = G/K$ is a symmetric space, the isotropy group K of a point p is the centralizer of a one-dimensional compact torus T. Since $gp = p$ for every $g \in T$, the lifting of the action of G to $L^{\otimes\mu}$ induces then an isomorphism $\lambda_g : L_p^{\otimes\mu} \to L_p^{\otimes\mu}$ of the fiber $L_p^{\otimes\mu}$, which yields a one-dimensional representation $\rho : T \to Aut(L_p^{\otimes\mu}) \simeq \mathbb{C}^*$ of the torus T. Being $U(1)$ the only non-trivial connected compact subgroup of $\mathbb{C}^*$, it must be either $\rho(T) = \{1\}$ or $\rho(T) = U(1)$. But it cannot be $\rho(T) = \{1\}$ since it is known (see, for example, [42]) that every complex line bundle on G/K is completely determined by the representation of T on a fiber, and this representation is trivial if and only if the line bundle is the trivial bundle, which is not the case.

Then, we have $\rho(T) = U(1)$ and so for every $e^{i\phi}$ there exists $g \in T$ such that

$$\frac{\lambda_g^{-1}(F(gp))}{F_k(p)} = e^{i\phi}\frac{F(p)}{F_k(p)}$$

which proves (43) and concludes the proof of the lemma. □

Proposition 3.5. *$P_{\Omega_c,\mu}$ is a cohomogeneity 1 G-space, the action of G is ordinary and the flag associated to the regular orbit of this action is $\Omega_c = G/K$. Moreover, $g_{P,\mu}$ is a G-invariant Kähler metric.*

Proof. Notice that the regular orbits are the orbits of the points such that $W_0, W_1 \neq 0$. Indeed, on the one hand we have seen in Lemma 3.4 that these orbits identify with $p^{-1}(F(\Omega_c))$, and then their (real) dimension is

$$2\dim_{\mathbb{C}}(F(\Omega_c)) + 1 = 2\dim_{\mathbb{C}}(\Omega_c) + 1.$$

On the other hand, since P_{Ω_c} is a $\mathbb{C}P^1$-bundle on Ω_c we have

$$\dim_{\mathbb{R}}(P_{\Omega_c,\mu}) = 2(\dim_{\mathbb{C}}(\Omega_c) + 1) = 2\dim_{\mathbb{C}}(\Omega_c) + 2.$$

and then the real codimension of a regular orbit is exactly 1.

In order to prove that the action is ordinary, let $S = \{g \in G \mid gu_0 = u_0\}$ be the stabilizer of a regular point $u_0 \in f(\mathcal{O}) \subset S^{2N+1}(r)$ (see (42)). Notice that

$$N_G(S) = Y,$$

where

$$Y := \{g \in G \mid gu_0 = \lambda u_0, \text{ for some } \lambda \in \mathbb{C}\ \}.$$

Indeed, if $gu_0 = \lambda u_0$ we have, for any $s \in S$,

$$gsg^{-1}u_0 = gs\lambda^{-1}u_0 = \lambda^{-1}gu_0 = u_0$$

and then $g \in S$. This proves that $Y \subseteq N_G(S)$. Now, clearly

$$Y = \{g \in G \mid g[u_0] = [u_0]\} = K, \quad [u_0] \in S^{2N+1}(r)/\{e^{i\phi}\},$$

where K is the isotropy subgroup of the symmetric space $F(\Omega_c) \simeq \Omega_c = G/K$.

So $K \subseteq N_G(S)$: but since $\Omega_c = G/K$ is symmetric we have that K is a maximal compact subgroup in G and then we must in fact have $K = N_G(S)$.

Now, the fact we proved in Lemma 3.4 that the orbit G/S identifies with $p^{-1}(G/K)$, implies that $\dim(G/S) = \dim(G/K) + 1$. This means that $\dim(K) = \dim(S) + 1$ which, combined with $K = N_G(S)$, shows that (37) holds true and hence $P_{\Omega_c,\mu}$ is an ordinary C1 space. Finally the fact that g_P is G-invariant follows by combining (36) and (41). □

4. THE PROOF OF THEOREM 1.3

In order to prove Theorem 1.3 we start with some remarks on Calabi's diastasis function and Kähler duality.

Recall (see [8]) that among all the potentials of a real analytic Kähler metric g on a complex manifold M, Calabi's diastasis function characterized by

$$D_0^g(z) = \sum_{|I|,|J|\geq 0} a_{IJ} z^I \bar{z}^J,\ a_{J0} = a_{0J} = 0. \tag{44}$$

for any choice of local complex coordinates (z) centred at p.

Now, if (U, g) is an exact domain with associated Kähler form $\omega = \frac{i}{2\pi}\partial\bar{\partial}D_0^g$ which admits a Kähler dual (U^*, g^*) then, by (8), $D_0^{g^*}(z, \bar{z}) = -D_0^g(z, -\bar{z})$ turns out to be real-valued. Viceversa, if there exists a neighbourhood of 0 such that

$$(D_0^g)^*(z, \bar{z}) := -D_0^g(z, -\bar{z})$$

is real-valued then (U, g) admits a Kähler dual. Indeed, notice that

$$\frac{\partial^2 (D_0^g)^*}{\partial z_\alpha \partial \bar{z}_\beta}(0) = \frac{\partial^2 D_0^g}{\partial z_\alpha \partial \bar{z}_\beta}(0)$$

and since the matrix on the right hand side is positive definite it follows that the matrix on the left hand side is positive definite on a neighborhood of 0. Thus $\omega^* = \frac{i}{2\pi}\partial\bar{\partial}(D_0^g)^*$ is a Kähler form in this neighborhood. If g^* is the Kähler metric associated to ω^* then (U^*, g^*) is a Kähler dual of (U, g), where U^* is the maximal domain of extension of $(D_0^g)^*$. Indeed the later turns out to be the diastasis function at 0 for the metric g^* on U^*, i.e.

$$(D_0^g)^*(z, \bar{z}) = D_0^{g^*}(z, \bar{z}).$$

This can be seen by noticing that

$$(D_0^g)^*(z) = \sum_{|I|,|J|\geq 0} a^*_{IJ} z^I \bar{z}^J = \sum_{|I|,|J|\geq 0} (-1)^{|J|} a_{IJ} z^I \bar{z}^J \tag{45}$$

and hence $a^*_{J0} = a^*_{0J} = 0$ which proves our claim by the very definition of the Calabi's diastasis function.

More generally, we have $a^*_{IJ} = -a_{IJ}(-1)^{|J|}$. Hence, the fact that $D_0^{g^*}(z, \bar{z}) = -D_0^g(z, -\bar{z})$ is real valued gives $a^*_{JI} = \overline{a^*_{IJ}}$, i.e.

$$-a_{JI}(-1)^{|I|} = -\overline{a_{IJ}}(-1)^{|J|}$$

which combined with $a_{JI} = \overline{a_{IJ}}$ yields

$$(-1)^{|I|} = (-1)^{|J|},\ \text{if}\ \ a_{IJ} \neq 0. \tag{46}$$

In view of (46) we give the following

Definition 4. *A monomial $a_{IJ} z^I \bar{z}^J$ with $a_{IJ} \neq 0$ in the expansion of the Calabi diastasis function $D_0^g(z, \bar{z})$ is called a* forbidden monomial of kind $(|I|, |J|)$ *if* $(-1)^{|I|} \neq (-1)^{|J|}$.

Thus we obtain

Lemma 4.1. *An exact domain (U, g) admits a Kähler dual (U^*, g^*) iff the expansion of $D_0^g(z, \bar{z})$ does not contain forbidden monomials.*

It is known (see [1] and references therein) that invariant complex structures on a flag manifold $F = G/K$ are in one-to-one correspondence with *maximal closed nonsymmetric subsets* Q of the set R_M (see [32, Def.5]) of the black roots of G/K. More precisely, the manifold G/K endowed with the complex structure J_Q corresponding to Q is biholomorphic to the complex homogeneous manifold $G^{\mathbb{C}}/K^{\mathbb{C}}G^Q$, where $G^Q = \exp(\mathfrak{g}^Q)$ and $\mathfrak{g}^Q = \sum_{\alpha\in Q}\mathbb{C}E_\alpha$, where E_α is the so called *root vector* associated to α. Since the product $G^{\mathbb{C}}_{reg} = G^{-Q}K^{\mathbb{C}}G^Q$ (where $G^{-Q} = \exp(\mathfrak{g}^{-Q})$ and $\mathfrak{g}^{-Q} = \sum_{\alpha\in -Q}\mathbb{C}E_\alpha$) defines an open dense subset in $G^{\mathbb{C}}$, its image in $G^{\mathbb{C}}/K^{\mathbb{C}}G^Q$ via the natural projection $G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}G^Q$ defines an open dense subset in G/K, denoted $\Omega_{reg} = G^{\mathbb{C}}_{reg}/K^{\mathbb{C}}G^Q$. Clearly, $\Omega_{reg} \simeq G^{-Q}$. Then, one can define a biholomorphism by

$$z = (z_\alpha)_{\alpha\in -Q} \in \mathbb{C}^n \mapsto \exp(Z(z)) \in G^{-Q} \simeq \Omega_{reg} \subseteq F \tag{47}$$

where

$$Z(z) = \sum_{\alpha\in -Q} z_\alpha E_\alpha \tag{48}$$

(where n is the cardinality of Q). We call the corresponding system of complex coordinates $z_1, \dots, z_n$ on Ω_{reg}, the *Alekseevsky–Perelomov coordinates*. We then define a holomorphic embedding

$$J_F : \mathbb{C}^n \to F, z \mapsto [\exp(Z(z))] \tag{49}$$

with dense image $J_F(\mathbb{C}^n) = \Omega_{reg}$. The embedding J_F is the natural extension of the embedding $J : \mathbb{C}^n \to \Omega_c$ given in (6) for a Hermitian symmetric space of compact type Ω_c (see [21, Proof of Th.7.1, Chapter VIII Section 7, p.392]).

Given any G-invariant Kähler metric g_F on F, one has

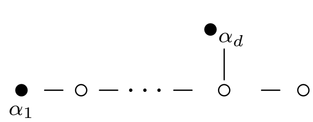
$$J_F^*\omega_F = \frac{i}{2\pi}\partial\bar\partial\sum_{j=1}^{p} c_j \log(\Delta_j)$$

where p is the number of the black nodes in the painted Dynkin diagram of G/K, $c_j > 0$, $j = 1, \dots, p$ and the Δ_j's are suitable minors of the matrix $A = {}^T\overline{\exp(Z)}\exp(Z)$, depending on the position of the black nodes in the diagram, called *admissible minors*.

In order to prove Theorem 1.3, namely that if the exact domain $(\mathbb{C}^n, J_F^* g_F)$ admits a Kähler dual then (F, g_F) is a Hermitian symmetric space of compact type, we need the following

Lemma 4.2. *(*[32, Th.1]*) Let (F, g_F) be an irreducible flag manifold of classical type with second Betti number $b_2(F) = p$, with associated Kähler form ω_F determined by coefficients $c_{i_1}, \dots, c_{i_p} > 0$ associated to the black nodes $\alpha_{i_1}, \dots, \alpha_{i_p}$ of its painted diagram. Then, the Alekseevsky–Perelomov coordinates $z_1, \dots, z_n$ on $(\mathbb{C}^n, J_F^* g_F)$ are Bochner, up to rescaling, only in the following cases:*

(i) $p = 1$, *for every G and every ω;*
(ii) $p = 2$, $G = SU(d)$ *and* $c_{i_1} = c_{i_2}$ *;*
(iii) $p = 2$, $G = SO(2d)$, *the painted diagram of F is*



and $c_1 = 2c_d$

Let us recall that the Bochner coordinates are those complex coordinates $w_1, \dots, w_n$ in a neighbourhood of the origin (uniquely defined up to a unitary transformation) such that

$$D_0^{J_F^* g_F}(w, \bar{w}) = |w|^2 + \sum_{|J|,|K| \geq 2} b_{JK} w^J \bar{w}^K$$

Notice that to prove Theorem 1.3 we can assume (F, g_F) is irreducible. Indeed if F is not irreducible, then its painted Dynkin diagram is given by the disjoint union of (connected) painted Dynkin diagrams of simple groups, and the coordinates are Bochner if and only if are the coordinates on each factor.

Thus Lemma 4.2 implies that, except for the cases (i)-(iii), the expansion of $D_0^{J_F^* g_F}(z, \bar{z})$ with respect to Alekseevsky–Perelomov coordinates contains forbidden monomials[3] of the kind $z_{i_1} \bar{z}_{i_2} \cdots \bar{z}_{i_k}$ (or their conjugates).

Then, in order to analyze when $(\mathbb{C}^n, J_F^* g_F)$ can admit a Kähler dual, we just need to consider cases (i)-(iii).

In fact, in both cases (ii) and (iii), the proof of Lemma 4.2 (see [32, cases 3.2.1 and 3.2.2 in Th.1]) shows that the diastasis is of the kind

$$D_0^{J_F^* g_F} = c_1 \log \Delta_1 + c_2 \log \Delta_2, \; c_1, c_2 \in \mathbb{R}^+ \tag{50}$$

where both Δ_1 and Δ_2 contain forbidden monomials. More precisely, write locally

$$\Delta_j = 1 + q_j + m_j + O_{\geq 4},$$

where q_j is the $(1,1)$-part and m_j denotes the sum of the forbidden terms of type $(1,2)$ (together with their conjugates). In cases (ii) and (iii) of Lemma 4.2, one has $q_1 \neq q_2$ and respectively

$$m_1 = -m_2 =: m$$

or

$$m_1 = -\frac{1}{2} m_2 =: n.$$

Since

$$\log(1 + q_j + m_j + \cdots) = q_j + m_j - q_j m_j + \cdots,$$

the terms of type $(1,2)$ cancel for $c_1 = c_2$ in case (ii), and for $c_1 = 2c_2$ in case (iii). However, the next contribution of forbidden bidegree is

$$c_1(-q_1 m_1) + c_2(-q_2 m_2),$$

which equals $c_1(q_2 - q_1)m$, up to a non-zero constant, in case (ii), and $2c_2(q_2 - q_1)n$, up to sign, in case (iii). Hence the diastasis still contains forbidden monomials of type $(2,3)$. Therefore, also in cases (ii) and (iii) the diastasis contains a forbidden monomial, and we are left with considering only case (i). When (F, g_F) is not symmetric, we are going to show that the admissible minor always contains forbidden monomials of type $(2,3)$ and then the proof of Theorem 1.3 will follow by Lemma 4.1. In order to do that, we pause to prove the following

Lemma 4.3. *Let $G = SU(n), Sp(n), SO(2n), SO(2n+1)$ and let $(F = G/K, g_F)$ be a nonsymmetric flag manifold represented by a painted diagram with only one black node with corresponding admissible minor Δ_r. Then the only forbidden monomials of types $(2,3)$ contained in Δ_r are of one of the following kinds*

---

[3] In fact, in the proof of [32, Th.1]) it is proven that, except for cases (i)-(iii), the expansion of the diastasis always contains forbidden monomial of type $(1,2)$.

(51) $$-\frac{1}{2} Z_{\gamma i} Z_{\alpha j} \bar{Z}_{\alpha i} \bar{Z}_{\beta j} \bar{Z}_{\gamma\beta}$$

(52) $$+\frac{1}{2} Z_{\alpha i} Z_{\gamma j} \bar{Z}_{\alpha i} \bar{Z}_{\beta j} \bar{Z}_{\gamma\beta}$$

where $1 \leq i, j \leq r$ *and* $\alpha, \beta, \gamma > r$ *and* $Z_{\delta\epsilon} := Z_{\delta\epsilon}(z)$.

Proof. It can be verified by a case-by-case analysis that, under the assumptions, we always have $Z^3 = 0$ and then $\exp(Z) = I + Z + \frac{1}{2}Z^2$. Then, by denoting $A = {}^T\overline{\exp(Z)}\exp(Z)$, we have

$$A = I + (Z + {}^T\bar{Z}) + (\frac{1}{2}Z^2 + {}^T\bar{Z}Z + \frac{1}{2}{}^T\bar{Z}^2) + (\frac{1}{2}{}^T\bar{Z}Z^2 + \frac{1}{2}{}^T\bar{Z}^2 Z) + \frac{1}{4}{}^T\bar{Z}^2 Z^2$$

Now, by this equality, the very definition of determinant

$$\Delta_r(A) = \sum_{\sigma \in S_r} s(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{r\sigma(r)}$$

and the fact that $Z_{ij} = 0$ for $i, j \leq r$ we get that a forbidden monomial of type $(2,3)$ can occur in $\Delta_r(A)$ only as one of the following addenda:

(53) $$(\frac{1}{2}{}^T\bar{Z}^2)_{ij}(\frac{1}{2}{}^T\bar{Z}Z^2)_{ji} = -\frac{1}{4} Z_{\gamma i} Z_{\beta\gamma} \bar{Z}_{\alpha i} \bar{Z}_{j\alpha} \bar{Z}_{\beta j}$$

(54) $$(\frac{1}{2}{}^T\bar{Z}^2)_{ii}(\frac{1}{2}{}^T\bar{Z}Z^2)_{jj} = +\frac{1}{4} Z_{\gamma j} Z_{\beta\gamma} \bar{Z}_{\alpha i} \bar{Z}_{i\alpha} \bar{Z}_{\beta j}$$

(55) $$({}^T\bar{Z}Z)_{ij}(\frac{1}{2}{}^T\bar{Z}^2 Z)_{ji} = -\frac{1}{2} Z_{\gamma i} Z_{\alpha j} \bar{Z}_{\alpha i} \bar{Z}_{\beta j} \bar{Z}_{\gamma\beta}$$

(56) $$({}^T\bar{Z}Z)_{ii}(\frac{1}{2}{}^T\bar{Z}^2 Z)_{jj} = +\frac{1}{2} Z_{\alpha i} Z_{\gamma j} \bar{Z}_{\alpha i} \bar{Z}_{\beta j} \bar{Z}_{\gamma\beta}$$

Monomials of kind (53) and (54) are in fact zero because $\bar{Z}_{j\alpha}$ and $\bar{Z}_{i\alpha}$ vanish for $i, j \leq r$ under the assumptions. Then only monomials of kind (55) and (56) are left, which proves the lemma. □

Now, by using the Lemma 4.3 it is not hard to show that for nonsymmetric flag manifolds represented by a painted diagram with only one black node the corresponding minor Δ_r always contains a forbidden monomial of type $(2,3)$.

More precisely, one has to consider the following cases.

(1) $G = Sp(n)$, diagram

$$\underset{\varepsilon_1-\varepsilon_2}{\circ} - \cdots - \underset{\varepsilon_r-\varepsilon_{r+1}}{\bullet} - \cdots - \underset{\varepsilon_{n-1}-\varepsilon_n}{\circ} \Leftarrow \underset{2\varepsilon_n}{\circ}$$

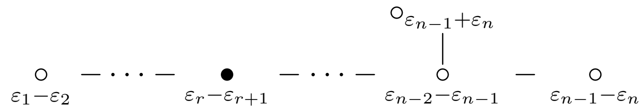
with $1 < r \leq n-1$. (notice that for $r = 1$ we get the symmetric space $\frac{Sp(n)}{Sp(1)\times Sp(n-1)} = \mathbb{H}P^{n-1} \simeq \mathbb{C}P^{2n-1}$).

(2) $G = SO(2n+1)$, diagram

$$\underset{\varepsilon_1-\varepsilon_2}{\circ} - \cdots - \underset{\varepsilon_r-\varepsilon_{r+1}}{\bullet} - \cdots - \underset{\varepsilon_{n-1}-\varepsilon_n}{\circ} \Rightarrow \underset{2\varepsilon_n}{\circ}$$

with $1 < r \leq n-1$.

(3) $G = SO(2n)$, diagram



with $1 < r \leq n-1$.

Then, one sees that in all these cases the forbidden monomial of type (2,3)

$$\frac{1}{2} Z_{n1} Z_{n+1,2} \bar{Z}_{n1} \bar{Z}_{2n,2} \bar{Z}_{n+1,2n} \tag{57}$$

is contained in Δ_r. Indeed, this monomial is of kind (52) with $\alpha = n, i = 1, \gamma = n+1, j = 2, \beta = 2n$ so is contained in Δ_r by Lemma 4.3. Moreover, the monomial in (57) cannot cancel with any other forbidden monomial appearing in the expansion of Δ_r. Indeed, its anti-holomorphic factor

$$\bar{Z}_{n1} \bar{Z}_{2n,2} \bar{Z}_{n+1,2n}$$

fixes the indices in Lemma 4.3. Hence the only possible competing contribution would have to come from a term of type (51) with the same anti-holomorphic factor. However, the corresponding holomorphic factor is different from $Z_{n1} Z_{n+1,2}$, and therefore no term with opposite coefficient can cancel (57). The proof of Theorem 1.3 is complete.

Remark 7. Note that for $c_1 = c_2$ the metric $J_F^* g_F$ on $\mathbb{C}^n$ whose diastasis is given by (50) is KE and hence one gets examples of exact domains $(\mathbb{C}^n, J_F^* g_F)$ even KE which have a Fubini-Study compactification and not admitting a Kähler dual. In order to give an explicit example consider the flag manifold $F = \frac{SU(3)}{S(U(1)^3)}$ of complex dimension 3. In this case the general homogeneous Kähler metric g_F is given on $\mathbb{C}^3$ by

$$D_0^{J_F^* g_F} = c_1 \log \Delta_1 + c_2 \log \Delta_2, \; c_1, c_2 \in \mathbb{R}^+,$$

where

$$\Delta_1 = \log \left[1 + |z_1|^2 + |z_2|^2 + \frac{|z_1|^2 |z_3|^2}{4} + \frac{z_2 \bar{z}_1 \bar{z}_3}{2} + \frac{\bar{z}_2 z_1 z_3}{2} \right]$$

and

$$\Delta_2 = \log \left[1 + |z_2|^2 + |z_3|^2 + \frac{|z_1|^2 |z_3|^2}{4} - \frac{z_2 \bar{z}_1 \bar{z}_3}{2} - \frac{\bar{z}_2 z_1 z_3}{2} \right]$$

and one can directly check that for $c_1 = c_2$ the exact domain $(\mathbb{C}^3, J_F^* g_F)$ has a Fubini-Study compactification given by (F, g_F), g_F is KE but $(\mathbb{C}^3, J_F^* g_F)$ does not admits a Kähler dual.

5. A CONJECTURE

In light of the Theorem 1.2, one might ask which additional properties on the metric g in Question A would yield a positive answer. We believe this extra property is the requirement that the metric g is KE. This is expressed by the following

Conjecture A. *Let (U, g) be an exact domain admitting a Kähler dual (U^*, g^*). Assume that there exists $\alpha > 0$ such that:*

(a) *$(U, \alpha g)$ has a Fubini-Study completion;*
(b) *$(U^*, \alpha g^*)$ has a Fubini-Study compactification.*

If the metric g is KE, then (U, g) is biholomorphically isometric to a bounded symmetric domain (Ω, g_Ω).

We will need the following result, interesting in its own right.

Lemma 5.1. *Let (U, g) be an exact domain admitting a Kähler dual (U^*, g^*). If g is KE with Einstein constant λ then g^* is KE with Einstein constant $\lambda^* = -\lambda$.*

Proof. A direct computation of the Ricci forms ρ_g and ρ_{g^*} for g and g^* show that they are related by

$$\rho_g(z, \bar{z}) = -\rho_{g^*}(z, -\bar{z}) \tag{58}$$

and the proof easily follows. □

Remark 8. From (58) one sees that the scalar curvatures scal_g and scal_{g^*} of the two metrics satisfy

$$\mathrm{scal}_g(z, \bar{z}) = -\mathrm{scal}_{g^*}(z, -\bar{z}).$$

Thus, one deduces that the metric g is extremal (in Calabi's sense[4] [9]), iff g^* is extremal. By this fact and since a finitely projectively induced extremal metric is conjecturally KE (see [33, Conjecture 1] and [33, Th,1.1] for the proof of the validity of this conjecture for radial metrics), we believe that the assumption that g is KE in Conjecture A can be be weakened by requiring only that g be extremal.

When $(U, g) = (M_{\Omega,\mu}, g_{\Omega,\mu})$, the following result shows that Conjecture A it is true by weakening condition (b), namely by only requiring the dual Kähler metric $\alpha g^*_{\Omega,\mu}$ on $\mathbb{C}^{n+1}$ to be projectively induced and the KE assumption by requiring g is extremal.

Corollary 5.2. *The Kähler metric $\alpha g^*_{\Omega,\mu}$ is projectively induced and extremal iff $M_{\Omega,\mu} = \mathbb{C}H^{n+1}$, $g_{\Omega,\mu} = g_{hyp}$, and $\alpha \in \mathbb{Z}^+$.*

Proof. If $g^*_{\Omega,\mu}$ is extremal then by Remark 8 also $g_{\Omega,\mu}$ is extremal. By [47, Th.1] $g_{\Omega,\mu}$ is forced to be KE and hence, by Theorem C in the introduction, $\mu = \frac{\gamma}{n+1}$, where γ is the genus of Ω. On the other hand, it is easily seen that

$$\frac{\gamma}{n+1} \leq 1 \tag{59}$$

and equality holds iff $\Omega = \mathbb{C}H^n$ and $\mu = 1$ (inequality (59) can be deduced by looking, for example, at the the table in [4, pag. 17])). Hence, the proof follows by Proposition 2.1. □

On the necessity of the KE assumption in Conjecture A. The following proposition combined with Theorem 1.2 shows the necessity of the KE assumption in Conjecture A.

Proposition 5.3. *The metric $\hat{g}_{\Omega,\mu}$ on the CH domain $M_{\Omega,\mu}$ is not KE for any value of $\mu > 0$.*

Proof. We use [44, Lemma 5] which asserts that a Kähler metric g with associated Kähler form $\omega = \frac{i}{2\pi}\partial\bar{\partial}\Phi$ on the CH domain $M_{\Omega,\mu}$ is KE, with Einstein constant $-(n+2)$, if and only if

$$\Phi(z, w) = h(X) - \frac{\gamma + \mu}{n+2} \log N_\Omega(z, \bar{z}), \tag{60}$$

[4] A metric g is extremal if the $(1, 0)$-part of the Hamiltonian vector field associated to the scalar curvature of g is holomorphic.

where $X = \frac{|w|^2}{N_\Omega(z,\bar z)^\mu}$ and h satisfies the differential equation

$$\left(\mu X h'(X) + \frac{\gamma+\mu}{d+2}\right)^d [Xh'(X)]' = k e^{(n+2)h(X)}, \tag{61}$$

for some $k \in \mathbb{R}$.

Now, on the one hand, it is easily seen that the Kähler potential

$$\Phi(z,w) := -\frac{\gamma+\mu}{(\mu+1)(d+2)}\left[\log\left(N_\Omega(z,\bar z)^\mu - |w|^2\right) + \log N_\Omega(z,\bar z)\right]$$

for the metric $-\frac{\gamma+\mu}{(\mu+1)(d+2)}\hat g_{\Omega,\mu}$ satisfies (60) with

$$h(X) := -\frac{\gamma+\mu}{(\mu+1)(d+2)}\log(1-X).$$

On the other hand, a simple computation shows that the latter does not satisfy (61), for any value of k. □

On the necessity of condition (b) in Conjecture A. We are indebted to Hishi Hideyuki for suggesting the following example, which shows that assumption (b) is necessary for the validity of Conjecture A.

Example 4. Let

$$U_{2,1} := \{(W,V) \in \mathrm{Sym}\,(2,\mathbb{C}) \times M_{2,1}\,(\mathbb{C}) \mid A\,(W,V) >> 0\} \subset \mathbb{C}^5,$$

where

$$A\,(W,V) = I_2 - W\overline{W} - \frac{1}{2}V\bar V^t - \frac{1}{2}\,(I_2 - W)\left(I_2 - \overline{W}\right)^{-1}\overline{V}V^t\,(I_2 - W)^{-1}\left(I_2 - \overline{W}\right)$$

It can be easily seen that $U_{2,1}$ is acted upon transitively by its automorphism group and the group $O(2,\mathbb{R}) \times U(1)$ acts on the domain $U_{2,1}$ as an isotropy at the origin, via the action $(W,V) \mapsto (AW^tA, AVB)$. Hence $U_{2,1}$ is a homogeneous bounded domain. One can also directly verify that $U_{2,1}$ is not symmetric and compute its Bergman Kernel, which is given by

$$K_{U_{2,1}}\,(W,V) = C\det A\,(W,V)^{-4}$$

where C is a constant (whose exact value is not important for our aims). Consider the Bergman metric g_B on $U_{2,1}$ whose associated Kähler form is given by:

$$\omega_B = -\frac{2i}{\pi}\partial\bar\partial\log\det A\,(W,V)\,.$$

Notice that g_B is homogeneous and hence KE [25, Th.4.1]. It is not hard to verify that the Calabi's diastasis function centred at the origin of g_B is given by

$$D_0^{g_B}(W,V) = -\log\det A\,(W,V)\,.$$

Notice that by applying the "dual trick" (8), one gets

$$(D_0^{g_B})^*(W,V) = \log\det A^*\,(W,V)$$

where

$$A^*\,(W,V) = I_2 + W\overline{W} + \frac{1}{2}V\overline{V}^t + \frac{1}{2}\,(I_2 - W)\left(I_2 + \overline{W}\right)^{-1}\overline{V}V^t\,(I_2 - W)^{-1}\left(I_2 + \overline{W}\right).$$

Since the later is a well-defined real-valued function on $\mathbb{C}^5$, by the remarks before the proof of Theorem 1.3 the exact domain $(U_{2,1}, g_B)$ admits a Kähler dual $(U^*_{2,1}, g^*_B)$, $U^*_{2,1} \subset \mathbb{C}^5$, such that $D_0^{g^*_B} = (D_0^{g_B})^*{}_{|U^*_{2,1}}$, where $U^*_{2,1}$ is the maximal domain of definition of $D_0^{g^*_B}$.

Consider the real-valued $(1,1)$-form ω on $\mathbb{C}^5$ given by

$$\omega = \frac{2i}{\pi}\partial\bar{\partial}\log\det A^*(W,V) \tag{62}$$

and let g be the corresponding symmetric two tensor (thus $g^*_B = g_{|U^*_{2,1}}$).

Since the Bergman metric g_B is infinitely projectively induced and it is complete (by homogeneity) it follows that $(U_{2,1}, \alpha g_B)$ admits a Fubini-Study completion (cf. Remark 1) for all $\alpha \in \mathbb{Z}^+$ (and hence (a) in Conjecture A is satisfied for such α).

We want to prove that the dual Kähler metric αg^*_B on $U^*_{2,1} \subset \mathbb{C}^5$ is never finitely projectively induced, for $\alpha > 0$. This will show that assumption (b) in Conjecture A is necessary for its validity. In order to achieve this result consider the holomorphic curve

$$\gamma : \mathbb{C} \ni z \mapsto \left(\begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix}, \begin{pmatrix} z \\ 0 \end{pmatrix} \right) \in \mathbb{C}^5.$$

Let $\Gamma \subset \mathbb{C}$ be the connected component of the origin, where $g_\gamma := \gamma^* g$ is a Kähler metric. Thus,

$$h(z) := \frac{\partial^2[\log\det A^*(\gamma(z))]}{\partial z \partial \bar{z}} > 0, \ \forall z \in \Gamma. \tag{63}$$

By a direct computation or with the aid of Mathematica one can show that

$$h(x + \frac{i}{2}x) = \frac{P(x)}{Q(x)}, \ \forall x \in \mathbb{R}.$$

where

$$P(x) = 4\left(12288 + 4096x^2 - 62592x^4 - 12320x^6 - 19800x^8 - 138750x^{10} + 78125x^{12}\right)$$

and

$$Q(x) = \left(128 + 288x^2 - 120x^4 - 50x^6 + 625x^8\right)^2 > 0.$$

Thus, if x_0 is the smallest positive root of $P(x)$ it follows by (63) that the line segment

$$\left\{ x + \frac{i}{2}x \mid x \in (-x_0, x_0) \right\}$$

is a subset of Γ. Let W be an open and simply-connected open neighborhood of the origin of $\mathbb{C}$ such that

$$\left\{ x + \frac{i}{2}x \mid x \in (-x_0, x_0) \right\} \subset W \subset \Gamma$$

Assume now, by a contradiction, that there exists $\alpha > 0$ such that αg^*_B is finitely projectively induced. Then, since $\alpha g^*_{|U_{2,1}} = \alpha g^*_B$ there exists an open neighborhood of the origin contained in W such that αg_γ is finitely projectively induced on this neighborhood.

By combining Calabi's resolvability [8, Th.10] and Calabi's extension [8, Th.11] (since W is simply-connected) also $\alpha g_{\gamma|W}$ is finitely projectively induced by the same finite complex projective space, say $(\mathbb{C}P^N, g_{FS})$. Now, again with the aid of Mathematica, the holomorphic sectional curvature $K(z)$ of $g_{\gamma|W}$ (which in this case is simply the Gaussian curvature) satisfies

$$K(x+\frac{i}{2}x) = -\frac{1}{h}\frac{\partial^2 \log h}{\partial z \partial \bar{z}}_{|z=x+\frac{i}{2}x} = \frac{R(x)}{(P(x))^2},\ \forall x \in (-x_0, x_0),$$

with $R(x)$ a polynomial of degree 36 with positive coefficients and then satisfying $R(x) > 0$. Thus

$$\lim_{x \to x_0^-} K\left(x + \frac{i}{2}x\right) = +\infty.$$

This yields the desired contradiction, since by the Gauss-Codazzi equation for Kähler submanifolds (see e.g. [26, Prop. 9.2], the holomorphic sectional curvature of $\alpha g_{\gamma|W}$ must be bounded from above by the constant holomorphic sectional curvature of the finite dimensional complex projective space $(\mathbb{C}P^N, g_{FS})$. This concludes the example.

Remark 9. In the previous example we are showing that αg_B^* is not projectively induced, a condition which is weaker then (b) in Conjecture A (cf. Proposition 5.4 below).

Remark 10. Example 4 should be compared with the proofs of Proposition 2.1 above. By a straightforward computation the holomorphic sectional curvature K of the metric g in the proof of the proposition is bounded for $\mu \geq 2$ (for example, for $\mu = 5/2$, $\lim_{|z| \to \infty} K(\xi) = 4/5$). Thus, we cannot apply there the argument used in Example 4 to conclude that g is not projectively induced and we need to use the full strength of Calabi's theory. Similar considerations apply to the metric g in Proposition 3.2.

Remark 11. Notice that given any homogeneous bounded domain (Ω, g), with g not necessarily KE, then αg is infinitely projectively induced for all α sufficiently large (see [29, Th.1.1]) and so $(\Omega, \alpha g)$ admits a Fubini-Study completion (cf. Remark 1). Thus, Example 4 stimulates the following question, which the authors will consider in a future paper: if a homogeneous bounded domain (Ω, g) admits a Kähler dual (U^*, g^*) such that αg^* is finitely projectively induced for some $\alpha > 0$, is it true that (U, g) is biholomorphically isometric to a bounded symmetric domain?

On the necessity of condition (a) in Conjecture A. In order to understand the role of assumption (a) in Conjecture A we first recall the following two conjectures in the context of flag manifolds.

Conjecture B. *Let (M, g) be a compact KE manifold such that g is (finitely) projectively induced. Then (M, g) is biholomorphically isometric to a flag manifold (F, g_F).*

Conjecture C. *Let (F, g_F) be a flag manifold of complex dimension n. Then, for all $p \in F$ there exists a holomorphic embedding with dense image $J_F : \mathbb{C}^n \to F$, $J_F(0) = p$, such that if $(\mathbb{C}^n, J_F^* g_F)$ admits a Kähler dual then (F, g_F) is biholomorphically isometric to a Hermitian symmetric space of compact type.*

Conjecture B is a long-standing conjecture which has been affirmatively solved in codimension ≤ 2 ([11], [41]), for complete intersections [17], for toric varieties with $n \leq 4$ [6, Prop.4.2] and for $\mathbb{T}^n$-invariant Kähler metrics with $n \leq 6$ in the recent preprint [37]. The assumption of KE in Conjecture B can be weakened by requiring g is a Kähler-Ricci soliton, since a finitely projectively induced Kähler-Ricci solitons is trivial [30, Th.1.1]). Notice also that Conjecture B is not valid for infinitely projectively induced KE metrics (see [34, Th.1] and also [19]). For example, the metric $\alpha g_{\Omega,\mu}$, on the CH domain $M_{\Omega,\mu}$ is infinitely projectively induced and KE for $\mu = \frac{\gamma}{n+1}$ and sufficiently large α, as it follows by Theorem C and Proposition 2.3. Conjecture C represents an extension of Theorem 1.3 to all flag manifolds, not necessarily of classical types. Notice that for such manifolds the existence of the embedding J_F remains an open issue.

We now observe that the validity of Conjectures B and C implies the validity of Conjecture A, even without assumption (a) (hence we suspect assumption (a) can be dropped from Conjecture A). Indeed, assume that (U, g) is an exact domain of $\mathbb{C}^n$ admitting a Kähler dual (U^*, g^*). Then, the Fubini-Study compactification of $(U^*, \alpha g^*)$ (assumption (b)) is KE by Lemma 5.1). Thus, if Conjecture B is valid this compactification turns out to be a flag manifold (F, g_F) and if also Conjecture C is true, we have $U^* = J_F(\mathbb{C}^n)$ and (F, g_F) is biholomorphically isometric to a Hermitian symmetric space of compact type and hence (U, g) is biholomorphically isometric to a bounded symmetric domain.

One wonders if the conclusion of Conjecture A is still valid when weakening the assumption (b) from the existence of a Fubini-Study compactification of $(U^*, \alpha g^*)$, to the requirement that the Kähler metric αg^* is finitely projectively induced (cf. Remarks 5 and 9) . It is interesting to notice that this is indeed the case if one assumes the validity of Conjectures B and C as expressed by the following

Proposition 5.4. *Let (U, g) be an exact domain which admits a Kähler dual (U^*, g^*). Assume that there exists $\alpha > 0$ such that $(U, \alpha g)$ has a Fubini-Study completion, αg^* is finitely projectively induced and g is KE. If Conjectures B and C are true then (U, g) is biholomorphically isometric to a bounded symmetric domain.*

Proof. By Remark 5.1 we know that $(U^*, \alpha g^*)$ is KE with Einstein constant $\frac{\lambda^*}{\alpha}$, where $\lambda = -\lambda^*$ is the Einstein constant of g. By [22, Th.4.5] (see also [27, Th.1.2] for case of extremal Kähler metrics) since αg^* is finitely projectively induced, there exists a complete KE manifold (P, h) such that $U^* \subset P$, $h_{|U^*} = \alpha g^*$, h is KE (with Einstein constant $\frac{\lambda^*}{\alpha}$), and h is still finitely projectively induced.

We claim that $\lambda^* > 0$. Indeed $\lambda^* \neq 0$ by [7, Cor.1.7] asserting that a Ricci flat metric cannot be finitely projectively induced. Moreover, λ^* cannot be negative otherwise $\lambda > 0$ and the Fubini-Study completion of $(U, \alpha g)$ would be compact by Bonnet-Myers theorem, in contrast to the fact that αg is infinitely projectively induced. By applying again Bonnet-Myers theorem we deduce that P is compact. Thus, by the validity of Conjectures B and C, we deduce that (P, h) is biholomorphically isometric to a Hermitian symmetric space of compact type $(\Omega_c, \alpha g_{\Omega_c})$ and hence (U, g) is biholomorphically isometric to the bounded symmetric domain (Ω, g_Ω). ☐

Remark 12. We note that in the proof of the proposition we have used assumption (a) of Conjecture A only to deduce that $\lambda^* > 0$. This probably can be obtained by the fact that $\lambda^* < 0$ is conjecturally forbidden by [33, Conjecture 1] asserting that the finitely projectively induced KE metric αg^* has non-negative Einstein constant. Notice also that in the compact case a finitely projectively induced KE metric has positive Einstein constant (see [23] for a proof), a fact that we cannot use in our proof since U^* is not compact.

6. The case of generalized CH domains

Our results regarding CH domains can be suitably extended to generalized CH domains. By a *generalized CH domain based on a bounded symmetric* Ω we mean the bounded domain in $\mathbb{C}^{n+1}$

given by

$$M_{\Omega,\bar\mu} := \left\{ (z,w) \in \Omega \times \mathbb{C} \mid |w|^2 < \prod_{j=1}^{s} N_{\Omega_j}^{\mu_j}(z_j,\bar z_j) \right\}$$

where $\Omega = \Omega_1 \times \cdots \times \Omega_s$ and each $\Omega_j \subset \mathbb{C}^{n_j}$, $n = n_1 + \cdots + n_s$, is a Cartan domain of genus γ_j, $\bar\mu = (\mu_1, \ldots, \mu_s)$ is a s-vector with positive entries μ_j and $z = (z_1, \ldots, z_s) \in \Omega$, with $z_j \in \Omega_j \subset \mathbb{C}^{n_j}$, $j = 1, \ldots, s$. A generalized CH domain $M_{\Omega,\bar\mu}$ can be equipped with two natural complete Kähler metrics $g_{\Omega,\bar\mu}$ and $\hat g_{\Omega,\bar\mu}$ (generalizing $g_{\Omega,\mu}$ and $\hat g_{\Omega,\mu}$ when $\mu \in \mathbb{R}$ and Ω is irreducible) whose associated Kähler forms are

$$\omega_{\Omega,\bar\mu} = -\frac{i}{2\pi}\partial\bar\partial \log\left(N_\Omega^{\bar\mu}(z,\bar z) - |w|^2\right)$$

$$\hat\omega_{\Omega,\bar\mu} = -\frac{i}{2\pi}\partial\bar\partial \log\left(N_\Omega^{\bar\mu}(z,\bar z) - |w|^2\right) - \frac{i}{2\pi}\partial\bar\partial \log N_\Omega^{\bar\mu}(z,\bar z),$$

where $N_\Omega^{\bar\mu}(z,\bar z) = \prod_{j=1}^{s} N_{\Omega_j}^{\mu_j}(z_j,\bar z_j)$. It is not hard to see that exact domains $(M_{\Omega,\bar\mu}, g_{\Omega,\bar\mu})$ and $(M_{\Omega,\bar\mu}, \hat g_{\Omega,\bar\mu})$ admit Kähler duals $(\mathbb{C}^{n+1}, g^*_{\Omega,\bar\mu})$ and $(\mathbb{C}^{n+1}, \hat g^*_{\Omega,\bar\mu})$, respectively, where the corresponding associated Kähler forms are given by

$$\omega^*_{\Omega,\bar\mu} = \frac{i}{2\pi}\partial\bar\partial \log\left(N_\Omega^{\bar\mu}(z,-\bar z) + |w|^2\right)$$

$$\hat\omega^*_{\Omega,\bar\mu} = \frac{i}{2\pi}\partial\bar\partial \log\left(N_\Omega^{\bar\mu}(z,-\bar z) + |w|^2\right) + \frac{i}{2\pi}\partial\bar\partial \log N_\Omega^{\bar\mu}(z,-\bar z),$$

The following proposition extends Propositions 2.1, 2.3, 3.1 and 3.2. Its proof is omitted since it can be easily obtained by reducing to the irreducible case.

Proposition 6.1. *The Kähler metrics $\alpha g_{\Omega,\bar\mu}$ and $\alpha \hat g_{\Omega,\bar\mu}$ are infinitely projectively induced for sufficiently large α. The Kähler metrics $\alpha g^*_{\Omega,\bar\mu}$ and $\alpha \hat g^*_{\Omega,\bar\mu}$ are finitely projectively induced iff $\alpha \in \mathbb{Z}^+$ and $\mu_j \in \mathbb{Z}^+$, for $j = 1, \ldots, s$.*

Using this proposition and by noticing that the proofs of Theorem 1.1,Theorem 1.2 and Proposition 3.5 work also when Ω is an arbitrary bounded symmetric domain, not necessarily irreducible, one gets the following results for generalized CH domains.

Theorem 6.2. *The generalized CH domain $(M_{\Omega,\alpha\bar\mu}, g_{\Omega,\bar\mu})$ admits a Fubini-Study completion for all sufficiently large α. Additionally, $(\mathbb{C}^{n+1}, \alpha g^*_{\Omega,\bar\mu})$ admits a Fubini-Study compactification for some α iff $\bar\mu = \mu = 1$ and $\Omega = \mathbb{C}H^n$.*

Theorem 6.3. *For $\mu_j \in \mathbb{Z}^+$, $j = 1, \ldots, s$, and sufficiently large integer α, the generalized CH domain $(M_{\Omega,\bar\mu}, \alpha \hat g_{\Omega,\bar\mu})$ admits a Fubini-Study completion and $(\mathbb{C}^{n+1}, \alpha \hat g^*_{\Omega,\bar\mu})$ admits a Fubini-Study compactification $(P_{\Omega_c,\bar\mu}, g_{P,\bar\mu})$. Moreover, if one writes $\Omega_c = G/K$ then $P_{\Omega_c,\mu}$ is a cohomogeneity 1 G-space, the action of G is ordinary (with Ω_c as the flag associated to the regular orbit) and $g_{P,\bar\mu}$ is a G-invariant Kähler metric.*

Note that for generalized CH domains, the extremality of the Kähler metric $g_{\Omega,\bar\mu}$ does not imply it is KE, as in the case of CH domains (see the proof of Corollary 5.2). Nevertheless, Corollary 5.2 also extends to generalized CH domains, as expressed by the following

Corollary 6.4. *The Kähler metric $\alpha g^*_{\Omega,\bar\mu}$ is projectively induced and extremal iff $\bar\mu = \mu = 1$, $\Omega = \mathbb{C}H^n$, and $\alpha \in \mathbb{Z}^+$.*

Proof. By Remark 8 $g^*_{\Omega,\bar\mu}$ is extremal iff $g_{\Omega,\bar\mu}$ is extremal. On the other hand, by [18, Th.1.1] the metric $g_{\Omega,\bar\mu}$ is extremal (iff it has constant scalar curvature) iff

$$\sum_{j=1}^{s}(n+1-\frac{\gamma_j}{\mu_j})n_j=0. \tag{64}$$

Assuming that $\alpha g^*_{\Omega,\bar\mu}$ is projectively induced, we have

$$\frac{\gamma_j}{\mu_j(n+1)}\leq\frac{\gamma_j}{n+1}\leq\frac{\gamma_j}{n_j+1}\leq 1,\ j=1,\ldots,s. \tag{65}$$

where the first inequality follows by the fact that $\mu_j\in\mathbb{Z}^+$, for $j=1,\ldots,s$ (by Proposition 6.1) and the third inequality is given by (59), as Ω_j is irreducible. By combining (64) and (65), we deduce that

$$\mu_j=\frac{\gamma_j}{n+1},\ j=1,\ldots,s, \tag{66}$$

and $n_j=n$, i.e., Ω is irreducible. Hence, we can apply Corollary 5.2 to get the conclusion. □

Remark 13. Notice that conditions (66) are equivalent to the assumption that the metric $g_{\Omega,\bar\mu}$ (and hence $g^*_{\Omega,\bar\mu}$, by Lemma 5.1) is KE (see [18, Th.1.2]).

REFERENCES

[1] D. V. Alekseevskiĭ and A. M. Perelomov, *Invariant Kähler-Einstein metrics on compact homogeneous spaces*, Funktsional. Anal. i Prilozhen. 20 (1986), 1–16, 96.

[2] D. Alekseevsky and F. Zuddas, *Cohomogeneity one Kähler and Kähler-Einstein manifolds with one singular orbit I*, Ann. Global Anal. Geom. 52 (2017), 99–128.

[3] D. Alekseevsky and F. Zuddas, *Cohomogeneity one Kähler and Kähler-Einstein manifolds with one singular orbit II*, Ann. Global Anal. Geom. 57 (2020), 153–174.

[4] J. Arazy, *A survey of invariant Hilbert spaces of analytic functions on bounded symmetric domains*, In: "Multivariable operator theory (Seattle, WA, 1993)", Contemp. Math. 185, Amer. Math. Soc., Providence, RI, 1995, 7–65.

[5] C. Arezzo and A. Loi, *Quantization of Kähler manifolds and the asymptotic expansion of Tian-Yau-Zelditch*, J. Geom. Phys. 47 (2003), 87–99.

[6] C. Arezzo, A. Loi, and F. Zuddas, *On homothetic balanced metrics*, Ann. Global Anal. Geom. 41 (2012), 473–491.

[7] C. Arezzo, C. Li, and A. Loi, *Gromov-Hausdorff limits and Holomorphic isometries*, to appear in Math. Res. Lett., arXiv:2306.16113, 2024.

[8] E. Calabi, *Isometric imbedding of complex manifolds*, Ann. of Math. (2) 58 (1953), 1–23.

[9] E. Calabi, *Extremal Kähler metrics*, In: "Seminar on Differential Geometry", Ann. of Math. Stud., No. 102, Princeton Univ. Press, Princeton, NJ, 1982, 259–290.

[10] S. T. Chan and N. Mok, *Holomorphic isometries of $\mathbb{B}^m$ into bounded symmetric domains arising from linear sections of minimal embeddings of their compact duals*, Math. Z. 286 (2017), 679–700.

[11] S. Chern, *Einstein hypersurfaces in a Kählerian manifold of constant holomorphic curvature*, J. Differential Geom. 1 (1967), 21–31.

[12] A. J. Di Scala and A. Loi, *Symplectic duality of symmetric spaces*, Adv. Math. 217 (2008), 2336–2352.

[13] A. J. Di Scala, A. Loi, and G. Roos, *The bisymplectomorphism group of a bounded symmetric domain*, Transform. Groups 13 (2008), 283–304.

[14] A. J. Di Scala, A. Loi, and F. Zuddas, *Symplectic duality between complex domains*, Monatsh. Math. 160 (2010), 403–428.

[15] A. J. Di Scala, H. Ishi, and A. Loi, *Kähler immersions of homogeneous Kähler manifolds into complex space forms*, Asian J. Math. 16 (2012), 479–487.

[16] M. P. do Carmo, "Riemannian geometry", Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992, xiv+300. Translated from the second Portuguese edition by Francis Flaherty.
[17] J. Hano, *Einstein complete intersections in complex projective space*, Math. Ann. 216 (1975), 197–208.
[18] Y. Hao, *Canonical metrics on generalized Cartan-Hartogs domains*, Chinese Ann. Math. Ser. B 37 (2016), 357–366.
[19] Y. Hao, A. Wang, and L. Zhang, *On holomorphic isometric immersions of nonhomogeneous Kähler-Einstein manifolds into the infinite dimensional complex projective space*, J. Math. Anal. Appl. 423 (2015), 547–560.
[20] R. Hartshorne, "Algebraic geometry", Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977, xvi+496.
[21] S. Helgason, "Differential geometry, Lie groups, and symmetric spaces", Graduate Studies in Mathematics 34, American Mathematical Society, Providence, RI, 2001, xxvi+641. Corrected reprint of the 1978 original.
[22] D. Hulin, *Sous-variétés complexes d'Einstein de l'espace projectif*, Bull. Soc. Math. France 124 (1996), 277–298.
[23] D. Hulin, *Kähler-Einstein metrics and projective embeddings*, J. Geom. Anal. 10 (2000), 525–528.
[24] K. Knopp, "Infinite sequences and series", Dover Publications, Inc., New York, 1956, v+186. Translated by Frederick Bagemihl.
[25] S. Kobayashi, *Geometry of bounded domains*, Trans. Amer. Math. Soc. 92 (1959), 267–290.
[26] S. Kobayashi and K. Nomizu, "Foundations of differential geometry. Vol. II", Wiley Classics Library, John Wiley & Sons, Inc., New York, 1996, xvi+468. Reprint of the 1969 original, A Wiley-Interscience Publication.
[27] C. Li, *Extensions of extremal Kähler submanifolds of complex projective spaces*, arXiv:2306.16151, 2023.
[28] A. Loi and R. Mossa, *The diastatic exponential of a symmetric space*, Math. Z. 268 (2011), 1057–1068.
[29] A. Loi and R. Mossa, *Some remarks on homogeneous Kähler manifolds*, Geom. Dedicata 179 (2015), 377–383.
[30] A. Loi and R. Mossa, *Kähler immersions of Kähler-Ricci solitons into definite or indefinite complex space forms*, Proc. Amer. Math. Soc. 149 (2021), 4931–4941.
[31] A. Loi, R. Mossa, and F. Zuddas, *Symplectic capacities of Hermitian symmetric spaces of compact and noncompact type*, J. Symplectic Geom. 13 (2015), 1049–1073.
[32] A. Loi, R. Mossa, and F. Zuddas, *Bochner coordinates on flag manifolds*, Bull. Braz. Math. Soc. (N.S.) 50 (2019), 497–514.
[33] A. Loi, F. Salis, and F. Zuddas, *Extremal Kähler metrics induced by finite or infinite-dimensional complex space forms*, J. Geom. Anal. 31 (2021), 7842–7865.
[34] A. Loi and M. Zedda, *Kähler-Einstein submanifolds of the infinite dimensional projective space*, Math. Ann. 350 (2011), 145–154.
[35] A. Loi and M. Zedda, "Kähler immersions of Kähler manifolds into complex space forms", Lecture Notes of the Unione Matematica Italiana 23, Springer, Cham; Unione Matematica Italiana, [Bologna], 2018, x+99.
[36] A. Loi and G. Placini, *Ricci iterations of well-behaved Kähler metrics*, arXiv:2307.11500, 2023.
[37] G. Manno and F. Salis, *$\mathbb{T}^n$-invariant Kaehler-Einstein manifolds immersed in complex projective spaces*, arXiv:2407.12685, 2024.
[38] R. Mossa and M. Zedda, *Symplectic geometry of Cartan-Hartogs domains*, Ann. Mat. Pura Appl. (4) 201 (2022), 2315–2339.
[39] H. Nakagawa and R. Takagi, *On locally symmetric Kaehler submanifolds in a complex projective space*, J. Math. Soc. Japan 28 (1976), 638–667.
[40] F. Podestà and A. Spiro, *Kähler manifolds with large isometry group*, Osaka J. Math. 36 (1999), 805–833.
[41] B. Smyth, *Differential geometry of complex hypersurfaces*, Ann. of Math. (2) 85 (1967), 246–266.
[42] D. M. Snow, *Homogeneous vector bundles*, In: "Group actions and invariant theory (Montreal, PQ, 1988)", CMS Conf. Proc. 10, Amer. Math. Soc., Providence, RI, 1989, 193–205.
[43] M. Takeuchi, *Homogeneous Kähler submanifolds in complex projective spaces*, Japan. J. Math. (N.S.) 4 (1978), 171–219.
[44] A. Wang, W. Yin, L. Zhang, and G. Roos, *The Kähler-Einstein metric for some Hartogs domains over symmetric domains*, Sci. China Ser. A 49 (2006), 1175–1210.
[45] A. Wang, W. Yin, L. Zhang, and W. Zhang, *The Einstein-Kähler metric with explicit formulas on some non-homogeneous domains*, Asian J. Math. 8 (2004), 39–49.
[46] W. Yin, K. Lu, and G. Roos, *New classes of domains with explicit Bergman kernel*, Sci. China Ser. A 47 (2004), 352–371.

[47] M. Zedda, *Canonical metrics on Cartan-Hartogs domains*, Int. J. Geom. Methods Mod. Phys. 9 (2012), 1250011, 13.

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