

A LANE-EMDEN SYSTEM OF FREE BOUNDARY TYPE: EXISTENCE, UNIQUENESS AND MONOTONICITY OF SOLUTIONS

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ABSTRACT. We study a Hamiltonian system of free boundary type. We first establish uniform bounds and prove the existence of solutions as well as of the free boundary. Next, for any smooth and bounded domain, we prove uniqueness of positive solutions within a suitable interval and show that the associated energies and boundary values have a monotonic behavior. Some consequences are discussed about the parametrization of the unbounded Rabinowitz continuum for a class of superlinear elliptic systems.

Keywords: Free boundary problems, Hamiltonian elliptic systems, bifurcation analysis, existence, uniqueness, monotonicity.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open and bounded domain of class C^3 , we are concerned with the following constrained Hamiltonian system of free boundary type,

$$\left\{ \begin{array}{l} -\Delta v_1 = \lambda(v_2)_+^{p_2} \quad \text{in } \Omega \\ -\Delta v_2 = \lambda(v_1)_+^{p_1} \quad \text{in } \Omega \\ -\int_{\partial\Omega} \frac{\partial v_1}{\partial \nu} = 1 = -\int_{\partial\Omega} \frac{\partial v_2}{\partial \nu} \\ v_1 = \alpha_1, \quad v_2 = \alpha_2 \quad \text{on } \partial\Omega \end{array} \right. \quad (\mathbf{F})_\lambda$$

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for the unknowns $\alpha_i \in \mathbb{R}$, and $v_i \in C^{2,r}(\bar{\Omega})$, $i = 1, 2$, for some fixed $r \in (0, 1)$. Here, $(v)_+$ is the positive part of v , ν is the exterior unit normal, $\lambda > 0$ and

$$\frac{1}{p_1 + 1} + \frac{1}{p_2 + 1} > \frac{N - 2}{N - 1}, \quad p_i \in (0, +\infty), \quad i = 1, 2. \quad (1.1)$$

The relevance of the limiting hyperbola in (1.1) for classical Hamiltonian elliptic systems of Lane-Emden type was first noticed in [34], see [20] for more details. Putting,

$$p_N = \begin{cases} +\infty, & N = 2 \\ \frac{N}{N-2}, & N \geq 3, \end{cases}$$

we remark that, as far as $N \geq 3$, $p_1 = p_N = p_2$ satisfy $\frac{1}{p_1+1} + \frac{1}{p_2+1} = \frac{N-2}{N-1}$. To simplify the exposition, by a suitable scaling of λ we assume that $|\Omega| = 1$.

The system $(\mathbf{F})_\lambda$ is a vectorial generalization of the classical ([35, 36, 9]) "scalar" free boundary problem which is obtained in the particular case $p_1 = p_2$, $\alpha_1 = \alpha_2$, $v_1 = v_2$, whose study is motivated by Tokamak's plasma physics ([23, 39]). In the scalar case p_N turns out to be a natural critical exponent see [9, 28].

Another motivation to pursue the analysis of $(\mathbf{F})_\lambda$ is to find a parametrization of solutions of the Hamiltonian elliptic system,

$$\begin{cases} -\Delta u_1 = \mu_2(1 + u_2)^{p_2} & \text{in } \Omega \\ -\Delta u_2 = \mu_1(1 + u_1)^{p_1} & \text{in } \Omega \\ u_1 > 0, \quad u_2 > 0 & \text{in } \Omega \\ u_1 = 0, \quad u_2 = 0 & \text{on } \partial\Omega. \end{cases} \quad (\mathbf{H})$$

For $\mu_1 = \mu_2$ the existence of an unbounded continuum of "scalar" ($u_1 = u_2$) solutions of (\mathbf{H}) follows from the classical result in [31]. The analysis of Hamiltonian elliptic systems is a classical subject and we refer to [20] for a comprehensive introduction about this topic, see also [33] and references therein. Minimal solutions branches (in the sense of Crandall-Rabinowitz ([18])) and multiplicity results for general systems including (\mathbf{H}) has been described in [12, 13], see also [29] and references therein.

Our main motivation stems from the lack of results on the uniqueness and qualitative behavior of solution branches of $(\mathbf{F})_\lambda$ as well as on the qualitative properties of non minimal solutions of (\mathbf{H}) . Note that if $p_1 \neq p_2$ then $(\mathbf{F})_\lambda$ admits no scalar solutions. In the context of (\mathbf{H}) , and inspired by recent works [3], [4], [5], we seek integral quantities naturally associated with $(\mathbf{P})_\lambda$ that can be used to characterize the monotonic behavior of solutions.

First of all let us consider the auxiliary problem,

$$\begin{cases} -\Delta \psi_1 = (\alpha_2 + \lambda \psi_2)_+^{p_2} & \text{in } \Omega \\ -\Delta \psi_2 = (\alpha_1 + \lambda \psi_1)_+^{p_1} & \text{in } \Omega \\ \int_{\Omega} (\alpha_2 + \lambda \psi_2)_+^{p_2} = 1 = \int_{\Omega} (\alpha_1 + \lambda \psi_1)_+^{p_1} & (\mathbf{P})_\lambda \\ \psi_i = 0 & \text{on } \partial\Omega, \quad i = 1, 2, \\ \alpha_i \in \mathbb{R}, \quad i = 1, 2, \end{cases}$$

for the unknowns $\alpha_i \in \mathbb{R}$ and $\psi_i \in C_0^{2,r}(\bar{\Omega})$, $i = 1, 2$. Here $\lambda \geq 0$, (p_1, p_2) satisfy (1.1) and, for some fixed $r \in (0, 1)$, we set,

$$C_0^{2,r}(\bar{\Omega}) = \{\psi \in C^{2,r}(\bar{\Omega}) : \psi = 0 \text{ on } \partial\Omega\}, \quad C_{0,+}^{2,r}(\bar{\Omega}) = \{\psi \in C_0^{2,r}(\bar{\Omega}) : \psi > 0 \text{ in } \Omega\}.$$

Throughout this paper, we refer to solutions of $(\mathbf{P})_\lambda$ of this type as classical solutions. Interestingly enough, problem $(\mathbf{P})_\lambda$ is also of independent interest, as it characterizes stationary solutions arising in models of two-species chemotaxis with nonlinear diffusion, recently studied in [14]; see also Remark 1.1 below.

For fixed $\lambda \geq 0$, it is convenient to denote a solution of $(\mathbf{P})_\lambda$ by $\alpha_\lambda = (\alpha_{1,\lambda}, \alpha_{2,\lambda})$, $\psi_\lambda = (\psi_{1,\lambda}, \psi_{2,\lambda})$. We then define positive/non negative solutions as follows,

Definition. *We say that $(\alpha_\lambda, \psi_\lambda)$ is a positive/non negative solution of $(\mathbf{P})_\lambda$ if $\alpha_{i,\lambda} > 0/\alpha_{i,\lambda} \geq 0$, $i = 1, 2$, respectively.*

Clearly, if $(\alpha_\lambda, \psi_\lambda)$ is a positive solution, then by the strong maximum principle $\psi_{i,\lambda} > 0$ in Ω , $i = 1, 2$. Remark that for $\lambda > 0$, $(\alpha_\lambda, \psi_\lambda)$ is a solution of $(\mathbf{P})_\lambda$ if and only if $((\alpha_{1,\lambda}, \alpha_{2,\lambda}), (v_{1,\lambda}, v_{2,\lambda}))$ solves $(\mathbf{F})_\lambda$, with $v_{i,\lambda} = \alpha_{i,\lambda} + \lambda\psi_{i,\lambda}$, $i = 1, 2$.

We point out that, since $|\Omega| = 1$ and $\lambda \geq 0$ by assumption, then if $(\alpha_\lambda, \psi_\lambda)$ is a non negative solution of $(\mathbf{P})_\lambda$ then necessarily,

$$\alpha_{i,\lambda} \leq 1, \quad i = 1, 2,$$

and the equality $\alpha_{1,\lambda} = 1 = \alpha_{2,\lambda}$ holds if and only if $\lambda = 0$. We will frequently use this fact without further comments. Actually, if $\lambda = 0$, then $(\mathbf{P})_\lambda$ takes the form

$$\begin{cases} -\Delta\psi_{1,0} = 1 & \text{in } \Omega \\ -\Delta\psi_{2,0} = 1 & \text{in } \Omega \\ \psi_{1,0} = 0 = \psi_{2,0} & \text{on } \partial\Omega \end{cases}$$

and admits a unique (in fact scalar) solution $(\alpha_0, \psi_0) = ((1, 1), (G[1], G[1]))$, where we define,

$$G[\rho](x) = \int_{\Omega} G_{\Omega}(x, y)\rho(y) dy, \quad x \in \Omega.$$

Here G_{Ω} is the Green function of $-\Delta$ with Dirichlet boundary conditions on Ω . Obviously, to say that $(\alpha_\lambda, \psi_\lambda)$ is a solution of $(\mathbf{P})_\lambda$ is the same as to say that $\psi_\lambda = (G[\rho_{2,\lambda}], G[\rho_{1,\lambda}])$ and $\int_{\Omega} \rho_{1,\lambda} = 1 = \int_{\Omega} \rho_{2,\lambda}$, where, unless otherwise specified, we set

$$\rho_{i,\lambda} = (\alpha_{i,\lambda} + \lambda\psi_{i,\lambda})_+^{p_i}, \quad i = 1, 2.$$

By a standard fixed point argument (see Appendix A), one shows that for all sufficiently small $\lambda > 0$, problem $(\mathbf{P})_\lambda$ admits at least one solution. Moreover, for any such solution, $\alpha_i > \frac{1}{3}$, $i = 1, 2$. Using a classical argument from [9], one can also prove existence for all $\lambda > 0$, provided $p_i < p_N$, $i = 1, 2$.

Here we refine the variational approach of [9] using the weak Young inequality ([25]); see section 4. We obtain existence of at least one solution to $(\mathbf{P})_\lambda$ whenever (p_1, p_2) satisfy (1.1). Under the same condition, any strong solution $(\psi_{1,\lambda}, \psi_{2,\lambda}) \in W_0^{2,p_2}(\Omega) \times W_0^{2,p_1}(\Omega)$ actually belongs to $(\psi_{1,\lambda}, \psi_{2,\lambda}) \in C_0^{2,r_0}(\Omega) \times C_0^{2,r_0}(\Omega)$ and satisfies a uniform bound for bounded λ ; see Lemma 2.1. These estimates appear to be new and are crucial for our analysis. In the scalar case $p_1 = p = p_2$, $\psi_1 = \psi_2$, the estimates are sharp. If $p \geq p_N$, solutions may blow up for large λ ([37],[38]). In this regime, the variational functional (see (4.3) below) is no longer coercive.

Remark 1.1. *After completing this work, we became aware of the recent paper [14] where essentially the same variational functional (see (4.3) below) is studied in the whole space \mathbb{R}^N . It would be interesting to extend our uniform estimates (Lemma 2.1 below) to a larger region, defined as follows,*

$$p_1 \left(p_2 - \frac{2}{N-2} \right) < p_N \quad \text{or} \quad p_2 \left(p_1 - \frac{2}{N-2} \right) < p_N,$$

as far as $N \geq 3$. Although not stated explicitly in [14], these inequalities follow from the range of parameters considered there. We also do not rule out that some arguments in [14] adapted to prove existence of solutions to $(\mathbf{P})_\lambda$ in this larger region. Remark that the intersection of the two hyperbolas is the symmetric boundary point (p_N, p_N) in (1.1).

It would be of interest to examine the role of the well-known critical hyperbolas introduced in [16] and [17] for problem $(\mathbf{P})_\lambda$.

Our main focus is the uniqueness and qualitative behavior of $(\alpha_\lambda, \psi_\lambda)$ as function of λ . This is delicate for two reasons. First, there are four unknowns: $\alpha_{i,\lambda}$ and $\psi_{i,\lambda}$, $i = 1, 2$. Second, the constraints in $(\mathbf{P})_\lambda$, create competing effects. In the scalar case $\psi_\lambda = \psi_{1,\lambda} = \psi_{2,\lambda}$, $\alpha_\lambda = \alpha_{1,\lambda} = \alpha_{2,\lambda}$, it was shown in [5] that for small $\lambda > 0$, α_λ is strictly decreasing. On the other hand, for fixed $\alpha_\lambda = \alpha$ and without constraints, standard arguments ([18]), show that ψ_λ is strictly increasing for small λ . The same holds for $\psi_{1,\lambda}$ and $\psi_{2,\lambda}$ for fixed (α_1, α_2) , by the maximum principle for cooperative elliptic systems ([22]). Therefore, unlike classical scalar problems ([18]) there is a competition between the monotonic behavior of $(\alpha_{i,\lambda} + \lambda\psi_{i,\lambda})$ as a function of λ .

For this reason, we do not use arguments based on classical maximum principles. Instead, we follow ideas developed in [5, 7], see also [2, 4, 8] for related problems in different settings.

We prove existence, uniqueness, and monotonicity using a refined dual spectral formulation. This approach is well suited to analyze positive solutions of the constrained problem $(\mathbf{P})_\lambda$. The first eigenvalue in this spectral framework is denoted by $\sigma_1(\alpha_\lambda, \psi_\lambda)$, see section 3. In particular, we establish the monotonicity of two naturally defined variational quantities associated with $(\mathbf{P})_\lambda$ (see section 4), which are the energy,

$$E_\lambda := \int_\Omega \rho_{1,\lambda} G[\rho_{2,\lambda}] \equiv \int_\Omega \rho_{i,\lambda} \psi_{i,\lambda} = \int_\Omega (\nabla \psi_{1,\lambda}, \nabla \psi_{2,\lambda}), \quad i = 1, 2,$$

and the free energy,

$$F_\lambda = \frac{1}{r_1} \int_\Omega (\rho_{1,\lambda})^{r_1} + \frac{1}{r_2} \int_\Omega (\rho_{2,\lambda})^{r_2} - \lambda \int_\Omega \rho_{1,\lambda} G[\rho_{2,\lambda}],$$

where $r_i = 1 + \frac{1}{p_i}$, $i = 1, 2$. Furthermore, we can prove the monotonicity of the linear combination

$$\frac{p_1 \alpha_{1,\lambda}}{p_1 + 1} + \frac{p_2 \alpha_{2,\lambda}}{p_2 + 1}.$$

Set $\mathbf{p} = (p_1, p_2)$ and

$$\lambda^*(\Omega, \mathbf{p}) = \sup\{\lambda > 0 : \sigma_1(\alpha_\lambda, \psi_\lambda) > 0, \alpha_{i,\mu} > 0, i = 1, 2, \text{ for any solution of } (\mathbf{P})_\mu, \forall \mu < \lambda\}.$$

It can be shown, see Lemma A.1 in Appendix A and Proposition 3.4 in section 3, that $\lambda^*(\Omega, \mathbf{p})$ is well defined and strictly positive. Our first goal is to prove that $\lambda^*(\Omega, \mathbf{p}) < +\infty$. More precisely, we study the existence of a free boundary inside Ω for solutions of $(\mathbf{F})_\lambda$ with (p_1, p_2) satisfying (1.1). The key question is whether, for a fixed λ , $\min\{\alpha_{1,\lambda}, \alpha_{2,\lambda}\}$ is negative. This would imply that at least one of the sets $\Omega_{i,-} := \{x \in \Omega : v_i < 0\}$, $i = 1, 2$ is not empty. In the scalar case with $p = 1$, this problem is well understood, see [9, 30, 36]. For $p > 1$, the existence of a multiply connected free boundary was proved in [38] for λ large, under assumptions on non degenerate critical points of a suitably Kirchoff-Routh type functional. For large λ , a similar result was obtained in [27], but only for $N = 2$ and for domains with non trivial topology. Other sufficient conditions ensuring solutions with $\alpha < 0$ were given in [1]. However, those results replace the nonlinearity v_+^p with a function $g_+(x, v)$ satisfying $g(x, t) \geq ct$, for some $c > 0$, and thus do not apply to our scalar problem. More recently it has been shown in [3] that there are no scalar solutions with $\alpha_\lambda \geq 0$ for large λ . Here, we extend that result to cooperative free boundary systems of type $(\mathbf{F})_\lambda$.

Theorem 1.2. *Let (p_1, p_2) satisfy (1.1). Then we have:*

- (a) (Existence) *For any $\lambda > 0$ there exists at least one solution of $(\mathbf{F})_\lambda$.*

- (b) (Existence of free boundary) *Suppose either $N = 2$ or $N \geq 3$ and Ω convex. Then, there exists $\bar{\lambda} = \bar{\lambda}(\Omega, \mathbf{p}, N) > 0$ depending only on \mathbf{p}, N and Ω such that if $(\alpha_\lambda, \psi_\lambda)$ is a non negative solution of $(\mathbf{P})_\lambda$, then $\lambda \leq \bar{\lambda}$. In particular, for any $\lambda > \bar{\lambda}$ we have $\min\{\alpha_{1,\lambda}, \alpha_{2,\lambda}\} < 0$ for any solution of $(\mathbf{F})_\lambda$.*

As mentioned above, part (a) follows from a refinement of the variational argument used in the scalar case for $(\mathbf{P})_\lambda$, see [9]. We prove (b) by a blow up argument. This is a standard tool for deriving a priori estimates in Lane-Emden systems, see [20] and references therein. However, our setting differs slightly from the classical one. For this reason, we provide a self-contained proof.

Let $\mathcal{G}(\Omega)$ denote the set of solutions of $(\mathbf{P})_\lambda$ for $\lambda \in [0, \lambda^*(\Omega, \mathbf{p})]$. Our next goal is to prove uniqueness of these solutions, see Theorem 1.3 below. Under the assumption of Theorem 1.2-(b), we have $\lambda^*(\Omega, \mathbf{p}) < +\infty$. Since uniqueness is not expected for all $\lambda \in (0, +\infty)$, we conjecture that Theorem 1.2-(b) also holds for non-convex domains with $N \geq 3$.

Let $B_r = \{x \in \mathbb{R}^N : |x| < r\}$ with volume $|B_r|$, \mathbb{D}_N be the N -dimensional ball of unit volume and let us denote by,

$$\Lambda(\Omega, t) = \inf_{w \in H_0^1(\Omega), w \neq 0} \frac{\int_\Omega |\nabla w|^2}{\left(\int_\Omega |w|^t\right)^{\frac{2}{t}}}, \quad (1.2)$$

which provides the best constant in the Sobolev embedding $\|w\|_p \leq S_p(\Omega) \|\nabla w\|_2$, $S_p(\Omega) = \Lambda^{-2}(\Omega, p)$, $p \in [1, 2p_N]$. Let us set

$$\sigma_{1,*} = \liminf_{\lambda \rightarrow \lambda^*(\Omega, \mathbf{p})^-} \sigma_1(\alpha_\lambda, \psi_\lambda), \quad \alpha_{i,*} = \liminf_{\lambda \rightarrow \lambda^*(\Omega, \mathbf{p})^-} \alpha_{i,\lambda}, \quad i = 1, 2.$$

For the sake of clarity, we point out that if a map \mathcal{M} from an interval $[a, b] \subset \mathbb{R}$ to a Banach space X is said to be real analytic, then it is understood that \mathcal{M} can be extended in an open neighborhood of a and b where it admits a power series expansion, totally convergent in the X -norm.

Here $\|\cdot\|_{i,\lambda}$ stands for the weighted norms naturally associated to the problem, see section 3 for definitions.

Theorem 1.3. *Let (p_1, p_2) satisfy (1.1). Then we have:*

- (Uniqueness) *For any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p})]$ there exists a unique solution $(\alpha_\lambda, \psi_\lambda)$ of $(\mathbf{P})_\lambda$ and $\mathcal{G}(\Omega)$ is a real analytic simple curve of positive solutions $[0, \lambda^*(\Omega, \mathbf{p})] \ni \lambda \mapsto (\alpha_\lambda, \psi_\lambda)$. As $\lambda \rightarrow 0^+$ we have,*
 $\alpha_\lambda = (1, 1) + O(\lambda)$, $\psi_\lambda = (\psi_{1,0}, \psi_{2,0}) + O(\lambda)$, $E_\lambda = E_0(\Omega) + O(\lambda)$, where,

$$E_0(\Omega) = \int_\Omega \int_\Omega G_\Omega(x, y) dx dy \leq E_0(\mathbb{D}_N) = \frac{|B_1|^{-\frac{2}{N}}}{2(N+2)}.$$

- (Monotonicity) *For any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p})]$ it holds,*

$$\frac{dF_\lambda}{d\lambda} < 0, \quad \frac{dE_\lambda}{d\lambda} \geq 0, \quad \frac{d}{d\lambda} \left(\frac{p_1 \alpha_{1,\lambda}}{p_1 + 1} + \frac{p_2 \alpha_{2,\lambda}}{p_2 + 1} \right) < 0. \quad (1.3)$$

Moreover,

$$\frac{dE_\lambda}{d\lambda} \geq p_1 \|[\psi_{1,\lambda}]_{1,\lambda}\|_{1,\lambda}^2 + p_2 \|[\psi_{2,\lambda}]_{2,\lambda}\|_{2,\lambda}^2.$$

- (Spectral estimates) *If either $\sigma_{1,*} = 0$ or if $\alpha_{i,*} = 0$, $i = 1, 2$, then*

$$\lambda^*(\Omega, \mathbf{p}) \geq \frac{1}{p_2} \Lambda(\Omega, 2p_2).$$

Clearly, $E_0(\Omega)$ coincides with the torsional rigidity of Ω . The above theorem holds for any smooth and bounded domain, in any dimension and for any subcritical exponent (in the sense of (1.1)). The result is sharp in the scalar case $p = p_1 = p_2$, $\alpha_\lambda = \alpha_{1,\lambda} = \alpha_{2,\lambda}$, $\psi_\lambda = \psi_{1,\lambda} = \psi_{2,\lambda}$. In this case $\alpha_\lambda > 0$ and $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$ for any $\lambda < \frac{1}{p} \Lambda(\Omega, 2p)$, see [5]. In particular, it has been shown

in [5] that if $p < p_N$ there exists a positive solution for any $\lambda < \frac{1}{p}\Lambda(\Omega, 2p)$, then by Theorem 1.3 we immediately deduce the following corollary about the case $1 \leq p_1 = p_2 < p_N$.

Corollary 1.4. *Let $1 \leq p_1 = p_2 < p_N$. For any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$ the unique solution of $(\mathbf{P})_\lambda$ is the scalar solution $\alpha_\lambda = \alpha_{1,\lambda} = \alpha_{2,\lambda}$, $\psi_\lambda = \psi_{1,\lambda} = \psi_{2,\lambda}$.*

It is still unknown whether $\sigma_{1,*} = 0$. When $p_1 \neq p_2$, the condition $\alpha_{i,*} = 0$, $i = 1, 2$ does not seem natural. A more plausible scenario is the following. If $\sigma_{1,*} > 0$, then either $\alpha_{1,*} = 0, \alpha_{2,*} > 0$ or $\alpha_{1,*} > 0, \alpha_{2,*} = 0$. This situation is of particular interest. In this case, one obtains a parametrization of an unbounded branch of solutions of (\mathbf{H}) .

For any classical solution $\mathbf{u} = (u_1, u_2)$ of (\mathbf{H}) for some $\boldsymbol{\mu} = (\mu_1, \mu_2) \in ([0, +\infty))^2$ we define,

$$\begin{aligned}\gamma(\boldsymbol{\mu}, \mathbf{u}) &= \frac{p_1}{p_1 + 1} \frac{1}{\|1 + u_1\|_{p_1}} + \frac{p_2}{p_2 + 1} \frac{1}{\|1 + u_2\|_{p_2}}, \\ E(\boldsymbol{\mu}, \mathbf{u}) &= \frac{1}{2\mu_1\mu_2} \int_{\Omega} \frac{\mu_1(1 + u_1)^{p_1}u_1 + \mu_2(1 + u_2)^{p_2}u_2}{\|1 + u_1\|_{p_1}^{p_1} \|1 + u_2\|_{p_2}^{p_2}}, \\ F(\boldsymbol{\mu}, \mathbf{u}) &= \gamma(\boldsymbol{\mu}, \mathbf{u}) + \frac{p_1 p_2 - 1}{(p_2 + 1)(p_1 + 1)} E(\boldsymbol{\mu}, \mathbf{u}).\end{aligned}$$

Then we have,

Theorem 1.5. *Let (p_1, p_2) satisfy (1.1) and $\mathcal{G}(\Omega)$ be the set of unique solutions $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ of $(\mathbf{P})_\lambda$ for $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$. Then*

$$\mathbf{u}_\lambda = (u_{1,\lambda}, u_{2,\lambda}) = \left(\frac{\lambda}{\alpha_{1,\lambda}} \psi_{1,\lambda}, \frac{\lambda}{\alpha_{2,\lambda}} \psi_{2,\lambda} \right),$$

is a solution of (\mathbf{H}) with,

$$\boldsymbol{\mu}_\lambda = (\mu_{1,\lambda}, \mu_{2,\lambda}) = \left(\lambda \frac{\alpha_{1,\lambda}^{p_1}}{\alpha_{2,\lambda}}, \lambda \frac{\alpha_{2,\lambda}^{p_2}}{\alpha_{1,\lambda}} \right),$$

and for any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$ it holds,

$$\frac{dF}{d\lambda}(\boldsymbol{\mu}_\lambda, \mathbf{u}_\lambda) < 0, \quad \frac{dE}{d\lambda}(\boldsymbol{\mu}_\lambda, \mathbf{u}_\lambda) \geq 0, \quad \frac{d\gamma}{d\lambda}(\boldsymbol{\mu}_\lambda, \mathbf{u}_\lambda) < 0. \quad (1.4)$$

In particular, if $\alpha_{1,*} = 0, \alpha_{2,*} > 0$ then $(\boldsymbol{\mu}_\lambda, \mathbf{u}_\lambda)$ is a real analytic and unbounded curve and, possibly along a subsequence, we have that

$$\mu_{1,\lambda} \rightarrow 0, \quad \mu_{2,\lambda} \rightarrow +\infty, \quad \|1 + u_{1,\lambda}\|_{p_1} \rightarrow +\infty.$$

Moreover, under the assumption of Theorem 1.2-(b), we have $\frac{u_{1,\lambda}}{\|1 + u_{1,\lambda}\|_{p_1}} \rightarrow \lambda^* \psi_{1,*}, \frac{u_{2,\lambda}}{\|1 + u_{2,\lambda}\|_{p_2}} \rightarrow \lambda^* \psi_{2,*}$, with convergence in $C^2(\overline{\Omega})$ where $(\psi_{1,*}, \psi_{2,*})$ is a solution of $(\mathbf{P})_\lambda$ with $\lambda = \lambda^* = \lambda^*(\Omega, \mathbf{p})$, $\alpha_1 = 0$ for some $\alpha_2 > 0$. The conclusion is analogous in the case $\alpha_{1,*} > 0, \alpha_{2,*} = 0$.

Regarding Theorem 1.3, the competition between the monotonic behavior of $\alpha_{i,\lambda}$ and $\psi_{i,\lambda}$ makes the problem difficult. In particular, standard arguments based on the maximum principle do not seem effective. On the other hand, the monotonicity of E_λ, F_λ can be derived for variational solutions of $(\mathbf{P})_\lambda$, once uniqueness is known. However, this is not sufficient to establish the smoothness of E_λ . It is also not enough to prove the monotonicity or smoothness of $\left(\frac{p_1 \alpha_{1,\lambda}}{p_1 + 1} + \frac{p_2 \alpha_{2,\lambda}}{p_2 + 1} \right)$. The situation is more delicate for E_λ . Variational methods seem to provide no useful information in this case.

The proof of Theorem 1.3 relies on the interplay between a refined bifurcation analysis and the variational formulation of $(\mathbf{P})_\lambda$.

The key step point is the construction of a dual "Hamiltonian" spectral theory for the linearized

operator of $(\mathbf{P})_\lambda$. See the definition (3.9) of $\mathbf{L}_\lambda = (L_{1,\lambda}, L_{2,\lambda})$ and the related eigenvalue equation (3.10) in section 3. This eigenvalue problem is non standard. For this reason, we refer to it as a "Hamiltonian" eigenvalue problem. The operator \mathbf{L}_λ is delicate to handle. It arises as the linearization of a constrained vector problem with $\int_\Omega \rho_{i,\lambda} = 1, i = 1, 2$, taken with respect to $(\alpha_\lambda, \psi_\lambda)$. This leads to a non-local structure. As a consequence, even in the scalar case, standard spectral properties may fail. In general, the first eigenvalue $\sigma_1(\alpha_\lambda, \psi_\lambda)$ is not simple. Moreover, its positivity does not imply the validity of the maximum principle. This phenomenon already appears in the scalar case with $\lambda = 0$ on \mathbb{D}_2 . In that setting, $\sigma_1(\alpha_0, \psi_0)$ can be evaluated explicitly (see [4]). One finds that $\sigma_1(\alpha_0, \psi_0) = \lambda^{(2,0)}(\mathbb{D}_2) \simeq \pi(3, 83)^2$, and there are three eigenfunctions. Two of them change sign. Here, $\lambda^{(2,0)}(\Omega)$ denotes the first nonzero eigenvalue of $-\Delta$ on Ω , restricted to functions in $H^1(\Omega)$ with zero mean and constant boundary trace. See also [6] for a related results.

Let $(\alpha_\lambda, \psi_\lambda)$ be a positive solution with $\lambda \geq 0$. Assume that $0 \notin \sigma(\mathbf{L}_\lambda)$, where $\sigma(\mathbf{L}_\lambda)$ denotes the spectrum of \mathbf{L}_λ . Then, by the real analytic implicit function theorem ([10]), the set of solutions of $(\mathbf{P})_\lambda$ is locally a real analytic curve of positive solutions. In particular, such a curve exists near (α_0, ψ_0) . Since solutions of $(\mathbf{P})_\lambda$ are uniformly bounded (see Lemma 2.1) then for any $\lambda < \lambda^*(\Omega, \mathbf{p})$ there is a unique solution. These solutions form a real analytic curve, denoted by $\mathcal{G}(\Omega)$. At this stage, one can derive a positive lower bound for $\sigma_1(\alpha_\lambda, \psi_\lambda)$ for positive solutions, see Proposition 3.4. One also obtains an estimate on the range where $\alpha_{1,\lambda}$ and $\alpha_{2,\lambda}$ may vanish simultaneously see Proposition 3.5. Since F_λ, E_λ and $\alpha_{i,\lambda}$ are real analytic in λ for $\lambda < \lambda^*(\Omega, \mathbf{p})$, their monotonicity follows from the variational characterization of $(\alpha_\lambda, \psi_\lambda)$. The estimate for the derivative of E_λ requires a more refined analysis. It relies on a Fourier expansion of $\frac{dE_\lambda}{d\lambda}$ in the "Hamiltonian" Fourier basis, see section 4.

It would be interesting to find a fourth monotone quantity associated with the problem. For example the self-interaction energy

$$E_{\lambda,s} := \frac{1}{2} \int_\Omega \rho_{2,\lambda} G[\rho_{2,\lambda}] + \frac{1}{2} \int_\Omega \rho_{1,\lambda} G[\rho_{1,\lambda}].$$

However, its qualitative behavior is unclear. We postpone this problem to a future work.

The paper is organized as follows. Section 2 addresses the existence of the free boundary and proves Theorem 1.2. In section 3 we set up the spectral and bifurcation analysis with the needed spectral estimates. In section 4 we prove existence of variational solutions, uniqueness and monotonicity, which yield the proof of Theorem 1.3 and, as a corollary, that of Theorem 1.5. The proof of the existence of solutions and of the positivity of $\alpha_{i,\lambda}$ for λ small is discussed in Appendix A.

2. Existence of the free boundary

For later purposes, see either Theorem 4.2 below, we prove a regularity result of independent interest, showing that if $(\alpha_\lambda, \psi_\lambda)$ is a solution of $(\mathbf{P})_\lambda$ such that $(\psi_{1,\lambda}, \psi_{2,\lambda}) \in W_0^{2,p_2}(\Omega) \times W_0^{2,p_1}(\Omega)$ is just assumed to be a strong solution of $(\mathbf{P})_\lambda$, then $(\psi_{1,\lambda}, \psi_{2,\lambda}) \in C_0^{2,r_0}(\Omega) \times C_0^{2,r_0}(\Omega)$ and is uniformly bounded in $C_0^{2,r_0}(\Omega) \times C_0^{2,r_0}(\Omega)$, as far as λ is bounded as well. Indeed we have,

Lemma 2.1. *Let $\mathbf{p} = (p_1, p_2)$ satisfy (1.1). For any $\bar{\lambda} > 0$ there exists a positive constant $C_1 = C_1(r, \Omega, \bar{\lambda}, \mathbf{p}, N)$ depending only on $\Omega, \bar{\lambda}, \mathbf{p}, N$ and $r \in (0, 1)$ such that $\|\psi_{i,\lambda}\|_{C_0^{2,r_0}(\bar{\Omega})} \leq C_1$, $i = 1, 2$ for any strong solution $(\alpha_\lambda, \psi_\lambda)$ of $(\mathbf{P})_\lambda$ with $\lambda \in [0, \bar{\lambda}]$, where $r_0 = \min\{p_1, p_2, r\}$.*

Proof. To simplify the notations we set $(\alpha_i, \psi_i) = (\alpha_{i,\lambda}, \psi_{i,\lambda})$. Since Ω is of class C^3 , by standard elliptic estimates and a bootstrap argument it is enough to prove that either ψ_1 or ψ_2 is bounded. We prove only the case $N \geq 3$ which is more delicate.

Let (p_1, p_2) satisfy (1.1) and assume w.l.o.g. that $p_1 < p_N$. Since $\int_{\Omega} (\alpha_i + \lambda\psi_i)_+^{p_i} = 1$, $i = 1, 2$, then it is well known ([26]) that for any $t \in [1, \frac{N}{N-1})$ there exists $C = C(t, N, \Omega)$ such that $\|\psi_i\|_{W_0^{1,t}(\Omega)} \leq C(t, N, \Omega)$, $i = 1, 2$ for any solution of $(\mathbf{P})_{\lambda}$. Thus, by the Sobolev inequality, for any $1 \leq s < \frac{N}{N-2}$ we have $\|\psi_i\|_{L^s(\Omega)} \leq C(s, N, \Omega)$, $i = 1, 2$ for some $C(s, N, \Omega)$. By the maximum principle $\psi_i \geq 0$, $i = 1, 2$. Thus, either $\alpha_i > 0$ and then $\int_{\Omega} (\alpha_i + \lambda\psi_i) = 1$ and (recall $|\Omega| = 1$) $\alpha_i \leq 1$, or $\alpha_i < 0$ and then $(\alpha_i + \psi_i)_+ \leq \psi_i$. Since $p_1 < p_N$, then for any $1 < m < \frac{p_N}{p_1}$, $\|(\alpha_1 + \lambda\psi_1)^{p_1}\|_{L^m(\Omega)} \leq C(p_1, N, \lambda, s, \Omega)$.

From now on we suppress the indications of the properties of the various constants involved in the estimates, being understood that C is just a suitable uniform constant which do not depend by the solutions.

By standard elliptic theory we have that $\|\psi_2\|_{W_0^{2,m}(\Omega)} \leq C$, for any $m < \frac{p_N}{p_1}$ and by the Sobolev embedding either $p_1 \leq \frac{2}{N-2}$ and then $\|\psi_2\|_{L^r(\Omega)} \leq C$ for any $r \geq 1$ and then the desired conclusion follows in a standard way by a bootstrap argument, or $\frac{2}{N-2} < p_1 < p_N$ and then $\|\psi_2\|_{L^r(\Omega)} \leq C$ for any $r < \frac{N}{p_1 - \frac{2}{N-2}}$. By (1.1) we have

$$p_2 < \frac{p_1 + N}{p_1(N-2) - 1},$$

and observe that, putting $p_1 = \frac{x}{N-2}$, $x \in (2, N)$,

$$m_{N,p_1} := \frac{\frac{N}{p_1 - \frac{2}{N-2}}}{\frac{p_1 + N}{p_1(N-2) - 1}} = \frac{N(N-2)}{p_1 + N} \frac{p_1(N-2) - 1}{p_1(N-2) - 2} = (N-2)f_N(x),$$

where $f_N(x) = \frac{x-1}{(x-2)(1+a_N x)}$, $a_N = \frac{1}{N(N-2)}$. Therefore, since f_N is decreasing, we see that m_{N,p_1} is monotonic decreasing, with $m_{N,p_1} \rightarrow +\infty$ as $p_1 \rightarrow \left(\frac{2}{N-2}\right)^+$, $m_{N,p_1} \rightarrow (N-2)^+$ as $p_1 \rightarrow \left(\frac{N}{N-2}\right)^-$. In particular $\|(\alpha_2 + \lambda\psi_2)^{p_2}\|_{L^n(\Omega)} \leq C$ for any $n < m_{N,p_1}$, whence by standard elliptic theory $\|\psi_1\|_{W_0^{2,n}(\Omega)} \leq C$, for any $n < m_{N,p_1}$ and either $m_{N,p_1} \geq \frac{N}{2}$ and then $\|\psi_1\|_{L^r(\Omega)} \leq C$ for any $r \geq 1$ and then the desired conclusion follows in a standard way by a bootstrap argument, or $m_{N,p_1} < \frac{N}{2}$ and then $\|\psi_1\|_{L^r(\Omega)} \leq C$ for any $r < \frac{N}{N-2m_{N,p_1}}$. On the other side, as deduced above, we have that $m_{N,p_1} \geq (N-2)$ whence in particular $m_{N,p_1} \geq \frac{N}{2}$, as far $N \geq 4$ and we are just left with the case $N = 3$ which requires a different argument.

With the notations adopted above, we have that $\|(\alpha_2 + \lambda\psi_2)^{p_2}\|_{L^n(\Omega)} \leq C$ for any $n < m_{3,p_1}^{(1)}$, where

$$m_{3,p_1}^{(1)} = \frac{1}{1 + \frac{p_1}{3}} \frac{p_1 - 1}{p_1 - 2} \quad \text{and} \quad 1 < m_{3,p_1}^{(1)} < \frac{3}{2}.$$

Here the assumption about p_1 is that $p_1 \in (\bar{p}, 3)$ where $m_{3,\bar{p}}^{(1)} = \frac{3}{2}$ (actually $\bar{p} > \frac{5}{2}$), where we recall that $m_{3,p_1}^{(1)}$ is decreasing in p_1 and $m_{3,3} = 1$. In particular, as mentioned above, by standard elliptic theory $\|\psi_1\|_{W_0^{2,n}(\Omega)} \leq C$ for any $n < m_{3,p_1}^{(1)}$ and by the Sobolev embedding $\|\psi_1\|_{L^s(\Omega)} \leq C$ for any $s < s_2 := \frac{3}{3-2m_{3,p_1}^{(1)}}$. We define

$$\sigma_1 := \frac{s_2}{3} = \frac{1}{3 - 2m_{3,p_1}^{(1)}} > 1, \quad \delta := \sigma_1 - 1,$$

whence $\|(\alpha_2 + \lambda\psi_2)^{p_2}\|_{L^m(\Omega)} \leq C$ for any $m < \frac{3}{p_1} \sigma_1$. Therefore, assuming w.l.o.g. that

$$\sigma_1 < \frac{p_1}{2},$$

by standard elliptic theory $\|\psi_2\|_{W_0^{2,m}(\Omega)} \leq C$ for any $m < \frac{3}{p_1} \sigma_1$ and by the Sobolev embedding $\|\psi_2\|_{L^r(\Omega)} \leq C$ for any $r < \frac{3\sigma_1}{p_1 - 2\sigma_1}$. As a consequence $\|(\alpha_2 + \lambda\psi_2)^{p_2}\|_{L^n(\Omega)} \leq C$ for any $n < m_{3,p_1}^{(2)}$

where, by using (1.1) once more, we define,

$$m_{3,p_1}^{(2)} = \frac{3\sigma_1}{p_1 - 2\sigma_1} \frac{p_1 - 1}{p_1 + 3} = \sigma_1 \frac{p_1 - 2}{p_1 - 2\sigma_1} m_{3,p_1}^{(1)} > \sigma_1 m_{3,p_1}^{(1)} = \frac{m_{3,p_1}^{(1)}}{3 - 2m_{3,p_1}^{(1)}}.$$

Obviously we can assume w.l.o.g. that $m_{3,p_1}^{(2)} < \frac{3}{2}$, that is, using the last equality,

$$1 < m_{3,p_1}^{(1)} < \frac{9}{8} \quad \text{and consequently} \quad 1 < \sigma_1 < \frac{4}{3}.$$

Therefore, as above $\|\psi_1\|_{L^s(\Omega)} \leq C$ for any $s < s_3 := \frac{3}{3 - 2m_{3,p_1}^{(2)}}$ and we define

$$\sigma_2 := \frac{s_3}{s_2} = \frac{3 - 2m_{3,p_1}^{(1)}}{3 - 2m_{3,p_1}^{(2)}}.$$

At this point, since $m_{3,p_1}^{(2)} > \sigma_1 m_{3,p_1}^{(1)}$, elementary arguments show that

$$\sigma_2 > \frac{3 - 2m_{3,p_1}^{(1)}}{3 - 2\sigma_1 m_{3,p_1}^{(1)}} = 1 + \frac{2m_{3,p_1}^{(1)}(\sigma_1 - 1)}{3 - 2\sigma_1 m_{3,p_1}^{(1)}} > 1 + \delta = \sigma_1.$$

Consequently $\|(\alpha_2 + \lambda\psi_1)^{p_1}\|_{L^m(\Omega)} \leq C$ for any $m < \frac{s_3}{p_1} = \frac{3}{p_1} \sigma_1 \sigma_2 > \frac{3}{p_1} \sigma_1^2$ and by standard elliptic theory and the Sobolev embedding we have that $\|(\alpha_2 + \lambda\psi_2)^{p_2}\|_{L^n(\Omega)} \leq C$ for any $n < m_{3,p_1}^{(3)}$ where, by using (1.1) once more, we define,

$$m_{3,p_1}^{(3)} := \frac{3\sigma_1^2}{p_1 - 2\sigma_1^2} \frac{p_1 - 1}{p_1 + 3} = \sigma_1 \frac{3\sigma_1}{p_1 - 2\sigma_1^2} \frac{p_1 - 1}{p_1 + 3} > \sigma_1 m_{3,p_1}^{(2)},$$

where we can assume w.l.o.g. that,

$$\sigma_1^2 < \frac{p_1}{2}.$$

At this point, by induction, it is not too difficult to prove that after a finite number of iterations, either $p_1 - 2\sigma_1^{k-1} \leq 0$ or $m_{3,p_1}^{(k)} = \frac{3\sigma_1^{k-1}}{p_1 - 2\sigma_1^{k-1}} \frac{p_1 - 1}{p_1 + 3}$ will become larger than $\frac{3}{2}$ and the desired conclusion follows, in this case as well. \square

Next we present the proof of Theorem 1.2 about a priori estimates and the existence of a free boundary.

Proof. We postpone the proof of (a), i.e. the existence of at least one solution of $(\mathbf{P})_\lambda$ for any $\lambda > 0$, to Theorem 4.2 in section 4.

Therefore we are going to prove (b). We argue by contradiction and assume that there exists a sequence of non negative solutions $((\alpha_{1,n}, \alpha_{2,n}), (\psi_{1,n}, \psi_{2,n}))$ of $(\mathbf{P})_\lambda|_{\lambda=\lambda_n}$ such that $\lambda_n \rightarrow +\infty$. By Lemma 2.1, for any fixed n it holds $\|\psi_{i,n}\|_\infty \leq C_n$, $i = 1, 2$. Let $m_{i,n} = \sup_{\Omega} (\alpha_{i,n} + \lambda_n \psi_{i,n})$, $i = 1, 2$. We split the proof in various steps.

STEP 1. Along a subsequence we have $\max\{m_{1,n}, m_{2,n}\} \rightarrow +\infty$, as $n \rightarrow +\infty$.

We will need the following lemma whose proof can be found in [3].

Lemma 2.2. *Let ψ be any solution of*

$$\begin{cases} -\Delta\psi = f & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega \end{cases}$$

where $\int_{\Omega} f = 1$ and $\int_{\Omega} |f|^N \leq C$. Then,

$$\int_{\Omega} |\nabla\psi|^2 \geq c > 0,$$

for some positive constant $c > 0$ depending only by C , N and Ω .

At this point we argue by contradiction and assume that there exists $C > 0$ such that

$$\sup_n \max\{m_{1,n}, m_{2,n}\} \leq C,$$

so that $\sup_n \max\{\|\psi_{1,n}\|_\infty, \|\psi_{2,n}\|_\infty\} \leq \frac{C}{\lambda_n}$. Therefore, along a subsequence we have,

$$\int_\Omega |\nabla \psi_{1,n}|^2 = \int_\Omega (\alpha_{2,n} + \lambda_n \psi_{2,n})^{p_2} \psi_{1,n} \leq \frac{C}{\lambda_n} \rightarrow 0, \text{ as } n \rightarrow +\infty,$$

which contradicts Lemma 2.2 since $f_{i,n} = (\alpha_{i,n} + \lambda_n \psi_{i,n})^{p_i}$ obviously satisfies the needed assumptions for any $i = 1, 2$.

Therefore, along a subsequence we have $\max\{m_{1,n}, m_{2,n}\} \rightarrow +\infty$, as $n \rightarrow +\infty$.

STEP 2. Let $x_{i,n}$ be any maximum point of $\psi_{i,n}$, $i = 1, 2$, we prove that

$$\text{dist}(x_{i,n}, \partial\Omega) \geq d_0,$$

for some positive constant $d_0 > 0$.

Suppose first $N = 2$. We argue as in [24] p.223. Let $x_0 \in \partial\Omega$, ν_0 be the outer unit normal at x_0 and $B_r(x_1) \subset \mathbb{R}^N \setminus \bar{\Omega}$ such that $\bar{B}_r(x_1) \cap \partial\Omega = \{x_0\}$. This is always possible since Ω is of class C^3 . After a translation and a rotation we can assume w.l.o.g. that $x_1 = 0$, $\nu_0 = (-1, 0, \dots, 0)$. Also, after a dilation and a suitable scaling of λ_n , we can assume that $r = 1$, whence $x_0 = (1, 0, \dots, 0)$. At this point if $N = 2$ we define (see [24] p.223) $y = \frac{x}{|x|^2}$ and $u_{i,n}(y) = \psi_{i,n}(x)$, $i = 1, 2$. The image of Ω under this Kelvin transform, say $\tilde{\Omega}$, lies inside $B_1(0)$ and its closure touches the boundary only at x_0 . Also $(u_{1,n}(y), u_{2,n}(y))$ satisfies

$$\begin{cases} -\Delta u_{1,n} = h(y)(\alpha_{2,n} + \lambda_n u_{2,n})^{p_2} & \text{in } \tilde{\Omega} \\ -\Delta u_{2,n} = h(y)(\alpha_{1,n} + \lambda_n u_{1,n})^{p_1} & \text{in } \tilde{\Omega} \\ u_{i,n} = 0 & \text{on } \partial\tilde{\Omega}, \quad i = 1, 2, \\ \alpha_{i,n} \geq 0, & i = 1, 2, \end{cases}$$

where $h(y) = \frac{1}{|y|^4}$. Let $y = (y_1, y_2)$, clearly $\frac{\partial}{\partial y_1} h(y) < 0$ as far as $y_1 > 0$, whence we can apply the argument of Theorem 2.1' in [24], just replacing the classical strong maximum principle and Hopf lemma used there (see Lemma H in [24]) with the corresponding results for cooperative linear elliptic systems, see Theorem 2.2 in [19]. Therefore, if $N = 2$, as in [24] p.223 we deduce that in a neighborhood of x_0 depending only by the geometry of Ω there are no critical points of $\psi_{i,n}$, $i = 1, 2$. Since $\partial\Omega$ is compact the desired conclusion follows by a covering argument.

Suppose now $N \geq 3$ and Ω convex. As above, using the results for cooperative linear elliptic systems we can exploit the argument of Theorem 2.1' in [24]. It is well known that the convexity condition ensures then that there are no critical points of $\psi_{i,n}$, $i = 1, 2$ in a sufficiently small uniform neighborhood of the boundary, see [21] for further details in the scalar case.

STEP 3. We obtain a contradiction assuming that $m_{2,n} \leq m_{1,n} \rightarrow +\infty$, as $n \rightarrow +\infty$. We will never use the fact that $p_1 \leq p_2$, whence a contradiction arises in the same way in the case where, along a subsequence, $m_{1,n} \leq m_{2,n} \rightarrow +\infty$, as $n \rightarrow +\infty$.

Let $x_{1,n}$ be such that $\psi_{1,n}(x_{1,n}) = m_{1,n}$. Then by step 2 we have $x_{i,n} \rightarrow \bar{x} \in \Omega$ and after a translation we can assume w.l.o.g. that $x_{1,n} = 0$, $\forall n \in \mathbb{N}$. There are only two possibilities: either,

$$(j) \sup_n \frac{m_{2,n}^{p_2+1}}{m_{1,n}^{p_1+1}} \leq C, \text{ or, passing to a further subsequence if necessary,}$$

$$(jj) \frac{m_{1,n}^{p_1+1}}{m_{2,n}^{p_2+1}} \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

We discuss (j) first and define $\delta_n^2 = \frac{m_{2,n}}{\lambda m_{1,n}^{p_1}}$ and, for $i = 1, 2$,

$$v_{i,n}(y) = \frac{1}{m_{i,n}}(\alpha_{i,n} + \lambda \psi_{i,n}(\delta_n y)), \quad y \in \Omega_n = \{y \in \mathbb{R}^N : \delta_n y \in \Omega\}$$

which satisfy

$$\begin{cases} -\Delta v_{1,n} = \frac{m_{2,n}^{p_2+1}}{m_{1,n}^{p_1+1}} v_{2,n}^{p_2} & \text{in } \Omega_n \\ -\Delta v_{2,n} = v_{1,n}^{p_1} & \text{in } \Omega_n \\ v_{i,n}(y) \leq v_{1,n}(0) = 1 & \text{in } \Omega_n, \quad i = 1, 2, \\ v_{i,n}(y) \geq \frac{\alpha_{i,n}}{m_{i,n}} \geq 0 & \text{in } \Omega_n, \quad i = 1, 2. \end{cases}$$

Since $0 \in \Omega$, then for any $R \geq 1$ we have that for any n large enough it holds $B_R(0) \subset \Omega_n$. Along a subsequence we can assume that $\frac{m_{2,n}^{p_2+1}}{m_{1,n}^{p_1+1}} \rightarrow \mu \in [0, +\infty)$. Since $\|v_{i,n}\|_{L^\infty(\Omega_n)} \leq C$, then by standard elliptic estimates there exists a subsequence such that $v_{i,n}$, $i = 1, 2$ converge in $C_{\text{loc}}^2(\mathbb{R}^N)$ to v_i , $i = 1, 2$ which are classical solutions of

$$\begin{cases} -\Delta v_1 = \mu v_2^{p_2} & \text{in } \mathbb{R}^N \\ -\Delta v_2 = v_1^{p_1} & \text{in } \mathbb{R}^N \\ 0 \leq v_i(y) \leq v_1(0) = 1 & \text{in } \mathbb{R}^N, \quad i = 1, 2, \end{cases} \quad (2.1)$$

At this point observe that if $\mu = 0$ then necessarily $v_1 \equiv 1$ in \mathbb{R}^N and then v_2 would solve,

$$\begin{cases} -\Delta v_2 = 1 & \text{in } \mathbb{R}^N \\ 0 \leq v_2(y) \leq 1 & \text{in } \mathbb{R}^N, \quad i = 1, 2. \end{cases}$$

By the maximum principle we would have $v_2(y) \geq \frac{1}{2N}(R^2 - |y|^2)$, for any $R > 0$ and in particular $v_2(0) \geq \frac{1}{2N}R^2$, for any $R > 0$, which is impossible. Therefore $\mu \in (0, +\infty)$ which however is also impossible since it is well known by the result in [34] that (1.1) implies that the unique solution of (2.1) is $v_i \equiv 0$, $i = 1, 2$.

Therefore (jj) holds and in this case we choose $\delta_n^2 = \frac{m_{1,n}}{\lambda m_{2,n}^{p_2}}$ so that $(v_{1,n}, v_{2,n})$ satisfies

$$\begin{cases} -\Delta v_{1,n} = v_{2,n}^{p_2} & \text{in } \Omega_n \\ -\Delta v_{2,n} = \frac{m_{1,n}^{p_1+1}}{m_{2,n}^{p_2+1}} v_{1,n}^{p_1} & \text{in } \Omega_n \\ v_{i,n}(y) \leq v_{1,n}(0) = 1 & \text{in } \Omega_n, \quad i = 1, 2, \\ v_{i,n}(y) \geq \frac{\alpha_{i,n}}{m_{i,n}} \geq 0 & \text{in } \Omega_n, \quad i = 1, 2. \end{cases}$$

This case is easily seen to lead to the same situation described above for $\mu = 0$, whence a contradiction arise in this case as well.

As mentioned above, by symmetry, the discussion of the case in which along a subsequence $m_{1,n} \leq m_{2,n} \rightarrow +\infty$, as $n \rightarrow +\infty$, is exactly the same. \square

3. Spectral and bifurcation analysis

In this section we develop the spectral and bifurcation analysis for positive solutions of $(\mathbf{P})_\lambda$ with $\lambda \geq 0$ and (p_1, p_2) satisfying (1.1). From now on and unless otherwise specified, $(\alpha_\lambda, \psi_\lambda)$ is assumed to be a positive solution of $(\mathbf{P})_\lambda$.

By the maximum principle $\psi_{i,\lambda} \geq 0$, $i = 1, 2$ in Ω for any solution, whence for non negative solutions ($\alpha_{i,\lambda} \geq 0$) we have $\alpha_{i,\lambda} + \psi_{i,\lambda} \equiv (\alpha_{i,\lambda} + \psi_{i,\lambda})_+$. Therefore from now on and unless otherwise specified we will denote by,

$$\tau_{i,\lambda} = \lambda p_i, \rho_{i,\lambda,\alpha_i}(\psi_i) = (\alpha_i + \lambda\psi_i)^{p_i}, \rho_{i,\lambda} = (\alpha_{i,\lambda} + \lambda\psi_{i,\lambda})^{p_i}, i = 1, 2,$$

$$V_{i,\lambda,\alpha_i}(\psi_i) = (\alpha_i + \lambda\psi_i)^{p_i-1} \text{ and } V_{i,\lambda} = (\alpha_{i,\lambda} + \lambda\psi_{i,\lambda})^{p_i-1},$$

where $\alpha_{i,\lambda}, \psi_{i,\lambda}$, $i = 1, 2$ denote the components of a non negative solution $(\alpha_\lambda, \psi_\lambda)$ of $(\mathbf{P})_\lambda$ and q_i denotes the conjugate exponent of p_i , that is,

$$\frac{1}{p_i} + \frac{1}{q_i} = 1, \quad i = 1, 2.$$

For $(\alpha_\lambda, \psi_\lambda)$ a non negative solution of $(\mathbf{P})_\lambda$ we denote,

$$\langle \eta \rangle_{i,\lambda} = \frac{\int_\Omega V_{i,\lambda} \eta}{\int_\Omega V_{i,\lambda}} \quad \text{and} \quad [\eta]_{i,\lambda} = \eta - \langle \eta \rangle_{i,\lambda}, \quad i = 1, 2,$$

and define,

$$\langle \eta, \phi \rangle_{i,\lambda} := \int_\Omega V_{i,\lambda} \eta \phi \quad \text{and} \quad \|\phi\|_{i,\lambda}^2 := \langle \phi, \phi \rangle_{i,\lambda} = \int_\Omega V_{i,\lambda} \phi^2, \quad i = 1, 2,$$

where $\{\eta, \phi\} \subset L^2(\Omega)$. For non negative solutions $(\alpha_\lambda, \psi_\lambda)$ of $(\mathbf{P})_\lambda$, by the strong maximum principle we have that $\rho_{i,\lambda}$, $i = 1, 2$, is strictly positive in Ω , whence $\langle \cdot, \cdot \rangle_{i,\lambda}$, $i = 1, 2$ define scalar products on $L^2(\Omega)$ whose norms are denoted by $\|\cdot\|_{i,\lambda}$, $i = 1, 2$. We will also adopt the useful shorthand notation,

$$m_{i,\lambda} = \int_\Omega V_{i,\lambda}, \quad i = 1, 2.$$

In the sequel we aim to describe possible branches of solutions of $(\mathbf{P})_\lambda$ around a positive solution, i.e. with $\alpha_{i,\lambda} > 0$, $i = 1, 2$. To this end, it is not difficult to construct an open subset A_Ω of the Banach space of triples $(\lambda, \alpha, \psi) \in \mathbb{R} \times \mathbb{R}^2 \times (C_0^{2,r_0}(\bar{\Omega}))^2$ such that, on A_Ω , the densities $\rho_{i,\lambda,\alpha_i} = \rho_{i,\lambda,\alpha_i}(\psi_i) = (\alpha_i + \lambda\psi_i)^{p_i}$ are well defined and

$$\alpha_{i,\lambda} + \lambda\psi_{i,\lambda} \geq \frac{\alpha_{i,\lambda}}{2} \quad \text{in } \bar{\Omega}, \quad i = 1, 2, \quad (3.1)$$

in a sufficiently small open neighborhood in A_Ω of any triple of the form $(\lambda, \alpha_\lambda, \psi_\lambda)$ whenever $(\alpha_\lambda, \psi_\lambda)$ is a positive ($\alpha_{i,\lambda} > 0$) solution of $(\mathbf{P})_\lambda$. At this point we introduce the maps,

$$\mathbf{F} : A_\Omega \rightarrow (C^r(\bar{\Omega}))^2, \quad \mathbf{F}(\lambda, \alpha, \psi) := \begin{pmatrix} -\Delta\psi_1 - \rho_{2,\lambda,\alpha_2}(\psi_2) \\ -\Delta\psi_2 - \rho_{1,\lambda,\alpha_1}(\psi_1) \end{pmatrix} \quad (3.2)$$

and

$$\Phi : A_\Omega \rightarrow \mathbb{R}^2 \times (C^r(\bar{\Omega}))^2, \quad \Phi(\lambda, \alpha, \psi) := \begin{pmatrix} \mathbf{F}(\lambda, \alpha, \psi) \\ -1 + \int_\Omega \rho_{1,\lambda,\alpha_1}(\psi_1) \\ -1 + \int_\Omega \rho_{2,\lambda,\alpha_2}(\psi_2) \end{pmatrix}, \quad (3.3)$$

and, for positive solutions and for a fixed $(\lambda, \alpha, \psi) \in A_\Omega$, their differentials with respect to (α, ψ) , that is the linear operators,

$$D_{\alpha,\psi} \Phi(\lambda, \alpha, \psi) : \mathbb{R}^2 \times (C_0^{2,r}(\bar{\Omega}))^2 \rightarrow \mathbb{R}^2 \times (C^r(\bar{\Omega}))^2,$$

which acts as follows on $(\mathbf{s}, \boldsymbol{\phi}) = (s_1, s_2, \phi_1, \phi_2) \in \mathbb{R}^2 \times (C_0^{2,r}(\overline{\Omega}))^2$,

$$D_{\boldsymbol{\alpha}, \boldsymbol{\psi}} \Phi(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi})[\mathbf{s}, \boldsymbol{\phi}] = \begin{pmatrix} D_{\boldsymbol{\psi}} \mathbf{F}(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi})[\boldsymbol{\phi}] + d_{\boldsymbol{\alpha}} \mathbf{F}(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi})[\mathbf{s}] \\ \int_{\Omega} \left(D_{\psi_1} \rho_{1,\lambda,\alpha_1}(\psi_1)[\phi_1] + d_{\alpha_1} \rho_{1,\lambda,\alpha_1}(\psi_1)[s_1] \right) \\ \int_{\Omega} \left(D_{\psi_2} \rho_{2,\lambda,\alpha_2}(\psi_2)[\phi_2] + d_{\alpha_2} \rho_{2,\lambda,\alpha_2}(\psi_2)[s_2] \right) \end{pmatrix}, \quad (3.4)$$

where we have introduced the differential operators,

$$D_{\boldsymbol{\psi}} \mathbf{F}(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi})[\boldsymbol{\phi}] = \begin{pmatrix} -\Delta \phi_1 - \tau_{2,\lambda} V_{2,\lambda,\alpha_2}(\psi_2) \phi_2 \\ -\Delta \phi_2 - \tau_{1,\lambda} V_{1,\lambda,\alpha_1}(\psi_1) \phi_1 \end{pmatrix}, \quad \boldsymbol{\phi} = (\phi_1, \phi_2) \in (C_0^{2,r}(\overline{\Omega}))^2, \quad (3.5)$$

$$D_{\psi_i} \rho_{i,\lambda,\alpha_i}[\phi_i] = \tau_{i,\lambda} V_{i,\lambda,\alpha_i}(\psi_i) \phi_i, \quad \phi_i \in C_0^{2,r}(\overline{\Omega}), \quad i = 1, 2, \quad (3.6)$$

and

$$d_{\boldsymbol{\alpha}} \mathbf{F}(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi})[\mathbf{s}] = \begin{pmatrix} -p_2 V_{2,\lambda,\alpha_2}(\psi_2) s_2 \\ -p_1 V_{1,\lambda,\alpha_1}(\psi_1) s_1 \end{pmatrix} \quad \mathbf{s} = (s_1, s_2) \in \mathbb{R}^2, \quad (3.7)$$

$$d_{\alpha_i} \rho_{i,\lambda,\alpha_i}[s_i] = p_i V_{i,\lambda,\alpha_i}(\psi_i) s_i, \quad s_i \in \mathbb{R}, \quad i = 1, 2, \quad (3.8)$$

where we recall $\tau_{i,\lambda} = \lambda p_i$.

By the construction of A_{Ω} , see in particular (3.1), relying on known techniques about real analytic functions on Banach spaces ([10]), it can be shown that $\Phi(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi})$ is jointly real analytic in an open neighborhood of A_{Ω} around any triple of the form $(\lambda, \boldsymbol{\alpha}_{\lambda}, \boldsymbol{\psi}_{\lambda})$ whenever $(\boldsymbol{\alpha}_{\lambda}, \boldsymbol{\psi}_{\lambda})$ is a positive solution of $(\mathbf{P})_{\lambda}$.

For fixed $\lambda > 0$ the pair $(\boldsymbol{\alpha}_{\lambda}, \boldsymbol{\psi}_{\lambda})$ is a non negative solution of $(\mathbf{P})_{\lambda}$ in the classical sense as defined in the introduction if and only if $\Phi(\lambda, \boldsymbol{\alpha}_{\lambda}, \boldsymbol{\psi}_{\lambda}) = (0, 0, 0, 0)$, and, for positive solutions, we define the linearized operator,

$$\mathbf{L}_{\lambda}[\boldsymbol{\phi}] = \begin{pmatrix} L_{1,\lambda}[\boldsymbol{\phi}] \\ L_{2,\lambda}[\boldsymbol{\phi}] \end{pmatrix} = \begin{pmatrix} -\Delta \phi_1 - \tau_{2,\lambda} V_{2,\lambda}[\phi_2]_{2,\lambda} \\ -\Delta \phi_2 - \tau_{1,\lambda} V_{1,\lambda}[\phi_1]_{1,\lambda} \end{pmatrix}. \quad (3.9)$$

We say that $\sigma = \sigma(\boldsymbol{\alpha}_{\lambda}, \boldsymbol{\psi}_{\lambda}) \in \mathbb{R}$ is an eigenvalue of \mathbf{L}_{λ} if the "Hamiltonian" eigenvalue equation,

$$\begin{cases} -\Delta \phi_1 - \tau_{2,\lambda} V_{2,\lambda}[\phi_2]_{2,\lambda} = \sigma p_2 V_{2,\lambda}[\phi_2]_{2,\lambda}, \\ -\Delta \phi_2 - \tau_{1,\lambda} V_{1,\lambda}[\phi_1]_{1,\lambda} = \sigma p_1 V_{1,\lambda}[\phi_1]_{1,\lambda}, \end{cases} \quad (3.10)$$

admits a non-trivial weak solution $(\phi_1, \phi_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$, which is by definition an eigenfunction of σ .

We show that, with this particular definition, the eigenvalues of \mathbf{L}_{λ} share the usual properties of a general self-adjoint elliptic operator.

Let us define the Hilbert space,

$$\mathbf{Y}_0 := \{ \boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \{(L^2(\Omega))^2, \langle \cdot, \cdot \rangle_{\lambda}\} : \langle \varphi_i \rangle_{i,\lambda} = 0, i = 1, 2 \}, \quad (3.11)$$

where

$$\langle \boldsymbol{\varphi}, \boldsymbol{\eta} \rangle_{\lambda} := p_1 \langle \varphi_1, \eta_1 \rangle_{1,\lambda} + p_2 \langle \varphi_2, \eta_2 \rangle_{2,\lambda}, \quad \forall (\boldsymbol{\varphi}, \boldsymbol{\eta}) \in (\mathbf{Y}_0)^2$$

and the linear operator $\mathbf{T}_0 : \mathbf{Y}_0 \rightarrow \mathbf{Y}_0$, which acts on $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \mathbf{Y}_0$ as follows

$$\mathbf{T}_0(\boldsymbol{\varphi}) := \begin{pmatrix} \tau_{2,\lambda} G[V_{2,\lambda} \varphi_2] - \langle \tau_{2,\lambda} G[V_{2,\lambda} \varphi_2] \rangle_{1,\lambda} \\ \tau_{1,\lambda} G[V_{1,\lambda} \varphi_1] - \langle \tau_{1,\lambda} G[V_{1,\lambda} \varphi_1] \rangle_{2,\lambda} \end{pmatrix} \equiv \begin{pmatrix} \tau_{2,\lambda} [G[V_{2,\lambda} \varphi_2]]_{1,\lambda} \\ \tau_{1,\lambda} [G[V_{1,\lambda} \varphi_1]]_{2,\lambda} \end{pmatrix}. \quad (3.12)$$

By standard elliptic theory $G[V_{i,\lambda}\varphi_i] \subset W^{2,2}(\Omega)$, $i = 1, 2$, whence \mathbf{T}_0 is compact by the Sobolev embedding. Also for any $(\varphi, \eta) \in (\mathbf{Y}_0)^2$, we have,

$$\begin{aligned} \langle \eta, \mathbf{T}_0(\varphi) \rangle_{\lambda} &= p_1 \langle \eta_1, [\tau_{2,\lambda}G[V_{2,\lambda}\varphi_2]]_{1,\lambda} \rangle_{1,\lambda} + p_2 \langle \eta_2, [\tau_{1,\lambda}G[V_{1,\lambda}\varphi_1]]_{2,\lambda} \rangle_{2,\lambda} = \\ &= \lambda p_1 p_2 \int_{\Omega} V_{1,\lambda} \eta_1 [G[V_{2,\lambda}\varphi_2]]_{1,\lambda} + \lambda p_2 p_1 \int_{\Omega} V_{2,\lambda} \eta_2 [G[V_{1,\lambda}\varphi_1]]_{2,\lambda} = \\ &= \lambda p_1 p_2 \int_{\Omega} V_{1,\lambda} \eta_1 G[V_{2,\lambda}\varphi_2] + \lambda p_1 p_2 \int_{\Omega} V_{2,\lambda} \eta_2 G[V_{1,\lambda}\varphi_1] = \\ &= \lambda p_1 p_2 \int_{\Omega} G[V_{1,\lambda}\eta_1] V_{2,\lambda} \varphi_2 + \lambda p_1 p_2 \int_{\Omega} G[V_{2,\lambda}\eta_2] V_{1,\lambda} \varphi_1 = \\ &= p_2 \int_{\Omega} [\tau_{1,\lambda}G[V_{1,\lambda}\eta_1]]_{2,\lambda} V_{2,\lambda} \varphi_2 + p_1 \int_{\Omega} [\tau_{2,\lambda}G[V_{2,\lambda}\eta_2]]_{1,\lambda} V_{1,\lambda} \varphi_1 = \end{aligned}$$

$$p_1 \langle [\tau_{2,\lambda}G[V_{2,\lambda}\eta_2]]_{1,\lambda}, \varphi_1 \rangle_{1,\lambda} + p_2 \langle [\tau_{1,\lambda}G[V_{1,\lambda}\eta_1]]_{2,\lambda}, \varphi_2 \rangle_{2,\lambda} = \langle \mathbf{T}_0(\eta), \varphi \rangle_{\lambda},$$

which shows that \mathbf{T}_0 is also self-adjoint. Remark also that $\langle \varphi, \mathbf{T}_0(\varphi) \rangle_{\lambda} > 0$ if $\varphi \neq (0, 0)$, as is readily verified observing that $\psi_i = G[V_{i,\lambda}\varphi_i]$ satisfies $\int_{\Omega} |\nabla \psi_i|^2 = \langle \varphi_i, G[V_{i,\lambda}\varphi_i] \rangle_{i,\lambda}$, $i = 1, 2$.

As a consequence, by the spectral Theorem for self-adjoint, compact, linear operators on Hilbert spaces, we have that \mathbf{Y}_0 is the Hilbertian direct sum of the eigenfunctions of \mathbf{T}_0 , which can be represented as follows,

$$\varphi_k = (\varphi_{1,k}, \varphi_{2,k}), \varphi_{i,k} = [\phi_{i,k}]_{i,\lambda}, \quad i = 1, 2, \quad k \in \mathbb{N} = \{1, 2, \dots\},$$

$$\mathbf{Y}_0 = \overline{\text{Span}\{([\phi_{1,k}]_{1,\lambda}, [\phi_{2,k}]_{2,\lambda}), \quad k \in \mathbb{N}\}},$$

for some $(\phi_{1,k}, \phi_{2,k}) \in (H_0^1(\Omega))^2$, $k \in \mathbb{N} = \{1, 2, \dots\}$. In fact, any eigenfunction φ_k , whose eigenvalue is $\mu_k \in \mathbb{R} \setminus \{0\}$, satisfies, $\mu_k \varphi_k = \mathbf{T}_0(\varphi_k)$ and, by defining,

$$\frac{\lambda}{\lambda + \sigma_k} = \mu_k \in (0, +\infty), \quad 0 > \mu_1 \geq \mu_2 \geq \dots \geq \mu_k \rightarrow 0,$$

$$\phi_{1,k} := (\tau_{2,\lambda} + p_2 \sigma_k) G[V_{2,\lambda}\varphi_{2,k}], \quad \phi_{2,k} := (\tau_{1,\lambda} + p_1 \sigma_k) G[V_{1,\lambda}\varphi_{1,k}],$$

it is easy to see that φ_k is an eigenfunction of \mathbf{T}_0 with eigenvalue μ_k if and only if $\phi_k = (\phi_{1,k}, \phi_{2,k}) \in (H_0^1(\Omega))^2$ weakly solves,

$$\begin{cases} -\Delta \phi_{1,k} = (\tau_{2,\lambda} + \sigma_k p_2) V_{2,\lambda} [\phi_{2,k}]_{2,\lambda}, \\ -\Delta \phi_{2,k} = (\tau_{1,\lambda} + \sigma_k p_1) V_{1,\lambda} [\phi_{1,k}]_{1,\lambda}. \end{cases} \quad (3.13)$$

In particular the first eigenvalue $\sigma_1 = \sigma_1(\alpha_{\lambda}, \psi_{\lambda})$ is uniquely defined by the spectral radius of \mathbf{T}_0 , $r(\mathbf{T}_0) \equiv \mu_1 = \frac{\lambda}{\lambda + \sigma_1}$ where

$$\mu_1 = r(\mathbf{T}_0) = \sup_{\varphi \in \mathbf{Y}_0 \setminus \{0\}} \frac{\langle \varphi, \mathbf{T}_0(\varphi) \rangle_{\lambda}}{\langle \varphi, \varphi \rangle_{\lambda}}.$$

Since $\sigma_1 + \lambda = \frac{\lambda}{\mu_1}$ and since μ_1 is positive, then

$$\lambda + \sigma_1 > 0. \quad (3.14)$$

By the Fredholm alternative, if $0 \notin \{\sigma_j\}_{j \in \mathbb{N}}$, then $\mathbf{I} - \mathbf{T}_0$ is an isomorphism of \mathbf{Y}_0 onto itself. Clearly, we can construct an orthonormal base of eigenfunctions $\{\phi_k\}_{k \in \mathbb{N}}$ with respect to the scalar product $\langle \cdot, \cdot \rangle_{\lambda}$. However we need a refined property which is the following,

Lemma 3.1. *There exists a complete orthonormal base $\{\varphi_k\}_{k \in \mathbb{N}}$ of \mathbf{Y}_0 , with the property that $\varphi_k = ([\phi_{1,k}]_{1,\lambda}, [\phi_{2,k}]_{2,\lambda})$ satisfies,*

$$\langle [\phi_{1,k}]_{1,\lambda}, [\phi_{1,j}]_{1,\lambda} \rangle_{1,\lambda} = 0 = \langle [\phi_{2,k}]_{2,\lambda}, [\phi_{2,j}]_{2,\lambda} \rangle_{2,\lambda}, \quad \forall k \neq j. \quad (3.15)$$

In particular $\{[\phi_{i,k}]\}_{k \in \mathbb{N}}$, $i = 1, 2$ is a complete orthonormal base of

$$Y_{i,0} := \{\varphi_i \in \{L^2(\Omega), \langle \cdot, \cdot \rangle_{i,\lambda}\} : \langle \varphi_i \rangle_{i,\lambda} = 0\}, \quad i = 1, 2.$$

Proof. If σ_k, σ_j denote the eigenvalues of ϕ_k, ϕ_j respectively, and if $\sigma_k \neq \sigma_j$, then for fixed k we can multiply the first equation in (3.13) by $\phi_{2,j}$, the second by $\phi_{1,j}$ and integrate by parts to conclude that

$$\begin{cases} (\tau_{1,\lambda} + \sigma_j p_1) \int_{\Omega} V_{1,\lambda}[\phi_{1,j}]_{1,\lambda}[\phi_{1,k}]_{1,\lambda} = (\tau_{2,\lambda} + \sigma_k p_2) \int_{\Omega} V_{2,\lambda}[\phi_{2,k}]_{2,\lambda}[\phi_{2,j}]_{2,\lambda}, \\ (\tau_{2,\lambda} + \sigma_j p_2) \int_{\Omega} V_{2,\lambda}[\phi_{2,j}]_{2,\lambda}[\phi_{2,k}]_{2,\lambda} = (\tau_{1,\lambda} + \sigma_k p_1) \int_{\Omega} V_{1,\lambda}[\phi_{1,k}]_{1,\lambda}[\phi_{1,j}]_{1,\lambda}. \end{cases}$$

Putting

$$x = p_1 \int_{\Omega} V_{1,\lambda}[\phi_{1,j}]_{1,\lambda}[\phi_{1,k}]_{1,\lambda}, \quad y = p_2 \int_{\Omega} V_{2,\lambda}[\phi_{2,k}]_{2,\lambda}[\phi_{2,j}]_{2,\lambda},$$

this is equivalent to the system

$$\begin{cases} (\lambda + \sigma_j)x - (\lambda + \sigma_k)y = 0 \\ -(\lambda + \sigma_k)x + (\lambda + \sigma_j)y = 0 \end{cases},$$

whose determinant is not zero as far as $\sigma_k \neq \sigma_j$. Therefore in particular

$$\langle [\phi_{i,j}]_{i,\lambda}[\phi_{i,k}]_{i,\lambda} \rangle_{i,\lambda} = \int_{\Omega} V_{i,\lambda}[\phi_{i,j}]_{1,\lambda}[\phi_{i,k}]_{i,\lambda} = 0, \quad i = 1, 2,$$

whenever $\sigma_j \neq \sigma_k$. If $\sigma_k = \sigma_j$, since the eigenspace is of finite dimension, a standard component-wise orthonormalization argument shows that in fact the basis can be chosen to satisfy (3.15). At this point, since $\{\varphi_k\}_{k \in \mathbb{N}}$ is a complete orthonormal base then also $\{[\phi_{i,k}]_{i,\lambda}\}_{k \in \mathbb{N}}$ must be a complete orthonormal base of $Y_{i,0}$ for any $i = 1, 2$. \square

Concerning $D_{\alpha,\psi}\Phi(\lambda, \alpha, \psi)$ we have,

Proposition 3.2. *For any positive solution $(\alpha_\lambda, \psi_\lambda)$ of $(\mathbf{P})_\lambda$ with $\lambda \geq 0$, the kernel of $D_{\alpha,\psi}\Phi(\lambda, \alpha_\lambda, \psi_\lambda)$ is empty if and only if the system,*

$$\begin{cases} -\Delta\phi_1 - \tau_{2,\lambda}V_{2,\lambda}[\phi_2]_{2,\lambda} = 0, \\ -\Delta\phi_2 - \tau_{1,\lambda}V_{1,\lambda}[\phi_1]_{1,\lambda} = 0, \end{cases} \quad (\phi_1, \phi_2) \in (C_0^{2,r}(\overline{\Omega}))^2 \quad (3.16)$$

admits only the trivial solution, or equivalently, if and only if 0 is not an eigenvalue of \mathbf{L}_λ .

Proof. If $(\phi_1, \phi_2) \in (H_0^1(\Omega))^2$ solves (3.16) and since Ω is of class C^3 , then by standard elliptic regularity theory and a bootstrap argument we have $(\phi_1, \phi_2) \in (C_0^{2,r}(\overline{\Omega}))^2$. Therefore, in particular 0 is not an eigenvalue of \mathbf{L}_λ if and only if (3.16) admits only the trivial solution.

Suppose first that there exists a non-vanishing pair $(\mathbf{s}, \phi) \in \mathbb{R}^2 \times (C_0^{2,r}(\overline{\Omega}))^2$ such that

$$D_{\alpha,\psi}\Phi(\lambda, \alpha_\lambda, \psi_\lambda)[\mathbf{s}, \phi] = (\mathbf{0}, \mathbf{0}).$$

Then the second pair of equations in (3.4), that is $\int_{\Omega} \left(D_\psi \rho_{i,\lambda}[\phi_i] + d_{\alpha_i} \rho_{i,\lambda}[s_i] \right) \Big|_{(\alpha,\psi)=(\alpha_\lambda,\psi_\lambda)} = 0$, take the form,

$$p_i \int_{\Omega} (\lambda V_{i,\lambda} \phi_i + V_{i,\lambda} s_i) = 0, \quad i = 1, 2$$

which is equivalent to

$$s_i = s_{i,\lambda} = -\lambda \langle \phi_i \rangle_{i,\lambda}.$$

Substituting this relations into the first pair of equations,

$$\mathbf{L}_\lambda[\phi] = D_\psi F(\lambda, \alpha_\lambda, \psi_\lambda)[\phi] + d_{\alpha} F(\lambda, \alpha_\lambda, \psi_\lambda)[s_\lambda] = 0,$$

we conclude that $\phi = (\phi_1, \phi_2)$ is a non-trivial, classical solution of (3.16).

This shows one part of the claim, while on the other side, if a non-trivial, classical solution of (3.16) exists, then by arguing the other way around, obviously we can find some $(\mathbf{s}, \phi) \neq (\mathbf{0}, \mathbf{0})$ such that $D_{\alpha,\psi}\Phi(\lambda, \alpha_\lambda, \psi_\lambda)[\mathbf{s}, \phi] = (\mathbf{0}, \mathbf{0})$, as claimed. \square

We state now the result needed to describe branches of solutions of $(\mathbf{P})_\lambda$ at regular points.

Lemma 3.3. *Let $(\alpha_{\lambda_0}, \psi_{\lambda_0})$ be a positive solution of $(\mathbf{P})_\lambda$ with $\lambda = \lambda_0 \geq 0$.*

If 0 is not an eigenvalue of \mathbf{L}_{λ_0} , then:

(i) $D_{\alpha, \psi} \Phi(\lambda_0, \alpha_{\lambda_0}, \psi_{\lambda_0})$ is an isomorphism;

(ii) There exists an open neighborhood $\mathcal{U} \subset A_\Omega$ of $(\lambda_0, \alpha_{\lambda_0}, \psi_{\lambda_0})$ such that the set of solutions of $(\mathbf{P})_\lambda$ in \mathcal{U} is a real analytic curve of positive solutions $J \ni \lambda \mapsto (\alpha_\lambda, \psi_\lambda) \in B$, for suitable neighborhoods J of λ_0 and B of $(\alpha_{\lambda_0}, \psi_{\lambda_0})$ in $(0, +\infty)^2 \times (C_{0,+}^{2,r}(\overline{\Omega}))^2$.

(iii) In particular if $(\alpha_{\lambda_0}, \psi_{\lambda_0}) = (\alpha_0, \psi_0)$, then $(\alpha_\lambda, \psi_\lambda) = (\alpha_0, \psi_0) + O(\lambda)$ as $\lambda \rightarrow 0$.

Proof. By the construction of A_Ω , the map F as defined in (3.2) is jointly analytic in a suitable neighborhood of $(\lambda, \alpha_\lambda, \psi_\lambda)$. As a consequence, whenever (i) holds, then (ii) is an immediate consequence of the real analytic implicit function theorem, see for example Theorem 4.5.4 in [10]. In particular (iii) is a straightforward consequence of (ii). Therefore, we are just left with the proof of (i).

Concerning (i) we observe that, although the differential of the constrained equations (which are the last two differentials in (3.4)), do not define a Fredholm operator (since obviously the dimension of their kernel is not finite dimensional), however a simple inspection shows that in fact $D_{\alpha, \psi} \Phi(\lambda_0, \alpha_{\lambda_0}, \psi_{\lambda_0})$ is a Fredholm operator, see for example Lemma 2.4 in [5]. As a consequence (i) follows from Proposition 3.2 and the Fredholm alternative. \square

We conclude this section with some spectral estimates about $\sigma_1(\alpha_\lambda, \psi_\lambda)$ and $\lambda^*(\Omega, \mathbf{p})$.

Proposition 3.4. *Let (p_1, p_2) satisfy (1.1), assume without loss of generality $p_1 \leq p_2$ and suppose that $(\alpha_\lambda, \psi_\lambda)$ is a positive solution of $(\mathbf{P})_\lambda$ with $\lambda \leq \frac{1}{p_2} \Lambda(\Omega, 2p_2)$. Then $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$.*

Proof. For $k = 1$ and to ease the notations, in this proof we set $(\phi_1, \phi_2) = (\phi_{1,1}, \phi_{2,1})$, where $(\phi_{1,1}, \phi_{2,1})$ is any eigenvector of σ_1 . Clearly we can just consider $\lambda > 0$. We multiply the first equation in (3.13) by ϕ_1 , the second by ϕ_2 and deduce that

$$\int_{\Omega} |\nabla \phi_1|^2 = (\lambda + \sigma_1) p_2 \int_{\Omega} V_{2,\lambda} [\phi_2]_{2,\lambda} [\phi_1]_{1,\lambda},$$

and,

$$\int_{\Omega} |\nabla \phi_2|^2 = (\lambda + \sigma_1) p_1 \int_{\Omega} V_{1,\lambda} [\phi_1]_{1,\lambda} [\phi_2]_{2,\lambda},$$

which in view of (3.14) shows that

$$\int_{\Omega} V_{2,\lambda} [\phi_2]_{2,\lambda} [\phi_1]_{1,\lambda} > 0, \int_{\Omega} V_{1,\lambda} [\phi_1]_{1,\lambda} [\phi_2]_{2,\lambda} > 0. \quad (3.17)$$

In particular we readily deduce that

$$\sigma_1 = \frac{N(\phi_1, \phi_2)}{\int_{\Omega} (p_1 V_{1,\lambda} + p_2 V_{2,\lambda}) [\phi_1]_{1,\lambda} [\phi_2]_{2,\lambda}},$$

where

$$N(\phi_1, \phi_2) = \int_{\Omega} |\nabla \phi_1|^2 + \int_{\Omega} |\nabla \phi_2|^2 - \lambda p_2 \int_{\Omega} V_{2,\lambda} [\phi_2]_{2,\lambda} [\phi_1]_{1,\lambda} - \lambda p_1 \int_{\Omega} V_{1,\lambda} [\phi_2]_{2,\lambda} [\phi_1]_{1,\lambda}.$$

By using $ab \leq \frac{1}{2}(a^2 + b^2)$ we see that,

$$\begin{aligned} N(\phi_1, \phi_2) &\geq \frac{1}{2} \sum_{i=1}^2 \left(\int_{\Omega} |\nabla \phi_i|^2 - \lambda p_1 \int_{\Omega} V_{1,\lambda} [\phi_i]_{i,\lambda}^2 \right) + \\ &\quad \frac{1}{2} \sum_{i=1}^2 \left(\int_{\Omega} |\nabla \phi_i|^2 - \lambda p_2 \int_{\Omega} V_{2,\lambda} [\phi_i]_{i,\lambda}^2 \right) \geq \\ &\quad \frac{1}{2} \sum_{i=1}^2 \left(\int_{\Omega} |\nabla \phi_i|^2 - \lambda p_1 \int_{\Omega} V_{1,\lambda} \phi_i \right) + \frac{1}{2} \sum_{i=1}^2 \left(\int_{\Omega} |\nabla \phi_i|^2 - \lambda p_2 \int_{\Omega} V_{2,\lambda} \phi_i \right), \end{aligned} \quad (3.18)$$

where the last inequality is an equality if and only if $\langle \phi_1 \rangle_{1,\lambda} = 0 = \langle \phi_2 \rangle_{2,\lambda}$.

Next observe that for any $\phi \in H_0^1(\Omega) \setminus \{0\}$ and for any $\{i, j\} \in \{1, 2\}$, by the Holder inequality and (1.2) we have,

$$\int_{\Omega} |\nabla \phi|^2 - \lambda p_i \int_{\Omega} V_{i,\lambda} \phi^2 \geq \int_{\Omega} |\nabla \phi|^2 - \lambda p_i \left(\int_{\Omega} \rho_{i,\lambda} \right)^{\frac{1}{q_i}} \left(\int_{\Omega} \phi^{2p_i} \right)^{\frac{1}{p_i}} = \quad (3.19)$$

$$\int_{\Omega} |\nabla \phi|^2 - \lambda p_i \left(\int_{\Omega} \phi^{2p_i} \right)^{\frac{1}{p_i}} \geq \left(\int_{\Omega} \phi^{2p_i} \right)^{\frac{1}{p_i}} (\Lambda(\Omega, 2p_i) - \lambda p_i), \quad i = 1, 2. \quad (3.20)$$

Since $|\Omega| = 1$ and $p_1 \leq p_2$, it is well known (see Theorem 3 in [15]) that $\Lambda(\Omega, 2p_2) \leq \Lambda(\Omega, 2p_1)$ and that the inequality is strict as far as $p_1 < p_2$. Therefore we have proved that if $\lambda \leq \frac{1}{p_2} \Lambda(\Omega, 2p_2)$ then $\sigma_1 \geq 0$.

At this point we argue by contradiction and assume that $\sigma_1 = 0$ for some $\lambda \leq \frac{1}{p_2} \Lambda(\Omega, 2p_2)$. Following the equality sign in all the inequalities used so far we see that if $\sigma_1 = 0$ then we would necessarily have $\lambda = \frac{1}{p_2} \Lambda(\Omega, 2p_2)$, $p_1 = p_2$, $\langle \phi_1 \rangle_{1,\lambda} = 0 = \langle \phi_2 \rangle_{2,\lambda}$, and in particular $\phi_1 = \phi_2$ a.e. in Ω . By (3.16) this implies also $V_{1,\lambda} = V_{2,\lambda}$ a.e. in Ω . As a consequence we would also have,

$$0 = p_1 \sigma_1 = \frac{\int_{\Omega} |\nabla \phi_1|^2 - \lambda p_1 \int_{\Omega} V_{1,\lambda} \phi_1^2}{\int_{\Omega} V_{1,\lambda} \phi_1^2} > \nu_1 := \inf_{\phi \in H_0^1(\Omega)} \frac{\int_{\Omega} |\nabla \phi|^2 - \lambda p_1 \int_{\Omega} V_{1,\lambda} \phi^2}{\int_{\Omega} V_{1,\lambda} \phi^2},$$

where the strict inequality is due to the fact that any function which attains the inf is a first eigenfunction and consequently does not change sign. On the other side if $\phi \in C_0^1(\bar{\Omega})$, $\phi \neq 0$, then, once more by the Holder inequality and (1.2), we have,

$$\frac{\int_{\Omega} |\nabla \phi|^2}{\int_{\Omega} V_{2,\lambda} \phi^2} \geq \frac{1}{\left(\int_{\Omega} \rho_{2,\lambda} \right)^{\frac{1}{q_2}}} \frac{\int_{\Omega} |\nabla \phi|^2}{\left(\int_{\Omega} \phi^{2p_2} \right)^{\frac{1}{p_2}}} = \frac{\int_{\Omega} |\nabla \phi|^2}{\left(\int_{\Omega} \phi^{2p_1} \right)^{\frac{1}{p_1}}} \geq \Lambda(\Omega, 2p_1)$$

which immediately implies that,

$$\nu_1 \geq \Lambda(\Omega, 2p_1) - \lambda p_1 \geq 0, \quad \forall \lambda p_1 \leq \Lambda(\Omega, 2p_1),$$

which is a contradiction to $\nu_1 < 0$. \square

Proposition 3.5. *Let (p_1, p_2) satisfy (1.1), assume without loss of generality $p_1 \leq p_2$ and suppose that $\alpha_{i,*} = 0$, $i = 1, 2$. Then $\lambda^*(\Omega, \mathbf{p}) \geq \frac{1}{p_2} \Lambda(\Omega, 2p_2)$.*

Proof. We can assume w.l.o.g. $\lambda^*(\Omega, \mathbf{p}) \in (0, +\infty)$. By definition there exists a sequence $\lambda_n \rightarrow (\lambda^*(\Omega, \mathbf{p}))^-$, such that $\alpha_{i,\lambda_n} \rightarrow 0^+$. By Lemma 2.1 and passing to a further subsequence if necessary we can assume that $u_{i,n} = \lambda_n \psi_{i,\lambda_n}$, $i = 1, 2$, converge smoothly to u_i , $i = 1, 2$, which are classical solutions of

$$\begin{cases} -\Delta u_1 = \lambda^* u_2^{p_2} & \text{in } \Omega \\ -\Delta u_2 = \lambda^* u_1^{p_1} & \text{in } \Omega \\ u_i \geq 0 & \text{in } \Omega, \quad i = 1, 2 \\ u_i = 0 & \text{on } \partial\Omega, \quad i = 1, 2. \end{cases}$$

By the monotonicity of the energy E_λ we have $E_{\lambda^*} \geq E_0(\Omega)$, whence both u_1 and u_2 cannot vanish identically. Let us set $V_i = u_i^{p_i-1}$, $\phi_i = u_i$, $i = 1, 2$, then we have,

$$\begin{cases} -\Delta \phi_1 = \lambda^* V_2 \phi_2 & \text{in } \Omega \\ -\Delta \phi_2 = \lambda^* V_1 \phi_1 & \text{in } \Omega \\ \phi_i \geq 0 & \text{on } \Omega, \quad i = 1, 2 \\ \phi_i = 0 & \text{on } \partial\Omega, \quad i = 1, 2. \end{cases}$$

By the strong maximum principle for cooperative linear elliptic systems (see Theorem 2.2 in [19]) we have $\phi_i > 0$ in Ω , $i = 1, 2$ and we deduce that,

$$\begin{cases} -\Delta\phi_1 \leq \lambda^* p_2 V_2 \phi_2 & \text{in } \Omega \\ -\Delta\phi_2 \leq \lambda^* p_1 V_1 \phi_1 & \text{in } \Omega \\ \phi_i = 0 & \text{on } \partial\Omega, \quad i = 1, 2, \end{cases}$$

where the equality holds if and only if $p_1 = 1 = p_2$. Multiplying the first equation by ϕ_1 , the second by ϕ_2 and integrating by parts we deduce that

$$0 \geq Q(\phi_1, \phi_2) := \int_{\Omega} |\nabla\phi_1|^2 + \int_{\Omega} |\nabla\phi_2|^2 - \lambda^* p_2 \int_{\Omega} V_2 \phi_2 \phi_1 - \lambda^* p_1 \int_{\Omega} V_1 \phi_2 \phi_1. \quad (3.21)$$

By using $ab \leq \frac{1}{2}(a^2 + b^2)$ we see that,

$$Q(\phi_1, \phi_2) \geq \frac{1}{2} \sum_{i=1}^2 \left(\int_{\Omega} |\nabla\phi_i|^2 - \lambda^* p_1 \int_{\Omega} V_1 \phi_i^2 \right) + \frac{1}{2} \sum_{i=1}^2 \left(\int_{\Omega} |\nabla\phi_i|^2 - \lambda^* p_2 \int_{\Omega} V_2 \phi_i^2 \right),$$

and then by using (3.19), (3.20), $|\Omega| = 1$, $p_1 \leq p_2$ and $\Lambda(\Omega, 2p_2) \leq \Lambda(\Omega, 2p_1)$ (this is well known, see for example [15]) as above we deduce that that

$$Q(\phi_1, \phi_2) \geq \left(\frac{1}{2} \sum_{i=1}^2 \int_{\Omega} \int_{\Omega} V_1 \phi_i^2 + \frac{1}{2} \sum_{i=1}^2 \int_{\Omega} \int_{\Omega} V_2 \phi_i^2 \right) (\Lambda(\Omega, 2p_2) - \lambda^* p_2),$$

and we see from (3.21) that $\lambda^* p_2 \geq \Lambda(\Omega, 2p_2)$, as claimed. \square

4. Existence, uniqueness and monotonicity

Let $\mathcal{G}(\Omega)$ denote the set of solutions of $(\mathbf{P})_{\lambda}$ for $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$.

Lemma 4.1. *Let (p_1, p_2) satisfy (1.1). For any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$ there exists one and only one solution $(\alpha_{\lambda}, \psi_{\lambda})$ of $(\mathbf{P})_{\lambda}$. In particular $\mathcal{G}(\Omega)$ is a real analytic simple curve of positive solutions $[0, \lambda^*(\Omega, \mathbf{p})) \ni \lambda \mapsto (\alpha_{\lambda}, \psi_{\lambda})$.*

Proof. By Lemma A.1 and Proposition 3.4 we have that $\lambda^*(\Omega, \mathbf{p}) > 0$ and there exists a unique solution of $(\mathbf{P})_{\lambda}$ for $\lambda \leq \lambda_0$ for some λ_0 small enough. In particular these unique solutions are positive and we can assume, possibly taking a smaller λ_0 , that $\lambda^*(\Omega, \mathbf{p}) > \lambda_0$. However by definition we have $\sigma_1(\alpha_{\lambda}, \psi_{\lambda}) > 0$ for any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$ and then by Lemma 2.1 and Lemma 3.3 any solution whose λ is less than λ_0 generates a real analytic curve of positive solutions which can be continued to a real analytic curve of positive solutions defined in $[(-\delta, \lambda^*(\Omega, \mathbf{p}))$ for some small $\delta > 0$.

If at any point in $(\lambda_0, \lambda^*(\Omega, \mathbf{p}))$ there exist two solutions, then by definition of $\lambda^*(\Omega, \mathbf{p})$ they would be both positive and each one would generate in the same way another curve of positive solutions in $(-\delta, \lambda^*(\Omega, \mathbf{p}))$, for some small $\delta > 0$. Obviously for both curves at $\lambda = 0$ we have $(\alpha_{\lambda}, \psi_{\lambda}) = (\alpha_0, \psi_0)$ which is the unique solution of $(\mathbf{P})_{\lambda}$ for $\lambda = 0$. This is obviously impossible since then (α_0, ψ_0) would be a bifurcation point, in contradiction with Lemma 3.3. \square

Problem $(\mathbf{P})_{\lambda}$ is the Euler-Lagrange equation of the constrained minimization principle (\mathbf{VP}) below for the densities (ρ_1, ρ_2) . As far as $1 \leq p_1 \leq p_2 < p_N$, existence of solutions could be proved by an adaptation of an argument in [9], worked out there for a more general "scalar" problem, based on the theory of conjugate convex function. We adopt here a different argument based on the weak Young inequality ([25]), which yields existence for (p_1, p_2) satisfying (1.1), that is,

$$\frac{1}{p_1 + 1} + \frac{1}{p_2 + 1} > \frac{N - 2}{N - 1}, \quad p_i \in (0, +\infty), \quad i = 1, 2, \quad (4.1)$$

whose relevance for elliptic systems of Lane-Emden type was first noticed in [34].

Theorem 4.2. *Let (p_1, p_2) satisfy (1.1). For any $\lambda > 0$ there exists a solution $(\alpha_\lambda, \psi_\lambda)$ of $(\mathbf{P})_\lambda$.*

Proof. We discuss only the case $N \geq 3$, the case $N = 2$ is easier. Here $\|\rho\|_p$ denotes the standard $L^p(\Omega)$ norm. We will denote with C various constants depending only by (p_1, p_2) , Ω and N .

Let us define $r_i = 1 + \frac{1}{p_i}$, $i = 1, 2$ and

$$\mathcal{P}_{\Omega, i} := \left\{ \rho \in L^{r_i}(\Omega) \mid \rho \geq 0 \text{ a.e. in } \Omega, \int_{\Omega} \rho = 1 \right\}, \quad i = 1, 2.$$

It is readily seen that (1.1) is equivalent to

$$\frac{1}{r_1} + \frac{1}{r_2} < \frac{N}{N-1}, \quad r_i \in (1, +\infty), \quad i = 1, 2, \quad (4.2)$$

and for any (r_1, r_2) satisfying (4.2), for any $\lambda > 0$ and

$$(\rho_1, \rho_2) \in \mathcal{P}_{\Omega} \equiv \mathcal{P}_{\Omega, 1} \times \mathcal{P}_{\Omega, 2},$$

we define the free energy,

$$J_\lambda(\rho_1, \rho_2) = \frac{1}{r_1} \int_{\Omega} (\rho_1)^{r_1} + \frac{1}{r_2} \int_{\Omega} (\rho_2)^{r_2} - \lambda \int_{\Omega} \rho_1 G[\rho_2]. \quad (4.3)$$

We split the proof in several steps.

STEP 1. We prove that if (r_1, r_2) satisfies (4.2), then J_λ is coercive for any $\lambda > 0$, that is, if $\lambda > 0$ and $(\rho_{1,n}, \rho_{2,n}) \in \mathcal{P}_{\Omega, 1} \times \mathcal{P}_{\Omega, 2}$ and $\max\{\|\rho_{1,n}\|_{r_1}, \|\rho_{2,n}\|_{r_2}\} \rightarrow +\infty$, then $J_\lambda(\rho_{1,n}, \rho_{2,n}) \rightarrow +\infty$, as $n \rightarrow +\infty$.

Let us recall the weak Young inequality ([25]),

$$\int_{\Omega} \rho_1 G[\rho_2] \leq C \|\rho_1\|_{s_1} \|\rho_2\|_{s_2}, \quad \frac{1}{s_1} + \frac{1}{s_2} = \frac{N+2}{N}, \quad s_i \in (0, +\infty). \quad (4.4)$$

Assume for the time being that for any (r_1, r_2) which satisfies (4.2), there exists (s_1, s_2) which satisfies (4.4) and

$$s_i \in (1, r_i), \quad i = 1, 2. \quad (4.5)$$

Then by the (4.4) and standard interpolation inequalities (recall $\int_{\Omega} \rho_i = 1$, $i = 1, 2$) we would have that,

$$\int_{\Omega} \rho_1 G[\rho_2] \leq C \|\rho_1\|_{s_1} \|\rho_2\|_{s_2} \leq C \|\rho_1\|_{r_1}^{\gamma_1 r_1} \|\rho_2\|_{r_2}^{\gamma_2 r_2}, \quad (4.6)$$

where

$$\gamma_i = \left(1 - \frac{1}{s_i}\right) \frac{1}{r_i - 1}, \quad i = 1, 2.$$

Based on (4.6), elementary arguments show that J_λ is coercive for any $\lambda > 0$ as far as $\gamma_1 + \gamma_2 < 1$. Therefore we are left with showing that for any (r_1, r_2) which satisfies (4.2), there exists (s_1, s_2) which satisfies (4.4), (4.5) and in particular,

$$\left(1 - \frac{1}{s_1}\right) \frac{1}{r_1 - 1} + \left(1 - \frac{1}{s_2}\right) \frac{1}{r_2 - 1} < 1. \quad (4.7)$$

Obviously there is no loss of generality in assuming

$$r_2 \geq r_1.$$

Observe that, as far as,

$$(r_1, r_2) \in \left(1, \frac{N}{2}\right] \times \left[\frac{N}{2}, +\infty\right) \cup \{r_2 \geq r_1 : \left[\frac{N}{2}, +\infty\right) \times \left[\frac{N}{2}, +\infty\right)\},$$

then putting $\frac{1}{s_2} = \frac{2}{N} + \varepsilon$ for some small enough $\varepsilon > 0$, from (4.4) we would have $\frac{1}{s_1} = 1 - \varepsilon$ and then

$$1 < s_2 < \frac{N}{2} \leq r_2, \quad 1 < s_1 = \frac{1}{1 - \varepsilon} < r_1.$$

Thus (4.5) is satisfied. On the other side, since here $r_2 \geq \frac{N}{2}$, we also have that,

$$\begin{aligned} \left(1 - \frac{1}{s_1}\right) \frac{1}{r_1 - 1} + \left(1 - \frac{1}{s_2}\right) \frac{1}{r_2 - 1} &= \frac{\varepsilon}{r_1 - 1} + \left(1 - \frac{2}{N} - \varepsilon\right) \frac{1}{r_2 - 1} = \\ \varepsilon \left(\frac{1}{r_1 - 1} - \frac{1}{r_2 - 1}\right) + \frac{N-2}{N} \frac{1}{r_2 - 1} &\leq \varepsilon \left(\frac{1}{r_1 - 1} - \frac{1}{r_2 - 1}\right) + \frac{2}{N} < 1, \end{aligned}$$

for any $\varepsilon > 0$ small enough, showing that (4.7) is satisfied as well.

Therefore we are left with showing that (s_1, s_2) which satisfies (4.4), (4.5) and (4.7) exists in the region

$$\frac{1}{r_1} + \frac{1}{r_2} < \frac{N}{N-1}, \quad r_1 \in (r_N^*, \frac{N}{2}), \quad r_1 \leq r_2 \leq \frac{N}{2}, \quad (4.8)$$

where $r_N^* = \frac{N(N-1)}{N^2-N+2}$ is the intersection of $\frac{1}{r_1} + \frac{1}{r_2} < \frac{N}{N-1}$ with the line $r_2 = \frac{N}{2}$. Remark that the lowest possible value of r_2 in this region is the one at the corner point where $r_2 = r_1 = 2\frac{N-1}{N}$. Put

$$\frac{1}{s_1} = \frac{1}{r_1} + \varepsilon,$$

which in view of (4.4) implies that

$$1 - \frac{1}{s_2} = \frac{1}{r_1} - \frac{2}{N} + \varepsilon.$$

Then, in view of (4.2), and since $r_2 - 1 \geq r_1 - 1 \geq 2\frac{N-1}{N} - 1 = \frac{N-2}{N}$, concerning (4.7) we have,

$$\begin{aligned} \left(1 - \frac{1}{s_1}\right) \frac{1}{r_1 - 1} + \left(1 - \frac{1}{s_2}\right) \frac{1}{r_2 - 1} &= \\ \varepsilon \left(\frac{1}{r_2 - 1} - \frac{1}{r_1 - 1}\right) + \frac{1}{r_1} + \left(\frac{1}{r_1} - \frac{2}{N}\right) \frac{1}{r_2 - 1} &\leq \\ \frac{1}{r_1} + \left(\frac{1}{r_1} + \frac{1}{r_2} - 1 - \frac{2}{N}\right) \frac{1}{r_2 - 1} + \left(1 - \frac{1}{r_2}\right) \frac{1}{r_2 - 1} &= \\ \frac{1}{r_1} + \frac{1}{r_2} + \left(\frac{1}{r_1} + \frac{1}{r_2} - 1 - \frac{2}{N}\right) \frac{1}{r_2 - 1} &< \\ \frac{1}{r_1} + \frac{1}{r_2} + \left(\frac{1}{N-1} - \frac{2}{N}\right) \frac{1}{r_2 - 1} &\leq \\ \frac{1}{r_1} + \frac{1}{r_2} + \frac{2-N}{N(N-1)} \frac{N}{N-2} &= \\ \frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{N-1} &< 1. \end{aligned}$$

Therefore we are left with showing that if (r_1, r_2) satisfies (4.8), then we can find (s_1, s_2) which satisfies (4.5) as well as $\frac{1}{s_1} = \frac{1}{r_1} + \varepsilon$, $1 - \frac{1}{s_2} = \frac{1}{r_1} - \frac{2}{N} + \varepsilon$ for some $\varepsilon > 0$. We split the discussion in three regions which could be possibly empty for some $N \geq 3$.

We recall that the symmetric point of the critical hyperbola in (4.4) is just $s_1 = \frac{2N}{N+2} = s_2$ and that

$$\frac{1}{r_1} + \frac{1}{r_2} < \frac{N}{N-1} < \frac{N+2}{N}. \quad (4.9)$$

We start with the domain,

$$\Omega_1 = \left\{ r_1 \in \left[2\frac{N-1}{N}, \frac{N}{2}\right], r_1 \leq r_2 \leq \frac{N}{2} \right\}.$$

If $(r_1, r_2) \in \Omega_1$, since $r_1 \in [2\frac{N-1}{N}, \frac{N}{2}]$, then for some $\sigma_\varepsilon \rightarrow 0$, as $\varepsilon \rightarrow 0^+$, we have that if $\frac{1}{s_1} = \frac{1}{r_1} + \varepsilon$ then $r_1 > s_1 > 2\frac{N-1}{N} - \sigma_\varepsilon > \frac{2N}{N+2}$, whence in particular

$$s_2 < \frac{2N}{N+2} < 2\frac{N-1}{N} \leq r_2,$$

where we use (4.9). Therefore (4.5) is satisfied and then we are done with Ω_1 . Next we consider the case,

$$\Omega_2 = \left\{ \frac{1}{r_1} + \frac{1}{r_2} < \frac{N}{N-1}, r_1 \in \left[\frac{2N}{N+2}, 2\frac{N-1}{N} \right), r_2 \leq \frac{N}{2} \right\}.$$

If $(r_1, r_2) \in \Omega_2$, since $r_1 \in \left[\frac{2N}{N+2}, 2\frac{N-1}{N} \right)$, then for some $\sigma_\varepsilon \rightarrow 0$, $h_\varepsilon \rightarrow 0$, as $\varepsilon \rightarrow 0^+$, we have that if $\frac{1}{s_1} = \frac{1}{r_1} + \varepsilon$ then $r_1 > s_1 \geq r_1 - \sigma_\varepsilon 2\frac{2N}{N+2} - \sigma_\varepsilon$, whence in particular

$$s_2 < \frac{2N}{N+2} + h_\varepsilon < 2\frac{N-1}{N} \leq r_2,$$

where we use again (4.9). Therefore (4.5) is satisfied and then we are done with Ω_2 as well. At last we discuss the case,

$$\Omega_3 = \left\{ \frac{1}{r_1} + \frac{1}{r_2} < \frac{N}{N-1}, r_1 \in [r_N^*, \frac{2N}{N+2}), r_2 \leq \frac{N}{2} \right\}.$$

If $(r_1, r_2) \in \Omega_3$, since $r_1 \in [r_N^*, \frac{2N}{N+2})$, then for some $\sigma_\varepsilon \rightarrow 0$, $h_\varepsilon \rightarrow 0$, as $\varepsilon \rightarrow 0^+$, we have that if $\frac{1}{s_1} = \frac{1}{r_1} + \varepsilon$ then $r_1 > s_1 > 2\frac{2N}{N+2} - \sigma_\varepsilon$, whence in particular

$$s_2 < 2\frac{N-1}{N} + h_\varepsilon < r_N^{**} \leq r_2,$$

where r_N^{**} is the intersection of $\frac{1}{r_1} + \frac{1}{r_2} < \frac{N}{N-1}$ with the line $r_1 = \frac{N}{2}$. Therefore (4.5) is satisfied in this case as well, which concludes the proof of the claim of STEP 1.

STEP 2. We prove the existence of the minimum of J_λ by the direct method.

Let $(\rho_{1,n}, \rho_{2,n})$ be a minimizing sequence, by STEP 1 $\|\rho_{1,n}\|_{r_1}$ and $\|\rho_{2,n}\|_{r_2}$ are bounded and we assume that $\rho_{2,n}$ weakly converges in $L^{r_2}(\Omega)$ and $\rho_{1,n}$ weakly converges in $L^{r_1}(\Omega)$ to some $(\rho_{1,\lambda}, \rho_{2,\lambda})$. Clearly $\frac{1}{r_1} \int_\Omega (\rho_1)^{r_1} + \frac{1}{r_2} \int_\Omega (\rho_2)^{r_2}$ is convex whence lowersemicontinuous with respect to the weak topology in $L^{r_1}(\Omega) \times L^{r_2}(\Omega)$. We will conclude the proof showing that, possibly along a subsequence, we have

$$\lim_{n \rightarrow +\infty} \int_\Omega \rho_{1,n} G[\rho_{2,n}] = \int_\Omega \rho_{1,\lambda} G[\rho_{2,\lambda}].$$

Indeed, writing

$$\begin{aligned} \int_\Omega \rho_{1,n} G[\rho_{2,n}] - \rho_{1,\lambda} G[\rho_{2,\lambda}] &= \int_\Omega \rho_{1,n} (G[\rho_{2,n} - \rho_{2,\lambda}]) + \int_\Omega (\rho_{1,n} - \rho_{1,\lambda}) G[\rho_{2,\lambda}] = \\ &= \int_\Omega \rho_{1,n} (G[\rho_{2,n} - \rho_{2,\lambda}]) + \int_\Omega \rho_{2,\lambda} (G[\rho_{1,n} - \rho_{1,\lambda}]), \end{aligned}$$

it is enough to prove that the embeddings of $W^{2,r_2}(\Omega)$ in $L^{r_1'}(\Omega)$ and of $W^{2,r_1}(\Omega)$ in $L^{r_2'}(\Omega)$ are compact, where r_i' is the exponent conjugate to r_i . Clearly it is just enough to prove the former. By (1.1) we readily deduce that,

$$r_2 > \frac{N - r_1 - 1}{N(r_1 - 1) + 1},$$

whence by the Sobolev embedding we find that $W^{2,r_2}(\Omega)$ is compactly embedded in $W^{1,t}(\Omega)$ for any

$$t < t_2 = r_1' \frac{N^2 - N}{N^2 - N + r_1'} = r_1' - \frac{r_1'}{N^2 - N + r_1'},$$

where we remark that $N^2 - N + r_1' > r_1' > 0$. Thus, again by the Sobolev embedding, we find that $W^{2,r_2}(\Omega)$ is compactly embedded in $L^k(\Omega)$ for any

$$k < k_2 = \frac{Nr_1' \left(1 - \frac{1}{N^2 - N + r_1'} \right)}{N - r_1' \left(1 - \frac{1}{N^2 - N + r_1'} \right)},$$

where we assume without loss of generality that $N - r'_1 \left(1 - \frac{1}{N^2 - N + r'_1}\right) > 0$. Therefore it is enough to prove that,

$$\frac{Nr'_1 \left(1 - \frac{1}{N^2 - N + r'_1}\right)}{N - r'_1 \left(1 - \frac{1}{N^2 - N + r'_1}\right)} > r'_1,$$

which, after a straightforward evaluation, takes the form,

$$N^2 - \frac{1}{r_1}N + \frac{1}{r_1 - 1} > 0.$$

The determinant of this polynomial is $\frac{1}{r_1^2} - \frac{4}{r_1 - 1}$ which is readily seen to be always negative. Thus we have proved the existence of at least one minimizer of J_λ .

STEP 3 We prove that any minimizer $(\rho_{1,\lambda}, \rho_{2,\lambda})$ defines a solution of $(\mathbf{P})_\lambda$.

Let $\Omega_+ = \{x \in \Omega : \rho_{2,\lambda} > 0, \text{ a.e.}\}$ and $\Omega_0 = \{x \in \Omega : \rho_{2,\lambda} = 0, \text{ a.e.}\}$. Since $\int_\Omega \rho_{2,\lambda} = 1$, then $0 < |\Omega_+| \leq |\Omega|$. For any $n \in \mathbb{N}$ and for any variation of the form $(\rho_{1,\lambda}, \rho_{2,\lambda} + \varepsilon\eta)$, with $\text{supp}(\eta) \subseteq \{x \in \Omega_+ : \rho_{2,\lambda} > \frac{1}{n}, \text{ a.e.}\}$, $\eta \in L^\infty(\Omega)$ and $\int_\Omega \eta = 0$, by using the minimality of $(\rho_{1,\lambda}, \rho_{2,\lambda})$ we have,

$$\int_{\{x \in \Omega_+ : \rho_{2,\lambda} > \frac{1}{n}\}} \left((\rho_{2,\lambda})^{\frac{1}{p_2}} - \lambda G[\rho_{1,\lambda}] \right) \eta \geq o(1), \text{ as } \varepsilon \rightarrow 0.$$

Since this is true for any such η with $\int_\Omega \eta = 0$ and any $n \in \mathbb{N}$, then we have that,

$$(\rho_{2,\lambda}(x))^{\frac{1}{p_2}} - \lambda G[\rho_{1,\lambda}](x) = \alpha_2 \text{ a.e. in } \Omega_+, \quad (4.10)$$

for a suitable constant $\alpha_2 \in \mathbb{R}$. Next, let χ_A denote the characteristic function of the set A , and assume that $|\Omega_0| > 0$. For any variation of the form $(\rho_{1,\lambda}, \rho_{2,\lambda} + \varepsilon\eta)$, with

$$\eta = \varphi \chi_{\Omega_0} - \left(\int_{\Omega_0} \varphi \right) \frac{\chi_{\Omega_+}}{|\chi_{\Omega_+}|}, \quad \eta \in L^\infty(\Omega), \varphi \geq 0,$$

we have,

$$\int_{\Omega_0} \left(-\alpha_2 - \lambda G[\rho_{1,\lambda}] \right) \varphi \geq o(1), \text{ as } \varepsilon \rightarrow 0.$$

Therefore we conclude that,

$$\alpha_2 + \lambda G[\rho_{1,\lambda}](x) \leq 0 \text{ a.e. in } \Omega_0, \quad (4.11)$$

and in particular that $\psi_{2,\lambda} = G[\rho_{1,\lambda}] \in W_0^{2,r_1}(\Omega)$ is a strong solution of the first equation in $(\mathbf{P})_\lambda$ with $\psi_{1,\lambda} = G[\rho_{2,\lambda}] \in W_0^{2,r_2}(\Omega)$, where $(\alpha_2 + \lambda\psi_{2,\lambda})_+^{p_2} \equiv (\alpha_2 + \lambda\psi_{2,\lambda})^{p_2}$ as far as $|\Omega_0| = 0$. The same argument shows that $\psi_{1,\lambda}$ is a strong solution of the second equation in $(\mathbf{P})_\lambda$.

At last, by Lemma 2.1 any strong solution determined in STEP 3 is a classical solution. This fact concludes the proof. \square

Theorem 4.3. *For any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$ F_λ is real analytic, decreasing with $\frac{dE_\lambda}{d\lambda} < 0$, and concave and we have $F'_\lambda = -E_\lambda$. In particular $\frac{dE_\lambda}{d\lambda} \geq 0$ and $\frac{d}{d\lambda} \left(\frac{p_1 \alpha_{1,\lambda}}{p_1 + 1} + \frac{p_2 \alpha_{2,\lambda}}{p_2 + 1} \right) < 0$ for any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$.*

Proof. We first observe that, by the uniqueness in Lemma 4.1, we have that F_λ is the same as $F(\lambda)$ in **(VP)**. It is useful at this stage to introduce the vectorial density,

$$\underline{\rho} = (\rho_1, \rho_2)$$

and the corresponding entropy,

$$\mathcal{S}(\underline{\rho}) = \frac{1}{r_1} \int_{\Omega} (\rho_1)^{r_1} + \frac{1}{r_2} \int_{\Omega} (\rho_2)^{r_2},$$

and energy

$$\mathcal{E}(\underline{\rho}) = \int_{\Omega} \rho_1 G[\rho_2],$$

whence $J_\lambda(\underline{\rho}) = \mathcal{S}(\underline{\rho}) - \lambda \mathcal{E}(\underline{\rho})$.

By Lemma 4.1 F_λ is real analytic. If $\lambda_2 > \lambda_1 \geq 0$, then $J_{\lambda_2}(\underline{\rho}) < J_{\lambda_1}(\underline{\rho})$, whence $F_{\lambda_2} \leq F_{\lambda_1}$. Thus F_λ is decreasing for $\lambda > 0$. Moreover, setting $\lambda = t\lambda_1 + (1-t)\lambda_2$, $t \in [0, 1]$, and letting $\underline{\rho}_\lambda$ be any minimizer of J_λ , we find that,

$$\begin{aligned} F_\lambda &= \mathcal{S}(\underline{\rho}_\lambda) - \lambda \mathcal{E}(\underline{\rho}_\lambda) = -\mathcal{S}(\underline{\rho}_\lambda)(t + (1-t)) - (t\lambda_1 + (1-t)\lambda_2) \mathcal{E}(\underline{\rho}_\lambda) = \\ & \quad tJ_{\lambda_1}(\underline{\rho}_\lambda) + (1-t)J_{\lambda_2}(\underline{\rho}_\lambda) \geq tF_{\lambda_1} + (1-t)F_{\lambda_2}. \end{aligned}$$

Therefore F_λ is concave with $\frac{d^2 F_\lambda}{d\lambda^2} \leq 0$.

At this point we use a well known trick about canonical variational principles (see for example [11]). Let $\lambda_1 \neq \lambda_2$ in $[0, \lambda^*(\Omega, \mathbf{p}))$ and let $\underline{\rho}_1, \underline{\rho}_2$ be the minimizers of $J_{\lambda_1}, J_{\lambda_2}$ respectively. Clearly we have

$$F_{\lambda_1} \leq \mathcal{S}(\underline{\rho}_2) - \lambda_1 \mathcal{E}(\underline{\rho}_2) = F_{\lambda_2} - (\lambda_1 - \lambda_2) \mathcal{E}(\underline{\rho}_2), \quad (4.12)$$

$$F_{\lambda_2} \leq \mathcal{S}(\underline{\rho}_1) - \lambda_2 \mathcal{E}(\underline{\rho}_1) = F_{\lambda_1} - (\lambda_2 - \lambda_1) \mathcal{E}(\underline{\rho}_1). \quad (4.13)$$

and we deduce from (4.12), (4.13) that

$$\begin{aligned} -\mathcal{E}(\underline{\rho}_1) &\leq \frac{F_{\lambda_1} - F_{\lambda_2}}{\lambda_1 - \lambda_2} \leq -\mathcal{E}(\underline{\rho}_2) \quad \text{if } \lambda_1 > \lambda_2, \\ -\mathcal{E}(\underline{\rho}_2) &\leq \frac{F_{\lambda_2} - F_{\lambda_1}}{\lambda_2 - \lambda_1} \leq -\mathcal{E}(\underline{\rho}_1) \quad \text{if } \lambda_2 > \lambda_1. \end{aligned}$$

By Lemma 4.1, as $\lambda_2 \rightarrow \lambda_1$ we have $\underline{\rho}_2 \rightarrow \underline{\rho}_1$ smoothly, whence

$$\frac{dF_{\lambda_1}}{d\lambda} = -\mathcal{E}(\underline{\rho}_1) \equiv -E_\lambda|_{\lambda=\lambda_1}.$$

Remark that $E_\lambda > 0$ in $[0, \lambda^*(\Omega, \mathbf{p}))$, whence $\frac{dF_\lambda}{d\lambda} < 0$. In particular, since F_λ is real analytic and concave, then $\frac{dE_\lambda}{d\lambda} = -\frac{d^2 F_\lambda}{d\lambda^2} \geq 0$, for any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$.

At this point observe that,

$$\begin{aligned} F_\lambda &= \frac{1}{r_1} \int_{\Omega} (\rho_{1,\lambda})^{r_1} + \frac{1}{r_2} \int_{\Omega} (\rho_{2,\lambda})^{r_2} - \lambda \int_{\Omega} \rho_{1,\lambda} G[\rho_{2,\lambda}] = \\ & \frac{1}{r_1} \int_{\Omega} \rho_{1,\lambda} (\alpha_{1,\lambda} + \lambda_1 \psi_{1,\lambda}) + \frac{1}{r_2} \int_{\Omega} \rho_{2,\lambda} (\alpha_{2,\lambda} + \lambda_2 \psi_{2,\lambda}) - \lambda \int_{\Omega} \rho_{1,\lambda} G[\rho_{2,\lambda}] = \\ & \frac{\alpha_{1,\lambda}}{r_1} + \frac{\lambda}{r_1} \int_{\Omega} \rho_{1,\lambda} G[\rho_{2,\lambda}] + \frac{\alpha_{2,\lambda}}{r_2} + \frac{\lambda}{r_2} \int_{\Omega} \rho_{2,\lambda} G[\rho_{1,\lambda}] - \lambda \int_{\Omega} \rho_{1,\lambda} G[\rho_{2,\lambda}] = \\ & \quad \frac{p_1 \alpha_{1,\lambda}}{p_1 + 1} + \frac{p_2 \alpha_{2,\lambda}}{p_2 + 1} + \frac{p_1 p_2 - 1}{(p_2 + 1)(p_1 + 1)} E_\lambda. \end{aligned}$$

By Lemma 4.1, $\alpha_{i,\lambda}$, $i = 1, 2$ are real analytic, and then we deduce that,

$$\frac{d}{d\lambda} \left(\frac{p_1 \alpha_{1,\lambda}}{p_1 + 1} + \frac{p_2 \alpha_{2,\lambda}}{p_2 + 1} \right) = \frac{dF_\lambda}{d\lambda} - \frac{p_1 p_2 - 1}{(p_2 + 1)(p_1 + 1)} \frac{dE_\lambda}{d\lambda} < 0,$$

where the strict inequality follows from $\frac{dF_\lambda}{d\lambda} < 0$. \square

Finally we prove the estimate about the derivative of E_λ .

Proposition 4.4. *Let $(\alpha_\lambda, \psi_\lambda) \in \mathcal{G}(\Omega)$ be the unique positive solutions of $(\mathbf{P})_\lambda$ for $\lambda \in [0, \lambda^*(\Omega, \mathbf{p})]$. Then,*

$$\frac{dE_\lambda}{d\lambda} \geq p_1 \|[\psi_{1,\lambda}]_{1,\lambda}\|_{1,\lambda}^2 + p_2 \|[\psi_{2,\lambda}]_{2,\lambda}\|_{2,\lambda}^2. \quad (4.14)$$

Proof. By Lemma 4.1 $(\alpha_\lambda, \psi_\lambda)$ is a real analytic function of λ and then, by standard elliptic estimates, we have that $\eta_\lambda \in (C_0^2(\bar{\Omega}))^2$ where

$$\eta_\lambda = (\eta_{1,\lambda}, \eta_{2,\lambda}) = \frac{d\psi_\lambda}{d\lambda} = \left(\frac{d\psi_{1,\lambda}}{d\lambda}, \frac{d\psi_{2,\lambda}}{d\lambda} \right),$$

is a classical solution of,

$$\begin{cases} -\Delta \eta_{1,\lambda} = \tau_{2,\lambda} V_{2,\lambda} \eta_{2,\lambda} + p_2 V_{2,\lambda} \psi_{2,\lambda} + p_2 V_{2,\lambda} \frac{d\alpha_{2,\lambda}}{d\lambda}, \\ -\Delta \eta_{2,\lambda} = \tau_{1,\lambda} V_{1,\lambda} \eta_{1,\lambda} + p_1 V_{1,\lambda} \psi_{1,\lambda} + p_1 V_{1,\lambda} \frac{d\alpha_{1,\lambda}}{d\lambda}, \end{cases}$$

and $\frac{d\alpha_{i,\lambda}}{d\lambda}$ can be computed by the unit mass constraints in $(\mathbf{P})_\lambda$, that is

$$p_2 \frac{d\alpha_{2,\lambda}}{d\lambda} = -\tau_{2,\lambda} \langle \eta_{2,\lambda} \rangle_{2,\lambda} - p_2 \langle \psi_{2,\lambda} \rangle_{2,\lambda}, \quad p_1 \frac{d\alpha_{1,\lambda}}{d\lambda} = -\tau_{1,\lambda} \langle \eta_{1,\lambda} \rangle_{1,\lambda} - p_1 \langle \psi_{1,\lambda} \rangle_{1,\lambda}.$$

Therefore, we conclude that η_λ is a solution of,

$$\begin{cases} -\Delta \eta_{1,\lambda} = \tau_{2,\lambda} V_{2,\lambda} [\eta_{2,\lambda}]_{2,\lambda} + p_2 V_{2,\lambda} [\psi_{2,\lambda}]_{2,\lambda}, \\ -\Delta \eta_{2,\lambda} = \tau_{1,\lambda} V_{1,\lambda} [\eta_{1,\lambda}]_{1,\lambda} + p_1 V_{1,\lambda} [\psi_{1,\lambda}]_{1,\lambda}. \end{cases} \quad (4.15)$$

Since $E_\lambda = \int_\Omega (\nabla \psi_{1,\lambda}, \nabla \psi_{2,\lambda})$, then by using $(\mathbf{P})_\lambda$ and (4.15) we also have,

$$\begin{aligned} \frac{d}{d\lambda} E_\lambda &= \int_\Omega (\nabla \eta_{1,\lambda}, \nabla \psi_{2,\lambda}) + \int_\Omega (\nabla \psi_{1,\lambda}, \nabla \eta_{2,\lambda}) = \\ &= \sum_{i=1}^2 \tau_{i,\lambda} \langle [\eta_{i,\lambda}]_{i,\lambda} [\psi_{i,\lambda}]_{i,\lambda} \rangle_{i,\lambda} + \sum_{i=1}^2 p_i \|[\psi_{i,\lambda}]_{i,\lambda}\|_{i,\lambda}^2. \end{aligned}$$

Let us denote, for $i = 1, 2$,

$$[\psi_{i,\lambda}]_{i,\lambda} = \sum_{k=1}^{+\infty} \xi_{i,k} [\phi_{i,k}]_{i,\lambda}, \quad [\eta_{i,\lambda}]_{i,\lambda} = \sum_{k=1}^{+\infty} \beta_{i,k} [\phi_{i,k}]_{i,\lambda},$$

$$\xi_{i,k} = \langle [\phi_{i,k}]_{i,\lambda} [\psi_{i,\lambda}]_{i,\lambda} \rangle_{i,\lambda}, \quad \beta_{i,k} = \langle [\phi_{i,k}]_{i,\lambda} [\eta_{i,\lambda}]_{i,\lambda} \rangle_{i,\lambda},$$

the Fourier expansions of $[\psi_{i,\lambda}]_{i,\lambda}$ and $[\eta_{i,\lambda}]_{i,\lambda}$ in $Y_{i,0}$ (see Lemma 3.1), with respect to the normalized eigenfunctions $[\phi_{i,k}]_{i,\lambda}$, satisfying $\|[\phi_{i,k}]_{i,\lambda}\|_{i,\lambda} = 1$, $i = 1, 2$. Then, by Lemma 3.1, we have,

$$\frac{d}{d\lambda} E_\lambda = \tau_{2,\lambda} \sum_{k=1}^{+\infty} \beta_{2,k} \xi_{2,k} + \tau_{1,\lambda} \sum_{k=1}^{+\infty} \beta_{1,k} \xi_{1,k} + p_2 \sum_{k=1}^{+\infty} \xi_{2,k}^2 + p_1 \sum_{k=1}^{+\infty} \xi_{1,k}^2. \quad (4.16)$$

We can now consider $\lambda > 0$. On the other side, multiplying the first equation in (4.15) by $\phi_{2,k}$, the second by $\phi_{1,k}$, using (3.13) and integrating by parts, we have,

$$\begin{cases} (\tau_{1,\lambda} + p_1 \sigma_k) \beta_{1,k} = \tau_{2,\lambda} \beta_{2,k} + p_2 \xi_{2,k}, \\ (\tau_{2,\lambda} + p_2 \sigma_k) \beta_{2,k} = \tau_{1,\lambda} \beta_{1,k} + p_1 \xi_{1,k}, \end{cases} \quad (4.17)$$

where $\sigma_k = \sigma_k(\alpha_\lambda, \psi_\lambda)$. Since by (3.14) we also have $2\lambda + \sigma_k > \lambda$, then (4.17) admits the unique solution,

$$\beta_{1,k} = \frac{\lambda(p_1 \xi_{1,k} + p_2 \xi_{2,k}) + \sigma_k p_2 \xi_{2,k}}{p_1 \sigma_k (2\lambda + \sigma_k)}, \quad \beta_{2,k} = \frac{\lambda(p_1 \xi_{1,k} + p_2 \xi_{2,k}) + \sigma_k p_1 \xi_{1,k}}{p_2 \sigma_k (2\lambda + \sigma_k)},$$

which we can substitute in (4.16) to deduce that

$$\frac{d}{d\lambda} E_\lambda = \lambda \sum_{k=1}^{+\infty} \frac{\lambda(p_1 \xi_{1,k} + p_2 \xi_{2,k})(\xi_{1,k} + \xi_{2,k}) + 2\sigma_k \xi_{1,k} \xi_{2,k}}{\sigma_k (2\lambda + \sigma_k)} + p_2 \sum_{k=1}^{+\infty} \xi_{2,k}^2 + p_1 \sum_{k=1}^{+\infty} \xi_{1,k}^2. \quad (4.18)$$

At this point observe that the equations in $(\mathbf{P})_\lambda$ can be written in the following form,

$$\begin{cases} -\Delta\psi_{1,\lambda} = V_{2,\lambda}(\alpha_{2,\lambda} + \lambda\psi_{2,\lambda}) & \text{in } \Omega \\ -\Delta\psi_{2,\lambda} = V_{1,\lambda}(\alpha_{1,\lambda} + \lambda\psi_{1,\lambda}) & \text{in } \Omega \\ \int_{\Omega} V_{i,\lambda}(\alpha_{i,\lambda} + \lambda\psi_{i,\lambda}) = 1 & i = 1, 2. \end{cases}$$

Therefore we can evaluate

$$\alpha_{i,\lambda} = \frac{1}{m_{i,\lambda}} - \lambda < \psi_{i,\lambda} >_{i,\lambda}, \quad i = 1, 2,$$

and deduce that

$$\begin{cases} -\Delta\psi_{1,\lambda} = (m_{2,\lambda})^{-1}V_{2,\lambda} + \lambda V_{2,\lambda}[\psi_{2,\lambda}]_{2,\lambda} & \text{in } \Omega \\ -\Delta\psi_{2,\lambda} = (m_{1,\lambda})^{-1}V_{1,\lambda} + \lambda V_{1,\lambda}[\psi_{1,\lambda}]_{1,\lambda} & \text{in } \Omega \\ \psi_{i,\lambda} = 0 & \text{on } \partial\Omega, \quad i = 1, 2. \end{cases} \quad (4.19)$$

Multiplying the first equation in (4.19) by $\phi_{2,k}$, the second by $\phi_{1,k}$, using (3.13) and integrating by parts, we have,

$$\begin{cases} (\tau_{1,\lambda} + p_1\sigma_k)\xi_{1,k} = \langle \phi_{2,k} \rangle_{2,\lambda} + \lambda\xi_{2,k}, \\ (\tau_{2,\lambda} + p_2\sigma_k)\xi_{2,k} = \langle \phi_{1,k} \rangle_{1,\lambda} + \lambda\xi_{1,k}. \end{cases} \quad (4.20)$$

Since $p_i \geq 1$, $i = 1, 2$ and since by (3.14) we also have $2\lambda + \sigma_k > \lambda$, then (4.20) admits the unique solution,

$$\begin{aligned} \xi_{1,k} &= \frac{(\tau_{2,\lambda} + p_2\sigma_k) \langle \phi_{2,k} \rangle_{2,\lambda} + \lambda \langle \phi_{1,k} \rangle_{1,\lambda}}{p_1 p_2 (\lambda + \sigma_k)^2 - \lambda^2}, \\ \xi_{2,k} &= \frac{(\tau_{1,\lambda} + p_1\sigma_k) \langle \phi_{1,k} \rangle_{1,\lambda} + \lambda \langle \phi_{2,k} \rangle_{2,\lambda}}{p_1 p_2 (\lambda + \sigma_k)^2 - \lambda^2}, \end{aligned}$$

and since of course we can assume that $\langle \phi_{i,k} \rangle_{i,\lambda} \geq 0$, $i = 1, 2$, for any $k \in \mathbb{N}$, then we deduce that

$$\xi_{i,k} \geq 0, \quad i = 1, 2, \forall k \in \mathbb{N}. \quad (4.21)$$

Remark that all the relations above, including (4.18), (4.21), hold just by assuming that $0 \notin \sigma(\mathbf{L}_\lambda)$. However at this point we use the fact that for any $\lambda < \lambda^*(\Omega, \mathbf{p})$ it holds $\sigma_k \geq \sigma_1 > 0$, $\forall k \in \mathbb{N}$, which, together with (4.18) and (4.21) implies that

$$\frac{d}{d\lambda} E_\lambda \geq p_2 \sum_{k=1}^{+\infty} \xi_{2,k}^2 + p_1 \sum_{k=1}^{+\infty} \xi_{1,k}^2 = p_2 \|[\psi_{2,\lambda}]_{2,\lambda}\|_{2,\lambda}^2 + p_1 \|[\psi_{1,\lambda}]_{1,\lambda}\|_{1,\lambda}^2.$$

which is (4.14). \square

At last we present the proofs of Theorems 1.3 and 1.5.

The proof of Theorem 1.3. Uniqueness and regularity follow immediately from Lemma 4.1 and then a straightforward evaluation yields the behavior in the claim as $\lambda \rightarrow 0$. The inequality about the energy for $\lambda = 0$ is a well known torsional inequality, see [15]. The monotonicity in the claim follows from Theorem 4.3 and Proposition 4.4. Finally, if either $\sigma_{1,*} = 0$ or if $\alpha_{i,*} = 0$, $i = 1, 2$, it follows from Propositions 3.4 and 3.5 that $\lambda^*(\Omega, \mathbf{p}) \geq \frac{1}{p_2} \Lambda(\Omega, 2p_2)$. \square

The proof of Theorem 1.5. For $(\boldsymbol{\mu}_\lambda, \mathbf{u}_\lambda)$ as defined in the claim, from the constrains in $(\mathbf{P})_\lambda$ we have

$$\alpha_{i,\lambda} = \|1 + u_{i,\lambda}\|_{p_i}^{-1}, \quad i = 1, 2, \quad (4.22)$$

which immediately shows that

$$\frac{p_1 \alpha_{1,\lambda}}{p_1 + 1} + \frac{p_2 \alpha_{2,\lambda}}{p_2 + 1} = \gamma(\boldsymbol{\mu}, \mathbf{u}_\lambda).$$

Next observe that

$$E_\lambda = \frac{1}{2} \int_\Omega \rho_{1,\lambda} \psi_{1,\lambda} + \frac{1}{2} \int_\Omega \rho_{2,\lambda} \psi_{2,\lambda} = \frac{\alpha_{1,\lambda}^{p_1+1}}{2\lambda} \int_\Omega (1+u_1)^{p_1} u_1 + \frac{\alpha_{2,\lambda}^{p_2+1}}{2\lambda} \int_\Omega (1+u_2)^{p_2} u_2,$$

which together with (4.22) and

$$\lambda = \mu_{1,\lambda} \frac{\alpha_{2,\lambda}}{\alpha_{1,\lambda}^{p_1}}, \quad \lambda = \mu_{2,\lambda} \frac{\alpha_{1,\lambda}}{\alpha_{2,\lambda}^{p_2}}, \quad (4.23)$$

yields

$$E_\lambda = \frac{\alpha_{1,\lambda}^{p_1} \alpha_{2,\lambda}^{p_2}}{2\mu_{2,\lambda}} \int_\Omega (1+u_1)^{p_1} u_1 + \frac{\alpha_{2,\lambda}^{p_2} \alpha_{1,\lambda}^{p_1}}{2\mu_{1,\lambda}} \int_\Omega (1+u_2)^{p_2} u_2 = E(\boldsymbol{\mu}_\lambda, \mathbf{u}_\lambda).$$

Moreover, we immediately have $F(\boldsymbol{\mu}_\lambda, \mathbf{u}_\lambda) = F_\lambda$ as well.

Therefore, as far as $\lambda < \lambda^*(\Omega, \mathbf{p})$ the monotonicity properties in the claim follow immediately from Theorem 1.3.

Clearly, as far as $\lambda < \lambda^*(\Omega, \mathbf{p})$, by definition we also have $\sigma_1(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda) > 0$ and $\alpha_{i,\lambda} > 0$, $i = 1, 2$, whence by Lemma 4.1 we see that $(\boldsymbol{\mu}_\lambda, \mathbf{u}_\lambda)$ is a continuous real analytic curve.

Finally, assume that $\alpha_{1,*} = 0$ and $\alpha_{2,*} > 0$. By definition there exists a sequence $\lambda_n \rightarrow (\lambda^*(\Omega, \mathbf{p}))^-$, such that $\alpha_{1,\lambda_n} \rightarrow 0^+$, $\alpha_{2,\lambda_n} \rightarrow \alpha_2 > 0$. The curve is obviously unbounded since by (4.23) we have $\mu_{2,\lambda} \rightarrow +\infty$.

Now, under the assumption of Theorem 1.2-(b), we have $\lambda^*(\Omega, \mathbf{p}) < +\infty$ and then by Lemma 2.1 and passing to a further subsequence if necessary we can assume that ψ_{i,λ_n} , $i = 1, 2$ converge smoothly to $\psi_{i,*}$, $i = 1, 2$, which are classical solutions of $(\mathbf{P})_\lambda$ for $\lambda = \lambda^*(\Omega, \mathbf{p})$ and $\alpha_1 = 0$, $\alpha_2 > 0$. Since by definition $\alpha_{i,\lambda_n} u_{i,\lambda_n} = \lambda_n \psi_{i,\lambda_n}$, $i = 1, 2$, then the convergence in the claim follows from (4.22).

The conclusion in case $\alpha_{2,*} = 0$ and $\alpha_{1,*} > 0$ follows in the same way. \square

APPENDIX A. UNIQUENESS OF SOLUTIONS FOR λ SMALL

The following lemma is proved by a standard application of the contraction mapping principle and we prove it here just for reader's convenience. We will denote by C_1 the constant in Lemma 2.1 and by C_2, C_3 other positive constants depending only by $r_0, \Omega, \mathbf{p}, N$.

Lemma A.1. *There exists $\lambda_0 > 0$ such that:*

(j) *for any $\lambda \in [0, \lambda_0]$ there exist at least one solution $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ of $(\mathbf{P})_\lambda$.*

(jj) *for any solution of $(\mathbf{P})_\lambda$ we have $\alpha_{i,\lambda} > \frac{1}{3}$, $i = 1, 2$ for any $\lambda \in [0, \lambda_0]$.*

Proof. (j)

Putting $u_1 = \lambda \psi_1, u_2 = \lambda \psi_2$ the proof is an immediate consequence of the following lemmas. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\mathbf{u} = (u_1, u_2)$ and let us define,

$$\mathbf{B}_\infty = \left\{ \mathbf{u} \in (L^\infty(\Omega))^2 \mid \|\mathbf{u}\|_{L^\infty(\Omega)} := \max_{i=1,2} \|u_i\|_{L^\infty(\Omega)} \leq C_1, u_i \geq 0, \text{ a.e. in } \Omega, i = 1, 2 \right\}$$

where $C_1(r, \Omega, 1, \mathbf{p}, N)$ is the constant obtained in Lemma 2.1 evaluated with $\bar{\lambda} = 1$.

Lemma A.2. *Let (p_1, p_2) satisfy (1.1). There exists $\lambda_0 \in (0, 1]$ such that for any $\lambda \in [0, \lambda_0]$ and for any $\alpha_i \in (-\infty, 1]$ there exists one and only one solution $\mathbf{u} = (u_{1,\lambda,\alpha}, u_{2,\lambda,\alpha}) \in (C_0^{2,r}(\bar{\Omega}))^2$ of*

the problem

$$\begin{cases} -\Delta u_1 = \lambda(\alpha_2 + u_2)_+^{p_2} & \text{in } \Omega \\ -\Delta u_2 = \lambda(\alpha_1 + u_1)_+^{p_1} & \text{in } \Omega \\ u_1 = 0 = u_2 & \text{on } \partial\Omega \\ u_i \in B_\infty, i = 1, 2. \end{cases} \quad (\text{A.1})$$

Moreover, for fixed $\lambda \in [0, \lambda_0]$, the maps $(-\infty, 1] \ni \alpha_1 \rightarrow u_{2,\lambda}[\alpha_1] = u_{2,\lambda,\alpha} \in B_\infty$, $(-\infty, 1] \ni \alpha_2 \rightarrow u_{1,\lambda}[\alpha_2] = u_{1,\lambda,\alpha} \in B_\infty$, are continuous and $u_{1,\lambda,\alpha} \equiv 0 \equiv u_{2,\lambda,\alpha}$ if either $\lambda = 0$ or if $\alpha_i \leq 0$, $i = 1, 2$.

Proof. First of all, if either $\lambda = 0$ or if $\alpha_i \leq 0$, $i = 1, 2$, then $(u_1, u_2) \equiv (0, 0)$ is a solution, whence the last part of the statement will follow immediately from the uniqueness.

For $\lambda_0 \in (0, 1]$ to be fixed later on and for fixed $\lambda \in [0, \lambda_0]$ and $\alpha_i \in (-\infty, 1]$, $i = 1, 2$, we define

$$\mathbf{T}_{\lambda,\alpha}(\mathbf{u}) = \lambda(G[(\alpha_2 + u_2)_+^{p_2}], G[(\alpha_1 + u_1)_+^{p_1}]), \quad u_i \in B_\infty, i = 1, 2.$$

Recall that if $\alpha_i > 0$ then $\alpha_i \leq 1$, while if $\alpha_i < 0$ then $(\alpha_i + u_i)_+ \leq (u_i)_+$, whence we have,

$$\|\mathbf{T}_{\lambda,\alpha}(\mathbf{u})\|_{L^\infty(\Omega)} := \lambda \max_{i=1,2} \|G[(\alpha_i + u_i)_+^{p_i}]\|_{L^\infty(\Omega)} \leq \lambda C_2,$$

and we readily see that $\mathbf{T}_{\lambda,\alpha} : \mathbf{B}_\infty \rightarrow \mathbf{B}_\infty$ for any $\lambda \leq \frac{C_1}{C_2}$. Also,

$$\begin{aligned} \|\mathbf{T}_{\lambda,\alpha}(\mathbf{u}) - \mathbf{T}_{\lambda,\alpha}(\mathbf{v})\|_{L^\infty(\Omega)} &\leq \lambda \max_{i=1,2} \|p_i G[(\alpha_i + w_i)_+^{p_i-1} |u_i - v_i]|\|_{L^\infty(\Omega)} \leq \\ &\lambda \max_{i=1,2} \|p_i G[(\alpha_i + w_i)_+^{p_i-1}]\|_{L^\infty(\Omega)} \|u_i - v_i\|_{L^\infty(\Omega)} \leq \lambda C_3 \|\mathbf{u} - \mathbf{v}\|_{L^\infty(\Omega)} \end{aligned}$$

where $w_i \in B_\infty$ satisfies $u_i \leq w_i \leq v_i$, $i = 1, 2$.

Therefore, we also have $\|\mathbf{T}_{\lambda,\alpha}(\mathbf{u}) - \mathbf{T}_{\lambda,\alpha}(\mathbf{v})\|_{L^\infty(\Omega)} \leq \frac{1}{2} \|\mathbf{u} - \mathbf{v}\|_{L^\infty(\Omega)}$, for any $\lambda \leq \frac{1}{2C_3}$. As a consequence putting $\lambda_0 = \min\{1, \frac{C_1}{C_2}, \frac{1}{2C_3}\}$, we have that $\mathbf{T}_{\lambda,\alpha}$ is a contraction in \mathbf{B}_∞ for any $\lambda \leq \lambda_0$. Whence, in particular, for any fixed $\alpha \in ((-\infty, 1])^2$, we have that for any $\lambda \in [0, \lambda_0]$ there exists a unique solution of $\mathbf{u} = \mathbf{T}_{\lambda,\alpha}(\mathbf{u})$. The existence and uniqueness claim follows since, by standard elliptic estimates, $(u_{1,\lambda,\alpha}, u_{2,\lambda,\alpha}) \in (C_0^{2,r}(\bar{\Omega}))^2$ solves the problem in the statement of the lemma if and only if $\mathbf{u} \in \mathbf{B}_\infty$ satisfies $\mathbf{u} = \mathbf{T}_{\lambda,\alpha}(\mathbf{u})$.

Concerning the continuity of $(u_{1,\lambda}[\alpha_2], u_{2,\lambda}[\alpha_1]) = (u_{1,\lambda,\alpha}, u_{2,\lambda,\alpha})$ for $\alpha_i \in (-\infty, 1]$, $i = 1, 2$, we observe that if $\alpha_n = (\alpha_{1,n}, \alpha_{1,n}) \rightarrow \alpha = (\alpha_1, \alpha_2)$, then

$$\begin{aligned} \|u_{2,\lambda}[\alpha_{1,n}] - u_{2,\lambda}[\alpha_1]\|_{L^\infty(\Omega)} &= \|G[(\alpha_{1,n} + u_{1,\lambda,\alpha_n})_+^{p_1}] - G[(\alpha_1 + u_{1,\lambda,\alpha})_+^{p_1}]\|_{L^\infty(\Omega)} \leq \\ &\|G[(\alpha_{1,n} + u_{1,\lambda,\alpha_n})_+^{p_1}] - G[(\alpha_{1,n} + u_{1,\lambda,\alpha})_+^{p_1}]\|_{L^\infty(\Omega)} + \\ &\|G[(\alpha_{1,n} + u_{1,\lambda,\alpha})_+^{p_1}] - G[(\alpha_1 + u_{1,\lambda,\alpha})_+^{p_1}]\|_{L^\infty(\Omega)} \leq \\ &\lambda C_3 \|u_{1,\lambda}[\alpha_{2,n}] - u_{1,\lambda}[\alpha_2]\|_{L^\infty(\Omega)} + p_1 \lambda \|G[(s + u_{1,\lambda,\alpha})_+^{p_1-1}]\|_{L^\infty(\Omega)} |\alpha_{1,n} - \alpha_1| \leq \\ &\frac{1}{2} \|u_{1,\lambda}[\alpha_{2,n}] - u_{1,\lambda}[\alpha_2]\|_{L^\infty(\Omega)} + \lambda_0 C_3 |\alpha_{1,n} - \alpha_1|. \end{aligned}$$

Clearly the same argument applies to the other component and then we deduce that

$$\|u_{2,\lambda}[\alpha_{1,n}] - u_{2,\lambda}[\alpha_1]\|_{L^\infty(\Omega)} + \|u_{1,\lambda}[\alpha_{2,n}] - u_{1,\lambda}[\alpha_2]\|_{L^\infty(\Omega)} \leq 4\lambda_0 C_3 (|\alpha_{1,n} - \alpha_1| + |\alpha_{2,n} - \alpha_2|),$$

which readily implies the claim. \square

For fixed $\lambda \in [0, \lambda_0]$ we consider the continuous map

$$((-\infty, 1])^2 \ni \alpha = (\alpha_1, \alpha_2) \rightarrow \mathbf{u}_\alpha = (u_{1,\lambda}[\alpha_2], u_{2,\lambda}[\alpha_1]) \in (B_\infty)^2,$$

where $u_{1,\lambda}[\alpha_2] = u_{1,\lambda,\alpha}$, $u_{2,\lambda}[\alpha_1] = u_{2,\lambda,\alpha}$. Then we have,

Lemma A.3. *By taking a smaller λ_0 if necessary, for any fixed $\lambda \in [0, \lambda_0]$ we have:*

(i) *The maps $u_{1,\lambda}[\alpha_2], u_{2,\lambda}[\alpha_1]$ are monotonic increasing,*

$$u_{1,\lambda}[\alpha_2] \leq u_{1,\lambda}[\beta_2], \quad \forall 0 < \alpha_2 < \beta_2 \leq 1, \quad u_{2,\lambda}[\alpha_1] \leq u_{2,\lambda}[\beta_1], \quad \forall 0 < \alpha_1 < \beta_1 \leq 1.$$

(ii) *There exists at least one $\alpha_\lambda = (\alpha_{1,\lambda}, \alpha_{2,\lambda}) \in ((\frac{1}{3}, 1])^2$ such that,*

$$\int_{\Omega} (\alpha_{1,\lambda} + u_{1,\lambda}[\alpha_{2,\lambda}])^{p_1} = 1 = \int_{\Omega} (\alpha_{2,\lambda} + u_{2,\lambda}[\alpha_{1,\lambda}])^{p_2}.$$

Proof. (i) If $\lambda = 0$ we have $u_{1,0,\alpha} = 0$ for any α and the conclusion is trivial. For any fixed $-\infty < \alpha_2 < \beta_2 \leq 1$ let us set,

$$(w_1, w_2) = (u_{1,\lambda}[\beta_2] - u_{1,\lambda}[\alpha_2], u_{2,\lambda}[\beta_1] - u_{2,\lambda}[\alpha_1]) \in (C_0^{2,r}(\bar{\Omega}))^2,$$

then

$$-\Delta w_1 = \lambda(\beta_2 + u_{2,\lambda}[\beta_1])_+^{p_2} - \lambda(\alpha_2 + u_{2,\lambda}[\alpha_1])_+^{p_2} \geq \lambda(\alpha_2 + u_{2,\lambda}[\beta_1])_+^{p_2} - \lambda(\alpha_2 + u_{2,\lambda}[\alpha_1])_+^{p_2} \geq \lambda p_2(\alpha_2 + u_{2,\lambda}[\alpha_1])_+^{p_2-1}(u_{2,\lambda}[\beta_1] - u_{2,\lambda}[\alpha_1]) = \lambda p_2(\alpha_2 + u_{2,\lambda}[\alpha_1])_+^{p_2-1} w_2,$$

by the convexity of $f(t) = (\alpha + t)_+^p$ for $t \in \mathbb{R}$. By applying the same argument to w_2 we deduce that

$$\begin{cases} -\Delta w_1 \geq V_2 w_2 \\ -\Delta w_2 \geq V_1 w_1 \end{cases}, \quad V_1 = \lambda p_1(\alpha_1 + u_{1,\lambda}[\alpha_2])_+^{p_1-1}, \quad V_2 = \lambda p_2(\alpha_2 + u_{2,\lambda}[\alpha_1])_+^{p_2-1}.$$

and then, in view of Lemma 2.1 and possibly taking a smaller λ_0 , well known results for cooperative elliptic systems ([22]) show that $w_i \geq 0$, $i = 1, 2$, as claimed.

(ii) For $\lambda = 0$ we have $u_{i,0,\alpha} = 0$, $i = 1, 2$ and then necessarily $\alpha_{i,\lambda} = 1$, $i = 1, 2$. For fixed $\lambda \in (0, \lambda_0]$, by Lemma A.2 and (i) the functions

$$g_1(\alpha) = \int_{\Omega} (\alpha_1 + u_{1,\lambda}[\alpha_2])_+^{p_1}, \quad g_2(\alpha) = \int_{\Omega} (\alpha_2 + u_{2,\lambda}[\alpha_1])_+^{p_2}, \quad \alpha \in ((-\infty, 1])^2,$$

are continuous as a function of (α_1, α_2) and increasing in α_1 and α_2 . Clearly $\|u_{1,\lambda}[\alpha_2]\|_{\infty} \leq \lambda C_2$, $\|u_{1,\lambda}[\alpha_2]\|_{\infty} \leq \lambda C_2$ for any $\lambda \leq \lambda_0$, and then, possibly taking λ_0 small enough to guarantee that

$$(2\lambda_0 C_2)^{p_i} \leq \frac{1}{4}, \quad i = 1, 2, \tag{A.2}$$

we have,

$$\begin{aligned} g_1((0, \alpha_2)) &= \int_{\Omega} (u_{1,\lambda}[\alpha_2])^{p_1} \leq \frac{1}{4}, \quad \forall \lambda \in (0, \lambda_0], \quad \forall \alpha_2 \in (-\infty, 1], \\ g_2((\alpha_1, 0)) &\leq \int_{\Omega} (u_{2,\lambda}[\alpha_1])^{p_2} \leq \frac{1}{4}, \quad \forall \lambda \in (0, \lambda_0], \quad \forall \alpha_1 \in (-\infty, 1], \end{aligned}$$

while we also have,

$$\begin{aligned} g_1((1, \alpha_2)) &= \int_{\Omega} (1 + u_{1,\lambda}[\alpha_2])^{p_1} > 1, \quad \forall \lambda \in (0, \lambda_0], \quad \forall \alpha_2 \in (-\infty, 1], \\ g_2((\alpha_1, 1)) &= \int_{\Omega} (1 + u_{2,\lambda}[\alpha_1])^{p_2} > 1, \quad \forall \lambda \in (0, \lambda_0], \quad \forall \alpha_1 \in (-\infty, 1]. \end{aligned}$$

As a consequence, for any $\lambda \in (0, \lambda_0]$ there exists at least one $\alpha_\lambda = (\alpha_{1,\lambda}, \alpha_{2,\lambda})$ such that $g_i(\alpha_\lambda) = 1$, $i = 1, 2$ which, by the monotonicity of g_i , must necessarily satisfy

$$\alpha_{i,\lambda} \in (0, 1), \quad i = 1, 2. \tag{A.3}$$

This concludes the proof of Lemma A.1-(j).

Proof of (jj)

Let $(\alpha_\lambda, \psi_\lambda)$ be any solution of $(\mathbf{P})_\lambda$ for $\lambda \in [0, \lambda_0]$. If $\lambda = 0$ then necessarily $\alpha_\lambda = (1, 1)$.

Otherwise let $\lambda \in (0, \lambda_0]$ and observe that $\lambda\psi_\lambda = \mathbf{u}_\lambda$ where \mathbf{u}_λ is the unique solution of (A.1) found in (j). As a consequence, by (A.2) and (A.3) we have that,

$$1 = \int_{\Omega} (\alpha_{1,\lambda} + u_{1,\lambda}[\alpha_{2,\lambda}]_+^{p_1}) \leq 2^{p_1} (\alpha_{1,\lambda})_+^{p_1} + (2\lambda C_2)^{p_1} \leq 2^{p_1} \alpha_{1,\lambda}^{p_1} + \frac{1}{4},$$

whence $\alpha_{1,\lambda} \geq (\frac{3}{4})^{\frac{1}{p_1}} \frac{1}{2} > \frac{1}{3}$. □

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