HIGHER DISCRIMINANTS OF VECTOR BUNDLES AND SCHUR FUNCTORS

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ABSTRACT. We derive closed formulas for the first logarithmic Chern characters of a complex vector bundle. Our argument is representation-theoretic and relates these formulas to the action of certain Casimir elements for $\mathfrak{sl}_r$. As an application, we provide a method for constructing slope-polystable modular bundles on hyper-Kähler manifolds starting from given ones.

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INTRODUCTION

Let X be a complex manifold and E a complex vector bundle on X. Computing the Chern classes $\mathrm{c}_i(E) \in \mathrm{H}^{2i}(X, \mathbb{Z})$ is, in general, both important and difficult.

Similarly, if a vector bundle E is obtained from another bundle E' through tensor operations, it is often nontrivial to express the Chern classes of E in terms of those of E'. Typical examples are symmetric and exterior powers: in [18] a combinatorial expression for the Chern character of these operations is given. However, extracting general formulas can be quite involved even for tensor products of two bundles (see [12]).

Our main result provides explicit formulas for the first three *logarithmic Chern characters* of a Schur functor applied to a vector bundle E. As a consequence, we obtain explicit expressions for the first three Chern characters of Schur bundles (see Appendix B.3). Let us recall the setting: if E is a vector bundle of rank r, we set

$$\begin{aligned}\Delta_1(E) &= \mathrm{c}_1(E)\\ \Delta_2(E) &= \mathrm{c}_1(E)^2 - 2r\,\mathrm{ch}_2(E)\\ \Delta_3(E) &= \mathrm{c}_1(E)^3 - 3r\,\mathrm{c}_1(E)\,\mathrm{ch}_2(E) + 3r^2\,\mathrm{ch}_3(E)\,.\end{aligned}$$

These classes arise naturally from the expansion of the logarithmic of the Chern character (see Section 2.2).

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Theorem A (Theorem 4.2). *Let X be a complex manifold and E be a complex vector bundle of rank r on X. Denote by $\mathbf{S}^\alpha E$ the Schur functor associated with a partition $\alpha = (\alpha_1, \dots, \alpha_r)$, and by r_α its rank. Then*

$$\Delta_1(\mathbf{S}^\alpha E) = |\alpha| \frac{r_\alpha}{r} \Delta_1(E)$$

$$\Delta_2(\mathbf{S}^\alpha E) = \frac{\dot{\delta}_r^{(2)}(\alpha)}{(r-1)(r+1)} \left(\frac{r_\alpha}{r}\right)^2 \Delta_2(E)$$

$$\Delta_3(\mathbf{S}^\alpha E) = \frac{\dot{\delta}_r^{(3)}(\alpha)}{(r-2)(r-1)(r+1)(r+2)} \left(\frac{r_\alpha}{r}\right)^3 \Delta_3(E)\,,$$

where $\dot{\delta}_r^{(2)}(\alpha)$ and $\dot{\delta}_r^{(3)}(\alpha)$ are explicit polynomials in $\alpha = (\alpha_1, \dots, \alpha_r)$ introduced in Theorem 3.1.

The Theorem above applies more generally to any context where Chern classes are defined and the splitting principle holds. To the best of our knowledge, this is the first instance where explicit formulas for Chern classes of Schur bundles are presented at this level of generality. Note also that from classes $\Delta_1(\mathbf{S}^\alpha E)$, $\Delta_2(\mathbf{S}^\alpha E)$ and $\Delta_3(\mathbf{S}^\alpha E)$ one can recursively also recover $c_1(\mathbf{S}^\alpha E)$, $\mathrm{ch}_2(\mathbf{S}^\alpha E)$, and $\mathrm{ch}_3(\mathbf{S}^\alpha E)$. For convenience, we collect these expressions in Section B.3. The polynomials $\dot{\delta}_r^{(2)}(\alpha)$ and $\dot{\delta}_r^{(3)}(\alpha)$ admit a purely algebraic interpretation: they arise as characters of two explicit Casimir elements for $\mathfrak{sl}_r$ (see Section 3).

The proof of Theorem A is representation-theoretic. Using the splitting principle, we reduce to the formal setting of a complex vector space V of dimension r and its Schur modules $\mathbf{S}^\alpha V$, i.e. irreducible polynomial representations of $\mathrm{GL}(V)$. Section 1 reviews the necessary background on polynomial representations. The key results in this framework are Theorem 2.6 and Theorem 3.1, proved in Section 2 and Section 3 respectively. Their combination yields Theorem A.

Extending these results beyond degree 3 appears to be difficult. As discussed in Remark 2.5, one possible approach is to rigidify the situation by fixing certain numerical invariants.

Beyond their intrinsic combinatorial interest, higher discriminants also have geometric significance, as they are closely related to the notion of stability. For example, the class Δ_2 is the usual discriminant, and a celebrated theorem by Bogomolov and Gieseker states that if a vector bundle E is ω-polystable, then $\Delta_2(E).\omega^{\dim X - 2} \geq 0$, where ω is a Kähler class. Additional connections appear in [3, 7].

Another motivation comes from the geometry of compact hyper-Kähler manifolds. O'Grady recently introduced the notion of *modular* vector bundles on compact hyper-Kähler manifolds as an attempt to generalise the theory of moduli spaces of bundles on K3 surfaces to higher dimensions (see [15]). In this context, a vector bundle E is called *modular* if $\Delta_2(E)$ belongs to a distinguished line in $\mathrm{H}^4(X, \mathbb{Q})$ intrinsically attached to X (see Definition 4.8). Because of the numerical nature of this definition, constructing examples of modular bundles is particularly challenging. Recent contributions in this direction include [1, 4, 5, 6, 16].

A direct corollary of Theorem A yields the following result, which allows one to construct infinitely many polystable modular bundles starting from a given one.

Theorem B (Theorem 4.9). *Let X be a compact hyper-Kähler manifold. If E is a slope-polystable modular vector bundle, then for every partition α the Schur functor $\mathbf{S}^\alpha E$ is also a slope-polystable modular vector bundle.*

Even when E is slope-stable, the bundle $\mathbf{S}^\alpha E$ would not be stable in general (cf. Remark 4.7).

The article ends with two appendices. Appendix A discusses a property of discriminants that fails starting from degree 4, offering a different viewpoint on higher discriminants. Appendix B lists explicit formulas for the Chern character (up to degree three) of symmetric powers, exterior powers, and Schur bundles, derived as consequences of Theorem A.

After completing this paper, the authors discovered an alternative approach [2] to computing the full Chern character of Schur bundles. This method yields a closed combinatorial formula for the coefficients in the linear combination expressing $\mathrm{ch}_n(\mathbf{S}^\alpha E)$ in terms of products $\mathrm{ch}_{i_1}(E)\cdots\mathrm{ch}_{i_k}(E)$; in particular, it recovers Theorem A in an alternative way. However, it does not provide a representation-theoretic interpretation of these coefficients, as developed in the present work, which therefore remains of independent interest.

Formulas in [2] show that $\Delta_n(\mathbf{S}^\alpha V)$ equals a certain multiple of $\Delta_n(V)$ up to corrections that are indexed by partitions of n with minimal part size ≥ 2. As there are no such partitions when $n \leq 3$, one obtains the claims in Theorem A. However, when $n \geq 4$ there have to be corrections, and no higher discriminant $\Delta_n, n \geq 4$, is an *eigenclass* of all Schur functors. Inapplicability of our techniques to degree higher than 3 is thus inevitable.

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1. Review of polynomial representations of GL_r

Let V be a complex vector space of dimension r, and let $\mathrm{GL}_r = \mathrm{GL}(V)$ denote the corresponding algebraic group.

Choosing a basis of V provides a choice of a maximal subtorus $T \subset \mathrm{GL}_r$ consisting of diagonal matrices. Every polynomial group homomorphism $T \to \mathbb{C}^*$ is of the form

$$\lambda\colon\ \mathrm{diag}(t_1,\ldots,t_r) \mapsto t_1^{\lambda_1}\ldots t_r^{\lambda_r},$$

so that the set $\widehat{T}$ of all polynomial group homomorphisms $T \to \mathbb{C}^*$ can be identified with $\mathbb{N}^r$ via $\lambda \mapsto (\lambda_1,\ldots,\lambda_r)$. Restricting a finite-dimensional polynomial representation of GL_r to the torus T yields a direct sum decomposition

$$W = \bigoplus_{\lambda\in\widehat{T}} W(\lambda),$$

where

$$W(\lambda) = \{v \in W \mid t.v = \lambda(t)v\}$$

is the *weight space* corresponding to λ.

Polynomial actions of GL_r are completely reducible and each irreducible polynomial representation — called *Schur representations* — occurs as a direct summand of some $V^{\otimes d}$. Recall that Schur representations are indexed by *partitions*, i.e. non increasing r-tuples $\alpha = (\alpha_1,\ldots,\alpha_r)$ of non-negative integers, and are denoted by $\mathbf{S}^\alpha V$. The *size* of α is $|\alpha| = \alpha_1 + \cdots + \alpha_r$ and if $\mathbf{S}^\alpha V \subset V^{\otimes d}$, then $|\alpha| = d$, and in this case we say that $\mathbf{S}^\alpha V$ is a Schur representation of degree d.

With each polynomial representation W of GL_r, one associates its character

$$\mathrm{Char}(W) = \sum_{\lambda\in\mathbb{N}^r} \dim W(\lambda)\, q^\lambda \in \mathbb{Z}[q_1,\ldots,q_r],$$

where $q^\lambda = q_1^{\lambda_1}\cdots q_r^{\lambda r}$. It is immediate that

$$\mathrm{Char}(W_1 \oplus W_2) = \mathrm{Char}(W_1) + \mathrm{Char}(W_2) \quad \text{and} \quad \mathrm{Char}(W_1 \otimes W_2) = \mathrm{Char}(W_1)\cdot\mathrm{Char}(W_2).$$

The character of a polynomial representation is always a symmetric polynomial in $q_1, \dots, q_r$, and the character of a Schur representation of degree d is homogeneous of degree d.

Example 1.1. Well-known examples of Schur functors are the symmetric powers $S^k V$ and the exterior powers $\bigwedge^k V$. Their characters give rise to three important families of symmetric polynomials:

- elementary symmetric polynomials $\sigma_k = \mathrm{Char}(\bigwedge^k V)$;
- complete symmetric polynomials $h_k = \mathrm{Char}(S^k V)$;
- Schur polynomials $s_\alpha = \mathrm{Char}(\mathbf{S}^\alpha V)$.

Finally, let us also recall that every action of GL_r induces a corresponding representation of the Lie algebra $\mathfrak{gl}_r$. Each $\mathbf{S}^\alpha V$ is a highest-weight representation of $\mathfrak{gl}_r$ and its highest weight is precisely α.

The universal enveloping algebra $\mathfrak{U}(\mathfrak{gl}_r)$ acts naturally on each Schur representation, and carries a canonical cocommutative Hopf algebra structure in which the elements of $\mathfrak{gl}_r \subset \mathfrak{U}(\mathfrak{gl}_r)$ are primitive. This fact will be used in Section 3 to study certain special characters.

1.1. **The Grothendieck group of representations.** Let $\mathrm{Rep} = \mathrm{Rep}(\mathrm{GL}_r)$ denote the Grothendieck ring of polynomial representations of GL_r, i.e. the abelian group of formal $\mathbb{Z}$-linear combinations of isomorphism classes of irreducible finite-dimensional polynomial representations of GL_r endowed with the unique multiplication extending the tensor product of representations. We also consider the $\mathbb{Q}$-linear extension of Rep, namely the $\mathbb{Q}$-algebra $\mathrm{Rep}_\mathbb{Q} = \mathrm{Rep} \otimes_\mathbb{Z} \mathbb{Q}$. Schur representations provide free bases of both algebras.

The character map Char extends $\mathbb{Q}$-linearly to a $\mathbb{Q}$-algebra homomorphism

$$\mathrm{Char}\colon\ \mathrm{Rep}_\mathbb{Q} \to \mathbb{Q}[q_1, \dots, q_r] \tag{1.1}$$

which identifies $\mathrm{Rep}_\mathbb{Q}$ with the subalgebra of symmetric polynomials.

Moreover, $\mathrm{Char}(U)$ is homogeneous of degree d if and only if U is a $\mathbb{Q}$-linear combination of Schur representations of degree d. This induces a grading of $\mathrm{Rep}_\mathbb{Q}$ by degree,

$$\mathrm{Rep}_\mathbb{Q} = \bigoplus_{d \geq 0} \mathrm{Rep}^d_\mathbb{Q},$$

with respect to which Char becomes a homogeneous homomorphism.

It is well known that the elementary symmetric polynomials $\sigma_1, \dots, \sigma_r$ generate the ring $\mathbb{Z}[q_1, \dots, q_r]^{\mathfrak{S}_r}$ of symmetric polynomials with integral coefficients. It follows that the exterior powers $\wedge^k V$, for $k = 1, \dots, r$, generate Rep as a $\mathbb{Z}$-algebra (and hence generate $\mathrm{Rep}_\mathbb{Q}$ as a $\mathbb{Q}$-algebra). The same holds for the symmetric powers $S^k V$, $k = 1, \dots, r$, since the complete homogeneous symmetric polynomials $h_1, \dots, h_r$ also generate $\mathbb{Z}[q_1, \dots, q_r]^{\mathfrak{S}_r}$. In particular, the symmetric power representations $S^1 V, \dots, S^r V$ generate Rep over $\mathbb{Z}$, and similarly generate $\mathrm{Rep}_\mathbb{Q}$ over $\mathbb{Q}$.

Remark 1.2. Each $S^k V$ is a representation of degree k. The graded summand Rep^d is generated by monomials in symmetric power representations of total degree d.

To introduce notation to be used later, for any $\alpha = (\alpha_1, \dots, \alpha_t) \in \mathbb{N}^t$ we set

$$\mathrm{Sym}^\alpha V := S^{\alpha_1} V \otimes \cdots \otimes S^{\alpha_t} V \in \mathrm{Rep}\,.$$

Then Rep^d is $\mathbb{Z}$-linearly generated by all $\mathrm{Sym}^\alpha V$ such that $|\alpha| = d$.

Remark 1.3. The power sum polynomials

$$p_k = q_1^k + \cdots + q_r^k, \tag{1.2}$$

for $k = 1, \dots, r$, do not generate $\mathbb{Z}[\sigma_1, \dots, \sigma_r]$, but they do generate $\mathbb{Q}[\sigma_1, \dots, \sigma_r]$ as a $\mathbb{Q}$-algebra. If we define $P_k := \mathrm{Char}^{-1}(p_k) \in \mathrm{Rep}_{\mathbb{Q}}$, then the elements P_k, for $k = 1, \dots, r$, generate $\mathrm{Rep}_{\mathbb{Q}}$ as a $\mathbb{Q}$-algebra. As above, setting

$$P_\alpha := P_{\alpha_1} \cdot \ldots \cdot P_{\alpha_t}$$

the space Rep^d gets $\mathbb{Q}$-linearly generated by all P_α such that $|\alpha| = d$.

For instance, $P_1 = V$ and $P_2 = S^2V - \wedge^2 V$. In general P_k is not an honest representation of GL_r: it is known that P_n is the alternating sum of all Schur *hook representations*[1] of degree n, although we will not use this fact. Note also that P_0 equals r times the trivial one-dimensional representation.

1.2. **Chern character of a representation.** In analogy with vector bundles, we now define the Chern character $\mathrm{ch}(W)$ of a polynomial representation W.

First, denote by

$$e\colon \mathbb{Q}[q_1, \dots, q_r] \longrightarrow \mathbb{Q}[[a_1, \dots, a_r]]$$

the evaluation homomorphism obtained by mapping $q_i \mapsto e^{a_i} \in \mathbb{Q}[[a_i]] \subset \mathbb{Q}[[a_1, \dots, a_r]]$. The indeterminates $a_1, \dots, a_r$ are called the *Chern roots* (of V).

Definition 1.4. *The* Chern character *morphism is defined by*

$$\mathrm{ch} := e \circ \mathrm{Char}\colon\ \mathrm{Rep}_{\mathbb{Q}} \to \mathbb{Q}[[a_1, \dots, a_r]].$$

Given $U \in \mathrm{Rep}_{\mathbb{Q}}$, we group terms in $\mathrm{ch}(U)$ by degree, so that

$$\mathrm{ch}(U) = \sum_{k \geq 0} \mathrm{ch}_k(U),$$

where $\mathrm{ch}_k(U)$ is a homogeneous symmetric polynomial in $a_1, \dots, a_r$ of degree k.

Example 1.5. Let P_d be the elements defined in Remark 1.3. By definition,

$$\mathrm{ch}_k(P_d) = \frac{(da_1)^k + \cdots + (da_r)^k}{k!} = d^k \frac{p_k(a_1, \dots, a_r)}{k!}.$$

In particular, $\mathrm{ch}_0(P_d) = r$ for every $d \geq 1$. Consequently,

$$\mathrm{ch}_k(P_d) = d^k \, \mathrm{ch}_k(P_1). \tag{1.3}$$

More generally, if $\mathsf{c} = \mathrm{ch}_{k_1} \ldots \mathrm{ch}_{k_s}$, then

$$\mathsf{c}(P_d) = d^{k_1 + \cdots + k_s} \mathsf{c}(P_1).$$

Example 1.6 (Svrtan). In [18], closed formulas are given for $\mathrm{ch}(S^m V)$ and $\mathrm{ch}(\bigwedge^m V)$ (see Sections B.1 and B.2). Although the statements are phrased in terms of Chern characters of vector bundles, the proofs are purely combinatorial and rely only on the splitting principle, and therefore apply equally well in the present algebraic framework.

As an illustration, we state the formula for the symmetric power ([18, Theorem 4.8]):

$$\mathrm{ch}(S^m V) = \sum_{\alpha} \frac{1}{\|\alpha\|} \sum_{\beta \leq \alpha} \binom{m + r - 1}{m - |\beta|} \overleftarrow{\beta}\,! S(\alpha, \beta) \, \mathrm{ch}_\alpha(V). \tag{1.4}$$

The first sum runs over all partitions $\alpha = (\alpha_1, \dots, \alpha_t)$ such that $\alpha_i \neq 0$ for every $i = 1, \dots, t$, and $\mathrm{ch}_\alpha(\tilde{V})$ denotes the product $\prod_i \mathrm{ch}_{\alpha_i}(V)$. The norm $\|\alpha\|$ is defined as $a_1! a_2! \cdots a_t!$, where $a_i = \sharp\{j \mid \alpha_j = i\}$. The second sum runs over all $\beta \in \mathbb{N}^s$ such that $\beta \leq \alpha$ and $\ell(\beta) = \ell(\alpha)$ (in particular $\beta_i \neq 0$ for every i). The

[1] i.e. those corresponding to a hook partition $(k, 1, 1, \dots, 1, 0, \dots, 0)$.

symbol $\overleftarrow{\beta}$ denotes the partition $(\beta_1 - 1, \dots, \beta_t - 1)$ and $\overleftarrow{\beta}! = \prod_i (\beta_i - 1)!$. Finally, $S(\alpha, \beta) = \prod_i S(\alpha_i, \beta_i)$, where $S(x, y)$ is the Stirling number of the second kind.

Remark 1.7. If U is a representation of GL_r, then

$$\mathrm{ch}_0(U) = \dim U.$$

The same holds for $U \in \mathrm{Rep}_{\mathbb{Q}}$; equivalently, one may take this identity as the definition of the dimension of U.

Finally, for any $U_1, U_2 \in \mathrm{Rep}_{\mathbb{Q}}$ we have

$$\mathrm{ch}(U_1 + U_2) = \mathrm{ch}(U_1) + \mathrm{ch}(U_2) \qquad \text{and} \qquad \mathrm{ch}(U_1 \cdot U_2) = \mathrm{ch}(U_1)\,\mathrm{ch}(U_2). \tag{1.5}$$

2. Discriminants

2.1. **Slope.** From (1.5) we obtain the additivity relation

$$\mathrm{ch}_1(U_1 + U_2) = \mathrm{ch}_1(U_1) + \mathrm{ch}_1(U_2)$$

and the identity

$$\mathrm{ch}_1(U_1 \cdot U_2) = \mathrm{ch}_0(U_2)\,\mathrm{ch}_1(U_1) + \mathrm{ch}_0(U_1)\,\mathrm{ch}_1(U_2),$$

which we refer to as *log-multiplicativity*, since it can be rewritten as

$$\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(U_1 \cdot U_2) = \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(U_1) + \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(U_2). \tag{2.1}$$

Proposition 2.1. *If* $U \in \mathrm{Rep}^d_{\mathbb{Q}}$*, then*

$$\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(U) = d\,\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(V).$$

Proof. By Example 1.5, we know that $\mathrm{ch}_1(P_d) = d\,\mathrm{ch}_1(P_1)$ and $\mathrm{ch}_0(P_d) = r = \mathrm{ch}_0(P_1)$, which proves the claim when $U = P_d$ (recall that $P_1 = V$).

Let now $\alpha = (\alpha_1, \dots, \alpha_t)$ with $|\alpha| = d$. Applying (2.1) we obtain

$$\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(P_\alpha) = d\,\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(P_1), \tag{2.2}$$

where $P_\alpha = P_{\alpha_1} \cdots P_{\alpha_t}$. The conclusion follows from the fact that the P_α, with $|\alpha| = d$, generate $\mathrm{Rep}^d_{\mathbb{Q}}$ as a $\mathbb{Q}$-vector space. □

The ratio $\mathrm{ch}_1/\mathrm{ch}_0$ is known as *slope*. Proposition 2.1 shows in particular that the slope of each Schur representation $S^\alpha V$ is $|\alpha|$ times the slope of V. Thus, Rep is (essentially) graded by the slope.

2.2. **Logarithmic Chern character.** The key ingredient behind Proposition 2.1 is the logarithmic identity (2.1). To extend this idea to higher degrees, we introduce the logarithmic Chern character.

Definition 2.2. *For any* $U \in \mathrm{Rep}_{\mathbb{Q}}$*, the* logarithmic Chern character *is defined by*

$$\mathrm{L}_+(U) = \log\frac{\mathrm{ch}(U)}{\mathrm{ch}_0(U)} = \log\left(1 + \frac{\mathrm{ch}_1(U)}{\mathrm{ch}_0(U)} + \frac{\mathrm{ch}_2(U)}{\mathrm{ch}_0(U)} + \cdots\right) = \sum_{k>0} (-1)^{k+1} \frac{\Delta_k(U)}{k\,\mathrm{ch}_0(U)^k}.$$

As an illustration, the first values of Δ_k expand to:

$$\begin{aligned} \Delta_1(U) &= \mathrm{ch}_1(U) \\ \Delta_2(U) &= \mathrm{ch}_1(U)^2 - 2\,\mathrm{ch}_0(U)\,\mathrm{ch}_2(U) \\ \Delta_3(U) &= \mathrm{ch}_1(U)^3 - 3\,\mathrm{ch}_0(U)\,\mathrm{ch}_1(U)\,\mathrm{ch}_2(U) + 3\,\mathrm{ch}_0(U)^2\,\mathrm{ch}_3(U)\,. \end{aligned} \tag{2.3}$$

The multiplicativity of ch translates into the *log-multiplicativity* of L_+,

$$\mathrm{L}_+(U_1 \cdot U_2) = \mathrm{L}_+(U_1) + \mathrm{L}_+(U_2)\,. \tag{2.4}$$

Before proceeding to the main result of this section, we introduce another useful definition.

Definition 2.3. *For any $U \in \mathrm{Rep}_{\mathbb{Q}}$, define*

$$\mathsf{d}_k = (-1)^{k+1} \frac{\Delta_k(U)}{k\,\mathrm{ch}_0(U)^{k-1}}.$$

Equation (2.4) translates to

$$\frac{\mathsf{d}_k}{\mathrm{ch}_0}(U_1 \cdot U_2) = \frac{\mathsf{d}_k}{\mathrm{ch}_0}(U_1) + \frac{\mathsf{d}_k}{\mathrm{ch}_0}(U_2) \tag{2.5}$$

or, equivalently, to

$$\mathsf{d}_k(U_1 \cdot U_2) = \mathrm{ch}_0(U_2)\,\mathsf{d}_k(U_1) + \mathrm{ch}_0(U_1)\,\mathsf{d}_k(U_2). \tag{2.6}$$

Moreover, $\mathsf{d}_1(U) = \mathrm{ch}_1(U)$ is additive with respect to direct sums. On the other hand, for $k \geq 2$ the map d_k is not additive in general. Nevertheless, a *weak additivity* property holds for $\mathsf{d}_2, \mathsf{d}_3$.

Proposition 2.4. *Fix $d \geq 0$. If $U_1, U_2 \in \mathrm{Rep}_{\mathbb{Q}}^d$, then*

$$\mathsf{d}_k(U_1 + U_2) = \mathsf{d}_k(U_1) + \mathsf{d}_k(U_2)$$

for $k = 2, 3$.

Proof. Since

$$\mathrm{L}_+ = \sum_{k>0} \frac{\mathsf{d}_k}{\mathrm{ch}_0}$$

and $\mathrm{ch}_0 \cdot \exp(\mathrm{L}_+) = \mathrm{ch}$, each coefficient in the expansion

$$\mathrm{ch}_0 \cdot \exp(\mathrm{L}_+) = \mathrm{ch}_0 \cdot \left(1 + \frac{\mathsf{d}_1}{\mathrm{ch}_0} + \left(\frac{\mathsf{d}_2}{\mathrm{ch}_0} + \frac{\mathsf{d}_1^2}{2\,\mathrm{ch}_0^2}\right) + \left(\frac{\mathsf{d}_3}{\mathrm{ch}_0} + \frac{\mathsf{d}_1\,\mathsf{d}_2}{\mathrm{ch}_0^2} + \frac{\mathsf{d}_1^3}{6\,\mathrm{ch}_0^3}\right) + \dots\right) \tag{2.7}$$

must be additive. In particular, both

$$\mathsf{d}_2 + \frac{1}{2} \cdot \frac{\mathsf{d}_1}{\mathrm{ch}_0} \cdot \mathsf{d}_1 \qquad \text{and} \qquad \mathsf{d}_3 + \frac{\mathsf{d}_1}{\mathrm{ch}_0} \cdot \mathsf{d}_2 + \left(\frac{\mathsf{d}_1}{\mathrm{ch}_0}\right)^2 \cdot \frac{\mathsf{d}_1}{6}$$

are additive on $\mathrm{Rep}_{\mathbb{Q}}$. When restricted to $\mathrm{Rep}_{\mathbb{Q}}^d$, the first expression becomes

$$\mathsf{d}_2 + d/2 \cdot \mathsf{d}_1$$

by Proposition 2.1. Since $\mathsf{d}_1 = \mathrm{ch}_1$ is additive, it follows that d_2 is additive on $\mathrm{Rep}_{\mathbb{Q}}^d$.

Similarly, on $\mathrm{Rep}_{\mathbb{Q}}^d$ the second expression becomes

$$\mathsf{d}_3 + d \cdot \mathsf{d}_2 + d^2/6 \cdot \mathsf{d}_1$$

Since d_1 and d_2 are additive on $\mathrm{Rep}_{\mathbb{Q}}^d$, it follows that d_3 is also additive, which completes the proof. □

Recall that additivity on a rational vector space, such as $\mathrm{Rep}_{\mathbb{Q}}^d$, is equivalent to $\mathbb{Q}$-linearity.

Remark 2.5. Proposition 2.4 stops working from degree 4 on: it is not difficult to find counterexamples. Following the lines of the proof, it is clear that one may rigidify the situation for example by considering the subspace of $\mathrm{Rep}_{\mathbb{Q}}^d$ where also d_2 is fixed. We decided though not to follow this road.

Our main result is the following generalisation of Proposition 2.1.

Theorem 2.6. *If $U \in \mathrm{Rep}_{\mathbb{Q}}^d$, then for $k = 2, 3$*

$$\mathsf{d}_k(U) = f_U^{(k)}(r)\, \mathsf{d}_k(V)$$

for certain polynomials $f_U^{(k)} \in \mathbb{Q}[x]$ of degree at most $d-1$.

Proof. By Example 1.5, we have $\mathrm{ch}_k(P_\ell) = \ell^k \,\mathrm{ch}(P_1)$ and, more generally, $\mathsf{c}(P_\ell) = \ell^{k_1+\cdots+k_t}\mathsf{c}(P_1)$, where $\mathsf{c} = \mathrm{ch}_{k_1}\cdots\mathrm{ch}_{k_t}$. In particular we have that

$$\frac{\mathsf{d}_k}{\mathrm{ch}_0}(P_\ell) = \ell^k \frac{\mathsf{d}_k}{\mathrm{ch}_0}(V)$$

for all k and all $\ell \geq 1$.

As usual, writing $P_\alpha = P_{\alpha_1}\cdots P_{\alpha_t}$, then log-multiplicativity yields

$$\frac{\mathsf{d}_k}{\mathrm{ch}_0}(P_\alpha) = \left(\sum_i \alpha_i^k\right)\frac{\mathsf{d}_k}{\mathrm{ch}_0}(P_1).$$

In particular $\mathsf{d}_k(P_\alpha)$ is a scalar multiple of $\frac{\mathrm{ch}_0(P_\alpha)}{\mathrm{ch}_0(V)}\,\mathsf{d}_k(V) = r^{t-1}\,\mathsf{d}_k(V)$.

Finally, for $k = 2$ or $k = 3$ and fixed degree d, the monomials P_α generate $\mathrm{Rep}_{\mathbb{Q}}^d$. The conclusion then follows from weak additivity (Proposition 2.4), which implies $\mathbb{Q}$-linearity of the restriction of d_k to $\mathrm{Rep}_{\mathbb{Q}}^d$. □

Example 2.7. We have $\frac{\mathsf{d}_2}{\mathrm{ch}_0}(P_2) = 4\frac{\mathsf{d}_2}{\mathrm{ch}_0}(V)$. As $\mathrm{ch}_0(V) = \mathrm{ch}_0(P_2)$, we obtain $\mathsf{d}_2(P_2) = 4\,\mathsf{d}_2(V)$. Also, $\frac{\mathsf{d}_2}{\mathrm{ch}_0}(V^{\otimes 2}) = 2\frac{\mathsf{d}_2}{\mathrm{ch}_0}(V)$. Here, $\mathrm{ch}_0(V^{\otimes 2}) = r^2$ so that $\mathsf{d}_2(V^{\otimes 2}) = 2r\,\mathsf{d}_2(V)$.

Now, $V^{\otimes 2} = S^2V \oplus \wedge^2 V$ and $P_2 = S^2V - \wedge^2V$, so that

$$S^2V = \frac{V^{\otimes 2}+P_2}{2} \qquad \text{and} \qquad \wedge^2\, V = \frac{V^{\otimes 2}-P_2}{2}.$$

Using weak additivity, we obtain

$$\mathsf{d}_2(S^2V) = (r+2)\,\mathsf{d}_2(V) \qquad \text{and} \qquad \mathsf{d}_2(\wedge^2V) = (r-2)\,\mathsf{d}_2(V),$$

and equivalently

$$\frac{\mathsf{d}_2}{\mathrm{ch}_0}(S^2V) = 2\frac{r+2}{r+1}\cdot\frac{\mathsf{d}_2}{\mathrm{ch}_0}(V) \qquad \text{and} \qquad \frac{\mathsf{d}_2}{\mathrm{ch}_0}(\wedge^2V) = 2\frac{r-2}{r-1}\cdot\frac{\mathsf{d}_2}{\mathrm{ch}_0}(V).$$

It follows that

$$\Delta_2(S^2V) = \frac{(r+1)(r+2)}{2}\Delta_2(V) \qquad \text{and} \qquad \Delta_2(\wedge^2V) = \frac{(r-1)(r-2)}{2}\Delta_2(V).$$

Example 2.8. Using formula (1.4) in Example 1.6 (see Section B.1), one can explicitly compute $f_{S^mV}^{(k)}(r)$ in Theorem 2.6. The result is:

$$f_{S^mV}^{(2)}(r) = \frac{m(m+r)}{(r+1)}\frac{r_m}{r} \tag{2.8}$$

and

$$f_{S^mV}^{(3)}(r) = \frac{m(m+r)(2m+r)}{(r+1)(r+2)}\frac{r_m}{r}, \tag{2.9}$$

where in both cases $r_m = \binom{m+r-1}{r-1}$ denotes the dimension of S^mV.

We conclude with the following easy remark on the additivity properties of the coefficients $f_U^{(k)}(r)$, which will be useful later.

Proposition 2.9. *Let $k = 2, 3$.*

(1) *If $U_1, U_2 \in \mathrm{Rep}_{\mathbb{Q}}$, then*

$$f^{(k)}_{U_1 \cdot U_2}(r) = \mathrm{ch}_0(U_2) f^{(k)}_{U_1}(r) + \mathrm{ch}_0(U_1) f^{(k)}_{U_2}(r).$$

(2) *If $U_1, U_2 \in \mathrm{Rep}^d_{\mathbb{Q}}$, then*

$$f^{(k)}_{U_1 + U_2}(r) = f^{(k)}_{U_1}(r) + f^{(k)}_{U_2}(r).$$

Proof. The first identity follows from log-multiplicativity (2.6), while the second follows from weak additivity (Proposition 2.4). □

3. Casimir actions

In this section we determine the expressions $f_U^{(k)}(r)$ appearing in Theorem 2.6 in the case where $U = \mathbf{S}^\alpha V$ is a Schur functor. The key observation is that these expressions can be identified with the action of the quadratic and cubic Casimir elements of the *universal enveloping algebra* $\mathfrak{U}(\mathfrak{gl}_r)$.

We now state the main result of this section. Recall that $\alpha = (\alpha_1, \ldots, \alpha_r)$ is a partition and that r_α denotes the dimension of the Schur module $\mathbf{S}^\alpha V$.

Theorem 3.1. *Keep notations as in Theorem 2.6.*

(1) *For the operator d_2 we have*

$$f^{(2)}_{\mathbf{S}^\alpha V}(r) = \frac{\dot{\delta}^{(2)}_r(\alpha)}{(r-1)(r+1)} \frac{r_\alpha}{r},$$

where

$$\dot{\delta}^{(2)}_r(\alpha) = (r-1)\sum_{i=1}^r \alpha_i^2 - 2 \sum_{1 \le i < j \le r} \alpha_i \alpha_j + r \sum_{i=1}^r (r+1-2i)\alpha_i\,.$$

(2) *For the operator d_3 we have*

$$f^{(3)}_{\mathbf{S}^\alpha V}(r) = \frac{\dot{\delta}^{(3)}_r(\alpha)}{(r-2)(r-1)(r+1)(r+2)} \frac{r_\alpha}{r},$$

where

$$\begin{aligned}\dot{\delta}^{(3)}_r(\alpha) =& 2(r-2)(r-1)\sum_{i=1}^r \alpha_i^3 - 6(r-2)\sum_{1 \le i \ne j \le r} \alpha_i^2 \alpha_j + 24 \sum_{1 \le i < j < k \le r} \alpha_i \alpha_j \alpha_k \\ &+ 3r(r-2)\sum_{i=1}^r (r+1-2i)\alpha_i^2 - 12r \sum_{i<j}(r+1-i-j)\alpha_i\alpha_j \\ &+ r^2 \sum_{i=1}^r (6i^2 - 6i(r+1) + r^2 + 3r + 2)\alpha_i\,.\end{aligned}$$

Before proving Theorem 3.1 in Section 3.2 below, let us recall some facts about universal enveloping algebras and Casimir elements.

3.1. **Universal enveloping algebra and Casimir elements.** We refer to [8] for background on Lie algebras and their universal enveloping algebras.

Let $\mathfrak{g}$ be a Lie algebra, and denote by $\mathfrak{U}(\mathfrak{g})$ its universal enveloping algebra. Recall that $\mathfrak{U}(\mathfrak{g})$ is obtained as a quotient of the tensor algebra $T(\mathfrak{g})$ by the ideal generated by the Lie bracket relations. This construction endows $\mathfrak{U}(\mathfrak{g})$ with both a natural filtration

$$\mathfrak{U}_0(\mathfrak{g}) \subset \mathfrak{U}_1(\mathfrak{g}) \subset \mathfrak{U}_2(\mathfrak{g}) \subset \cdots$$

and a canonical Hopf algebra structure uniquely determined by

$$\epsilon(g) = 0, \qquad \Delta(g) = g \otimes 1 + 1 \otimes g,$$

for every $g \in \mathfrak{g} \subset \mathfrak{U}(\mathfrak{g})$. By the Poincaré-Birkhoff-Witt theorem, the graded pieces $\mathfrak{U}_m(\mathfrak{g})/\mathfrak{U}_{m-1}(\mathfrak{g})$ may be identified with symmetric m-tensors. The coproduct $\Delta : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$ encodes the action on tensor product of representations. Finally, we denote by $\mathfrak{U}(\mathfrak{g})^+ = \mathfrak{g}\mathfrak{U}(\mathfrak{g})$ the *augmentation ideal*, i.e. the kernel of the counit ϵ. Both Δ and ϵ are algebra homomorphisms.

Our first observation is that the trace of the action of certain elements of $\mathfrak{U}(\mathfrak{gl}_r)$ satisfies additivity and log-multiplicativity identities. If $C \in \mathfrak{U}(\mathfrak{gl}_r)$ and W is a $\mathfrak{gl}_r$-representation, we denote by $\mathrm{Tr}_W(C)$ the trace of the action of C on W.

Lemma 3.2. *Let $C \in \mathfrak{U}(\mathfrak{sl}_r) \subset \mathfrak{U}(\mathfrak{gl}_r)$ be an element such that $C \in \mathfrak{U}(\mathfrak{gl}_r)^+ \cap \mathfrak{U}_3(\mathfrak{sl}_r)$.*
For every $U_1, U_2 \in \mathrm{Rep}_{\mathbb{Q}}$, the trace of the action of C satisfies

$$\mathrm{Tr}_{U_1+U_2} C = \mathrm{Tr}_{U_1} C + \mathrm{Tr}_{U_2} C \qquad \textit{and} \qquad \mathrm{Tr}_{U_1 \cdot U_2} C = \mathrm{Tr}_{U_1} C \cdot \dim(U_2) + \dim(U_1) \cdot \mathrm{Tr}_{U_2} C.$$

Proof. Additivity always holds, so we focus on log-multiplicativity.

First observe that if C has degree 1, i.e. $C \in \mathfrak{sl}_r \subset \mathfrak{U}(\mathfrak{sl}_r)$, then it is primitive. By this we mean that $\Delta(C) = C \otimes 1 + 1 \otimes C$, from which the claim follows immediately.

Next assume that $C \in \mathfrak{U}(\mathfrak{gl}_r)^+ \cap \mathfrak{U}_2(\mathfrak{sl}_r)$ is quadratic, and let U_1, U_2 be polynomial representations of GL_r. If C is of the form $C = \sum_i x_i y_i$, where $x_i, y_i \in \mathfrak{sl}_r$, then

$$\Delta(C) = C \otimes 1 + 1 \otimes C + \sum_i (x_i \otimes y_i + y_i \otimes x_i).$$

The trace of the action of C on $U_1 \otimes U_2$, which is described by $\Delta(C)$, coincides with the trace of $C \otimes 1 + 1 \otimes C$, since $\mathrm{Tr}(x_i) = \mathrm{Tr}(y_i) = 0$ for all i, as every element of $\mathfrak{sl}_r$ is traceless. Moreover, $\mathrm{Tr}_{U_1 \otimes U_2}(C \otimes 1) = (\mathrm{Tr}_{U_1} C) \cdot \dim(U_2)$ and similarly $\mathrm{Tr}_{U_1 \otimes U_2}(1 \otimes C) = \dim(U_1) \cdot (\mathrm{Tr}_{U_2} C)$. This proves log-multiplicativity.

The same argument applies to cubic elements. □

Remark 3.3. If C has degree 1 and is primitive, the above proof also applies when $C \in \mathfrak{U}(\mathfrak{gl}_r)$.

Remark 3.4. The argument above does not extend to elements of degree greater than or equal to four, for reasons analogous to those discussed in Remark 2.5.

Remark 3.5. We stated Lemma 3.2 for elements in $\mathfrak{U}(\mathfrak{sl}_r) \subset \mathfrak{U}(\mathfrak{gl}_r)$, but in fact only elements in the centre were needed. In fact let us notice that the trace of the corresponding action is invariant under the Adjoint action of GL_r, with respect to which we may average, thus obtaining an element in the centre of $\mathfrak{U}(\mathfrak{gl}_r)$.

By *Casimir element* for a Lie algebra $\mathfrak{g}$ we mean every element in the centre of its universal enveloping algebra. Let us describe two special Casimir elements that will be useful in the next section.

Denote by $E_{ij} \in \mathfrak{gl}_r$ the elementary matrix whose only non-zero entry is 1 in position (i,j): these matrices form the standard basis of $\mathfrak{gl}_r$. Consider now the following three elements of $\mathfrak{U}(\mathfrak{gl}_r)$:

$$I = \sum_{i=1}^{r} E_{ii}, \qquad C_2 = \sum_{i,j=1}^{r} E_{ij}E_{ji} \qquad \text{and} \qquad C_3 = \sum_{i,j,k=1}^{r} E_{ij}E_{jk}E_{ki}.$$

It is immediate to show that I, C_2 and C_3 belong to the centre of $\mathfrak{U}(\mathfrak{gl}_r)$ (see for example [13, Example 4.2.5]). Consider the correction $C^{(2)} = \sum_{i,j=1}^{r}(E_{ij} - \delta_{ij}I/r)(E_{ji} - \delta_{ji}I/r) \in \mathfrak{U}(\mathfrak{sl}_r) \subset \mathfrak{U}(\mathfrak{gl}_r)$. Then $C^{(2)}$ still lies in the centre of $\mathfrak{U}(\mathfrak{gl}_r)$ and one easily computes

$$C^{(2)} = C_2 - \frac{1}{r}I^2. \tag{3.1}$$

Similarly, if $C_3^0 = \sum_{i,j,k=1}^{r}(E_{ij} - \delta_{ij}I/r)(E_{jk} - \delta_{jk}I/r)(E_{ki} - \delta_{ki}I/r)$ then

$$C_3^0 = C_3 - \frac{3}{r}C_2 I + \frac{2}{r^2}I^3 \in \mathfrak{U}(\mathfrak{sl}_r) \subset \mathfrak{U}(\mathfrak{gl}_r)\,.$$

Finally, we set

$$C^{(3)} = 2C_3^0 - rC^{(2)} \in \mathfrak{U}(\mathfrak{sl}_r). \tag{3.2}$$

3.2. **Proof of Theorem 3.1.** The strategy of the proof is to interpret the coefficients $f^{(k)}_{\mathbf{S}^\alpha V}(r)$ as the actions of certain Casimir elements.

Definition 3.6. *Let $C^{(2)}$ and $C^{(3)}$ be the quadratic and cubic Casimir elements of $\mathfrak{U}(\mathfrak{sl}_r) \subset \mathfrak{U}(\mathfrak{gl}_r)$ as defined in (3.1) and (3.2). Polynomials $\phi_U^{(2)}(r), \phi_U^{(3)}(r)$ are defined as follows:*

$$\phi_U^{(2)}(r) = \mathrm{Tr}_U\left(\frac{C^{(2)}}{(r-1)(r+1)}\right) \qquad \textit{and} \qquad \phi_U^{(3)}(r) = \mathrm{Tr}_U\left(\frac{r\,C^{(3)}}{(r-2)(r-1)(r+1)(r+2)}\right).$$

The main result of the section is the following proposition.

Proposition 3.7. *For every $U \in \mathrm{Rep}^d_{\mathbb{Q}}$ and $k = 2,3$,*

$$f_U^{(k)}(r) = \phi_U^{(k)}(r)\,.$$

Proposition 3.7 is a consequence of the following remark.

Lemma 3.8. *For $k = 2,3$, we have the equalities*

$$f^{(k)}_{S^mV}(r) = \phi^{(k)}_{S^mV}(r)$$

for all $m \geq 0$.

Proof. Recall that the $C^{(k)}$'s are constructed starting from the standard Casimir elements I, C_2 and C_3. One can explicitly compute the actions of the latter on the symmetric representation S^mV:

$$I|_{S^mV} = m\cdot\mathrm{id}, \qquad C_2|_{S^mV} = m(m+r-1)\cdot\mathrm{id} \qquad \text{and} \qquad C_3|_{S^mV} = m(m+r-1)^2\cdot\mathrm{id}\;.$$

By the definition we see that $C^{(2)} = C_2 - \frac{1}{r}I^2$ acts as

$$C^{(2)}|_{S^mV} = m(m+r)\frac{r-1}{r}\cdot\mathrm{id},$$

hence

$$\phi^{(2)}_{S^mV}(r) = \frac{m(m+r)}{r+1}\frac{r_m}{r} = f^{(2)}_{S^mV}(r),$$

where $r_m = \dim S^mV$ and the last equality can be found in Example 2.8.

Continuing with the degree 3 case, since $C_3^0 = C_3 - \frac{3}{r}C_2 I + \frac{2}{r^2}I^3$, so then its action on $S^m V$ is

$$C_3^0|_{S^m V} = m(m+r)\frac{(r-1)}{r^2}[(r-2)m + r(r-1)] \cdot \mathrm{id}\,.$$

Similarly, the action of $C^{(3)} = 2C_3^0 - rC^{(2)}$ on $S^m V$ is given by

$$C^{(3)}|_{S^m V} = \frac{m(m+r)(2m+r)(r-1)(r-2)}{r^2} \cdot \mathrm{id}\,.$$

Finally,

$$\phi^{(3)}_{S^m V}(r) = \frac{m(m+r)(2m+r)}{(r+1)(r+2)}\frac{r_m}{r} = f^{(3)}_{S^m V}(r),$$

where again the last equality follows from Example 2.8. □

Proof of Proposition 3.7. By Lemma 3.2 we know that, for $k = 2, 3$, $\phi_U^{(k)}(r)$ is additive and log-multiplicative. Similarly, by Proposition 2.9, $f_U^{(k)}(r)$ is weakly additive and log-multiplicative. Moreover, by Lemma 3.8 they agree on symmetric representations $S^m V$.

It follows that $\phi_U^{(k)}(r)$ and $f_U^{(k)}(r)$ also agree when $U = \mathrm{Sym}^\alpha V = S^{\alpha_1} V \cdots S^{\alpha_t} V$ is a monomial in symmetric representation, i.e.

$$f^{(k)}_{\mathrm{Sym}^\alpha V}(r) = \phi^{(k)}_{\mathrm{Sym}^\alpha V}(r).$$

We conclude the proof by recalling that the vector space $\mathrm{Rep}^d_{\mathbb{Q}}$ is $\mathbb{Q}$-linearly generated by monomials $\mathrm{Sym}^\alpha V$, for $|\alpha| = d$. □

Remark 3.9. The maps $\phi^{(k)}$ are fully additive, whereas the operators d_k are only weakly additive. The fact that $\mathsf{d}_k(U) = \phi_U^{(k)}(r)\,\mathsf{d}_2(V)$ seems to hint at the fact that they coincide on the whole $\mathrm{Rep}_{\mathbb{Q}}$. However, this is false: as is evident from the proof of Proposition 2.4 the operators d_k are not fully additive.

3.3. **The Harish-Chandra isomorphism.** Let us now describe the centre $\mathsf{Z}(\mathfrak{gl}_r)$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{gl}_r)$ using the Harish-Chandra Theorem. We follow the conventions in [13, Chapter 4].

Every element $z \in \mathsf{Z}(\mathfrak{gl}_r)$ acts on each Schur representation $\mathbf{S}^\alpha V$ via multiplication by a scalar $\chi^\alpha(z)$ which depends polynomially on α. The action of $\mathfrak{gl}_r$ on a highest-weight vector $v \in \mathbf{S}^\alpha V$ satisfies

$$E_{ii} v = \alpha_i v, \qquad \text{and} \qquad E_{ij}.v = 0$$

if $i < j$. If $\chi^\alpha(z) = p(\alpha_1, \ldots, \alpha_r)$, then the shifted polynomial $p(x_1, x_2 + 1, \ldots, x_r + r - 1)$ is symmetric under permutations of the x_i and the corresponding Harish-Chandra map

$$\mathcal{H}\colon \mathsf{Z}(\mathfrak{gl}_r) \xrightarrow{\sim} \mathbb{C}[x_1, \ldots, x_r]^{\mathfrak{S}_r}$$

is an isomorphism (see [13, isomorphism (4.15)]).

The variable x_i describes the action of the element $E_{ii} - i + 1$ on the highest-weight vector of $\mathbf{S}^\alpha V$. Thus, if $\mathcal{H}(z) = q(x)$, then the action of z on $\mathbf{S}^\alpha V$ is given by multiplication by $q(\alpha_1, \alpha_2 - 1, \ldots, \alpha_r - r + 1)$. We shall denote the polynomial $q(x_1, x_2 - 1, \ldots x_r - r + 1)$ by $\dot{q}(x)$.

Remark 3.10. Central elements in $\mathfrak{U}(\mathfrak{sl}_r)$ are also central in $\mathfrak{U}(\mathfrak{gl}_r)$. One checks that $z \in \mathsf{Z}(\mathfrak{gl}_r)$ lies in $\mathsf{Z}(\mathfrak{sl}_r)$ if and only if $p(x) = \mathcal{H}(z)$ is translation invariant, i.e. it satisfies $p(x_1, \ldots, x_r) = p(x_1 + c, \ldots, x_r + c)$ for all $c \in \mathbb{Z}$, and hence for all $c \in \mathbb{C}$.

Indeed, if α and β are partitions such that $\beta - \alpha = (c, \ldots, c)$, then the $\mathfrak{gl}_r$-representations $V(\alpha)$ and $V(\beta)$ differ by tensoring with a one-dimensional $\mathfrak{gl}_r$-representation which restricts to the trivial $\mathfrak{sl}_r$-representation. Thus any $z \in \mathsf{Z}(\mathfrak{sl}_r)$ must act by the same scalar on both representations, which shows that $\mathcal{H}(z)$ takes the same value on α and β.

Conversely, every element of $\mathsf{Z}(\mathfrak{gl}_r)$ can be written as a polynomial in the identity element $I \in \mathfrak{gl}_r$, with coefficients in $\mathsf{Z}(\mathfrak{sl}_r)$, from which the claim follows easily.

Lemma 3.11. *If $q(x) \in \mathbb{C}[x_1, \dots, x_r]^{\mathfrak{S}_r}$ is a symmetric polynomial, then for every Schur module we have*

$$\mathcal{H}^{-1}(q(x))|_{\mathbf{S}^\alpha V} = \dot{q}(\alpha) \cdot \mathrm{id}_{\mathbf{S}^\alpha V} .$$

Proof. The module $\mathbf{S}^\alpha V$ is a highest-weight representation of $\mathfrak{gl}_r$ with highest weight α, and the statement follows from the definition of the Harish–Chandra isomorphism. □

Let us define the following symmetric polynomials:

$$\delta_r^{(2)}(x) = (r-1)\sum_{i=1}^r x_i^2 - 2\sum_{1\le i<j\le r} x_i x_j - \frac{r^2(r^2-1)}{12} \tag{3.3}$$

and

$$\delta_r^{(3)}(x) = 2(r-2)(r-1)\sum_{i=1}^r x_i^3 - 6(r-2)\sum_{1\le i\neq j\le r} x_i^2 x_j + 24\sum_{1\le i<j<k\le r} x_i x_j x_k . \tag{3.4}$$

Lemma 3.12. *After the change of variables $\alpha_i = x_i + i - 1$, we have that*

$$\dot{\delta}_r^{(2)}(\alpha) = (r-1)\sum_{i=1}^r \alpha_i^2 - 2\sum_{1\le i<j\le r}\alpha_i\alpha_j + r\sum_{i=1}^r (r+1-2i)\alpha_i .$$

and

$$\begin{aligned}\dot{\delta}_r^{(3)}(\alpha) =& 2(r-2)(r-1)\sum_{i=1}^r \alpha_i^3 - 6(r-2)\sum_{1\le i\neq j\le r}\alpha_i^2\alpha_j + 24\sum_{1\le i<j<k\le r}\alpha_i\alpha_j\alpha_k \\ &+3r(r-2)\sum_{i=1}^r (r+1-2i)\alpha_i^2 - 12r\sum_{i<j}(r+1-i-j)\alpha_i\alpha_j \\ &+r^2\sum_{i=1}^r (6i^2 - 6i(r+1) + r^2 + 3r + 2)\alpha_i .\end{aligned}$$

Proof. This is a direct substitution of variables. □

Notice that $\dot{\delta}_r^{(2)}$ and $\dot{\delta}_r^{(3)}$ coincide with the polynomials defined in Theorem 3.1.

Lemma 3.13. *For $k = 2, 3$, let $\delta_r^{(k)}$ be as above. Then $\mathcal{H}^{-1}(\delta_r^{(k)}) \in \mathfrak{U}(\mathfrak{gl}_r)^+ \cap \mathfrak{U}(\mathfrak{sl}_r)$.*

Proof. By Lemma 3.12, $\dot{\delta}_r^{(k)}(\alpha)$ has no constant term, so that $\mathcal{H}^{-1}(\delta_r^{(k)}(x))$ belongs to the augmentation ideal $\mathfrak{U}(\mathfrak{gl}_r)^+$. An easy computation shows that

$$\delta_r^{(k)}(x_1 + c, \dots, x_r + c) = \delta_r^{(k)}(x_1, \dots, x_r),$$

for all $c \in \mathbb{C}$, so that $\mathcal{H}^{-1}(\delta_r^{(k)}) \in \mathsf{Z}(\mathfrak{sl}_r)$ by Remark 3.10. □

The following proposition completes the proof of Theorem 3.1.

Proposition 3.14. *Let $\phi^{(k)}$ be the characters defined in Definition 3.6, and let $\delta_r^{(k)}$ be the quadratic and cubic symmetric polynomials defined in (3.3) and (3.4). Then for every Schur module $\mathbf{S}^\alpha V$ we have*

$$\phi^{(2)}_{\mathbf{S}^\alpha V}(r) = \frac{\dot{\delta}_r^{(2)}(\alpha)}{(r-1)(r+1)}\frac{r_\alpha}{r} \qquad \textit{and} \qquad \phi^{(3)}_{\mathbf{S}^\alpha V}(r) = \frac{\dot{\delta}_r^{(3)}(\alpha)}{(r-2)(r-1)(r+1)(r+2)}\frac{r_\alpha}{r}.$$

Proof. By the proof of Lemma 3.8, together with a direct verification using Lemma 3.12, both identities hold for symmetric representations $S^m V$.

By Lemma 3.2 (which applies in view of Lemma 3.13), the functions $\phi^{(k)}_{\mathbf{S}^\alpha V}(r)$ and

$$\frac{\dot\delta_2(\alpha)}{(r-1)(r+1)}\frac{r_\alpha}{r}, \qquad \frac{\dot\delta_3(\alpha)}{(r-2)(r-1)(r+1)(r+2)}\frac{r_\alpha}{r}$$

are additive and log-multiplicative. Hence they coincide on $\mathrm{Rep}_{\mathbb{Q}}$ if and only if they coincide on a set of generators. Since symmetric representations generate $\mathrm{Rep}_{\mathbb{Q}}$, the claim follows. □

Proof of Theorem 3.1. The proof follows at once from Proposition 3.7 and Proposition 3.14. □

Remark 3.15. Let us consider the central element $I = \sum E_{ii} \in \mathfrak{U}(\mathfrak{gl}_r)$ of degree 1. It is not an element in $\mathfrak{U}(\mathfrak{sl}_r)$, but by Remark 3.3 the proof of Lemma 3.2 continues to work for it. Therefore if we define

$$\phi^{(1)}_U(r) = \mathrm{Tr}_U\left(\frac{1}{r}I\right),$$

then $\phi^{(1)}(r)$ is additive and log-multiplicative. Explicitly we have

$$\phi^{(1)}_{\mathbf{S}^\alpha V}(r) = |\alpha|\frac{r_\alpha}{r}.$$

In this way we understand Proposition 2.1 as a Casimir action. In fact, it is this observation that led us to think of the characters $f^{(k)}(r)$ as Casimir actions for $k=2$ and 3.

Remark 3.16. The dimension of a Schur module $\mathbf{S}^\alpha V$, as a polynomial $f^{(0)}_\alpha(r)$ in $r = \dim V$, has the following well-known property: if we denote by α' the conjugate partition, then $\dim \mathbf{S}^{\alpha'} V = f^{(0)}_{\alpha'}(r) = (-1)^{|\alpha|} f^{(0)}_\alpha(-r)$. A similar claim holds for $f^{(2)}_\alpha(r)$ and $f^{(3)}_\alpha(r)$:

$$f^{(2)}_{\alpha'}(r) = (-1)^{|\alpha|+1} f^{(2)}_\alpha(-r) \qquad \text{and} \qquad f^{(3)}_{\alpha'}(r) = (-1)^{|\alpha|+1} f^{(3)}_\alpha(-r).$$

Our more recent paper [2] provides a different combinatorial description of such polynomials which can (and will) be used to give a short proof of this observation.

Remark 3.17. The function

$$f^{(2)}_{S^m V}(r)\cdot\left(\frac{r_m}{r}\right) = \frac{m(m+r)}{(r+1)}\left(\frac{r_m}{r}\right)^2$$

counts the dimension of the representation $\mathbf{S}^{m-1,m-1}V_{r+2}$, as it can be directly seen from Weyl's formula. Here V_{r+2} is a vector space of dimension $r+2$.

Equivalently, it counts the number of standard tableaux of size $(m-1, m-1)$ for $\mathrm{Mat}(2, r+2)$. These form a basis for the degree $m-1$ component of the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}(2, r+2)$ (see [14, Lemma 8.14, Remark 8.16]), i.e.

$$f(m) = h^0(\mathrm{Gr}(2, r+2), \mathcal{O}(m-1)).$$

We point out that $f^{(2)}_{S^m V}(r)$ has another interesting Lie-theoretical connection when $r=4$, being the dimension of $V^{(k)}$, the distinguished module in the Severi variety with $a=4$ in [10, Theorem 7.3]. We have not explored a connection between these two facts yet. On the other hand, for $f^{(3)}_{S^m V}(r)$ we have not found an interpretation valid for every r.

We do not have an explanation for such phenomena, but a natural question is to understand if the polynomials appearing in Theorem 3.1 admit in general a geometrical interpretation.

4. Geometric applications

4.1. **Schur functors of complex vector bundles.** In this section X is a complex manifold.

Let E be a complex vector bundle of rank r on X. Define the following characteristic classes:

$$\begin{aligned}\Delta_1(E) &= \mathrm{c}_1(E)\\ \Delta_2(E) &= \mathrm{c}_1(E)^2 - 2r\,\mathrm{ch}_2(E)\\ \Delta_3(E) &= \mathrm{c}_1(E)^3 - 3r\,\mathrm{c}_1(E)\,\mathrm{ch}_2(E) + 3r^2\,\mathrm{ch}_3(E).\end{aligned}$$

Remark 4.1. These classes are the same as defined in (2.3). We refer to [3, 7] for other appearances of logarithmic Chern classes.

For any partition $\alpha = (\alpha_1, \dots, \alpha_r)$, we denote by $\mathbf{S}^\alpha E$ the *Schur bundle* associated to E, i.e. the corresponding Schur functor of E. In particular, it is a vector bundle of rank r_α such that

$$(\mathbf{S}^\alpha E)_x = \mathbf{S}^\alpha E_x \qquad \forall x \in X.$$

As a consequence of the results in the previous section, we get the following statement.

Theorem 4.2. *Let X and E be as above. Then*

$$\begin{aligned}\Delta_1(\mathbf{S}^\alpha E) &= |\alpha|\frac{r_\alpha}{r}\Delta_1(E)\\ \Delta_2(\mathbf{S}^\alpha E) &= \frac{\dot{\delta}_r^{(2)}(\alpha)}{(r-1)(r+1)}\left(\frac{r_\alpha}{r}\right)^2\Delta_2(E)\\ \Delta_3(\mathbf{S}^\alpha E) &= \frac{\dot{\delta}_r^{(3)}(\alpha)}{(r-2)(r-1)(r+1)(r+2)}\left(\frac{r_\alpha}{r}\right)^3\Delta_3(E)\,,\end{aligned}$$

where $\dot{\delta}_r^{(2)}(\alpha)$ and $\dot{\delta}_r^{(3)}(\alpha)$ are as in Theorem 3.1.

Proof. Let us assume that $E = L_1 \oplus \cdots \oplus L_r$, where the L_i's are line bundles with $\mathrm{c}(L_i) = a_i$, i.e. the a_i's are the Chern roots of E. Then by definition

$$\mathrm{ch}(E) = \sum_{i=1}^{r} e^{a_i},$$

which is the same expression as in Definition 1.4. In particular every expression involving the Chern character of E and $\mathbf{S}^\alpha E$ is equivalent to the corresponding expression for the vector spaces E_x and $\mathbf{S}^\alpha E_x$, for any point $x \in X$.

The claim then follows at once from Proposition 2.1, Theorem 2.6 and Theorem 3.1.

If E is any vector bundle, then by the splitting principle we can formally reduce to the case above, thus concluding the proof. □

Remark 4.3. The case $\Delta_1 = \mathrm{c}_1$ was already proved in [17]. Rubei's proof uses a double-induction argument to reduce to the symmetric case, where the result is easy to prove. Also our proof is deduced from the symmetric case, but our reduction argument is not inductive and is instead based on representation-theoretic arguments.

Remark 4.4. The expressions in Theorem 4.2 can be used to deduce closed expressions for the first three Chern characters of a Schur functor, see Section B.3. Similar expressions have been investigated in [9, Section 4]. More precisely, in [9, Theorem 4.3] there is an asymptotic version of the expression for $\mathrm{ch}_2(\mathbf{S}^\alpha E)$ under some (strong) assumptions on the partition α.

Remark 4.5. It is interesting to notice that for any partition α, the polynomial $\dot{\delta}_r^{(2)}(\alpha)$ is non-negative. It follows that if X is a Kähler manifold of dimension N, then for any Kähler class ω on X

$$\Delta_2(\mathbf{S}^\alpha E).\omega^{N-2} \geq 0 \qquad \Longleftrightarrow \qquad \Delta_2(E).\omega^{N-2} \geq 0.$$

The comment above should be read with an eye on the Bogomolov inequality, which says that if a vector bundle E is slope ω-semistable on a compact Kähler manifold, then $\Delta_2(E).\omega^{N-2} \geq 0$. The vice versa is not true, but it is known that we can still deduce polystability of $\mathbf{S}^\alpha E$ from that of E. We state here the following well-known result for lack of a better reference.

Proposition 4.6. *Let X be a compact Kähler manifold, ω a Kähler class and E a slope ω-polystable vector bundle on X. Then for any partition α the Schur functor $\mathbf{S}^\alpha E$ is slope ω-polystable.*

Proof. If we put $\ell = |\alpha|$, then $\mathbf{S}^\alpha E \subset E^{\otimes \ell}$ as a direct summand. It follows from the Donaldson–Uhlenbeck–Yau Theorem (see for example [11]) that $E^{\otimes \ell}$ is slope ω-polystable, so that $\mathbf{S}^\alpha E$ is also slope ω-polystable. □

Remark 4.7. In Proposition 4.6, even if E is slope stable, the polystability of $\mathbf{S}^\alpha E$ cannot in general be improved. As an example, consider $E = S^2 Q$, where Q is the quotient tautological bundle on a grassmannian $\mathrm{Gr}(k,n)$. Then E is slope stable, but $S^2E = S^2(S^2Q)$ is not indecomposable, hence it is strictly polystable. On the other hand, indecomposability of $\mathbf{S}^\alpha E$ is the only obstruction to stability: if $\mathbf{S}^\alpha E$ is indecomposable, then it is stable.

4.2. **Modular bundles on hyper-Kähler manifolds.** Let X be a compact hyper-Kähler manifold. Recall that the second integral cohomology group $\mathrm{H}^2(X,\mathbb{Q})$ is torsion free and endowed with a non-degenerate quadratic form q_X. With an abuse of notation, we keep denoting by q_X the class in $\mathrm{H}^4(X,\mathbb{Q})$ dual to this quadratic form.

We denote by $\mathrm{SH}(X) \subset \mathrm{H}(X,\mathbb{Q})$ the Verbitsky component, namely the submodule generated by $\mathrm{H}^2(X,\mathbb{Q})$ via cup product. Notice that $\mathsf{q}_X \in \mathrm{SH}^4(X)$.

Definition 4.8 ([15, Definition 1.1])**.** *A vector bundle E on X is called* modular *if the projection of $\Delta_2(E)$ onto $\mathrm{SH}^4(X)$ is a multiple of q_X.*

As a corollary of Theorem 4.2 we obtain the following result that allows one to construct new modular bundles on hyper-Kähler manifolds from existing ones.

Theorem 4.9. *Let X be a compact hyper-Kähler manifold. If E is a slope polystable modular vector bundle, then $\mathbf{S}^\alpha E$ is a slope polystable modular vector bundle.*

Proof. The slope polystability follows from Proposition 4.6. The modularity follows from Theorem 4.2: in fact the projection into $\mathrm{SH}^4(X)$ of $\Delta_2(\mathbf{S}^\alpha E)$ is a multiple of the projection of $\Delta_2(E)$, which is a multiple of q_X by hypothesis. □

Example 4.10. Let S be a K3 surface, i.e. a hyper-Kähler surface, and let E be a stable vector bundle with Mukai vector $v(E) = (r, \mathrm{c}, s)$, where r is the rank of E. By dimension reasons, the bundle E is modular.

For the same dimension reasons, any Schur functor $\mathbf{S}^\alpha E$ is also modular. The Mukai vector of $\mathbf{S}^\alpha E$ is

$$v(\mathbf{S}^\alpha E) = \left(r_\alpha, |\alpha| \frac{r_\alpha}{r} \mathrm{c}, \frac{(|\alpha|^2 - \tilde{\delta}_r^{(2)}(\alpha))}{r} \frac{r_\alpha}{r} \frac{\mathrm{c}^2}{2} + \tilde{\delta}_r^{(2)}(\alpha)(s-r)\frac{r_\alpha}{r} + r_\alpha \right),$$

where we have used the same notations as in Section B.3, so that $\tilde{\delta}_r^{(2)}(\alpha) = \frac{\dot{\delta}_r^{(2)}(\alpha)}{(r-1)(r+1)}$.

The Mukai vector $v(\mathbf{S}^\alpha E) \in \mathrm{H}^*(S,\mathbb{Z})$ can be primitive or not depending on α, r and the bundle E itself. For example, let us suppose that $\mathrm{Pic}(S) = \mathbb{Z}\cdot H$, where H is ample and $H^2 = 2d \geq 6$. Then there exists a

H-stable vector bundle E of rank 2 with $\mathrm{c}_1(E) = H$ and $\mathrm{ch}_2(E) = 0$. Such a vector bundle has a primitive Mukai vector $v(E) = (2, H, 2)$. The symmetric power S^2E has Mukai vector

$$v(S^2E) = (3, 3H, d + 3),$$

therefore it is primitive for all $3 \nmid d$, but it is not primitive for $d \in 3\mathbb{Z}$.

Example 4.11. Examples of stable modular vector bundles on hyper-Kähler fourfolds of type $\mathrm{K3}^{[2]}$ have been constructed in [16, 1, 4, 5, 6]. By Theorem 4.9, each of these examples gives rise to many more (polystable) examples.

Appendix A. Discriminants for low-rank bundles

Let us consider the logarithmic classes defined in (2.3). As already remarked in the introduction, these logarithmic classes $\Delta_k(F)$ for a coherent sheaf F are linked to the the stability of F itself. This is well known for $k = 1, 2$, and there is evidence starting to build up for $k = 3$, see [7]. It is therefore interesting to study not only the first logarithmic classes, but also the higher-degree terms. As a first observation, notice that *all* logarithmic classes behave well with respect to twist with line bundles: for example (see [3, Proposition 2.1]) we have for E a vector bundle and L a line bundle

$$\Delta_k(E \otimes L) = \Delta_k(E),\ k \geq 2$$

The first three logarithmic classes satisfy another nice property.

Proposition A.1. *Let E be a vector bundle of rank $r < k$, for $k = 1, 2, 3$. Then $\Delta_k(E) = 0$.*

Proof. For $k = 1$ the claim is trivially true. For $k = 2, 3$ this is not evident from the definition. However, if we write Δ_k in terms of the Chern classes, we find that

$$\Delta_2(E) = -(r-1)\,\mathrm{c}_1^2(E) + 2r\,\mathrm{c}_2(E)$$

and

$$\Delta_3(E) = \frac{1}{2}\left((r-1)(r-2)\,\mathrm{c}_1(E)^3 - 3r(r-2)\,\mathrm{c}_1(E)\,\mathrm{c}_2(E) + 3r^2\,\mathrm{c}_3(E)\right).$$

The result then follows by simply noting that $\mathrm{c}_i(E) = 0$ for $i > r$. □

The above result does not hold for $k \geq 4$, highlighting another difference between lower and higher degree discriminants. In fact, $\Delta_4(E)$ is already not zero on rank 2 bundles with $\mathrm{c}_2 \neq 0$. However, it is possible to find suitable modifications for both Δ_4 and Δ_5 that behave well with respect to this property. For $k = 4, 5$ we expand the expression in Definition 2.2:

$$\begin{aligned}\Delta_4(E) &= \mathrm{ch}_1(E)^4 - 4r\,\mathrm{ch}_1(E)^2\,\mathrm{ch}_2(E) + 2r^2(\mathrm{ch}_2(E)^2 + 2\,\mathrm{ch}_1(E)\,\mathrm{ch}_3(E)) - 4r^3\,\mathrm{ch}_4(E)\\ \Delta_5(E) &= \mathrm{ch}_1(E)^5 - 5r\,\mathrm{ch}_1(E)^3\,\mathrm{ch}_2(E) + 5r^2\,\mathrm{ch}_1(E)(\mathrm{ch}_2(E)^2 + \mathrm{ch}_1(E)\,\mathrm{ch}_3(E))\\ &\quad - 5r^3(\mathrm{ch}_2(E)\,\mathrm{ch}_3(E) + \mathrm{ch}_1(E)\,\mathrm{ch}_4(E)) + 5r^4\,\mathrm{ch}_5(E)\,.\end{aligned}$$

We have the following result.

Proposition A.2. *Let E be a vector bundle of rank r and let us consider the two modified classes*

$$\begin{aligned}\widetilde{\Delta}_4(E) &= (r+1)\Delta_4(E) - \Delta_2(E)^2\\ \widetilde{\Delta}_5(E) &= (r+5)\Delta_5(E) - 5\Delta_2(E)\Delta_3(E).\end{aligned}$$

Then $\widetilde{\Delta}_k(E) = 0$ for $r < k$ and $k = 4, 5$.

Proof. Once again, it is enough to expand the two classes in terms of the Chern classes. We have:

$$\begin{aligned}\widetilde{\Delta}_4 = &\frac{1}{6} r(-\mathrm{c}_1^4(-3+r)(-2+r)(-1+r) + 4\,\mathrm{c}_1^2\,\mathrm{c}_2(-3+r)(-2+r)r \\ &- 2\,\mathrm{c}_2^2(-3+r)(-2+r)r - 4\,\mathrm{c}_1\,\mathrm{c}_3(-3+r)r(1+r) + 4\,\mathrm{c}_4\,r^2(1+r))\end{aligned}$$

and

$$\begin{aligned}\widetilde{\Delta}_5 = &\frac{1}{24} r(\mathrm{c}_1^5(-4+r)(-3+r)(-2+r)(-1+r) - 5\,\mathrm{c}_1^3\,\mathrm{c}_2(-4+r)(-3+r)(-2+r)r \\ &+ 5\,\mathrm{c}_1^2\,\mathrm{c}_3(-4+r)(-3+r)r(2+r) + 5\,\mathrm{c}_1(-4+r)r(\mathrm{c}_2^2(-3+r)(-2+r) - \mathrm{c}_4\,r(5+r)) \\ &+ 5r^2(-\mathrm{c}_2\,\mathrm{c}_3(-4+r)(-3+r) + \mathrm{c}_5\,r(5+r))).\end{aligned}$$

The result then follows. □

One can of course try to generalise this to higher discriminants, but we could not find any clear pattern. It would in any case be very interesting to have an explicit expression for classes $\widetilde{\Delta}_k$, for any $k \geq 1$, with the property that $\widetilde{\Delta}_k(E) = 0$ for every vector bundle E of rank $r < k$.

APPENDIX B. EXPLICIT FORMULAS

In this appendix we list some explicit formulas for the computation of Chern classes of Schur bundles. Sections B.1 and B.2 follow from [18], while Section B.3 is a consequence of Theorem 4.2.

From now on E is a vector bundle of rank r, $\alpha = (\alpha_1, \dots, \alpha_r)$ a partition of size $|\alpha| = \alpha_1 + \cdots + \alpha_r$ and r_α the rank of the Schur bundle $\mathbf{S}^\alpha E$.

Finally, for simplicity, we use the shorthand

$$\mathrm{c}_1 := \mathrm{c}_1(E) \qquad \text{and} \qquad \mathrm{ch}_i := \mathrm{ch}_i(E).$$

B.1. **The symmetric bundle.**

The following is an expansion of [18, Theorem 4.8], see also (1.4).

$$\begin{aligned}
\mathrm{rk}(S^m E) &= r_m := \binom{m+r-1}{r-1} \\
\mathrm{c}_1(S^m E) &= m\frac{r_m}{r}\,\mathrm{c}_1 \\
\mathrm{ch}_2(S^m E) &= \frac{1}{2}\frac{(m-1)m}{r+1}\frac{r_m}{r}\,\mathrm{c}_1^2 + \frac{m(m+r)}{r+1}\frac{r_m}{r}\,\mathrm{ch}_2 \\
\mathrm{ch}_3(S^m E) &= \frac{1}{6}\frac{(m-2)(m-1)m}{(r+1)(r+2)}\frac{r_m}{r}\,\mathrm{c}_1^3 + \frac{(m-1)m(m+r)}{(r+1)(r+2)}\frac{r_m}{r}\,\mathrm{c}_1\,\mathrm{ch}_2 + \frac{m(m+r)(2m+r)}{(r+1)(r+2)}\frac{r_m}{r}\,\mathrm{ch}_3 \\
\Delta_2(S^m E) &= \frac{m(m+r)}{r+1}\left(\frac{r_m}{r}\right)^2 \Delta_2(E) \\
\Delta_3(S^m E) &= \frac{m(m+r)(2m+r)}{(r+1)(r+2)}\left(\frac{r_m}{r}\right)^3 \Delta_3(E)
\end{aligned}$$

B.2. **The exterior bundle.**

The following is an expansion of [18, Theorem 4.2].

$$
\begin{aligned}
\mathrm{rk}(\bigwedge^n E) &= r_n := \binom{r}{n} \\
\mathrm{c}_1(\bigwedge^n E) &= n\frac{r_n}{r}\,\mathrm{c}_1 \\
\mathrm{ch}_2(\bigwedge^n E) &= \frac{1}{2}\,\frac{(n-1)n}{r-1}\,\frac{r_n}{r}\,\mathrm{c}_1^2 + \frac{n(r-n)}{r-1}\,\frac{r_n}{r}\,\mathrm{ch}_2 \\
\mathrm{ch}_3(\bigwedge^n E) &= \frac{1}{6}\,\frac{(n-2)(n-1)n}{(r-2)(r-1)}\,\frac{r_n}{r}\,\mathrm{c}_1^3 + \frac{(n-1)n(r-n)}{(r-2)(r-1)}\,\frac{r_n}{r}\,\mathrm{c}_1\,\mathrm{ch}_2 + \frac{n(2n^2-3rn+r^2)}{(r-2)(r-1)}\,\frac{r_n}{r}\,\mathrm{ch}_3 \\
\Delta_2(\bigwedge^n E) &= \frac{n(r-n)}{r-1}\left(\frac{r_m}{r}\right)^2 \Delta_2(E) \\
\Delta_3(\bigwedge^n E) &= \frac{n(2n^2-3nr+r^2)}{(r-2)(r-1)}\left(\frac{r_m}{r}\right)^3 \Delta_3(E)
\end{aligned}
$$

B.3. **The Schur bundles.** The functions $\dot{\delta}_2(\alpha, r)$ and $\dot{\delta}_3(\alpha, r)$ are the same as defined in Theorem 3.1. For the sake of readability, let us introduce this notation:

$$
\tilde{\delta}_r^{(2)} := \frac{\dot{\delta}_r^{(2)}(\alpha)}{(r-1)(r+1)} \qquad \text{and} \qquad \tilde{\delta}_r^{(3)} := \frac{\dot{\delta}_r^{(3)}(\alpha)}{(r-2)(r-1)(r+1)(r+2)}.
$$

$$
\begin{aligned}
\mathrm{rk}(\mathbf{S}^\alpha E) &= r_\alpha := \prod_{1\le i<j\le r} \frac{\alpha_i - \alpha_j + j - i}{j-i} \\
\mathrm{c}_1(\mathbf{S}^\alpha E) &= |\alpha|\frac{r_\alpha}{r}\,\mathrm{c}_1 \\
\mathrm{ch}_2(\mathbf{S}^\alpha E) &= \frac{1}{2r}\left(|\alpha|^2 - \tilde{\delta}_r^{(2)}\right)\frac{r_\alpha}{r}\,\mathrm{c}_1^2 + \tilde{\delta}_r^{(2)}\frac{r_\alpha}{r}\,\mathrm{ch}_2 \\
\mathrm{ch}_3(\mathbf{S}^\alpha E) &= \frac{1}{6r^2}\left(|\alpha|^3 - 3|\alpha|\tilde{\delta}_r^{(2)} + 2\tilde{\delta}_r^{(3)}\right)\frac{r_\alpha}{r}\,\mathrm{c}_1^3 + \frac{1}{r}\left(|\alpha|\tilde{\delta}_r^{(2)} - \tilde{\delta}_r^{(3)}\right)\frac{r_\alpha}{r}\,\mathrm{c}_1\,\mathrm{ch}_2 + \tilde{\delta}_r^{(3)}\frac{r_\alpha}{r}\,\mathrm{ch}_3 \\
\Delta_2(\mathbf{S}^\alpha E) &= \tilde{\delta}_r^{(2)}\left(\frac{r_\alpha}{r}\right)^2 \Delta_2(E) \\
\Delta_3(\mathbf{S}^\alpha E) &= \tilde{\delta}_r^{(3)}\left(\frac{r_\alpha}{r}\right)^3 \Delta_3(E)
\end{aligned}
$$

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