THE GREEN FUNCTION FOR $p-$LAPLACE OPERATORS

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Abstract. On a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, we consider existence, uniqueness and "regularity" issues for the Green function G_λ of the quasi-linear operator $u \to -\Delta_p u - \lambda |u|^{p-2}u$ with $1 < p \leq N$, homogeneous Dirichlet boundary condition and $\lambda < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta_p$.

1. Introduction

Given $1 < p \leq N$ and a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, for $x_0 \in \Omega$ we are interested in nonnegative solutions G_λ of

$$-\Delta_p G - \lambda G^{p-1} = 0 \qquad \text{in } \Omega \setminus \{x_0\},$$

where $\Delta_p(\cdot) = \operatorname{div}\big(|\nabla(\cdot)|^{p-2}\nabla(\cdot)\big)$ is the p-Laplace operator and $\lambda < \lambda_1$. Here $G_\lambda \in W^{1,p}_{\text{loc}}(\Omega \setminus \{x_0\})$ and λ_1 is the first eigenvalue of $-\Delta_p$ given by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}.$$

When $\lambda = 0$, by elliptic regularity theory a nonnegative $p-$harmonic function G_0 in $\Omega \setminus \{x_0\}$ belongs to $C^{1,\alpha}_{\text{loc}}(\Omega \setminus \{x_0\})$ for some $\alpha \in (0,1)$ and, according to [32], behaves - if singular - like the fundamental solution

$$\Gamma(x) = \begin{cases} \frac{C_0}{|x-x_0|^{\frac{N-p}{p-1}}} & \text{if } 1 < p < N \\ -(N\omega_N)^{-\frac{1}{N-1}} \log|x-x_0| & \text{if } p = N \end{cases}$$

of $-\Delta_p \Gamma = \delta_{x_0}$ in $\mathbb{R}^N$, where $C_0 = \frac{p-1}{N-p}(N\omega_N)^{-\frac{1}{p-1}}$ and ω_N is the measure of the unit ball in $\mathbb{R}^N$. By a combination of scaling arguments and regularity estimates, Kichenassamy and Veron [24] showed that, in the singular situation, up to a re-normalization, G_0 is a solution of

$$-\Delta_p G = \delta_{x_0} \qquad \text{in } \Omega \tag{1.1}$$

and differs from Γ by a locally bounded function $H_0 = G_0 - \Gamma$ in Ω. Given $g \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, a solution $G_0 \in W^{1,p}_{\text{loc}}(\Omega \setminus \{x_0\}) \cap W^{1,p-1}(\Omega)$ to (1.1) with $G_0\Big|_{\partial\Omega} = g$ can be found in many different ways (see for example [24, 32]) and turns out to be unique thanks to the property $\nabla H_0 = o(|\nabla \Gamma|)$ as $x \to x_0$. As noticed in [24], the same approach via scaling arguments leads to a continuity property of H_0 at x_0.

The aim of the present paper is to establish the Hölder continuity of $H_\lambda = G_\lambda - \Gamma$ at x_0 when $\lambda = 0$ and to include the case $\lambda < \lambda_1$. Notice that such Hölder property is new already when $\lambda = 0$ and is relevant since Green's functions naturally arise in the description of concentration phenomena for quasi-linear PDE's, see for example [2], even if representation formulas are no-longer available in a quasi-linear context. Since the seminal works [26, 32, 33] in the sixties, the regularity theory for quasi-linear elliptic problems has been first refined in [18, 27] in the $p-$harmonic setting, see also [35], and then in [11, 28, 34] for general $p-$Laplace type equations. To treat the case of a Radon measure as right hand side, a general existence and uniqueness theory has been developed, both in the scalar and vectorial case, through different approaches: renormalized solutions, see for instance [9, 30]; entropy solutions or SOLA (solutions obtained as limit of approximations) in [1, 3, 4, 5]; in weak Lebesgue spaces [12, 13, 14]; in grand Sobolev spaces [21]. A powerful and general approach has also been developed through a potential theory in nonlinear form, see for example [23, 25] for an overview on old and recent achievements. Also in the simplest case $\lambda = 0$ the problem we are interested in does not fit into these general theories and a different approach, based on a new but rather simple idea, is necessary. The main point is to consider H_λ as a solution of

$$-\Delta_p(\Gamma + H_\lambda) + \Delta_p\Gamma = \lambda G_\lambda^{p-1} \qquad \text{in } \Omega \setminus \{x_0\} \tag{1.2}$$

for any $G_\lambda = \Gamma + H_\lambda$ solving (1.5) below and to apply the Moser iterative scheme in [32] to derive Hölder estimates on H_λ thanks to the coercivity of the difference operator, as expressed by the estimate

$$\inf_{X \neq Y} \frac{\langle |X+Y|^{p-2}(X+Y) - |X|^{p-2}X, Y\rangle}{(|X|+|Y|)^{p-2}|Y|^2} > 0. \tag{1.3}$$

When $p \geq 2$ gradient L^p-estimates on H_λ can be derived for the difference equation (1.2) as in the pure $p-$Laplace case and the only difficulty, when performing local estimates, comes from the failure of good upper estimates on $|\nabla\Gamma + \nabla H_\lambda|^{p-2}(\nabla\Gamma + \nabla H_\lambda) - |\nabla\Gamma|^{p-2}\nabla\Gamma$, caused by the singular behavior of $\nabla\Gamma$ at x_0. Since the inequality $(|X|+|Y|)^{p-2}|Y|^2 \geq \delta|Y|^p$, $\delta > 0$, is no longer true for $1 < p < 2$, one realizes that equation (1.2) differs from the pure $p-$Laplace case and weighted gradient L^2-estimates on H_λ are the natural ones one can hope for.

Let us first discuss the case $\lambda = 0$, which is the most relevant since it concerns the behavior of $p-$harmonic functions at isolated singularities. In the two-dimensional situation a very precise description has been provided in [29], whereas for $N \geq 2$ the only available result concerns the continuity of H_0 and has been given in [24], as already discussed. A special attention is paid here to avoid any restrictions on p and our first main result below improves in full generality what was previously known:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $x_0 \in \Omega$ and $1 < p \leq N$. The unique nonnegative solution G_0 to*

$$\begin{cases} -\Delta_p G = \delta_{x_0} & \text{in } \Omega \\ G = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies

$$\nabla(G_0 - \Gamma) \in L^{\bar q}(\Omega), \quad \bar q = \frac{N(p-1)}{N-1}, \tag{1.4}$$

and the regular part $H_0 = G_0 - \Gamma$ is Hölder continuous at x_0.

Let us stress that the integrability condition (1.4) can be improved into $\nabla H_0 \in L^p(\Omega)$ if $p \geq 2$. Since $\nabla\Gamma \in L^q(\Omega)$ for all $q < \bar q$, the exponent $\bar q$ represents the threshold gradient-integrability which distinguishes the singular situation from the non-singular one and the property (1.4) is crucial, when running the Moser iterative scheme, to use appropriate test functions $\Psi(H_\lambda)$ into (1.2) as the equation were valid in the whole Ω. The validity of higher regularity properties for H_0 represents a challenging open question in this context.

Let us now address the case $\lambda \neq 0$ and consider the problem

$$\begin{cases} -\Delta_p G - \lambda G^{p-1} = \delta_{x_0} & \text{in } \Omega \\ G \geq 0 & \text{in } \Omega \\ G = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Our second main result is the following:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $x_0 \in \Omega$ and $2 \leq p \leq N$. If $\lambda < \lambda_1$ with $\lambda \neq 0$, problem* (1.5) *has a solution G_λ with*

$$\nabla(G_\lambda - \Gamma) \in L^{\bar q}(\Omega), \quad \bar q = \frac{N(p-1)}{N-1}, \tag{1.6}$$

which is unique in the class of solutions satisfying (1.6)*. Moreover, the regular part $H_\lambda = G_\lambda - \Gamma$ is Hölder continuous at x_0 if $p > \frac{N}{2}$.*

Some comments are in order. While (1.4) is proved to be true for G_0, for $\lambda \neq 0$ we cannot guarantee the validity of (1.6) for any solution G_λ. However, since (1.6) is generally valid for all solutions obtained through an approximation scheme, assumption (1.6) in Theorem 1.2 is a rather natural request which - at the same time - allows us to show uniqueness of G_λ when $p \geq 2$ and Hölder continuity of H_λ when $p > \frac{N}{2}$. In view of $H_\lambda \in L^\infty(\Omega)$ and

$$\Gamma \in L^q(\Omega) \quad \text{for } 1 \leq q < \bar{q}^*,\ \bar{q}^* = \begin{cases} \frac{N(p-1)}{N-p} & \text{if } 1 < p < N \\ +\infty & \text{if } p = N, \end{cases}$$

notice that condition $p > \frac{N}{2}$ ensures $G_\lambda^{p-1} \in L^q(\Omega)$ for some $q > \frac{N}{p}$ in (1.2), a natural condition arising in [32] to prove $L^\infty-$bounds. In this respect, observe that also in the semilinear case $p = 2$ the function H_λ is no longer regular at x_0 when $2 = p \leq \frac{N}{2}$.

Let us emphasize that the main idea in the paper is to derive a-priori bounds on H_λ as a solution of the difference equation (1.2). These are achieved by the Moser iterative scheme and the main difficulty is the appearance of two qualitatively different terms in the estimates when using the coercivity property (1.3) in the iterative scheme. The results of the present paper are crucial in [2] to discuss existence results for a quasi-linear elliptic equation of critical Sobolev growth [6, 22] in the low-dimensional case as in [15, 16].

The paper is organized as follows. Section 2 is devoted to establish the existence part in Theorems 1.1 and 1.2 along with some $L^\infty-$estimates, while uniqueness issues are addressed in Section 3. Harnack inequalities and Hölder estimates for H_λ are established in Section 4. For ease of notation, we will just consider the case $x_0 = 0$.

2. Existence of Green's functions

Given $g \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, set $W_g^{1,q}(\Omega) = g + W_0^{1,q}(\Omega)$ for all $q \geq 1$ and consider

$$\lambda_{1,g} = \inf_{u \in W_g^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}.$$

Since the minimizer $\tilde{g}$ of $\int_\Omega |\nabla u|^p$ in $W_g^{1,p}(\Omega)$ is a $p-$harmonic function in Ω so that $\|\tilde{g}\|_\infty \leq \|g\|_\infty$, we assume that either $g = 0$ or $g \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ is a $p-$harmonic and non-constant function in Ω so to guarantee $\lambda_{1,g} > 0$.

For $g \geq 0$ and $\lambda < \lambda_{1,g}$ let us discuss the problem

$$\begin{cases} -\Delta_p G - \lambda G^{p-1} = \delta_0 & \text{in } \Omega \\ G \geq 0 & \text{in } \Omega \\ G = g & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

with

$$g \in L^\infty(\Omega) \cap W^{1,p}(\Omega)\ p-\text{harmonic in } \Omega,\ g \text{ non-constant unless } g = 0. \tag{2.2}$$

Solutions of (2.1) are found by an approximation procedure based either on removing small balls $B_\epsilon(0)$ when $\lambda = 0$ as in [24] or on approximating δ_0 by smooth functions when $\lambda \neq 0$ as in [1, 3, 4, 5]. We have the following existence result.

Theorem 2.1. *Let $1 < p \leq N$, $g \geq 0$ satisfying (2.2), $\lambda < \lambda_{1,g}$ and assume $p \geq 2$ only when $\lambda \neq 0$. Then there exists a solution G_λ of problem (2.1) so that $H_\lambda = G_\lambda - \Gamma$ satisfies (1.6). Moroever, there holds $H_\lambda \in L^\infty(\Omega)$ whenever either $\lambda = 0$ or $\lambda \neq 0$, $p > \frac{N}{2}$.*

Proof. Consider first the case $\lambda = 0$. We repeat the argument in [24] and the only point is to establish suitable bounds on $H_0 = G_0 - \Gamma$. Let G_ϵ be the $p-$harmonic function in $\Omega_\epsilon = \Omega \setminus B_\epsilon(0)$ so that $G_\epsilon = g$ on $\partial\Omega$ and $G_\epsilon = \Gamma$ on $\partial B_\epsilon(0)$. Since Γ is a positive $p-$harmonic function in $\Omega \setminus \{0\}$, by comparison principle we deduce that $G_\epsilon \geq 0$ and $|G_\epsilon - \Gamma| \leq C_0$ in Ω_ϵ, with $C_0 = \|g\|_\infty + \|\Gamma\|_{\infty,\partial\Omega}$. By elliptic estimates [18, 27, 35] for $p-$harmonic functions we deduce that G_ϵ is uniformly bounded in $C^{1,\alpha}_{\rm loc}(\Omega\setminus\{0\})$. By Ascoli-Arzelá's Theorem we can find a sequence $\epsilon_n\to 0$ so that $G_n:=G_{\epsilon_n}\to G_0$ in $C^1_{\rm loc}(\Omega\setminus\{0\})$ as $n\to+\infty$, where $G_0\geq 0$ is a $p-$harmonic function in $\Omega\setminus\{0\}$ so that

$$H_0=G_0-\Gamma\in L^\infty(\Omega). \tag{2.3}$$

Letting η be a cut-off function with $\eta=1$ near $\partial\Omega$ and $\eta=0$ near 0, use $\eta^p(G_\epsilon-g)\in W^{1,p}_0(\Omega_\epsilon)$ as a test function for $\Delta_p G_\epsilon=0$ in Ω_ϵ to get

$$\int_{\Omega_\epsilon}\eta^p\langle|\nabla G_\epsilon|^{p-2}\nabla G_\epsilon,\nabla(G_\epsilon-g)\rangle=-p\int_{\Omega_\epsilon}\eta^{p-1}(G_\epsilon-g)\langle|\nabla G_\epsilon|^{p-2}\nabla G_\epsilon,\nabla\eta\rangle\leq C \tag{2.4}$$

in view of $\nabla\eta=0$ near $\partial\Omega$ and 0. Since

$$\int_{\Omega_\epsilon}\eta^p|\nabla G_\epsilon|^{p-1}|\nabla g|\leq\frac{1}{2}\int_{\Omega_\epsilon}\eta^p|\nabla G_\epsilon|^p+C\int_{\Omega_\epsilon}\eta^p|\nabla g|^p$$

for some $C>0$ in view of the Young inequality, by (2.4) we deduce that G_ϵ is uniformly bounded in $W^{1,p}$ near $\partial\Omega$. Then $G_0=g$ on $\partial\Omega$ and G_0 solves (2.1) with $\lambda=0$ in view of (2.3) and [24, 32].

Moreover, use $(1-\eta)(G_\epsilon-\Gamma)\in W^{1,p}_0(\Omega_\epsilon)$ as a test function for $-\Delta_p G_\epsilon+\Delta_p\Gamma=0$ in Ω_ϵ to get

$$\int_{\Omega_\epsilon}(1-\eta)\langle|\nabla G_\epsilon|^{p-2}\nabla G_\epsilon-|\nabla\Gamma|^{p-2}\nabla\Gamma,\nabla(G_\epsilon-\Gamma)\rangle\leq C \tag{2.5}$$

in view of $\nabla\eta=0$ near $\partial\Omega$ and 0. By the coercivity estimate (1.3) and the uniform $W^{1,p}-$bound on G_ϵ and Γ away from 0 we deduce that (2.5) implies

$$\int_{\Omega_\epsilon}(|\nabla\Gamma|+|\nabla H_\epsilon|)^{p-2}|\nabla H_\epsilon|^2\leq C \tag{2.6}$$

for some uniform constant $C>0$, where $H_\epsilon=G_\epsilon-\Gamma$. When $p\geq 2$ estimate (2.6) implies

$$\nabla H_0\in L^p(\Omega)$$

thanks to the Fatou convergence Theorem along the sequence ϵ_n. For $1<p<2$ by (2.6) and the Hölder inequality we get

$$\begin{aligned}\int_{\Omega_\epsilon}|\nabla H_\epsilon|^{\bar q} &= \int_{\Omega_\epsilon}(|\nabla\Gamma|+|\nabla H_\epsilon|)^{\frac{(p-2)\bar q}{2}}|\nabla H_\epsilon|^{\bar q}(|\nabla\Gamma|+|\nabla H_\epsilon|)^{\frac{(2-p)\bar q}{2}}\leq C(\|\nabla\Gamma\|_{s,\Omega_\epsilon}^{\frac{(2-p)\bar q}{2}}+\|\nabla H_\epsilon\|_{s,\Omega_\epsilon}^{\frac{(2-p)\bar q}{2}})\\ &\leq C(\|\nabla\Gamma\|_{s,\Omega_\epsilon}^{\frac{(2-p)\bar q}{2}}+|\Omega_\epsilon|^{\frac{(2-p)(\bar q-s)}{2s}}\|\nabla H_\epsilon\|_{\bar q,\Omega_\epsilon}^{\frac{(2-p)\bar q}{2}})\end{aligned}$$

for some $C>0$ and $s=\frac{N(p-1)(2-p)}{3N-2-Np}$, thanks to $s<\bar q$ in view of $p<2\leq N$. By $\nabla\Gamma\in L^q(\Omega)$ for all $q<\bar q$ and the Young inequality we finally obtain $\int_{\Omega_\epsilon}|\nabla H_\epsilon|^{\bar q}\leq C$ for some uniform constant $C>0$ and then

$$\nabla H_0\in L^{\bar q}(\Omega)$$

does hold in the case $1<p<2$ thanks to the Fatou convergence Theorem.

Once the case $\lambda=0$ has been treated, assume $p\geq 2$ and follow the approach in [1, 3, 4, 5]. Notice that for $\lambda=0$ we provide below an efficient approximation scheme which is different from the previous one. Consider a sequence $0\leq f_n\in C_0^\infty(\Omega)$ so that $f_n\rightharpoonup\delta_0$ weakly in the sense of measures in Ω with $\sup_n\|f_n\|_1<+\infty$ and $f_n\to 0$ locally uniformly in $\Omega\setminus\{0\}$ as $n\to+\infty$. Since $\lambda<\lambda_{1,g}$ and $g,f_n\geq 0$, the minimization of

$$\frac{1}{p}\int_\Omega|\nabla u|^p-\frac{\lambda}{p}\int_\Omega|u|^p-\int_\Omega f_n u,\quad u\in W^{1,p}_g(\Omega),$$

provides a nonnegative solution $G_n\in W^{1,p}_g(\Omega)$ to

$$-\Delta_p G_n-\lambda G_n^{p-1}=f_n\quad\text{ in }\Omega. \tag{2.7}$$

We use here Lemmas 2.2 and 2.3 below to show first that G_n^{p-1} is uniformly bounded in $L^1(\Omega)$ and then, up to a subsequence, $G_n \to G_\lambda$ in $W^{1,q}_g(\Omega)$ as $n \to +\infty$ for some G_λ and for all $1 \le q < \bar q$. By the Sobolev embedding Theorem we have that $G_n \to G_\lambda$ in $L^q(\Omega)$ as $n \to +\infty$ for all $1 \le q < \bar q^*$ and in particular in $L^{p-1}(\Omega)$. Therefore one can pass to the limit in (2.7) and get that $G_\lambda \ge 0$ solves (2.1) in view of $\bar q > p-1$.

In order to establish suitable bounds on $H_\lambda = G_\lambda - \Gamma$, let $0 \le \tilde G_n \in W^{1,p}_g(\Omega)$ be the solution of

$$-\Delta_p \tilde G_n = f_n \quad \text{ in } \Omega,$$

obtained as a minimizer of $\frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega f_n u$ in $W^{1,p}_g(\Omega)$ in view of $\lambda_{1,g} > 0$. Arguing as for (2.7), we deduce that, up to a subsequence, $\tilde G_n \to \tilde G$ in $W^{1,q}_g(\Omega)$ as $n \to +\infty$ for all $1 \le q < \bar q$, where $\tilde G \ge 0$ solves $-\Delta_p \tilde G = \delta_0$ in Ω. By [32] and the uniqueness result in [24] we have that $\tilde G = G_0$ and $\tilde H = \tilde G - \Gamma = H_0$. Since $-\Delta_p G_n + \Delta_p \tilde G_n = \lambda G_n^{p-1}$ in Ω with $G_n = \tilde G_n$ on $\partial\Omega$, by Lemma 2.3 we deduce that $\sup_n \|\nabla(G_n - \tilde G_n)\|_{\bar q} < +\infty$ in view of $\sup_n \|G_n^{p-1}\|_m < +\infty$ for all $1 \le m < \frac{\bar q^*}{p-1}$. Since $\nabla(G_n - \tilde G_n) \to \nabla(H_\lambda - H_0)$ a.e. in Ω as $n \to +\infty$ and ∇H_0 satisfies (1.4), by the Fatou convergence Theorem we obtain that ∇H_λ satisfies (1.6). If either $\lambda = 0$ or $\lambda \ne 0$, $p > \frac{N}{2}$ a $L^\infty-$bound on H_λ follows by Theorem 2.6 below and the proof is complete. □

The following result has been crucially used in the proof of Theorem 2.1 and in its proof we closely follow a tricky idea in [31] combined with some apriori estimates given in Lemma 2.3 below.

Lemma 2.2. *Let $2 \le p \le N$. Assume that $a_n \in L^\infty(\Omega)$, $f_n \in L^1(\Omega)$, g_n satisfy (2.2) and*

$$\lim_{n\to+\infty} \|a_n - a\|_\infty = 0, \quad \sup_\Omega a < \lambda_1, \quad \sup_{n\in\mathbb{N}} [\|f_n\|_1 + \|g_n\|_\infty] < +\infty. \tag{2.8}$$

If $u_n \in W^{1,p}_{g_n}(\Omega)$ is a sequence of solutions to

$$-\Delta_p u_n - a_n|u_n|^{p-2}u_n = f_n \quad \text{ in } \Omega,$$

then $\sup_{n\in\mathbb{N}} \|u_n\|_{p-1} < +\infty$.

Proof. Assume by contradiction that

$$\|u_n\|_{p-1} \to +\infty \qquad \text{as } n \to +\infty. \tag{2.9}$$

Setting $\hat u_n = \frac{u_n}{\|u_n\|_{p-1}}$, $\hat f_n = \frac{f_n}{\|u_n\|_{p-1}^{p-1}}$ and $\hat g_n = \frac{g_n}{\|u_n\|_{p-1}}$, we have that $\hat u_n$ solves

$$\begin{cases} -\Delta_p \hat u_n - a_n|\hat u_n|^{p-2}\hat u_n = \hat f_n & \text{in } \Omega \\ \hat u_n = \hat g_n & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

with

$$\|\hat u_n\|_{p-1} = 1,\ \sup_{n\in\mathbb{N}} \|a_n\|_\infty < \infty,\ \|\hat f_n\|_{L^1(\Omega)} + \|\hat g_n\|_\infty \to 0 \text{ as } n \to +\infty \tag{2.11}$$

in view of (2.8)-(2.9). Fix $p-1 < p_0 < \bar q$ and define $p_j = \frac{N^2(p-1)p_{j-1}}{(N+1)[N(p-1)-p_{j-1}]}$ in a recursive way for $j \ge 1$. Notice that $\frac{N(p-1)}{N+1} < p_j < p_{j+1}$ by induction and there exists a unique $J \ge 0$ so that $p_0, \dots, p_{J-1} \le \frac{Np(p-1)}{Np-N+p} < p_J$. Since $\Delta_p \hat g_n = 0$ in Ω, by Lemma 2.3 with $m = 1$ we get that $\hat u_n - \hat g_n$ is uniformly bounded in $W^{1,q}_0(\Omega)$ for all $1 \le q < \bar q$ in view of (2.10)-(2.11) and then, up to a subsequence, $\hat u_n - \hat g_n \rightharpoonup v^0$ in $W^{1,p_0}(\Omega)$ as $n \to +\infty$. Define $v_n^0 = \hat u_n$ and $v_n^j \in W^{1,p}_{\hat g_n}$ as the solution of $-\Delta_p v_n^j = a_n|v_n^{j-1}|^{p-2}v_n^{j-1}$ in Ω in view of $\lambda_{1,\hat g_n} = \lambda_{1,g_n} > 0$. Lemma 2.3, applied to $v_n^1 - \hat g_n$ with $m = \frac{p_0}{p-1} \le \frac{Np}{Np-N+p}$, $q = \frac{N}{N+1}\frac{mN(p-1)}{N-m}$ and to $v_n^1 - v_n^0$ with $m = 1$, $q = p_0$ in view of (2.10)-(2.11), provides that, up to a subsequence, $v_n^1 - \hat g_n \rightharpoonup v^1$ in $W^{1,p_1}_0(\Omega)$ and $v_n^1 - v_n^0 \to 0$ in $W^{1,p_0}_0(\Omega)$ as $n \to +\infty$. By iterating we deduce that, up to a subsequence, $v_n^j - \hat g_n \rightharpoonup v^j$ in $W^{1,p_j}_0(\Omega)$ and $v_n^j - v_n^{j-1} \to 0$ in $W^{1,p_{j-1}}_0(\Omega)$ as $n \to +\infty$ for all $j = 1, \dots, J$. Since $a_n|v_n^J|^{p-2}v_n^J$ is uniformly bounded in $L^m(\Omega)$

with $m=\frac{p_J}{p-1}>\frac{Np}{Np-N+p}$, by Lemma 2.3 we deduce that, up to a subsequence, $v_n^{J+1}-\hat g_n\rightharpoonup v^{J+1}$ in $W_0^{1,p}(\Omega)$ as $n\to+\infty$. At the same time, by Lemma 2.3 $v_n^{J+1}-v_n^J\to 0$ in $W_0^{1,p_J}(\Omega)$ as $n\to+\infty$. Since $v_n^j-v_n^{j-1}\to 0$ in $W_0^{1,p_0}(\Omega)$ and $v_n^j-v_n^{j-1}=(v_n^j-\hat g_n)-(v_n^{j-1}-\hat g_n)\rightharpoonup v^j-v^{j-1}$ weakly in $W_0^{1,p_0}(\Omega)$ as $n\to+\infty$ for all $j=1,\dots,J+1$, we deduce that $v^0=\ldots=v^{J+1}$ and then $\hat u_n-\hat g_n\rightharpoonup v^0$ in $W_0^{1,p_0}(\Omega)$ as $n\to+\infty$ with $v^0=v^{J+1}\in W_0^{1,p}(\Omega)$.

Let us compare $\hat u_n$ with $z_n\in W_0^{1,p}(\Omega)$, solution to

$$-\Delta_p z_n=a_n|\hat u_n|^{p-2}\hat u_n+\hat f_n \quad \text{in } \Omega. \tag{2.12}$$

Since $|\hat u_n-z_n|\le\|\hat g_n\|_\infty$ on $\partial\Omega$, by the weak maximum principle we deduce that $\|\hat u_n-z_n\|_\infty\le\|\hat g_n\|_\infty$. By (2.11)-(2.12) and Lemma 2.3 we deduce that, up to a subsequence and for some z^0, there holds

$$z_n\to z^0 \quad \text{in } W_0^{1,q}(\Omega),\ 1\le q<\bar q. \tag{2.13}$$

By testing $-\Delta_p\hat u_n+\Delta_p z_n=0$ in Ω against $\eta^p(\hat u_n-z_n)$, $0\le\eta\in C_0^\infty(\Omega)$, one gets

$$\begin{aligned}\int_\Omega \eta^p|\nabla(\hat u_n-z_n)|^p &\le C'\int_\Omega \eta^{p-1}|\nabla\eta|(|\nabla(\hat u_n-z_n)|^{p-2}+|\nabla z_n|^{p-2})|\nabla(\hat u_n-z_n)||\hat u_n-z_n|\\ &\le \frac12\int_\Omega \eta^p|\nabla(\hat u_n-z_n)|^p+C\left(\|\hat g_n\|_\infty^p+\|\hat g_n\|_\infty^{\frac{p}{p-1}}\|\nabla z_n\|^{p-2}_{\frac{p(p-2)}{p-1}}\right)\to 0\end{aligned}$$

as $n\to+\infty$ in view of the Young's inequality and (2.11). We have used that $\sup_n\|\nabla z_n\|_{\frac{p(p-2)}{p-1}}<+\infty$ thanks to (2.13) and $\frac{p(p-2)}{p-1}<\bar q$. Since $\nabla(\hat u_n-z_n)\to 0$ locally in L^p-norm as $n\to+\infty$, by (2.13) we deduce that

$$\hat u_n\to v^0 \quad \text{in } L^{p-1}(\Omega) \text{ and } W^{1,q}(\Omega'),\quad \forall\ \Omega'\subset\subset\Omega,\ \ \forall\ 1\le q<\bar q, \tag{2.14}$$

in view of $\|\hat u_n-z_n\|_\infty\le\|\hat g_n\|_\infty\to 0$ and $\hat u_n-\hat g_n\rightharpoonup v^0$ in $W_0^{1,p_0}(\Omega)$ as $n\to+\infty$ for $p_0\ge p-1$.

By (2.10) and (2.14) we have that $v^0\in W_0^{1,p}(\Omega)$ solves

$$-\Delta_p v^0-a|v^0|^{p-2}v^0=0 \qquad \text{in } \Omega \tag{2.15}$$

in view of (2.8) and (2.11). Since

$$\int_\Omega|\nabla v^0|^p-\int_\Omega a|v^0|^p=0$$

by integration of (2.15) against $v^0\in W_0^{1,p}(\Omega)$, by $\sup_\Omega a<\lambda_1$ one finally deduces that $v^0=0$ and then $\hat u_n\to 0$ in $L^{p-1}(\Omega)$, in contradiction with $\|\hat u_n\|_{p-1}=1$. □

The results in [1, 4, 5], valid for homogeneous boundary values, can be easily extended to non-homogeneous ones when $p\ge 2$, as discussed for instance in the Appendix of [1] when $p=N$. For the sake of completeness, we reproduce here a relevant Lemma in the following simplest form, sufficient for our purposes:

Lemma 2.3. *Let $2\le p\le N$. Assume $\|f_1-f_2\|_m\le C_0$ for some $C_0>0$ and either $1\le m\le\frac{Np}{Np-N+p}$, $1\le q<\frac{mN(p-1)}{N-m}$ or $m>\frac{Np}{Np-N+p}$, $1\le q\le p$. Then there exists $C>0$ so that $\|\nabla(u_1-u_2)\|_q\le C\|f_1-f_2\|_m^{\frac1p}$ for all solutions $u_1,u_2\in W^{1,p}(\Omega)$ of $-\Delta_p u_i=f_i$, $i=1,2$, in Ω with $u_1=u_2$ on $\partial\Omega$. Moreover, given g satisying (2.2) the set of solutions $u\in W_g^{1,p}(\Omega)$ of $-\Delta_p u=f$ in Ω with $\|f\|_1\le C_0$ is relatively compact in $W^{1,q}(\Omega)$ for all $1\le q<\bar q$.*

Proof. Let $u_1,u_2\in W^{1,p}(\Omega)$ be solutions of $-\Delta_p u_i=f_i$, $i=1,2$, in Ω with $u_1=u_2$ on $\partial\Omega$. Take $T_{k,l}$, $0\le k\le l$, as the odd function so that

$$T_{k,l}(s)=\min\{\max\{s-k,0\},l-k\} \quad \text{in } [0,+\infty) \tag{2.16}$$

and use $T_{k,k+1}(u_1-u_2)$ as a test function to get

$$\int_{\{k\le |u_1-u_2|<k+1\}} \langle |\nabla u_1|^{p-2}\nabla u_1-|\nabla u_2|^{p-2}\nabla u_2,\nabla(u_1-u_2)\rangle=\int_\Omega (f_1-f_2)T_{k,k+1}(u_1-u_2),$$

which implies

$$\int_{\{k\le |u_1-u_2|<k+1\}} |\nabla(u_1-u_2)|^p\le C\|f_1-f_2\|_m |\{|u_1-u_2|\ge k\}|^{\frac{m-1}{m}} \tag{2.17}$$

in view of (1.3) and $p\ge 2$. By (2.17) the function $v=u_1-u_2\in W_0^{1,p}(\Omega)$ satisfies

$$\int_{B_k} |\nabla v|^p\le c_0|E_k|^{\frac{m-1}{m}},\ k\ge 0, \tag{2.18}$$

with $c_0=C\|f_1-f_2\|_m$, where $E_k=\{|v|\ge k\}$ and $B_k=E_k\setminus E_{k+1}$.

Consider first the case $1\le m\le \frac{Np}{Np-N+p}$, $1\le q<\frac{mN(p-1)}{N-m}$ and set $q^*=\frac{Nq}{N-q}$. Since $q<\frac{mN(p-1)}{N-m}\le p$ thanks to $m\le \frac{Np}{Np-N+p}$ and

$$\int_{B_k}|\nabla v|^q\le (\int_{B_k}|\nabla v|^p)^{\frac{q}{p}}|B_k|^{\frac{p-q}{p}} \tag{2.19}$$

in view of the Hölder inequality, by (2.18) we obtain that

$$\int_{B_k}|\nabla v|^q\le c_0^{\frac{q}{p}}\|v\|_{q^*}^{\frac{qq^*(m-1)}{pm}}(\int_{B_k}|v|^{q^*})^{\frac{p-q}{p}}\frac{1}{k^{\frac{q^*(pm-q)}{pm}}}$$

for all $k\ge 1$ thanks to

$$|B_k|\le k^{-q^*}\int_{B_k}|v|^{q^*},\quad |E_k|\le k^{-q^*}\int_\Omega |v|^{q^*}.$$

Summing up, still by Hölder's inequality one deduces

$$\int_{\{|v|\ge k_0\}}|\nabla v|^q\le c_0^{\frac{q}{p}}\|v\|_{q^*}^{\frac{qq^*(m-1)}{pm}}(\sum_{k=k_0}^\infty\int_{B_k}|v|^{q^*})^{\frac{p-q}{p}}(\sum_{k=k_0}^\infty \frac{1}{k^{\frac{q^*(pm-q)}{mq}}})^{\frac{q}{p}}$$

and then

$$\int_\Omega|\nabla v|^q\le k_0c_0^{\frac{q}{p}}|\Omega|^{\frac{pm-q}{pm}}+c_0^{\frac{q}{p}}\|v\|_{q^*}^{\frac{q^*(pm-q)}{pm}}(\sum_{k=k_0}^\infty \frac{1}{k^{\frac{q^*(pm-q)}{mq}}})^{\frac{q}{p}} \tag{2.20}$$

for a given $k_0\in\mathbb{N}$ in view of (2.18)-(2.19) for $k=0,\dots,k_0-1$. Since $\frac{q^*(pm-q)}{pm}\le q$, by Young's inequality (2.20) implies in turn that

$$\int_\Omega|\nabla v|^q\le k_0c_0^{\frac{q}{p}}|\Omega|^{\frac{pm-q}{pm}}+Cc_0^{\frac{q}{p}}(\|v\|_{q^*}^q+1)(\sum_{k=k_0}^\infty \frac{1}{k^{\frac{q^*(pm-q)}{mq}}})^{\frac{q}{p}}. \tag{2.21}$$

Since $\frac{q^*(pm-q)}{mq}>1$ thanks to $q<\frac{mN(p-1)}{N-m}$, the series in (2.21) is convergent and we can choose k_0 sufficienty large (depending on C_0) so that $\|v\|_{q^*}\le C'c_0^{\frac{1}{p}}$ and then $\|\nabla v\|_q\le Cc_0^{\frac{1}{p}}$ in view of the Sobolev embedding Theorem, where the last estimate gets rewritten as

$$\|\nabla(u_1-u_2)\|_q\le C\|f_1-f_2\|_m^{\frac{1}{p}}. \tag{2.22}$$

Consider now the case $m>\frac{Np}{Np-N+p}$, $1\le q\le p$. Use u_1-u_2 as a test function to get

$$\|\nabla(u_1-u_2)\|_p^p\le C\|u_1-u_2\|_{\frac{m}{m-1}}\|f_1-f_2\|_m$$

in view of the Hölder inequality and then $\|\nabla(u_1-u_2)\|_p\le C\|f_1-f_2\|_m^{\frac{1}{p-1}}$ by the Sobolev embedding Theorem in view of $\frac{m}{m-1}<p^*$. Notice that such last argument works as well as $m=\frac{Np}{Np-N+p}$ for $p<N$ since $\frac{Np}{Np-N+p}>1$ in this case.

Fix now $m=1$ and let $u_1,u_2\in W^{1,p}_g(\Omega)$ be solutions of $-\Delta_p u_i=f_i$, $i=1,2$, in Ω with $\|f_i\|_1\le C_0$. Use $T_{0,\epsilon}(u_1-u_2)$, $T_{k,l}$ given by (2.16), as a test function to get

$$\int_{\{|u_1-u_2|\le\epsilon\}}|\nabla(u_1-u_2)|^p\le C\epsilon\|f_1-f_2\|_1\le 2CC_0\epsilon \tag{2.23}$$

in view of (1.3) and $p\ge 2$. Given $1\le q<\bar q$, by (2.22) and Hölder's inequality (2.23) implies

$$\begin{aligned}\int_\Omega|\nabla(u_1-u_2)|^q &\le C'\epsilon^{\frac{q}{p}}+\Big(\int_{\{|u_1-u_2|>\epsilon\}}|\nabla(u_1-u_2)|^s\Big)^{\frac{q}{s}}|\{|u_1-u_2|>\epsilon\}|^{\frac{s-q}{s}}\\ &\le C(\epsilon^{\frac{q}{p}}+\{|u_1-u_2|>\epsilon\}|^{\frac{s-q}{s}})\end{aligned} \tag{2.24}$$

for some $q<s<\bar q$ in view of $\bar q<p$. Since g is $p-$harmonic in Ω, taking now a sequence of solutions $u_n\in W^{1,p}_g(\Omega)$ to $-\Delta_p u_n=f_n$ in Ω with $\sup_n\|f_n\|_1<+\infty$, by the first part we know that u_n-g is bounded in $W^{1,q}_0(\Omega)$ and then, up to a subsequence, we have that $u_n\rightharpoonup u$ in $W^{1,q}_g(\Omega)$ for all $1\le q<\bar q$ and strongly in $L^s(\Omega)$ for all $1\le s<\bar q^*$. Applying (2.24) to u_n-u_m it is easily seen that u_n is a Cauchy sequence in $W^{1,q}_g(\Omega)$ and then converges to u in $W^{1,q}_g(\Omega)$ for all $1\le q<\bar q$. The proof is complete. □

Let us push further the analysis in Lemma 2.2 towards an L^∞-estimate when $p>\frac N2$.

Proposition 2.4. *Let $2\le p\le N$ with $p>\frac N2$ and $M>0$. Then there exists $C>0$ so that $\|u_1-u_2\|_\infty\le C$ for any pair $u_i\in W^{1,p}_{g_i}(\Omega)$, $i=1,2$, of solutions to*

$$-\Delta_p u_i-\lambda^i|u_i|^{p-2}u_i=f\quad\text{in }\Omega, \tag{2.25}$$

where $\|f\|_1+\sup_{i=1,2}\left[\frac{1}{(\lambda_1-\lambda^i)_+}+\|g_i\|_\infty\right]\le M$ *and g_1, g_2 satisfy* (2.2).

Proof. By Lemma 2.2 we get an universal bound on $\|f+\lambda^i|u_i|^{p-2}u_i\|_1$. Since g_i is $p-$harmonic function in Ω, Lemma 2.3 and the Sobolev embedding Theorem provide an universal bound on u_i-g_i in $W^{1,q}_0(\Omega)$ for all $1\le q<\bar q$ and u_i in $L^q(\Omega)$ for all $1\le q<\bar q^*$. Since $\frac{\bar q^*}{p-1}>\frac Np$ thanks to $p>\frac N2$, we can find $q_0>\frac Np$ so that $\hat f=\lambda^1|u_1|^{p-2}u_1-\lambda^2|u_2|^{p-2}u_2$ satisfies

$$\|\hat f\|_{q_0}\le C \tag{2.26}$$

for some universal $C>0$. Thanks to (2.25) we can write

$$\begin{cases}-\Delta_p u_1+\Delta_p u_2=\hat f & \text{in }\Omega\\ u_1-u_2=g_1-g_2 & \text{on }\partial\Omega.\end{cases} \tag{2.27}$$

Since $q_0>\frac Np$ let us fix $\beta_0>0$ sufficiently small so that $p_0:=\frac{q_0(\beta_0-1+p)}{q_0-1}<\bar q^*$. Set $u=u_1-u_2$, $C_0=\|g_1\|_\infty+\|g_2\|_\infty$ and define $\Psi(s)=[T_{0,l}(s\mp C_0)_\pm+\epsilon]^\beta-\epsilon^\beta$, with $l,\epsilon>0$ and $\beta\ge\beta_0$, where $T_{k,l}$ is given by (2.16). Notice that $l<+\infty$ and $\epsilon>0$ guarantee the boundedness and the differentiability of Ψ in $\mathbb R$, respectively. Use $\Psi(u)\in W^{1,p}_0(\Omega)$ as a test function in (2.27) to get

$$\beta\int_{\{(u\mp C_0)_\pm\le l\}}[T_{0,l}(u\mp C_0)_\pm+\epsilon]^{\beta-1}(|\nabla u_2|+|\nabla u|)^{p-2}|\nabla u|^2\le C\int_\Omega|\hat f|[T_{0,l}(u\mp C_0)_\pm+\epsilon]^\beta \tag{2.28}$$

in view of (1.3). Since $p\ge 2$, by Hölder's inequality with exponents $\frac{q_0(\beta-1+p)}{(q_0-1)(p-1)}$, q_0 and $\frac{q_0(\beta-1+p)}{(q_0-1)\beta}$ estimate (2.28) implies the following estimate:

$$\frac{\delta p^p\beta}{(\beta-1+p)^p}\int_\Omega|\nabla w_{l,\epsilon}|^p\le|\Omega|^{\frac{(q_0-1)(p-1)}{q_0(\beta-1+p)}}\||\hat f\|_{q_0}\|w_{l,\epsilon}\|^{\frac{\beta p}{\beta-1+p}}_{\frac{pq_0}{q_0-1}}\le C\|w_\epsilon\|^{\frac{\beta p}{\beta-1+p}}_{\frac{pq_0}{q_0-1}}$$

for some $C>0$, where

$$w_{l,\epsilon}=[T_{0,l}(u\mp C_0)_\pm+\epsilon]^{\frac{\beta-1+p}{p}},\quad w_\epsilon=[(u\mp C_0)_\pm+\epsilon]^{\frac{\beta-1+p}{p}},\quad w=(u\mp C_0)_\pm^{\frac{\beta-1+p}{p}}.$$

By the Sobolev embedding Theorem on $w_{l,\epsilon}-\epsilon^{\frac{\beta-1+p}{p}}\in W_0^{1,p}(\Omega)$ and the Fatou convergence Theorem as $l\to+\infty$ we deduce that

$$\|w_\epsilon-\epsilon^{\frac{\beta-1+p}{p}}\|_{p^*}\le C(\beta-1+p)\|w_\epsilon\|_{\frac{pq_0}{q_0-1}}^{\frac{\beta}{\beta-1+p}} \tag{2.29}$$

for some $C>0$ provided the R.H.S. is finite, where $p^*=\frac{Np}{N-p}$ if $p<N$ and $p^*\in(\frac{pq_0}{q_0-1},+\infty)$ if $p=N$. By using again the Fatou convergence Theorem on the L.H.S. and the Lebesgue convergence Theorem on the R.H.S. in (2.29), as $\epsilon\to0$ we deduce that

$$\|w\|_{p^*}\le C(\beta-1+p)\|w\|_{\frac{pq_0}{q_0-1}}^{\frac{\beta}{\beta-1+p}}$$

for some $C>0$, provided $\|w\|_{\frac{pq_0}{q_0-1}}<+\infty$. By the definition of w and taking the $\frac{p}{\beta-1+p}-$power we then deduce that

$$\|(u\mp C_0)_\pm\|_{\frac{(\beta-1+p)p^*}{p}}\le[C(\beta-1+p)]^{\frac{p}{\beta-1+p}}\|(u\mp C_0)_\pm\|_{\frac{q_0(\beta-1+p)}{q_0-1}}^{\frac{\beta}{\beta-1+p}},$$

or equivalently

$$\|(u\mp C_0)_\pm\|_{\kappa\mu}\le[C\frac{q_0-1}{q_0}\mu]^{\frac{pq_0}{\mu(q_0-1)}}\|(u\mp C_0)_\pm\|_\mu^{1-\frac{(p-1)q_0}{\mu(q_0-1)}}, \tag{2.30}$$

where $\mu=\frac{q_0(\beta-1+p)}{q_0-1}$ and $\kappa=\frac{(q_0-1)p^*}{pq_0}>1$ in view of $q_0>\frac{N}{p}$. Setting $\mu_j=\kappa^jp_0$, we can perform $j+1$ iterations of (2.30) to get

$$\begin{aligned}\|(u\mp C_0)_\pm\|_{\mu_{j+1}}&\le[C(\beta_0-1+p)\kappa^j]^{\frac{p}{(\beta_0-1+p)\kappa^j}}\|(u\mp C_0)_\pm\|_{\mu_j}^{1-\frac{p-1}{(\beta_0-1+p)\kappa^j}}\le\dots\\&\le[C(\beta_0-1+p)+1]^{\sum\limits_{s=0}^{j}\frac{1}{\kappa^s}}\kappa^{\sum\limits_{s=0}^{j}\frac{s}{\kappa^s}}\|(u\mp C_0)_\pm\|_{p_0}^{\prod\limits_{s=0}^{j}(1-\frac{p-1}{(\beta_0-1+p)\kappa^s})}\end{aligned}$$

in view of $[C(\beta_0-1+p)+1]\kappa^j\ge1$ and $1-\frac{p-1}{(\beta_0-1+p)\kappa^s}\le1$. By letting $j\to+\infty$ we deduce that

$$\|(u\mp C_0)_\pm\|_\infty\le C'\|(u\mp C_0)_\pm\|_{p_0}^{\theta_0}\le C'_M$$

in view of

$$\theta_0:=\prod_{s=0}^{\infty}(1-\frac{p-1}{(\beta_0-1+p)\kappa^s})<+\infty,\quad\sum_{s=0}^{\infty}\frac{1}{\kappa^s}+\sum_{s=0}^{\infty}\frac{s}{\kappa^s}<+\infty.$$

In conclusion, $\|u_1-u_2\|_\infty\le C'_M+C_0\le C_M$ and the proof is complete. □

The aim now is to extend Proposition 2.4 to H_λ as a solution of (1.2) (to be compared with (2.27)) and to include the case $1<p<2$. Since it is no longer a matter of universal estimates, the argument is potentially simpler but the singular character of equation (1.2) has to be controlled thanks to the assumption $\nabla H_\lambda\in L^{\bar q}(\Omega)$. For later convenience, let us write the following result in a sufficiently general way.

Lemma 2.5. *Let $1<p\le N$ and $u\in W^{1,p}_{loc}(\Omega\setminus\{0\})$ be a solution of*

$$-\Delta_p(\mathit{\Gamma}+u)+\Delta_p\mathit{\Gamma}=f\quad\text{in }\Omega\setminus\{0\} \tag{2.31}$$

with $f\in L^1(\Omega)$, $\nabla u\in L^{\bar q}(\Omega)$ and

$$\begin{aligned}\frac{1}{C}|\nabla\Gamma|\le|\nabla\mathit{\Gamma}|\le C|\nabla\Gamma|&\quad\text{if } 1<p<2\\|\nabla\mathit{\Gamma}|\le C|\nabla\Gamma|&\quad\text{if } p\ge2\end{aligned} \tag{2.32}$$

in Ω *for some* $C>1$. *Let* $\eta \in C^1(\bar\Omega)$ *and* $\Psi\colon \mathbb{R}\to\mathbb{R}$ *be a bounded monotone Lipschitz function. Assuming either* $\eta=0$ *or* $\Psi(u)=0$ *on* $\partial\Omega$, *then there holds*

$$\int_\Omega \eta^2|\Psi'(u)|(|\nabla\mathit{\Gamma}|+|\nabla u|)^{p-2}|\nabla u|^2 \le C\Big(\int_\Omega |\eta||\nabla\eta||\Psi(u)|(|\nabla\mathit{\Gamma}|+|\nabla u|)^{p-2}|\nabla u| + \int_\Omega \eta^2|f||\Psi(u)|\Big)$$

for some $C>0$.

Proof. Consider a sequence $\eta_\epsilon\in C^1(\bar\Omega)$ so that

$$\eta_\epsilon=\eta \text{ in } \Omega\setminus B_\epsilon(0),\quad \eta_\epsilon=0 \text{ in } B_{\frac{\epsilon}{2}}(0),\quad |\eta_\epsilon|+\epsilon|\nabla\eta_\epsilon|\le C \text{ in } B_\epsilon(0)\setminus B_{\frac{\epsilon}{2}}(0) \tag{2.33}$$

for some $C>0$. Since $\eta_\epsilon^2\Psi(u)$ vanishes in $B_{\frac{\epsilon}{2}}(0)$ and on $\partial\Omega$, it can be used a test function in (2.31):

$$\int_\Omega \eta_\epsilon^2|\Psi'(u)|(|\nabla\mathit{\Gamma}|+|\nabla u|)^{p-2}|\nabla u|^2 \le C\int_\Omega \Big[|\eta_\epsilon||\nabla\eta_\epsilon|(|\nabla\mathit{\Gamma}|+|\nabla u|)^{p-2}|\nabla u| + \eta_\epsilon^2|f|\Big]|\Psi(u)| \tag{2.34}$$

for some $C>0$ since Ψ' has given sign. We have used here (1.3) and the estimate

$$\Big||x+y|^{p-2}(x+y)-|x|^{p-2}x\Big| = (|x|+|y|)^{p-2}O(|y|).$$

Since $(|\nabla\mathit{\Gamma}|+|\nabla u|)^{p-2}=O(|\nabla\Gamma|^{p-2}+|\nabla u|^{p-2})$ in view of (2.32), by the Hölder inequality we have that

$$\begin{aligned}\int_{B_\epsilon(0)\setminus B_{\frac{\epsilon}{2}}(0)} |\eta_\epsilon||\nabla\eta_\epsilon||\Psi(u)|(|\nabla\mathit{\Gamma}|+|\nabla u|)^{p-2}|\nabla u| &\le C\int_{B_\epsilon(0)\setminus B_{\frac{\epsilon}{2}}(0)} \Big(\frac{|\nabla u|}{\epsilon^{\frac{N(p-1)-(N-1)}{p-1}}}+\frac{|\nabla u|^{p-1}}{\epsilon}\Big)\\ &\le C\left[\Big(\int_{B_\epsilon(0)\setminus B_{\frac{\epsilon}{2}}(0)}|\nabla u|^{\bar q}\Big)^{\frac{1}{\bar q}}+\Big(\int_{B_\epsilon(0)\setminus B_{\frac{\epsilon}{2}}(0)}|\nabla u|^{\bar q}\Big)^{\frac{N-1}{N}}\right]\to 0 \end{aligned}\tag{2.35}$$

as $\epsilon\to 0$, in view of $\|\Psi\|_\infty<+\infty$ and $\nabla u\in L^{\bar q}(\Omega)$. By inserting (2.35) into (2.34) and by using the Lebesgue convergence Theorem for $\int_\Omega \eta_\epsilon^2|f||\Psi(u)|$ we get the validity of Lemma 2.5 in view of the monotone convergence Theorem. □

We are now ready to complete the proof of Theorem 2.1 by establishing $L^\infty-$bounds on H_λ.

Theorem 2.6. *Let* $1<p\le N$ *and assume either* $\lambda=0$ *or* $\lambda\neq 0$ *and* $p\ge 2$ *with* $p>\frac{N}{2}$. *Then* $H_\lambda=G_\lambda-\Gamma\in L^\infty(\Omega)$, *where* G_λ *is any solution to* (2.1) *satisfying* (1.6).

Proof. By (2.1) the function $u=H_\lambda$ solves (2.31) with $\mathit{\Gamma}=\Gamma$ and $f=\lambda G_\lambda^{p-1}$. Given $0<\beta_0<1$ to be fixed later, by Lemma 2.5 with $\eta=1$ and $\Psi(s)=[T_{0,l}(s\mp C_0)_\pm+\epsilon]^\beta-\epsilon^\beta$, with $l,\epsilon>0$, $\beta\ge\beta_0$, $C_0=\|g\|_\infty+\|\Gamma\|_{\infty,\partial\Omega}$ and $T_{k,l}$ given by (2.16), we get that

$$\beta\int_{\{(u\mp C_0)_\pm\le l\}}[T_{0,l}(u\mp C_0)_\pm+\epsilon]^{\beta-1}(|\nabla\Gamma|+|\nabla u|)^{p-2}|\nabla u|^2\le C\int_\Omega |f|[T_{0,l}(u\mp C_0)_\pm+\epsilon]^\beta \tag{2.36}$$

in view of $\Psi(u)=0$ on $\partial\Omega$ thanks to $H_\lambda=g-\Gamma$ on $\partial\Omega$.

Let us first consider the case $\lambda=0$. Then $f=0$ and the choice $\beta=1$ in (2.36) gives

$$\int_\Omega(|\nabla\Gamma|+|\nabla u|)^{p-2}|\nabla T_{0,l}(u\mp C_0)_\pm|^2\le 0.$$

Then $T_{0,l}(u\mp C_0)_\pm=0$ a.e. in Ω for any $l>0$, which implies $|H_0|\le C_0$ a.e. in Ω.

Consider now the case $\lambda\neq 0$ and assume $p\ge 2$ with $p>\frac{N}{2}$. Since $\nabla G_\lambda=\nabla\Gamma+\nabla H_\lambda\in L^q(\Omega)$ for all $1\le q<\bar q$ in view of (1.6), by the Sobolev embedding Theorem $G_\lambda\in L^q(\Omega)$ for all $1\le q<\bar q^*$ and in particular f satisfies

$$\|f\|_{q_0}<\infty \tag{2.37}$$

for some $q_0>\frac{N}{p}$ in view of $p>\frac{N}{2}$.

Notice that (2.36)-(2.37) are the analogue of (2.26) and (2.28), and then the argument now goes exactly as in the proof of Proposition 2.4. □

For the case $g=0$ let us collect here some useful facts which will be used in the next two sections. Given $1<p<N$, an important ingredient is given by the estimate

$$|\nabla H_\lambda|=O(|\nabla \Gamma|) \quad \text{in } \Omega \tag{2.38}$$

for any solution $G_\lambda=\Gamma+H_\lambda$ of $(2.1)_{g=0}$. Indeed, by [33] any solution G_λ of $(2.1)_{g=0}$ satisfies

$$\frac{\Gamma}{C}\le G_\lambda \le C\Gamma \quad \text{in } B_{2R_0}(0) \tag{2.39}$$

for some $C>1$, where $R_0=\frac{1}{4}\text{dist}(0,\partial\Omega)$. For $0<R\le R_0$ consider the scaling $G_{\lambda,R}(y)=R^{\frac{N-p}{p-1}}G_\lambda(Ry)$ of G_λ in $\Omega_R=\frac{\Omega}{R}$ which satisfies

$$\begin{cases} -\Delta_p G_{\lambda,R}-\lambda R^p G_{\lambda,R}^{p-1}=\delta_0 & \text{in } \Omega_R\\ G_{\lambda,R}\ge 0 & \text{in } \Omega_R\\ G_{\lambda,R}=0 & \text{on } \partial\Omega_R. \end{cases} \tag{2.40}$$

Since $\Gamma_R(y)=R^{\frac{N-p}{p-1}}\Gamma(Ry)=\Gamma(y)$ in view of $1<p<N$, we have that condition (2.39) is scaling invariant:

$$\frac{\Gamma}{C}\le G_{\lambda,R}\le C\Gamma \quad \text{in } B_{\frac{2R_0}{R}}(0). \tag{2.41}$$

Since $G_{\lambda,R}$ is uniformly bounded in $L^\infty_{\text{loc}}(B_2(0)\setminus\{0\})$ thanks to (2.41), elliptic estimates [11, 34] for (2.40) imply that

$$G_{\lambda,R} \text{ uniformly bounded in } C^{1,\alpha}_{\text{loc}}(B_2(0)\setminus\{0\})$$

for some $\alpha\in(0,1)$. Since in particular $\|\nabla G_{\lambda,R}\|_{\infty,\partial B_1(0)}\le C$, setting $H_{\lambda,R}(y)=R^{\frac{N-p}{p-1}}H_\lambda(Ry)$ we deduce that $\|\nabla H_{\lambda,R}\|_{\infty,\partial B_1(0)}\le C'$ in view of $\nabla G_{\lambda,R}=\nabla\Gamma+\nabla H_{\lambda,R}$, which can be re-written as

$$|\nabla H_\lambda|\le \frac{C'}{|x|^{\frac{N-1}{p-1}}}=C|\nabla\Gamma| \quad \text{on } \partial B_R(0) \tag{2.42}$$

for all $0<R\le\frac{1}{4}\text{dist}(0,\partial\Omega)$. Away from the origin ∇H_λ is bounded thanks to [11, 28, 34] and $|\nabla\Gamma|$ is bounded from below, and then estimate (2.38) follows by (2.42). Moreover, notice that for $1<p\le N$ there holds

$$\|H_\lambda\|_\infty<+\infty \quad\Rightarrow\quad |\nabla H_\lambda(x)|=o(|\nabla\Gamma(x)|) \quad \text{as } x\to 0. \tag{2.43}$$

Indeed, for $1<p<N$ we have that $\|H_{\lambda,R}\|_{\infty,\Omega_R}\to 0$ and then $\|\nabla H_{\lambda,R}\|_{\infty,\partial B_1(0)}\to 0$ as $R\to 0$, which provides the validity of (2.43). When $p=N$ the function $G_{\lambda,R}(y)=G_\lambda(Ry)+(N\omega_N)^{-\frac{1}{N-1}}\log R=\Gamma(y)+H_\lambda(Ry)$ is uniformly bounded in $L^\infty_{\text{loc}}(\mathbb{R}^N\setminus\{0\})$ and satisfies

$$-\Delta_N G_{\lambda,R}-\lambda R^N\Big[G_{\lambda,R}-(N\omega_N)^{-\frac{1}{N-1}}\log R\Big]^{N-1}=\delta_0 \quad \text{in } \Omega_R.$$

We argue as above to show that, up to a subsequence, $H_{\lambda,R}(y)=H_\lambda(Ry)\to H_0$ in $C^1_{\text{loc}}(\mathbb{R}^N\setminus\{0\})$ as $R\to 0$, where $\|H_0\|_\infty<+\infty$ and $\Gamma+H_0$ is a $N-$harmonic function in $\mathbb{R}^N\setminus\{0\}$. It follows that H_0 is a constant function, see for example Lemma 4.3 in [17]. Since this is true along any such subsequence, then $\nabla H_{\lambda,R}\to 0$ in $C_{\text{loc}}(\mathbb{R}^N\setminus\{0\})$ as $R\to 0$ and (2.43) does hold also in the case $p=N$.

Once we have $\delta|\nabla\Gamma|^{p-2}\le(|\nabla\Gamma|+|\nabla H_\lambda|)^{p-2}$ for $1<p<2$ in view of (2.38), it becomes clear the usefulness of the following weigthed Sobolev inequalities of Caffarelli-Kohn-Nirenberg type [7]: given $1<p<2$, there exists $C>0$ so that

$$\left(\int_{\mathbb{R}^N}|\nabla\Gamma|^{p-2}|u|^{\frac{2(N-2+p)}{N-p}}\right)^{\frac{N-p}{N-2+p}}\le C\int_{\mathbb{R}^N}|\nabla\Gamma|^{p-2}|\nabla u|^2 \tag{2.44}$$

for any compactly supported $u\in L^\infty(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}|\nabla\Gamma|^{p-2}|\nabla u|^2<+\infty$. Valid in $C_0^\infty(\mathbb{R}^N)$, (2.44) can be first extended to $W^{1,2}-$functions with compact support in view of $|\nabla\Gamma|^{p-2}\in L^\infty_{\text{loc}}(\mathbb{R}^N)$ and then to compactly supported $u \in L^\infty(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} |\nabla \Gamma|^{p-2}|\nabla u|^2 < +\infty$ through the sequence $\eta_\epsilon u \in W^{1,2}(\mathbb{R}^N)$, η_ϵ being given by (2.33) with $\eta = 1$ in $\mathbb{R}^N$, since

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} |\nabla \Gamma|^{p-2} |\nabla \eta_\epsilon|^2 u^2 \to 0.$$

For later convenience, when either $2 \le p < N$ or $p = N \ge 3$ observe also the validity of the following inequality

$$\left(\int_{\mathbb{R}^N} |u|^{\frac{2N(p-1)}{N(p-1)-p}} \right)^{\frac{N(p-1)-p}{N(p-1)}} \le C \int_{\mathbb{R}^N} |x|^{\frac{p-2}{p-1}} |\nabla u|^2 \tag{2.45}$$

for any compactly supported $u \in L^\infty(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} |x|^{\frac{p-2}{p-1}} |\nabla u|^2 < +\infty$.

3. Weak comparison principle and uniqueness results

This section is devoted to discuss the uniqueness part in Theorem 1.2 when $2 \le p \le N$ among solutions satisfying the natural condition (1.6). When $\lambda = 0$ maximum and comparison principles in weak or strong form are well known, see for example [36], and have been extended in various forms to the case $\lambda < \lambda_1$ in connection with existence and uniqueness results, see [8, 10, 19, 20] just to quote a few.

To extend the previous uniqueness results to the singular situation, the crucial property is given by the convexity of the functional

$$I(w) = \begin{cases} \int_\Omega |\nabla w^{\frac{1}{p}}|^p & \text{if } w \ge 0 \text{ and } \nabla(w^{\frac{1}{p}}) \in L^p(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Proved in [10] for $p > 1$, a quantitative form is established here giving a positive lower bound for I'' when $2 \le p \le N$, crucial to be applied on $\Omega_\epsilon = \Omega \setminus B_\epsilon(0)$ as $\epsilon \to 0$.

Lemma 3.1. *Let $w \ge 0$ a.e. in Ω so that $\nabla(w^{\frac{1}{p}}) \in L^p(\Omega)$. Let ϕ be a variation so that $w_t = w + t\phi \ge 0$ a.e. in Ω and $\nabla(w_t^{\frac{1}{p}}) \in L^p(\Omega)$ for $t \ge 0$ small. Letting $\rho(w,\phi)$ be given in (3.7), there hold*

$$I'(w)[\phi] = \int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2} \langle \nabla w^{\frac{1}{p}}, \nabla (w^{\frac{1-p}{p}} \phi) \rangle, \quad I''(w)[\phi,\phi] = \int_\Omega \rho(w,\phi) \tag{3.1}$$

with

$$\begin{aligned} \rho(w,\phi) \quad \ge \quad & \frac{p-1}{p}(p^3 - 3p^2 + 5p - 2) |\nabla w^{\frac{1}{p}}|^p \left(\frac{\phi}{w} - \frac{p(p^2 - 2p + 2)\langle \nabla w, \nabla \phi \rangle}{(p^3 - 3p^2 + 5p - 2)|\nabla w|^2} \right)^2 \\ & + \frac{(p-1)(p-2)}{p(p^3 - 3p^2 + 5p - 2)} w^{\frac{2(1-p)}{p}} |\nabla w^{\frac{1}{p}}|^{p-2} |\nabla \phi|^2, \end{aligned} \tag{3.2}$$

where $I'(w)[\phi] = \frac{d}{dt} I(w_t)\Big|_{t=0^+}$ and $I''(w)[\phi,\phi] = \frac{d}{dt} I'(w_t)[\phi]\Big|_{t=0^+}$.

Proof. Since $\frac{d}{dt} w_t^{\frac{1}{p}} = \frac{1}{p} w_t^{\frac{1-p}{p}} \phi$, we have that

$$I'(w_t)[\phi] = \int_\Omega |\nabla w_t^{\frac{1}{p}}|^{p-2} \langle \nabla w_t^{\frac{1}{p}}, \nabla (w_t^{\frac{1-p}{p}} \phi) \rangle,$$

providing, when evaluated at $t = 0$, the validity of the first in formula (3.1). Differentiating once more in t at 0^+, we have that

$$\begin{aligned} I''(w)[\phi,\phi] \quad = \quad & (p-2) \int_\Omega |\nabla w^{\frac{1}{p}}|^{p-4} \langle \nabla w^{\frac{1}{p}}, \nabla (w^{\frac{1-p}{p}} \phi) \rangle^2 + \frac{1}{p} \int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2} |\nabla (w^{\frac{1-p}{p}} \phi)|^2 \\ & - \frac{p-1}{p} \int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2} \langle \nabla w^{\frac{1}{p}}, \nabla (w^{\frac{1-2p}{p}} \phi^2) \rangle. \end{aligned} \tag{3.3}$$

Writing $\langle \nabla w, \nabla \phi\rangle = \cos\alpha |\nabla w||\nabla \phi|$ the first, second and third term in (3.3) produce, respectively,

$$\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-4}\langle \nabla w^{\frac{1}{p}}, \nabla (w^{\frac{1-p}{p}}\phi)\rangle^2 = \int_\Omega \frac{|\nabla w^{\frac{1}{p}}|^{p-2}}{w^{\frac{2(p-1)}{p}}}\Big[\frac{(p-1)^2}{p^2}\frac{|\nabla w|^2}{w^2}\phi^2 + \cos^2\alpha |\nabla \phi|^2 - \frac{2(p-1)}{p}\cos\alpha \frac{|\nabla w|}{w}\phi|\nabla\phi|\Big], \tag{3.4}$$

$$\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}|\nabla (w^{\frac{1-p}{p}}\phi)|^2 = \int_\Omega \frac{|\nabla w^{\frac{1}{p}}|^{p-2}}{w^{\frac{2(p-1)}{p}}}\Big[\frac{(p-1)^2}{p^2}\frac{|\nabla w|^2}{w^2}\phi^2 + |\nabla \phi|^2 - \frac{2(p-1)}{p}\cos\alpha \frac{|\nabla w|}{w}\phi|\nabla\phi|\Big], \tag{3.5}$$

$$\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\langle \nabla w^{\frac{1}{p}}, \nabla (w^{\frac{1-2p}{p}}\phi^2)\rangle = \int_\Omega \frac{|\nabla w^{\frac{1}{p}}|^{p-2}}{w^{\frac{2(p-1)}{p}}}\Big[-\frac{2p-1}{p^2}\frac{|\nabla w|^2}{w^2}\phi^2 + \frac{2}{p}\cos\alpha \frac{|\nabla w|}{w}\phi|\nabla\phi|\Big]. \tag{3.6}$$

Collecting (3.4)-(3.6), the expression of (3.3) becomes $I''(w)[\phi,\phi] = \int_\Omega \rho(w,\phi)$, with

$$\begin{aligned} \rho(w,\phi) &= w^{\frac{2(1-p)}{p}}|\nabla w^{\frac{1}{p}}|^{p-2}\Big[C_1\frac{|\nabla w|^2}{w^2}\phi^2 - C_2\cos\alpha\frac{|\nabla w|}{w}\phi|\nabla\phi| + C_3|\nabla\phi|^2\Big] \\ &= w^{\frac{2(1-p)}{p}}|\nabla w^{\frac{1}{p}}|^{p-2}\Big[C_1(\frac{|\nabla w|}{w}\phi - \frac{C_2}{2C_1}\cos\alpha|\nabla\phi|)^2 + \frac{4C_1C_3 - C_2^2\cos^2\alpha}{4C_1}|\nabla\phi|^2\Big] \end{aligned} \tag{3.7}$$

by a square completion in view of $C_1 > 0$, where

$$C_1 = \frac{p-1}{p^3}(p^3 - 3p^2 + 5p - 2), \quad C_2 = \frac{2(p-1)}{p^2}(p^2 - 2p + 2), \quad C_3 = \frac{1}{p} + (p-2)\cos^2\alpha.$$

Since

$$4\frac{p-1}{p^3}(p^3 - 3p^2 + 5p - 2)(p-2) - \frac{4(p-1)^2}{p^4}(p^2 - 2p + 2)^2 = -4\frac{p-1}{p^4}(p^3 - 4p^2 + 8p - 4) < 0,$$

then $4C_1C_3 - C_2^2\cos^2\alpha \geq 4\frac{(p-1)^2(p-2)}{p^4}$ and (3.2) follows by (3.7). □

As a first application, we deduce the validity of a weak comparison principle for positive solutions.

Proposition 3.2. *Let $2 \leq p \leq N$ and $a, f_1, f_2 \in L^\infty(\Omega)$. Let $u_i \in C^1(\bar\Omega)$, $i = 1,2$, be solutions to*

$$-\Delta_p u_i - a u_i^{p-1} = f_i \quad \text{in } \Omega \tag{3.8}$$

so that

$$u_i > 0 \text{ in } \Omega, \quad \frac{u_1}{u_2} \leq C \text{ near } \partial\Omega \tag{3.9}$$

for some $C > 0$. If $f_1 \leq f_2$ with $f_2 \geq 0$ in Ω and $u_1 \leq u_2$ on $\partial\Omega$, then $u_1 \leq u_2$ in Ω.

Proof. Setting $w_1 = u_1^p$, $w_2 = u_2^p$ and $\phi = (w_1 - w_2)_+$, consider $w_s = s w_1 + (1-s) w_2$ for $s \in [0,1]$. Since

$$w_s + t\phi = u_2^p\Big[s(\frac{u_1}{u_2})^p + (1-s) + t\left((\frac{u_1}{u_2})^p - 1\right)_+\Big],$$

by (3.9) there exists $t_0 > 0$ small so that $w_s + t\phi \geq 0$ in Ω and $\nabla(w_s + t\phi)^{\frac{1}{p}} \in L^p(\Omega)$ for each $s \in [0,1]$ and $|t| \leq t_0$. Then we can apply (3.1) at $s = 0, 1$ to get

$$\begin{aligned} I'(w_1)[\phi] - I'(w_2)[\phi] &= \int_\Omega |\nabla w_1^{\frac{1}{p}}|^{p-2}\langle \nabla w_1^{\frac{1}{p}}, \nabla (w_1^{\frac{1-p}{p}}\phi)\rangle - \int_\Omega |\nabla w_2^{\frac{1}{p}}|^{p-2}\langle \nabla w_2^{\frac{1}{p}}, \nabla (w_2^{\frac{1-p}{p}}\phi)\rangle \\ &= \int_\Omega |\nabla u_1|^{p-2}\langle \nabla u_1, \nabla \frac{\phi}{u_1^{p-1}}\rangle - \int_\Omega |\nabla u_2|^{p-2}\langle \nabla u_2, \nabla \frac{\phi}{u_2^{p-1}}\rangle. \end{aligned}$$

Since $\phi \in W_0^{1,p}(\Omega)$ we deduce that

$$I'(w_1)[\phi] - I'(w_2)[\phi] = \int_\Omega \left(\frac{f_1}{u_1^{p-1}} - \frac{f_2}{u_2^{p-1}} \right) (u_1^p - u_2^p)^+ \leq 0$$

in view of (3.8) and $f_1 \leq f_2$ with $f_2 \geq 0$. Since

$$I'(w_1)[\phi] - I'(w_2)[\phi] = \int_0^1 I''(w_s)[w_1 - w_2, \phi] ds = \int_0^1 I''(w_s)[\phi, \phi] ds$$

in view of $I''(w_s)[w_1 - w_2, \phi] = I''(w_s)[\phi, \phi]$, by Lemma 3.1 $I''(w_s)[\phi, \phi] = \int_\Omega \rho(w_s, \phi)$ with $\rho(w_s, \phi) \geq 0$ thanks to (3.2) when $p \geq 2$. Then, we deduce that $\rho(w_s, \phi) = 0$ for all $s \in [0, 1]$ and then

- $\nabla\phi = 0$ in Ω if $p > 2$
- $\langle \nabla w_s, \nabla \phi \rangle = \phi \frac{|\nabla w_s|^2}{w_s}$ if $p = 2$, which implies $\langle \nabla(w_1 - w_2), \nabla\phi \rangle = s\phi \frac{|\nabla(w_1 - w_2)|^2}{w_s}$ for all $0 \leq s \leq 1$.

In both cases $\nabla\phi = 0$ in Ω and then $w_1 \leq w_2$ in Ω, or equivalently $u_1 \leq u_2$ in Ω. □

Finally, we use Lemma 3.1 to show the uniqueness part in Theorem 1.2.

Theorem 3.3. *Let $2 \leq p \leq N$. If $\lambda < \lambda_1$ with $\lambda \neq 0$ and $p > \frac{N}{2}$, problem $(2.1)_{g=0}$ has exactly one solution G_λ so that $H_\lambda = G_\lambda - \Gamma$ satisfies (1.6). Moreover, if $H_\lambda \in C(\Omega)$ for all $\lambda < \lambda_1$, then the map $\lambda \in (-\infty, \lambda_1) \to H_\lambda(x)$ is strictly increasing at any given $x \in \Omega$.*

Proof. We follow the same argument as in the proof of Proposition 3.2. Letting G_1 and G_2 be two solutions of $(2.1)_{g=0}$ satisfying (1.6), by elliptic regularity theory [11, 28, 32, 34] we know that $G_i \in C^{1,\alpha}(\bar{\Omega} \setminus \{0\})$, $i = 1, 2$, for some $\alpha > 0$. By [33] we know that G_i, $i = 1, 2$, satisfies (2.39) and by the strong maximum principle [36] $\partial_\nu G_i < 0$, $i = 1, 2$, on $\partial\Omega$, where ν denotes the outward unit normal vector. Set $w_1 = G_1^p$, $w_2 = G_2^p$, $\phi = w_1 - w_2$ and $w_s = s w_1 + (1 - s) w_2$ for $s \in [0, 1]$. We have that for each $s \in [0, 1]$ there hold $w_s + t\phi \geq 0$ in Ω and $\nabla(w_s + t\phi)^{\frac{1}{p}} \in L^p(\Omega)$ for t small, in view of the properties of G_1 and G_2. Letting I_ϵ be the functional I defined on $\Omega_\epsilon = \Omega \setminus B_\epsilon(0)$, by (3.1) at $s = 0, 1$ we have that

$$\begin{aligned} I'_\epsilon(w_1)[\phi] - I'_\epsilon(w_2)[\phi] &= \int_{\Omega_\epsilon} |\nabla G_1|^{p-2} \langle \nabla G_1, \nabla \frac{\phi}{G_1^{p-1}} \rangle - \int_\Omega |\nabla G_2|^{p-2} \langle \nabla G_2, \nabla \frac{\phi}{G_2^{p-1}} \rangle \\ &= \int_{\partial B_\epsilon(0)} \left(\frac{|\nabla G_2|^{p-2} \partial_\nu G_2}{G_2^{p-1}} - \frac{|\nabla G_1|^{p-2} \partial_\nu G_1}{G_1^{p-1}} \right) (G_1^p - G_2^p) \end{aligned}$$

in view of $\phi = 0$ on $\partial\Omega$ and the equation $(2.1)_{g=0}$ satisfied by G_1, G_2. Notice that

$$I'_\epsilon(w_1)[\phi] - I'_\epsilon(w_2)[\phi] = \int_0^1 I''_\epsilon(w_s)[\phi, \phi] ds$$

with $I''_\epsilon(w_s)[\phi, \phi] = \int_{\Omega_\epsilon} \rho(w_s, \phi)$ in view of Lemma 3.1. Since $\rho(w_s, \phi) \geq 0$ when $p \geq 2$ in view of (3.2), by the Fatou convergence Theorem we deduce that

$$\int_0^1 ds \int_\Omega \rho(w_s, \phi) \leq \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(0)} \left(\frac{|\nabla G_2|^{p-2} \partial_\nu G_2}{G_2^{p-1}} - \frac{|\nabla G_1|^{p-2} \partial_\nu G_1}{G_1^{p-1}} \right) (G_1^p - G_2^p). \tag{3.10}$$

We claim that the R.H.S. in (3.10) vanishes and then $\rho(w_s, \phi) = 0$ for all $s \in [0, 1]$, which implies, as already discussed in the proof of Proposition 3.2, $\nabla\phi = 0$ in Ω and then $G_1 = G_2$ in Ω.

In order to prove the previous claim, for $i = 1, 2$ notice that $H_i = G_i - \Gamma \in L^\infty(\Omega)$ follows by Theorem 2.6 in view of the assumption (1.6) for G_i. Once $H_i \in L^\infty(\Omega)$, we have that H_i satisfies (2.43) and then

$$G_i^q = \Gamma^q + O(\Gamma^{q-1}), \quad |\nabla G_i|^{p-2} \partial_\nu G_i = |\nabla\Gamma|^{p-2} \partial_\nu \Gamma + o(|\nabla\Gamma|^{p-1}) \tag{3.11}$$

as $x \to 0$ for $q > 0$. By (3.11) we deduce that $G_1^p - G_2^p = O(\Gamma^{p-1})$ and

$$\frac{|\nabla G_i|^{p-2}\partial_\nu G_i}{G_i^{p-1}} = \frac{|\nabla \Gamma|^{p-2}\partial_\nu \Gamma}{\Gamma^{p-1}} + o(\frac{|\nabla \Gamma|^{p-1}}{\Gamma^{p-1}}),$$

which imply

$$\Big|\int_{\partial B_\epsilon(0)} (\frac{|\nabla G_2|^{p-2}\partial_\nu G_2}{G_2^{p-1}} - \frac{|\nabla G_1|^{p-2}\partial_\nu G_1}{G_1^{p-1}})(G_1^p - G_2^p)\Big| = o(\int_{\partial B_\epsilon(0)} |\nabla \Gamma|^{p-1}) = o(1)$$

as $\epsilon \to 0$, as claimed.

Finally, assume $H_\lambda \in C(\Omega)$ for all $\lambda < \lambda_1$ to have well defined values $H_\lambda(x)$ for all $x \in \Omega$ (at $x = 0$ too) and take $\mu_1 < \mu_2$. Letting $0 \le G_n^1, G_n^2 \in W_0^{1,p}(\Omega)$ be the solutions of (2.7) corresponding to $\lambda = \mu_1$ and $\lambda = \mu_2$, respectively, by the proof of Theorem 2.1 recall that $G_{\mu_1} = \lim\limits_{n\to+\infty} G_n^1$ and $G_{\mu_2} = \lim\limits_{n\to+\infty} G_n^2$ a.e. in Ω, where $f_n \ge 0$ is a suitable smooth approximating sequence for the measure δ_0. Since $G_n^i > 0$ in Ω and $\partial_\nu G_n^i < 0$ on $\partial\Omega$ by the strong maximum principle [36], we can apply Proposition 3.2 to get $G_n^1 \le G_n^2$ in view of $0 \le f_n \le f_n + (\mu_2 - \mu_1)(G_n^2)^{p-1}$ with $f_n, G_n^2 \in L^\infty(\Omega)$, and then $G_{\mu_1} \le G_{\mu_2}$ in Ω as $n \to +\infty$. Since

$$-\Delta_p G_{\mu_1} = \mu_1 (G_{\mu_1})^{p-1} < \mu_2 (G_{\mu_2})^{p-1} = -\Delta_p G_{\mu_2} \quad \text{in } \Omega \setminus B_\epsilon(0),$$

apply once again the strong maximum principle [36] to deduce $G_{\mu_1} < G_{\mu_2}$ in $\Omega \setminus B_\epsilon(0)$ for all $\epsilon > 0$, and the strict monotonicity is established in $\Omega \setminus \{0\}$. Given $0 < \epsilon < \text{dist}\,(0, \partial\Omega)$, we can find $\eta \in C_0^1(\Omega)$ with $\eta = 1$ in $B_\epsilon(0)$ and $\delta > 0$ so that $H_{\mu_1} - H_{\mu_2} + \delta \le 0$ on $\text{supp}(\eta) \setminus B_\epsilon(0)$. Observe that $u = H_{\mu_1} - H_{\mu_2}$ and $\mathit{\Gamma} = \Gamma + H_{\mu_2}$ satisfy $\nabla u \in L^{\bar q}(\Omega)$, (2.32) and

$$-\Delta_p(\mathit{\Gamma} + u) + \Delta_p(\mathit{\Gamma}) = f \quad \text{in } \Omega \setminus \{0\}$$

with $f = \mu_1 (G_{\mu_1})^{p-1} - \mu_2 (G_{\mu_2})^{p-1} \le 0$. We can apply a variant of Lemma 2.5 with η and $\Psi(u) = (u+\delta)_+$ to get

$$\int_\Omega \eta^2 |\nabla (u+\delta)_+|^p \le C \int_\Omega |\eta||\nabla \eta|(u+\delta)_+ (|\nabla \mathit{\Gamma}| + |\nabla u|)^{p-2}|\nabla u| + \int_\Omega \eta^2 f (u+\delta)_+ \le 0$$

and then $(u+\delta)_+ = 0$ in $B_\epsilon(0)$, providing $H_{\mu_1} - H_{\mu_2} \le -\delta < 0$ in $B_\epsilon(0)$ too. The proof is complete. □

4. Harnack inequalities and Hölder continuity of H_λ at the pole

In this section we will use the Moser iterative scheme in [32] to establish local estimates for the solution H_λ of (1.2) at 0, leading to an Harnack inequality for $H_\lambda + c$ which is the crucial tool to show Hölder estimates at 0. The function $\mathcal{H}(x) = R^{\frac{N-p}{p-1}}(\pm H_\lambda(Rx) + c)$, $0 < R < \frac{1}{2}\text{dist}\,(0, \partial\Omega)$, satisfies

$$-\Delta_p(\mathit{\Gamma} + \mathcal{H}) + \Delta_p \mathit{\Gamma} = \mathcal{G} \quad \text{in } B_2(0) \setminus \{0\} \tag{4.1}$$

in view of (1.2), where $\mathit{\Gamma} = \pm R^{\frac{N-p}{p-1}} \Gamma(Rx)$ with $\nabla \mathit{\Gamma} = \pm \nabla \Gamma$ and $\mathcal{G} = \pm \lambda R^N G_\lambda^{p-1}(Rx)$. Differently from Proposition 2.4 and Theorem 2.6, we need to perform homogeneuos estimates on $\mathcal{H}$ and to this aim for $2 \le p \le N$ assume

$$\Lambda = \|\mathcal{G}\|_{q_0, B_2(0)}^{\frac{1}{p-1}} < +\infty \tag{4.2}$$

for some $q_0 > \frac{N}{p}$. Consider the weight function $\rho = |\nabla \Gamma|^{p-2}$ when $1 < p < 2$, $\mathcal{G} = 0$ and $\rho = 1$ otherwise, and introduce the weighted integrals $\Phi_\rho(s,h) = \left(\int_{B_h(0)} \rho |u|^s\right)^{\frac{1}{s}}$, $h, s > 0$. Define κ as

$$\kappa = \begin{cases} \frac{N-2+p}{N-p} & \text{if } 1 < p < 2 \text{ and } \mathcal{G} = 0 \\ \frac{N(p-1)}{N(p-1)-p} & \text{if either } 2 \le p < N \text{ or } p = N \ge 3 \\ 2 & \text{if } p = N = 2. \end{cases} \tag{4.3}$$

We are now ready to establish the main estimates in the section.

Proposition 4.1. *Let $\mathcal{H} \in L^\infty(B_2(0))$ be a solution of (4.1) so that $\nabla \mathcal{H} \in L^{\bar q}(B_2(0))$, Γ satisfies (2.32) and (4.2) holds. Assume $\mathcal{G}=0$, $|\nabla \mathcal{H}| \le M|\nabla \Gamma|$ in $B_2(0)$ when $1<p<2$ and $\|\mathcal{H}\|_\infty + \Lambda \le M$, $|x|^{\frac{1}{p-1}} \le M|\nabla \Gamma|$ in $B_2(0)$ when $2 \le p \le N$, for some $M>0$. Given $\mu \in \mathbb{R}\setminus\{0\}$, there exist $\nu, \beta \ge 0$ and $C>0$ so that the function $u=|\mathcal{H}|+\Lambda+\epsilon$ satisfies*

$$\pm \Phi_\rho(\kappa\mu, h_1) \le \pm [C|\mu|^\nu (h_2-h_1)^{-\beta}]^{\frac{1}{\mu}} \Phi_\rho(\mu, h_2) \tag{4.4}$$

for all $1 \le h_1 < h_2 \le 2$ and $0<\epsilon \le 1$, uniformly for μ away from $2-p$, 0 and 1, where $\kappa>1$ is given in (4.3) and $\pm$ simply denotes the sign of μ.

Remark 4.2. *The assumption $|x|^{\frac{1}{p-1}} \le M|\nabla \Gamma|$ when $2 \le p \le N$ is sufficiently general in order to establish the validity of Corollary 4.5, which will be used in a crucial way in [2].*

Proof. Given $T_{k,l}$ in (2.16), introduce the bounded monotone Lipschitz function

$$\Psi(s) = \text{sign } s \left([T_{0,l}(|s|+\Lambda+\epsilon)]^\beta - [T_{0,l}(\Lambda+\epsilon)]^\beta \right), \beta \in \mathbb{R}\setminus\{0\}.$$

Let $\eta \in C_0^\infty(B_{h_2}(0))$ be a cut-off function so that $0 \le \eta \le 1$, $\eta=1$ in $B_{h_1}(0)$ and $|\nabla \eta| \le \frac{2}{h_2-h_1}$. Since $\eta=0$ on $\partial B_2(0)$ and $\nabla \mathcal{H} \in L^{\bar q}(B_2(0))$ we can apply Lemma 2.5 to $\mathcal{H}$, solution of (4.1), to get

$$\begin{aligned} \int \eta^2 |\Psi'(\mathcal{H})|(|\nabla \Gamma|+|\nabla \mathcal{H}|)^{p-2}|\nabla \mathcal{H}|^2 \ &\le \ C\int \eta|\nabla \eta||\Psi(\mathcal{H})|(|\nabla \Gamma|+|\nabla \mathcal{H}|)^{p-2}|\nabla \mathcal{H}| \\ &\quad + C\int \eta^2 |\mathcal{G}||\Psi(\mathcal{H})| \end{aligned} \tag{4.5}$$

for some $C>0$. Define $v=u^{\frac{\beta+1}{2}}$ and $w=u^{\frac{\beta-1+p}{p}}$ with $u=|\mathcal{H}|+\Lambda+\epsilon$. Since $\Psi'(\mathcal{H})=\beta u^{\beta-1}$ and $|\Psi(\mathcal{H})| \le u^\beta$ for $l>M+1$, by (4.5) we deduce that

$$|\beta| \int \eta^2 u^{\beta-1}(|\nabla \Gamma|+|\nabla u|)^{p-2}|\nabla u|^2 \le C\left(\int \eta |\nabla \eta| u^\beta (|\nabla \Gamma|+|\nabla u|)^{p-2}|\nabla u| + \int \eta^2 |\mathcal{G}| u^\beta \right) \tag{4.6}$$

in view of $|\nabla \mathcal{H}|=|\nabla u|$.

Consider first the case $1<p<2$. Since $|\nabla u| \le M|\nabla \Gamma|$ in $B_2(0)$, we have that

$$(1+M)^{p-2}|\nabla \Gamma|^{p-2} \le (|\nabla \Gamma|+|\nabla u|)^{p-2} \le |\nabla \Gamma|^{p-2} \quad \text{in } B_2(0)$$

and then (4.6) implies

$$\int \eta^2 |\nabla \Gamma|^{p-2}|\nabla v|^2 \le C' \frac{|\beta+1|}{|\beta|} \int \eta|\nabla \eta||\nabla \Gamma|^{p-2} v|\nabla v| \le C\int \eta|\nabla \eta||\nabla \Gamma|^{p-2} v|\nabla v| \tag{4.7}$$

uniformly for β away from 0 in view of $\mathcal{G}=0$. Since

$$C\int \eta|\nabla \eta||\nabla \Gamma|^{p-2} v|\nabla v| \le \frac{1}{2}\int \eta^2|\nabla \Gamma|^{p-2}|\nabla v|^2 + C' \int |\nabla \eta|^2 |\nabla \Gamma|^{p-2} v^2$$

thanks to the Young inequality, we can re-write (4.7) as

$$\int |\nabla \Gamma|^{p-2}|\nabla(\eta v)|^2 \le C\int |\nabla \eta|^2|\nabla \Gamma|^{p-2} v^2. \tag{4.8}$$

Thanks to (2.32) and making use of (2.44), by (4.8) we deduce for $\mu=\beta+1$ that

$$\pm\Phi_\rho(\kappa\mu, h_1) \le \pm\left(\frac{C}{(h_2-h_1)^2}\right)^{\frac{1}{\mu}} \Phi_\rho(\mu, h_2)$$

does hold uniformly for μ away from 1, where κ is given by (4.3). Observe that the $(\beta+1)-$th root of (4.8) for $\beta<-1$ reverses the inequality causing the presence of $\pm$ in (4.4).

Consider now the case $2 \le p \le N$. Since

$$\begin{aligned} & C \int \eta^{\frac{p}{2}} |\nabla \eta^{\frac{p}{2}}| u^\beta (|\nabla \Gamma| + |\nabla u|)^{p-2} |\nabla u| \\ \le & \frac{|\beta|}{4} \int \eta^p u^{\beta-1} (|\nabla \Gamma| + |\nabla u|)^{p-2} |\nabla u|^2 + \frac{C'}{|\beta|} \int |\nabla \eta|^2 u^{\beta+1} |\nabla \Gamma|^{p-2} + \frac{C'}{|\beta|} \int \eta^{p-2} |\nabla \eta|^2 u^{\beta+1} |\nabla u|^{p-2} \\ \le & \frac{|\beta|}{2} \int \eta^p u^{\beta-1} (|\nabla \Gamma| + |\nabla u|)^{p-2} |\nabla u|^2 + \frac{C}{|\beta|} \int |\nabla \eta|^2 v^2 + \frac{C}{|\beta|^{p-1}} \int |\nabla \eta|^p w^p \end{aligned}$$

in view of the Young inequality, (2.32) and $\sup\limits_{B_2 \setminus B_1} |\nabla \Gamma|^{p-2} < +\infty$, by replacing η with $\eta^{\frac{p}{2}}$ (4.6) implies

$$\int \eta^p |\nabla \Gamma|^{p-2} |\nabla v|^2 + \frac{1}{|\beta|^{p-2}} \int \eta^p |\nabla w|^p \le C\Big(\int |\nabla \eta|^2 v^2 + \frac{1}{|\beta|^{p-2}} \int |\nabla \eta|^p w^p + |\beta| \int \eta^p |\mathcal{G}| u^\beta\Big) \quad (4.9)$$

uniformly for β away from $1-p$ and 0. Since $q_0 > \frac{N}{p}$, fix α and γ so that $\alpha \in (\frac{q_0}{q_0-1}, \frac{pq_0}{N-p})$ and $\frac{1}{\alpha} + \frac{1}{\gamma} = \frac{q_0-1}{q_0}$. By the Hölder inequality with exponents q_0, γ and α we have that

$$\int \eta^p |\mathcal{G}| u^\beta \le \frac{1}{\Lambda^{p-1}} \int |\mathcal{G}| (\eta w)^{\frac{p}{\gamma} + \frac{p(q_0+\alpha)}{\alpha q_0}} \le \frac{1}{\Lambda^{p-1}} \|\mathcal{G}\|_{q_0, B_2(0)} \|\eta w\|_p^{\frac{p}{\gamma}} \|\eta w\|_{\frac{p(q_0+\alpha)}{q_0}}^{\frac{p(q_0+\alpha)}{\alpha q_0}} = \|\eta w\|_p^{\frac{p}{\gamma}} \|\eta w\|_{\frac{p(q_0+\alpha)}{q_0}}^{\frac{p(q_0+\alpha)}{\alpha q_0}}$$

in view of (4.2) and then

$$\begin{aligned} C|\beta| \int \eta^p |\mathcal{G}| u^\beta & \le C'|\beta| \|\eta w\|_p^{\frac{p}{\gamma}} (\|\eta \nabla w\|_p^{\frac{p(q_0+\alpha)}{\alpha q_0}} + \|w \nabla \eta\|_p^{\frac{p(q_0+\alpha)}{\alpha q_0}}) \\ & \le \frac{1}{2|\beta|^{p-2}} \|\eta \nabla w\|_p^p + C''|\beta|^{\frac{\alpha q_0 + (p-2)(q_0+\alpha)}{\alpha q_0 - \alpha - q_0}} \|\eta w\|_p^p + \frac{1}{|\beta|^{p-2}} \|w \nabla \eta\|_p^p \end{aligned} \quad (4.10)$$

by the Sobolev embedding Theorem in view of $(N-p)(q_0+\alpha) < Nq_0$ and the Young inequality. Inserting (4.10) into (4.9) we get that

$$\int |x|^{\frac{p-2}{p-1}} |\nabla (\eta^{\frac{p}{2}} v)|^2 \le C \left(\int |\nabla \eta|^2 v^2 + |\beta|^{\frac{\alpha q_0 + (p-2)(q_0+\alpha)}{\alpha q_0 - \alpha - q_0}} \int \eta^p |w|^p + \frac{1}{|\beta|^{p-2}} \int |\nabla \eta|^p |w|^p \right) \quad (4.11)$$

in view of $|x|^{\frac{1}{p-1}} \le M |\nabla \Gamma|$ in $B_2(0)$. Since $\|\mathcal{H}\|_\infty + \Lambda \le M$ if $p \ge 2$, we have that $\|u\|_\infty \le M+1$ when $0 < \epsilon \le 1$ and then $w^p = u^{\beta+1} u^{p-2} \le (M+1)^{p-2} v^2$. By using the Sobolev embedding Theorem when $p = N = 2$ or (2.45) otherwise, for $\mu = \beta + 1$ estimate (4.11) gives that

$$\pm \Phi_1(\kappa \mu, h_1) \le \pm [C \frac{|\mu|^{\frac{\alpha q_0 + (p-2)(q_0+\alpha)}{\alpha q_0 - \alpha - q_0}}}{(h_2 - h_1)^p}]^{\frac{1}{\mu}} \Phi_1(\mu, h_2)$$

does hold uniformly for μ away from $2-p$ and 1, where κ is given by (4.3). Estimate (4.4) is then established in all the cases and the proof is complete. □

Hereafter we specialize the argument to $\mathcal{H} = R^{\frac{N-p}{p-1}} (\pm H_\lambda(Rx) + c)$, $R > 0$. Let us consider now the case $\beta = -1$ in the proof of Proposition 4.1 when $\mathcal{H} \ge 0$ and the result we have is the following.

Proposition 4.3. *Let $1 < p \le N$ if $\lambda = 0$ and $p \ge 2$ with $p > \frac{N}{2}$ if $\lambda \neq 0$. Assume $\frac{N}{p} < q_0 < \frac{N}{N-p}$ if $\lambda \neq 0$ and $\mathcal{H} = R^{\frac{N-p}{p-1}} (\pm H_\lambda(Rx) + c) \ge 0$. There exist $R_0 > 0$ and $C > 0$ so that $v = \log u$, where $u = \mathcal{H} + \Lambda + \epsilon$ and $\epsilon > 0$, satisfies*

$$⨍_B |v - \bar{v}| \le C$$

for all open ball $B \subset B_1(0)$, $0 < R \le R_0$ and $0 < \epsilon \le 1$, where ⨍ denotes an integral mean and $\bar{v} = ⨍_B v$.

Proof. First of all, observe that $p \geq 2$ and $p > \frac{N}{2}$ imply $p^2 \geq 2p > N$. Let $B = B_h(x_0) \subset B_1(0)$. Since $|x_0| + h < 1$ implies $|x| \leq |x - x_0| + |x_0| < \frac{3}{2}h + |x_0| < 2$ for all $x \in B_{\frac{3}{2}h}(x_0)$, we have that $B_{\frac{3}{2}h}(x_0) \subset B_2$. Let $\eta \in C_0^\infty(B_{\frac{3}{2}h}(x_0))$ be a cut-off function with $0 \leq \eta \leq 1$, $\eta = 1$ in $B_h(x_0)$ and $|\nabla \eta| \leq \frac{4}{h}$. Since $\mathcal{H}$ solves (4.1) with $\nabla \mathit{\Gamma} = \pm \nabla \Gamma$ and $\mathcal{G} = \pm \lambda R^N G^{p-1}(Rx)$, we can apply Lemma 2.5 with the bounded monotone Lipschitz function $\Psi(s) = \text{sign } s \left([T_{0,l}(|s| + \Lambda + \epsilon)]^{-1} - [T_{0,l}(\Lambda + \epsilon)]^{-1}\right)$, for $l > \|\mathcal{H}\|_\infty + \Lambda + 1$ and $T_{k,l}$ given by (2.16), and a cut-off function $\eta_\delta = \eta(\delta + |x|^2)^{\frac{(N-1)(p-2)}{4(p-1)} - 1}|x|^{\frac{5}{2}}$, $\delta > 0$, to get

$$\int \eta_\delta^2 |\nabla \Gamma|^{p-2}|\nabla v|^2 \leq C\left(\int \eta_\delta |\nabla \eta_\delta||\nabla \Gamma|^{p-2}|\nabla v| + \int \eta_\delta^2 \frac{|\mathcal{G}|}{u}\right)$$

in view of (2.38) (which follows by (2.43) and $\|H_\lambda\|_\infty < +\infty$ when $p = N$) and then by the Young inequality

$$\begin{aligned}\int \eta_\delta^2 |\nabla \Gamma|^{p-2}|\nabla v|^2 \quad &\leq C'\left(\int |\nabla \eta_\delta|^2 |\nabla \Gamma|^{p-2} + \int \eta_\delta^2 \frac{|\mathcal{G}|}{u}\right) \leq C\left(\int |x|\left(\frac{|x|^2}{\delta + |x|^2}\right)^{-\frac{(N-1)(p-2)}{2(p-1)}}|\nabla \eta|^2\right.\\ &\left. + \int \left(\frac{|x|^2}{\delta + |x|^2}\right)^{2 - \frac{(N-1)(p-2)}{2(p-1)}}\frac{\eta^2}{|x|} + \int |x|(\delta + |x|^2)^{\frac{(N-1)(p-2)}{2(p-1)}}\eta^2 \frac{|\mathcal{G}|}{u}\right) \qquad (4.12)\end{aligned}$$

for universal constants in R, δ and c. Since $(\frac{|x|^2}{\delta + |x|^2})^\alpha \leq C|x|^{-\max\{-2\alpha, 0\}}$, we have that

$$\begin{aligned}|x|\left(\frac{|x|^2}{\delta + |x|^2}\right)^{-\frac{(N-1)(p-2)}{2(p-1)}} &\leq C|x|^{-\max\{\frac{(N-1)(p-2)}{p-1} - 1, -1\}} \in L^1_{\text{loc}}(\mathbb{R}^N)\\ \left(\frac{|x|^2}{\delta + |x|^2}\right)^{2 - \frac{(N-1)(p-2)}{2(p-1)}}\frac{1}{|x|} &\leq C|x|^{-\max\{\frac{(N-1)(p-2)}{p-1} - 3, 1\}} \in L^1_{\text{loc}}(\mathbb{R}^N)\end{aligned} \qquad (4.13)$$

in view of $\frac{(N-1)(p-2)}{p-1} < N$. Since $\mathcal{G} = \pm \lambda R^p \Gamma^{p-1}(x)[1 + O(R^{\frac{N-p}{p-1}})]$ when $2 \leq p < N$ in view of $\|H_\lambda\|_\infty < +\infty$, for $\lambda \neq 0$ there holds $\Lambda \geq CR^{\frac{p}{p-1}}$ for some $C > 0$ and all R small in view of $q_0 < \frac{N}{N-p}$, where Λ is given by (4.2), and then

$$\begin{aligned}\int |x|(\delta + |x|^2)^{\frac{(N-1)(p-2)}{2(p-1)}}\eta^2 \frac{|\mathcal{G}|}{u} \quad &\leq \quad \frac{1}{\Lambda}\int |x|(\delta + |x|^2)^{\frac{(N-1)(p-2)}{2(p-1)}}\eta^2 |\mathcal{G}|\\ &\leq \quad C\int |x|^{p+1-N}(\delta + |x|^2)^{\frac{(N-1)(p-2)}{2(p-1)}}\eta^2. \qquad (4.14)\end{aligned}$$

On the other hand, since $\mathcal{G} = \pm \lambda R^N |\log R|^{N-1}[1 + O(\frac{\log |x|}{\log R})]$ in $B_2(0)$ when $p = N$ thanks to $\|H_\lambda\|_\infty < +\infty$, for $\lambda \neq 0$ there holds $\Lambda \geq CR^{\frac{N}{N-1}}|\log R|$ for some $C > 0$ and for all R small and then

$$\int |x|(\delta + |x|^2)^{\frac{N-2}{2}}\eta^2 \frac{|\mathcal{G}|}{u} \leq \frac{1}{\Lambda}\int |x|(\delta + |x|^2)^{\frac{N-2}{2}}\eta^2 |\mathcal{G}| \leq C\int |x||\log |x||(\delta + |x|^2)^{\frac{N-2}{2}}\eta^2. \qquad (4.15)$$

Since

$$|x|^{p+1-N}|\log |x||(\delta + |x|^2)^{\frac{(N-1)(p-2)}{2(p-1)}} \leq C|x|^{p+1-N}|\log |x|| \in L^1_{\text{loc}}(\mathbb{R}^N) \qquad (4.16)$$

when $\lambda \neq 0$ in view of $p \geq 2$, we can use (4.13), (4.16) and the Lebesgue convergence Theorem in (4.12) and (4.14)-(4.15) to get

$$\int \eta^2 |x||\nabla v|^2 \leq C\left(\int |x||\nabla \eta|^2 + \int \frac{\eta^2}{|x|} + \underbrace{\int |x|^{p - \frac{N-1}{p-1}}|\log |x||\eta^2}_{\lambda \neq 0}\right) \qquad (4.17)$$

thanks to the Fatou convergence Theorem. Since $p-\frac{N-1}{p-1}>-1$ if $\lambda\neq 0$ and

$$\begin{aligned}\int_B |v-\bar v| &\leq C'h\int_B |\nabla v|\leq C'h(\int_B \frac{1}{|x|})^{\frac12}(\int_B |x||\nabla v|^2)^{\frac12}\\ &\leq Ch(\int_B \frac{1}{|x|})^{\frac12}\left(\int |x||\nabla \eta|^2+\int \frac{\eta^2}{|x|}+\underbrace{\int |x|^{p-\frac{N-1}{p-1}}|\log |x||\eta^2}_{\lambda\neq 0}\right)^{\frac12}\end{aligned}$$

in view of (4.17), for $|x_0|<3h$ one has that

$$\int_B |v-\bar v|\leq Ch^{\frac{N+1}{2}}\left(h^{N-1}+\underbrace{h^{p-\frac{N-1}{p-1}+N}|\log h|}_{\lambda\neq 0}\right)^{\frac12}=O(h^N)$$

in view of $B_{\frac32 h}(x_0)\subset B_{5h}(0)$, while for $|x_0|\geq 3h$ there holds

$$\begin{aligned}\int_B |v-\bar v| &\leq C'\left[h^2(\int_B \frac{1}{|x|})(\int |x||\nabla\eta|^2)+h^{N+1}(h^{N-1}+|\log h|h^{\min\{p-\frac{N-1}{p-1}+N,N\}})\right]^{\frac12}\\ &\leq C\left[h^2(\frac{h^N}{|x_0|})(|x_0|h^{N-2})+h^{2N}\right]^{\frac12}=O(h^N)\end{aligned}$$

in view of $\frac{3h}{2}\leq\frac{|x_0|}{2}\leq |x|\leq\frac32|x_0|$ for all $x\in B_{\frac32 h}(x_0)$. The proof is complete. $\square$

We are now ready to establish an Harnack inequality for $\mathcal{H}=R^{\frac{N-p}{p-1}}(\pm H_\lambda(Rx)+c)$ when $\mathcal{H}\geq 0$, a crucial tool to establish the Hölder continuity of H_λ at 0.

Theorem 4.4. *Let $1<p\leq N$ if $\lambda=0$ and $p\geq 2$ with $p>\frac N2$ if $\lambda\neq 0$. Assume that $\mathcal{H}=R^{\frac{N-p}{p-1}}(\pm H_\lambda(Rx)+c)\geq 0$ in $B_2(0)$. Then there exist $R_0>0$ and $C>0$ so that*

$$\sup_{B_1(0)}\mathcal{H}\leq C(\inf_{B_1(0)}\mathcal{H}+\Lambda)\tag{4.18}$$

for all $0<R\leq R_0$, where Λ is given in (4.2) in terms of $\mathcal{G}=\pm\lambda R^N G_\lambda^{p-1}(Rx)$.

Proof. Given $p_0>0$ to be specified below, let us fix $0<p_1<p_0$ so that $\kappa^j p_1\neq 2-p,1$ for all $j\geq 0$. Consider first the case $\mu>0$ in Proposition 4.1 to get

$$\Phi_\rho(\kappa\mu,h_1)\leq[\tilde C\mu^\nu(h_2-h_1)^{-\beta}]^{\frac1\mu}\Phi_\rho(\mu,h_2)\tag{4.19}$$

for all $\mu\neq 2-p,1$ and for suitable $\nu,\beta\geq 0$, where $u=|\mathcal{H}|+\Lambda+\epsilon\geq 0$. Starting from p_1 along $\mu_j=\kappa^j p_1$ estimate (4.19) with $1\leq h_1^j=1+2^{-(j+1)}<h_2^j=1+2^{-j}\leq 2$ gives

$$\Phi_\rho(\mu_{j+1},h_1^j)\leq[C(2^\beta\kappa^\nu)^j]^{\frac{1}{\kappa^j p_1}}\Phi_\rho(\mu_j,h_2^j)$$

and then

$$\sup_{B_1(0)}u\leq\lim_{j\to+\infty}\Phi_\rho(\mu_{j+1},h_1^j)\leq C_1\Phi_\rho(p_1,2),\quad C_1=C^{\frac{\kappa}{p_1(\kappa-1)}}(2^\beta\kappa^\nu)^{\frac{1}{p_1}\sum_j\frac{j}{\kappa^j}}\tag{4.20}$$

via an iteration argument as in the proof of Proposition 2.4. Since $\rho>0$ in $B_1(0)\setminus\{0\}$, notice that

$$\|u\|_{\infty,B_1(0)\setminus B_\epsilon(0)}\leq\liminf_{\mu\to+\infty}\Phi_\rho(\mu,1)\leq\limsup_{\mu\to+\infty}\Phi_\rho(\mu,1)\leq\|u\|_{\infty,B_1(0)}$$

and then as $\epsilon\to 0$

$$\lim_{\mu\to+\infty}\Phi_\rho(\mu,1)=\|u\|_{\infty,B_1(0)}=\sup_{B_1(0)}u.\tag{4.21}$$

Consider the case $\mu<0$ in Proposition 4.1 to get

$$\Phi_\rho(\kappa\mu,h_1)\geq [\tilde C|\mu|^\nu(h_2-h_1)^{-\beta}]^{\frac{1}{\mu}}\Phi_\rho(\mu,h_2) \tag{4.22}$$

for all $\mu\neq 2-p$. Starting from $-p_1$ along $\mu_j=\kappa^j(-p_1)$, one can use estimate (4.22) with h_1^j and h_2^j to get

$$\Phi_\rho(\mu_{j+1},h_1^j)\geq [C(2^\beta\kappa^\nu)^j]^{-\frac{1}{\kappa^j p_1}}\Phi_\rho(\mu_j,h_2^j)$$

and then, arguing as we did to show (4.21), one deduces that

$$\inf_{B_1(0)} u\geq \lim_{j\to+\infty}\Phi_\rho(\mu_{j+1},h_1^j)\geq C_2\Phi_\rho(-p_1,2),\quad C_2=C^{-\frac{\kappa}{p_1(\kappa-1)}}(2^\beta\kappa^\nu)^{-\frac{1}{p_1}\sum_j\frac{j}{\kappa^j}}, \tag{4.23}$$

in view of $\mu_j\to-\infty$ as $j\to+\infty$.

Assume now $\mathcal{H}\geq 0$ in $B_2(0)$. Let us finally use Proposition 4.3 to compare (4.20) and (4.23). Indeed, as a consequence of John-Nirenberg Lemma (see Lemma 7 in [32]), Proposition 4.3 shows the existence of $p_0>0$ so that

$$\left(\int_{B_2(0)}\rho e^{p_0 v}\int_{B_2(0)}\rho e^{-p_0 v}\right)^{\frac{1}{p_0}}\leq \|\rho\|_{\infty,B_2(0)}^{\frac{2}{p_0}}\left(\int_{B_2(0)}e^{p_0 v}\int_{B_2(0)}e^{-p_0 v}\right)^{\frac{1}{p_0}}\leq C_3$$

for some universal $C_3>0$, or equivalently

$$\Phi_\rho(p_0,2)\leq C_3\Phi_\rho(-p_0,2) \tag{4.24}$$

in terms of $u=e^v=\mathcal{H}+\Lambda+\epsilon$. The use of (4.24) along with (4.20) and (4.23) gives

$$\sup_{B_1(0)} u\leq C_1\Phi_\rho(p_1,2)\leq C_1'\Phi_\rho(p_0,2)\leq C_1'C_3\Phi_\rho(-p_0,2)\leq C_1'C_3'\Phi_\rho(-p_1,2)\leq \frac{C_1'C_3'}{C_2}\inf_{B_1(0)} u$$

thanks to the Hölder estimate in view of $p_1<p_0$ and $\rho\in L^\infty(B_2(0))$. Since $u=\mathcal{H}+\Lambda+\epsilon$, one then deduces

$$\sup_{B_1(0)}\mathcal{H}\leq C(\inf_{B_1(0)}\mathcal{H}+\Lambda+\epsilon)$$

for some $C>0$ and (4.18) follows by letting $\epsilon\to 0$. □

In particular, for $p\geq 2$ we have the following a-priori $L^\infty-$estimate.

Corollary 4.5. *Let $2\leq p\leq N$. Given $M>0$ and $p_0\geq 1$ there exists $C>0$ so that*

$$\|h+c\|_{\infty,B_R(0)}\leq C(R^{-\frac{N}{p_0}}\|h+c\|_{p_0,B_{2R}(0)}+R^{\frac{pq_0-N}{q_0(p-1)}}\|f\|_{q_0,B_{2R}(0)}^{\frac{1}{p-1}}) \tag{4.25}$$

for all $\epsilon^{p-1}\leq R\leq R_0=\frac14 dist(0,\partial\Omega)$ and all solution h to

$$-\Delta_p(\gamma+h)+\Delta_p\gamma=f\qquad in\ \Omega\setminus\{0\}$$

so that $\nabla h\in L^{\bar q}(\Omega)$, $\frac{|x|^{\frac{1}{p-1}}}{M(\epsilon^p+|x|^{\frac{p}{p-1}})^{\frac{N}{p}}}\leq|\nabla\gamma|\leq M|\nabla\Gamma|$ for some $\epsilon>0$ and $|c|+\|h\|_\infty+\|f\|_{q_0}^{\frac{1}{p-1}}\leq M$ for some $q_0>\frac{N}{p}$.

Proof. Set $\mathcal{H}(x)=R^{\frac{N-p}{p-1}}(h(Rx)+c)$, $0<R<2R_0$. We have that $\mathcal{H}\in L^\infty(B_2(0))$ solves (4.1) with $\mathit{\Gamma}=R^{\frac{N-p}{p-1}}\gamma(Rx)$, $\mathcal{G}=R^Nf(Rx)$ and satisfies $\nabla\mathcal{H}\in L^{\bar q}(B_2(0))$. Since $\|\mathcal{H}\|_{\infty,B_2(0)}\leq 2MR^{\frac{N-p}{p-1}}$ and

$$\|\mathcal{G}\|_{q_0,B_2(0)}^{\frac{1}{p-1}}=R^{\frac{N(q_0-1)}{q_0(p-1)}}\|f\|_{q_0,B_{2R}(0)}^{\frac{1}{p-1}}\leq MR^{\frac{N(q_0-1)}{q_0(p-1)}}, \tag{4.26}$$

we have that

$$\|\mathcal{H}\|_{\infty,B_2(0)}+\|\mathcal{G}\|_{q_0,B_2(0)}^{\frac{1}{p-1}}\leq\tilde M$$

for some $\tilde{M}$ and all $0 < R \leq R_0$. Since

$$\frac{|x|^{\frac{1}{p-1}}}{M2^{\frac{N}{p-1}+\frac{N}{p}}} \leq \frac{|x|^{\frac{1}{p-1}}}{M((\epsilon^{p-1}R^{-1})^{\frac{p}{p-1}}+|x|^{\frac{p}{p-1}})^{\frac{N}{p}}} \leq |\nabla \Gamma| \leq M|\nabla \Gamma|$$

in $B_2(0)$ for $\epsilon^{p-1} \leq R \leq R_0$, Proposition 4.1 gives the validity of (4.4) for all $\mu \neq 0$ and we can argue as in (4.20) to get

$$\sup_{B_1(0)} u \leq C_1 \Phi_1(p_1, 2) \tag{4.27}$$

for a given $0 < p_1 < p_0$ so that $\kappa^j p_1 \neq 1$ for all $j \in \mathbb{N}$, where $u = |\mathcal{H}| + \|\mathcal{G}\|_{q_0,B_2(0)}^{\frac{1}{p-1}} + \epsilon'$. Since $\Phi_1(p_1, 2) \leq |B_2(0)|^{\frac{p_0-p_1}{p_0p_1}} \Phi_1(p_0, 2)$ by Hölder estimate, by (4.27) we deduce that

$$\begin{aligned} \|h+c\|_{\infty,B_R(0)} &= R^{-\frac{N-p}{p-1}} \sup_{B_1(0)} |\mathcal{H}| \leq C' R^{-\frac{N-p}{p-1}} (\|\mathcal{H}\|_{p_0,B_2(0)} + \|\mathcal{G}\|_{q_0,B_2(0)}^{\frac{1}{p-1}} + \epsilon') \\ &\leq C\left(R^{-\frac{N}{p_0}}\|h+c\|_{p_0,B_{2R}(0)} + R^{\frac{pq_0-N}{q_0(p-1)}}\|f\|_{q_0,B_{2R}(0)}^{\frac{1}{p-1}} + \epsilon' R^{-\frac{N-p}{p-1}}\right) \end{aligned} \tag{4.28}$$

in view of (4.26) and

$$\|\mathcal{H}\|_{p_0,B_2(0)} = R^{\frac{N-p}{p-1}} R^{-\frac{N}{p_0}} \|h+c\|_{p_0,B_{2R}(0)}.$$

Letting $\epsilon' \to 0$ in (4.28) we deduce the validity of (4.25) and the proof is complete. □

Finally, let us discuss the Hölder regularity of H_λ at the pole 0. Given Λ in (4.2) in terms of $\mathcal{G} = \pm\lambda R^N G_\lambda^{p-1}(Rx)$, let us re-write the Harnack inequality (4.18) for $\mathcal{H} = R^{\frac{N-p}{p-1}}(\pm H_\lambda(Rx)+c) \geq 0$ in $B_{2R}(0)$ as

$$\sup_{B_R(0)} (\pm H_\lambda + c) \leq C\left(\inf_{B_R(0)} (\pm H_\lambda + c) + R^\sigma\right) \tag{4.29}$$

for all $0 < R \leq R_0$, in view of (4.26) with $f = \pm\lambda G_\lambda^{p-1}$. Since we assume $p \geq 2$ with $p > \frac{N}{2}$ if $\lambda \neq 0$, notice that $\sigma = \frac{pq_0-N}{q_0(p-1)} > 0$ when $\lambda \neq 0$ in view of (2.37) with $q_0 > \frac{N}{p}$, while the term R^σ is not present when $\lambda = 0$. In this second case, we can assume $\sigma \in (0, +\infty)$.

We are now in position to follow the argument in [32] and establish the following Hölder property.

Theorem 4.6. *Let $1 < p \leq N$ if $\lambda = 0$ and $p \geq 2$ with $p > \frac{N}{2}$ if $\lambda \neq 0$. Then $H_\lambda \in C(\bar{\Omega})$ and there exists $C > 0$ such that*

$$|H_\lambda(x) - H_\lambda(0)| \leq C|x|^\alpha \quad \forall\, x \in \Omega \tag{4.30}$$

for some $\alpha \in (0,1)$.

Proof. Setting $M(R) = \sup_{B_R(0)} H_\lambda$ and $\mu(R) = \inf_{B_R(0)} H_\lambda$ for $R > 0$, we claim that the oscillation $\omega(R) = M(R) - \mu(R)$ of H in $B_R(0)$ satisfies

$$\omega(R) \leq C_0 R^\alpha \tag{4.31}$$

for all $0 < R \leq R_0$, for some $\alpha, C_0, R_0 > 0$.

Indeed, apply (4.29) on $B_{\frac{R}{2}}(0)$ either with $c = M(R)$ and the $-$ sign or with $c = -\mu(R)$ and the $+$ sign to get

$$M(R) - \mu'(R) \leq C[M(R) - M'(R)] + CR^\sigma, \quad M'(R) - \mu(R) \leq C[\mu'(R) - \mu(R)] + CR^\sigma \tag{4.32}$$

for all $0 < R \leq 2R_0$, where $M'(R) = M(\frac{R}{2})$ and $\mu'(R) = \mu(\frac{R}{2})$. By adding the two inequalities in (4.32) we get that

$$\omega(\frac{R}{2}) \leq \theta\omega(R) + C_0 R^\sigma \tag{4.33}$$

for all $0 < R \leq 2R_0$, where $\theta = \frac{C-1}{C+1} < 1$ and $C_0 = \frac{2C}{C+1}$. If $\theta \leq 0$, then (4.33) implies the validity of (4.31) with $\alpha = \sigma > 0$ for all $0 < R \leq R_0$ and some $C_0 > 0$. In the case $\theta > 0$, for $S \geq 2$ (4.33) gives that

$$\omega(\frac{R}{S}) \leq \omega(\frac{R}{2}) \leq \theta(\omega(R) + \tau R^\sigma), \quad 0 < R \leq R_0,$$

for some $\tau > 0$ and an iteration starting from $r = R_0$ leads to

$$\omega(\frac{R_0}{S^j}) \leq \theta^j[\omega(R_0) + \tau R_0^\sigma \sum_{k=0}^{j-1} (\theta S^\sigma)^{-k}]. \tag{4.34}$$

Since $\theta \in (0,1)$ and $\sigma > 0$ in (4.33) can be taken smaller than 1, the choice $S = (\frac{2}{\theta})^{\frac{1}{\sigma}} \geq 2$ is admissible in (4.34) yielding

$$\omega(\frac{R_0}{S^j}) \leq \theta^j(\omega(R_0) + 2\tau R_0^\sigma). \tag{4.35}$$

Given $0 < R \leq \frac{R_0}{S}$, let $j_0 \geq 1$ be so that $\frac{R_0}{S^{j_0+1}} < R \leq \frac{R_0}{S^{j_0}}$ and by (4.35) we have

$$\omega(R) \leq \omega(\frac{R_0}{S^{j_0}}) \leq \theta^{j_0}(\omega(R_0) + 2\tau R_0^\sigma) \leq C\theta^{j_0} \tag{4.36}$$

with $C = \omega(R_0) + 2\tau R_0^\sigma$. Setting $\gamma = -\frac{\log \theta}{\log 2} > 0$, then $\theta = 2^{-\gamma} = S^{-\alpha}$ with $\alpha = \frac{\sigma\gamma}{\gamma+1} \in (0,1)$ and (4.36) implies

$$\omega(R) \leq C(\frac{S}{R_0})^\alpha R^\alpha$$

for all $0 < R \leq R_0 2^{-\frac{\gamma+1}{\sigma}}$, and (4.31) is established in this case too.

Since (4.31) gives that $\lim_{R \to 0} \omega(R) = 0$, we deduce that $H_\lambda \in C(\bar{\Omega})$ in view of $G_\lambda \in C^{1,\beta}(\bar{\Omega} \setminus \{0\})$ by elliptic regularity theory [11, 28, 32, 34]. Setting $R = |x|$, (4.31) implies

$$|H_\lambda(x) - H_\lambda(0)| \leq \omega(R) \leq C_0|x|^\alpha$$

for all $x \in B_{R_0}(0)$. Since (4.30) clearly holds in $\Omega \setminus B_{R_0}(0)$ in view of the boundedness of H_λ, we get the validity of (4.30) in the whole Ω and the proof is complete. □

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