

# STABLE PROPER BIHARMONIC MAPS IN EUCLIDEAN SPHERES

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ABSTRACT. We construct an explicit family of stable proper weak biharmonic maps from the closed Euclidean unit ball  $B^m$ ,  $m \geq 5$ , to Euclidean spheres. To the best of the authors knowledge this is the first example of a stable proper weak biharmonic map from a compact domain. To achieve our result we first establish the second variation formula of the bienergy for maps from the unit ball into a Euclidean sphere. Employing this result, we examine the stability of the proper weak biharmonic map  $q : B^m \rightarrow \mathbb{S}^{m_\ell}$ ,  $m, \ell \in \mathbb{N}$  with  $\ell \leq m$ , which we recently constructed in [5] and thus deduce the existence of an explicit family of stable proper biharmonic maps to Euclidean spheres.

## 1. INTRODUCTION

The study of energy functionals is a central direction of research within geometric analysis. Such energy functionals can include various geometric objects such as maps between manifolds or sections of specific vector bundles. Typical questions arising here are the classification of critical points or understanding the properties of a given critical point as for example questions concerning its stability.

Recall that the *energy* of a smooth map  $\phi : M \rightarrow N$  between two Riemannian manifolds  $(M, g)$  and  $(N, h)$ , is given by

$$E(\phi) := \frac{1}{2} \int_M |d\phi|^2 dv,$$

where  $dv := dv_g$  denotes the volume form of  $M$  with respect to  $g$ . Critical points of this energy functional are called *harmonic maps* and they are characterized by the vanishing of the tension field  $\tau$ , which is defined by

$$\tau(\phi) := \text{Tr}_g \bar{\nabla} d\phi.$$

Here,  $\bar{\nabla}$  denotes the connection on the pull-back bundle  $\phi^*TN$ . The harmonic map equation  $\tau(\phi) = 0$  comprises a semilinear elliptic partial differential equation of second order for which many results could be achieved over the last sixty years. For more details on harmonic maps one can consult the book [1]. In recent years many mathematicians got attracted by the study of higher order energy functionals which extend the energy of a map. The most prominent generalization to mention here is the (intrinsic) bienergy functional, or simply the bienergy, which is defined by

$$E_2(\phi) := \frac{1}{2} \int_M |\tau(\phi)|^2 dv. \tag{1.1}$$

Critical points of the bienergy functional  $E_2$  are called *intrinsic biharmonic maps*, or simply *biharmonic maps*, and are characterized by the vanishing of the *bitension field*  $\tau_2$ , which is given by

$$\tau_2(\phi) := \bar{\Delta} \tau(\phi) + \text{Tr}_g R^N(\tau(\phi), d\phi) d\phi. \tag{1.2}$$

Here,  $\bar{\Delta}$  denotes the rough Laplacian acting on sections of the pullback bundle  $\phi^*TN$ . Trivially, harmonic maps are examples of biharmonic maps. Henceforth we will restrict our considerations to non-harmonic biharmonic maps, so called *proper* biharmonic maps. In the case that  $M$  is closed and  $N$  has non-positive curvature the maximum principle implies that

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all biharmonic maps are harmonic [12]. In contrast, there exist many examples of proper biharmonic maps in the Euclidean sphere, see for example [10].

For more details on biharmonic maps in Riemannian geometry we refer to the recent book [22].

In the specific case of maps to the sphere the bienergy (1.1) acquires the form

$$E_2(u) = \frac{1}{2} \int_M (|\Delta u|^2 - |\nabla u|^4) dv, \quad (1.3)$$

where  $u: M \rightarrow \mathbb{S}^n \subset \mathbb{R}^{n+1}$  denotes the composition  $u := \iota \circ \phi: M \rightarrow \mathbb{R}^{n+1}$  of the map  $\phi$  and the canonical inclusion  $\iota: \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$ . The critical points of (1.3) are given by

$$\Delta^2 u + 2 \operatorname{div} (|\nabla u|^2 \nabla u) - (\langle \Delta^2 u, u \rangle - 2|\nabla u|^4)u = 0. \quad (1.4)$$

Although being natural from the geometric point of view, the bienergy (1.1) is not coercive and for this reason it is a challenging task to prove the existence of critical points. In this regard we want to point out the recent result of Laurain and Lin [16] who showed the existence of a biharmonic map from the unit ball in four dimensions to the sphere using the heat flow method.

Recall that the equator map is given by

$$\begin{aligned} u_\star^{(1)}: B^m &\rightarrow \mathbb{S}^m \subset \mathbb{R}^m \times \mathbb{R}, \\ x &\mapsto (u^{(1)}(x), 0), \end{aligned} \quad (1.5)$$

where  $r := r(x) := \|x\|$  denotes the Euclidean distance to the origin and  $u^{(1)}: B^m \rightarrow \mathbb{S}^{m-1}$ ,  $x \mapsto \frac{x}{r}$  is the well-studied radial projection map. Here, the close unit ball is defined by  $B^m = \{x \in \mathbb{R}^m \mid \sum_{i=1}^m x_i^2 \leq 1\}$ . The equator map  $u_\star^{(1)}$  is a well-known example of a harmonic map, see for example [15], and thus does not yield an example of a proper biharmonic map.

Recently, Nakauchi [21] provided a recursive definition of a generalized radial projection map  $u^{(\ell)}: B^m \rightarrow \mathbb{S}^{m^\ell-1}$ , where  $\ell \in \mathbb{N}$  is a parameter and  $u^{(1)}$  coincides with the radial projection map  $u^{(1)}$  introduced above.

For  $\ell = 2$  this generalized radial projection map has the form

$$\begin{aligned} u^{(2)} = (u_{ij})_{1 \leq i, j \leq m}: B^m &\rightarrow \mathbb{S}^{m^2-1} \subset \mathbb{R}^{m^2}, \\ x &\mapsto \frac{1}{\sqrt{m(m-1)}} \left( -\delta_{ij} + m \frac{x_i x_j}{r^2} \right) \end{aligned}$$

and it can be checked that it solves the equation for harmonic maps to spheres.

The generalized radial projection map gives rise to a generalized equator map as follows

$$\begin{aligned} u_\star^{(\ell)}: B^m &\rightarrow \mathbb{S}^{m^\ell} \subset \mathbb{R}^{m^\ell} \times \mathbb{R}, \\ x &\mapsto (u^{(\ell)}(x), 0), \end{aligned} \quad (1.6)$$

which we studied extensively in [6]. It turns out that, by employing a suitable deformation, the generalized equator map (1.6) can be turned into a biharmonic map, compare [5, Theorem 3.2], [7]. More precisely, the map  $q: B^m \rightarrow \mathbb{S}^{m^\ell}$  given by

$$q := (\sin \alpha \cdot u^{(\ell)}, \cos \alpha), \quad \alpha \in (0, \frac{\pi}{2}) \quad (1.7)$$

is a proper biharmonic map if and only if the following equation is satisfied

$$\sin^2 \alpha = \frac{\ell(\ell + m - 2) + 2m - 8}{2\ell(\ell + m - 2)}. \quad (1.8)$$

It turns out that for  $m \geq 5, \ell \geq 2$  equation (1.8) always admits a solution and leads to a weak biharmonic map. Note that these biharmonic maps would be smooth if we would consider  $B^m \setminus \{0\}$  as domain.

In this manuscript we will study a specific property of biharmonic maps, namely their stability: A biharmonic map  $u: M \rightarrow \mathbb{S}^n$  is called *stable* if the second variation of (1.2) is non-negative,

i.e. the inequality

$$\frac{d^2}{dt^2} E_2(u_t)|_{t=0} \geq 0 \quad (1.9)$$

holds for all variations  $u_t$  of  $u$  with  $u_t - u \in W_0^{2,2}(M, \mathbb{R}^{n+1})$ . Since we are working in a weak setting (1.9) is supposed to hold a.e., see Subsection 2.2 for the precise definition of the Sobolev spaces we employ. If inequality (1.9) is strict for all variations  $u_t$  of  $u$  with  $u_t - u \in W_0^{2,2}(M, \mathbb{R}^{n+1})$ , the biharmonic map  $u : M \rightarrow \mathbb{S}^n$  is called *strictly stable*. If the map  $u$  is not stable (and thus not strictly stable), it is called *unstable*.

The main achievement of this manuscript is the construction of proper stable biharmonic maps in spheres. In order to set up our main result we proceed as follows: First we establish the second variation of the bienergy for maps from compact manifolds with boundary to spheres. Employing this identity we deduce a sufficient condition for the stability of sphere-valued biharmonic maps. We then apply this result to examine the stability of the biharmonic maps  $q : B^m \rightarrow \mathbb{S}^{m^\ell}$ ,  $m, \ell \in \mathbb{N}$  with  $\ell \leq m$ , which we have constructed in [5] and recalled above. Combining all these steps we finally arrive at the following result:

**Theorem 1.1.** *The proper weak biharmonic map  $q : B^m \rightarrow \mathbb{S}^{m^\ell}$ ,  $m \geq 5$  given by (1.7) is strictly stable if*

$$m > 2(\sqrt{12\ell^2 + 30\ell + 12} + 3\ell + 6). \quad (1.10)$$

Consequently, for each pair of integers  $m, \ell \in \mathbb{N}$  such that (1.10) is satisfied and (1.8) admits a solution, we get an explicit strictly stable proper biharmonic map  $q : B^m \rightarrow \mathbb{S}^{m^\ell}$ .

**Remark 1.2.** Let us connect the condition (1.10) to the results that have already been established in the literature:

- (1) For  $\ell = 1$  it is known that we obtain a biharmonic map for  $m = 5, 6$  as these are the only dimensions in which (1.8) admits a solution, see [7]. Moreover, in that reference it was shown by constructing an explicit equivariant variation that these biharmonic maps are unstable. This is consistent with Theorem 1.1 as for  $\ell = 1, m = 5, 6$  inequality (1.10) is not satisfied: We would need  $m \geq 33$  in order for (1.10) to be satisfied when  $\ell = 1$ .
- (2) For  $\ell = 2$  the condition (1.8) can be solved if  $m \geq 5$ , see [2, Theorem 1.2]. Moreover, in [2, Theorem 1.3] it was shown that this biharmonic map is unstable if  $5 \leq m \leq 12$ , again by constructing an explicit equivariant variation. This is still consistent with (1.10) which, for  $\ell = 2$ , yields stability if  $m \geq 46$ .
- (3) Moreover, for  $\ell = 3$  we also have that (1.8) admits a solution for  $m \geq 5$ , see [2, Theorem 1.4]. This map is again unstable when  $5 \leq m \leq 18$  being in agreement with (1.10) which requires  $m \geq 59$  in order to obtain a stable biharmonic map.
- (4) In [19, Theorem 1.16] the first example of a strictly stable, proper biharmonic map into a sphere has been provided by Montaldo, Oniciuc and Ratto: Let  $\varphi : \mathbb{R} \rightarrow \mathbb{S}^2$  be given by

$$\gamma \mapsto (\cos(A(\gamma)), \sin(A(\gamma)), 0) \in \mathbb{S}^2 \hookrightarrow \mathbb{R}^3, \quad (1.11)$$

where

$$A(\gamma) = a\gamma^3 + b\gamma^2 + c\gamma + d,$$

and  $a, b, c, d \in \mathbb{R}$  with  $a^2 + b^2 > 0$ . This provides a proper biharmonic curve on  $\mathbb{S}^2$  with a non-compact domain. Assume that either

$$a = 0 \quad \text{or} \quad \{a \neq 0 \text{ and } b^2 - 3ac \leq 0\}.$$

Then  $\varphi$  is strictly stable with respect to compactly supported variations.

- (5) In [18] biharmonic, rotationally symmetric, conformal maps between four-dimensional models of constant sectional curvature have been constructed. In particular, in [18, Theorem 5.3] the authors showed that an explicitly given rotationally symmetric proper biharmonic conformal diffeomorphism  $\varphi : B^4 \rightarrow \mathbb{H}^4$  is equivariantly stable.

Here, equivariant stability refers to the fact that only variations which are invariant under a group action are allowed, see e.g. [4] and the references therein for more details.

One often sees the phenomenon that a critical point of an energy functional is unstable in general but is stable with respect to equivariant variations. This can be explained by the fact that only a special class of directions in the formula for the second variation of the energy is considered.

- (6) Concerning the stability of (1.5) we want to mention the seminal result of Jäger and Kaul [15] who showed that the equator map is minimizing if  $m \geq 7$  and unstable if  $3 \leq m < 7$  when considered as a harmonic map. Later, in [14] Hong and Thompson studied the stability of (1.5) as an extrinsic biharmonic map and showed that the equator map is minimizing if  $m \geq 10$  and unstable if  $5 \leq m < 10$ . Recently, Fardoun, Montaldo and Ratto extended this stability analysis to the case of the equator map as a critical point of the extrinsic  $k$ -energy for  $k \geq 3$  in [8]. Very recently, we extended all of the above results to the case of the generalized equator map (1.6) in [6].
- (7) For proper biharmonic maps which are given explicitly it is often possible to determine the complete spectrum of the corresponding Jacobi operator, see for example [19]. However, these calculations are usually very demanding as one needs to solve a spectral problem for a Jacobi operator of fourth order.
- (8) In Theorem 18 of [12] it is shown that a smooth biharmonic map from a compact Riemannian manifold to the sphere is unstable under the assumption that it satisfies the so-called conservation law. The proof of this fact employs the Hessian of the bienergy and chooses the tension field  $\tau(\phi)$  as variational vector field. This is well-defined as long as one considers a smooth solution of the biharmonic map equation. Once can check that the map (1.7) also satisfies the so-called conservation law at least in a distributional sense. However, as we are considering a weak solution of the biharmonic map equation we cannot insert the tension field of the map into the Hessian of the bienergy as the resulting expression would no longer be well-defined in the Sobolev spaces that we make use of. Thus, our stability result Theorem 1.1 does not contradict [12, Theorem 18].
- (9) Finally, we want to mention that all of the above results only seem to hold for the case of a compact domain manifold with boundary and that one cannot expect corresponding statements on closed manifolds, see [3] for more details.
- (10) Biharmonic curves (biharmonic maps with a one-dimensional domain) from an interval with boundary have been studied by Mou in [20].

**Notation and Conventions:** Throughout this article we will employ the following sign conventions: For the Riemannian curvature tensor field we use

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z,$$

where  $X, Y, Z$  are vector fields.

For the rough Laplacian on the pull-back bundle  $\phi^*TN$  we employ the analysts sign convention, i.e.

$$\bar{\Delta} = \text{Tr}(\bar{\nabla}\bar{\nabla} - \bar{\nabla}_{\nabla}).$$

In particular, this implies that the Laplace operator has a negative spectrum.

By  $dv$  we represent the volume element of an arbitrary Riemannian manifold  $(M, g)$ . In case that the manifold  $(M, g)$  has a non-empty boundary  $\partial M$  we denote the volume element on the boundary by  $dv^{\partial M}$ . Moreover,  $\nu$  will always denote the outward unit normal of  $\partial M$ . In the specific case that  $(M, g)$  is the Euclidean ball, the volume element is simply written as  $dx$ .

Throughout this manuscript we employ the summation convention, i.e. we tacitly sum over repeated indices.

**Organization:** Section 2 contains preliminaries on the generalized radial projection map introduced by Nakauchi in [21]. In Section 3 the first and the second variation of the bienergy for

maps from domain manifolds with non-empty boundary are derived. We provide stable proper biharmonic maps in Section 4, in particular Theorem 1.1 is proven there.

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## 2. PRELIMINARIES

In this section we provide basic facts on the generalized radial projection map introduced by Nakauchi in [21], which appear in the definition (1.7) of the map  $q$ . Here, we closely follow the presentation from [6, Section 2] and also show that the map  $q: B^m \rightarrow \mathbb{S}^{m^\ell}$  defined in (1.7) is a weak biharmonic map.

**2.1. Generalized radial projection.** Recall that in [17, 21] a ‘generalized radial projection’ has been introduced by Nakauchi. More precisely, in [21, Main Theorem, p.1] Nakauchi showed that for any  $\ell, m \in \mathbb{N}$  with  $\ell \leq m$  there exists a harmonic map

$$\begin{aligned} u^{(\ell)}: \mathbb{R}^m \setminus \{0\} &\rightarrow \mathbb{S}^{m^\ell-1}, \\ x = (x_1, \dots, x_m) &\mapsto u^{(\ell)}(x) = (u_{i_1 \dots i_\ell}^{(\ell)}(x))_{1 \leq i_1, \dots, i_\ell \leq m}. \end{aligned} \quad (2.1)$$

We set  $y_i = \frac{x_i}{r}$  and have the recursive definition

$$\begin{aligned} u_{i_1}^{(1)}(x) &= y_{i_1}, \\ u_{i_1 \dots i_\ell}^{(\ell)}(x) &= C_{\ell, m} (y_{i_1} u_{i_1 \dots i_{\ell-1}}^{(\ell-1)}(x) - \frac{1}{\ell + m - 3} r \frac{\partial}{\partial x_{i_\ell}} u_{i_1 \dots i_{\ell-1}}^{(\ell-1)}(x)), \end{aligned}$$

where

$$C_{\ell, m} = \sqrt{\frac{\ell + m - 3}{2\ell + m - 4}}.$$

In particular, we can use (2.1) to define a generalized equator map as we mentioned in the introduction, see (1.6).

For  $\ell = 1$  the equator map (1.5) is recovered. Further, note that we consider  $u_\star^{(\ell)}$  as one-parameter family of maps and use the singular to refer to it.

We will now give some more background on the map defined in (2.1). Let  $\ell, m \in \mathbb{N}$  such that  $\ell \leq m$ . Nakauchi [21] proved that the map  $u^{(\ell)}: \mathbb{R}^m \setminus \{0\} \rightarrow \mathbb{S}^{m^\ell-1}$  has the following properties:

- (1)  $u^{(\ell)}$  satisfies the equation for harmonic maps to spheres

$$\Delta u^{(\ell)} + |\nabla u^{(\ell)}|^2 u^{(\ell)} = 0;$$

- (2)  $u^{(\ell)}$  is a polynomial in  $u_{i_1}, \dots, u_{i_\ell}$  of degree  $\ell$ , where  $u_{i_j} = \frac{x_{i_j}}{r}$ ;

- (3)  $|\nabla u^{(\ell)}|^2 = \frac{\ell(\ell+m-2)}{r^2}$ .

Furthermore, Nakauchi [21, Proposition 1, (1)] showed that the map  $u^{(\ell)}$  satisfies the identity

$$\sum_{j=1}^m y_j \cdot \nabla_j u^{(\ell)} = 0. \quad (2.2)$$

The following Lemma, which is [5, Lemma 2.3], gives an explicit expression for  $\Delta^k u^{(\ell)}$  for any  $k \in \mathbb{N}$ .

**Lemma 2.1.** *For each  $k \in \mathbb{N}$  the map  $u^{(\ell)}$  defined in (2.1) satisfies*

$$\Delta^k u^{(\ell)} = \prod_{j=1}^k (2j + \ell - 2)(2j - \ell - m) \frac{u^{(\ell)}}{r^{2k}}.$$

**Remark 2.2.** The equator map, i.e.  $u_\star^{(1)}$ , has been studied thoroughly in the last decades, see e.g. [8, 15]. Recently, the map  $u^{(\ell)}$  with  $\ell \geq 1$  has been a valuable tool to construct extrinsic polyharmonic maps and to study their stability: The map  $u^{(\ell)}$  has been employed in [2, 5] to provide proper biharmonic maps to spheres. Further, the map  $u^{(\ell)}$  has been used in [5] to construct proper triharmonic maps to spheres. A detailed stability analysis of the generalized equator map (1.6), considered as a critical point of both the extrinsic  $k$ -energy and the  $p$ -energy, has been carried out in [6]. For the notions of extrinsic polyharmonic maps and the extrinsic  $k$ -energy we refer the reader to [5, 6] and the references therein.

**2.2. Weak biharmonic maps to spheres.** In many nonlinear problems in geometric analysis it is favourable to not only consider the class of smooth maps but to also allow for maps of lower regularity and to consider a weak version of the problem at hand. In order to approach the notion of weak solutions of the harmonic map equation let us recall the definition of the Sobolev space for maps to the sphere

$$W^{p,q}(M, \mathbb{S}^n) := \{u \in W^{p,q}(M, \mathbb{R}^{n+1}) \mid u(x) \in \mathbb{S}^n \text{ a.e.}\}.$$

In the case that  $\partial M \neq \emptyset$  we let  $u_0 \in W^{p,q}(M, \mathbb{S}^n)$  and define

$$W_{u_0}^{p,q}(M, \mathbb{S}^n) := \{u \in W^{p,q}(M, \mathbb{S}^n) \mid \nabla^k(u - u_0)|_{\partial M} = 0, 0 \leq k \leq p - 1\}.$$

Here, the boundary condition is to be understood in the sense of traces.

For more details concerning Sobolev spaces on manifolds with boundary we refer to [23, Chapter 4]. Now, we recall the following definition as it is central within this manuscript.

**Definition 2.3.** A map  $u \in W^{2,2}(B^m, \mathbb{S}^n)$  is called a weak biharmonic map if it solves (1.4) in the sense of distributions.

In order to show that  $q: B^m \rightarrow \mathbb{S}^{m^\ell}$  is a weak biharmonic map we have to check if  $q \in W^{2,2}(B^m, \mathbb{S}^{m^\ell})$ . We find that

$$\begin{aligned} \int_{B^m} |\nabla q|^2 dv &= \frac{1}{2}(\ell(\ell + m - 2) + 2m - 8)\text{vol}(\mathbb{S}^{m-1}) \int_0^1 r^{m-3} dr, \\ \int_{B^m} |\Delta q|^2 dv &= \frac{1}{2}\ell(\ell + m - 2)(\ell(\ell + m - 2) + 2m - 8)\text{vol}(\mathbb{S}^{m-1}) \int_0^1 r^{m-5} dr. \end{aligned}$$

We realize that we need to require  $m \geq 5$  in order for the second integral to be finite, hence  $q$  belongs to  $W^{2,2}(B^m, \mathbb{S}^{m^\ell})$  whenever  $m \geq 5$ .

### 3. VARIATIONAL FORMULAS FOR MANIFOLDS WITH BOUNDARY

In this section we derive the formulas for the first and second variation of the bienergy on compact domain manifolds  $M$  with non-empty, smooth boundary  $\partial M$  without embedding the target manifold  $N$  into some Euclidean space. Here, we work exclusively in the category of smooth maps and derive the variational formulas for the bienergy in this setup.

Throughout this section let  $\phi_t: (-\varepsilon, \varepsilon) \times M \rightarrow N, \varepsilon > 0$  be a smooth variation of the map  $\phi: M \rightarrow N$  with associated variational vector field

$$\frac{\bar{\nabla} \phi_t}{\partial t} \Big|_{t=0} = V. \quad (3.1)$$

Moreover,  $\{e_i\}, i = 1, \dots, \dim M$  is a local orthonormal frame of  $TM$  which, at a fixed but arbitrary point  $p \in M$  satisfies  $\nabla_{e_j} e_i = 0, 1 \leq i, j \leq \dim M$  and  $\nabla_{\partial_t} e_j = 0, 1 \leq j \leq \dim M$ .

Throughout the manuscript we will frequently make use of the following Green's formula for the rough Laplacian.

**Lemma 3.1.** *Let  $(M, g)$  be a compact Riemannian manifold with non-empty boundary  $\partial M$  and  $E$  a vector bundle over  $M$ . Then, the following formula holds*

$$\int_M \langle \Delta^E V, W \rangle dv - \int_M \langle V, \Delta^E W \rangle dv = \int_{\partial M} (\langle \nabla_\nu^E V, W \rangle - \langle V, \nabla_\nu^E W \rangle) dv^{\partial M}, \quad (3.2)$$

where  $V, W \in \Gamma(E)$  are smooth sections,  $\nu$  represents the unit outward normal of  $\partial M$  and  $\Delta^E, \nabla^E$  the rough Laplacian and the connection on  $E$  respectively.

*Proof.* This follows from

$$\langle V, \Delta^E W \rangle + \langle \nabla^E V, \nabla^E W \rangle = \operatorname{div} \langle V, \nabla^E W \rangle$$

and the divergence theorem

$$\int_M \operatorname{div} \langle V, \nabla^E W \rangle \, dv = \int_{\partial M} \langle V, \nabla_\nu^E W \rangle \, dv^{\partial M},$$

see [1, Section 2.2] for more details.  $\square$

**Theorem 3.2** (First variation formula with boundary). *Consider a variation of a smooth map  $\phi: M \rightarrow N$  as described above. Then, the first variation of the bienergy functional acquires the form*

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_M |\tau(\phi_t)|^2 \, dv &= \int_M \langle \tau_2(\phi_t), d\phi_t(\partial_t) \rangle \, dv \\ &+ \int_{\partial M} (\langle \bar{\nabla}_\nu d\phi_t(\partial_t), \tau(\phi_t) \rangle - \langle d\phi_t(\partial_t), \bar{\nabla}_\nu \tau(\phi_t) \rangle) \, dv^{\partial M}. \end{aligned} \quad (3.3)$$

*Proof.* We calculate

$$\frac{d}{dt} \frac{1}{2} \int_M |\tau(\phi_t)|^2 \, dv = \int_M \langle R^N(d\phi_t(\partial_t), d\phi_t(e_j))d\phi_t(e_j), \tau(\phi_t) \rangle \, dv + \int_M \langle \bar{\Delta} d\phi_t(\partial_t), \tau(\phi_t) \rangle \, dv.$$

Now, we apply Green's formula for manifolds with boundary, that is (3.2) and find

$$\begin{aligned} \int_M \langle \bar{\Delta} d\phi_t(\partial_t), \tau(\phi_t) \rangle \, dv &= \int_M \langle d\phi_t(\partial_t), \bar{\Delta} \tau(\phi_t) \rangle \, dv \\ &+ \int_{\partial M} (\langle \bar{\nabla}_\nu d\phi_t(\partial_t), \tau(\phi_t) \rangle - \langle d\phi_t(\partial_t), \bar{\nabla}_\nu \tau(\phi_t) \rangle) \, dv^{\partial M}. \end{aligned}$$

Combining these two identities and using (1.2) completes the proof.  $\square$

A direct consequence of Theorem 3.2 is the following fact:

**Proposition 3.3.** *Let  $(M, g)$  be a compact Riemannian manifold with non-empty boundary  $\partial M$  and  $(N, h)$  a Riemannian manifold. Then, a smooth map  $\phi: M \rightarrow N$  is biharmonic if it satisfies*

$$\begin{cases} \tau_2(\phi) = 0 & \text{on } M, \\ \phi = \xi & \text{on } \partial M, \\ d\phi(\nu) = \chi & \text{on } \partial M, \end{cases} \quad (3.4)$$

where  $\xi: \partial M \rightarrow N, \chi: M \rightarrow TN$  are given smooth boundary data.

*Proof.* First of all, we note that the second and third conditions in (3.4) ensure that the map and its normal derivative do not contribute to the variation of the map  $\phi$  on the boundary. In particular, we have that

$$\begin{aligned} d\phi_t(\partial_t)|_{t=0} &= \frac{\nabla}{\partial t} \xi|_{t=0} = 0, \\ \bar{\nabla}_\nu d\phi_t(\partial_t)|_{t=0} &= \bar{\nabla}_{\partial_t} d\phi_t(\nu)|_{t=0} = \bar{\nabla}_{\partial_t} \chi|_{t=0} = 0 \end{aligned}$$

holds on  $\partial M$  which shows that the boundary terms in (3.3) do not contribute to the bitension field.  $\square$

**Remark 3.4.** We would like to point out that the choice of boundary conditions in (3.4) is not unique. This is to be expected as already the boundary problem for the biharmonic equation in a domain of Euclidean space admits several distinguished boundary conditions, see for example [11, Section 2.3] for further details.

**Theorem 3.5.** *Let  $(M, g)$  be a compact Riemannian manifold with non-empty boundary  $\partial M \neq \emptyset$  and  $(N, h)$  a second Riemannian manifold. Assume that  $\phi: M \rightarrow N$  is a smooth biharmonic map satisfying the boundary conditions (3.4). Then, the second variation of the bienergy acquires the form*

$$\begin{aligned} \text{Hess } E_2(\phi)(V, V) &= \int_M |\bar{\Delta}V - R^N(d\phi(e_i), V)d\phi(e_i)|^2 dv & (3.5) \\ &- \int_M \langle (\nabla_{d\phi(e_i)} R^N)(d\phi(e_i), \tau(\phi))V, V \rangle dv \\ &- \int_M \langle (\nabla_{\tau(\phi)} R^N)(d\phi(e_i), V)d\phi(e_i), V \rangle dv \\ &- \int_M \langle R^N(\tau(\phi), V)\tau(\phi), V \rangle dv \\ &- 2 \int_M \langle R^N(d\phi(e_i), V)\bar{\nabla}_{e_i}\tau(\phi), V \rangle dv \\ &- 2 \int_M \langle R^N(d\phi(e_i), \tau(\phi))\bar{\nabla}_{e_i}V, V \rangle dv, \end{aligned}$$

where  $V \in \Gamma(\phi^*TN)$  represents the variational vector field.

*Proof.* We consider a variation of the smooth map  $\phi: M \rightarrow N$  as detailed in (3.1). Then, we find

$$\begin{aligned} \frac{\bar{\nabla}}{\partial t} \Big|_{t=0} \bar{\Delta}\tau(\phi) &= R^N(V, d\phi(e_i))\bar{\nabla}_{e_i}\tau(\phi) + \bar{\nabla}_{e_i}(R^N(V, d\phi(e_i))\tau(\phi)) \\ &+ \bar{\Delta}(\bar{\Delta}V - R^N(d\phi(e_i), V)d\phi(e_i)) \\ &= 2R^N(V, d\phi(e_i))\bar{\nabla}_{e_i}\tau(\phi) + (\nabla_{d\phi(e_i)} R^N)(V, d\phi(e_i))\tau(\phi) \\ &+ R^N(\bar{\nabla}_{e_i}V, d\phi(e_i))\tau(\phi) + R^N(V, \tau(\phi))\tau(\phi) \\ &+ \bar{\Delta}(\bar{\Delta}V - R^N(d\phi(e_i), V)d\phi(e_i)), \end{aligned}$$

where we used that

$$\frac{\bar{\nabla}}{\partial t} \Big|_{t=0} \tau(\phi) = \bar{\Delta}V - R^N(d\phi(e_i), V)d\phi(e_i) \quad (3.6)$$

in the first step. Moreover, we get

$$\begin{aligned} \frac{\bar{\nabla}}{\partial t} \Big|_{t=0} R^N(\tau(\phi), d\phi(e_i))d\phi(e_i) &= (\nabla_V R^N)(\tau(\phi), d\phi(e_i))d\phi(e_i) + R^N(\bar{\Delta}V, d\phi(e_i))d\phi(e_i) \\ &- R^N(R^N(d\phi(e_k), V)d\phi(e_k), d\phi(e_i))d\phi(e_i) \\ &+ R^N(\tau(\phi), \bar{\nabla}_{e_i}V)d\phi(e_i) + R^N(\tau(\phi), d\phi(e_i))\bar{\nabla}_{e_i}V \\ &= -(\nabla_{d\phi(e_i)} R^N)(V, \tau(\phi))d\phi(e_i) - (\nabla_{\tau(\phi)} R^N)(d\phi(e_i), V)d\phi(e_i) \\ &+ R^N(\bar{\Delta}V, d\phi(e_i))d\phi(e_i) \\ &- R^N(R^N(d\phi(e_k), V)d\phi(e_k), d\phi(e_i))d\phi(e_i) \\ &+ R^N(\tau(\phi), \bar{\nabla}_{e_i}V)d\phi(e_i) + R^N(\tau(\phi), d\phi(e_i))\bar{\nabla}_{e_i}V, \end{aligned}$$

where we used (3.6) to establish the first equality. For the second equality we employed the first and the second Bianchi identity.

To determine the boundary contributions in the second variation formula of the bienergy we calculate

$$\begin{aligned} \frac{\bar{\nabla}}{\partial t} \Big|_{t=0} \langle d\phi_t(\partial_t), \bar{\nabla}_\nu \tau(\phi_t) \rangle &= \left\langle \frac{\bar{\nabla}}{\partial t} d\phi_t(\partial_t) \Big|_{t=0}, \bar{\nabla}_\nu \tau(\phi) \right\rangle + \langle R^N(V, d\phi(\partial_\nu))\tau(\phi), V \rangle \\ &+ \langle V, \bar{\nabla}_\nu \bar{\Delta}V \rangle - \langle V, \bar{\nabla}_\nu (R^N(d\phi(e_i), V)d\phi(e_i)) \rangle \end{aligned}$$

and

$$\begin{aligned} \frac{\bar{\nabla}}{\partial t} \Big|_{t=0} \langle \bar{\nabla}_\nu d\phi_t(\partial_t), \tau(\phi) \rangle &= \langle \frac{\bar{\nabla}}{\partial t} \bar{\nabla}_\nu d\phi_t(\partial_t) \Big|_{t=0}, \tau(\phi_t) \rangle \\ &\quad + \langle \bar{\nabla}_\nu V, \bar{\Delta} V \rangle - \langle \bar{\nabla}_\nu V, R^N(d\phi(e_i), V)d\phi(e_i) \rangle. \end{aligned}$$

Thus, we arrive at

$$\begin{aligned} \text{Hess } E_2(\phi)(V, V) &= \int_M \langle \frac{\bar{\nabla}}{\partial t} d\phi_t(\partial_t) \Big|_{t=0}, \tau_2(\phi) \rangle dv \\ &\quad - \int_{\partial M} \langle \frac{\bar{\nabla}}{\partial t} d\phi_t(\partial_t) \Big|_{t=0}, \bar{\nabla}_\nu \tau(\phi) \rangle dv^{\partial M} + \int_{\partial M} \langle \frac{\bar{\nabla}}{\partial t} \bar{\nabla}_\nu d\phi_t(\partial_t) \Big|_{t=0}, \tau(\phi) \rangle dv^{\partial M} \\ &\quad + \int_M \langle V, \bar{\Delta}(\bar{\Delta}V - R^N(d\phi(e_i), V)d\phi(e_i)) \rangle dv \\ &\quad + \int_M \langle R^N(\bar{\Delta}V, d\phi(e_i))d\phi(e_i), V \rangle dv \\ &\quad + \int_M |R^N(d\phi(e_i), V)d\phi(e_i)|^2 dv \\ &\quad + \int_M \langle R^N(V, \tau(\phi))\tau(\phi), V \rangle dv \\ &\quad + 2 \int_M \langle R^N(V, d\phi(e_i))\bar{\nabla}_{e_i} \tau(\phi), V \rangle dv \\ &\quad + 2 \int_M \langle R^N(\tau(\phi), d\phi(e_i))\bar{\nabla}_{e_i} V, V \rangle dv \\ &\quad + \int_M \langle (\nabla_{d\phi(e_i)} R^N)(\tau(\phi), d\phi(e_i))V, V \rangle dv \\ &\quad - \int_M \langle (\nabla_{\tau(\phi)} R^N)(d\phi(e_i), V)d\phi(e_i), V \rangle dv \\ &\quad - \int_{\partial M} \langle R^N(V, d\phi(\partial_\nu))\tau(\phi), V \rangle dv^{\partial M} \\ &\quad - \int_{\partial M} \langle \bar{\nabla}_\nu \bar{\Delta}V, V \rangle dv^{\partial M} + \int_{\partial M} \langle \bar{\nabla}_\nu (R^N(d\phi(e_i), V)d\phi(e_i)), V \rangle dv^{\partial M} \\ &\quad + \int_{\partial M} \langle \bar{\nabla}_\nu V, \bar{\Delta}V \rangle dv^{\partial M} - \int_{\partial M} \langle \bar{\nabla}_\nu V, R^N(d\phi(e_i), V)d\phi(e_i) \rangle dv^{\partial M}, \end{aligned}$$

where we used the first Bianchi identity as follows

$$\begin{aligned} &R^N(\tau(\phi), \bar{\nabla}_{e_i} V)d\phi(e_i) + R^N(\tau(\phi), d\phi(e_i))\bar{\nabla}_{e_i} V + R^N(\bar{\nabla}_{e_i} V, d\phi(e_i))\tau(\phi) \\ &= 2R^N(\tau(\phi), d\phi(e_i))\bar{\nabla}_{e_i} V, \\ &\quad - (\nabla_{d\phi(e_i)} R^N)(V, \tau(\phi))d\phi(e_i) + (\nabla_{d\phi(e_i)} R^N)(V, d\phi(e_i))\tau(\phi) = (\nabla_{d\phi(e_i)} R^N)(\tau(\phi), d\phi(e_i))V. \end{aligned}$$

Concerning the fourth term in the above equation for  $\text{Hess } E_2(\phi)(V, V)$  we again use Green's formula (3.2) to deduce

$$\begin{aligned} \int_M \langle V, \bar{\Delta}(\bar{\Delta}V - R^N(d\phi(e_i), V)d\phi(e_i)) \rangle dv &= \int_M \langle \bar{\Delta}V, \bar{\Delta}V - R^N(d\phi(e_i), V)d\phi(e_i) \rangle dv \\ &\quad - \int_{\partial M} \langle \bar{\nabla}_\nu V, \bar{\Delta}V - R^N(d\phi(e_i), V)d\phi(e_i) \rangle dv^{\partial M} \\ &\quad + \int_{\partial M} \langle V, \bar{\nabla}_\nu (\bar{\Delta}V - R^N(d\phi(e_i), V)d\phi(e_i)) \rangle dv^{\partial M}. \end{aligned}$$

The claim now follows from adding up all the contributions and using that  $\phi: M \rightarrow N$  is a biharmonic map satisfying the boundary data.  $\square$

**Remark 3.6.** (1) If  $\partial M = \emptyset$ , the formula for the second variation (3.5) agrees with the formula provided by Jiang [12, Theorem 16] who studied the case of a closed domain manifold only. Note that a different sign convention for the curvature tensor is used in [12].

(2) We want to highlight an interesting difference between biharmonic maps from closed manifolds and from manifolds with boundary. In the case of a closed manifold every harmonic map is automatically biharmonic. However, this fact may no longer hold in the case of a non-vanishing boundary: As we need to impose additional boundary conditions in the case of the biharmonic map equation, it is not automatic that a harmonic map also provides a solution for the boundary value problem associated with biharmonic maps.

#### 4. STABLE WEAK BIHARMONIC MAPS

In this section we employ the generalized radial projection (2.1) to obtain the first example of a strictly stable, proper biharmonic map, i.e. we prove Theorem 1.1.

We proceed as follows: First, we establish the second variation of the bienergy for sphere-valued maps. From this identity we deduce a sufficient condition for the stability of sphere-valued weak biharmonic maps. In a second step we then apply this result to examine the stability of the sphere-valued biharmonic maps  $q : B^m \rightarrow \mathbb{S}^{m^\ell}$ ,  $m, \ell \in \mathbb{N}$ , which we recently constructed in [5]. Combining these results we can then deduce the existence of strictly stable, proper biharmonic maps which are given explicitly.

We would like to point out that in this section we consider weak solutions of the biharmonic map equation unless stated otherwise.

As already mentioned in the introduction, in the specific case of maps to the sphere  $u : M \rightarrow \mathbb{S}^n$ , the bienergy  $E_2$  acquires the form

$$E_2(u) = \frac{1}{2} \int_M (|\Delta u|^2 - |du|^4) dv. \quad (4.1)$$

First, we recall the precise structure of the critical points of (4.1).

**Proposition 4.1.** *Let  $(M, g)$  be a compact Riemannian manifold with non-empty boundary  $\partial M$ . Then, a smooth map  $u : M \rightarrow \mathbb{S}^n \subset \mathbb{R}^{n+1}$  is biharmonic if it satisfies*

$$\begin{cases} \tau_2(u) = 0 & \text{on } M, \\ u = \xi & \text{on } \partial M, \\ du(\partial_\nu) = \chi & \text{on } \partial M, \end{cases} \quad (4.2)$$

where  $\xi : \partial M \rightarrow \mathbb{S}^n, \chi : \partial M \rightarrow T\mathbb{S}^n$  are given smooth boundary data and  $\tau_2(u)$  is defined by

$$\tau_2(u) = \Delta^2 u + 2 \operatorname{div}(|du|^2 du) - (\langle \Delta^2 u, u \rangle - 2|du|^4)u.$$

In the following Lemma we provide the second variation of the functional (4.1) on Euclidean balls:

**Lemma 4.2.** *Let  $u : B^m \rightarrow \mathbb{S}^n \subset \mathbb{R}^{n+1}$  be a solution of (4.2). Then, the Hessian of the bienergy is given by*

$$\begin{aligned} \operatorname{Hess} E_2(u)(\eta, \eta) &= \int_{B^m} (|\Delta \eta|^2 + 2|du|^4 |\eta|^2) dx - \int_{B^m} \langle \Delta^2 u, u \rangle |\eta|^2 dx \\ &\quad - 2 \int_{B^m} |du|^2 |d\eta|^2 dx - 4 \int_{B^m} |\langle du, d\eta \rangle|^2 dx, \end{aligned} \quad (4.3)$$

where  $\eta \in C_0^\infty(B^m, \mathbb{R}^{n+1})$  with  $\langle \eta, u \rangle = 0$  almost everywhere.

*Proof.* This follows from [8, Proof of Proposition 2.2] and the application of the second variation formula for the  $p$ -energy (4.5) with  $p = 4$ : To determine the second variation of  $E_2$ , we calculate the second variations of the functionals  $\frac{1}{2} \int_{B^m} |\Delta u|^2 dx$  and  $-\frac{1}{2} \int_{B^m} |du|^4 dx$  separately.

The second variation of the first term of (4.1) was determined in the proof of Proposition 2.2 in [8], that is, it was shown that

$$\frac{d^2}{dt^2} \Big|_{t=0} \frac{1}{2} \int_{B^m} |\Delta u_t|^2 dx = \int_{B^m} |\Delta \eta|^2 dx - \int_{B^m} \langle \Delta^2 u, u \rangle |\eta|^2 dx. \quad (4.4)$$

The second term in (4.1) corresponds, up to a constant factor, to the  $p$ -energy with  $p = 4$ . Therefore, we recall that the second variation formula of the  $p$ -energy

$$E_p(u) := \frac{1}{p} \int_{B^m} |du|^p dx$$

for maps from the ball to the sphere, which is given by

$$\text{Hess } E_p(u)(\eta, \eta) = \int_{B^m} |du|^{p-2} (|d\eta|^2 - |du|^2 |\eta|^2) dx + (p-2) \int_{B^m} |du|^{p-4} |\langle du, d\eta \rangle|^2 dx, \quad (4.5)$$

where  $\eta \in C_0^\infty(B^m, \mathbb{R}^{n+1})$  with  $\langle \eta, u \rangle = 0$  almost everywhere. A derivation of (4.5) can be found in [13, p. 143].

The claim follows immediately from identities (4.4) and (4.5).  $\square$

Before we proceed with the construction of a proper stable biharmonic map, we will show in the next Lemma that, for the specific setting under consideration, the formula for the second variation of the bienergy (4.3) coincides with the one derived by Jiang [12], which we recalled in Theorem 3.5. The purpose of presenting the equivalence of these two versions of the second variation is to build a bridge between the analytic and geometric versions of the biharmonic map equation.

Thus, let  $\phi: M \rightarrow N$  be a biharmonic map between Riemannian manifolds and assume that  $N$  has constant curvature. In this case, the second variation of the bienergy evaluated at a critical point (3.5) acquires the form

$$\begin{aligned} \text{Hess } E_2(\phi)(V, V) = & \int_M (|\bar{\Delta} V - R^N(d\phi(e_i), V)d\phi(e_i)|^2 - \langle R^N(\tau(\phi), V)\tau(\phi), V \rangle \\ & - 2\langle R^N(d\phi(e_i), V)\bar{\nabla}_{e_i}\tau(\phi), V \rangle - 2\langle R^N(d\phi(e_i), \tau(\phi))\bar{\nabla}_{e_i}V, V \rangle) dv, \end{aligned} \quad (4.6)$$

where  $V \in \Gamma(\phi^*TN)$ . Note that in [12] a different sign convention for the Riemann curvature tensor is used.

**Lemma 4.3.** *Let  $\phi: B^m \rightarrow \mathbb{S}^n$  be a smooth biharmonic map satisfying the boundary conditions, i.e. a solution of (3.4). Then equation (4.6) is equivalent to a smooth solution  $u: B^m \rightarrow \mathbb{S}^n \subset \mathbb{R}^{n+1}$  of (4.3).*

*Proof.* Since  $N = \mathbb{S}^n$ , the Riemann curvature tensor  $R^N$  is given by

$$R^N(X, Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y.$$

Thus, (4.6) acquires the form

$$\begin{aligned} \text{Hess } E_2(\phi)(V, V) = & \int_{B^m} (|\bar{\Delta} V + |d\phi|^2 V - \langle V, d\phi(e_j) \rangle d\phi(e_j)|^2 \\ & - |\langle V, \tau(\phi) \rangle|^2 + |\tau(\phi)|^2 |V|^2 - 2\langle V, \bar{\nabla}_{e_i}\tau(\phi) \rangle \langle d\phi(e_i), V \rangle \\ & + 2\langle d\phi(e_i), \bar{\nabla}_{e_i}\tau(\phi) \rangle |V|^2 - 2\langle \tau(\phi), \bar{\nabla}_{e_i}V \rangle \langle d\phi(e_i), V \rangle \\ & + 2\langle d\phi(e_i), \bar{\nabla}_{e_i}V \rangle \langle \tau(\phi), V \rangle) dx. \end{aligned}$$

Now, for  $u := \iota \circ \phi: B^m \rightarrow \mathbb{S}^n \subset \mathbb{R}^{n+1}$ , with  $\iota$  being the canonical embedding of  $\mathbb{S}^n$  into  $\mathbb{R}^{n+1}$ , we have  $\tau(u) = \Delta u + |du|^2 u$  and

$$\begin{aligned} \bar{\nabla}_{e_j} V &= \nabla_{e_j} V + \langle du(e_j), V \rangle u, \\ \bar{\Delta} V &= \Delta V - \langle u, \Delta V \rangle u + \langle du, V \rangle du, \end{aligned}$$

where we used that  $\langle u, V \rangle = 0$ . In the following, we will not distinguish between  $V$  and  $du(V)$  to shorten the presentation.

Furthermore, direct computations yield the following identities

$$\begin{aligned}
|\langle V, \tau(\phi) \rangle|^2 &= |\langle V, \Delta u \rangle|^2, \\
|\tau(\phi)|^2 |V|^2 &= (|\Delta u|^2 - |du|^4) |V|^2, \\
\langle V, \bar{\nabla}_{e_j} \tau(\phi) \rangle \langle d\phi(e_j), V \rangle &= (\langle V, \nabla \Delta u \rangle + \langle V, du \rangle |du|^2) \langle du, V \rangle, \\
\langle d\phi(e_j), \bar{\nabla}_{e_j} \tau(\phi) \rangle |V|^2 &= \langle \nabla u, \nabla \Delta u \rangle |V|^2 + |du|^4 |V|^2, \\
\langle \tau(\phi), \bar{\nabla}_{e_j} V \rangle \langle d\phi(e_j), V \rangle &= (\langle \Delta u, \nabla V \rangle - \langle du, V \rangle |du|^2) \langle du, V \rangle, \\
\langle d\phi(e_j), \bar{\nabla}_{e_j} V \rangle \langle V, \tau(\phi) \rangle &= \langle V, \Delta u \rangle \langle du, \nabla V \rangle
\end{aligned}$$

such that

$$\begin{aligned}
\text{Hess } E_2(u)(V, V) &= \int_{B^m} (|\Delta V|^2 + 2|du|^4 |V|^2 + (|\Delta u|^2 + 2\langle du, d\Delta u \rangle) |V|^2 \\
&\quad - |\langle u, \Delta V \rangle|^2 - |\langle V, \Delta u \rangle|^2 + 2|du|^2 \langle \Delta V, V \rangle \\
&\quad - 2\langle V, \nabla \Delta u \rangle \langle du, V \rangle - 2\langle \Delta u, \nabla V \rangle \langle du, V \rangle + 2\langle V, \Delta u \rangle \langle du, \nabla V \rangle) dx.
\end{aligned}$$

Now, we employ the following equations

$$\begin{aligned}
0 &= \langle \Delta u, V \rangle + 2\langle du, \nabla V \rangle + \langle u, \Delta V \rangle, \\
0 &= \Delta |du|^2 + |\Delta u|^2 + 2\langle d\Delta u, du \rangle + \langle \Delta^2 u, u \rangle
\end{aligned}$$

which follow from differentiating  $\langle u, V \rangle = 0$  and  $|u|^2 = 1$ . Hence, we get

$$\begin{aligned}
\text{Hess } E_2(u)(V, V) &= \int_{B^m} (|\Delta V|^2 + 2|du|^4 |V|^2 - \langle \Delta^2 u, u \rangle |V|^2 - 4|\langle du, \nabla V \rangle|^2 \\
&\quad - |\langle u, \Delta V \rangle|^2 - |\langle V, \Delta u \rangle|^2 - (\Delta |du|^2) |V|^2 + 2|du|^2 \langle \Delta V, V \rangle \\
&\quad - 2\langle V, \nabla \Delta u \rangle \langle du, V \rangle - 2\langle \Delta u, \nabla V \rangle \langle du, V \rangle - 2\langle \Delta V, u \rangle \langle du, \nabla V \rangle) dx.
\end{aligned}$$

Using Green's formula for the Laplacian we obtain

$$\begin{aligned}
\int_{B^m} (2|du|^2 \langle \Delta V, V \rangle - (\Delta |du|^2) |V|^2) dv &= -2 \int_{B^m} |du|^2 |\nabla V|^2 dx - \int_{\partial B^m} |V|^2 \partial_\nu |du|^2 dv^{\partial B^m} \\
&\quad + \int_{\partial B^m} |du|^2 \partial_\nu |V|^2 dv^{\partial B^m}.
\end{aligned}$$

Thus, it remains to show that

$$\begin{aligned}
0 &= \int_{B^m} (-|\langle u, \Delta V \rangle|^2 - |\langle V, \Delta u \rangle|^2 \\
&\quad - 2\langle V, d\Delta u \rangle \langle du, V \rangle - 2\langle \Delta u, \nabla V \rangle \langle du, V \rangle - 2\langle \Delta V, u \rangle \langle du, \nabla V \rangle) dx.
\end{aligned} \tag{4.7}$$

Again using the divergence theorem we get

$$\begin{aligned}
\int_{B^m} (\langle V, \nabla \Delta u \rangle \langle du, V \rangle + \langle \Delta u, \nabla V \rangle \langle du, V \rangle) dx &= \int_{B^m} (-|\langle V, \Delta u \rangle|^2 - \langle V, \Delta u \rangle \langle du, \nabla V \rangle) dv \\
&\quad + \int_{\partial B^m} \langle V, \Delta u \rangle \langle du(\partial_\nu), V \rangle dv^{\partial B^m}.
\end{aligned}$$

Moreover, we have

$$-|\langle u, \Delta V \rangle|^2 - |\langle V, \Delta u \rangle|^2 = -2|\langle V, \Delta u \rangle|^2 - 4\langle \nabla u, \nabla V \rangle \langle \Delta u, V \rangle - 4|\langle \nabla u, \nabla V \rangle|^2$$

such that (4.7) indeed holds true completing the proof.  $\square$

We now turn to the construction of stable proper biharmonic maps to the Euclidean sphere. More precisely, from Lemma 4.2 we deduce the following sufficient condition for the stability of a sphere-valued biharmonic map:

**Corollary 4.4.** *A biharmonic map  $u: B^m \rightarrow \mathbb{S}^n \subset \mathbb{R}^{n+1}$  satisfying the boundary conditions is strictly stable if the inequality*

$$\int_{B^m} (|\Delta\eta|^2 + 2|du|^4|\eta|^2) dx - \int_{B^m} \langle \Delta^2 u, u \rangle |\eta|^2 dx - 6 \int_{B^m} |du|^2 |d\eta|^2 dx > 0 \quad (4.8)$$

holds, where  $\eta \in C_0^\infty(B^m, \mathbb{R}^{n+1})$  with  $\langle \eta, u \rangle = 0$  almost everywhere.

*Proof.* The Cauchy-Schwarz inequality yields

$$-2 \int_{B^m} |du|^2 |d\eta|^2 dx - 4 \int_{B^m} |\langle du, d\eta \rangle|^2 dx \geq -6 \int_{B^m} |du|^2 |d\eta|^2 dx.$$

Plugging this inequality into the stability condition (4.3) provides the claim.  $\square$

We now return to the construction of stable biharmonic maps. More precisely, we will use Corollary 4.4 to examine the stability of specific biharmonic maps to spheres which we recently constructed in [5]. To this end, we first recall this existence result, see [5, Theorem 3.2]:

**Theorem 4.5.** *The map  $q: B^m \rightarrow \mathbb{S}^{m^\ell}$  given by*

$$q := (\sin \alpha \cdot u^{(\ell)}, \cos \alpha), \quad (4.9)$$

where  $u^{(\ell)}$  is defined in (2.1), is a proper weak biharmonic map if and only if the following equation is satisfied

$$\sin^2 \alpha = \frac{\ell(\ell + m - 2) + 2m - 8}{2\ell(\ell + m - 2)}, \quad \ell \leq m.$$

Note that [5, Theorem 3.2] is formulated for maps from  $\mathbb{R}^m \setminus \{0\}$  but, apart from including the boundary conditions, all arguments work the same way when considering  $B^m$  instead of  $\mathbb{R}^m \setminus \{0\}$  and changing to the weak formulation of the biharmonic map equation.

Straightforward calculations yield the following identities for (4.9):

$$\begin{aligned} |dq|^2 &= \frac{1}{2}(\ell(\ell + m - 2) + 2m - 8) \frac{1}{r^2}, \\ \int_{B^m} |dq|^4 |\eta|^2 dx &= \frac{1}{4}(\ell(\ell + m - 2) + 2m - 8)^2 \int_{B^m} \frac{|\eta|^2}{r^4} dx, \\ \int_{B^m} \langle \Delta^2 q, q \rangle |\eta|^2 dx &= \frac{1}{2}(\ell + 2)(m + \ell - 4)(\ell(\ell + m - 2) + 2m - 8) \int_{B^m} \frac{|\eta|^2}{r^4} dx, \\ \int_{B^m} |dq|^2 |d\eta|^2 dx &= \frac{1}{2}(\ell(\ell + m - 2) + 2m - 8) \int_{B^m} \frac{|d\eta|^2}{r^2} dx. \end{aligned} \quad (4.10)$$

Our main tool will be the following Hardy-type inequality:

**Lemma 4.6.** *For  $\eta \in W_0^{2,2}(B^m, \mathbb{S}^n)$  with  $m \geq 5$  the following Hardy-type inequality holds*

$$\int_{B^m} |\Delta\eta|^2 dx \geq \frac{m^2}{4} \int_{B^m} \frac{|d\eta|^2}{r^2} dx. \quad (4.11)$$

*Proof.* The statement follows from [24, Theorem 1.1] and a summation over the components, see [9, p. 272] for a similar discussion.  $\square$

With these preparations at hand we are now able to examine the stability of the proper biharmonic map (4.9).

**Proposition 4.7.** *Let  $m \geq 5$ . The proper biharmonic map  $q: B^m \rightarrow \mathbb{S}^{m^\ell}$  defined as in (4.9) is strictly stable if the following inequality is satisfied*

$$m^2 - 12(\ell(\ell + m - 2) + 2m - 8) > 0. \quad (4.12)$$

*Proof.* Plugging the identities (4.10) into the stability condition (4.8) we obtain

$$\begin{aligned} & \int_{B^m} (|\Delta\eta|^2 + 2|dq|^4|\eta|^2) dx - \int_{B^m} \langle \Delta^2 q, q \rangle |\eta|^2 dx - 6 \int_{B^m} |dq|^2 |d\eta|^2 dx \\ &= \int_{B^m} |\Delta\eta|^2 dx - \delta \int_{B^m} \frac{|d\eta|^2}{r^2} dx, \end{aligned}$$

where

$$\delta = 3(\ell(\ell + m - 2) + 2m - 8).$$

Now, we employ (4.11) and get

$$\begin{aligned} & \int_{B^m} (|\Delta\eta|^2 + 2|dq|^4|\eta|^2) dx - \int_{B^m} \langle \Delta^2 q, q \rangle |\eta|^2 dx - 6 \int_{B^m} |dq|^2 |d\eta|^2 dx \\ & \geq \left(\frac{m^2}{4} - \delta\right) \int_{B^m} \frac{|d\eta|^2}{r^2} dx. \end{aligned}$$

As spheres do not admit parallel vector fields we have  $\int_{B^m} \frac{|d\eta|^2}{r^2} dx > 0$ . Therefore, and by inequality (4.12), the map  $q$  is strictly stable.  $\square$

Using Proposition 4.7 we can now prove the main result of this article.

*Proof of Theorem 1.1.* The statement now immediately follows from solving the quadratic equation (4.12).  $\square$

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