

# EULER CHARACTERISTICS OF HIGHER RANK DOUBLE RAMIFICATION LOCI IN GENUS ONE

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ABSTRACT. Double ramification loci parametrise marked curves where a weighted sum of the markings is linearly trivial; higher-rank loci are obtained by imposing several such conditions simultaneously. We obtain closed formulae for the orbifold Euler characteristics of double ramification loci, and their higher-rank generalisations, in genus one. The rank-one formula is a polynomial, while the higher-rank formula involves greatest common divisors of matrix minors. The proof is based on a recurrence relation, which allows for induction on the rank and number of markings.

## INTRODUCTION

Fix  $g \geq 0$  and a vector  $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$  with  $\sum_{i=1}^n a_i = 0$ . The associated (open) double ramification locus is given on the level of closed points by:

$$\mathrm{DR}_{g,n}(a) = \{(C, p_1, \dots, p_n) : \mathcal{O}_C(\sum_{i=1}^n a_i p_i) \cong \mathcal{O}_C\} \subseteq \mathcal{M}_{g,n}.$$

Higher-rank double ramification loci are produced by intersecting several of the above loci. We determine the orbifold Euler characteristics of double ramification loci and their higher-rank generalisations in the first nontrivial genus, namely  $g = 1$ .

Recently, strata of differentials have attracted considerable attention due to their position at the interface of dynamics and algebraic geometry [17–19, 21]. Yet, little is known about their global topology [11, 38]. In genus one, double ramification loci exhaust the strata of meromorphic  $k$ -differentials, due to the triviality of the canonical bundle.

**0.1. Results.** We begin in Section 1 with the classical (rank-one) case. The main result is:

**Theorem X** (Theorem 1.1). *Given  $a = (a_1, \dots, a_n)$  the orbifold Euler characteristic of  $\mathrm{DR}_{1,n}(a)$  is given by:*

$$\chi_{\mathrm{orb}}(\mathrm{DR}_{1,n}(a)) = \frac{(-1)^{n-1} (n-1)!}{24} \left( \sum_{i=1}^n a_i^2 - 2 \right).$$

The combinatorial prefactor equals  $-\chi_{\mathrm{orb}}(\mathcal{M}_{1,n})/2$  (see Lemma 1.3). The above formula is obtained independently (and with a different proof) in the upcoming [13].

In Section 2 we proceed to the higher-rank case. Here the input data is an  $r \times n$  matrix  $A$  such that each row sums to zero. The associated higher-rank double ramification locus

$$\mathrm{DR}_{g,n}^r(A) \subseteq \mathcal{M}_{g,n}$$

is the intersection of the  $r$  double ramification loci associated to the rows of  $A$ . We determine its orbifold Euler characteristic.

**Theorem Y** (Theorem 2.2). *The orbifold Euler characteristic of the higher-rank double ramification locus  $\mathrm{DR}_{1,n}^r(A)$  is given by:*

$$\chi_{\mathrm{orb}}(\mathrm{DR}_{1,n}^r(A)) = \frac{(-1)^n}{12} \sum_{k=0}^r \sum_{\substack{\mathcal{I} \vdash [n] \\ \ell(\mathcal{I})=k+1}} (-1)^k (\#I_1 - 1)! \cdots (\#I_{k+1} - 1)! \cdot G_{k \times k}(A_{\mathcal{I}})^2.$$

The sum is over partitions  $\mathcal{I} = \{I_1, \dots, I_{k+1}\}$  of the set  $[n] = \{1, \dots, n\}$ , the contraction matrix  $A_{\mathcal{I}}$  is the  $r \times (k+1)$  matrix obtained by summing the columns of  $A$  associated to each part of  $\mathcal{I}$ , and  $G_{k \times k}(A_{\mathcal{I}})$  denotes the greatest common divisor of all the  $k \times k$  minors of  $A_{\mathcal{I}}$ .

We note in particular that in higher rank, the orbifold Euler characteristic is not polynomial in the entries of the matrix  $A$ .

In Section 2.6 we simplify the  $k = r$  term in the above sum, expressing it in terms of the  $r \times r$  minors of the original matrix  $A$  (Proposition 2.10). When  $r = 1$  the formulae in Theorems X and Y are superficially different, but in Section 2.7 we identify them using symmetric function theory.

The following example families convey the flavour of Theorem Y:

$$\chi_{\text{orb}} \text{DR}_{1,3}^2 \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \frac{1}{12} (-2 + \gcd(a_1, b_1)^2 + \gcd(a_2, b_2)^2 + \gcd(a_3, b_3)^2 - (a_1 b_2 - a_2 b_1)^2),$$

$$\chi_{\text{orb}} \text{DR}_{1,4}^2 \begin{bmatrix} a & -a & 0 & 0 \\ 0 & 0 & b & -b \end{bmatrix} = \frac{1}{12} (6 - 4a^2 - 4b^2 - 2 \gcd(a, b)^2 + 4(ab)^2),$$

$$\begin{aligned} \chi_{\text{orb}} \text{DR}_{1,4}^3 \begin{bmatrix} a & -a & 0 & 0 \\ 0 & b & -b & 0 \\ 0 & 0 & c & -c \end{bmatrix} &= \frac{1}{12} (6 - 2a^2 - b^2 - 2c^2 - 2 \gcd(a, b)^2 - \gcd(a, c)^2 - 2 \gcd(b, c)^2 \\ &\quad - \gcd(a, b, c)^2 + (ab)^2 + (ac)^2 + (bc)^2 + 3 \gcd(ab, ac, bc)^2 - (abc)^2). \end{aligned}$$

**0.2. Proof strategy.** The proof hinges on a recurrence relation, which we illustrate in rank one. Fix  $a = (a_1, \dots, a_n, a_{n+1})$  with  $\sum_{i=1}^{n+1} a_i = 0$  and suppose without loss of generality that  $a_{n+1} \neq 0$ . Consider the forgetful morphism:

$$(1) \quad \text{DR}_{1,n}(a_1, \dots, a_n, a_{n+1}) \rightarrow \mathcal{M}_{1,n}.$$

Given  $(C, p_1, \dots, p_n) \in \mathcal{M}_{1,n}$  a choice of lift is a choice of  $p_{n+1} \in C \setminus \{p_1, \dots, p_n\}$  such that:

$$\mathcal{O}_C(a_{n+1} p_{n+1}) \cong \mathcal{O}_C(-\sum_{i=1}^n a_i p_i).$$

Since we work in genus one, the simple expectation is that there are precisely  $a_{n+1}^2$  such lifts. However there is a complication: we must exclude the possibility that  $p_{n+1} = p_i$  are valid lifts. This amounts to removing the following double ramification loci from  $\mathcal{M}_{1,n}$ :

$$(2) \quad \text{DR}_{1,n}(a_1, \dots, a_i + a_{n+1}, \dots, a_n) \subseteq \mathcal{M}_{1,n}.$$

The proof now proceeds by cut-and-paste. The double ramification loci (2) define a stratification of  $\mathcal{M}_{1,n}$  which pulls back to a stratification of  $\text{DR}_{1,n}(a_1, \dots, a_n, a_{n+1})$ . We then study the map (1) stratum by stratum; on each locally-closed stratum it is étale of calculable degree.

Higher-rank double ramification loci inevitably enter into this argument, since they arise as deeper strata. However, these higher-rank loci do not appear in the final statement of the recursion: after assembling the contributions we observe a remarkable collection of terms, collapsing the formula and producing a purely rank-one statement:

**Theorem Z** (Theorem 1.2). *The orbifold Euler characteristic of  $\text{DR}_{1,n+1}(a)$  satisfies the following recurrence:*

$$\chi_{\text{orb}}(\text{DR}_{1,n+1}(a)) = a_{n+1}^2 \chi_{\text{orb}}(\mathcal{M}_{1,n}) - \sum_{i=1}^n \chi_{\text{orb}}(\text{DR}_{1,n}(a_1, \dots, a_i + a_{n+1}, \dots, a_n)).$$

The proof of Theorem X then proceeds by induction on  $n$ , and is straightforward once the correct formula is guessed.

The higher-rank recursion (Theorem 2.7) is no more complicated, however it only applies to matrices  $A$  of a special form. We reduce to such matrices using  $\text{GL}_r(\mathbb{Z})$ -invariance of the final formula (Lemma 2.4 and Proposition 2.6). The proof then proceeds by induction on  $(r, n)$  in lexicographic order. Again, the difficult step is guessing the correct formula.

Our proof in fact establishes a recurrence in the étale Grothendieck ring of orbifolds, the quotient of the Grothendieck ring of orbifolds by the relations

$$[Y] = [X] \cdot [F]$$

for any étale morphism  $Y \rightarrow X$  with fibre  $F$ . However, the étale Grothendieck ring of orbifolds is isomorphic to  $\mathbb{Q}$ , the isomorphism being given by the orbifold Euler characteristic: see [53] for a proximate argument. This refinement thus contains no additional information.

**Remark 0.1.** The orbifold Euler characteristic, also known as the Euler–Satake characteristic, was introduced by Satake in his foundational work on orbifolds [51, Section 3]. It is additive with respect to stratifications and multiplicative with respect to étale covers. It is closely related to the Euler characteristic of classifying spaces [57], a perspective adopted by Harer–Zagier [27].

The above notion is not to be confused with the **stringy orbifold Euler characteristic** (confusingly, sometimes also called the orbifold Euler characteristic) arising from physics [16, 28]. This is not multiplicative with respect to étale covers, and typically differs from both the Euler–Satake characteristic and the naive Euler characteristic.

**0.3. Intersection theory.** Given a normal crossings compactification  $\mathrm{DR}_{g,n}^r(A) \subseteq \overline{\mathrm{DR}}_{g,n}^r(A)$  the logarithmic Gauss–Bonnet–Poincaré–Hopf formula identifies

$$(3) \quad \chi_{\mathrm{orb}}(\mathrm{DR}_{g,n}^r(A)) = \int_{\overline{\mathrm{DR}}_{g,n}^r(A)} c_{\mathrm{top}}(\Theta)$$

where  $\Theta$  is the logarithmic tangent bundle of the compactification (see e.g. [11, Proposition 2.1]).

Unfortunately, no such normal crossings compactification has yet been constructed. While compactifications of (higher-rank) double ramification loci have been studied extensively, and play a central role in the (logarithmic) intersection theory of the moduli space of curves and enumerative geometry, these compactifications either contain spurious components or are highly singular at the boundary [1–3, 6, 7, 9, 10, 20, 22, 24–26, 29–37, 39, 40, 42–47, 50, 54, 56].

Focusing on genus one, the triviality of the canonical bundle identifies the double ramification locus with the space of  $k$ -differentials for any  $k$ . In rank one, a normal crossings compactification is then given by the space of multiscale differentials [4]. This can be interpreted [5, 8] in terms of logarithmic structures and Gorenstein curves, based on the ideas of [48, 49]. These same ideas can then be used to produce a normal crossings compactification in higher rank, by combining [49, Theorem B] with rigidification arguments [43, 50].

Once such a general compactification is constructed, its fundamental class will push forward to a class on a logarithmic blowup of the moduli space of curves. This will differ from the logarithmic double ramification cycle [31, 45] by boundary corrections arising from the spurious components of the latter.

Reversing the logic, our Theorems X and Y calculate the specific class (3) of tautological integrals. Recent work of Toh [55] studies tautological integrals on the full logarithmic double ramification cycle in rank two, obtaining formulae which also involve greatest common divisors and matrix minors, but not matrix contractions.

This intersection-theoretic strategy has been applied in other contexts. In [23] it is used to give a new proof of the Harer–Zagier formula for the orbifold Euler characteristic of the moduli space of curves. Here the description

$$\Theta = R^1 \pi_* \omega_C^\vee(-\sum_{i=1}^n p_i)$$

identifies the orbifold Euler characteristic with an integral on  $\overline{\mathcal{M}}_{g,n}$  involving lambda and psi classes. The same strategy also applies to strata of differentials with a single marking [14, Theorem 1.5]. For a calculation strategy closer to our own, involving universal compactified Jacobians, see [58].

**0.4. Higher genus: the Hurwitz stratification.** We describe an in-principle method for computing the orbifold Euler characteristic of the double ramification locus in all genus. This method is significantly less efficient than the genus-one recursion employed above, and we are unable to use it to obtain a closed formula. Moreover it does not generalise to higher rank.

The locus  $\text{DR}_{g,n}(a)$  is stratified by Hurwitz spaces, which fix the entire ramification profile. A Hurwitz space with  $m$  branch points is an étale cover of  $\mathcal{M}_{0,m}$  of degree equal to the associated Hurwitz number. Using  $\chi(\mathcal{M}_{0,m}) = (-1)^{m-3}(m-3)!$  and accounting for the labelling of the branch points, this expresses each  $\chi_{\text{orb}}(\text{DR}_{g,n}(a))$  as a weighted sum of Hurwitz numbers.

In the upcoming [13] this method is computer implemented using the packages `admcycles` and `diffstrata` [12, 15] where it is in particular used to experimentally verify Theorem X.

**Notation.** For an integer  $n \geq 1$  we write  $[n] := \{1, \dots, n\}$ .

**Acknowledgements.** This project began at the University of Bologna, which we thank for excellent working conditions. Across the years we have benefited from discussions on double ramification loci with Francesca Carocci, Robert Crumplin, David Holmes, Sam Molcho, and Dhruv Ranganathan. We thank Johannes Schmitt for interesting conversations on the upcoming related work [13]; Terry Dekun Song and Evgeny Shinder for useful clarifications on the (étale) Grothendieck ring; and the anonymous referee for valuable expositional comments.

**Funding.** L.B. is partially supported by the European Union – NextGenerationEU under the National Recovery and Resilience Plan (PNRR) – Mission 4: Education and Research – Component 2: From Research to Business – Investment 1.1 Notice PRIN 2022 – DD N. 104 del 2/2/2022, from title “Symplectic varieties: their interplay with Fano manifolds and derived categories”, proposal code 2022PEKYBJ – CUP J53D23003840006. L.B. is a member of INdAM group GNSAGA.

## 1. RANK ONE

Fix  $g = 1$ ,  $n \geq 1$  and a ramification vector

$$a = (a_1, \dots, a_n)$$

with  $\sum_{i=1}^n a_i = 0$ . Let  $\text{Jac}_{1,n} \rightarrow \mathcal{M}_{1,n}$  denote the universal Jacobian and consider the sections:

$$\begin{aligned} 0: \mathcal{M}_{1,n} &\rightarrow \text{Jac}_{1,n} & \text{aj}_a: \mathcal{M}_{1,n} &\rightarrow \text{Jac}_{1,n} \\ (C, p_1, \dots, p_n) &\mapsto \mathcal{O}_C, & (C, p_1, \dots, p_n) &\mapsto \mathcal{O}_C(\sum_{i=1}^n a_i p_i). \end{aligned}$$

The double ramification locus  $\text{DR}_{1,n}(a)$  is the fibre product:

$$\begin{array}{ccc} \text{DR}_{1,n}(a) & \longrightarrow & \mathcal{M}_{1,n} \\ \downarrow & \square & \downarrow \text{aj}_a \\ \mathcal{M}_{1,n} & \xrightarrow{0} & \text{Jac}_{1,n}. \end{array}$$

By [52, Proposition 1.2] this is a smooth hypersurface in  $\mathcal{M}_{1,n}$  (though possibly empty, and excluding the trivial case  $a = (0, \dots, 0)$ ). At the level of closed points:

$$\text{DR}_{1,n}(a) = \{(C, p_1, \dots, p_n) \in \mathcal{M}_{1,n} \mid \mathcal{O}_C(\sum_{i=1}^n a_i p_i) \cong \mathcal{O}_C\}.$$

The main result of this section is:

**Theorem 1.1** (Theorem X). *The orbifold Euler characteristic of  $\text{DR}_{1,n}(a)$  is given by:*

$$(4) \quad \chi_{\text{orb}}(\text{DR}_{1,n}(a)) = \frac{(-1)^{n-1}(n-1)!}{24} \left( \sum_{i=1}^n a_i^2 - 2 \right).$$

This result will be deduced from the following recurrence relation.

**Theorem 1.2** (Theorem Z). Fix  $n \geq 1$  and a length  $n+1$  ramification vector  $a = (a_1, \dots, a_n, a_{n+1})$ . For each  $i \in [n]$  define the following length  $n$  ramification vector:

$$a(i) := (a_1, \dots, a_{i-1}, a_i + a_{n+1}, a_{i+1}, \dots, a_n).$$

Then the orbifold Euler characteristic of  $\mathrm{DR}_{1,n+1}(a)$  satisfies the following recurrence relation:

$$(5) \quad \chi_{\mathrm{orb}}(\mathrm{DR}_{1,n+1}(a)) = a_{n+1}^2 \chi_{\mathrm{orb}}(\mathcal{M}_{1,n}) - \sum_{i=1}^n \chi_{\mathrm{orb}}(\mathrm{DR}_{1,n}(a(i))).$$

*Proof.* For  $a_{n+1} = 0$  the result follows immediately by studying the fibres of the smooth and representable morphism:

$$\mathrm{DR}_{1,n+1}(a_1, \dots, a_n, 0) \rightarrow \mathrm{DR}_{1,n}(a_1, \dots, a_n).$$

Thus assume  $a_{n+1} \neq 0$  and consider the representable morphism forgetting the final marking:

$$(6) \quad \mathrm{DR}_{1,n+1}(a_1, \dots, a_n, a_{n+1}) \rightarrow \mathcal{M}_{1,n}.$$

We begin with the  $n = 1$  case, which is instructive. We claim that the morphism (6) is étale of degree  $a_2^2 - 1$ . Indeed we have  $a_2 = -a_1$ , and given  $(C, p_1) \in \mathcal{M}_{1,1}$  a lift consists of a choice of  $p_2 \in C$  such that  $\mathcal{O}_C(a_2 p_2) \cong \mathcal{O}_C(a_2 p_1)$  and  $p_2 \neq p_1$ . There are precisely  $a_2^2 - 1$  of these, and we obtain the relation

$$(7) \quad \chi_{\mathrm{orb}}(\mathrm{DR}_{1,2}(a_1, a_2)) = (a_2^2 - 1) \chi_{\mathrm{orb}}(\mathcal{M}_{1,1}).$$

In this case we only have  $a(1) = (a_1 + a_2) = (0)$  and so  $\mathrm{DR}_{1,1}(a(1)) = \mathcal{M}_{1,1}$ . Therefore (7) is equivalent to (5) and this completes the  $n = 1$  case.

For  $n \geq 2$ , however, the morphism (6) is not étale. We will now define stratifications of the source and target, and show that the morphism is étale on each locally-closed stratum, with degree depending on the stratum.

We first define the stratification on  $\mathcal{M}_{1,n}$ . Recall for  $i \in [n]$  the length  $n$  ramification vector:

$$a(i) := (a_1, \dots, a_{i-1}, a_i + a_{n+1}, a_{i+1}, \dots, a_n).$$

We define the depth-1 closed strata in  $\mathcal{M}_{1,n}$  to be:

$$\mathrm{DR}_{1,n}(a(1)), \dots, \mathrm{DR}_{1,n}(a(n)).$$

More generally, the depth- $k$  closed strata are indexed by subsets  $I = \{i_1, \dots, i_k\} \subseteq [n]$  of size  $k$  and given by:

$$\mathrm{DR}_{1,n}(a(I)) = \mathrm{DR}_{1,n} \begin{bmatrix} a(i_1) \\ \vdots \\ a(i_k) \end{bmatrix} := \bigcap_{j=1}^k \mathrm{DR}_{1,n}(a(i_j)).$$

Note that these intersections are often not dimensionally transverse, for instance if two rows admit a common divisor. Therefore a depth- $k$  stratum may have smaller codimension than  $k$ .

We put a partial order on the closed strata  $\mathrm{DR}_{1,n}(a(I))$  using the inclusion order on the index sets  $I$  (note that inside  $\mathcal{M}_{1,n}$  there may be additional containments of closed strata beyond those forced by the index sets). From the closed strata we obtain locally-closed strata by removing the deeper strata:

$$\mathrm{DR}_{1,n}^\circ(a(I)) := \mathrm{DR}_{1,n}(a(I)) \setminus \bigcup_{I \subsetneq J \subseteq [n]} \mathrm{DR}_{1,n}(a(J)).$$

Some locally-closed strata may be empty, but this does not affect the argument.

This stratification of  $\mathcal{M}_{1,n}$  pulls back along (6) to give a stratification of  $\mathrm{DR}_{1,n+1}(a)$  which we describe explicitly. The depth-1 closed strata in this stratification are

$$\mathrm{DR}_{1,n+1} \begin{bmatrix} a_1 & \cdots & a_n & a_{n+1} \\ \hline & & a(i) & 0 \end{bmatrix}$$

while the depth- $k$  closed strata consist of the intersections. We use the same notation as before for the locally-closed strata. Restricting (6) to a locally-closed stratum in  $\mathrm{DR}_{1,n+1}(a)$  of depth  $k$  we obtain a map:

$$\mathrm{DR}_{1,n+1}^\circ \begin{bmatrix} a_1 & \cdots & a_n & a_{n+1} \\ \text{---}a(i_1)\text{---} & & & 0 \\ & \vdots & & \vdots \\ \text{---}a(i_k)\text{---} & & & 0 \end{bmatrix} \rightarrow \mathrm{DR}_{1,n}^\circ \begin{bmatrix} \text{---}a(i_1)\text{---} \\ \vdots \\ \text{---}a(i_k)\text{---} \end{bmatrix}.$$

We claim that this map is étale of degree  $a_{n+1}^2 - k$ . Fix a point  $(C, p_1, \dots, p_n)$  in the target. Then a lift consists of a choice of point  $p_{n+1} \in C$  such that

$$(8) \quad \mathcal{O}_C(a_{n+1}p_{n+1}) \cong \mathcal{O}_C(-\sum_{i=1}^n a_i p_i).$$

There are  $a_{n+1}^2$  of these, but we need to examine the possibilities  $p_{n+1} = p_i$ . Let  $I = \{i_1, \dots, i_k\} \subseteq [n]$  be the set indexing the given stratum. For  $i \in I$  the point  $(C, p_1, \dots, p_n)$  in the target satisfies

$$\mathcal{O}_C(a_1 p_1 + \dots + a_{i-1} p_{i-1} + (a_i + a_{n+1}) p_i + a_{i+1} p_{i+1} + \dots + a_n p_n) \cong \mathcal{O}_C$$

because  $a(i)$  appears as a row in the target matrix. It follows that the choice  $p_{n+1} = p_i$  satisfies the ramification condition (8). On the other hand for  $i \in [n] \setminus I$  the fact that we have removed the intersections with deeper strata ensures that

$$\mathcal{O}_C(a_1 p_1 + \dots + a_{i-1} p_{i-1} + (a_i + a_{n+1}) p_i + a_{i+1} p_{i+1} + \dots + a_n p_n) \not\cong \mathcal{O}_C$$

and therefore the choice  $p_{n+1} = p_i$  does not satisfy the ramification condition (8). It follows that we must remove  $|I| = k$  of the choices of  $p_{n+1}$  and that the remaining choices give valid lifts. This shows that the map is étale of degree  $a_{n+1}^2 - k$  as claimed. We obtain an identity:

$$\chi_{\mathrm{orb}}(\mathrm{DR}_{1,n+1}^\circ \begin{bmatrix} a_1 & \cdots & a_n & a_{n+1} \\ \text{---}a(i_1)\text{---} & & & 0 \\ & \vdots & & \vdots \\ \text{---}a(i_k)\text{---} & & & 0 \end{bmatrix}) = (a_{n+1}^2 - k) \chi_{\mathrm{orb}}(\mathrm{DR}_{1,n}^\circ \begin{bmatrix} \text{---}a(i_1)\text{---} \\ \vdots \\ \text{---}a(i_k)\text{---} \end{bmatrix}).$$

The above formula is also valid when  $a_{n+1}^2 - k < 0$ , for in this case we claim that both source and target strata are empty. Indeed, suppose otherwise and choose a point  $(C, p_1, \dots, p_n)$  in the target stratum. The ramification conditions imply that

$$\mathcal{O}_C(a_{n+1} p_i) \cong \mathcal{O}_C(a_{n+1} p_j)$$

for all  $i, j \in I$ . There are at most  $a_{n+1}^2$  such points of  $C$ , and it follows that  $k \leq a_{n+1}^2$  which contradicts the assumption. We conclude that if  $a_{n+1}^2 - k < 0$  then the source and target strata are empty, so the above formula trivially holds.

We now employ the scissor relations with respect to the stratification of the source of (6):

$$(9) \quad \begin{aligned} \chi_{\mathrm{orb}}(\mathrm{DR}_{1,n+1}(a)) &= \sum_{k=0}^n \sum_{\{i_1, \dots, i_k\} \subseteq [n]} \chi_{\mathrm{orb}}(\mathrm{DR}_{1,n+1}^\circ \begin{bmatrix} a_1 & \cdots & a_n & a_{n+1} \\ \text{---}a(i_1)\text{---} & & & 0 \\ & \vdots & & \vdots \\ \text{---}a(i_k)\text{---} & & & 0 \end{bmatrix}) \\ &= \sum_{k=0}^n \sum_{\{i_1, \dots, i_k\} \subseteq [n]} (a_{n+1}^2 - k) \chi_{\mathrm{orb}}(\mathrm{DR}_{1,n}^\circ \begin{bmatrix} \text{---}a(i_1)\text{---} \\ \vdots \\ \text{---}a(i_k)\text{---} \end{bmatrix}). \end{aligned}$$

Having used the scissor relations to deconstruct the source, we now use them to reconstruct the target. The key relations are:

$$\chi_{\text{orb}}(\mathcal{M}_{1,n}) = \sum_{k=0}^n \sum_{\{i_1, \dots, i_k\} \subseteq [n]} \chi_{\text{orb}}(\text{DR}_{1,n}^{\circ} \begin{bmatrix} \text{---}a(i_1)\text{---} \\ \vdots \\ \text{---}a(i_k)\text{---} \end{bmatrix}),$$

$$\chi_{\text{orb}}(\text{DR}_{1,n}(a(i))) = \sum_{k=1}^n \sum_{\substack{\{i_1, \dots, i_k\} \subseteq [n] \\ i \in \{i_1, \dots, i_k\}}} \chi_{\text{orb}}(\text{DR}_{1,n}^{\circ} \begin{bmatrix} \text{---}a(i_1)\text{---} \\ \vdots \\ \text{---}a(i_k)\text{---} \end{bmatrix}).$$

Examining (9) we see that each term of the form

$$k \cdot \chi_{\text{orb}}(\text{DR}_{1,n}^{\circ} \begin{bmatrix} \text{---}a(i_1)\text{---} \\ \vdots \\ \text{---}a(i_k)\text{---} \end{bmatrix})$$

participates in precisely  $k$  of the  $\chi_{\text{orb}}(\text{DR}_{1,n}(a(i)))$ . Assembling, we conclude from (9) that

$$\chi_{\text{orb}}(\text{DR}_{1,n+1}(a)) = a_{n+1}^2 \chi_{\text{orb}}(\mathcal{M}_{1,n}) - \sum_{i=1}^n \chi_{\text{orb}}(\text{DR}_{1,n}(a(i)))$$

as required. □

*Proof of Theorem 1.1.* We induct on  $n$ . The base case  $n = 1$  is trivial; we must have  $a_1 = 0$  and then

$$\chi_{\text{orb}}(\text{DR}_{1,1}(0)) = \chi_{\text{orb}}(\mathcal{M}_{1,1}) = -1/12$$

by the Harer–Zagier formula (Lemma 1.3), which agrees with (4). Now assume the claim holds for ramification vectors of length  $n$  and consider a ramification vector  $a = (a_1, \dots, a_n, a_{n+1})$  of length  $n+1$ . We then have

$$\begin{aligned} \chi_{\text{orb}}(\text{DR}_{1,n+1}(a)) &= a_{n+1}^2 \chi_{\text{orb}}(\mathcal{M}_{1,n}) - \sum_{i=1}^n \chi_{\text{orb}}(\text{DR}_{1,n}(a(i))) \\ &= a_{n+1}^2 \left( \frac{(-1)^n (n-1)!}{12} \right) - \sum_{i=1}^n \frac{(-1)^{n-1} (n-1)!}{24} (\sum_{j \neq i} a_j^2 + (a_i + a_{n+1})^2 - 2) \\ &= \frac{(-1)^n (n-1)!}{24} \left( 2a_{n+1}^2 + \sum_{i=1}^n (\sum_{j=1}^{n+1} a_j^2 + 2a_i a_{n+1} - 2) \right) \\ &= \frac{(-1)^n (n-1)!}{24} \left( n (\sum_{j=1}^{n+1} a_j^2 - 2) + 2a_{n+1}^2 + 2a_{n+1} \sum_{i=1}^n a_i \right) \\ &= \frac{(-1)^n n!}{24} \left( \sum_{i=1}^{n+1} a_i^2 - 2 \right) \end{aligned}$$

where the first equality follows from Theorem 1.2, the second equality follows from the Harer–Zagier formula (Lemma 1.3), and the induction hypothesis, and the last equality follows from the fact that  $\sum_{i=1}^{n+1} a_i = 0$ . This completes the induction step. □

The above proof uses the Harer–Zagier formula [27] for the orbifold Euler characteristic of the moduli space of curves. In genus one this admits an elementary proof, presumably well-known to experts, which we include for completeness.

**Lemma 1.3** (Harer–Zagier in genus one). *For  $n \geq 1$  we have:*

$$\chi_{\text{orb}}(\mathcal{M}_{1,n}) = \frac{(-1)^n (n-1)!}{12}.$$

*Proof.* We proceed by induction on  $n$ . For the base case we note that  $\mathcal{M}_{1,1}$  has  $\mathbb{A}^1$  as its coarse moduli space, with the general point having an automorphism group of order 2, and two special points  $\xi_{1728}$  and  $\xi_0$  having automorphism groups of orders 4 and 6. We thus have:

$$\chi_{\text{orb}}(\mathcal{M}_{1,1}) = \frac{1}{2} \chi(\mathbb{A}^1 \setminus \{\xi_{1728}, \xi_0\}) + \frac{1}{4} \chi(\xi_{1728}) + \frac{1}{6} \chi(\xi_0) = -\frac{1}{2} + \frac{1}{4} + \frac{1}{6} = -\frac{1}{12}.$$

For the induction step, consider the forgetful morphism  $\mathcal{M}_{1,n+1} \rightarrow \mathcal{M}_{1,n}$ . This morphism is representable, and each fibre is a genus-one curve  $C$  with the points  $p_1, \dots, p_n$  removed. We conclude:

$$\begin{aligned} \chi_{\text{orb}}(\mathcal{M}_{1,n+1}) &= \chi(C \setminus \{p_1, \dots, p_n\}) \cdot \chi_{\text{orb}}(\mathcal{M}_{1,n}) \\ &= (-n) \cdot \frac{(-1)^n (n-1)!}{12} \\ &= \frac{(-1)^{n+1} ((n+1)-1)!}{12}. \end{aligned} \quad \square$$

## 2. HIGHER RANK

We proceed to the higher-rank case. The recursion generalises directly (Theorem 2.7) and the induction strategy still applies. The difficulty is guessing the correct formula.

Fix  $g = 1$ ,  $r \geq 1$ ,  $n \geq 1$  and an  $r \times n$  integer matrix

$$A = \begin{bmatrix} a_1^{(1)} & \cdots & a_n^{(1)} \\ \vdots & & \vdots \\ a_1^{(r)} & \cdots & a_n^{(r)} \end{bmatrix}$$

such that each row sums to zero:  $\sum_{i=1}^n a_i^{(j)} = 0$  for all  $j \in [r]$ . We refer to this as a **double ramification matrix**. Given an  $r \times n$  double ramification matrix  $A$ , the associated **rank- $r$  double ramification locus** is denoted and defined:

$$\text{DR}_{1,n}^r(A) := \bigcap_{i=1}^r \text{DR}_{1,n}(a^{(i)}) \subseteq \mathcal{M}_{1,n}$$

where  $a^{(i)}$  denotes the  $i$ th row of  $A$ . Its closed points correspond to marked curves  $(C, p_1, \dots, p_n)$  satisfying the  $r$  simultaneous equations:

$$\begin{aligned} \mathcal{O}_C(\sum_{i=1}^n a_i^{(1)} p_i) &\cong \mathcal{O}_C, \\ &\vdots \\ \mathcal{O}_C(\sum_{i=1}^n a_i^{(r)} p_i) &\cong \mathcal{O}_C. \end{aligned}$$

The main result of this section (Theorem 2.2) gives a formula for the orbifold Euler characteristic of this locus.

**Remark 2.1.** While  $\text{DR}_{1,n}^r(A) \subseteq \mathcal{M}_{1,n}$  has expected dimension  $n - r$  its actual dimension may be larger, for instance if some rows of  $A$  are linearly dependent over  $\mathbb{Z}$ . It can also be empty, for instance if  $A$  contains a row of the form  $(1, -1, 0, \dots, 0)$ . The formula below for the orbifold Euler characteristic holds in all cases.

In Section 2.1 we state the formula (Theorem 2.2). In Section 2.2 we use  $\text{GL}_r(\mathbb{Z})$ -invariance to reduce to a special class of double ramification matrices (Proposition 2.6), and in Section 2.3 we establish an orbifold Euler characteristic recursion for these matrices (Theorem 2.7). In Section 2.4 we establish an important lemma on the linear algebra of double ramification matrices (Corollary 2.9). This is used in

the proof of the formula, which is given in Section 2.5. Having obtained the formula, in Section 2.6 we provide a simplification of its leading term, and finally in Section 2.7 we compare it to the rank-one formula obtained in the previous section.

**2.1. Formula.** We establish the necessary notation. A **partition**  $\mathcal{I} \vdash [n]$  is an unordered collection of subsets

$$\mathcal{I} = \{I_1, \dots, I_{\ell(\mathcal{I})}\}$$

with each  $I_j \neq \emptyset$  and  $[n] = I_1 \sqcup \dots \sqcup I_{\ell(\mathcal{I})}$ . Given a partition  $\mathcal{I} \vdash [n]$  the associated **contraction** of  $A$  is obtained by summing the columns associated to each part of  $\mathcal{I}$ ,

$$A_{\mathcal{I}} := \begin{bmatrix} a_{I_1}^{(1)} & \cdots & a_{I_{\ell(\mathcal{I})}}^{(1)} \\ \vdots & & \vdots \\ a_{I_1}^{(r)} & \cdots & a_{I_{\ell(\mathcal{I})}}^{(r)} \end{bmatrix}$$

where  $a_I^{(j)} := \sum_{i \in I} a_i^{(j)}$  for any subset  $I \subseteq [n]$ . The contraction  $A_{\mathcal{I}}$  is well-defined up to permutation of the columns and is an  $r \times \ell(\mathcal{I})$  double ramification matrix. For  $0 \leq k \leq \min(r, \ell(\mathcal{I}))$  we then define

$$G_{k \times k}(A_{\mathcal{I}}) \in \mathbb{Z}$$

to be the greatest common divisor of all the  $k \times k$  minors of  $A_{\mathcal{I}}$ . This is well-defined up to sign. By convention we take:

$$G_{0 \times 0}(A_{\mathcal{I}}) = 1, \quad \gcd(m_1, \dots, m_l, 0) = \gcd(m_1, \dots, m_l), \quad \gcd(\emptyset) = 0.$$

We are now ready to state the main result.

**Theorem 2.2 (Theorem Y).** *Fix an  $r \times n$  double ramification matrix  $A$ . The orbifold Euler characteristic of the associated rank  $r$  double ramification locus is:*

$$(10) \quad \chi_{\text{orb}}(\text{DR}_{1,n}^r(A)) = \frac{(-1)^n}{12} \sum_{k=0}^r \sum_{\substack{\mathcal{I} \vdash [n] \\ \ell(\mathcal{I})=k+1}} (-1)^k (\#I_1 - 1)! \cdots (\#I_{k+1} - 1)! \cdot G_{k \times k}(A_{\mathcal{I}})^2.$$

**Remark 2.3.** If  $n \leq r$ , then there are no partitions  $\mathcal{I} \vdash [n]$  of length  $k+1$  for  $n \leq k \leq r$ . The associated terms in the above formula simply vanish.

**2.2. Reduction via  $\text{GL}_r(\mathbb{Z})$ -invariance.** Given an  $r \times n$  double ramification matrix  $A$  and a matrix  $M \in \text{GL}_r(\mathbb{Z})$ , the product  $MA$  is again an  $r \times n$  double ramification matrix, since elementary row operations preserve this property. Clearly we have

$$(11) \quad \text{DR}_{1,n}^r(A) = \text{DR}_{1,n}^r(MA)$$

as substacks of  $\mathcal{M}_{1,n}$ . We will use this  $\text{GL}_r(\mathbb{Z})$ -invariance to reduce to a special class of double ramification matrices. The key fact is the following:

**Lemma 2.4.** *The right-hand side of (10) is  $\text{GL}_r(\mathbb{Z})$ -invariant.*

*Proof.* Taking contractions commutes with the action of  $\text{GL}_r(\mathbb{Z})$ , that is

$$(MA)_{\mathcal{I}} = M(A_{\mathcal{I}})$$

for every  $\mathcal{I} \vdash [n]$  and  $M \in \text{GL}_r(\mathbb{Z})$ . Each  $k \times k$  minor of  $M(A_{\mathcal{I}})$  is a  $\mathbb{Z}$ -linear combination of  $k \times k$  minors of  $A_{\mathcal{I}}$ . Therefore

$$G_{k \times k}(A_{\mathcal{I}}) \mid G_{k \times k}(M(A_{\mathcal{I}})).$$

But the same argument applied to  $M^{-1}$  shows the reverse divisibility, so in fact

$$G_{k \times k}(A_{\mathcal{I}}) = G_{k \times k}(M(A_{\mathcal{I}})). \quad \square$$

Consequently, to prove Theorem 2.2 for a double ramification matrix  $A$  it is sufficient to prove it for  $MA$  for a single  $M \in \text{GL}_r(\mathbb{Z})$ . We use the following reduction.

**Lemma 2.5.** *Given an  $r \times n$  integer matrix  $A$ , there exists  $M \in \mathrm{GL}_r(\mathbb{Z})$  such that  $MA$  takes the following special form:*

$$(12) \quad \begin{bmatrix} a_1^{(1)} & \cdots & a_{n-1}^{(1)} & a_n^{(1)} \\ a_1^{(2)} & \cdots & a_{n-1}^{(2)} & 0 \\ \vdots & & \vdots & \vdots \\ a_1^{(r)} & \cdots & a_{n-1}^{(r)} & 0 \end{bmatrix}.$$

*Proof.* We prove the result for  $r = 2$ ; the general case proceeds by induction on the rows. We have:

$$A = \begin{bmatrix} a_1 & \cdots & a_{n-1} & a_n \\ b_1 & \cdots & b_{n-1} & b_n \end{bmatrix}.$$

Take  $d := \gcd(a_n, b_n)$  with an arbitrary choice of sign. There exist  $p, q \in \mathbb{Z}$  with

$$pa_n + qb_n = d.$$

Dividing through by  $d$ , we obtain  $s, t \in \mathbb{Z}$  with

$$(13) \quad ps + qt = 1.$$

Consider the matrix

$$M' := \begin{bmatrix} p & q \\ -t & s \end{bmatrix}$$

which belongs to  $\mathrm{GL}_2(\mathbb{Z})$  by (13). The matrix  $M'A$  takes the following form:

$$\begin{bmatrix} \text{---} \star \text{---} & pa_n + qb_n \\ \text{---} \star \text{---} & -ta_n + sb_n \end{bmatrix} = \begin{bmatrix} \text{---} \star \text{---} & d \\ \text{---} \star \text{---} & -ta_n + sb_n \end{bmatrix}.$$

Since  $d \mid a_n$  and  $d \mid b_n$  we have  $d \mid -ta_n + sb_n$ . Thus we may add an appropriate multiple of the first row to the second row to obtain a matrix of the form

$$\begin{bmatrix} \text{---} \star \text{---} & d \\ \text{---} \star \text{---} & 0 \end{bmatrix}$$

as required. □

**Proposition 2.6.** *To prove Theorem 2.2, it is sufficient to prove it for matrices of the form (12).*

*Proof.* Combine (11) with Lemmas 2.4 and 2.5. □

**2.3. Geometric recursion.** Having reduced to matrices of the form (12), we now establish the following recursion generalising Theorem 1.2:

**Theorem 2.7.** *Fix  $n \geq 1$  and consider an  $r \times (n + 1)$  double ramification matrix  $A$  of the special form (12), writing  $a = (a_1, \dots, a_n, a_{n+1})$  for the first row and  $B$  for the  $(r - 1) \times n$  submatrix in the bottom left corner:*

$$A = \begin{bmatrix} a_1 & \cdots & a_n & a_{n+1} \\ & & & 0 \\ & B & & \vdots \\ & & & 0 \end{bmatrix}.$$

For each  $i \in [n]$  define the length  $n$  ramification vector:

$$a^{(i)} := (a_1, \dots, a_{i-1}, a_i + a_{n+1}, a_{i+1}, \dots, a_n).$$

Then the orbifold Euler characteristic of  $\mathrm{DR}_{1,n+1}^r(A)$  satisfies the following recurrence:

$$(14) \quad \chi_{\mathrm{orb}}(\mathrm{DR}_{1,n+1}^r(A)) = a_{n+1}^2 \chi_{\mathrm{orb}}(\mathrm{DR}_{1,n}^{r-1}(B)) - \sum_{i=1}^n \chi_{\mathrm{orb}}(\mathrm{DR}_{1,n}^r \left[ \begin{array}{c} \text{---} a^{(i)} \text{---} \\ B \end{array} \right]).$$

*Proof.* The proof of Theorem 1.2 applies *mutatis mutandis*. The analogue of the forgetful morphism  $\mathrm{DR}_{1,n+1}(a) \rightarrow \mathcal{M}_{1,n}$  is

$$\mathrm{DR}_{1,n+1}^r(A) \rightarrow \mathrm{DR}_{1,n}^{r-1}(B)$$

and the construction of the stratification is identical.  $\square$

**2.4. Matrix lemmas.** The proof will proceed by induction, using Proposition 2.6 and Theorem 2.7. We require a basic result on the linear algebra of double ramification matrices.

**Lemma 2.8.** *Consider an  $r \times (r + 1)$  double ramification matrix  $A$ . Then all the  $r \times r$  minors of  $A$  coincide up to sign.*

*Proof.* Consider such a double ramification matrix:

$$A = \begin{bmatrix} a_1^{(1)} & \cdots & a_{r+1}^{(1)} \\ \vdots & & \vdots \\ a_1^{(r)} & \cdots & a_{r+1}^{(r)} \end{bmatrix}.$$

An  $r \times r$  minor is obtained by removing a single column. Given indices  $i, j \in [r + 1]$  with  $i < j$  we consider the contraction  $B$  of  $A$  obtained by summing the  $i$ th and  $j$ th columns:

$$B := \left[ \begin{array}{c|c|c|c|c|c|c} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & \cdots & \widehat{a}_i & \cdots & \widehat{a}_j & \cdots & a_r \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_r & \cdots & \widehat{a}_i & \cdots & \widehat{a}_j & \cdots & a_r + a_j \\ \vdots & & \vdots & & \vdots & & \vdots \end{array} \right].$$

This is an  $r \times r$  double ramification matrix, so the sum of the columns is equal to the zero vector and so  $\det B = 0$ . But on the other hand by multilinearity of the determinant  $\det B$  is equal to:

$$\det \left[ \begin{array}{c|c|c|c|c|c|c} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & \cdots & \widehat{a}_i & \cdots & \widehat{a}_j & \cdots & a_r \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_r & \cdots & \widehat{a}_i & \cdots & \widehat{a}_j & \cdots & a_i \\ \vdots & & \vdots & & \vdots & & \vdots \end{array} \right] + \det \left[ \begin{array}{c|c|c|c|c|c|c} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & \cdots & \widehat{a}_i & \cdots & \widehat{a}_j & \cdots & a_r \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_r & \cdots & \widehat{a}_i & \cdots & \widehat{a}_j & \cdots & a_j \\ \vdots & & \vdots & & \vdots & & \vdots \end{array} \right].$$

Up to signs determined by the appropriate column permutations, these two terms are the  $r \times r$  minors of  $A$  corresponding to  $i$  and  $j$ .  $\square$

**Corollary 2.9.** *For  $k \in \{0, \dots, r\}$  consider an  $r \times (k + 1)$  double ramification matrix  $A$ , so that the  $k \times k$  minors of  $A$  are obtained by deleting  $r - k$  rows and 1 column. Then up to sign, the  $k \times k$  minor depends only on the choice of rows and not on the choice of column.*

*Proof.* This follows immediately from Lemma 2.8: deleting  $r - k$  rows produces a  $k \times (k + 1)$  double ramification matrix.  $\square$

## 2.5. Proof.

*Proof of Theorem 2.2.* We induct on the pair  $(r, n)$  using the lexicographic order. Given  $(r, n)$  we assume that the formula has already been established for pairs  $(r', n')$  such that either:

- (i)  $r' < r$ ; or
- (ii)  $r' = r$  and  $n' < n$ .

The base case is when  $r$  is arbitrary and  $n = 1$ . Then  $A$  is a column vector consisting of  $r$  zeros, so  $\mathrm{DR}_{1,1}^r(A) = \mathcal{M}_{1,1}$ . In this case by the Harer–Zagier formula:

$$\chi_{\mathrm{orb}}(\mathrm{DR}_{1,1}^r(A)) = -1/12.$$

On the other hand in the formula (10) there is a single partition  $\mathcal{I} \vdash [1]$  and by convention we have  $G_{0 \times 0}(A_{\mathcal{I}}) = 1$ . The total contribution is  $-1/12$ , verifying the base case.

For the induction step, consider an  $r \times (n + 1)$  double ramification matrix  $A$ . By Proposition 2.6 we may assume  $A$  takes the following form

$$(15) \quad A = \begin{bmatrix} a_1 & \cdots & a_n & a_{n+1} \\ & & & 0 \\ & B & & \vdots \\ & & & 0 \end{bmatrix}.$$

and then Theorem 2.7 gives:

$$(16) \quad \chi_{\text{orb}}(\text{DR}_{1,n+1}^r(A)) = a_{n+1}^2 \chi_{\text{orb}}(\text{DR}_{1,n}^{r-1}(B)) - \sum_{i=1}^n \chi_{\text{orb}}(\text{DR}_{1,n}^r \left[ \begin{array}{c} \text{---} a(i) \text{---} \\ B \end{array} \right]).$$

We apply the induction hypothesis to the right-hand side. The following definition will be useful. A partition  $\mathcal{I} \vdash [n+1]$  is **lonely** if  $n+1$  constitutes an entire part, and **friendly** otherwise. In the right-hand side above, the first term will provide the contributions of the lonely partitions, while the second term will provide the contributions of the friendly partitions.

We begin with  $\chi_{\text{orb}}(\text{DR}_{1,n}^{r-1}(B))$ . The contributions are indexed by partitions  $\mathcal{I} = \{I_1, \dots, I_{k+1}\} \vdash [n]$  of length  $k+1$  for  $k = 0, \dots, r-1$ . These correspond bijectively with lonely partitions

$$\mathcal{I}' := \{I_1, \dots, I_{k+1}, \{n+1\}\} \vdash [n+1]$$

of length  $k+2$ . Ranging over all  $k$ , we obtain all lonely partitions  $\mathcal{I} \vdash [n+1]$  of length  $k+1$  for  $k = 1, \dots, r$  (and since there are no lonely partitions of length 1, we may in fact say for  $k = 0, \dots, r$ ).

Given such an  $\mathcal{I} \vdash [n]$  and corresponding lonely partition  $\mathcal{I}' \vdash [n+1]$ , the associated contractions are related as follows:

$$A_{\mathcal{I}'} = \begin{bmatrix} a_{I_1} & \cdots & a_{I_{k+1}} & a_{n+1} \\ & & & 0 \\ & B_{\mathcal{I}} & & \vdots \\ & & & 0 \end{bmatrix}.$$

We must now compare the  $k \times k$  minors of  $B_{\mathcal{I}}$  with the  $(k+1) \times (k+1)$  minors of  $A_{\mathcal{I}'}$ .

The  $(k+1) \times (k+1)$  minors of  $A_{\mathcal{I}'}$  are obtained by selecting  $(k+1)$  rows and  $(k+1)$  columns, but up to sign the choice of columns does not matter by Corollary 2.9. If the first row is not among the  $(k+1)$  selected rows, then we may include the final column among the  $(k+1)$  selected columns: the resulting submatrix has a column of zeros, and hence the minor vanishes.

To obtain a nonzero minor of  $A_{\mathcal{I}'}$  we must therefore include the first row among the  $(k+1)$  selected rows. This amounts to choosing  $k$  rows of  $B_{\mathcal{I}}$ . Once this is done, we can make an arbitrary choice of  $(k+1)$  columns of  $A_{\mathcal{I}'}$  by Corollary 2.9. We choose  $k$  columns of  $B_{\mathcal{I}}$  together with the final column of  $A_{\mathcal{I}'}$ .

In this way we obtain a bijection between the nonzero  $(k+1) \times (k+1)$  minors of  $A_{\mathcal{I}'}$  and the  $k \times k$  minors of  $B_{\mathcal{I}}$ . Expanding along the final column we see that these are related, up to sign, by the factor  $a_{n+1}$ . We conclude:

$$G_{k+1 \times k+1}(A_{\mathcal{I}'}) = a_{n+1} G_{k \times k}(B_{\mathcal{I}}).$$

Examining the first term on the right-hand side of (16) we obtain precisely the lonely contributions:

$$\begin{aligned}
a_{n+1}^2 \chi_{\text{orb}}(\text{DR}_{1,n}^{r-1}(B)) &= \frac{(-1)^n}{12} \sum_{k=0}^{r-1} \sum_{\substack{\mathcal{I} \vdash [n] \\ \ell(\mathcal{I})=k+1}} (-1)^k (\#I_1 - 1)! \cdots (\#I_{k+1} - 1)! \cdot a_{n+1}^2 G_{k \times k}(B_{\mathcal{I}})^2 \\
&= \frac{(-1)^n}{12} \sum_{k=0}^r \sum_{\substack{\mathcal{I} \vdash [n+1] \\ \ell(\mathcal{I})=k+1 \\ \mathcal{I} \text{ lonely}}} (-1)^{k-1} (\#I_1 - 1)! \cdots (\#I_{k+1} - 1)! \cdot G_{k \times k}(A_{\mathcal{I}})^2 \\
(17) \quad &= \frac{(-1)^{n+1}}{12} \sum_{k=0}^r \sum_{\substack{\mathcal{I} \vdash [n+1] \\ \ell(\mathcal{I})=k+1 \\ \mathcal{I} \text{ lonely}}} (-1)^k (\#I_1 - 1)! \cdots (\#I_{k+1} - 1)! \cdot G_{k \times k}(A_{\mathcal{I}})^2.
\end{aligned}$$

We now turn to the second term on the right-hand side of (16). Given  $i \in [n]$  write  $A(i)$  for the matrix:

$$A(i) := \left[ \begin{array}{c} \text{---}a(i)\text{---} \\ B \end{array} \right].$$

The contributions to  $\chi_{\text{orb}}(\text{DR}_{1,n}^r(A(i)))$  are indexed by partitions  $\mathcal{I} \vdash [n]$  of length  $k+1$  for  $k = 0, \dots, r$ . For each such partition, the associated contraction satisfies:

$$A(i)_{\mathcal{I}} = A_{\mathcal{I}_i}$$

where  $\mathcal{I}_i \vdash [n+1]$  is the partition obtained by appending  $n+1$  to the part of  $\mathcal{I}$  containing  $i \in [n]$ . Note that  $\mathcal{I}$  and  $\mathcal{I}_i$  have the same length. In this way, we enumerate all the friendly partitions of  $[n+1]$ , and each such partition appears  $(\#I_{k+1} - 1)$  times, where without loss of generality  $I_{k+1}$  is the part containing  $n+1$ . We conclude:

$$\begin{aligned}
-\sum_{i=1}^n \chi_{\text{orb}}(\text{DR}_{1,n}^r(A(i))) &= -\sum_{i=1}^n \frac{(-1)^n}{12} \sum_{k=0}^r \sum_{\substack{\mathcal{I} \vdash [n] \\ \ell(\mathcal{I})=k+1}} (-1)^k (\#I_1 - 1)! \cdots (\#I_{k+1} - 1)! \cdot G_{k \times k}(A(i)_{\mathcal{I}})^2 \\
(18) \quad &= \frac{(-1)^{n+1}}{12} \sum_{k=0}^r \sum_{\substack{\mathcal{I} \vdash [n+1] \\ \ell(\mathcal{I})=k+1 \\ \mathcal{I} \text{ friendly}}} (-1)^k (\#I_1 - 1)! \cdots (\#I_{k+1} - 1)! \cdot G_{k \times k}(A_{\mathcal{I}})^2.
\end{aligned}$$

Combining (17) and (18) we obtain precisely the desired formula for  $\chi_{\text{orb}}(\text{DR}_{1,n+1}^r(A))$ . This completes the induction step.  $\square$

**2.6. Simplifying the leading term.** We refer to the  $k = r$  term in (10) as the **leading term**:

$$(19) \quad \frac{(-1)^{n+r}}{12} \sum_{\substack{\mathcal{I} \vdash [n] \\ \ell(\mathcal{I})=r+1}} (\#I_1 - 1)! \cdots (\#I_{r+1} - 1)! \cdot G_{r \times r}(A_{\mathcal{I}})^2.$$

We now simplify the leading term, expressing it in terms of minors of the original matrix  $A$  rather than its contractions  $A_{\mathcal{I}}$ .

Fix an  $r \times n$  double ramification matrix  $A$  and assume  $n \geq r+1$  (otherwise the leading term vanishes). The  $r \times r$  minors arise by selecting  $r$  columns of  $A$ . Given a subset  $I \subseteq [n]$  of size  $r$  we let

$$M_I(A)$$

denote the associated  $r \times r$  minor.

**Proposition 2.10.** *The leading term (19) is equal to:*

$$(20) \quad L_{1,n}^r(A) := \frac{(-1)^{n+r} (n-1)!}{12 (r+1)!} \sum_{I \in \binom{[n]}{r}} M_I(A)^2.$$

*Proof.* We repeat the proof of Theorem 2.2, replacing the old leading term (19) by the new leading term (20) in the formula. We adopt the same notation as before.

The base of the induction is straightforward. For the induction step, we note that  $L_{1,n+1}^r(A)$  is  $\mathrm{GL}_r(\mathbb{Z})$ -invariant, so we may reduce to matrices of the form (12) and apply Theorem 2.7 to obtain (16). We saw in the proof of Theorem 2.2 that the leading terms on both sides of (16) are identified, hence we can focus exclusively on these. It remains to prove:

$$(21) \quad L_{1,n+1}^r(A) = a_{n+1}^2 \cdot L_{1,n}^{r-1}(B) - \sum_{i=1}^n L_{1,n}^r(A(i)).$$

The rest of the argument consists of algebraic manipulations. We say a subset

$$I \in \binom{[n+1]}{r}$$

is **nouveau** if  $n+1 \in I$  and **ancien** if  $n+1 \notin I$ . In the nouveau case, we have

$$M_I(A) = a_{n+1} \cdot M_{I \setminus \{n+1\}}(B).$$

Thus the first term on the right-hand side of (21) is equal to a sum of nouveau contributions, which we can write as:

$$(22) \quad \frac{(-1)^{n+1+r} (n-1)!}{12 (r+1)!} (r+1) \sum_{\substack{I \in \binom{[n+1]}{r} \\ n+1 \in I}} M_I(A)^2.$$

The second term contains both nouveau and ancien contributions which we now disentangle. We can write:

$$(23) \quad - \sum_{i=1}^n L_{1,n}^r(A(i)) = \frac{(-1)^{n+1+r} (n-1)!}{12 (r+1)!} \sum_{i=1}^n \left( \sum_{\substack{I \in \binom{[n]}{r} \\ i \notin I}} M_I(A(i))^2 + \sum_{\substack{I \in \binom{[n]}{r} \\ i \in I}} M_I(A(i))^2 \right).$$

For  $i \in [n]$  and  $I \in \binom{[n]}{r}$  we have

$$M_I(A(i)) = \begin{cases} M_I(A) & \text{if } i \notin I, \\ M_I(A) + (-1)^{s(I,i)} M_{I \setminus \{i\} \cup \{n+1\}}(A) & \text{if } i \in I, \end{cases}$$

where  $s(I,i) := \#\{j \in I : j > i\}$ . Then the sum in (23) over  $I \not\ni i$  produces the following ancien contributions:

$$(24) \quad \frac{(-1)^{n+1+r} (n-1)!}{12 (r+1)!} (n-r) \sum_{\substack{I \in \binom{[n+1]}{r} \\ n+1 \notin I}} M_I(A)^2.$$

We now turn to the sum in (23) over  $I \ni i$ . Squaring  $M_I(A(i))$  produces the mixed term

$$2(-1)^{s(I,i)} \cdot M_I(A) \cdot M_{I \setminus \{i\} \cup \{n+1\}}(A).$$

Summing over  $i \in [n]$  and setting  $I' := I \setminus \{i\}$  we can rewrite the sum of the mixed terms as follows

$$\begin{aligned} \sum_{i=1}^n \sum_{\substack{I \in \binom{[n]}{r} \\ i \in I}} 2(-1)^{s(I,i)} \cdot M_I(A) \cdot M_{I \setminus \{i\} \cup \{n+1\}}(A) &= \sum_{I' \in \binom{[n]}{r-1}} \sum_{i \in [n] \setminus I'} 2(-1)^{s(I,i)} \cdot M_{I' \cup \{i\}}(A) \cdot M_{I' \cup \{n+1\}}(A) \\ &= 2 \sum_{I' \in \binom{[n]}{r-1}} M_{I' \cup \{n+1\}}(A) \sum_{i \in [n] \setminus I'} (-1)^{s(I,i)} \cdot M_{I' \cup \{i\}}(A) \\ &= -2 \sum_{I' \in \binom{[n]}{r-1}} M_{I' \cup \{n+1\}}(A)^2 \end{aligned}$$

where the final equality follows from a basic property of double ramification matrices, similar to Corollary 2.9. On the other hand, the square of  $M_{I' \cup \{n+1\}}(A)$  also appears in the summation once for every  $i \in [n] \setminus I'$ , of which there are  $n - (r - 1)$ . Assembling, we obtain

$$\begin{aligned} \sum_{i=1}^n \sum_{\substack{I \in \binom{[n]}{r} \\ i \in I}} M_I(A(i))^2 &= \left( \sum_{i=1}^n \sum_{\substack{I \in \binom{[n]}{r} \\ i \in I}} M_I(A)^2 \right) + (-2 + (n - (r - 1))) \sum_{\substack{I \in \binom{[n+1]}{r} \\ n+1 \in I}} M_I(A)^2 \\ &= r \sum_{\substack{I \in \binom{[n+1]}{r} \\ n+1 \notin I}} M_I(A)^2 + (n - r - 1) \sum_{\substack{I \in \binom{[n+1]}{r} \\ n+1 \in I}} M_I(A)^2 \end{aligned}$$

so that the contribution is:

$$(25) \quad \frac{(-1)^{n+1+r}}{12} \frac{(n-1)!}{(r+1)!} \left( r \sum_{\substack{I \in \binom{[n+1]}{r} \\ n+1 \notin I}} M_I(A)^2 + (n-r-1) \sum_{\substack{I \in \binom{[n+1]}{r} \\ n+1 \in I}} M_I(A)^2 \right).$$

Combining (22), (24), and (25) we obtain the desired identity (21).  $\square$

**2.7. Comparing Theorems 1.1 and 2.2.** When  $r = 1$  we can directly match the formula appearing in Theorem 2.2 with the considerably simpler formula appearing in Theorem 1.1. We require the following:

**Lemma 2.11.** Fix  $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$  with  $\sum_{i=1}^n a_i = 0$ . For each  $m \in \{1, \dots, n-1\}$  we have

$$\sum_{I \in \binom{[n]}{m}} a_I^2 = \binom{n-2}{m-1} \sum_{i=1}^n a_i^2$$

where  $a_I := \sum_{i \in I} a_i$ .

*Proof.* The left-hand side is a homogeneous quadratic symmetric polynomial in  $a_1, \dots, a_n$  and hence can be written in terms of power sums [41, I (2.12)] as

$$\lambda_1 \cdot (\sum_{i=1}^n a_i)^2 + \lambda_2 \cdot \sum_{i=1}^n a_i^2$$

for some  $\lambda_1, \lambda_2 \in \mathbb{Q}$ . The first term vanishes, and to determine  $\lambda_2$  it suffices to evaluate at a single vector. Take  $a = (1, -1, 0, \dots, 0)$ . Then  $a_I = 0$  unless  $I$  and  $[n] \setminus I$  separate 1 and 2. Enumerating separately the cases  $1 \in I$  and  $2 \in I$  we obtain:

$$\sum_{I \in \binom{[n]}{m}} a_I^2 = 2 \binom{n-2}{m-1}.$$

On the other hand  $\sum_{i=1}^n a_i^2 = 2$ . We conclude that  $\lambda_2 = \binom{n-2}{m-1}$  as required.  $\square$

Now consider the formula in Theorem 2.2. Since  $r = 1$  we sum over  $k = 0$  and  $k = 1$ . For  $k = 0$  we have a single partition of length 1, and by convention  $G_{0 \times 0}(A_{\mathcal{I}}) = 1$ . The contribution is:

$$(26) \quad \frac{(-1)^n(n-1)!}{12}.$$

For  $k = 1$  we sum over partitions  $\mathcal{I} = \{I_1, I_2\}$  of length 2. This is equal to half the sum over subsets  $I \subseteq [n]$  of size  $m \in \{1, \dots, n-1\}$ . Each subset leads to a  $1 \times 2$  matrix giving:

$$G_{1 \times 1}(A_{\mathcal{I}}) = a_I.$$

The contribution is thus:

$$(27) \quad \begin{aligned} & \frac{(-1)^n}{12} \cdot \frac{(-1)^1}{2} \cdot \sum_{m=1}^{n-1} \left( (m-1)!(n-m-1)! \sum_{I \in \binom{[n]}{m}} a_I^2 \right) \\ &= \frac{(-1)^{n+1}}{24} \left( \sum_{m=1}^{n-1} (m-1)!(n-m-1)! \binom{n-2}{m-1} \right) \sum_{i=1}^n a_i^2 \\ &= \frac{(-1)^{n+1}(n-1)!}{24} \sum_{i=1}^n a_i^2. \end{aligned}$$

Combining (26) and (27) we obtain the formula in Theorem 1.1.

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