PRODUCT OF BRAUER–MANIN OBSTRUCTIONS FOR 0-CYCLES OVER NUMBER FIELDS AND FUNCTION FIELDS

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Abstract. According to [Col95], it is conjectured that the Brauer–Manin obstruction is expected to control the existence of 0-cycles of degree 1 on smooth proper varieties over number fields. In this paper, we prove that the existence of the Brauer–Manin obstruction to the Hasse principle for 0-cycles of degree 1 on the product of two smooth (non-necessarily proper) varieties is equivalent to the existence of such an obstruction on one of the two factors.

We also prove an analogous statement for smooth varieties defined over function fields of $\mathbb{C}((t))$-curves.

1. Introduction

Consider a number field k or the function field of a projective smooth geometrically integral $\mathbb{C}((t))$-curve. We call the latter a function field for short. All varieties appearing in this section will be smooth and geometrically integral.

For a class of k-varieties $\mathcal{F}$, we say that the class $\mathcal{F}$ satisfies the Hasse principle if for each k-variety $V \in \mathcal{F}$, $V(\mathbf{A}_k) \neq \emptyset$ implies $V(k) \neq \emptyset$. When $V(\mathbf{A}_k) \neq \emptyset$ and $V(k) = \emptyset$, one may want to explain the failure of the Hasse principle. To do so, Manin introduced a subset $V(\mathbf{A}_k)^{\mathrm{Br}}$ of $V(\mathbf{A}_k)$. The set $V(\mathbf{A}_k)^{\mathrm{Br}}$ is defined as the right-kernel of the Brauer–Manin pairing

$$\langle\cdot,\cdot\rangle_{\mathrm{BM}} : \mathrm{Br}(V) \times V(\mathbf{A}_k) \to \mathbb{Q}/\mathbb{Z}. \tag{1}$$

Here $\mathrm{Br}(V) = \mathrm{H}^2_{\text{ét}}(V, \mathbb{G}_m)$ is the cohomological Brauer group of V. Using the Brauer–Hasse–Noether exact sequence when k is a number field and by Weil's reciprocity law when k is a function field, one can obtain the inclusions

$$V(k) \subseteq V(\mathbf{A}_k)^{\mathrm{Br}} \subseteq V(\mathbf{A}_k).$$

Hence, if $V(\mathbf{A}_k)^{\mathrm{Br}}$ is empty, then this explains why $V(k)$ is empty. From this perspective, if for some k-variety V, we have $V(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$, then we say that a Brauer–Manin obstruction to the Hasse principle on V exists.

It is natural to consider the Brauer–Manin obstruction on the product $Z = X \times_k Y$ of two projective varieties. Skorobogatov and Zarhin, in their paper [SZ14, Thm. C], proved that when k is a number field, one has $Z(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k)^{\mathrm{Br}} \times Y(\mathbf{A}_k)^{\mathrm{Br}}$. As a direct corollary, the existence of Brauer–Manin obstruction to the Hasse principle for rational points on Z is equivalent to the existence of the same obstruction on X or Y. The case without projectivity assumption is due to Chang Lv, see [Lv20, Thm. 3.1]. In this paper we are going to prove a variant of this statement for 0-cycles of degree 1.

Following Manin, Colliot-Thélène introduced a similar obstruction associated to 0-cycles in the paper [Col95]. Precisely, he derived a similar Brauer–Manin pairing between the group $\mathrm{Z}_{0,\mathbf{A}}(V)$ of adelic 0-cycles and the Brauer group $\mathrm{Br}(V)$. Hence, one obtains a

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subgroup $\mathrm{Z}_{0,\mathbf{A}}(V)^{\mathrm{Br}}$ containing $V(\mathbf{A}_k)^{\mathrm{Br}}$ as a subset. One can also prove that $\mathrm{Z}_{0,\mathbf{A}}(V)^{\mathrm{Br}}$ contains the group $\mathrm{Z}_0(V)$ of global 0-cycles on V. We denote by $\mathrm{Z}^r_{0,\mathbf{A}}(V)^{\mathrm{Br}}$ the subset consisting of adelic 0-cycles of degree r at each place. The group $\mathrm{Z}_{0,\mathbf{A}}(V)^{\mathrm{Br}}$ (resp. the subset $\mathrm{Z}^r_{0,\mathbf{A}}(V)^{\mathrm{Br}}$) allows one to define a Brauer–Manin obstruction to the Hasse principle for 0-cycles (resp. of degree r) similarly to the case of rational points. For a k-variety V, we say that the Brauer–Manin obstruction for 0-cycles on V (resp. of degree 1) exists if $\mathrm{Z}_{0,\mathbf{A}}(V)^{\mathrm{Br}} = \emptyset$ (resp. if $\mathrm{Z}^1_{0,\mathbf{A}}(V)^{\mathrm{Br}} = \emptyset$).

The second author considered the property of weak approximation in his paper [Lia23, Thm. 3.1]. Roughly speaking, he proved that, for rationally connected varieties X and Y defined over a number field k, the product $Z = X \times_k Y$ satisfies weak approximation with Brauer–Manin obstruction for 0-cycles if and only if both X and Y satisfy the same property.

In this paper, we consider the existence of Brauer–Manin obstruction to the Hasse principle for 0-cycles on the product of varieties which are neither necessarily rationally connected nor proper. Moreover, compared to the main result of [Lia23] where it was required to assume arithmetic properties over any finite extensions of the ground field, here we always stay on the ground field.

Given a 0-cycle x on X and a 0-cycle y on Y, one can determine a 0-cycle z on Z called the product of x and y. Precisely, if x and y are simply closed points, then z is the 0-cycle assocated to the finite k-scheme $\mathrm{Spec}(k(x)\otimes_k k(y))$. For general 0-cycles, one extends the product bilinearly and obtains a bilinear map

$$\mathrm{Z}_0(X) \times \mathrm{Z}_0(Y) \to \mathrm{Z}_0(Z).$$

Similarly, defined component by component, we obtain a bilinear map on the groups of adelic 0-cycles

$$\Phi : \mathrm{Z}_{0,\mathbf{A}}(X) \times \mathrm{Z}_{0,\mathbf{A}}(Y) \to \mathrm{Z}_{0,\mathbf{A}}(Z).$$

When one restricts Φ on the subset $\mathrm{Z}^r_{0,\mathbf{A}}(X) \times \mathrm{Z}^s_{0,\mathbf{A}}(Y)$, we denote it by $\Phi^{r,s}$ or simply Φ if there is no confusion. It is immediate that $\Phi^{1,1}$ is a section of the product of natural projections $\mathrm{Z}^1_{0,\mathbf{A}}(Z) \to \mathrm{Z}^1_{0,\mathbf{A}}(X) \times \mathrm{Z}^1_{0,\mathbf{A}}(Y)$.

Theorem 1.1. *Let k be either a number field or a function field. Let X and Y be smooth and geometrically integral varieties over k and let $Z = X \times_k Y$ be the product of X and Y. Then the section*

$$\Phi = \Phi^{1,1} : \mathrm{Z}^1_{0,\mathbf{A}}(X) \times \mathrm{Z}^1_{0,\mathbf{A}}(Y) \to \mathrm{Z}^1_{0,\mathbf{A}}(Z)$$

of $\mathrm{Z}^1_{0,\mathbf{A}}(Z) \to \mathrm{Z}^1_{0,\mathbf{A}}(X) \times \mathrm{Z}^1_{0,\mathbf{A}}(Y)$ gives rise to a section

$$\mathrm{Z}^1_{0,\mathbf{A}}(X)^{\mathrm{Br}} \times \mathrm{Z}^1_{0,\mathbf{A}}(Y)^{\mathrm{Br}} \to \mathrm{Z}^1_{0,\mathbf{A}}(Z)^{\mathrm{Br}}$$

of $\mathrm{Z}^1_{0,\mathbf{A}}(Z)^{\mathrm{Br}} \to \mathrm{Z}^1_{0,\mathbf{A}}(X)^{\mathrm{Br}} \times \mathrm{Z}^1_{0,\mathbf{A}}(Y)^{\mathrm{Br}}$.

Corollary 1.2. *The existence of Brauer–Manin obstruction to the Hasse principle for 0-cycles of degree 1 on $Z = X \times_k Y$ is equivalent to the existence of such an obstruction on X or Y.*

Notations

Here are several basic notations and conventions.

• A *function field* in this article always refers the function field of a projective smooth geometrically integral $\mathbb{C}((t))$-curve.

• The base field k is either a number field or a function field. Let $\overline{k}$ be a fixed algebraic closure of k. Let Γ_k (or simply Γ) be the absolute Galois group of k. In either case, the field k has characteristic 0. In the case of function fields, the cohomological dimension of k is always ≤ 2.

In either case, one can consider the set Ω_k of places associated to k. In the case of number fields, Ω_k comes from class field theory. In the case of function fields, Ω_k consists of closed points of the associated $\mathbb{C}((t))$-curve. For each $v \in \Omega_k$, let k_v be the completion of k at v.

• Let T be a finite subset of Ω_k. When k is a number field and T contains all archimedean places of k, let $\mathcal{O}_T$ denote the canonical integer ring associated to T. When k is a function field and $T \neq \emptyset$, let $\mathcal{O}_T$ denote the global sections of $C \setminus T$ where C is the associated $\mathbb{C}((t))$-curve. Note that in the case of function fields, $C \setminus T$ is affine if $T \neq \emptyset$.

• A variety over a field k is a separated scheme of finite type over k. Set $\overline{X} = X_{\overline{k}}$ and $X_v = X_{k_v}$ for each $v \in \Omega_k$.

• Given a field k and a k-variety X, the group $\mathrm{Z}_0(X)$ represents the abelian group of 0-cycles on X. If r is a given integer, the symbol $\mathrm{Z}_0^r(X)$ represents the 0-cycles on X of degree r.

• Given a number field or a function field k, a k-variety X and a place $v \in \Omega_k$, the group $\mathrm{Z}_{0,v}(X)$ represents the group $\mathrm{Z}_0(X_v)$. Furthermore, we use symbol $\mathrm{Z}_{0,\mathbf{A}}(X)$ to represent the subset $\big\{(z_v)_{v\in\Omega_k} \in \prod_{v\in\Omega_k} \mathrm{Z}_{0,v}(X)$: there is a finite subset S of Ω_k and a model $\mathcal{X}_T$ of X over $\mathcal{O}_T$ so that for each $v \notin T$, z_v comes from a relative 0-cycle $\mathcal{Z}_v$ of $\mathcal{X}_T \otimes_{\mathcal{O}_T} \mathcal{O}_v$ over $\mathcal{O}_v$ (or a formal finite sum of integral sub-schemes of $\mathcal{X}_T \otimes_{\mathcal{O}_T} \mathcal{O}_v$ which is finite over $\mathcal{O}_v)\big\}$. When X is proper, we have that $\mathrm{Z}_{0,\mathbf{A}}(X) = \prod_{v\in\Omega_k} \mathrm{Z}_{0,v}(X)$. We call elements of $\mathrm{Z}_{0,\mathbf{A}}(X)$ *adelic 0-cycles* on X over k.

• Given a scheme X, the Brauer group of X refers to the étale cohomology $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$. If X is a variety over a field k, the first Brauer group $\mathrm{Br}_1(X)$ of X refers to the kernel of the homomorphism $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$ and the algebraic Brauer group $\mathrm{Br}_a(X)$ of X refers to the cokernel of the homomorphism $\mathrm{Br}(k) \to \mathrm{Br}_1(X)$.

• Let k be a number field or a function field. We mention that there is an invariant map $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$. In the case of number fields, it comes from local class field theory. In the case of function fields, it is provided by the residue homomorphism $\mathrm{Br}(k_v) \simeq \mathbb{Q}/\mathbb{Z}(-1)$, composed with a (non-canonical) isomorphism $\mathbb{Q}/\mathbb{Z}(-1) \simeq \mathbb{Q}/\mathbb{Z}$ associated to a pre-fixed compatible system of roots of unity in $\overline{k}$.

These invariant maps fit into the following exact sequence

$$\mathrm{Br}(k) \longrightarrow \bigoplus_{v\in\Omega_k} \mathrm{Br}(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0\ . \tag{2}$$

For the case of number fields, the first map is also injective due to the Brauer–Hasse–Noether exact sequence. For the case of function fields, a proof can be found in [CH15, Prop. 2.1 (v)].

• We only deal with étale cohomologies. So, sheaves always refer to étale sheaves and the subscript ét in the symbol $\mathrm{H}^i_{\text{ét}}(*, \mathcal{F})$ will be omitted for convenience. If $\mathcal{F}^\bullet$ is a complex of étale sheaves, then $\mathrm{H}^i(*, \mathcal{F}^\bullet)$ always refers to the hyper-cohomology of $\mathcal{F}^\bullet$.

Let k be a number field or a function field and $\mathcal{F}$ be an object of the derived category $\mathrm{D}(k)$ of k, then we define $Ш^i(\mathcal{F}) = \mathrm{Ker}(\mathrm{H}^i(k, \mathcal{F}) \to \prod_{v\in\Omega_k} \mathrm{H}^i(k_v, \mathcal{F}))$.

2. Preliminaries

Let V be a variety over k and $z \in V$ be a closed point. Given an element $A \in \mathrm{Br}(V)$, we can define a pairing $(A, z) = \mathrm{Cor}_{k(z)/k}(A(z)) \in \mathrm{Br}(k)$ where $k(z)$ is the residue field of V at z and $A(z)$ is the restriction of A in $\mathrm{Br}(k(z))$.

We linearly extend the second entry of pairing (A, z) and obtain a pairing

$$(\cdot, \cdot) : \mathrm{Br}(V) \times \mathrm{Z}_0(V) \to \mathrm{Br}(k)$$

by setting $(A, z) = \sum n_i.(A, z_i)$ where $z = \sum n_i.z_i$ is a 0-cycle of V and each z_i is a closed point of V.

Let $f : W \to V$ be a morphism between two k-varieties. Given a 0-cycle z on W, we have the direct image $f_*(z)$ of z along f: It is the 0-cycle defined by the formula that $f_*(\sum n_i.y_i) = \sum n_i.[k(y_i) : k(x_i)].f(y_i)$. Notice that the direct image does not change the degree of a 0-cycle. With this definition, one obtains immediately the following formula

$$(f^*(A), z) = (A, f_*(z)) \tag{3}$$

where z is a 0-cycle on W and A is an element of $\mathrm{Br}(V)$.

Assume that k is a number field or a function field from now on. We may define the Brauer–Manin pairing

$$\langle \cdot, \cdot \rangle_{\mathrm{BM}} : \mathrm{Br}(V) \times \mathrm{Z}_{0,\mathbf{A}}(V) \to \mathbb{Q}/\mathbb{Z} \tag{4}$$

by setting $\langle A, z \rangle_{\mathrm{BM}} = \sum\limits_{v \in \Omega_k} \mathrm{inv}_v(A, z_v)$ where $z = (z_v) \in \mathrm{Z}_{0,\mathbf{A}}(V)$ is an adelic 0-cycle on V. Due to the definition of $\mathrm{Z}_{0,\mathbf{A}}(V)$, we know that $\sum\limits_{v \in \Omega_k} \mathrm{inv}_v(A, z_v)$ is a finite sum, hence, it is well-defined. We denote by $\mathrm{Z}_{0,\mathbf{A}}(V)^{\mathrm{Br}}$ the right-kernel of the pairing (4) and by $\mathrm{Z}^r_{0,\mathbf{A}}(V)^{\mathrm{Br}}$ the subset of $\mathrm{Z}_{0,\mathbf{A}}(V)^{\mathrm{Br}}$ given by the adelic 0-cycles on V of constant degree r.

We say that the Brauer–Manin obstruction for 0-cycles on V (resp. of degree r) exists if $\mathrm{Z}_{0,\mathbf{A}}(V)^{\mathrm{Br}} = \emptyset$ (resp. $\mathrm{Z}^r_{0,\mathbf{A}}(V)^{\mathrm{Br}} = \emptyset$).

With these notations, we are ready to prove Theorem 1.1. To do so, we will proceed in several steps. In section 3, we will prove the existence of universal torsors of n-torsion under the assumptions of the theorem. With the help of universal torsors of n-torsion, we will give a decomposition of the n-torsion part of the Brauer group in section 4 and at the end of section 4, we will finish the proof of Theorem 1.1 with the help of this decomposition.

We finish this section with the following immediate facts proving the easy part of the theorem.

Lemma 2.1. *The diagonal (injective) image of* $\mathrm{Z}^r_0(X)$ *in* $\prod\limits_{v \in \Omega_k} \mathrm{Z}^r_{0,v}(X)$ *is contained in* $\mathrm{Z}^r_{0,\mathbf{A}}(X)^{\mathrm{Br}}$.

Proof. It is clear that for each field extension K/k, the restriction homomorphism $\mathrm{Z}^r_0(X) \to \mathrm{Z}^r_0(X \otimes_k K)$ is injective, therefore, the diagonal homomorphism $\mathrm{Z}^r_0(X) \to \prod\limits_{v \in \Omega_k} \mathrm{Z}^r_{0,v}(X)$ is injective. Take $z = \sum n_i.x_i \in \mathrm{Z}_0^r(X)$ and consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Br}(X) \arrow[r] \arrow[d] & \mathrm{Br}(X_v) \arrow[d] \\
\mathrm{Br}(k(x_i)) \arrow[r] \arrow[d, "\mathrm{Cor}_{k(x_i)|k}"] & \mathrm{Br}(k(x_i)\otimes_k k_v) \arrow[d, "\mathrm{Cor}"] \\
\mathrm{Br}(k) \arrow[r, "\mathrm{Res}_v"'] & \mathrm{Br}(k_v)
\end{tikzcd}$$

Let z_v (resp. $x_{i,v}$) be the image of z (resp. x_i) in $\mathrm{Z}_{0,v}^r(X)$, then for each $A \in \mathrm{Br}(X)$, $(A, z_v) = \sum_i n_i.(A, x_{i,v}) = \sum_i n_i.\mathrm{Res}_v((A, x_i)) = \mathrm{Res}_v((A, z))$ and

$$\sum_v \mathrm{inv}_v(A, z_v) = \sum_v \mathrm{inv}_v\Big(\mathrm{Res}_v((A, z))\Big) = 0$$

according to the exact sequence (2). □

Proposition 2.2. *Let $f : Y \to X$ be a morphism of two k-varieties. Take an adelic 0-cycles $y \in \mathrm{Z}_{0,\mathbf{A}}^r(Y)^{\mathrm{Br}}$ and let $x = f_*(y) \in \mathrm{Z}_{0,\mathbf{A}}^r(X)$ be the direct image of y, then x lies in $\mathrm{Z}_{0,\mathbf{A}}^r(X)^{\mathrm{Br}}$.*

Proof. It is enough to notice that for each $A \in \mathrm{Br}(X)$, $\sum \mathrm{inv}_v(A, x_v) = \sum \mathrm{inv}_v(f^*(A), y_v) = 0$ by applying the formula (3). □

3. Existence of universal torsors of n-torsion

In this section, we recall the concept of universal torsor of n-torsion, which can be found in [Cao20], and give a sufficient condition for the existence of universal torsors of n-torsion.

Let S be an arbitrary scheme, X be a scheme over S and π be the structural morphism $X \xrightarrow{\pi} S$. Let $\Delta(X)$ and $\Delta_n(X)$ be the respective mapping cones of the morphisms

$$\mathbb{G}_{m,S}[1] \to (\tau_{\leq 1} R\pi_*(\mathbb{G}_{m,X}))[1] \tag{5}$$

$$\mu_{n,S}[1] \to (\tau_{\leq 1} R\pi_*(\mu_{n,X}))[1]. \tag{6}$$

If X and S are clear, we denote them by Δ and Δ_n for convenience. We mention that when $S = \mathrm{Spec}(k)$ is the spectrum of a field k, then one obtains the following identification

$$\mathrm{Br}_1(X) = \mathrm{H}^2(k, \tau_{\leq 1} R\pi_*(\mathbb{G}_{m,X})). \tag{7}$$

Indeed, the distinguished triangle

$$\tau_{\leq 1} R\pi_*(\mathbb{G}_{m,X}) \to R\pi_*(\mathbb{G}_{m,X}) \to \tau_{\geq 2} R\pi_*(\mathbb{G}_{m,X}) \to \tau_{\leq 1} R\pi_*(\mathbb{G}_{m,X})[1]$$

yields an exact sequence

$$0 \to \mathrm{H}^2(k, \tau_{\leq 1} R\pi_*(\mathbb{G}_{m,X})) \to \mathrm{H}^2(k, R\pi_*(\mathbb{G}_{m,X})) \to \mathrm{H}^2(k, \tau_{\geq 2} R\pi_*(\mathbb{G}_{m,X}))$$

in which we have identifications $\mathrm{H}^2(k, R\pi_*(\mathbb{G}_{m,X})) = \mathrm{Br}(X)$ and $\mathrm{H}^2(k, \tau_{\geq 2} R\pi_*(\mathbb{G}_{m,X})) = \mathrm{Br}(\overline{X})^\Gamma$. Hence, we have the following lemma.

Lemma 3.1. *Assume that k is a number field or a function field, then $Ш^1(\Delta)$ is the kernel of the homomorphism*

$$\mathrm{Br}_a(X) \longrightarrow \prod_{v\in\Omega_k} \mathrm{Br}_a(X_v)\ .$$

Proof. Due to the identification (7), the cohomology groups $\mathrm{H}^1(k,\Delta)$ and $\mathrm{H}^1(k_v,\Delta)$ (for any $v \in \Omega_k$) fit into exact sequences

$$\mathrm{Br}(k) \longrightarrow \mathrm{Br}_1(X) \longrightarrow \mathrm{H}^1(k,\Delta) \longrightarrow \mathrm{H}^3(k,\mathbb{G}_{m,k})$$

$$\mathrm{Br}(k_v) \longrightarrow \mathrm{Br}_1(X_v) \longrightarrow \mathrm{H}^1(k_v,\Delta) \longrightarrow \mathrm{H}^3(k_v,\mathbb{G}_{m,k_v})\ .$$

Since $\mathrm{H}^3(k,\mathbb{G}_{m,k}) = 0$ is a well-known fact (for k a number field or a function field, see [Mil06, Cor. I.4.21] or [CH15, Prop. 2.1 (iii)] respectively), we have the following diagram with two exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Br}_a(X) \arrow[r] \arrow[d] & \mathrm{H}^1(k,\Delta) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod \mathrm{Br}_a(X_v) \arrow[r] & \prod \mathrm{H}^1(k_v,\Delta) \arrow[r] & \prod \mathrm{H}^3(k_v,\mathbb{G}_{m,k_v}).
\end{tikzcd}$$

We conclude the result by the snake lemma. □

Lemma 3.2. *Let S be an integral regular noetherian scheme, let X be a faithfully flat S-scheme of finite type, and let G be a S-group scheme of multiplicative type, then there exists an exact sequence*

$$\mathrm{H}^1(S,G) \longrightarrow \mathrm{H}^1(X,G) \xrightarrow{\chi} \mathrm{Hom}_{\mathrm{D}(S)}(\hat{G},\Delta) \xrightarrow{\partial} \mathrm{H}^2(S,G) \longrightarrow \mathrm{H}^2(X,G) \tag{8}$$

where $\hat{G}$ is the Cartier dual of G and $\mathrm{D}(S)$ is the derived category of sheaves over S.

Proof. See [HS13, Prop. 1.1]. Note that in *loc. cit*, ∂ is induced from the canonical homomorphism $R^0\mathrm{Hom}_{\mathrm{D(S)}}(\hat{G},\Delta) \to R^2\mathrm{Hom}_{\mathrm{D(S)}}(\hat{G},\mathbb{G}_m) \simeq \mathrm{H}^2(S,G)$. □

Definition 3.3. *Let $S = \mathrm{Spec}(k)$ be the spectrum of a field k, let X be a geometrically integral k-variety, and let G be the k-group variety of multiplicative type whose Cartier dual is the locally constant k-group variety corresponding to the finite Γ_k-module $\mathrm{H}^1(\overline{X},\mu_n)$.*

Mention that there is a canonical morphism $\psi : \hat{G} = \Delta_n \to \Delta$ in $\mathrm{D}(k)$. We call any pre-image $\mathcal{T}_X \in \mathrm{H}^1(X,G)$ of ψ along χ (denoted in Lemma 3.2) a universal torsor of n-torsion of X.

Before we provide a sufficient condition for the existence of universal torsors of n-torsion, we prove some lemmas at first.

Definition 3.4. *Assume that k is a number field or a function field and X is a variety over k so that $\mathrm{Z}^1_{0,\mathbf{A}}(X) \neq \emptyset$. Consider the following commutative diagram*

$$\begin{tikzcd}
 & & & Ш^1(\Delta) \arrow[d] & \\
 & \mathrm{Br}(k) \arrow[r] \arrow[d] & \mathrm{Br}_1(X) \arrow[r] \arrow[d] & \mathrm{Br}_a(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod \mathrm{Br}(k_v) \arrow[r] \arrow[d] & \prod \mathrm{Br}_1(X_v) \arrow[r] & \prod \mathrm{Br}_a(X_v) \arrow[r] & 0 \\
 & Q & & &
\end{tikzcd}$$

Here Q is the cokernel of the homomorphism $\mathrm{Br}(k) \to \prod_{v\in\Omega_k} \mathrm{Br}(k_v)$ *and* $\mathrm{Br}(k_v) \to \mathrm{Br}_1(X_v)$ *is injective due to the existence of a retraction induced by an adelic 0-cycle of degree 1. According to the exact sequence (2), we may treat* $\mathbb{Q}/\mathbb{Z}$ *as a subgroup of* Q*. By the snake lemma, we have a homomorphism* $Ш^1(\Delta) \to Q$*. The following lemma shows that the image of this homomorphism is indeed contained in* $\mathbb{Q}/\mathbb{Z}$*. Thus, it defines a homomorphism* $Ш^1(\Delta) \to \mathbb{Q}/\mathbb{Z}$*. We denote this homomorphism by* inv *as well.*

Lemma 3.5. *Keeping the notations in Definition 3.4, assume that X is smooth and geometrically integral over k so that* $\mathrm{Z}^1_{0,\mathbf{A}}(X) \neq \emptyset$*, then the homomorphism* $Ш^1(\Delta) \xrightarrow{\mathrm{inv}} \mathbb{Q}/\mathbb{Z}$ *is well-defined.*

Proof. Fix an adelic 0-cycle $z = (z_v)_{v\in\Omega_k}$ of degree 1 on X. There is a non-empty finite subset T of Ω_k so that there is a smooth model $\mathcal{X}$ of X over $\mathcal{O}_T$. By the definition of adelic 0-cycles, we may enlarge T so that for each $v \notin T$, $z_v = \sum n_{v,i}.x_{v,i}$ comes from a finite formal sum $\sum n_{v,i}.R_{v,i}$ where $R_{v,i}$ is integral finite $\mathcal{O}_v$-algebra, hence, free over $\mathcal{O}_v$ of rank $[k(x_{v,i}) : k_v]$. Notice that such a formal sum also gives a retraction $\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(\mathcal{O}_v)$ by corestrictions and it is compatible with the retraction s_v given by z_v. Namely, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Br}(\mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Br}(\mathcal{X} \otimes_{\mathcal{O}_T} \mathcal{O}_v) \arrow[r] \arrow[d] & \mathrm{Br}(\mathcal{O}_v) = 0 \arrow[d] \\ \mathrm{Br}(X) \arrow[r] & \mathrm{Br}(X_v) \arrow[r, "s_v"'] & \mathrm{Br}(k_v). \end{tikzcd} \tag{9}$$

By the definition of the homomorphism $Ш^1(\Delta) \to Q$, given $\overline{\alpha} \in Ш^1(\Delta)$, we take a pre-image $\alpha \in \mathrm{Br}_1(X)$ of $\overline{\alpha}$. By enlarging T if necessary, we may assume that α comes from some α_T in $\mathrm{Br}(\mathcal{X})$. Since the restriction $\alpha_v \in \mathrm{Br}_1(X_v)$ maps to zero in $\mathrm{Br}_a(X_v)$, we have that $\alpha_v \in \mathrm{Br}(k_v)$. But for each $v \notin T$, α_v coincides the image of α_T along the path from the left-up corner of diagram (9) to the right-down corner. This implies that $\alpha_v = 0$ for each $v \notin T$. Thus, $(\alpha_v) \in \bigoplus_{v\in\Omega_k} \mathrm{Br}(k_v)$ and the invariant $\mathrm{inv}(\overline{\alpha})$ lies in $\mathbb{Q}/\mathbb{Z}$. □

Next, we discuss the relation between the Poitou–Tate pairing and the invariant homomorphism defined above.

Let G be a k-group of multiplicative type, there is a perfect Poitou–Tate pairing of finite groups.

$$\langle\, , \rangle_{\mathrm{PT}} : Ш^2(G) \times Ш^1(\hat{G}) \to \mathbb{Q}/\mathbb{Z}.$$

For k a number field, see [Dem11, Thm. 5.7]. For k a function field, see [Izq16, Thm. 2.4]. It is defined as follows. Take $a \in Ш^1(\hat{G})$ and $b \in Ш^2(G)$ and let $\mathcal{G}$ be a smooth model of G over a non-empty open subset W of $\mathrm{Spec}(\mathcal{O}_k)$ or the $\mathbb{C}((t))$-curve C associated to k. By a limit argument, $\mathrm{H}^2(k, G)$ is the direct limit of the groups $\mathrm{H}^2(U, \mathcal{G})$ where U runs over non-empty open subsets of W. For U sufficiently small, a comes from some $a'_U \in \mathrm{H}^1(U, \hat{\mathcal{G}})$ where $\hat{\mathcal{G}} = \mathrm{Hom}(\mathcal{G}, \mathbb{G}_{m,W})$ and b from $b_U \in \mathrm{H}^2(U, \mathcal{G})$. For any object $\mathcal{C}$ of $\mathrm{D}(U)$, we have hyper-cohomology groups with compact support $\mathrm{H}^i_c(U, \mathcal{C})$. (For k a function field, it is just $\mathrm{H}^i(U, j_!(\mathcal{C}))$ where j is the embedding $j : U \to X$. But for k a number field, it is more subtle. See [HS05, Section 3] for the detailed definition.) By the localization exact sequence

$$\cdots \to \mathrm{H}^i_c(U, \mathcal{C}) \to \mathrm{H}^i(U, \mathcal{C}) \to \bigoplus_{v\notin U} \mathrm{H}^i(\hat{k}_v, \mathcal{C}) \to \cdots ,$$

since a is locally trivial everywhere, a'_U comes from some $a_U \in \mathrm{H}^1_c(U, \hat{\mathcal{G}})$ along the natural map $\mathrm{H}^1_c(U, \hat{\mathcal{G}}) \to \mathrm{H}^1(U, \hat{\mathcal{G}})$. We define $\langle b, a\rangle_{\mathrm{PT}}$ as the cup-product $b_U \cup a_U \in \mathrm{H}^3_c(U, \mathbb{G}_{m,U}) \simeq \mathbb{Q}/\mathbb{Z}$ where the last isomorphism comes from the trace map, see [Mil06, Prop. II.2.6] for the case of number fields or [CH15, Prop. 2.1 (iii)] for the case of function fields. We do not know whether this definition of the Poitou–Tate pairing coincides with the classical definition in terms of cocycles, but we shall only use the fact that it leads to a perfect pairing.

Lemma 3.6. *Let X be a smooth and geometrically integral variety over a number field or function field k and let G be k-group variety of multiplicative type. Assume that for each $v \in \Omega_k$, the canonical homomorphism $\mathrm{H}^2(k_v, G) \to \mathrm{H}^2(X_v, G)$ is injective. Let $\psi \in \mathrm{Hom}_{\mathrm{D}(k)}(\hat{G}, \Delta)$ be a homomorphsim in the derived category of k and take $A \in \text{Ш}^1(\hat{G})$, then $\partial(\psi)$ lies in $\text{Ш}^2(G)$ and we have that*

$$\langle \partial(\psi), A\rangle_{\mathrm{PT}} = \mathrm{inv}(\psi_*(A)) \tag{10}$$

where ψ_* is the induced homomorphism $\mathrm{H}^1(k, \hat{G}) \to \mathrm{H}^1(k, \Delta)$ and ∂ is the homomorphism appearing in Lemma 3.2.

Proof. We include a proof for completeness following a similar argument of the proof in [HS13, Prop. 3.5].

We only prove for the case of number fields. One can obtain the proof of the case of function fields by simply replacing $\mathrm{Spec}(\mathcal{O}_k)$ with the associated $\mathbb{C}((t))$-curve C.

Note that the image of $\partial(\psi)$ in $\mathrm{H}^2(X, G)$ is zero by Lemma 3.2. Since $\mathrm{H}^2(k_v, G) \to \mathrm{H}^2(X_v, G)$ is injective by the assumption, the image of $\partial(\psi)$ in $\prod \mathrm{H}^2(k_v, G)$ is zero. Therefore, one obtains that $\partial(\psi) \in \text{Ш}^2(G)$.

Let w be the canonical homomorphism $\Delta \to \mathbb{G}_{m,k}[2]$ and π be the structural morphism $X \xrightarrow{\pi} \mathrm{Spec}(k)$. Since the exact sequence (8) is obtained by applying the functor $\mathrm{Hom}_{\mathrm{D}(k)}(\hat{G}, *)$ to the distinguished triangle

$$\Delta \xrightarrow{w} \mathbb{G}_{m,k}[2] \longrightarrow \tau_{\leq 1} R\pi_*(\mathbb{G}_{m,X})[2] \longrightarrow [1] \tag{11}$$

under the isomorphism $\mathrm{Hom}_{\mathrm{D}(k)}(\hat{G}, \mathbb{G}_{m,k}[2]) = \mathrm{H}^2(k, G)$, we have that $w \circ \psi = \partial(\psi)$.

Let $U \subseteq \mathrm{Spec}(\mathcal{O}_k)$ be a sufficiently small non-empty open subset so that there exists a smooth U-scheme $\mathcal{X}$ with geometrically integral fibres and generic fibre $X = \mathcal{X} \times_U k$, and a smooth U-group of multiplicative type $\mathcal{G}$ with generic fibre $G = \mathcal{G} \times_U k$. Let $w_U \in \mathrm{Hom}_{\mathrm{D}(U)}(\Delta(\mathcal{X}), \mathbb{G}_{m,U}[2])$ be defined in the same manner as w. There is a canonical restriction homomorphism

$$\mathrm{Hom}_{\mathrm{D}(U)}(\hat{\mathcal{G}}, \Delta(\mathcal{X})) \longrightarrow \mathrm{Hom}_{\mathrm{D}(U)}(\hat{G}, \Delta(X)) \tag{12}$$

and for each $V \subseteq U$ a non-empty open subset, we obtain a commutative diagram

$$\begin{array}{ccccccc} \mathrm{H}^1(\mathcal{X}_V, \mathcal{G}_V) & \longrightarrow & \mathrm{Hom}_{\mathrm{D}(V)}(\hat{\mathcal{G}}_V, \Delta(\mathcal{X}_V)) & \longrightarrow & \mathrm{H}^2(V, \mathcal{G}_V) & \longrightarrow & \mathrm{H}^2(\mathcal{X}_V, \mathcal{G}_V) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^1(X, G) & \longrightarrow & \mathrm{Hom}_{\mathrm{D}(k)}(\hat{G}, \Delta(X)) & \longrightarrow & \mathrm{H}^2(k, G) & \longrightarrow & \mathrm{H}^2(X, G). \end{array} \tag{13}$$

Note that the rows of this diagram are exact by Lemma 3.2. Passing to the limit, one obtains an isomorphism

$$\varinjlim_{V} \mathrm{Hom}_{\mathrm{D}(V)}(\hat{\mathcal{G}}_V, \Delta(\mathcal{X}_V)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(k)}(\hat{G}, \Delta(X)). \tag{14}$$

Hence, after shrinking U, ψ comes from some $\psi_U \in \mathrm{Hom}_{\mathrm{D}(U)}(\hat{\mathcal{G}}, \Delta(\mathcal{X}))$ and then, $w \circ \psi = \partial(\psi)$ is the image of $w_U \circ \psi_U \in \mathrm{Hom}_{\mathrm{D}(U)}(\hat{\mathcal{G}}, \mathbb{G}_{m,U}[2]) = \mathrm{H}^2(U, \mathcal{G})$.

Denote $\alpha = \psi_*(A) \in Ш^1(\Delta)$. By shrinking U further, A comes from some $A_U \in \mathrm{H}^1(U, \hat{\mathcal{G}})$. Since A vanishes locally everywhere, we have that $A_U \in \mathrm{H}^1_c(U, \hat{\mathcal{G}})$. Let $\alpha_U = \psi_{U*}(A_U)$, then α comes from α_U.

Consider the following commutative diagram whose rows are distinguished triangles

$$\begin{tikzcd} \mathbb{G}_{m,U}[1] \arrow[r] \arrow[d] & (\tau_{\leq 1} Rp_{U*}(\mathbb{G}_{m,\mathcal{X}}))[1] \arrow[r] \arrow[d] & \Delta(\mathcal{X}) \arrow[d] \\ \bigoplus_{v \notin U} j_{v*} j_v^* \mathbb{G}_{m,U}[1] \arrow[r] & \bigoplus_{v \notin U} j_{v*} j_v^* (\tau_{\leq 1} Rp_{U*}(\mathbb{G}_{m,\mathcal{X}}))[1] \arrow[r] & \bigoplus_{v \notin U} j_{v*} j_v^* \Delta(\mathcal{X}) \end{tikzcd} \tag{15}$$

where j_v is the natural map $\mathrm{Spec}(k_v) \to U$. Let $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$ be the mapping cones of three vertical homomorphisms, respectively. The group $\mathrm{H}^1(U, \mathcal{C}_2[-1])$ may be identified with $\mathrm{H}^1_c(U, \Delta(\mathcal{X}))$. So we may identify α_U as an element of $\mathrm{H}^1(U, \mathcal{C}_2[-1])$. Therefore, $w_{U*}(\alpha_U)$ is a class in $\mathrm{H}^2(U, \mathcal{C}_0[-1]) = \mathrm{H}^1(U, \mathcal{C}_0)$. After shrinking U and passing to the limit, this class yields an element in the cokernel of the map $\mathrm{Br}(k) \to \bigoplus_{v \in \Omega_k} \mathrm{Br}(k_v)$ (with the isomorphism $\mathrm{Br}(k_v) \simeq \mathrm{Br}(k_v^h)$ where k_v^h is the henselisation of k at v) which coincides precisely the snake-lemma definition of $\mathrm{inv}(\alpha)$.

Therefore, we can compare the cup-product pairing and the Yoneda pairing

$$\mathrm{inv}(\psi_*(A)) = \mathrm{inv}(\alpha) = w_{U*}(\alpha_U) = w_{U*}(\psi_{U*}(A_U)) = (w_U \circ \psi_U) \cup A_U. \tag{16}$$

Here $(w_U \circ \psi_U) \cup A_U$ is exactly the definition of $\langle \partial(\psi), A \rangle_{\mathrm{PT}}$. □

Remark 3.7. *Diagram (15) only takes into account the case when U is totally imaginary, that is when the compact support cohomology does not need to be modified because of infinite places. For the general case, one only needs to modify the second row in the diagram (15) at infinite places in the same manner. We omit the details here to avoid complicated notation.*

Proposition 3.8. *Let k be a number field or a function field and let X be a smooth and geometrically integral variety over k. Let $n \geq 2$ be a number and let G be the Cartier dual of $\mathrm{H}^1(\overline{X}, \mu_n)$. If $\mathrm{Z}^1_{0,\mathbf{A}}(X)^{\mathrm{Br}_1} \neq \emptyset$, then χ is surjective where χ is the homomorphism defined in Lemma 3.2. Hence, universal torsors of n-torsion on X always exist.*

Proof. Notice that the assumption $\mathrm{Z}^1_{0,\mathbf{A}}(X)^{\mathrm{Br}_1} \neq \emptyset$ guarantees the hypothesis of Lemma 3.6. We only need to show that $\partial(\psi) = 0$ for each $\psi \in \mathrm{Hom}_{\mathrm{D}(k)}(\hat{G}, \Delta)$ by the exact sequence (8). By Poitou–Tate exact sequence, we only need to show that for each $A \in Ш^1(\hat{S})$, $\langle \partial(\psi), A \rangle_{\mathrm{PT}} = 0$. By Lemma 3.6, we have that $\langle \partial(\psi), A \rangle_{\mathrm{PT}} = \mathrm{inv}(\psi_*(A))$. Since $\psi_*(A)$ lies in $Ш^1(\Delta) \subseteq \mathrm{Br}_a(X)$, we take α as a pre-image of $\psi_*(A)$ in $\mathrm{Br}_1(X)$. For each $v \in \Omega_k$, let s_v be the retraction of $\mathrm{Br}(k_v) \to \mathrm{Br}_1(X_v)$ defined by some adelic 0-cycle $(z_v) \in \mathrm{Z}^1_{0,\mathbf{A}}(X)^{\mathrm{Br}_1}$. Then we have that $\mathrm{inv}(\psi_*(A)) = \sum \mathrm{inv}_v(s_v(\alpha(X_v))) = \langle \alpha, (z_v) \rangle_{\mathrm{BM}} = 0$ by the snake lemma. (Here, the sum is finite due to the same reason as in the proof of Lemma 3.5.) □

4. A Decomposition of $\mathrm{H}^2(Z,\mu_n)$

In this section, we will give a decomposition of the Brauer group of the product Z and then use it to prove Theorem 1.1.

When $\mathrm{Z}^1_{0,\mathbf{A}}(X)^{\mathrm{Br}}\neq\emptyset$ and $\mathrm{Z}^1_{0,\mathbf{A}}(Y)^{\mathrm{Br}}\neq\emptyset$, universal torsors of n-torsion $\mathcal{T}_X$ and $\mathcal{T}_Y$ exist on X and Y respectively by Proposition 3.8.

Let

$$\begin{tikzcd} Z \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "q_0"] \\ Y \arrow[r, "p_0"'] & \mathrm{Spec}(k) \end{tikzcd} \tag{17}$$

be the morphisms between $\mathrm{Spec}(k), X, Y$ and Z.

We define the homomorphism

$$\begin{aligned} \epsilon : \mathrm{Hom}_k(G_X\otimes G_Y,\mu_n) &\longrightarrow \mathrm{H}^2(Z,\mu_n) \\ \psi &\mapsto \psi_*(p^*(\mathcal{T}_X)\cup q^*(\mathcal{T}_Y)) \end{aligned} \tag{18}$$

where ψ_* is the induced homomorphism $\mathrm{H}^2(Z,G_X\otimes G_Y)\to\mathrm{H}^2(Z,\mu_n)$.

Proposition 4.1. *Let k be a number field or a function field, let X and Y be two smooth and geometrically integral k-varieties, and let $Z=X\times_k Y$ be the product of X and Y. Denote canonical arrows between them as in the diagram (17). Assume that $\mathrm{Z}^1_{0,\mathbf{A}}(X)^{\mathrm{Br}}\neq\emptyset$ and $\mathrm{Z}^1_{0,\mathbf{A}}(Y)^{\mathrm{Br}}\neq\emptyset$. Let G_X (resp. G_Y) be the Cartier dual of $\mathrm{H}^1(\overline{X},\mu_n)$ (resp. $\mathrm{H}^1(\overline{Y},\mu_n)$) and let $\mathcal{T}_X$ (resp. $\mathcal{T}_Y$) be a universal torsor of n-torsion on X (resp. Y). Then we have the following decomposition*

$$\mathrm{H}^2(Z,\mu_n)=p^*\mathrm{H}^2(X,\mu_n)+q^*\mathrm{H}^2(Y,\mu_n)+\mathrm{Im}(\epsilon) \tag{19}$$

where ϵ is defined in (18).

Proof. Recall the spectral sequence

$$E_2^{i,j}=\mathrm{H}^i(k,\mathrm{H}^j(\overline{X},\mu_n))\Rightarrow\mathrm{H}^{i+j}(X,\mu_n). \tag{20}$$

We focus on the line $i+j=2$ of this spectral sequence:

$$\begin{tikzcd}[column sep=small] \mathrm{H}^2(\overline{X},\mu_n)^\Gamma \arrow[drr, "\delta_X"] & * & * & * \\ \mathrm{H}^1(\overline{X},\mu_n)^\Gamma \arrow[drr] & \mathrm{H}^1(k,\mathrm{H}^1(\overline{X},\mu_n)) \arrow[drr] & \mathrm{H}^2(k,\mathrm{H}^1(\overline{X},\mu_n)) & * \\ * & * & \mathrm{H}^2(k,\mu_n) & \mathrm{H}^3(k,\mu_n). \end{tikzcd} \tag{21}$$

Firstly, we prove that the homomorphism

$$\mathrm{H}^i(k,\mu_n)\to\mathrm{H}^i(X,\mu_n) \tag{22}$$

is injective for $i\geq 3$. In fact, in the case of function fields, it is the direct consequence of the fact that $\mathrm{cd}(k)\leq 2$. In the case of number fields, $\mathrm{H}^i(k,\mu_n)\to\bigoplus_{v\text{ real}}\mathrm{H}^i(k_v,\mu_n)$ is bijective for $i\geq 3$, see [Mil06, Thm. I.4.10 (c)]. Since any adelic 0-cycle of degree 1 defines a retraction of the homomorphism $\mathrm{H}^i(k_v, \mu_n) \to \mathrm{H}^i(X_v, \mu_n)$ for each $v \in \Omega_k$, this shows that the homomorphism

$$\mathrm{H}^i(k, \mu_n) \to \mathrm{H}^i(X, \mu_n) \to \bigoplus_{v \text{ real}} \mathrm{H}^i(X_v, \mu_n) \tag{23}$$

is injective. Hence, the homomorphism (22) is injective. This implies the triviality of all the homomorphisms in the spectral sequence (21) whose target is $E_2^{i,0} = \mathrm{H}^i(k, \mu_n)$ for $i \geq 3$.

Since the line $i + j = 2$ is stable after page 3 using the above fact, we obtain exact sequences for $\mathrm{H}^2(X, \mu_n)$:

$$0 \longrightarrow E_3^{2,0} \longrightarrow \mathrm{H}^2(X, \mu_n) \longrightarrow A \longrightarrow 0$$
$$0 \longrightarrow E_3^{1,1} \longrightarrow A \longrightarrow E_3^{0,2} \longrightarrow 0$$

where $E_3^{2,0} = \mathrm{H}^2(k, \mu_n)/\mathrm{Im}(\mathrm{H}^1(\overline{X}, \mu_n)^\Gamma)$, $E_3^{1,1} = \mathrm{H}^1(k, \mathrm{H}^1(\overline{X}, \mu_n))$, $E_3^{0,2} = \mathrm{Ker}(\delta_X)$ and δ_X is the homomorphism in diagram (21). Therefore, we obtain an identification $A = \mathrm{H}^2(X, \mu_n)/\mathrm{Im}(\mathrm{H}^2(k, \mu_n))$ and an exact sequence

$$0 \longrightarrow \mathrm{H}^1(k, \mathrm{H}^1(\overline{X}, \mu_n)) \longrightarrow \mathrm{H}^2(X, \mu_n)/\mathrm{Im}(\mathrm{H}^2(k, \mu_n)) \longrightarrow \mathrm{Ker}(\delta_X) \longrightarrow 0\,. \tag{24}$$

There are similar sequences for Y and Z linked by the homomorphisms p^* and q^*.

Let us denote

$$\mathcal{H} = p^*\mathrm{H}^2(X, \mu_n) + q^*\mathrm{H}^2(Y, \mu_n) + \mathrm{Im}(\epsilon) \subseteq \mathrm{H}^2(Z, \mu_n). \tag{25}$$

Note that the image of $\mathrm{H}^2(k, \mu_n)$ in $\mathrm{H}^2(Z, \mu_n)$ is contained in $\mathcal{H}$. It is enough to prove that the canonical homomorphism $\mathcal{H} \to \mathrm{H}^2(Z, \mu_n)/\mathrm{Im}(\mathrm{H}^2(k, \mu_n))$ is surjective.

By [Cao20, Prop. 2.6] (see also its corrigendum [Cao23, Thm. 2.1]), we have an isomorphism

$$\mathrm{H}^1(\overline{Z}, \mu_n) \simeq p^*\mathrm{H}^1(\overline{X}, \mu_n) \oplus q^*\mathrm{H}^1(\overline{Y}, \mu_n) \tag{26}$$

This implies that the image of $\mathrm{H}^1(k, \mathrm{H}^1(\overline{Z}, \mu_n))$ in $\mathrm{H}^2(Z, \mu_n)/\mathrm{Im}(\mathrm{H}^2(k, \mu_n))$ is contained in $\mathcal{H}$. By the exact sequence (24), it remains to prove that each element of $\mathrm{Ker}(\delta_Z)$ admits a pre-image from $\mathcal{H}$.

By [Cao20, Cor. 2.7] (see also its corrigendum [Cao23, Thm. 2.1]) and Galois descent, we have an isomorphism of Γ-modules

$$\mathrm{H}^2(\overline{Z}, \mu_n)^\Gamma \simeq \mathrm{H}^2(\overline{X}, \mu_n)^\Gamma \oplus \mathrm{H}^2(\overline{Y}, \mu_n)^\Gamma \oplus \mathrm{Hom}_k(G_X \otimes G_Y, \mu_n) \tag{27}$$

Each $\psi \in \mathrm{Hom}_k(G_X \otimes G_Y, \mu_n)$, considered as an element of $\mathrm{H}^2(\overline{Z}, \mu_n)^\Gamma$, comes from $\epsilon(\psi) \in \mathrm{H}^2(Z, \mu_n)$ as in *loc. cit.* Hence δ_Z is zero on the direct summand $\mathrm{Hom}_k(G_X \otimes G_Y, \mu_n)$ of $\mathrm{H}^2(\overline{Z}, \mu_n)^\Gamma$ by the exact sequence (24), so that δ_Z is the direct sum of the following homomorphisms

$$\begin{aligned} \delta_X : \mathrm{H}^2(\overline{X}, \mu_n)^\Gamma &\longrightarrow \mathrm{H}^2(k, \mathrm{H}^1(\overline{X}, \mu_n)) \\ \delta_Y : \mathrm{H}^2(\overline{Y}, \mu_n)^\Gamma &\longrightarrow \mathrm{H}^2(k, \mathrm{H}^1(\overline{Y}, \mu_n)) \\ \mathrm{Hom}_k(G_X \otimes G_Y, \mu_n) &\longrightarrow 0 \end{aligned} \tag{28}$$

Thus $\mathrm{Ker}(\delta_Z)$ is the surjective image of $\mathrm{H}^2(X, \mu_n) \oplus \mathrm{H}^2(Y, \mu_n) \oplus \mathrm{Im}(\epsilon)$ by the exact sequence (24). □

Remark 4.2. *Note that the definition of ϵ in diagram (18) relies on the choice of $\mathcal{T}_X$ or $\mathcal{T}_Y$. But the validity of Proposition 4.1 is independent on the choice of such universal torsors of n-torsion. In fact, any two universal torsors of n-torsion $\mathcal{T}_X$ and $\mathcal{T}'_X$ on X differ by the image of an element in $\mathrm{H}^1(k, G_X)$, and so $\psi_*(p^*(\mathcal{T}_X) \cup q^*(\mathcal{T}_Y))$ and $\psi_*(p^*(\mathcal{T}'_X) \cup q^*(\mathcal{T}_Y))$ differ by an element in $q^*\mathrm{H}^2(Y, \mu_n)$. Applying a similar argument on Y, the proposition holds for any choice of universal torsors of n-torsion.*

We are now ready to prove Theorem 1.1.

Proof of **Theorem 1.1**. According to Corollary 2.2, we only need to prove that if $x = (x_v)_{v\in\Omega_k} \in \mathrm{Z}^1_{0,\mathbf{A}}(X)^{\mathrm{Br}}$ and $y = (y_v)_{v\in\Omega_k} \in \mathrm{Z}^1_{0,\mathbf{A}}(Y)^{\mathrm{Br}}$, then $(z_v)_{v\in\Omega_k} = \Phi(x, y)$ is contained in $\mathrm{Z}^1_{0,\mathbf{A}}(Z)^{\mathrm{Br}}$.

Let G_X (resp. G_Y) be the Cartier dual of $\mathrm{H}^1(\overline{X}, \mu_n)$ (resp. $\mathrm{H}^1(\overline{Y}, \mu_n)$) and let $\mathcal{T}_X$ (resp. $\mathcal{T}_Y$) be a universal torsor of n-torsion on X (resp. Y).

Since the Brauer group of a smooth variety is a torsion group, and according to Proposition 4.1, for each positive integer n, the subgroup $\mathrm{Br}(X)[n]$ is generated by the images of $p^*\mathrm{H}^2(X, \mu_n)$, $q^*\mathrm{H}^2(Y, \mu_n)$ and $\mathrm{Im}(\epsilon)$ in $\mathrm{Br}(X)[n]$. Therefore, we only need to prove that z is orthogonal to $\mathrm{Im}(\epsilon)$.

The local cup-product couplings for each $v \in \Omega_k$,

$$\cup_v : \mathrm{H}^1(k_v, \hat{G}_Y) \times \mathrm{H}^1(k_v, G_Y) \to \mathrm{H}^2(k_v, \mu_n)$$

give rise to the global Poitou–Tate pairing

$$\langle\, ,\, \rangle_{\mathrm{PT}} : P^1(k, \hat{G}_Y) \times P^1(k, G_Y) \to \mathbb{Z}/n. \tag{29}$$

It is a perfect pairing, moreover, we have a Poitou–Tate exact sequence

$$\mathrm{H}^1(k, G_Y) \longrightarrow P^1(k, G_Y) \longrightarrow \mathrm{Hom}(\mathrm{H}^1(k, \hat{G}_Y)\ , \mathbb{Q}/\mathbb{Z}). \tag{30}$$

See [Mil06, Thm. 4.10] for the case of number fields and [Izq16, Thm. 2.7] for the case of function fields. Here for any finite Galois module F, $P^1(k, F)$ is the restricted product of groups $\mathrm{H}^1(k_v, F)$ for $v \in \Omega_k$, relative to the subgroups $\mathrm{H}^1(\mathcal{O}_v, F)$ where v is a non-archimedean place of k in the case of number fields. Take $\psi \in \mathrm{Hom}_k(G_X \otimes G_Y, \mu_n)$, then the Brauer–Manin pairing $\langle\epsilon(\psi), z\rangle_{\mathrm{BM}}$ is given by the Poitou–Tate pairing $\langle((\psi_*(\mathcal{T}_X), x_v)), ((\mathcal{T}_Y, y_v))\rangle_{\mathrm{PT}}$ by the following lemma:

Lemma 4.3. *Under the same hypothesis as in Proposition 4.1, take $x = (x_v)_{v\in\Omega_k} \in \mathrm{Z}^1_{0,\mathbf{A}}(X)$, $y = (y_v)_{v\in\Omega_k} \in \mathrm{Z}^1_{0,\mathbf{A}}(Y)$ and $\psi \in \mathrm{Hom}_k(G_X \otimes G_Y, \mu_n)$, and let $z = (z_v) = \Phi(x, y)$ be the product of adelic 0-cycles x and y, then we have that*

$$\langle\epsilon(\psi), z\rangle_{\mathrm{BM}} = \langle((\psi_*(\mathcal{T}_X), x_v))_{v\in\Omega_k}, ((\mathcal{T}_Y, y_v))_{y\in\Omega_k}\rangle_{\mathrm{PT}} \tag{31}$$

where ψ_* is the induced homomorphism $\mathrm{H}^1(X, G_X) \to \mathrm{H}^1(X, \hat{G}_Y)$.

Proof. Denote $\mathcal{A} = \psi_*(\mathcal{T}_X)$ for convenience.

We first consider that x and y are global 0-cycles on X and Y. Since the equation (31) is bilinear in entries x and y, we may assume that x and y are closed points. Since $k(x)$ is a separable extension of k, z correponds a formal sum of closed points of $\mathrm{Spec}(k(x) \otimes_k k(y))$. Write $z = \sum z_i$ as the sum of the closed points. Recall that $\epsilon(\psi)$ is the image of $p^*(\mathcal{T}_X) \cup q^*(\mathcal{T}_Y)$ along $\mathrm{H}^2(Z, G_X \otimes G_Y) \to \mathrm{H}^2(Z, \mu_n)$ and one obtains that $\epsilon(\psi) =$

$p^*(\psi_*(\mathcal{T}_X)) \cup q^*(\mathcal{T}_Y)$. Therefore, one obtains that

$$\begin{aligned}
\sum_i \mathrm{Cor}_{k(z_i)|k}(\mathcal{A}(z_i) \cup \mathcal{T}_Y(z_i)) &= \mathrm{Cor}_{k(y)|k}(\sum_i \mathrm{Cor}_{k(z_i)|k(y)}(\mathcal{A}(z_i)) \cup \mathcal{T}_Y(y)) \\
&= \mathrm{Cor}_{k(y)|k}(\mathrm{Res}_{k(y)|k}(\mathrm{Cor}_{k(x)|k}(\mathcal{A}(x))) \cup \mathcal{T}_Y(y)) \\
&= \mathrm{Cor}_{k(x)|k}(\mathcal{A}(x)) \cup \mathrm{Cor}_{k(y)|k}(\mathcal{T}_Y(y)) \\
&= (\mathcal{A}, x) \cup (\mathcal{T}_Y, y).
\end{aligned} \tag{32}$$

For adelic 0-cycles, with the formula (32), we have that

$$\langle \epsilon(\psi), z \rangle_{\mathrm{BM}} = \sum_{v \in \Omega_v} \mathrm{inv}_v((\mathcal{A}, x_v) \cup (\mathcal{T}_Y, y_v)) = \langle ((\psi_*(\mathcal{T}_X), x_v))_{v\in\Omega_k}, ((\mathcal{T}_Y, y_v))_{y\in\Omega_k} \rangle_{\mathrm{PT}}.$$

□

Return to the proof of Theorem 1.1. For each $a \in \mathrm{H}^1(k, \hat{G}_Y)$, one obtains that $\langle a, ((\mathcal{T}_Y, y_v))_{v\in\Omega_k} \rangle_{\mathrm{PT}} = \langle a \cup \mathcal{T}_Y, (y_v) \rangle_{\mathrm{BM}} = 0$. Hence, the element $((\mathcal{T}_Y, y_v))_{v\in\Omega_k}$, as an element of $P^1(k, G_Y)$, comes from some $b \in \mathrm{H}^1(k, G_Y)$ by the Poitou–Tate exact sequence (30). Finally, with the help of Lemma 4.3, one obtains that

$$\begin{aligned}
\langle \epsilon(\psi), z \rangle_{\mathrm{BM}} &= \langle ((\psi_*(\mathcal{T}_X), x_v)), ((\mathcal{T}_Y, y_v)) \rangle_{\mathrm{PT}} \\
&= \langle ((\psi_*(\mathcal{T}_X), x_v)), b \rangle_{\mathrm{PT}} \\
&= \langle \psi_*(\mathcal{T}_X) \cup b, (x_v) \rangle_{\mathrm{BM}} = 0.
\end{aligned}$$

This completes the proof. □

Remark 4.4. *We also remark that in the proof of Theorem 1.1, if one takes $z \in Z(\mathbf{A}_k)^{\mathrm{Br}} \subseteq \mathrm{Z}^1_{0,\mathbf{A}}(Z)^{\mathrm{Br}}$, then z is always of the form $\Phi(x, y)$ where $x \in X(\mathbf{A}_k)^{\mathrm{Br}}$ and $y \in Y(\mathbf{A}_k)^{\mathrm{Br}}$. One can also conclude that $X(\mathbf{A}_k)^{\mathrm{Br}} \times Y(\mathbf{A}_k)^{\mathrm{Br}} = Z(\mathbf{A}_k)^{\mathrm{Br}}$, which has been proved in [SZ14] when k is a number field. Now we also get this equality when k is a function field.*

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