NON DEGENERACY OF BLOW-UP SOLUTIONS OF NON-QUANTIZED SINGULAR LIOUVILLE-TYPE EQUATIONS AND THE CONVEXITY OF THE MEAN FIELD ENTROPY OF THE ONSAGER VORTEX MODEL WITH SINGULAR SOURCES

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ABSTRACT. We establish the non degeneracy of bubbling solutions for singular mean field equations when the blow-up points are either regular or involve non-quantized singular sources. This extends the results from Bartolucci-Jevnikar-Lee-Yang [7], which focused on regular blow-up points. As a consequence, we establish the strict convexity of the Entropy in the large energy limit for a specific class of two-dimensional domains in the Onsager mean field vortex model with singular sources.

1. INTRODUCTION

In this article, we shall investigate the non degeneracy property of the following mean field equation with singular sources defined on the Riemann surface M

$$\Delta_g v + \rho\left(\frac{he^v}{\int_M he^v \mathrm{d}\mu} - \frac{1}{vol_g(M)}\right) = \sum_{j=1}^N 4\pi\alpha_j\left(\delta_{q_j} - \frac{1}{vol_g(M)}\right) \text{ in } M, \tag{1.1}$$

along with the corresponding Dirichlet problem

$$\begin{cases} \Delta v + \rho\dfrac{he^v}{\int_\Omega he^v \mathrm{d}x} = \sum\limits_{j=1}^N 4\pi\alpha_j\delta_{q_j} & \text{in } \Omega, \\ \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where (M,g) in (1.1) represents a Riemann surface with metric g, and Ω in (1.2) is an open, bounded domain in $\mathbb{R}^2$ with a C^2 boundary $\partial\Omega$, Δ_g denotes the Laplace-Beltrami operator on M, while Δ stands for the classical Laplacian operator in Euclidean space, the potential function h is a positive C^5 function and the parameter ρ is strictly positive in both equations, the points $q_1, \cdots, q_N$ are distinct in the respective ambient space, the coefficients $\alpha_j > -1$, and δ_{q_j} represents the Dirac measure at q_j. The volume of

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(M,g) is denoted by $vol_g(M)$, and we assume throughout the article that $vol_g(M)=1$.

Both equations have a rich background in Differential Geometry and Physics, including topics such as conformal geometry [50, 80], Electroweak and Self-Dual Chern-Simons vortices [2, 71, 76, 77, 88], the statistical mechanics approach to turbulent Euler flows, plasmas, and self-gravitating systems [7, 25, 51, 38, 70, 81, 86], Cosmic Strings [18, 68], the theory of hyperelliptic curves and modular forms [26], and CMC immersions in hyperbolic 3-manifolds [79]. These factors motivated the significant efforts dedicated to studying (1.1), focusing on existence results [6, 11, 14, 20, 39, 27, 31, 32, 34, 35, 40, 41, 43, 59, 59, 62, 63, 64, 66], uniqueness [8, 9, 10, 12, 13, 28, 48, 53, 56, 58, 74, 87], and concentration-compactness and bubbling behavior [16, 17, 22, 45, 46, 47, 52, 54, 55, 75, 78, 82, 83, 84, 89, 90]. Notably, groundbreaking work initiated in [33, 60] has recently advanced the understanding of the flat two-torus with singular sources, see [36, 57] and related references.

By using the classical Green function $G_M(z,p)$, which satisfies

$$\begin{cases} -\Delta_g G_M(x,p)=\delta_p-1 \quad \text{in } M \\ \\ \int_M G_M(x,p)\mathrm{d}\mu=0, \end{cases}$$

we can rewrite (1.1) as follows,

$$\Delta_g \mathrm{w}+\rho\left(\frac{He^{\mathrm{w}}}{\int_M He^{\mathrm{w}}\mathrm{d}\mu}-1\right)=0 \quad \text{in } M, \tag{1.3}$$

where

(1.4)

$$\mathrm{w}(x)=v(x)+4\pi\sum_{j=1}^N \alpha_j G_M(x,q_j) \quad \text{and} \quad H(x)=h(x)\prod_{j=1}^N e^{-4\pi\alpha_j G_M(x,q_j)}.$$

Using the local isothermal coordinates $z=z(x)$ centered at p, $0=z(p)$, $G(z,0)=G_M(x(z),p)$ can be decomposed by

$$G(z,0)=-\frac{1}{2\pi}\log|z|+R(z,0),$$

where $R(z,0)$ denotes the regular part of $G(z,0)$, and $R_M(x(z),p)=R(z,0)$ stands for the regular part of $G_M(x(z),p)$. As a consequence, near each singular point q_j, $j=1,\cdots,N$, we can write

$$H(z)=h_j(z)|z|^{2\alpha_j}, \quad |z|\ll 1, \quad 1\le j\le N, \quad \text{where } h_j(z)>0 \text{ near } 0. \tag{1.5}$$

The equation (1.3) possesses a variational structure, as it can be derived as the Euler-Lagrange equation of the functional:

$$I_\rho(\mathrm{w})=\frac{1}{2}\int_M|\nabla \mathrm{w}|^2+\rho\int_M \mathrm{w}-\rho\log\int_M He^{\mathrm{w}}, \quad \mathrm{w}\in H^1(M).$$

Since adding a constant to any solution of (1.3) still yields a solution, we can, without loss of generality, define I_ρ on the following subspace

$$\mathring{H}^1(M) := \left\{ f \in H^1(M) \,|\, \int_M f d\mu = 0 \right\}.$$

A detailed analysis of the variational structure of (1.3) can be found in [64]. Typically, we say that w in (1.3) (or ν in (1.1)) is *non-degenerate* if the linearized equation corresponding to the solution w of (1.3) admits only the trivial solution. Specifically, the following equation

$$\Delta_g \phi + \rho \frac{He^{\mathrm{w}}}{\int_M He^{\mathrm{w}} \mathrm{d}\mu} \left(\phi - \int_M \frac{He^{\mathrm{w}}\phi}{\int_M He^{\mathrm{w}} \mathrm{d}\mu} \right) = 0 \quad \text{in } M \tag{1.6}$$

has only the solution $\phi \equiv 0$.

Similarly, for the Dirichlet problem (1.2), we recall the standard Green's function

$$\begin{cases} -\Delta G_\Omega(x,p) = \delta_p & \text{in } \Omega, \\ \\ G_\Omega(x,p) = 0 & \text{on } \partial\Omega. \end{cases}$$

Using $G_\Omega(x,p)$, equation (1.2) can be rewritten as follows

$$\begin{cases} \Delta \mathrm{w} + \rho \dfrac{He^{\mathrm{w}}}{\int_\Omega He^{\mathrm{w}} \mathrm{d}x} = 0 & \text{in } \Omega, \\ \\ \mathrm{w} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

with

$$\mathrm{w}(x) = \nu(x) + 4\pi \sum_{j=1}^N \alpha_j G_\Omega(x,q_j) \quad \text{and} \quad H(x) = h(x) \prod_{j=1}^N e^{-4\pi\alpha_j G_\Omega(x,q_j)}.$$

It is well known that $G_\Omega(x,p)$ can be decomposed by

$$G_\Omega(x,p) = -\frac{1}{2\pi} \log|x-p| + R_\Omega(x,p),$$

where $R_\Omega(x,p)$ denotes the regular part. Then near each singular point q_j, we have

$$H(z) = h_j(x)|x-q_j|^{2\alpha_j}, \quad x \neq q_j$$

with $h_j(x) > 0$ in a small neighbourhood of q_j. The linearized equation associated with a solution w of (1.7) takes the form

$$\Delta \phi + \rho \frac{He^{\mathrm{w}}}{\int_\Omega He^{\mathrm{w}} \mathrm{d}x} \left(\phi - \int_\Omega \frac{He^{\mathrm{w}}\phi}{\int_\Omega He^{\mathrm{w}} \mathrm{d}x} \right) = 0 \quad \text{in } \Omega. \tag{1.8}$$

As the equation (1.1) defined on the Riemann surface, we say that a solution ν of (1.2) is *non degenerate* if (1.8) admits only the trivial solution.

In two-dimensional space, the loss of compactness phenomenon occurs for both equations (1.1) (or (1.3)) and (1.2) (or (1.7)), primarily due to the exponential nonlinearity is critical in the sense of Sobolev embedding. Consequently, studying the qualitative behavior of blow-up solutions, such as their uniqueness and non-degenerate properties, becomes a natural and important question. For example, consider equation (1.1), we say that v_k (or w_k) is a sequence of bubbling solutions of (1.1) (or (1.3)) if the L^∞ norm of the corresponding w_k, defined by (1.4), tends to infinity as k approaches infinity. Through a series of seminal works (see [16, 17, 54]), it is well-known that once blow-up phenomenon occurs, the function w_k blows up at a finite number of points. To say that the set of blow-up points is $p_1, \cdots, p_m$, means that there exist m sequences of points $p_{k,1}, \cdots, p_{k,m}$ such that, possibly along a subsequence, $\mathrm{w}_k(p_{k,j}) \to +\infty$ and $p_{k,j} \to p_j$ as $k \to +\infty$ for $j = 1, \cdots, m$. If none of the blow-up points are singular sources, the local uniqueness of the bubbling solutions has been established in [8]. Here, the local uniqueness means that once the blow-up points of the bubbling solution are determined, the solution is unique. While the non degeneracy property was proved in [7]. A natural question arises: are the results on uniqueness and non degeneracy still true when the set of blow-up points contains singular points? In a recent paper, we have proved uniqueness in the case where at least one blow-up point is a singular source [19]; see also [9, 87] for related results. By assumption, the strength of the singular source at a point p is $4\pi\alpha_p$, where $\alpha_p = 0$ if p is a regular point. We use $\alpha_1, \cdots, \alpha_m$ to denote the strengths at the points $p_1, \cdots, p_m$, respectively, and assume that the first $m_1 \geq 1$ of these points are singular blow-up points:

$$\alpha_i > -1, \quad \alpha_i \notin \mathbb{N}, \quad 1 \leq i \leq m_1, \quad \alpha_{m_1+1} = \cdots = \alpha_m = 0. \tag{1.9}$$

Here, $\mathbb{N}$ represents the set of natural numbers including 0, and α_M denotes the maximum of $\alpha_1, \cdots, \alpha_m$. We say that the singular source at p is non-quantized if α_p is not a positive integer. Throughout this work, we assume all singular sources, when they occur at blow-up points, are non-quantized. In the first part of this work, we shall prove that the non degeneracy result holds true in this more general setting.

In order to state our results, we need the following definitions. First of all, let us set:

$$G_j^*(x) = 8\pi(1+\alpha_j)R_M(x, p_j) + 8\pi \sum_{l, l \neq j} (1+\alpha_l) G_M(x, p_l), \tag{1.10}$$

where $R_M(\cdot, p_j)$ is the regular part of $G_M(\cdot, p_j)$,

$$L(\mathbf{p}) = \sum_{j \in I_1} [\Delta \log h(p_j) + \rho_* - N^* - 2K(p_j)] h(p_j)^{\frac{1}{1+\alpha_M}} e^{\frac{G_j^*(p_j)}{1+\alpha_M}}, \tag{1.11}$$

and

$$\begin{cases} \alpha_M = \max_i \alpha_i, \quad I_1 = \{i \in \{1, \cdots, m\}; \quad \alpha_i = \alpha_M\}, \\ \\ \rho_* = 8\pi \sum_{j=1}^m (1+\alpha_j), \quad N^* = 4\pi \sum_{j=1}^m \alpha_j. \end{cases} \tag{1.12}$$

As mentioned above, some blow-up points being non-quantized singular sources while others being regular points, we can assume, without loss of generality, that $p_1, \cdots, p_{m_1}$ are singular sources and $p_{m_1+1}, \cdots, p_m$ are regular points. For $(x_{m_1+1}, \cdots, x_m) \in M \times \cdots \times M$, we define:

$$\begin{aligned} f^*(x_{m_1+1}, \cdots, x_m) = &\sum_{j=m_1+1}^{m} \left[\log h(x_j) + 4\pi R_M(x_j, x_j)\right] + 4\pi \sum_{l \neq j}^{m_1+1,\cdots,m} G_M(x_l, x_j) \\ &+ \sum_{j=m_1+1}^{m} \sum_{i=1}^{m_1} 8\pi(1+\alpha_i) G_M(x_j, p_i). \end{aligned}$$

We remark that for the set of regular blow-up points $(p_{m+1}, \cdots, p_m)$, Chen-Lin in [31] proved that it is a critical point of f^*. In this article, we will examine all possible combinations of blow-up points. Dividing them into two categories:

$$\textbf{Class One:} \quad \alpha_M > 0, \quad \textbf{Class Two:} \quad \alpha_M \leq 0.$$

In *Class One*, the set of blow-up points includes at least one positive singular source, while in *Class Two*, the blow-up set either consists of regular blow-up points and negative singular sources ($\alpha_M = 0$) or contains only negative singular sources ($\alpha_M < 0$). Our first result addresses *Class One*:

Theorem 1.1. *Let v_k be a sequence of bubbling solutions of* (1.1) *and assume that the blow-up set $\{p_1, \cdots, p_m\}$ satisfies $\{p_1, \cdots, p_m\} \cap \{q_1, \cdots, q_N\} \neq \emptyset$. Suppose $(\alpha_1, \cdots, \alpha_N)$ satisfies* (1.9), *$\alpha_M > 0$, $L(\mathbf{p}) \neq 0$ and, as far as $m_1 < m$,*

$$\det\left(D^2 f^*(p_{m_1+1}, \cdots, p_m)\right) \neq 0.$$

Then there exists $n_0 > 1$ such that v_k is non degenerate for all $k \geq n_0$.

Here $D^2 f^*$ denotes the Hessian tensor field on M. At each regular blow-up point p_j for $j = m_1+1, \cdots, m$, using conformal normal coordinates where χ_j is the conformal factor and satisfies the conditions

$$\Delta_g = e^{-\chi_j}\Delta, \quad \chi_j(0) = |\nabla \chi_j(0)| = 0, \quad e^{-\chi_j}\Delta\chi_j = -2K,$$

we can compute $\det(D^2 f^*)$ at these points. Here, h_j is interpreted as $h_j e^{\chi_j}$ in a small neighborhood of each p_j ($j = m_1+1, \cdots, m$). If $m_1 = m$, meaning all blow-up points are singular sources, then as long as $\alpha_M > 0$ we see that the unique relevant assumption is $L(\mathbf{p}) \neq 0$.

To present the result concerning *Class two*, we begin by noting that the set of blow-up points consists only of regular points and negative sources. We introduce some new quantities. Let $B(q, r)$ represent the geodesic ball of radius r centered at $q \in M$, and let $\Omega(q,r)$ denote the pre-image of the Euclidean ball of radius r, $B_r(q) \subset \mathbb{R}^2$, in a suitably defined isothermal coordinate system. If $m \geq 2$, we choose a collection of open, mutually disjoint sets M_j, with the closure of their union covering the entire manifold M.

In this case let us remark that if q_j is a regular blow-up point, then it is automatically a critical point of

$$G_j^*(x) - G_j^*(q_j) + \log h_j(x) - \log h_j(0)$$

and we make this assumption for any blow-up point (remark that if q_s is singular blow-up point, since $\alpha_M \leq 0$, then the corresponding $\alpha_s \in (-1,0)$). So for all blowup points we assume that,

$$\nabla \left(G_j^*(x) + \log h_j(x) \right) |_{x=q_j} = 0. \tag{1.13}$$

Then we define,

$$D(\mathbf{p}) = \lim_{r \to 0} \sum_{j=1}^{m} h_j(0) e^{G_j^*(q_j)} \left(\int_{M_j \setminus \Omega(q_j,r_j)} e^{\Phi_j(x,\mathbf{q})} d\mu(x) - \frac{\pi}{1+\alpha_j} r_j^{-2-2\alpha_j} \right), \tag{1.14}$$

where $M_1 = M$ if $m = 1$, h_j is defined in (1.5),

$$r_j = \begin{cases} r \left(8h_j(0) e^{G_j^*(q_j)} \right)^{1/2}, & m_1 < j \leq m, \\ \\ r, & 1 \leq j \leq m_1, \end{cases}$$

and

$$\begin{aligned} \Phi_j(x,\mathbf{q}) = &\sum_{l=1}^{m} 8\pi(1+\alpha_l) G_M(x,q_l) - G_j^*(q_j) \\ &+ \log h_j(x) - \log h_j(0) + 4\pi\alpha_j (R_M(x,q_j) - G_M(x,q_j)). \end{aligned} \tag{1.15}$$

Remark 1.1. *In local isothermal coordinates centered at q_j, $0 = z(q_j)$, we have $4\pi\alpha_j(R(z,0) - G(z,0)) = 2\alpha_j \log|z|$ and, as far as $\alpha_j \neq 0$, we have $\alpha_j \in (-1,0)$. Although the expansion term is not absolutely integrable, we can handle the leading-order term after integration by using the term $1/r_j^{2+2\alpha_j}$. The remaining non-integrable term in the definition of $D(\mathbf{p})$ cancels out due to* (1.13)*, ensuring that the limit in* (1.14) *is well-defined.*

Our second main result concerns *Class Two*, i.e. $\alpha_M \leq 0$.

Theorem 1.2. *Under the same hypothesis of Theorem 1.1 but assuming $\alpha_M \leq 0$ and (1.13), then in any one of the following situations:*

(1) $\alpha_M = 0$, $L(\mathbf{p}) \neq 0$, $\det\left(D^2 f^*(p_{m_1+1}, \cdots, p_m)\right) \neq 0$,
(2) $\alpha_M = 0$, $L(\mathbf{p}) = 0$, $D(\mathbf{p}) \neq 0$, $\det\left(D^2 f^*(p_{m_1+1}, \cdots, p_m)\right) \neq 0$,
(3) $\alpha_M < 0$, $D(\mathbf{p}) \neq 0$,

the same non degeneracy property as in Theorem 1.1 holds true.

The proof of the above two theorems is highly dependent on the refined estimates recently obtained in [19], which notably allow us for *Class One* problems to avoid the requirement that singular blow-up points must be critical points of specific Kirchoff-Routh type functionals. In the second part of this paper we apply the results about *Class Two* problems to the Onsager mean field vortex model with singular sources. Since long-lived vortex structures are generally expected to concentrate at the critical points of Kirchoff-Routh type functionals ([25],[61]), we will focus on the physically more interesting situation where a negative source (co-rotating vortex) satisfies this property. However we need the counterpart conclusions of the above results (in particular Theorem 1.2) for the Dirichlet problem (1.7) (or (1.2)). To present these results, we first introduce some local and global quantities that are appropriate for describing the combinations of blow-up points. We define the following:

$$G^*_{j,\Omega}(x) = 8\pi(1+\alpha_j)R_\Omega(x,p_j) + 8\pi\sum_{l\neq j}^{1,\cdots,m}(1+\alpha_l)G_\Omega(x,p_l),$$

and, similar to notations for the first part, assume (1.9) for $(\alpha_1,\cdots,\alpha_m)$, where we keep the same conventions about I_1 and α_M. Next let us define,

$$L_\Omega(\mathbf{p}) = \sum_{j\in I_1}\Delta\log h(p_j)h(p_j)^{\frac{1}{1+\alpha_M}}e^{\frac{G^*_{j,\Omega}(p_j)}{1+\alpha_M}},$$

$$\begin{aligned} f^*_\Omega(x_{m_1+1},\cdots,x_m) = &\sum_{j=m_1+1}^{m}\left[\log h(x_j) + 4\pi R(x_j,x_j)\right] + 4\pi\sum_{l\neq j}^{m_1+1,\cdots,m}G(x_l,x_j),\\ &+\sum_{j=m_1+1}^{m}\sum_{i=1}^{m_1}8\pi(1+\alpha_i)G(x_j,p_i),\end{aligned}$$

and let $D^2f^*_\Omega$ be the Hessian on Ω. Of course, in this case $(p_{m_1+1},\cdots,p_m)$ is a critical point of f^*_Ω ([61]). Concerning *Class one* we have,

Theorem 1.3. *Let v_k be a sequence of bubbling solutions of* (1.2) *and assume that the blow-up set $\{p_1,\cdots,p_m\}$ satisfies $\{p_1,\cdots,p_m\}\cap\{q_1,\cdots,q_N\}\neq\emptyset$. Suppose $(\alpha_1,\cdots,\alpha_N)$ satisfies* (1.9)*, $\alpha_M>0$, $L_\Omega(\mathbf{p})\neq 0$ and, as far as $m_1<m$, $\det\left(D^2f^*(p_{m_1+1},\cdots,p_m)\right)\neq 0$. Then there exists $n_0>1$ such that v_k is non degenerate for all $k\geq n_0$.*

Consider the case of *Class two* we set

$$\begin{aligned}\Phi_{j,\Omega}(\mathbf{q}) := &\sum_{l=1}^{m}8\pi(1+\alpha_l)G_\Omega(x,q_l) - G^*_{j,\Omega}(q_j) + \log h_j(x) - \log h_j(q_j)\\ &+2\alpha_j\log|x-q_j|,\end{aligned}$$

and

$$D_\Omega(\mathbf{p}) = \lim_{r\to 0}\sum_{j=1}^{m} h(p_j)e^{G^*_{j,\Omega}(p_j)}\left(\int_{\Omega_j\setminus B_{r_j}(p_j)} e^{\Phi_{j,\Omega}(\mathbf{q})}dx - \frac{\pi}{1+\alpha_j}r_j^{-2\alpha_j-2}\right),$$

where $r_j = r\left(8h(p_j)e^{G^*_{j,\Omega}(p_j)}\right)^{1/2}$ if $j > m_1$ and $r_j = r$ for $1 \le j \le m_1$; $\Omega_1 = \Omega$ if $m = 1$, otherwise we have $\Omega_l \cap \Omega_s = \emptyset$ for $l \ne s$ and $\cup_{j=m_1+1}^m \overline{\Omega}_j = \overline{\Omega}$. The counterpart result of Theorem 1.3 in the case of Class two reads as follows

Similar to the previous case we require

$$\nabla\left(G^*_{j,\Omega}(x) + \log h_j(x)\right)\Big|_{x=q_j} = 0, \quad 1 \le j \le m. \tag{1.16}$$

Then we have,

Theorem 1.4. *Under the same hypothesis of Theorem 1.3 but assuming $\alpha_M \le 0$ and (1.16), then for any one of the following situations:*

(1) $\alpha_M = 0$, $L_\Omega(\mathbf{p}) \ne 0$, $\det\left(D^2 f^*_\Omega\right)(p_{m_1+1}, \cdots, p_m) \ne 0$,
(2) $\alpha_M = 0$, $L_\Omega(\mathbf{p}) = 0$, $D_\Omega(\mathbf{p}) \ne 0$, $\det\left(D^2 f^*_\Omega\right)(p_{m_1+1}, \cdots, p_m) \ne 0$,
(3) $\alpha_M < 0$, $D_\Omega(\mathbf{p}) \ne 0$,

the same non degeneracy property as in Theorem 1.3 holds true.

In the second part of this paper, we apply our non degeneracy results to the analysis of mean field theory within the Onsager ([67] and [24],[25]) statistical mechanics description of the vortex model with singular sources. We consider the case where the mean field vorticity ω interacts with one fixed co-rotating vortex, whose total vorticity (proportional to $|\beta|$) has the same sign as that of ω, alongside N counter-rotating vortices. These singularities act as a fixed external potential in the model (see Remark 5.1 for further details). Compared to earlier findings in [25], [28], [11] and [12], a key nuance is that the critical threshold for the existence of solutions in the canonical mean field model is no longer 8π, but rather $8\pi(1+\beta)$. Physically, this suggests that entropy maximizers (i.e., thermodynamic equilibrium states satisfying the microcanonical variational principle, as discussed in section 5.2) are expected to concentrate at the vortex centered at the β-sink in the large energy limit. Consequently, we need to extend first the results from [25] and [28] regarding the existence of negative temperature thermodynamic equilibrium states and the equivalence of statistical ensembles for a specific class of domains (domains of the first kind) see Definition 6.1 and sections 5 and 6. This extension relies on adaptations of known ([85]) dual variational methods, a sharp singular version ([1]) of the Moser-Trudinger inequality [65], recent findings in [10] and [3], as well as arguments from [24] and [25], which is why we will be rather sketchy about some of these proof.

However, as in [25] and [7], the situation for domains of second kind (again see Definition 6.1) is much more subtle as it requires, among other things, the understanding of the existence/non existence of solutions of the mean field equation (which is (5.4) below) exactly at the critical (with respect to the singular Moser-Trudinger inequality [1]) threshold $8\pi(1+\beta)$ and a full description of the set of solutions in a small enough supercritical region behind $8\pi(1+\beta)$, where uniqueness of solutions fails. It is worth to remark that a full description of the thermodynamic equilibrium states is very involved and still open in general for domains of second kind also in the regular (without β-sources) case, see [3] for partial results concerning this point and the very interesting counterexamples recently found in [21]. Here we are able to generalize various results in [25], [28] and [7], [11], including the existence and asymptotic behavior of entropy maximizers for large energies (see Lemma 6.3) and the characterization of domains of second kind in terms of $D_\Omega(\mathbf{p})$ (see Theorem 6.2) which is in turn crucial to the existence of a full unbounded interval of strict convexity of the Entropy (negative specific heat states, see Theorem 6.5). Few rigorous proof of the existence of negative specific heat states are at hand for these models ([25],[7]) which in our case require the description of the monotonicity of the energy as a function of the inverse temperature in a supercritical (behind $8\pi(1+\beta)$) interval. This is not at all trivial and we succeed here by using the local uniqueness result in [19], Theorem 1.4 above and the local uniqueness and non degeneracy results in [44] about the naturally associated Gel'fand-type problem. These results are neither enough on their own which is why we purse a generalization (for β-sources) of independent interest of former estimates obtained in the regular case in [31], see Theorem 2.2, Proposition 2.1 and (2.27) below.

The proof of Theorem 6.2 requires careful consideration, especially in cases where $D_\Omega(\mathbf{p})$ might vanish. Theorem 6.2 itself is of independent interest as it generalizes results previously established for semilinear elliptic equations with critical nonlinearity in three dimensions ([42]) and for mean field type equations when $\beta = 0$ ([28], [11]). Additionally, we derive an intriguing sufficient condition for the non-existence of solutions at the critical parameter, which extends a similar result found in [11], as detailed in Corollary 6.2.

At last, interestingly enough, as an application of these refined analysis, we come up with the exact counting of the number of solutions for the mean field equation (5.4) in a small supercritical region behind $8\pi(1+\beta)$, see Theorem 6.6.

Throughout the article, $B_\tau = B_\tau(0)$ will always represent a ball centered at the origin in some local isothermal coordinates $y \in B_\tau$. Whenever B_τ refers to such a ball centered at $0 = y(p)$, $p \in M$, we will denote by $\Omega(p,\tau) \subset M$ the pre-image of B_τ. On the other hand, $B(p,\tau) \subset M$ will always refer to a geodesic ball. Additionally, many estimates will involve a small positive number $\epsilon_0 > 0$ and generic constant C, which may vary from line to line.

Note that while the letter ρ is commonly used in physics to denote density, we will use ω to represent the density starting from section 5, where we discuss the statistical mechanics of the Onsager model with singular sources. This choice is made because ρ is already used as a parameter in equations (1.3) and (1.7).

This paper is organized as follows: Section 2 presents preliminary results essential for proving non degeneracy, including a precise estimate for the difference between the parameter ρ_k and the critical value ρ_*, as well as a uniqueness lemma. Sections 3 and 4 revisit the asymptotic analysis from [19] and provide a proof of the non degeneracy result. The final two sections apply this non degeneracy result to demonstrate the strict convexity of the entropy in the large energy limit of the Onsager mean field vortex model with sinks.

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2. Preliminary Estimates

Since the proof of the main theorems requires delicate analysis, in this section we list some estimates established in [5, 16, 17, 18, 31, 34, 54, 89, 90].

Let w_k be a sequence of solutions of (1.3) with $\rho = \rho_k$ and assume that w_k blows up at m points $\{p_1, \cdots, p_m\}$. Since (1.3) is invariant after adding a constant to the solution. Without loss of generality, we may assume that

$$\int_M He^{\mathrm{w}_k} \mathrm{d}\mu = 1. \tag{2.1}$$

Then we can rewrite the equation for w_k as,

$$\Delta_g \mathrm{w}_k + \rho_k (He^{\mathrm{w}_k} - 1) = 0 \quad \text{in } \ M. \tag{2.2}$$

From well known results about Liouville-type equations ([18, 54]),

$$\mathrm{w}_k - \overline{\mathrm{w}_k} \to \sum_{j=1}^{m} 8\pi(1+\alpha_j) G(x, p_j) \quad \text{in} \quad \mathrm{C}^2_{\mathrm{loc}}(M \backslash \{p_1, \cdots, p_m\}),$$

where $\overline{w_k}$ is the average of w_k on M: $\overline{w_k} = \int_M w_k d\mu$. For later convenience we fix $r_0 > 0$ small enough and $M_j \subset M, 1 \leq j \leq m$ such that

$$M = \bigcup_{j=1}^{m} \overline{M}_j; \quad M_j \cap M_l = \varnothing, \text{ if } j \neq l; \quad B(p_j, 3r_0) \subset M_j, \quad j = 1, \cdots, m.$$

According to this definition $M_1 = M$, if $m = 1$.

The notation about local maximum is of particular relevance in this context. If p_j is a regular blow-up point (i.e. $\alpha_j = 0$) we define $p_{k,j}$ and $\lambda_{k,j}$ as follows,

$$\lambda_{k,j} = u_k(p_{k,j}) := \max_{B(p_j, r_0)} u_k,$$

while if p_j is a singular blow-up point (i.e. $\alpha_j \neq 0$), then we define,

$$p_{k,j} := p_j \quad \text{ and } \quad \lambda_{k,j} := u_k(p_{k,j}).$$

Next, let us define the so called "standard bubble" $U_{k,j}$ to be the solution of

$$\Delta U_{k,j} + \rho_k h_j(p_{k,j}) |x - p_{k,j}|^{2\alpha_j} e^{U_{k,j}} = 0 \quad \text{in } \mathbb{R}^2$$

which takes the form ([29, 30, 69]),

$$U_{k,j}(x) = \lambda_{k,j} - 2 \log \Big(1 + \frac{\rho_k h_j(p_{k,j})}{8(1+\alpha_j)^2} e^{\lambda_{k,j}} |x - p_{k,j}|^{2(1+\alpha_j)} \Big).$$

It is well-known ([5, 18, 54]) that u_k can be approximated by the standard bubbles $U_{k,j}$ near p_j up to a uniformly bounded error term:

$$\big| u_k(x) - U_{k,j}(x) \big| \leq C, \quad x \in B_r(p_j, r_0).$$

As a consequence, in particular we have,

$$|\lambda_{k,i} - \lambda_{k,j}| \leq C, \quad 1 \leq i, j \leq m,$$

for some C independent of k.

In case $m_1 < m$, it has been shown in [31] that,

$$\nabla (\log h + G_j^*)(p_{k,j}) = O(\lambda_{k,j} e^{-\lambda_{k,j}}), \quad m_1 + 1 \leq j \leq m, \tag{2.3}$$

which, in view of the non degeneracy condition

$$\det \big(D^2 f^*(p_{m_1+1}, \cdots, p_m) \big) \neq 0,$$

readily implies that,

$$|p_{k,j} - p_j| = O(\lambda_{k,j} e^{-\lambda_{k,j}}), \quad m_1 + 1 \leq j \leq m. \tag{2.4}$$

Later, sharper estimates were obtained in [34, 90] for $1 \leq j \leq m_1$ and in [31, 47, 89] for $m_1 + 1 \leq j \leq m$.

By using λ_i^k to denote the maximum of u_i^k, $i = 1, 2$ which share the same value of the parameter ρ_k, then it is a simple consequence of $L(\mathbf{p}) \neq 0$ (see [8, 87]) that

$$|\lambda_1^k - \lambda_2^k| \leq C e^{-\epsilon_0 \lambda_1^k} \quad \text{for some } \epsilon_0 > 0. \tag{2.5}$$

Let us also recall that it has been established in [16, 17] that (see (1.12)) $\rho_* = \lim_{k\to+\infty} \rho_k$. Concerning the difference between ρ_k and ρ_*, we set

$$\rho_{k,j} = \rho_k \int_{\Omega(p_j,\tau)} He^{\mathrm{w}_k}.$$

The following estimates hold (see [31], [34],[19]):

Theorem 2.1. *There exists $\epsilon_0 > 0$ and $d_j > 0$ such that,*

$$\begin{cases} \rho_{k,j} - 8\pi(1+\alpha_j) = 2\pi d_j e^{-\frac{\lambda_{k,j}}{1+\alpha_j}} + O(e^{-\frac{1+\epsilon_0}{1+\alpha_j}\lambda_{k,j}}), & \alpha_j > 0, \\ \rho_{k,j} - 8\pi(1+\alpha_j) = O(e^{-\lambda_{k,j}}) & \alpha_j < 0, \\ \rho_{k,j} - 8\pi = O\big(\lambda_{k,j} e^{-\lambda_{k,j}}\big), & \tau+1 \le j \le m. \end{cases}$$

It has been proved in [34] that

$$\rho_k - \rho_* = L(\mathbf{p}) e^{-\frac{\lambda_1^k}{1+\alpha_M}} + O(e^{-\frac{1+\epsilon_0}{1+\alpha_M}\lambda_{k,j}}) \tag{2.6}$$

for some $\epsilon_0 > 0$ and $L(\mathbf{p})$ as defined in (1.11).

For later study, we need a general theorem describing $\rho_k - \rho_*$ when the set of blow-up points is a mixture of regular points and singular sources with negative strength. Here we let $p_1, \cdots, p_{m_1}$ be singular sources with negative strength: $\alpha_l \in (-1, 0)$ $(l = 1, \cdots, m_1)$ and let $\alpha_l = 0$ for $l = m_1 + 1, \cdots, m$. That is, $p^k_{m_1+1}, \cdots, p^k_m$ are regular blow-up points. If $m > m_1$, we recall that

$$L(\mathbf{p}) = \sum_{j=m_1+1}^{m} [\Delta \log h(p_j) + \rho_* - N^* - 2K(p_j)] h(p_j) e^{G_j^*(p_j)},$$

where

$$\rho_* = 8\pi \sum_{j=1}^{m_1} (1+\alpha_j) + 8\pi(m - m_1), \quad N^* = \sum_{j=1}^{m_1} 4\pi\alpha_j.$$

Now we define two terms which involves global integration. Let

$$D_R := \lim_{r\to 0} \sum_{j=m_1+1}^{m} h_j(0) e^{G_j^*(q_j)} \bigg(\int_{M_j \setminus \Omega(q_j, r_k)} e^{\Phi_j(x,q)} d\mu - \pi r_j^{-2} \bigg),$$

where

$$r_j = r\left(8h_j(0)e^{G_j^*(q_j)}\right)^{\frac{1}{2}}, \quad j = m_1+1, \cdots, m,$$

and

$$D_s := \lim_{r\to 0} \sum_{j=1}^{m_1} h_j(0) e^{G_j^*(p_j)} \left(\int_{M_j \setminus \Omega(q_j, r)} e^{\Phi_j(x,q)} d\mu - \frac{\pi}{1+\alpha_j} r^{-2-2\alpha_j} \right).$$

Theorem 2.2. *Suppose $\alpha_l \in (-1,0)$ for $l = 1,\cdots,m_1$, $\alpha_l = 0$ for $l = m_1 + 1,\cdots,m$ and that* (1.13) *holds true. Then we have,*

$$\begin{aligned}\rho_k - \rho_* = &\frac{16\pi e^{-\lambda_m^k}}{\rho_* h_m^2(0)e^{G_m^*(p_m)}} L(\mathbf{p})[\lambda_m^k + \log(h_m^2(0)e^{G_m^*(p_m)}r^2\rho_*) - 2]\\ &+ \frac{64e^{-\lambda_m^k}}{\rho_* h_m^2(0)e^{G_m^*(p_m)}}(D_R + O(r^{\epsilon_0}))\\ &+ e^{-\lambda_1^k}\rho_* e^{-G_1^*(p_1)}\left(\frac{8(1+\alpha_1)^2}{\rho_* h_1(0)}\right)^2 (D_s + O(r^{\epsilon_0})) + O(e^{-(1+\epsilon_0)\lambda_1^k}),\end{aligned}$$

where ϵ_0 is a positive small number. If both regular and singular blow-up points exist, then λ_1^k and λ_m^k satisfy,

$$-\lambda_m^k = -\lambda_1^k + 2\log\frac{h_m(0)(1+\alpha_1)^2}{h_1(0)} + G_m^*(p_m) - G_1^*(p_1) + O(e^{-\epsilon_0\lambda_1^k}).$$

Remark 2.1. *The sum of the last two terms in the expansion of $\rho_k - \rho_*$ can be written either as follows,*

$$e^{-\lambda_1^k}\rho_* e^{-G_1^*(p_1)}\left(\frac{8(1+\alpha_1)^2}{\rho_* h_1(0)}\right)^2 (D(\mathbf{p}) + O(r^{\epsilon_0})),$$

whenever there exists at least one singular source or as follows,

$$\frac{64e^{-\lambda_m^k}}{\rho_* h_m^2(0)e^{G_m^*(p_m)}}(D(\mathbf{p}) + O(r^{\epsilon_0})),$$

if at least one regular blow-up point exists.

Proof of Theorem 2.2. Here for convenience we define

$$\Phi(x) = \sum_{j=1}^m 8\pi(1+\alpha_j)G(x,p_j), \tag{2.7}$$

and recall the equation for w_k in (2.2) and the normalization (2.1). We carry out the analysis around p_l. Let f_k be defined in a neighborhood of p_l by

$$\Delta_g f_k = \rho_k \text{ in } B(p_l,\tau), \quad f_k(p_l) = 0, \quad f_k = \text{constant on } \partial B(p_l,\tau). \tag{2.8}$$

Then we set

$$u_k = \mathrm{w}_k - f_k$$

and have

$$\Delta_g u_k + \rho_k H e^{f_k} e^{u_k} = 0, \quad \text{in} \quad B(p_l,\tau).$$

Using the conformal factor function χ, which in local coordinates centered at p_l satisfies,

$$\chi(0) = |\nabla\chi(0)| = 0, \quad \Delta_g = e^{-\chi}\Delta, \quad \Delta\chi(0) = -2K(p_l), \tag{2.9}$$

we have

$$\Delta u_k + \rho_k H e^{\chi + f_k} e^{u_k} = 0 \quad \text{in} \quad B_\tau.^{1} \tag{2.10}$$

To avoid cumbersome notations, for the time being we use the same notation for p_l in local and global coordinates. Recall the definition (1.5) of h_l, then in local coordinates around p_l we have, if $1 \le l \le m_1$,

$$h_l(x) = h(p_l + x) \exp\left(\sum_{j=1, j\neq l}^{m_1} (-4\pi\alpha_j) G(x, p_j) - 4\pi\alpha_l R(x, p_l) \right), \tag{2.11}$$

and

$$h_l(0) = h(p_l) \exp\left(\sum_{j=1, j\neq l}^{m_1} (-4\pi\alpha_j G(p_l, p_j)) - 4\pi\alpha_l R(p_l, p_l) \right),$$

while if $l > m_1$ we have,

$$h_l(x) = h(p_l + x) \exp\left(\sum_{j=1}^{m_1} (-4\pi\alpha_j G(x, p_j) \right), \quad m_1 < l \le m, \tag{2.12}$$

and

$$h_l(0) = h(p_l) \exp\left(\sum_{j=1}^{m_1} (-4\pi\alpha_j G(p_l, p_j)) \right), \quad m_1 < l \le m.$$

Here we also note that around p_l for $l > m_1$, by (1.12),

$$\Delta(\log H + \chi + f_k)(0) = \Delta \log h(p_l) + \rho_k - N^* - 2K(p_l). \tag{2.13}$$

Let ψ_k be the harmonic function that eliminates the oscillation of u_k on ∂B_τ, that is $\Delta\psi_k = 0$ in B_τ and

$$\psi_k = u_k - \frac{1}{2\pi\tau} \int_{\partial B_\tau} u_k dS \quad \text{on } \partial B_\tau.$$

Then, in local coordinates centered at p_l, a rough estimate for u_k in B_τ is,

$$u_k(x) = U_k + \psi_k + O(\epsilon_{l,k}^\delta), \quad x \in B_\tau$$

where

$$U_k(x) = \lambda_l^k - 2\log\left(1 + \frac{\rho_k h_l(0)}{8(1+\alpha_l)^2} e^{\lambda_l^k} |x|^{2+2\alpha_l} \right),$$

$\epsilon_{l,k} = e^{-\frac{\lambda_l^k}{2(1+\alpha_l)}}$. In particular for $x \in B_\tau \setminus B_{\tau/2}$, we have

$$\begin{aligned} u_k(x) = &- \lambda_l^k - 2\log \frac{\rho_k h_l(0)}{8(1+\alpha_l)^2} - 4(1+\alpha_l)\log|x| \\ &+ \psi_k(x) + O(\epsilon_{l,k}^\delta), \end{aligned} \tag{2.14}$$

[1]Here we use the same notation for u_k and f_k in both $B(p_l, \tau)$ (the geodesic ball on the Riemann manifold) and $B_\tau(0)$ (in the local isothermal coordinates).

for some $\delta > 0$. On the other hand, the Green representation of w_k gives

$$\mathrm{w}_k(x) = \overline{\mathrm{w}_k} + \Phi(x) + O(\epsilon_{l,k}^{\delta}), \tag{2.15}$$

for x away from singular sources, where Φ is defined in (2.7). By using the notation $G_l^*(x)$ in (1.10), this expression can be written as follows,

$$\mathrm{w}_k(x) = \overline{\mathrm{w}_k} - 4(1+\alpha_l)\log|x - p_l| + G_l^*(x) + O(\epsilon_{l,k}^{\delta}).$$

By the definition of f_k, in local coordinates around p_l we have,

$$\begin{aligned} u_k(x) = \overline{\mathrm{w}_k} - 4(1+\alpha_l)\log|x| + G_l^*(p_l) \\ + G_l^*(x) - G_l^*(p_l) - f_k + O(\epsilon_{l,k}^{\delta}). \end{aligned} \tag{2.16}$$

Comparing (2.16) and (2.14), by the definition of G_l^*, and recalling that $\psi_k(0) = 0$, we have that,

$$\psi_k(x) = G_l^*(x) - G_l^*(p_l) - f_k + O(\epsilon_{l,k}^{\delta})$$

and

$$\overline{\mathrm{w}_k} = -\lambda_l^k - 2\log\frac{\rho_k h_l(0)}{8(1+\alpha_l)^2} - G_l^*(p_l) + O(e^{-\epsilon_0\lambda_1^k}), \quad l = 1, \cdots, m. \tag{2.17}$$

In particular

$$\overline{\mathrm{w}_k} = -\lambda_1^k - 2\log\frac{\rho_k h_1(0)}{8(1+\alpha_1)^2} - G_1^*(p_1) + O(e^{-\epsilon_0\lambda_1^k}). \tag{2.18}$$

By using (2.18) and (2.17), for $l = 1, \cdots, m$, we have that

$$-\lambda_l^k = -\lambda_1^k + 2\log\frac{\rho_k h_l(0)}{8(1+\alpha_l)^2} - 2\log\frac{\rho_k h_1(0)}{8(1+\alpha_1)^2} + G_l^*(p_l) - G_1^*(p_1) + O(e^{-\epsilon_0\lambda_1^k}),$$

and consequently,

$$e^{-\lambda_l^k} = e^{-\lambda_1^k}\frac{(1+\alpha_1)^4 h_l^2(0)}{(1+\alpha_l)^4 h_1^2(0)} e^{G_l^*(p_l) - G_1^*(p_1)} + O(e^{-(1+\epsilon_0)\lambda_1^k}),\ l = 1, \cdots, m, \tag{2.19}$$

for some $\epsilon_0 > 0$. Remark that the estimate (2.19) holds for regular points as well. Next we evaluate $\rho_k = \int_M \rho_k H e^{\mathrm{w}_k} d\mu$ as follows,

$$\rho_k = \sum_{l=1}^{m} \int_{B(p_l,\tau_l)} \rho_k H e^{\mathrm{w}_k} d\mu + \int_E \rho_k H e^{\mathrm{w}_k} d\mu = \sum_{l=1}^{m} \rho_{k,l} + \int_E \rho_k H e^{\mathrm{w}_k} d\mu,$$

where $E = M \setminus (\bigcup_l B(p_l, \tau_l))$. First we evaluate $\rho_{k,j}$ for $j = 1, \cdots, m_1$. For each j let Ω_j be a neighborhood of p_j such that the Ω_j are mutually disjoint and their union is M. In each $B(p_l, \tau_l)$, $l = 1, \cdots, m_1$, we use Theorem 3.2 of [19] as follows,

$$\begin{aligned} \int_{B(p_l,\tau)} \rho_k H e^{\mathrm{w}_k} d\mu &= \int_{B_\tau} \rho_k |x|^{2\alpha_l} h_l e^{f_k + \chi} e^{u_k} d\mu \\ &= 8\pi(1+\alpha_l)\left(1 - \frac{8(1+\alpha_l)^2}{\rho_k h_l(0)}\tau^{-2-2\alpha_l} e^{-\lambda_l^k}\right) + l.o.t. \end{aligned} \tag{2.20}$$

Remark that the lower order terms in the r.h.s. of (2.20) include factors of order $e^{-\lambda_l^k}$ which are $o(1)$ in τ, as $\tau \to 0$. On the other side, by (2.15) and (2.17) we have that,

$$\begin{aligned}\int_{\Omega_l\setminus B(p_l,\tau)} \rho_k H e^{\mathrm{w}_k} d\mu &= \int_{\Omega_l\setminus B(p_l,\tau)} \rho_k H e^{\overline{\mathrm{w}_k}} e^{\Phi} d\mu + O(e^{-(1+\epsilon_0)\lambda_1^k}) \\ &= e^{-\lambda_l^k}\left(\frac{8(1+\alpha_l)^2}{\rho_k h_l(0)}\right)^2 e^{-G_l^*(p_l)} \int_{\Omega_l\setminus B(p_l,\tau)} \rho_k H e^{\Phi} d\mu \\ &\quad + O(e^{-(1+\epsilon_0)\lambda_1^k}).\end{aligned}$$

By using local coordinates around p_l to evaluate the last integral, we see that,

$$\begin{aligned}\int_{\Omega_l\setminus B(p_l,\tau)} \rho_k H e^{\Phi} d\mu &= \int_{\Omega_l\setminus B(p_l,\delta)} \rho_k H e^{\Phi} d\mu + \int_{B(p_l,\delta)\setminus B(p_l,\tau)} \rho_k h_l |x|^{-4-2\alpha_l} e^{G_l^*(x)} dx \\ &= \int_{\Omega_l\setminus B(p_l,\delta)} \rho_k H e^{\Phi} d\mu + \rho_k h_l(0) e^{G_l^*(p_l)}\left(\frac{\pi\tau^{-2-2\alpha_l}}{1+\alpha_l} + O(\delta^{-2-2\alpha_l})\right),\end{aligned}$$

where, in view of (1.13), the remaining non integrable term in the expansion vanishes, while the rest can be safely included in the $O(\delta^{-2-2\alpha_l})$. Therefore we have,

$$\int_{\Omega_l} \rho_k H e^{\mathrm{w}_k} d\mu = 8\pi(1+\alpha_l) + e^{-\lambda_l^k}\rho^* e^{-G_l^*(p_l)}\left(\frac{8(1+\alpha_l)^2}{\rho_* h_l(0)}\right)^2 \left(D_l + O(\delta^{-2\alpha_l})\right)$$

where we used once more (1.13) and

$$D_l = \lim_{\tau\to 0+} \int_{\Omega_l\setminus B(p_l,\tau)} H e^{\Phi} d\mu - \frac{h_l(0) e^{G_l^*(p_l)}\pi}{1+\alpha_l}\tau^{-2-2\alpha_l}, \quad l = 1, \cdots, m_1,$$

and we can replace λ_l^k by λ_1^k to deduce that,

$$\int_{\Omega_l} \rho_k H e^{\mathrm{w}_k} d\mu = 8\pi(1+\alpha_l) + e^{-\lambda_1^k}\rho^* e^{-G_1^*(p_1)}\left(\frac{8(1+\alpha_1)^2}{\rho_* h_1(0)}\right)^2 \left(D_l + O(\delta^{-2\alpha_l})\right).$$

At this point, for $l = m_1+1, \cdots, m$, we set $\tau_l = \sqrt{8e^{G_l^*(p_l)} h_l(0)}\tau$, and invoke Theorem 3.3 and Remark 3.2 of [19] as follows,

$$\begin{aligned}\int_{B(p_l,\tau_l)} \rho_k H e^{\mathrm{w}_k} d\mu &= \int_{B_{\tau_l}} \rho_k h_l e^{f_k+\chi} e^{u_k} dx \\ &= 8\pi - 8\pi\frac{e^{-\lambda_l^k}}{e^{-\lambda_l^k} + a_k\tau_l^2} + O(e^{-(2-\epsilon_0)\lambda_l^k}) \\ &\quad - \frac{\pi e^{-\lambda_l^k}}{2}(\Delta(\log h_l)(0) + \rho_k - 2K(p_l))\rho_k h_l(0) b_{0,k},\end{aligned}$$

where $\epsilon_0 > 0$ is a small constant and

$$b_{0,k} = \int_0^{\tau_l \bar{\epsilon}_k^{-1}} \frac{r^3(1-a_k r^2)}{(1+a_k r^2)^3} dr, \quad a_k = \frac{\rho_k h_l(0)}{8}.$$

Elementary arguments show that,

$$b_{0,k} = \frac{1}{2a_k^2}(-\lambda_l^k - \log(a_k \tau_l^2) + 2) + O(e^{-\lambda_l^k}).$$

For $m_1 + 1 \leq l \leq m$, using this expression of $b_{0,k}$ we have,

$$\begin{aligned}
&\int_{B(p_l,\tau_l)} \rho_k H e^{\mathrm{w}_k} d\mu \\
&= 8\pi - \frac{16\pi}{\rho_k h_l(0)}(\Delta \log h_l(0) + \rho_k - 2K(p_l))e^{-\lambda_l^k} \\
&\quad \times \left(-\lambda_l^k - \log \frac{\rho_k h_l(0)\tau_l^2}{8} + 2\right) - \frac{64\pi}{\rho_k h_l(0)} \tau_l^{-2} e^{-\lambda_l^k} + O(e^{-(2-\epsilon_0)\lambda_l^k}) \\
&= 8\pi - \frac{16\pi}{\rho_k h_m^2(0) e^{G_m^*(p_m)}} e^{-\lambda_m^k} h_l(0) e^{G_l^*(p_l)} (\Delta \log h_l(0) + \rho_k - 2K(p_l)) \\
&\quad \times \left(-\lambda_m^k - 2\log h_m(0) - G_m^*(p_m) + \log \frac{8h_l(0)e^{G_l^*(p_l)}}{\rho_k \tau_l^2} + 2\right) \\
&\quad - \frac{64\pi}{\rho_k h_m^2(0) e^{G_m^*(p_m)}} e^{-\lambda_m^k} h_l(0) e^{G_l^*(p_l)} \tau_l^{-2} + O(e^{-(2-\epsilon_0)\lambda_m^k}).
\end{aligned} \tag{2.21}$$

It is well known that ([31]) that the non degeneracy assumption $\det(D^2 f^*) \neq 0$ implies (2.4), that is,

$$|p_l^k - p_l| = O(\lambda_l^k e^{-\lambda_l^k})$$

so p_l^k can be replaced by p_l. Next we evaluate $\int_{\Omega_l \setminus B(p_l,\tau_l)} \rho_k H e^{\mathrm{w}_k} d\mu$ as follows,

$$\begin{aligned}
\int_{\Omega_l \setminus B(p_l,\tau_l)} \rho_k H e^{\mathrm{w}_k} d\mu &= \int_{\Omega_l \setminus B(p_l,\tau_l)} \rho_k h_l e^{\overline{\mathrm{w}_k} + \Phi} d\mu \\
&= \rho_k \int_{\Omega_l \setminus B(p_l,\tau_l)} h_l e^{-\lambda_l^k} \frac{64}{(\rho_k h_l(0))^2} e^{-G_l^*(p_l) + \Phi} d\mu \\
&= \frac{64 e^{-\lambda_l^k}}{\rho_k h_l(0)} \int_{\Omega_l \setminus B(p_l,\tau_l)} e^{\Phi_l(x,p_l)} d\mu,
\end{aligned} \tag{2.22}$$

where $\Phi_l(x, p_l)$ has been defined in (1.15). Concerning the last integral we have,

$$\begin{aligned}
&\int_{\Omega_l \setminus B(p_l,\tau_l)} e^{\Phi_l(x,p_l)} d\mu \\
&= \int_{\Omega_l \setminus B(p_l,\delta)} e^{\Phi_l(x,p_l)} d\mu + \int_{B(p_l,\delta) \setminus B(p_l,\tau_l)} e^{\Phi_l(x,p_l)} d\mu
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega_l \setminus B(p_l,\delta)} e^{\Phi_l(x,p_l)} d\mu + \int_{B(p_l,\delta)\setminus B(p_l,\tau_l)} \frac{h_l}{h_l(0)} |x|^{-4} e^{G_l^*(x)-G_l^*(p_l)} dx \\
&= \int_{\Omega_l \setminus B(p_l,\delta)} e^{\Phi_l(x,p_l)} d\mu + \pi(\tau_l^{-2} - \delta^{-2}) \\
&\quad + \frac{\pi}{2}(\Delta \log h_l(0) + \rho_k - 2K(p_l)) \log \frac{\delta}{\tau_l} + O(\tau_l).
\end{aligned}
$$

Then we see that

$$
\begin{aligned}
\int_{\Omega_l} \rho_k H e^{\mathrm{w}_k} d\mu = 8\pi &- \frac{16\pi}{\rho_k h_m^2(0) e^{G_m^*(p_m)}} e^{-\lambda_m^k} h_l(0) e^{G_l^*(p_l)} (\Delta \log h_l(0) + \rho_k - 2K(p_l)) \\
&\times \left(-\lambda_m^k - 2\log h_m(0) - G_m^*(p_m) + \log\left(\frac{\tau_l^{-2}}{\rho_k}\right) + 2 \right) \\
&+ \frac{64 e^{-\lambda_m^k} h_l(0) e^{G_l^*(p_l)}}{\rho_k h_m^2(0) e^{G_m^*(p_m)}} \left(\int_{\Omega_l \setminus B(p_l,\tau_l)} e^{\Phi_l(x,p_l)} d\mu - \pi \tau_l^{-2} + O(\tau_l) \right).
\end{aligned}
$$

Putting the estimates on regular points and singular sources together, we have

$$
\begin{aligned}
&\rho_k - \rho_* \\
&= \frac{16\pi}{\rho_* h_m^2(0) e^{G_m^*(p_m)}} e^{-\lambda_m^k} L(\mathbf{p}) \cdot (\lambda_m^k + 2\log h_m(0) + G_m^*(p_m) + \log(\tau^2 \rho_k) - 2) \\
&\quad + \frac{64 e^{-\lambda_m^k}}{\rho_* h_m^2(0) e^{G_*(p_m)}} \sum_{l=m_1+1}^{m} h_l(0) e^{G_l^*(p_l)} \left(\int_{\Omega_l \setminus B(p_l,\tau_l)} e^{\Phi_l(x,p_l)} d\mu - \frac{\pi}{\tau_l^2} + O(\tau_l) \right) \\
&\quad + 64 e^{-\lambda_1^k} \rho^* e^{-G_1^*(p_1)} \frac{(1+\alpha_1)^4}{\rho_*^2 h_1^2(0)} \left(\sum_{s=1}^{m_1} \left(\int_{\Omega_s \setminus B(p_s,\tau_s)} H e^{\Phi} d\mu - \frac{\pi h_s(0) e^{G_s^*(p_s)}}{(1+\alpha_s)\tau_s^{2+2\alpha_s}} \right) \right. \\
&\quad \left. + O(\tau^\sigma) e^{-\lambda_1^k} \right),
\end{aligned}
$$

where $\sigma > 0$ is a small positive number. By using the relation between λ_1^k and λ_m^k, we can rewrite the above as follows,

$$
\begin{aligned}
&\rho_k - \rho_* \\
&= \frac{16\pi}{\rho_* h_m^2(0) e^{G_m^*(p_m)}} e^{-\lambda_m^k} L(\mathbf{p}) \cdot (\lambda_m^k + \log(\rho_k h_m^2(0) e^{G_m^*(p_m)} \tau^2) - 2) \\
&\quad + \frac{64 e^{-\lambda_m^k}}{\rho_* h_m^2(0) e^{G_*(p_m)}} \sum_{l=1}^{m} h_l(0) e^{G_l^*(p_l)} \left(\int_{\Omega_l \setminus B(p_l,\tau_l)} e^{\Phi_l(x,p_l)} d\mu - \frac{\pi}{(1+\alpha_l)\tau_l^{2+2\alpha_l}} \right) \\
&\quad + o(\tau^\sigma) e^{-\lambda_m^k} + O(e^{-(1+\epsilon_0)\lambda_1^k}).
\end{aligned}
$$

Thus, Theorem 2.2 is established. □

2.1. **An estimate about a Dirichlet problem.** In this subsection we estimate an integral in the following Dirichlet problem:

$$\begin{cases} \Delta \mathrm{w}_k+\rho_k\frac{h(x)e^{\mathrm{w}_k(x)}}{\int_\Omega h(x)e^{\mathrm{w}_k(x)}}=0, & \text{in} \quad \Omega, \\ \\ \mathrm{w}_k(x)=0, & \text{on} \quad \partial\Omega. \end{cases} \tag{2.23}$$

where Ω is a bounded smooth domain in $\mathbb{R}^2$ and $h(x)$ is a positive C^5 function in Ω, ρ_k tends to a constant ρ and $\max\limits_{\overline{\Omega}} \mathrm{w}_k(x)\to\infty$. Set

$$I_{\mathrm{w}_k}=\log\int_{\Omega_k}h(x)e^{\mathrm{w}_k(x)}dx,$$

and we assume that $h(x)\geq 0$ may be zero at some blow-up points. Suppose $p_1,\cdots,p_m$ are blow-up points, $p_1,\cdots,p_{m_1}$ are singular sources and around p_l $(l=1,\cdots,m_1)$, $h(x)=|x-p_l|^{2\alpha_l}h_l$, while the remaining blow-up points, $p_{m_1+1},\cdots,p_m$, are regular blow-up points in Ω. So if we use α_l to denote the strength of singularity at each blow-up point, we have

$$\alpha_l\neq 0,\quad 1\leq l\leq m_1,\quad \alpha_l=0,\quad m_1+1\leq l\leq m.$$

We set

$$v_k=\mathrm{w}_k-I_{\mathrm{w}_k},$$

and λ_l^k to be the maximum of v_k around p_l, and α_M to be the largest index. Then we have

Proposition 2.1.

$$I_{\mathrm{w}_k}=\lambda_l^k+2\log\frac{\rho_k h_l(0)}{8(1+\alpha_l)^2}+G_l^*(p_l)+O(e^{-\epsilon_0\lambda_1^k}),\quad l=1,\cdots,m$$

for some small $\epsilon_0>0$.

Proof. Clearly $v_k=w_k-I_{\mathrm{w}_k}$ satisfies,

$$\Delta v_k+\rho_k h(x)e^{v_k}=0,\quad \text{in} \quad \Omega,$$

and $v_k=-I_{\mathrm{w}_k}$ on $\partial\Omega_k$. By the Green representation formula for v_k, for x away from the singular sources we have,

$$v_k(x)=\sum_{j=1}^m 8\pi(1+\alpha_j)G_\Omega(x,p_j)-I_{\mathrm{w}_k}+O(e^{-\epsilon_0\lambda_1^k}),\quad x\in\Omega\setminus\bigcup_{j=1}^m B_\epsilon(p_j)$$

for some small $\epsilon>0$. Around each p_j, the harmonic function that eliminates the oscillation of v_k on $\partial B_\epsilon(p_j)$ takes the form,

$$\phi_j^k(x)=G_j^*(x)-G_j^*(p_j)+O(e^{-\epsilon_0\lambda_1^k}).$$

On the other hand the expansion around p_j reads as follows,

$$v_k(x) = \log \frac{e^{\lambda_j^k}}{\left(1 + e^{\lambda_j^k} \frac{\rho_k h_j(0)}{8(1+\alpha_j)^2} |x - p_j|^{2\alpha_j + 2}\right)^2} + \phi_j^k(x) + \Pi_k$$

where (see either section 3 below or [47, 89, 19]),

$$\Pi_k = \begin{cases} O(e^{-\lambda_l^k/(1+\alpha_M)}), & \text{if} \quad \alpha_M > 0, \\ O(e^{-\lambda_l^k} \lambda_k^2), & \text{if} \quad \alpha_M = 0, \\ O(e^{-\lambda_l^k}), & \text{if} \quad \alpha_M < 0. \end{cases}$$

The comparison between the two expressions shows that,

$$I_{\mathrm{w}_k} = \lambda_l^k + 2\log \frac{\rho_k h_l(0)}{8(1+\alpha_l)^2} + G_l^*(p_l) + \Pi_k + O(e^{-\epsilon_0 \lambda_1^k}), \quad l = 1, \cdots, m. \tag{2.24}$$

Proposition 2.1 follows from (2.24). □

As particularly relevant case for applications to be discussed in sections 5 and 6 is when we have just one blow-up point placed on a negative source p with strength $\beta \in (-1, 0)$. Around the blow-up point p we write $h(x) = |x - p|^{2\beta} \bar{h}(x)$. We shall need an accurate estimate about $\rho_k - 8\pi(1+\beta)$ under the assumption (1.16). Let

$$c_k = I_{\mathrm{w}_k},$$

we recall that, for x far away from p, we have that,

$$v_k(x) = \int_\Omega G(x, \eta) \rho_k h e^{v_k} d\eta - c_k,$$

and a rough evaluation shows that,

$$v_k(x) = 8\pi(1+\beta) G_\Omega(x, p) - c_k + O(e^{-\lambda_k}), \quad x \in \Omega \setminus B_\tau(p) \tag{2.25}$$

On the other hand, since $\int_\Omega h e^{v_k} = 1$ and setting

$$G^*(x) = 8\pi(1+\beta) R_\Omega(x, p),$$

we can use Theorem 3.2 of [19] as follows,

$$\begin{aligned} \rho_k &= \int_{B_\tau(p)} \rho_k |x-p|^{2\beta} \bar{h}(x) e^{v_k} dx + \int_{\Omega \setminus B_\tau(p)} \rho_k |x-p|^{2\beta} \bar{h}(x) e^{v_k} \\ &= 8\pi(1+\beta) - \frac{64\pi(1+\beta)^3}{\rho_k \bar{h}(0)} \tau^{-2-2\beta} e^{-\lambda_k} \\ &\quad + e^{-c_k} \int_{\Omega \setminus B_\tau(p)} |x-p|^{-4-2\beta} e^{G^*(x)} \rho_k \bar{h}(x) dx + o_\tau(1)(e^{-\lambda_k}) \qquad (2.26) \\ &= 8\pi(1+\beta) - \frac{\pi \rho_k \bar{h}(0)}{1+\beta} e^{G^*(p)} \tau^{-2-2\beta} e^{-c_k} \\ &\quad + e^{-c_k} \int_{\Omega \setminus B_\tau(p)} |x-p|^{-4-2\beta} e^{G^*(x)} \rho_k \bar{h}(x) dx + o_\tau(1) e^{-\lambda_k}, \end{aligned}$$

where we have used (2.24)-(2.25) and $o_\tau(1)$ is a uniformly bounded quantity such that $o_\tau(1) \to 0$ as $\tau \to 0$. Thus, we get

$$\rho_k - 8\pi(1+\beta) = e^{-c_k}\rho_k(D_\beta + o_\tau(1)), \tag{2.27}$$

where D_β is defined as follows,

$$D_\beta := \lim_{\tau\to 0^+} \bar{h}(0)e^{G^*(p)}\left(\int_{\Omega\setminus B_\tau(p)} e^{\Phi_{\Omega,p}}dx - \frac{\pi}{1+\beta}\tau^{-2-2\beta}\right) \tag{2.28}$$

and

$$\Phi_{\Omega,p}(x) = 8\pi(1+\beta)G_\Omega(x,p) - G^*(p) + \log\frac{\bar{h}(x)}{\bar{h}(0)} + 2\beta\log|x-p|. \tag{2.29}$$

Remark that the limit defining D_β is well defined due to (1.16), see also Lemma 5.4 in section 5.

2.2. **A uniqueness lemma.** The proof of the following Lemma can be found in [34, 87], for $\alpha > 0$ and in [19] for $\alpha < 0$.

Lemma 2.1. *Let $\alpha > -1$, $\alpha \notin \mathbb{N}\cup\{0\}$ and ϕ be a C^2 solution of*

$$\begin{cases} \Delta\phi + 8(1+\alpha)^2|x|^{2\alpha}e^{U_\alpha}\phi = 0 \ \text{ in } \ \mathbb{R}^2, \\ \\ |\phi(x)| \le C(1+|x|)^\tau, \quad x \in \mathbb{R}^2, \end{cases}$$

where

$$U_\alpha(x) = \log\frac{1}{(1+|x|^{2\alpha+2})^2} \quad \textit{and} \quad \tau \in [0,1).$$

Then there exists some constant b_0 such that

$$\phi(x) = b_0\frac{1-|x|^{2(1+\alpha)}}{1+|x|^{2(1+\alpha)}}.$$

The following lemma has been proved in [31].

Lemma 2.2. *Let φ be a C^2 solution of*

$$\begin{cases} \Delta\varphi + e^U\varphi = 0 & \text{in } \ \mathbb{R}^2, \\ \\ |\varphi| \le c\big(1+|x|\big)^\kappa & \text{in } \ \mathbb{R}^2, \end{cases}$$

where $U(x) = \log\frac{8}{(1+|x|^2)^2}$ and $\kappa \in [0,1)$. Then there exist constants b_0, b_1, b_2 such that

$$\varphi = b_0\varphi_0 + b_1\varphi_1 + b_2\varphi_2,$$

where

$$\varphi_0(x) = \frac{1-|x|^2}{1+|x|^2}, \quad \varphi_1(x) = \frac{x_1}{1+|x|^2}, \quad \varphi_2(x) = \frac{x_2}{1+|x|^2}.$$

3. Some results in [19] about the local asymptotic analysis near a blow-up point

In this section we summarize the asymptotic expansions of a sequence of blow-up solutions near a blow-up point as recently derived in [19]. We assume that u_k denotes a sequence of blow-up solutions of

$$\Delta u_k + |x|^{2\alpha} h_k(x) e^{u_k} = 0 \quad \text{in} \quad B_{2\tau},$$

where $B_\tau \subset \mathbb{R}^2$ for $\tau > 0$ is the ball centered at the origin with radius $\tau > 0$,

$$\frac{1}{C} \le h_k(x) \le C, \quad \|D^m h_k\|_{L^\infty(B_{2\tau})} \le C \quad \forall x \in B_{2\tau}, \quad \forall m \le 5, \tag{3.1}$$

and

$$\int_{B_{2\tau}} |x|^{2\alpha} h_k(x) e^{u_k} < C, \tag{3.2}$$

for some uniform $C > 0$. The sequence $\{u_k\}$ has its only blow-up point at the origin:

$$\text{there exists} \quad z_k \to 0, \text{ such that} \quad \lim_{k\to\infty} u_k(z_k) \to \infty \tag{3.3}$$

and for any fixed $K \Subset B_{2\tau} \setminus \{0\}$, there exists $C(K) > 0$ such that

$$u_k(x) \le C(K), \quad x \in K. \tag{3.4}$$

It is also standard to assume that u_k has bounded oscillation on ∂B_τ

$$|u_k(x) - u_k(y)| \le C, \quad \forall x, y \in \partial B_\tau, \tag{3.5}$$

for some uniform $C > 0$. As before, we set ψ_k to be the harmonic function which encodes the boundary oscillation of u_k:

$$\begin{cases} \Delta \psi_k = 0 & \text{in} \quad B_\tau, \\ \\ \psi_k = u_k - \frac{1}{2\pi\tau} \int_{\partial B_\tau} u_k dS & \text{on} \quad \partial B_\tau. \end{cases} \tag{3.6}$$

Since u_k has a bounded oscillation on ∂B_τ (see (3.5)), $u_k - \psi_k$ is constant and uniformly bounded on ∂B_1 and $\|D^m \psi_k\|_{B_{\tau/2}} \le C(m)$ for any $m \in \mathbb{N}$. The mean value property of harmonic functions also gives $\psi_k(0) = 0$. For the sake of simplicity let us define,

$$V_k(x) = h_k(x) e^{\psi_k(x)}, \quad \varepsilon_k = e^{-\frac{u_k(0)}{2(1+\alpha)}}$$

and

$$v_k(y) = u_k(\varepsilon_k y) + 2(1+\alpha)\log \varepsilon_k - \psi_k(\varepsilon_k y), \quad y \in \Omega_k := B_{\tau/\varepsilon_k}. \tag{3.7}$$

Obviously we have $V_k(0) = h_k(0)$, $\Delta \log V_k(0) = \Delta \log h_k(0)$. Let

$$U_k(y) = -2\log\left(1 + \frac{h_k(0)}{8(\alpha+1)^2}|y|^{2\alpha+2}\right), \tag{3.8}$$

be the standard bubble which satisfies

$$\Delta U_k + |y|^{2\alpha} h_k(0) e^{U_k(y)} = 0 \quad \text{in} \quad \mathbb{R}^2.$$

In the remaining part of this section we split the results into three subsections according to the sign of α ($\alpha > 0$, $\alpha < 0$ and $\alpha = 0$).

3.1. **Asymptotic analysis for $\alpha > 0$.** We say that a function is separable if it is the product of a polynomial of two variables $P(x_1, x_2)$ times a radial function with certain decay at infinity, where P is the sum of monomials $x_1^n x_2^m$ where n, m are non-negative integers and at least one among them is odd. Let us define:

$$c_{1,k}(y) = \varepsilon_k g_k(r) \nabla(\log V_k)(0) \cdot \theta, \quad \theta = (\theta_1, \theta_2),\ \theta_j = y_j/r,\ j = 1,2, \tag{3.9}$$

where

$$g_k(r) = -\frac{2(1+\alpha)}{\alpha} \frac{r}{1 + \frac{V_k(0)}{8(1+\alpha)^2} r^{2\alpha+2}}, \tag{3.10}$$

$c_{0,k}$ to be the unique solution of

$$\begin{cases} \frac{d^2}{dr^2} c_{0,k} + \frac{1}{r}\frac{d}{dr} c_{0,k} + V_k(0) r^{2\alpha} e^{U_k} c_{0,k} \\ = -\frac{\varepsilon_k^2}{4} V_k(0) r^{2\alpha} e^{U_k} \Big((g_k + r)^2 |\nabla \log V_k(0)|^2 + \Delta \log V_k(0) r^2 \Big), \\ \\ c_{0,k}(0) = c_{0,k}'(0) = 0, \quad 0 < r < \tau/\varepsilon_k, \end{cases}$$

$c_{2,k}$ a separable function which satisfies,

$$\Delta c_{2,k} + V_k(0) r^{2\alpha} e^{U_k} c_{2,k} = -\varepsilon_k^2 r^{2\alpha} e^{U_k} \hat{\Theta}_2,$$

where

$$\begin{aligned} \varepsilon_k^2 \hat{\Theta}_2 &= \varepsilon_k^2 (E_{2,1}(r) \cos 2\theta + E_{2,2}(r) \sin 2\theta) \\ &= \varepsilon_k^2 \Bigg(\bigg(\frac{1}{4} \left(\partial_{11}(\log V_k)(0) - \partial_{22}(\log V_k)(0) \right) r^2 \\ &\quad + \frac{1}{4} \left((\partial_1 \log V_k(0))^2 - (\partial_2 \log V_k(0))^2 \right) (g_k + r)^2 \bigg) \cos 2\theta \\ &\quad + \left(\partial_{12}(\log V_k)(0) \frac{r^2}{2} + \frac{\partial_1 V_k(0)}{V_k(0)} \frac{\partial_2 V_k(0)}{V_k(0)} \frac{(g_k + r)^2}{2} \right) \sin 2\theta \Bigg). \end{aligned}$$

and

$$c_{2,k} = \varepsilon_k^2 f_{2a}(r) \cos(2\theta) + \varepsilon_k^2 f_{2b}(r) \sin(2\theta), \tag{3.11}$$

where

$$|f_{2a}(r)| + |f_{2b}(r)| = O(1+r)^{-2\alpha}, \quad 0 < r < \varepsilon_k^{-1}. \tag{3.12}$$

In the statement below $\epsilon_0 > 0$ is a suitable positive number allowed to change even line to line.

Theorem 3.1. *([19])*

$$\int_{B_\tau} h_k(x)|x|^{2\alpha}e^{u_k} = 8\pi(1+\alpha)\left(1-\frac{e^{-u_k(0)}}{e^{-u_k(0)}+\frac{h_k(0)}{8(1+\alpha)^2}\tau^{2\alpha+2}}\right)$$
$$+ d_{1,k}\Delta\log h_k(0)\varepsilon_k^2 + T_1(V_k)\varepsilon_k^4 + O(\varepsilon_k^{4+\epsilon_0}),$$

for some $\epsilon_0>0$, where,

$$d_{1,k} = -\frac{2\pi^2}{(1+\alpha)\sin\frac{\pi}{1+\alpha}}\left(\frac{8(1+\alpha)^2}{V_k(0)}\right)^{\frac{1}{1+\alpha}}$$
$$+\left(\frac{16\pi(1+\alpha)^2}{\alpha}\frac{|\nabla\log V_k(0)|^2}{V_k(0)\Delta\log V_k(0)}+\frac{16\pi(1+\alpha)^4}{\alpha V_k(0)}\right)\tau^{-2\alpha}\varepsilon_k^{2\alpha},$$

and $T_1(V_k)$ is a bounded function of $D^\beta V_k(0)$ for $|\beta|=0,1,2,3,4$.
Moreover for $|y|\le\tau/\varepsilon_k$, we have,

$$v_k(y) = U_k(y)+c_{1,k}(y)+O(\varepsilon_k^2(1+|y|)^{\epsilon_0}),$$
$$v_k(y) = U_k(y)+c_{1,k}(y)+c_{0,k}(y)+c_{2,k}(y)+O(\varepsilon_k^{2+\epsilon_0}(1+|y|)^{\epsilon_0}),$$
$$v_k(y) = U_k(y)+c_{1,k}(y)+c_{0,k}(y)+c_{2,k}(y)+c_{*,k}(y)+O(\varepsilon_k^{4+\epsilon_0}(1+|y|)^{\epsilon_0}),$$

for some $\epsilon_0>0$.

We refer to [19] for the definition of $c_{*,k}$.

3.2. **Asymptotic analysis around a negative pole.** In this case we use $\beta\in(-1,0)$ to denote the strength of a negative pole and define,

$$\tilde\varepsilon_k = e^{-\frac{\lambda^k}{2(1+\beta)}}.$$

We use the same notations to approximate v_k, where $c_{1,k}$ is defined as in (3.9) with α replaced by β in the definition of g_k in (3.10). In the statement below $\delta>0$ is a suitable small positive number allowed to change even line to line.

Theorem 3.2. ([19])

$$\int_{B_\tau} V_k(x)|x|^{2\beta}e^{u_k} = 8\pi(1+\beta)\left(1-\frac{e^{-u_k(0)}}{e^{-u_k(0)}+\frac{h_k(0)}{8(1+\beta)^2}\tau^{2\beta+2}}\right)$$
$$+ b_k(\tau,\beta)+\ell_k(\tau,\beta)+O(e^{(-2-\epsilon_0)u_k(0)})$$

for some $\epsilon_0>0$, where both $b_k(\tau,\beta)$ and $\ell_k(\tau,\beta)$ depend on $|\nabla\log V_k(0)|$ and $\Delta\log V_k(0)$, $b_k(\tau,\beta)\sim e^{-u_k(0)}$, $\ell_k(\tau,\beta)\sim e^{-2u_k(0)}$ for $\tau>0$, and

$$\lim_{\tau\to0}\frac{b_k(\tau,\beta)}{e^{-u_k(0)}}=0,\quad \lim_{\tau\to0}\frac{\ell_k(\tau,\beta)}{e^{-2u_k(0)}}=0.$$

Moreover for $|y|\le\tau/\varepsilon_k$, we have,

$$v_k(y) = U_k(y)+c_{1,k}(y)+O(\tilde\varepsilon_k^{2+2\beta-\delta}(1+|y|)^{\delta}),$$

$$v_k(y) = U_k(y) + c_{1,k}(y) + c_{0,k}(y) + c_{2,k}(y) + c_{*,k}(y) + O(\tilde{\varepsilon}_k^{4+4\beta-\delta}(1+|y|)^{\delta}),$$

for some small $\delta > 0$.

3.3. **Asymptotic analysis around a regular blow-up point.** In this subsection we analyze the solution near a regular blow-up point and consider the equation,

$$\Delta u_k + \bar{h}_k(x)e^{u_k} = 0 \quad \text{in} \quad B_{2\tau}.$$

We use $\bar{\lambda}_k = \max_{B_{2\tau}} u_k$ to denote the height of the bubble and set $\bar{\varepsilon}_k = e^{-\bar{\lambda}_k/2}$. We assume in particular (3.3), (3.4), (i.e. 0 is the only blow-up point in $B_{2\tau}$), that the standard uniform bound $\int_{B_{2\tau}} \bar{h}_k e^{u_k} \leq C$ holds and finally that (3.5) holds, implying by well known results ([54]) that the blow-up point is simple. We assume that (3.1), (3.3), (3.4), (3.5) and $\int_{B_{2\tau}} \bar{h}_k(x)e^{u_k} \leq C$ are satisfied and define ψ_k as in (3.6). Let $V_k(x) = h_k(x)e^{\psi_k(x)}$ and we have the following estimates, see [19, Theorem 3.3 and Remarks 3.2, 3.5]

Theorem 3.3.
(3.13)

$$\int_{B_\tau} \bar{h}_k(x)e^{u_k}dx = 8\pi - \frac{8\pi\bar{\varepsilon}_k^2}{\bar{\varepsilon}_k^2 + a_k\tau^2} - \frac{\pi\varepsilon_k^2}{2}\Delta(\log\bar{h}_k)(q_k)\bar{h}_k(q_k)b_{0,k} + O(\bar{\varepsilon}_k^{4-2\epsilon_0}).$$

for some $\epsilon_0 > 0$, where

$$b_{0,k} = \int_0^{\tau/\bar{\varepsilon}_k} \frac{r^3(1-a_kr^2)}{(1+a_kr^2)^3}dr, \quad a_k = \bar{h}_k(q_k)/8.$$

Moreover we have,

$$|\bar{v}_k(y) - (U_k + c_{0,k} + c_{2,k})(y)| \leq C\bar{\varepsilon}_k^{2+\epsilon_0}(1+|y|)^{2\epsilon_0} \quad \text{in} \quad \Omega_k,$$

and

$$|\nabla(\bar{v}_k - U_k - c_{0,k} - c_{2,k})(y)| \leq C\tau\bar{\varepsilon}_k^3, \quad y \in \Omega_k, \quad |y| \sim \bar{\varepsilon}_k^{-1}.$$

4. Proof of the non degeneracy result

In this section, we prove Theorems 1.1-1.4. Since the proof of these theorems are almost the same, we shall only focus on Theorem 1.1. To simplify our discussion, we assume there are only three blow-up points: q, p_1, p_2, where q is a regular blow-up point,p_1 is a positive singular source with $\alpha \equiv \alpha_M > 0$, p_2 is a negative singular source with $\beta \in (-1,0)$. Let v_k be a blow-up sequence of (1.1),

$$\mathrm{w}_k = v_k + 4\pi\alpha G(x,p_1) + 4\pi\beta G(x,p_2),$$

which satisfies

$$\Delta_g \mathrm{w}_k + \rho_k\left(\frac{He^{\mathrm{w}_k}}{\int_M He^{\mathrm{w}_k}} - 1\right) = 0,$$

where $\alpha = \alpha_M > 0$ and

$$H(x) = h(x)e^{-4\pi\alpha G(x,p_1) - 4\pi\beta G(x,p_2)}.$$

Since by adding any constant to w_k in (4) we still come up with a solution, without loss of generality we may assume that w_k satisfy

$$\int_M He^{\mathrm{w}_k} dV_g = 1.$$

We argue by contradiction and assume that ϕ^k is a non trivial sequence of solutions to

$$\Delta_g \phi^k + \rho He^{\mathrm{w}_k}\left(\phi^k - \int_M He^{\mathrm{w}_k}\phi^k\right) = 0 \quad \text{in } M.$$

Let us set $\lambda^k = \mathrm{w}_k(p_1)$, $\hat{\lambda}^k = \mathrm{w}_k(p_2)$ and $\bar{\lambda}^k = \max\limits_{B(q,\tau)} \mathrm{w}_k$, then it is well known that

$$\hat{\lambda}^k - \lambda^k = O(1), \quad \bar{\lambda}^k - \lambda^k = O(1). \tag{4.1}$$

We work in local isothermal coordinates around p_1, p_2, q, where the metric locally takes the form,

$$ds^2 = e^{\chi}((dx_1)^2 + (dx_2)^2),$$

for some χ that satisfies,

$$\Delta\chi + 2Ke^{\chi} = 0 \quad \text{in} \quad B_\tau, \quad \chi(0) = |\nabla\chi(0)| = 0. \tag{4.2}$$

Then we define f_k to be any solution of,

$$\Delta f_k = e^{\chi}\rho_k \quad \text{in} \quad B_\tau, \qquad f_k(0) = 0. \tag{4.3}$$

In particular we choose local coordinates so that p_i, q are locally the origin of a ball $x \in B_\tau = B_\tau(0)$. Remark that in our setting we have,

$$\begin{cases} \rho_* = 24\pi + 8\pi\alpha + 8\pi\beta, \quad N^* = 4\pi(\alpha+\beta), \\ \\ G_1^*(x) = 8\pi(1+\alpha)R(x,p_1) + 8\pi(1+\beta)G(x,p_2) + 8\pi G(x,q). \end{cases} \tag{4.4}$$

Remark 4.1. *Throughout this section, $B_\tau = B_\tau(0)$ (or $B_\tau(\bar{q})$) will represent the ball centered at the origin (or $\bar{q}$) of some local isothermal coordinates $x \in B_\tau$, while $\Omega(p,\tau) \subset M$ denotes a geodesic ball. Additionally, if B_τ is the ball in local isothermal coordinates centered at some point $0 = x(p), p \in M$, we will denote by $\Omega(p,\tau) \subset M$ the pre-image of B_τ. We can always choose τ small enough to ensure that $\Omega(p_i,\tau)$, $i = 1,2$ and $\Omega(q,\tau)$ are simply connected and at positive distance one from each other. After scaling, $x = \varepsilon y$ for some $\varepsilon > 0$, we will denote $B_{\tau/\varepsilon} = B_{\tau/\varepsilon}(0)$. By a slightly abuse of notations, we will use the same symbols, say w_k, ξ_k, to denote functions when expressed in different local coordinate systems. However, the context will clarify the meaning of the symbols as needed.*

Thus, working in these local coordinates centered at p_1, p_2, q respectively, we have the local variables $x \in B_\tau$ and we define

$$\tilde{h}_k = \rho_k h(x) e^{-4\pi\alpha R(x,p_1) - 4\pi\beta G(x,p_2) + \chi(x) + f_k}, \tag{4.5}$$

$$\hat{h}_k = \rho_k h(x) e^{-4\pi\alpha G(x,p_1) - 4\pi\beta R(x,p_2) + \chi(x) + f_k}, \tag{4.6}$$

$$\bar{h}_k = \rho_k h(x) e^{-4\pi\alpha G(x,p_1) - 4\pi\beta G(x,p_2) + \chi(x) + f_k}. \tag{4.7}$$

Obviously $\tilde{h}_k, \hat{h}_k, \bar{h}_k$ satisfy (3.1) and clearly (2.6) holds as well. Indeed from $\rho^k = \rho^k \int_M He^{u_k}$ we first evaluate the integrals around blow-up points and then check that the integrals outside the bubbling disks are of order $O(e^{-\lambda^k})$.

Remark 4.2. *Concerning* (4.7)*, for technical reasons, it will be useful to define* $\bar{q}^k \to 0$ *to be the maximum points of* u_k *in* B_τ *and work in the local coordinate system centered at* $\bar{q}^k$*. In particular* $q^k \in M$ *will denote the pre-images of these points via the local isothermal map and we will work sometime, possibly taking a smaller* τ*, with* (4.7) *where* $0 = x(q^k)$*.*

By using also (4.2) we see that the assumption $L(\mathbf{p}) \neq 0$ takes the form

$$\Delta_g \log h(p_1) + \rho_* - N^* - 2K(p_1) \neq 0,$$

which, for any k large enough, is equivalent to (see (4.4))

$$\Delta \log \tilde{h}_k(0) \neq 0.$$

At this point let us define,

$$< \phi^k >_k = \int_M \frac{He^{\mathrm{w}_k}\phi^k}{\int_M He^{\mathrm{w}_k} \mathrm{d}\mu}, \quad \sigma_k = \|\phi^k - < \phi^k >_k \|_{L^\infty(M)},$$

and

$$\xi_k := (\phi^k - < \phi^k >_k)/\sigma_k.$$

Then in local coordinates around p_1, so that $0 = x(p_1)$, $x \in B_\tau = B_\tau(0)$, ξ_k satisfies,

$$\Delta \xi_k(x) + |x|^{2\alpha} \tilde{h}_k(x) \tilde{c}_k(x) \xi_k(x) = 0 \quad \text{in} \quad B_\tau,$$

where $\tilde{h}_k$ is defined in (4.5) and $\tilde{c}_k(x)$ denotes the local coordinates expression of $e^{u_k(x)}$, that is,

$$\tilde{c}_k(x) = e^{u_k(x)}.$$

In local coordinates around p_2, so that $0 = x(p_2)$, $x \in B_\tau = B_\tau(0)$, ξ_k satisfies,

$$\Delta \xi_k(x) + |x|^{2\beta} \hat{h}_k(x) \hat{c}_k(x) \xi_k(x) = 0 \quad \text{in} \quad B_\tau,$$

where $\hat{h}_k$ is defined in (4.6) and $\hat{c}_k(x)$ denotes the local coordinates expression of $e^{u_k(x)}$, that is,

$$\hat{c}_k(x) = e^{u_k(x)}.$$

In local coordinates around q, so that $0 = x(q)$, $x \in B_\tau = B_\tau(0)$, ξ_k satisfies,

$$\Delta \xi_k(x) + \bar{h}_k(x) \bar{c}_k(x) \xi_k(x) = 0 \quad \text{in} \quad B_\tau,$$

where $\bar{h}_k$ is defined in (4.6) and $\bar{c}_k(x)$ denotes the local coordinates expression of $e^{u_k(x)}$, that is,

$$\bar{c}_k(x) = e^{u_k(x)}.$$

As in [19], after a suitable scaling in local coordinates, we see that the limits of ξ_k takes the form

$$b_0\frac{1-A|y|^{2\alpha+2}}{1+A|y|^{2\alpha+2}} \text{ near } p_1, \quad A=\lim_{k\to\infty}\frac{\tilde{h}_k(0)}{8(1+\alpha)^2}; \tag{4.8}$$

$$b_0\frac{1-B|y|^{2\beta+2}}{1+B|y|^{2\beta+2}} \text{ near } p_2, \quad B=\lim_{k\to\infty}\frac{\hat{h}_k(0)}{8(1+\beta)^2}; \tag{4.9}$$

$$b_0\frac{1-C|y|^2}{1+C|y|^2}+b_1\frac{y_1}{1+C|y|^2}+b_2\frac{y_2}{1+C|y|^2} \text{ near } q, \quad C=\lim_{k\to\infty}\frac{\bar{h}_k(0)}{8}. \tag{4.10}$$

In the following two subsections, we shall show that all these coefficients are zero.

4.1. **Proof of $b_0 = 0$.** First of all, locally around p_1 the rescaled function $\xi_k(\varepsilon_k y)$ converges on compact subsets of $\mathbb{R}^2$ to a solution of the following linearized equation

$$\Delta\xi+|y|^{2\alpha}\tilde{h}(0)e^U\xi=0, \quad \tilde{h}(0)=\lim_{k\to\infty}\tilde{h}_k(0),$$

where U is the limit of the standard bubble U_k, then we have that

$$\xi(y)=b_0\frac{1-c|y|^{2\alpha+2}}{1+c|y|^{2\alpha+2}}, \quad c=\frac{\tilde{h}(0)}{8(1+\alpha)^2}. \tag{4.11}$$

As a first approximation, we choose

$$w_{0,\xi}^k=b_0^k\frac{1-a_k|y|^{2\alpha+2}}{1+a_k|y|^{2\alpha+2}}, \quad a_k=\tilde{h}_k(0)/8(1+\alpha)^2. \tag{4.12}$$

It is well known that $w_{0,\xi}^k$ is a solution of

$$\Delta w_{0,\xi}^k+|y|^{2\alpha}\tilde{h}_k(0)e^{U_k}w_{0,\xi}^k=0, \tag{4.13}$$

where b_0^k is chosen such that $w_{0,\xi}^k(0)=\xi_k(0)$. Let ψ_ξ^k be the harmonic function which encodes the oscillation of ξ_k on ∂B_τ,

$$\Delta\psi_\xi^k=0 \text{ in } B_\tau, \quad \psi_\xi^k(x)=\xi_k(x)-\frac{1}{2\pi\tau}\int_{\partial B_\tau}\xi_k(s)ds \text{ for } x\in\partial B_\tau,$$

then we see that $\xi_k-\psi_\xi^k$ satisfies,

$$\Delta(\xi_k-\psi_\xi^k)+|x|^{2\alpha}\tilde{h}_k\tilde{c}_k(\xi_k-\psi_\xi^k)=-|x|^{2\alpha}\tilde{h}_k\tilde{c}_k\psi_\xi^k. \tag{4.14}$$

In order to show that the leading term of $(\xi_k-\psi_\xi^k)(\varepsilon_k y)$ is $w_{0,\xi}^k$, we write (4.13) as follows,

$$\begin{aligned}&\Delta w_{0,\xi}^k+|y|^{2\alpha}\tilde{h}_k(\varepsilon_k y)\varepsilon_k^{2+2\alpha}\tilde{c}_k(\varepsilon_k y)w_{0,\xi}^k\\ &=|y|^{2\alpha}\tilde{h}_k(0)\left(\frac{\tilde{h}_k(\varepsilon_k y)}{\tilde{h}_k(0)}\varepsilon_k^{2+2\alpha}\tilde{c}_k(\varepsilon_k y)-e^{U_k}\right)w_{0,\xi}^k(y).\end{aligned} \tag{4.15}$$

Next, we set

$$\tilde{w}_k(y) := \xi_k(\varepsilon_k y) - \psi^k_\xi(\varepsilon_k y) - w^k_{0,\xi}(y). \tag{4.16}$$

From (4.14) and (4.15) we have

$$\begin{aligned} &\Delta\tilde{w}_k + |y|^{2\alpha}\tilde{h}_k(\varepsilon_k y)\varepsilon_k^{2+2\alpha}\tilde{c}_k(\varepsilon_k y)\tilde{w}_k \\ &= \tilde{h}_k(0)|y|^{2\alpha}\left(e^{U_k} - \frac{\tilde{h}_k(\varepsilon_k y)}{\tilde{h}_k(0)}\varepsilon_k^{2+2\alpha}\tilde{c}_k(\varepsilon_k y)\right)w^k_{0,\xi}(y) \\ &\quad - |y|^{2\alpha}\tilde{h}_k(\varepsilon_k y)\varepsilon_k^{2+2\alpha}\tilde{c}_k(\varepsilon_k y)\psi^k_\xi(\varepsilon_k y). \end{aligned} \tag{4.17}$$

By the construction of $\tilde{w}_k$, it is readily seen that,

$$\tilde{w}_k(0) = 0, \quad \tilde{w}_k = \text{constant} \quad \text{on} \quad \partial B_{\tau/\varepsilon_k}.$$

Remark that the last term of (4.17) is of order $O(\varepsilon_k)(1+r)^{-3-2\alpha}$ whence we obtain a first estimate about $\tilde{w}_k$:

$$|\tilde{w}_k(y)| \le C(\delta)\varepsilon_k(1+|y|)^\delta \quad \text{in} \quad B_{\tau/\varepsilon_k}. \tag{4.18}$$

The analysis around p_2 is similar, we define (recall that now we have local coordinates where $0 = x(p_2)$),

$$\hat{w}_k(y) = \xi_k(\hat{\varepsilon}_k y) - \hat{\psi}^k_\xi(\hat{\varepsilon}_k y) - \hat{w}^k_{0,\xi}(y), \tag{4.19}$$

where $\hat{\varepsilon}_k = e^{-\frac{\hat{\lambda}_k}{2(1+\beta)}}$, $\hat{\psi}^k_\xi$ is the harmonic function that encodes the oscillation of ξ_k on ∂B_τ, $\hat{w}^k_{0,\xi}$ is the same as $w^k_{0,\xi}$ just with β replacing α and the local limit of $\xi_k(\hat{\varepsilon}_k y)$ is now (4.9) which replaces (4.11). By using the same argument adopted above for $\tilde{w}_k$, we have that,

$$|\hat{w}_k(y)| \le C(\delta)\hat{\varepsilon}_k^{1+\beta}(1+|y|)^\delta = C(\delta)e^{-\lambda^k/2}(1+|y|)^\delta \quad \text{in} \quad B_{\tau/\varepsilon_k}. \tag{4.20}$$

Next, we describe the expansion of ξ_k near q. In this case (4.11) is replaced by (4.10). As before, we denote $\bar{q}^k$ the maximum points of u^k in B_τ and work in the local coordinate system centered at $\bar{q}^k$. In particular $q^k \in M$ will denote the pre-image of these points via the local isothermal map and we will work with the equation with $0 = x(q^k)$. As usual the kernel functions are the first terms in the approximation of $\xi_k(\bar{q}^k + \bar{\varepsilon}_k y)$,

$$\bar{w}^k_{0,\xi}(y) = \bar{b}^k_0\frac{1 - \frac{\bar{h}^k_2(0)}{8}|y|^2}{1 + \frac{\bar{h}_k(0)}{8}|y|^2} + \bar{b}^k_1\frac{y_1}{1 + \frac{\bar{h}_k(0)}{8}|y|^2} + \bar{b}^k_2\frac{y_2}{1 + \frac{\bar{h}_k(0)}{8}|y|^2}.$$

Then we have

$$\xi_k(\bar{q}^k) - \bar{w}^k_{0,\xi}(0) = 0 \quad \text{and} \quad \nabla(\xi_k(\bar{q}^k + \bar{\varepsilon}_k y) - \bar{w}^k_{0,\xi}(\bar{\varepsilon}_k y))|_{y=0} = 0.$$

Next we set $\bar{\psi}^k_\xi$ to be the harmonic function which encodes the oscillation of $\xi_k(\bar{q}^k + \bar{\varepsilon}_k y) - \bar{w}^k_{0,\xi}(\bar{\varepsilon}_k y)$ on $\partial B_{\tau/\bar{\varepsilon}_k}$. Let us define,

$$\bar{w}_k(y) = \xi_k(\bar{q}^k + \bar{\varepsilon}_k y) - \bar{w}^k_{0,\xi}(y) - \bar{\psi}^k_\xi(\bar{\varepsilon}_k y), \tag{4.21}$$

then we have,

$$\begin{cases} \Delta \bar{w}_k(y) + \bar{\varepsilon}_k^2(\bar{h}_k \bar{c}_k)(\bar{q}^k + \bar{\varepsilon}_k y)\bar{w}_k(y) = \frac{O(\bar{\varepsilon}_k)}{(1+|y|)^3} \quad \text{in} \quad B_{\tau/\bar{\varepsilon}_k}, \\ \\ \bar{w}_k(0) = 0 \quad \text{and} \quad \bar{w}_k = \text{constant on } \partial B_{\tau/\bar{\varepsilon}_k}. \end{cases} \tag{4.22}$$

By using the fact that $\bar{\psi}_\xi^k$ is a harmonic, it is not difficult to check that $\bar{w}_k(0) = 0$ and $\nabla \bar{w}_k(0) = -\nabla \bar{\psi}_\xi^k(0) = O(\bar{\varepsilon}_k)$. Then, using the Green representation formula and standard potential estimates, we have that,

$$|\bar{w}_k(y)| \leq C(\delta)\bar{\varepsilon}_k(1+|y|)^\delta \quad \text{in} \quad B_{\tau/\bar{\varepsilon}_k}. \tag{4.23}$$

Next we present the following improved estimate on the oscillation of ξ_k

Lemma 4.1.

$$\xi_k(x_1) - \xi_k(x_2) = O(\varepsilon_k^2 + e^{-\lambda_1^k/2}) = O(\varepsilon_k^2 + \varepsilon_k^{1+\alpha}), \tag{4.24}$$

$\forall x_1, x_2 \in M \setminus \{\Omega(p_1,\tau) \cup \Omega(p_2,\tau) \cup \Omega(q,\tau)\}$.

Proof. We shall divide our proof into two steps, in the first step we derive the following estimate

$$\xi_k(x_1) - \xi_k(x_2) = O(\varepsilon_k), \tag{4.25}$$

for any $x_1, x_2 \in M \setminus \{\Omega(p_1,\tau) \cup \Omega(p_2,\tau) \cup \Omega(q,\tau)\}$. Then in the second step we shall improve (4.25) and obtain (4.24).

Step 1. Using the Green representation formula for ξ_k, we have, for $x \in M \setminus \{\Omega(p_1,\tau) \cup \Omega(p_2,\tau) \cup \Omega(q,\tau)\}$,

$$\begin{aligned} \xi_k(x) &= \xi_{k,a} + \int_M G(x,\eta)\rho_k H(\eta) e^{\mathrm{w}_k(\eta)} \xi_k(\eta) d\eta \\ &= \xi_{k,a} + \sum_{i=1}^2 \int_{\Omega(p_i,\tau/2)} G(x,\eta)\rho_k H(\eta) e^{\mathrm{w}_k(\eta)} \xi_k(\eta) d\eta \\ &\quad + \int_{\Omega(q,\tau/2)} G(x,\eta)\rho_k H(\eta) e^{\mathrm{w}_k(\eta)} \xi_k(\eta) d\eta \\ &\quad + \int_{M \setminus \{\Omega(p_1,\tau/2) \cup \Omega(p_2,\tau/2) \cup \Omega(q,\tau/2)\}} G(x,\eta)\rho_k H(\eta) e^{\mathrm{w}_k(\eta)} \xi_k(\eta) d\eta \\ &= \xi_{k,a} + I_1 + I_2 + I_3 + I_4, \end{aligned}$$

with obvious meaning and

$$\xi_{k,a} = \int_M \left(\int_M G(x,\eta)\rho_k H(\eta) e^{\mathrm{w}_k(\eta)} \xi_k(\eta) d\eta \right) H(x) e^{\mathrm{w}(x)} dx.$$

Clearly I_4 is of order $O(e^{-\lambda^k})$. For the other terms we write,

$$\begin{aligned} I_1 &= G(x,p_1) \int_{\Omega(p_1,\tau/2)} \rho_k H e^{\mathrm{w}_k} \xi_k + \int_{\Omega(p_1,\tau/2)} (G(x,\eta) - G(x,p_1))\rho_k H e^{\mathrm{w}_k} \xi_k \\ &= I_{11} + I_{12}. \end{aligned}$$

Concerning I_{12}, by the Mean Value Theorem, in local coordinates after scaling we have

$$G(x,\eta) - G(x,p_1) = \varepsilon_k(a_0 \cdot y) + O(\varepsilon_k^2|y|^2),$$

then in view of (4.18), we see that,

$$\begin{aligned}
I_{12} = \varepsilon_k \int_{B_{\frac{\tau}{2\varepsilon_k}}} & \tilde{h}_k(0)(a_0 \cdot y)|y|^{2\alpha} e^{U_k}(\psi_{\xi}^k(\varepsilon_k y) + \psi_k(\varepsilon_k y) + w_{0,\xi}^k(y) \\
& + O(\varepsilon_k(1+|y|)^{\delta})) + O(\varepsilon_k^2) \\
= \varepsilon_k \int_{B_{\frac{\tau}{2\varepsilon_k}}} & \tilde{h}_k(0)(a_0 \cdot y)|y|^{2\alpha} e^{U_k} w_{0,\xi}^k(y) + O(\varepsilon_k^2) = O(\varepsilon_k^2),
\end{aligned}$$

where we used $\psi_{\xi}^k(\varepsilon_k y) = O(\varepsilon_k|y|)$ and $\psi_k(\varepsilon_k y) = O(\varepsilon_k|y|)$. We notice that the first term on the right vanishes due to the fact that $w_{0,\xi}^k$ is a radial function and the integrand is separable. Consider I_{11} we have

$$\begin{aligned}
I_{11} &= G(x,p_1)\int_{\Omega(p_1,\tau/2)} \rho_k H e^{\mathrm{w}_k}(\xi_k - \psi_{\xi}^k) + G(x,p_1)\int_{\Omega(p_1,\tau/2)} \rho_k H e^{\mathrm{w}_k}\psi_{\xi}^k \\
&= I_{1a} + I_{1b},
\end{aligned}$$

where, writing $\psi_{\xi}^k(\varepsilon_k y) = \varepsilon_k a_k \cdot y + O(\varepsilon_k^2|y|^2)$, we have

$$I_{1b} = O(1)\int_{B_{\frac{\tau}{2\varepsilon_k}}} \tilde{h}_k(0)|y|^{2\alpha} e^{U_k}\psi_{\xi}^k(\varepsilon_k y) = O(\varepsilon_k^2),$$

again because the term of order $O(\varepsilon_k)$ is the integral of a separable function. On the other side, $I_{1a} = O(\varepsilon_k)$ since we can write,

$$\begin{aligned}
\int_{\Omega(p_1,\tau/2)} \rho_k H e^{\mathrm{w}_k}(\xi_k - \psi_{\xi}^k) &= \int_{B_{\frac{\tau}{2\varepsilon_k}}} |y|^{2\alpha}\tilde{h}_k(0)e^{U_k} w_{0,\xi}^k dy + O(\varepsilon_k) \\
&= -\int_{\partial B_{\frac{\tau}{2\varepsilon_k}}} \partial_\nu w_{0,\xi}^k + O(\varepsilon_k) \qquad (4.26)\\
&= O(\varepsilon_k^{2+\epsilon_0}) + O(\varepsilon_k) = O(\varepsilon_k),
\end{aligned}$$

where in the first equality we used (4.18) and at last the equation for $w_{0,\xi}^k$ and the fact that $\partial_\nu w_{0,\xi}^k = O(r^{-3-2\alpha})$ on the boundary. It is important to remark that I_{1a} is the only term of order ε_k, and all the other terms are either of order ε_k^2 or $e^{-\lambda^k/2} = \varepsilon_k^{1+\alpha}$. While for the integral I_2, by (4.20) and a similar argument we deduce that $I_2 = O(e^{-\lambda^k/2})$. Concerning I_3 we have,

$$\begin{aligned}
I_3 &= \int_{\Omega(q^k,\tau/2)} G(x,\eta)\rho_k H e^{\mathrm{w}_k}\xi_k + O(\varepsilon_k^2) \\
&= \int_{\Omega(q^k,\tau/2)} (G(x,\eta) - G(x,q^k))\rho_k H e^{\mathrm{w}_k}\xi_k + G(x,q^k)\int_{\Omega(q^k,\tau/2)} \rho_k H e^{\mathrm{w}_k}\xi_k
\end{aligned}$$

$$
\begin{aligned}
&+O(\varepsilon_k^2)\\
&=I_{31}+I_{32}+O(\varepsilon_k^2).
\end{aligned}
$$

where we also used (4.1) and $\bar{q}^k-\bar{q}=\bar{q}^k=O(e^{-\bar{\lambda}^k}\bar{\lambda}^k)$, see (2.4). By arguing as for I_{12} above we find that $I_{32}=O(e^{-\lambda^k/2})$. Also, by using (4.23) instead of (4.18), the same argument adopted for I_{11} shows that $I_{31}=O(e^{-\lambda^k/2})$. We skip these details to avoid repetitions. As a consequence, one can show that

$$|\xi_k(x_1)-\xi_k(x_2)|=O(\varepsilon_k)+O(e^{-\lambda^k/2})=O(\varepsilon_k+\varepsilon_k^{1+\alpha}),$$

for any $x_1,x_2\in M\setminus\{\Omega(p_1,\tau)\cup\Omega(p_2,\tau)\cup\Omega(q,\tau)\}$, that is exactly the estimation (4.25). Therefore, we see immediately that ψ_ξ^k satisfies,

$$|\psi_\xi^k(\varepsilon_k y)|\le C\varepsilon_k^2(1+|y|)\quad\text{in}\quad B_{\tau/\varepsilon_k}. \tag{4.27}$$

We will use these facts in a sort of bootstrap argument to improve the estimates about the integration of I_{1a} and the oscillation of ξ_k far away from blow-up points later.

Step 2. We invoke the intermediate estimate in Theorem 3.1 about the expansion of v_k near p_1:

$$v_k=U_k+c_{0,k}+c_{1,k}+c_{2,k}+O(\varepsilon_k^{2+\delta})(1+|y|)^\delta. \tag{4.28}$$

Around $0=x(p_1)$ we make the following expansion

$$
\begin{aligned}
&\tilde{h}_k(\varepsilon_k y)\varepsilon_k^{2+2\alpha}\tilde{c}_k(\varepsilon_k y)\\
&=\tilde{h}_k(0)\exp\left\{v_k+\log\frac{V_k(\varepsilon_k y)}{V_k(0)}\right\}(1+O(\varepsilon_k^{2+\epsilon_0}))\\
&=\tilde{h}_k(0)e^{U_k}\left(1+c_{0,k}+\tilde{c}_{1,k}+\tilde{c}_{2,k}+\frac{\Delta(\log V_k)(0)|y|^2\varepsilon_k^2}{4}+\frac{\tilde{c}_{1,k}^2}{2}\right)\\
&\quad+E_k,
\end{aligned}
\tag{4.29}
$$

where $c_{0,k}$ is a radial function corresponding to the ε_k^2 order term in the expansion of v_k (see section 3.1), $V_k=\tilde{h}_k e^{\psi_k}$ and ψ_k is the harmonic function that encodes the boundary oscillation of u_k on ∂B_τ, $\tilde{c}_{1,k}$ and $\tilde{c}_{2,k}$ are given below:

$$\tilde{c}_{1,k}=\varepsilon_k\nabla\log\tilde{h}_k(0)\cdot\theta\left(r-\frac{2(1+\alpha)}{\alpha}\frac{r}{1+a_k r^{2\alpha+2}}\right),\quad \tilde{c}_{2,k}=c_{2,k}+\Theta_2\varepsilon_k^2 r^2,$$

where $a_k=\frac{\tilde{h}_k(0)}{8(1+\alpha)^2}$ and Θ_2 are the collection of projections of $\log\frac{V_k(\varepsilon_k y)}{V_k(0)}$ onto non-radial modes. The left error term E_k in (4.29) satisfies,

$$|E_k|\le C\varepsilon_k^{2+\epsilon_0}(1+|y|)^{-2+\epsilon_0-2\alpha}.$$

At this point, around p_1 we define $w_{1,\xi}^k$, the next term in the approximation of ξ_k, to be a suitable solution of,

$$\Delta w_{1,\xi}^k+|y|^{2\alpha}\tilde{h}_k(0)e^{U_k}w_{1,\xi}^k=-|y|^{2\alpha}\tilde{h}_k(0)e^{U_k}\tilde{c}_{1,k}w_{0,\xi}^k. \tag{4.30}$$

It turns out that (4.30) can be explicitly solved in terms of a function. Indeed, let $c_{1,k}$ be defined as in (3.9) and we set,

$$\hat{c}_1^k(x) = c_{1,k}(x/\varepsilon_k),$$

that is,

$$\hat{c}_1^k(x) = -\frac{2(1+\alpha)}{\alpha}(\nabla \log \tilde{h}_k(0)\cdot\theta)\frac{e^{-\lambda^k}|x|}{\left(e^{-\lambda^k}+\frac{\tilde{h}_k(0)}{8(1+\alpha)^2}|x|^{2\alpha+2}\right)}.$$

Putting $\lambda = \lambda^k$ and differentiating with respect to λ we have that,

$$\frac{d}{d\lambda}\hat{c}_1^k(x) = \frac{\tilde{h}_k(0)}{4\alpha(1+\alpha)}(\nabla \log \tilde{h}_k(0)\cdot\theta)\frac{e^{-\lambda^k}|x|^{2\alpha+3}}{\left(e^{-\lambda^k}+\frac{\tilde{h}_k(0)}{8(1+\alpha)^2}|x|^{2\alpha+2}\right)^2}.$$

Setting

$$w_{1,\xi}^k(y) = b_0^k\left(\frac{d}{d\lambda}\hat{c}_1^k\right)(\varepsilon_k y) \quad \text{with} \quad \varepsilon_k = e^{-\frac{\lambda_1^k}{2(1+\alpha)}},$$

we have

$$w_{1,\xi}^k(y) = b_0^k\frac{\tilde{h}_k(0)}{4\alpha(1+\alpha)}(\nabla \log \tilde{h}_k(0)\cdot\theta)\frac{\varepsilon_k r^{2\alpha+3}}{\left(1+\frac{\tilde{h}_k(0)}{8(1+\alpha)^2}r^{2+2\alpha}\right)^2}. \tag{4.31}$$

It is not difficult to check that $w_{1,\xi}^k$ satisfies (4.30),

$$w_{1,\xi}^k(0) = |\nabla w_{1,\xi}^k(0)| = 0, \quad \text{and} \quad |w_{1,\xi}^k(y)| \le C\varepsilon_k(1+|y|)^{-1} \text{ in } B_{\tau/\varepsilon_k}.$$

Next, let us we write the equation for $w_{1,\xi}^k$ in the following form:

$$\begin{aligned}&\Delta w_{1,\xi}^k + |y|^{2\alpha}\tilde{h}_k(\varepsilon_k y)\varepsilon_k^{2+2\alpha}\tilde{c}_k(\varepsilon_k y)w_{1,\xi}^k\\ &= |y|^{2\alpha}\tilde{h}_k(0)\left[\left(\frac{\tilde{h}_k(\varepsilon_k y)}{\tilde{h}_k(0)}\varepsilon_k^{2+2\alpha}\tilde{c}_k(\varepsilon_k y) - e^{U_k}\right)w_{1,\xi}^k - e^{U_k}\tilde{c}_{1,k}w_{0,\xi}^k\right],\end{aligned} \tag{4.32}$$

and then set

$$z_k(y) = \tilde{w}_k(y) - w_{1,\xi}^k(y) = \xi_k(\varepsilon_k y) - \psi_\xi^k(\varepsilon_k y) - w_{0,\xi}^k(y) - w_{1,\xi}^k(y).$$

Therefore, in view of (4.17), (4.27) and (4.32), the equation for z_k reads:

$$\begin{aligned}&\Delta z_k + |y|^{2\alpha}\tilde{h}_k(\varepsilon_k y)\varepsilon_k^{2+2\alpha}\tilde{c}_k(\varepsilon_k y)z_k\\ &= \tilde{h}_k(0)|y|^{2\alpha}\left(e^{U_k} - \frac{\tilde{h}_k(\varepsilon_k y)}{\tilde{h}_k(0)}\varepsilon_k^{2+2\alpha}\tilde{c}_k(\varepsilon_k y)\right)(w_{0,\xi}^k + w_{1,\xi}^k)\\ &\quad + \tilde{h}_k(0)|y|^{2\alpha}e^{U_k}\tilde{c}_{1,k}w_{0,\xi}^k + O(\varepsilon_k^{2+\epsilon_0})(1+|y|)^{-2-2\alpha+\epsilon_0}\\ &= O\left(\varepsilon_k^2\right)(1+|y|)^{-2-2\alpha}.\end{aligned}$$

By using the usual argument based on Green's representation formula, we have

$$|z_k(y)| \le C(\delta)\varepsilon_k^2(1+|y|)^\delta, \tag{4.33}$$

it implies that the oscillation of z_k on $\partial B_{\tau/\varepsilon_k}$ is of order $O(\varepsilon_k^{2-\delta})$. In view of (4.33), I_{1a} can be improved as follows

$$\begin{aligned} I_{1a} &= G(x,p_1)\int_{B(p_1,\frac{\tau}{2})}\rho_k H e^{\mathrm{w}_k}(\xi_k-\psi_\xi^k)\\ &= G(x,p_1)\int_{B_{\frac{\tau}{2\varepsilon_k}}}\tilde{h}_k(\varepsilon_k y)e^{U_k}(w_{0,\xi}^k+w_{1,\xi}^k)dy+O(\varepsilon_k^2). \end{aligned}$$

Then we make a Taylor expansion for $\tilde{h}_k(\varepsilon_k y)$, its zero-th order term relative to $w_{0,\xi}^k$ has already been shown above to be of order $O(\varepsilon_k^{2+\epsilon_0})$ (see (4.26)), the one relative to $w_{1,\xi}^k$ vanishes since the integrand is separable, the term proportional to $\varepsilon_k w_{0,\xi}^k$ vanishes again because the integrand is separable, while the one proportional to $\varepsilon_k w_{1,\xi}^k$ is already of order $O(\varepsilon_k^2)$. On the other hand, we have already mentioned in Step 1 that all the terms except for I_{1a} are of order either ε_k^2 or $\varepsilon_k^{1+\alpha}$. Therefore, we derive that

$$|\xi_k(x_1)-\xi_k(x_2)| = O(\varepsilon_k^2)+O(e^{-\lambda^k/2}),$$

$\forall x_1,x_2\in M\setminus\{\Omega(p_1,\tau)\cup\Omega(p_2,\tau)\cup\Omega(q,\tau)\}$, which is (4.24). Hence, we finish the proof. □

Based on (4.24), we can get a better estimate for ψ_ξ^k:

$$|\psi_\xi^k(\varepsilon_k y)|\le C\varepsilon_k^{2+\epsilon_0}(1+|y|). \tag{4.34}$$

We will see that this estimate implies that all the terms in the expansions involving ψ_ξ^k are in fact negligible.

By using (4.29), the equation of z_k can be written as follows,

$$\begin{aligned} &\Delta z_k+|y|^{2\alpha}\tilde{h}_k(\varepsilon_k y)\varepsilon_k^{2+2\alpha}\tilde{c}_k(\varepsilon_k y)z_k\\ &= -\tilde{h}_k(0)|y|^{2\alpha}e^{U_k}\Big(w_{0,\xi}^k(c_{0,k}+\tilde{c}_{2,k}+\frac{1}{4}\Delta(\log\tilde{h}_k)(0)|y|^2\varepsilon_k^2+\frac{1}{2}(\tilde{c}_{1,k})^2)\\ &\quad + w_{1,\xi}^k\tilde{c}_{1,k}+O(\varepsilon_k^{2+\epsilon_0})(1+|y|)^{-2-2\alpha+\epsilon_0}\Big). \end{aligned}$$

By using separable functions to remove separable terms, we can write the equation of the radial part of z_k, which we denote $\tilde{z}_k$,

$$\begin{aligned} &\Delta\tilde{z}_k+|y|^{2\alpha}\tilde{h}_k(0)e^{U_k}\tilde{z}_k\\ &= -\tilde{h}_k(0)|y|^{2\alpha}e^{U_k}\left[w_{0,\xi}^k\left(c_{0,k}+\frac{\Delta(\log\tilde{h}_k)(0)|y|^2\varepsilon_k^2}{4}+\frac{(\tilde{c}_{1,k})_r^2}{2}\right)+(w_{1,\xi}^k\tilde{c}_{1,k})_r\right], \end{aligned}$$

where we let $(A)_\theta, (A)_r$ denote the angular part and radial part of A respectively. Corresponding to these terms we construct z_0^k to solve

$$\begin{cases} \dfrac{d^2}{dr^2} z_0^k + \frac{1}{r}\dfrac{d}{dr} z_0^k + r^{2\alpha}\tilde{h}_k(0)e^{U_k} z_0^k = F_0^k, \quad 0 < r < \tau/\varepsilon_k, \\ \\ z_0^k(0) = \frac{d}{dr} z_0^k(0) = 0, \end{cases} \tag{4.35}$$

where

$$F_0^k(r) = -\tilde{h}_k(0)|y|^{2\alpha} e^{U_k}\Big(w_{0,\xi}^k (c_{0,k} + \frac{1}{4}\Delta(\log\tilde{h}_k)(0)|y|^2\varepsilon_k^2 + \frac{1}{4}\varepsilon_k^2|\nabla\log\tilde{h}_k(0)|^2(g_k+r)^2) + (w_{1,\xi}^k \tilde{c}_{1,k})_r\Big)$$

and we used

$$\left(\frac{1}{2}(\tilde{c}_{1,k})^2\right)_r = \frac{1}{4}\varepsilon_k^2|\nabla\log\tilde{h}_k(0)|^2(g_k+r)^2.$$

By standard potential estimates it is not difficult to see that

$$|z_0^k| \le C\varepsilon_k^2(1+r)^{-2\alpha}\log(1+r).$$

Next, we introduce $w_{2,\xi}^k$, which is used for removing the separable terms of the order $O(\varepsilon_k^2)$ in $\tilde{w}_k$,

$$\begin{aligned} &\Delta w_{2,\xi}^k + |y|^{2\alpha} h_k(0) e^{U_k} w_{2,\xi}^k \\ &= -|y|^{2\alpha} h_k(0) e^{U_k}\Big(w_{0,\xi}^k(\tilde{c}_{2,k} + \frac{1}{2}(\tilde{c}_{1,k}^2)_\theta) + (\tilde{c}_{1,k} w_{1,\xi}^k)_\theta\Big). \end{aligned} \tag{4.36}$$

Using (3.12) we see that

$$|w_{2,\xi}^k| \le C\varepsilon_k^2(1+|y|)^{-2\alpha} \quad \text{in} \quad B_{\tau/\varepsilon_k}. \tag{4.37}$$

By standard potential estimates as usual we have

$$\left|\tilde{w}_k - \sum_{i=1}^{2} w_{i,\xi}^k - z_0^k(y)\right| \le C\varepsilon_k^{2+\epsilon_0}(1+|y|)^{-2\alpha+\epsilon_0}.$$

Next, following a similar argument about $w_{0,\xi}^k$ in (4.26), we evaluate the integral around p_1 as follows,

$$\begin{aligned} &\int_{B_{\tau/\varepsilon_k}} |y|^{2\alpha}\tilde{h}_k(\varepsilon_k y)\varepsilon_k^{2+2\alpha}\tilde{c}_k(\varepsilon_k y)\xi_k(\varepsilon_k y)dy \\ &= \tilde{h}_k(0)\int_{B_{\tau/\varepsilon_k}} |y|^{2\alpha}\exp\{U_k + c_{0,k} + \tilde{c}_{1,k} + \tilde{c}_{2,k} + \frac{1}{4}\Delta(\log\tilde{h}_k(0))\varepsilon_k^2|y|^2 \\ &\quad + \frac{1}{2}(\tilde{c}_{1,k})^2\}(w_{0,\xi}^k + w_{1,\xi}^k + z_0^k) + O(\varepsilon_k^{2+\epsilon_0}) \end{aligned}$$

$$= \int_{B_{\tau/\varepsilon_k}} \tilde{h}_k(0)|y|^{2\alpha} e^{U_k} \Bigg(\big(1 + c_{0,k} + \frac{1}{2}(\tilde{c}_{1,k})^2 + \frac{1}{4}\varepsilon_k^2 \Delta(\log \tilde{h}_k)(0)|y|^2\big) w_{0,\xi}^k + (\tilde{c}_{1,k} w_{1,\xi}^k)_r + z_0^k \Bigg) + O(\varepsilon_k^{2+\epsilon_0}).$$

Remark that, by using the equation for z_0^k in (4.35), we have,

$$\int_{B_{\tau/\varepsilon_k}} \varepsilon_k^{2+2\alpha} |y|^{2\alpha} \tilde{h}_k(\varepsilon_k y) \tilde{c}_k(\varepsilon_k y) \xi_k(\varepsilon_k y) dy = -\int_{\partial B_{\tau/\varepsilon_k}} \frac{\partial z_0^k}{\partial \nu} + O(\varepsilon_k^{2+\epsilon_0}). \tag{4.38}$$

We set $A_0^k(x) = b_0^k c_{0,k}(|x|/\varepsilon_k)$, then

$$\Delta A_0^k + \tilde{h}_k(0) \frac{|x|^{2\alpha} e^{-\lambda^k}}{\left(e^{-\lambda^k} + \frac{\tilde{h}_k(0)}{8(1+\alpha)^2}|x|^{2\alpha+2}\right)^2} A_0^k$$
$$= -b_0^k \frac{\tilde{h}_k(0)}{4} \frac{e^{-\lambda^k}|x|^{2\alpha+2}}{\left(e^{-\lambda^k} + \frac{\tilde{h}_k(0)}{8(1+\alpha)^2}|x|^{2\alpha+2}\right)^2}$$
$$\times \left(\left(1 - \frac{2(1+\alpha)}{\alpha} \frac{e^{-\lambda^k}}{e^{-\lambda^k} + \frac{\tilde{h}_k(0)}{8(1+\alpha)^2}|x|^{2\alpha+2}} \right)^2 |\nabla \log \tilde{h}_k(0)|^2 + \Delta \log \tilde{h}_k(0) \right).$$

Now we define $A_\lambda^k(x) = \frac{d}{d\lambda} A_0^k(x)$ and after a lengthy calculation we see that

$$z_0^k(y) = A_\lambda^k(\varepsilon_k y). \tag{4.39}$$

By using (4.39) we deduce that,

$$\int_{B_{\tau/\varepsilon_k}} \Delta z_0^k = \varepsilon_k^2 \int_{B_{\tau/\varepsilon_k}} \Delta A_\lambda^k(\varepsilon_k y) dy$$
$$= \int_{B_\tau} \Delta A_\lambda^k(x) dx = \frac{d}{d\lambda} \int_{B_\tau} \Delta A_0^k(x) dx.$$

Since

$$\Delta A_0^k(x) = b_0^k \Delta c_{0,k}\left(\frac{|x|}{\varepsilon_k}\right) \varepsilon_k^{-2},$$

we have

$$\frac{1}{b_0^k} \int_{B_\tau} \Delta A_0^k(x) dx = \int_{B_{\tau/\varepsilon_k}} \Delta c_{0,k}(y) dy$$
$$= \int_{\partial B_{\tau/\varepsilon_k}} \frac{\partial c_{0,k}}{\partial \nu} = d_{1,k} \Delta \log \tilde{h}_k(0) \varepsilon_k^2 + O(\varepsilon_k^{2+\epsilon_0}),$$

and

$$\int_{\partial B_{\tau/\varepsilon_k}} \frac{\partial z_0^k}{\partial \nu} = \int_{B_{\tau/\varepsilon_k}} \Delta z_0^k = -\frac{1}{1+\alpha} b_0^k d_{1,k} \Delta(\log \tilde{h}_k)(0) \varepsilon_k^2 + O(b_0^k \varepsilon_k^{2+\epsilon_0}),$$

which, in view of (4.38) eventually implies that

$$\int_{\Omega(p_1,\tau)} \rho_k H e^{\mathrm{w}_k} \xi_k = -\frac{1}{1+\alpha} b_0^k d_{1,k} \Delta(\log \tilde{h}_k)(0)\varepsilon_k^2 + O(b_0^k \varepsilon_k^{2+\epsilon_0}). \tag{4.40}$$

On the other side, the contribution of the integral around p_2 is very small,

$$\int_{\Omega(p_2,\tau)} \rho_k H e^{\mathrm{w}_k} \xi_k = O(\varepsilon_k^{2+\epsilon_0}). \tag{4.41}$$

The process is almost the same as what we did for p_1. Indeed, we make an expansion of ξ_k around p_2, $w_{1,\xi}^k$ is set for $c_{1,k}$ in the expansion of u_k, z_0^k is used for treating the second order radial term in the expansion of ξ_k and $c_{0,k}$ of u_k. The only difference is that the scaling is now with respect to $\hat{\varepsilon}_k = \varepsilon_k^{\frac{1+\alpha}{1+\beta}}$, then $O(\hat{\varepsilon}_k^2) = O(\varepsilon_k^{2+\epsilon_0})$. We skip these details to avoid repetitions.

Concerning the integral on $\Omega(q,\tau)$, we first improve the estimate of $\bar{w}_k$ (see (4.23)). At first, we can show that

$$\text{The oscillation of } \bar{\psi}_\xi^k(y) \text{ on } \partial B_{\tau/\bar{\varepsilon}_k} \text{ is of order } O(\varepsilon_k^2 + \bar{\varepsilon}_k),$$

where we used Lemma 4.1 and the explicit formula of $\bar{w}_{0,\xi}^k(y)$. As a consequence, by a standard argument we could derive an estimate like (4.34) holds for $\bar{\psi}_\xi^k(\varepsilon_k y)$ as well, i.e.,

$$|\bar{\psi}_\xi^k(\varepsilon_k y)| \le C(\varepsilon_k^2 \bar{\varepsilon}_k + \bar{\varepsilon}_k^2)(1+|y|), \quad y \in B_{\tau/\bar{\varepsilon}_k}. \tag{4.42}$$

While at $\bar{q}^k$, by standard elliptic estimate for harmonic function we have $|\nabla \bar{\psi}_\xi^k(0)| = O(\varepsilon_k^{2+\epsilon_0})$. Therefore, in local coordinates such that $0 = x(q)$ and after scaling $x = \bar{q}^k + \bar{\varepsilon}_k y$, where $\bar{\varepsilon}_k = e^{-\bar{\lambda}^k/2}$, we can write the equation for $\bar{w}_k$ as follows,

$$\Delta \bar{w}_k + \bar{h}_k(\bar{q}^k + \bar{\varepsilon}_k y)\bar{\varepsilon}_k^2 \bar{c}_k(\bar{q}^k + \bar{\varepsilon}_k y)\bar{w}_k = O(\varepsilon_k^{2+\epsilon_0})(1+|y|)^{-3} \quad \text{in} \quad B_{\tau/\bar{\varepsilon}_k}.$$

Here we remark that, compared with (4.22), the improvement in the estimates of the right hand side is obtained because of the vanishing rate of the gradient of the coefficient function for regular blow-up points (see (2.3)) and the improved estimate (4.42) for $\bar{\psi}_\xi^k$. On the other hand, we have

$$\bar{w}_k(0) = 0, \quad \nabla \bar{w}_k(0) = O(\varepsilon_k^{2+\epsilon_0}) \quad \text{and} \quad \bar{w}_k \text{ is a constant on } \partial B_{\tau/\bar{\varepsilon}_k}.$$

As a consequence, by the usual potential estimates, we conclude that,

$$|\bar{w}_k(y)| \le C(\delta)\varepsilon_k^{2+\epsilon_0}(1+|y|)^\delta, \quad |y| \le \tau/\bar{\varepsilon}_k.$$

Next, by using the expansion of $\bar{c}_k$ in we have,

$$\int_{\Omega(q,\tau)} \rho_k H(x) e^{\mathrm{w}_k(x)} \xi_k = \int_{B_{\tau/\bar{\varepsilon}_k}} \bar{h}_k(\bar{q}^k + \bar{\varepsilon}_k y) e^{U_k} \xi_k(\bar{q}^k + \bar{\varepsilon}_k y) + O(\varepsilon_k^{2+\epsilon_0}).$$

Remark that $e^{-\lambda^k/2}=O(\varepsilon_k^{1+\epsilon_0})$, then by (4.21)-(4.23) we see that all the terms including $\bar{\psi}_\xi^k(\bar{\varepsilon}_k y)$ are of order $O(\varepsilon_k^{2+\epsilon_0})$. Also, neglecting terms which vanish due to the integrand is separable, we have

$$\begin{aligned}
&\int_{B_{\tau/\bar{\varepsilon}_k}} \bar{h}_k(\bar{q}^k+\bar{\varepsilon}_k y)e^{U_k}\xi_k(\bar{q}^k+\bar{\varepsilon}_k y)\\
&=\int_{B_{\tau/\bar{\varepsilon}_k}} \bar{h}_k(\bar{q}^k)e^{U_k}(\bar{w}_{0,\xi}^k+\bar{w}_k)+\bar{\varepsilon}_k\int_{B_{\tau/\bar{\varepsilon}_k}}(\nabla\log\bar{h}_k(\bar{q}^k)\cdot y)\frac{e^{U_k}(\bar{b}_k\cdot y)}{1+\frac{\bar{h}_k(\bar{q}^k)}{8}|y|^2}\\
&\quad+\bar{\varepsilon}_k\int_{B_{\tau/\bar{\varepsilon}_k}}(\nabla\log\bar{h}_k(\bar{q}^k)\cdot y)e^{U_k}\bar{w}_k+O(\varepsilon_k^{2+\epsilon_0})\\
&=\int_{B_{\tau/\bar{\varepsilon}_k}} \bar{h}_k(\bar{q}^k)e^{U_k}\bar{w}_k+\bar{\varepsilon}_k\int_{B_{\tau/\bar{\varepsilon}_k}}(\nabla\log\bar{h}_k(\bar{q}^k)\cdot y)\frac{e^{U_k}(\bar{b}_k\cdot y)}{1+\frac{\bar{h}_k(\bar{q}^k)}{8}|y|^2}+O(\varepsilon_k^{2+\epsilon_0})\\
&=O(\varepsilon_k^{2+\epsilon_0}),
\end{aligned}$$

where $\bar{b}_k=(\bar{b}_1^k,\bar{b}_2^k)$ and we used the same argument as in (4.26) to show that $\int_{B_{\tau/\bar{\varepsilon}_k}}\bar{h}_k(\bar{q}^k)e^{U_k}\bar{w}_{0,\xi}^k$ is of order $O(\varepsilon_k^{2+\epsilon_0})$. Therefore we eventually deduce that,

$$\int_{\Omega(q,\tau)}\rho_k Hc_k(x)\xi_k=O(\varepsilon_k^{2+\epsilon_0}). \tag{4.43}$$

In view of (4.40), (4.41), (4.43), we come up with a contradiction as follows,

$$\begin{aligned}
0=\int_M\rho_k Hc_k\xi_k&=\int_{\Omega(p_1,\tau)}Hc_k\xi_k+\int_{\Omega(p_2,\tau)}Hc_k\xi_k+\int_{\Omega(q,\tau)}Hc_k\xi_k\\
&\quad+\int_{M\setminus\{\Omega(p_1,\tau)\cup\Omega(p_2,\tau)\cup\Omega(q,\tau)\}}Hc_k\xi_k\\
&=Cb_0\Delta\log h_k(p_1)\varepsilon_k^2+o(\varepsilon_k^2),
\end{aligned}$$

for some constant $C\neq 0$, since in particular the integrals on $\Omega(p_2,\tau)$, $\Omega(q,\tau)$ and $M\setminus\{\Omega(p_1,\tau)\cup\Omega(p_2,\tau)\cup\Omega(q,\tau)\}$ are all of order $o(\varepsilon_k^2)$. Since for k large $L(\mathbf{p})\neq 0$ is the same as $\Delta\log h_k(p_1)\neq 0$, we obtain a contradiction as far as $b_0\neq 0$.

4.2. **Proof of $b_1=b_2=0$.** In this subsection, we shall prove that $b_1=b_2=0$. At the beginning of this part we recall that in local coordinates around q, $0=x(q)$, u_k satisfy

$$\Delta u_k+\bar{h}_k(x)e^{u_k}=0 \quad \text{in} \quad B_\tau,$$

and ξ_k satisfies

$$\Delta\xi_k+\bar{h}_k\bar{c}_k\xi_k=0 \quad \text{in} \quad B_\tau.$$

We shall divide our argument into several steps as below, first, we would like to get some estimation on $\bar{v}_k$:

Step one: Intermediate estimates for $\bar{v}_k$

We work in local coordinates centered at q, $0 = x(q)$, with $\bar\varepsilon_k = e^{-\bar\lambda^k/2}$. As before, by a slightly abuse of notation we set ψ_k to be the function which encodes the boundary oscillation of u_k in $B_\tau(q)$. We set

$$\bar v_k(y) = u_k(\bar q^k + \bar\varepsilon_k y) + 2\log\bar\varepsilon_k - \psi_k(\bar\varepsilon_k y), \quad y \in \Omega_k := B_{\tau/\bar\varepsilon_k}, \tag{4.44}$$

and

$$\bar\xi_k(y) = \xi_k(\bar q^k + \bar\varepsilon_k y), \quad y \in \Omega_k. \tag{4.45}$$

Obviously, $\bar v_k$ is constant on $\partial\Omega_k$. By Theorem 3.3 we have

$$\bar v_k(y) = U_k + c_{0,k} + c_{2,k} + O(\bar\varepsilon_k^{2+\delta})(1+|y|)^\delta, \tag{4.46}$$

then we can write

$$\bar\varepsilon_k^2 \rho_k H \bar c_k(\bar q^k + \bar\varepsilon_k y)\xi_k = \bar h_{k,0}(\bar q^k + \bar\varepsilon_k y)e^{\bar v_k}\bar\xi_k,$$

where $\bar h_{k,0} = \bar h_k e^{\psi_k}$. According to our setting, $\bar\lambda^k = u_k(\bar q^k)$ and $\bar\varepsilon_k = e^{-\bar\lambda^k/2}$. Now we set

$$\bar w^k(y) = \bar v_k(y) - c_{0,k}(y) - c_{2,k}(y).$$

where $c_{i,k}$ $i = 0,1,2$, are defined as in the proof of Theorem 3.3. Different from the expansion of v_k in the neighborhood of p_1, the term $c_{1,k}$ can be regarded as error term, this is due to that $\nabla\bar h_{k,0}(\bar q^k) = O(\bar\lambda^k e^{-\bar\lambda^k})$. After direct calculation one can find that $\bar w^k$ satisfies,

$$\begin{cases} \Delta\bar w^k + \bar h_{k,0}(\bar q^k)e^{\bar v_k}\bar w^k = O(\bar\varepsilon_k^{3-\epsilon_0})(1+|y|)^{-1}, \\ \\ \bar w^k(0) = |\nabla\bar w^k(0)| = 0, \end{cases}$$

and the oscillation of $\bar w^k$ on $\partial\Omega_k$ is of order $O(\bar\varepsilon_k^2)$. By Theorem 3.3 we get

$$|\bar w^k(y)| \le C\bar\varepsilon_k^{2+\epsilon_0}(1+|y|)^{2\epsilon_0},$$

which implies

$$|\bar v_k - U_k - c_{0,k} - c_{2,k}| \le C\bar\varepsilon_k^{2+\epsilon_0}(1+|y|)^{2\epsilon_0}, \quad y \in \Omega_k. \tag{4.47}$$

As a consequence,

$$\nabla\bar v_k = \nabla U_k + \nabla c_{0,k} + \nabla c_{2,k} + O(\tau)\bar\varepsilon_k^3, \quad y \in \Omega_k, \quad |y| \sim \bar\varepsilon_k^{-1}. \tag{4.48}$$

Based on (4.46), we also have

$$\begin{aligned} \bar v_k(y) + \log\frac{\bar h_{k,0}(\bar q^k + \bar\varepsilon_k y)}{\bar h_{k,0}(\bar q^k)} &= U_k + c_{0,k} + \tilde c_{2,k} + \frac{\Delta(\log\bar h_{k,0})(\bar q^k)\bar\varepsilon_k^2 r^2}{4} \\ &\quad + O(\bar\varepsilon_k^3)r^3 + O(\bar\varepsilon_k^{2+\delta})(1+r)^\delta. \end{aligned} \tag{4.49}$$

Step two: Improved estimate on the oscillation of ξ_k away from blow-up points and vanishing rates of b_0^k, $\hat b_0^k$ and $\bar b_0^k$. From Lemma 4.1 we have

$$|\xi_k(x_1) - \xi_k(x_2)| \le C(\varepsilon_k^2 + \bar\varepsilon_k). \tag{4.50}$$

The new estimate (4.50) implies that the harmonic function ψ^k_ξ that encodes the oscillation of ξ_k on $\partial\Omega(p_1,\tau)$ satisfies

$$|\psi^k_\xi(\varepsilon_k y)| \le C(\varepsilon_k^3 + \varepsilon_k\bar{\varepsilon}_k)|y|, \quad |y| < \tau/\varepsilon_k. \tag{4.51}$$

In the following lemma, we shall prove that the oscillation of ξ_k far away from blow-up points is of order $O(\bar{\varepsilon}_k)$.

Proposition 4.1. *For any $x_1, x_2 \in M \setminus \{\Omega(p_1,\tau) \cup \Omega(p_2,\tau) \cup \Omega(q,\tau)\}$,*

$$|\xi_k(x_1) - \xi_k(x_2)| \le C\bar{\varepsilon}_k$$

for some $C > 0$ independent by k.

Remark 4.3. *Since the conclusion holds automatically by Lemma 4.1, it suffices to study the case for $\alpha > 1$.*

Proof. As we can see that, Proposition (4.1) can be seen as an improvement of Lemma 4.1. The crucial point is to get a better estimate for ψ^k_ξ around p_1. By (4.51) and the fact that $\psi^k_\xi(0) = 0$ we have

$$\psi^k_\xi(\varepsilon_k y) = \sum_j a_j(\varepsilon_k^3 + \varepsilon_k\bar{\varepsilon}_k)y_j + O(\varepsilon_k^4 + \varepsilon_k^2\bar{\varepsilon}_k)|y|^2, \quad |y| < \tau/\varepsilon_k,$$

for suitable $a_j, j = 1,2$. In local coordinates around p_1, $0 = x(p_1)$, the first term of the approximation of ξ_k around p_1 is still $w^k_{0,\xi}$ defined as in (4.12) that satisfies (4.13). Using the expansion of v_k around p_1 (4.28), we write

$$\begin{aligned}
&|y|^{2\alpha}\tilde{h}_k(0)\exp\left(v_k + \log\frac{\tilde{h}_k(\varepsilon_k y)}{\tilde{h}_k(0)}\right)\\
&= |y|^{2\alpha}\tilde{h}_k(0)\exp\{U_k + c_{0,k} + \tilde{c}_{1,k} + \tilde{c}_{2,k}\\
&\quad + \frac{1}{4}\Delta(\log\tilde{h}_k)(0)|y|^2\varepsilon_k^2 + O(\varepsilon_k^{2+\epsilon_0})(1+|y|)^{2+\epsilon_0})\}\\
&= |y|^{2\alpha}\tilde{h}_k(0)e^{U_k}(1 + c_{0,k} + \tilde{c}_{1,k} + \tilde{c}_{2,k} + \frac{1}{4}\Delta(\log\tilde{h}_k)(0)|y|^2\varepsilon_k^2\\
&\quad + \frac{1}{2}(\tilde{c}_{1,k})^2) + O(\varepsilon_k^{2+\epsilon_0})(1+|y|)^{-2-\epsilon_0-2\alpha},
\end{aligned}$$

for some small $\epsilon_0 > 0$ depending by $\alpha > 0$. Then we can write the equation of $w^k_{0,\xi}$ as follows,

$$\begin{aligned}
&\Delta w^k_{0,\xi} + \tilde{h}_k(0)|y|^{2\alpha}\exp\left(v_k + \log\frac{\tilde{h}_k(\varepsilon_k y)}{\tilde{h}_k(0)}\right)w^k_{0,\xi}\\
&= |y|^{2\alpha}\tilde{h}_k(0)e^{U_k}w^k_{0,\xi}\left(c_{0,k} + \tilde{c}_{1,k} + \tilde{c}_{2,k} + \frac{\Delta(\log\tilde{h}_k)(0)|y|^2\varepsilon_k^2}{4} + \frac{\tilde{c}_{1,k}^2}{2}\right)\\
&\quad + O(\varepsilon_k^{2+\epsilon_0})b_0^k(1+|y|)^{-2-\epsilon_0-2\alpha}.
\end{aligned} \tag{4.52}$$

The next term in the expansion is $w^k_{1,\xi}$ defined as in (4.30). Now we need to include b^k_0 in the estimate of $w^k_{1,\xi}$,

$$|w^k_{1,\xi}(y)| \le C b^k_0 \varepsilon_k (1+|y|)^{-1},$$

where we used $w^k_{0,\xi}(0) = b^k_0$. At this point we write the equation for $w^k_{1,\xi}$ in the following form:

$$\begin{aligned}
&\Delta w^k_{1,\xi} + \tilde{h}_k(0)|y|^{2\alpha} \exp\left(v_k + \log\frac{\tilde{h}_k(\varepsilon_k y)}{\tilde{h}_k(0)}\right) w^k_{1,\xi} \\
&= -\tilde{h}_k(0)|y|^{2\alpha} e^{U_k} \tilde{c}_{1,k} w^k_{0,\xi} + |y|^{2\alpha}\tilde{h}_k(0) e^{U_k}\tilde{c}_{1,k} w^k_{1,\xi} + \frac{O(b^k_0\varepsilon^3_k)}{(1+|y|)^{3+2\alpha}}.
\end{aligned}$$

Writing the last two term as $\frac{O(b^k_0\varepsilon^2_k)}{(1+|y|)^{3+2\alpha}}$, then the above equation can be rewritten as

$$\begin{aligned}
&\Delta w^k_{1,\xi} + \tilde{h}_k(0)|y|^{2\alpha} \exp\left(v_k + \log\frac{\tilde{h}_k(\varepsilon_k y)}{\tilde{h}_k(0)}\right) w^k_{1,\xi} \\
&= -\tilde{h}_k(0)|y|^{2\alpha} e^{U_k} \tilde{c}_{1,k} w^k_{0,\xi} + \frac{O(b^k_0\varepsilon^2_k)}{(1+|y|)^{3+2\alpha}}.
\end{aligned} \tag{4.53}$$

Let

$$\underline{w}^k_1 = \tilde{\xi}_k(y) - w^k_{0,\xi} - w^k_{1,\xi} - \psi^k_\xi(\varepsilon_k y),$$

then from (4.52) and (4.53) we see that the equation for $\underline{w}^k_1$ takes the form,

$$\begin{aligned}
&\left(\Delta + |y|^{2\alpha}\tilde{h}_k(0)\exp\left(v_k + \log\frac{\tilde{h}_k(\varepsilon_k y)}{\tilde{h}_k(0)}\right)\right)\underline{w}^k_1 \\
&= \frac{O(\bar{\varepsilon}_k)}{(1+|y|)^{4+2\alpha}} + \frac{O(b^k_0\varepsilon^2_k)}{(1+|y|)^{3+2\alpha}} + \frac{O(\varepsilon^3_k + \varepsilon_k\bar{\varepsilon}_k)}{(1+|y|)^{3+2\alpha}}.
\end{aligned} \tag{4.54}$$

The three terms on the right hand side of (4.54) come from the expansion of c_k, $w^k_{1,\xi}$ and ψ^k_ξ respectively. On the other hand, $w^k_{0,\xi}$ is constant on $\partial\Omega_k$ and $w^k_{1,\xi}$ has an oscillation of order $O(b^k_0\varepsilon^2_k)$. To eliminate this oscillation we use another harmonic function of order $O(b^k_0\varepsilon^3_k|y|)$. For simplicity and with an abuse of notations we include this function in ψ^k_ξ. As a consequence standard potential estimates show that,

$$|\underline{w}^k_1(y)| \le C(\delta)(b^k_0\varepsilon^2_k + \varepsilon^3_k + \bar{\varepsilon}_k)(1+|y|)^{\delta}. \tag{4.55}$$

In order to further improve the estimate about ψ^k_ξ we point out that, in view of (4.50), then around p_2 and q we have almost the same improved estimates. First of all, around p_2, we recall that $\hat{w}_k$ is defined in (4.19) and satisfies (4.20). Then we have,

$$\int_{B(p_2,\tau)} \rho_k H e^{\mathrm{w}_k}\xi_k = O(\bar{\varepsilon}_k).$$

Moreover, around q^k, we recall that $\bar{w}_k$ is defined in (4.21) and satisfies (4.23). By using (4.23) and the expansion of $\bar{w}_k$ we have

$$\int_{B(q^k,\tau)} \rho_k H e^{\mathrm{w}_k} \xi_k = O(\bar{\varepsilon}_k).$$

At this point we move back to the evaluation of the oscillation of ξ_k away from blow-up points. From the Green's representation formula of ξ_k, as in the proof of (4.24), by using (4.51) for ψ^k_ξ, (4.55) for $\underline{w}^k_1$, (4.20) for $\hat{w}_k$ and (4.23) for $\bar{w}_k$ we can further improve the estimate of ξ_k as follows,

$$|\xi_k(x) - \xi_k(y)| \le C(b^k_0 \varepsilon^2_k + \varepsilon^4_k + \bar{\varepsilon}_k), \tag{4.56}$$

for $x, y \in M \setminus \{\Omega(p_1,\tau) \cup \Omega(p_2,\tau) \cup \Omega(q,\tau)\}$. Then (4.56) further improves the estimate of $\underline{w}^k_1$, so that, by a bootstrap argument which does not involve the leading term proportional to $b^k_0 \varepsilon^2_k$ (which in fact comes just from the integration over $\Omega(p_1,\tau)$), after a finite number of iterations, we deduce that,

$$|\underline{w}^k_1(y)| \le C(\delta)(b^k_0 \varepsilon^2_k + \bar{\varepsilon}_k)(1+|y|)^\delta,$$

and the estimate about the oscillation of ξ_k takes the form,

$$|\xi_k(x_1) - \xi_k(x_2)| \le C(b^k_0 \varepsilon^2_k + \bar{\varepsilon}_k), \tag{4.57}$$

for all $x_1, x_2 \in M \setminus \{\Omega(p_1,\tau), \cup \Omega(p_2,\tau) \cup \Omega(q,\tau)\}$.

As an immediately consequence of (4.57) for ψ^k_ξ we have

$$|\psi^k_\xi(\varepsilon_k y)| \le C(b^k_0 \varepsilon^3_k + \varepsilon_k \bar{\varepsilon}_k)|y|. \tag{4.58}$$

In the following lemma, we shall prove that

$$|b^k_0| \le C e^{-\frac{\lambda^k}{2} + \frac{\lambda^k}{1+\alpha}} \quad \text{if} \quad \alpha > 1.$$

As a consequence, we deduce that Proposition 4.1 holds. □

Lemma 4.2. *There exists $C > 0$ independent of k such that*

$$|b^k_0| \le C e^{-\frac{\lambda^k}{2} + \frac{\lambda^k}{1+\alpha}} \quad \textit{if} \quad \alpha > 1. \tag{4.59}$$

Proof. Recall that $w^k_{2,\xi}$ was defined in (4.36) and the estimate of this term is given in (4.37), but now we include the dependence of b^k_0. So (4.37) takes the form,

$$|w^k_{2,\xi}(y)| \le C b^k_0 \varepsilon^2_k (1+|y|)^{-2\alpha}.$$

In order to take care of the radial part of order $O(\varepsilon^2_k) b^k_0$ in the expansion of ξ_k, we set $z_{0,k}$ to be the radial function satisfying

$$\begin{cases} \frac{d^2}{dr^2} z_{0,k} + \frac{1}{r}\frac{d}{dr} z_{0,k} + r^{2\alpha} \tilde{h}_k(0) e^{U_k} z_{0,k} = F_{0,k}, \quad 0 < r < \tau/\varepsilon_k, \\ \\ z_{0,k}(0) = \frac{d}{dr} z_{0,k}(0) = 0, \end{cases}$$

where

$$F_{0,k}(r) = -\tilde{h}_k(0)|y|^{2\alpha}e^{U_k}\Big(w^k_{0,\xi}(c_{0,k}+\frac{1}{4}\Delta(\log\tilde{h}_k)(0)|y|^2\varepsilon_k^2 + \frac{1}{4}\varepsilon_k^2|\nabla\log\tilde{h}_k(0)|^2(g_k+r)^2) + (w^k_{\xi,1}\tilde{c}_{1,k})_r\Big),$$

and

$$\frac{1}{4}\varepsilon_k^2|\nabla\log\tilde{h}_k(0)|^2(g_k+r)^2 = \left(\frac{1}{2}(\tilde{c}_{1,k})^2\right)_r.$$

Observe carefully that we already considered this function, see (4.35), but in that case the underlying assumption was $b_0^k \to b_0 \neq 0$. Concerning $z_{0,k}$ we have,

$$|z_{0,k}(r)| \leq C(\delta)b_0^k\varepsilon_k^2(1+r)^\delta,$$

and then, by defining $\underline{z}_k$ as follows,

$$\underline{z}_k(y) = \xi_k(\varepsilon_k y) - \psi^k_\xi(\varepsilon_k y) - w^k_{0,\xi}(y) - w^k_{1,\xi}(y) - w^k_{2,\xi}(y) - z_{0,k}(y),$$

we have

$$\Delta\underline{z}_k + |y|^{2\alpha}h_k(\varepsilon_k y)\varepsilon_k^{2+2\alpha}\tilde{c}_k(\varepsilon_k y)\underline{z}_k = E_{k,1}$$

with $z_k(0) = 0$ and

$$|E_{k,1}| \leq Cb_0^k(\varepsilon_k^{2+\epsilon_0}+\bar{\varepsilon}_k)(1+|y|)^{-2-\frac{\epsilon_0}{2}},$$

for some $\epsilon_0 > 0$. By standard potential estimates we have,

$$|\underline{z}_k(y)| \leq C(\delta)(b_0^k\varepsilon_k^{2+\epsilon_0}+\bar{\varepsilon}_k)(1+|y|)^\delta.$$

At this point, by using the estimate about $\underline{z}_k$, we deduce that,

$$\begin{aligned}\int_{\Omega(p_1,\tau)}\rho_k H c_k\xi_k &= \int_{\Omega_k}\tilde{h}_k(\varepsilon_k y)|y|^{2\alpha}e^{v_k}(\xi_k(\varepsilon_k y) - \psi^k_\xi(\varepsilon_k y))dy \\ &\quad + \int_{\Omega_k}\tilde{h}_k(\varepsilon_k y)|y|^{2\alpha}e^{U_k}\psi^k_\xi(\varepsilon_k y) \\ &= cb_0^k(\Delta\log h_k(0)+O(\varepsilon_k^{\epsilon_0}))\varepsilon_k^2 + O(\bar{\varepsilon}_k),\end{aligned}$$

for some suitable constant c. Here the derivation of the above is similar to the derivation of (4.40) except that we keep track of b_0^k in this estimate. Therefore the integral around p_1 reads,

$$\int_{\Omega(p_1,\tau)}\rho_k H c_k\xi_k = cb_0^k\varepsilon_k^2(\Delta(\log\tilde{h}_k)(0)+O(\varepsilon_k^{\epsilon_0})) + O(\bar{\varepsilon}_k), \quad c \neq 0.$$

From $\int_M \rho_k H c_k\xi_k = 0$ we readily obtain (4.59) by splitting the domain of integration in the three regions near the blow-up points plus the corresponding complement in M. Lemma 4.2 is established. □

With Proposition 4.1, we are able to get an estimate on $\bar{b}_0^k$ and $\hat{b}_0^k$ as well.

Lemma 4.3. *There exists $C>0$ independent by k, such that*

$$|\bar{b}_0^k|+|\hat{b}_0^k|\le C\bar{\varepsilon}_k\varepsilon_k^{-2}=Ce^{-\frac{\lambda^k}{2}+\frac{\lambda^k}{1+\alpha}}.$$

Proof. In the proof of Proposition 4.1 we already established (4.59) for b_0^k. Also the oscillation of ξ_k far away from blow-up points is of order $O(\bar{\varepsilon}_k)$ (Proposition 4.1). In the expansion of ξ_k around q^k we have

$$\xi_k(x)=-\bar{b}_0^k+O(\bar{\varepsilon}_k),\quad x\in\partial B(q^k,\tau).$$

On $\partial B(p_1^k,\tau)$ we have

$$\xi_k=-b_0^k+O(b_0^k\varepsilon_k^2)=-b_0^k+O(\bar{\varepsilon}_k),$$

and similarly on $\partial B(p_2,\tau)$,

$$\xi_k(x)=\hat{b}_0^k+O(\bar{\varepsilon}_k).$$

Thus $|b_0^k-\hat{b}_0^k|+|b_0^k-\bar{b}_0^k|\le C\bar{\varepsilon}_k$. Lemma 4.3 is established. $\square$

Step three: an improved expansion for $\bar{\xi}_k$.

As a consequence of Proposition 4.1, the fact that $\nabla\log\bar{h}_{k,0}(\bar{q}^k)=O(\bar{\lambda}^k e^{-\bar{\lambda}^k})$ and the expansion of $\bar{v}_k$ in (4.46), we have

$$\Delta\bar{\xi}_k(y)+\bar{h}_{k,0}(\bar{q}^k)e^{U_k}\bar{\xi}_k=O(\bar{\varepsilon}_k^{3/2})(1+|y|)^{-5/2}.$$

Then by standard potential estimates, we get

$$|\bar{\xi}_k(y)-\bar{w}_{0,\xi}^k|\le C(\delta)\bar{\varepsilon}_k(1+|y|)^{\delta}. \tag{4.60}$$

As before, we set $\bar{\psi}_\xi^k$ is set to be a harmonic function which encodes the bounded oscillation of $\bar{\xi}_k(y)-\bar{w}_{0,\xi}^k$ in $B_{\tau/\bar{\varepsilon}_k}$. Using Proposition 4.1, we have

$$|\bar{\psi}_\xi^k(\bar{\varepsilon}_k y)|\le C\bar{\varepsilon}_k^2|y|.$$

Let

$$z_{0,\xi}^k=\bar{\xi}_k-\bar{w}_{0,\xi}^k,$$

we have

$$(\Delta+\bar{h}_{k,0}(\bar{q}^k)e^{U_k})(z_{0,\xi}^k-\bar{\psi}_\xi^k(\bar{\varepsilon}_k y))=O(\bar{\varepsilon}_k^{1+\epsilon_0})(1+|y|)^{-\frac{5}{2}}\quad\text{in}\quad\Omega_k \tag{4.61}$$

for some $\epsilon_0>0$. Note that $z_{0,\xi}^k$ has the same value on the boundary, and

$$(z_{0,\xi}^k(y)-\bar{\psi}_\xi^k(\bar{\varepsilon}_k y))\,|_{y=0}=0\quad\text{and}\quad\nabla(z_{0,\xi}^k(y)-\bar{\psi}_\xi^k(\bar{\varepsilon}_k y))\,|_{y=0}=O(\bar{\varepsilon}_k^2).$$

By the standard argument (see [19, Proposition 3.1]) for equation (4.61) we deduce that

$$|z_{0,\xi}^k-\bar{\psi}_\xi^k(\bar{\varepsilon}_k y)|\le C(\delta)\bar{\varepsilon}_k^{1+\epsilon_0}(1+|y|)^{\delta}.\quad|y|<\tau/\bar{\varepsilon}_k, \tag{4.62}$$

for some $\epsilon_0>0$. Then we apply standard elliptic estimates based on (4.61) and (4.62) to obtain,

$$|\nabla(z_{0,\xi}^k-\bar{\psi}_\xi^k(\bar{\varepsilon}_k\cdot))|\le C\bar{\varepsilon}_k^{2+\epsilon_0}\quad\text{on}\quad\partial B_{\frac{\tau}{2\bar{\varepsilon}_k}} \tag{4.63}$$

for some $\varepsilon_0 > 0$. From (4.63) we see that for $r \sim \bar{\varepsilon}_k^{-1}$, the leading term in $\nabla \xi_k$ is $\bar{w}_{0,\xi}^k + c_{0,\xi}^k$:

$$\begin{cases} \partial_1 \bar{\xi}_k = \frac{b_1^k(1+a_k r^2) - 2a_k y_1(b_1^k y_1 + b_2^k y_2)}{(1+a_k r^2)^2} + O(\bar{\varepsilon}_k^{2+\delta_0}), \\ \\ \partial_2 \bar{\xi}_k = \frac{b_2^k(1+a_k r^2) - 2a_k y_2(b_2^k y_2 + b_1^k y_1)}{(1+a_k r^2)^2} + O(\bar{\varepsilon}_k^{2+\delta_0}), \end{cases} \tag{4.64}$$

where $a_k = \frac{\bar{h}_{k,0}(\bar{q}^k)}{8}$ and δ_0 is a small positive constant.

Step four: Evaluation of a simplified Pohozaev identity

In a local coordinates system around q, $0 = x(q)$, we have

$$\Delta u_k + \bar{h}_k e^{u_k} = 0,$$

where

$$\bar{h}_k = \rho_k h(x) e^{-4\pi\alpha G(x,p_1) - 4\pi\beta G(x,p_2) + \chi + f_k}$$

and f_k is any solution satisfying that

$$\Delta f_k = \rho_k e^{\chi} \quad \text{in} \quad B_\tau, \qquad f_k(0) = 0.$$

Here we can choose,

$$\begin{aligned} f_k(x) = & \frac{\rho_k}{3+\alpha+\beta}\Big(R(x,\bar{q}^k) - R(\bar{q}^k,\bar{q}^k)\Big) \\ & + \frac{\rho_k}{3+\alpha+\beta}\Big((1+\alpha)(G(x,p_1) - G(\bar{q}^k,p_1))\Big) \\ & + \frac{\rho_k}{3+\alpha+\beta}\Big((1+\beta)(G(x,p_2) - G(\bar{q}^k,p_2))\Big). \end{aligned} \tag{4.65}$$

Remark 4.4. *In the general case the corresponding definitions around a regular blow-up point q_j would be,*

$$\begin{aligned} f_{k,j}(x) = & \frac{\rho_k}{m + \sum\limits_{j=1}^{m_1} \alpha_j} \sum_{\ell \neq j, \ell = m_1+1}^{m} (G(x,\bar{q}_\ell^k) - G(\bar{q}_j^k,\bar{q}_\ell^k)) \\ & + \frac{\rho_k}{m + \sum\limits_{j=1}^{m_1} \alpha_j} \sum_{i=1}^{m_1} (1+\alpha_i)(G(x,p_i) - G(\bar{q}_j^k,p_i)) \\ & + \frac{\rho_k}{m + \sum\limits_{j=1}^{m_1} \alpha_j} \Big(R(x,\bar{q}_j^k) - R(\bar{q}_j^k,\bar{q}_j^k)\Big) \end{aligned}$$

In the following we shall apply an easier version of Pohozaev identity to verify that the coefficients b_1 and b_2 are zero. We first recall that, for the harmonic function ψ_k which was used to cancel out the boundary oscillation of u_k, by the expression of f_k as in (4.65), we have

$$|\psi_k(\bar{\varepsilon}_k y)| \leq C \bar{\varepsilon}_k \varepsilon_k^2 |y|.$$

Then for $j=1,2$, we use the standard Pohozaev identity for

$$\Delta\bar{\bar{\xi}}_k+\bar{h}_{k,0}e^{\bar{v}_k}\bar{\bar{\xi}}_k=0 \quad \text{in} \quad \Omega_{k,1}=B_{\frac{\tau}{2\bar{\varepsilon}_k}}.$$

Multiplying both sides by $\partial_j\bar{v}_k$ we get

$$\begin{aligned}
&-\int_{\partial\Omega_{k,1}}\partial_j\bar{v}_k\partial_\nu\bar{\bar{\xi}}_k+\int_{\Omega_{k,1}}\partial_j(\nabla\bar{v}_k)\nabla\bar{\xi}_k\\
&=\int_{\partial\Omega_{k,1}}\bar{h}_{k,0}(\bar{q}^k+\bar{\varepsilon}_k y)e^{\bar{v}_k}\nu_j dS-\bar{\varepsilon}_k\int_{\Omega_{k,1}}\partial_j\bar{h}_{k,0}(\bar{q}^k+\bar{\varepsilon}_k y)e^{\bar{v}_k}\\
&\quad-\int_{\Omega_{k,1}}\bar{h}_{k,0}(\bar{q}^k+\bar{\varepsilon}_k y)e^{\bar{v}_k}\partial_j\bar{\bar{\xi}}_k,
\end{aligned}\tag{4.66}$$

where ν_j is the j-th direction of outer normal vector on $\partial\Omega_{k,1}$. By the equation

$$\Delta\bar{v}_k+\bar{h}_{k,0}(\bar{q}^k+\bar{\varepsilon}_k y)e^{\bar{v}_k}=0,$$

we can rewrite the last term on the right hand side of (4.66) as $\int_{\Omega_{k,1}}\Delta\bar{v}_k\partial_j\bar{\bar{\xi}}_k$. Together with the integration by parts we get that

$$\begin{aligned}
&\frac{1}{\bar{\varepsilon}_k}\int_{\partial\Omega_{k,1}}(-\partial_\nu\bar{v}_k\partial_j\bar{\bar{\xi}}_k-\partial_\nu\bar{\bar{\xi}}_k\partial_j\bar{v}_k+(\nabla\bar{v}_k\cdot\nabla\bar{\bar{\xi}}_k)\nu_j)dS\\
&=-\int_{\Omega_{k,1}}\partial_j\bar{h}_{k,0}(\bar{q}^k+\bar{\varepsilon}_k y)e^{\bar{v}_k}\bar{\bar{\xi}}_k+\frac{1}{\bar{\varepsilon}_k}\int_{\partial\Omega_{k,1}}\bar{h}_{k,0}(\bar{q}^k+\bar{\varepsilon}_k y)e^{\bar{v}_k}\nu_j dS.
\end{aligned}\tag{4.67}$$

It is not difficult to see that the last term can be bounded by $\bar{\varepsilon}_k^{1+\delta_0}$. Concerning $\nabla\bar{v}_k$ we use

$$\nabla\bar{v}_k(y)=-(4+O(\lambda^k e^{-\lambda^k}))\left(\frac{y_1}{r^2},\frac{y_2}{r^2}\right), \quad |y|=\tau/(2\bar{\varepsilon}_k).$$

Then (4.67) is reduced to

$$\frac{1}{\bar{\varepsilon}_k}\int_{\partial\Omega_{k,1}}\frac{4}{r}\partial_j\bar{\bar{\xi}}_k=-\int_{\Omega_{k,1}}\partial_j\bar{h}_{k,0}(\bar{q}^k+\bar{\varepsilon}_k y)e^{\bar{v}_k}\bar{\bar{\xi}}_k+O(\bar{\varepsilon}_k^{1+\epsilon_0}).\tag{4.68}$$

To evaluate the left hand side of (4.68) we use (4.64) to deduce that,

$$\frac{1}{\bar{\varepsilon}_k}\int_{\partial\Omega_{k,1}}\frac{4}{r}\partial_j\bar{\bar{\xi}}_k=O(\bar{\varepsilon}_k^{1+\epsilon_0}).$$

To evaluate the right hand side of (4.68), we use,

$$\partial_j\bar{h}_{k,0}(\bar{q}^k+\bar{\varepsilon}_k y)=\partial_j\bar{h}_{k,0}(\bar{q}^k)+\sum_{s=1}^{2}\partial_{js}\bar{h}_{k,0}(\bar{q}^k)\bar{\varepsilon}_k y_s+O(\bar{\varepsilon}_k^2)|y|^2,$$

$$e^{\bar{v}_k}=\frac{1}{(1+a_k|y|^2)^2}(1+O((\log\bar{\varepsilon}_k)^2\bar{\varepsilon}_k^2))$$

and (4.62) for $\bar{\bar{\xi}}_k$. Then by a straightforward evaluation we have,

$$\bar{\varepsilon}_k\begin{pmatrix}\partial_{11}\bar{h}_{k,0}(\bar{q}_2^k) & \partial_{12}\bar{h}_{k,0}(\bar{q}_2^k)\\ \partial_{12}\bar{h}_{k,0}(\bar{q}_2^k) & \partial_{22}\bar{h}_{k,0}(\bar{q}_2^k)\end{pmatrix}\begin{pmatrix}b_1^k\\ b_2^k\end{pmatrix}=O(\bar{\varepsilon}_k^{1+\epsilon_0}).$$

Since the underlying assumption is $(b_1^k, b_2^k) \to (b_1, b_2) \neq (0,0)$ and since the non degeneracy condition is $\det(D^2\bar{h}_k(0)) \neq 0$, then we readily obtain a contradiction. Up to now, we have proved Theorem 1.1 under the assumption that $\alpha > 1$ and three different type of blow-up points consisting of one positive singular source, one negative singular source and one regular point. When $m_1 = m$, the situation is easier and we omit the details to avoid repetitions. The more general case can be addressed with minor adjustments, primarily involving changes in notation, which completes the proof of Theorem 1.1 as far as $\alpha > 1$. While for $\alpha \in (0,1)$, we observe that

$$\varepsilon_k^2 = e^{-\lambda^k/(1+\alpha)} = o(e^{-\lambda^k/2}),$$

then the proof of $b_1 = b_2 = 0$ around each regular blow-up point follows as in [7]. While the general case can be argued similarly. Hence we finish the whole proof. □

Proof of Theorem 1.1. Once we have shown that all the coefficients b_0, b_1 and b_2 are zero, then we can follow the arguments in [7] to show that $\xi_k \equiv 0$. Hence we finish the proof. □

Remark 4.5. *The proof of Theorem 1.2 follows by a similar approach. The key difference is that if $L(\mathbf{p}) = 0$, we use $D(\mathbf{p}) \neq 0$ to show that the projection on the radial kernel vanishes. If all blow-up points are negative singular points, there is no projection on the translation kernels, allowing us to omit the assumption* $\det(D^2 f^*(p_{m_1+1}, \cdots, p_m)) \neq 0$. *The proof of the non degeneracy result for the Dirichlet problem* (1.7) *(Theorems 1.3 and 1.4) closely mirrors the process used for the case on Riemann surfaces, so we omit the details.*

5. Mean field theory of the vortex model with singular sources

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class C^2, we define the mean field theory ([25]) of the vortex model with singular sources. Let $\mathcal{P}(\Omega)$ be the space of vorticity densities,

$$\mathcal{P}(\Omega) = \left\{ \omega \in L^1(\Omega) \mid \omega \geq 0 \; a.e., \int_\Omega \omega \mathrm{d}x = 1, \int_\Omega \omega \log(\omega) < +\infty \right\}, \tag{5.1}$$

by definition the energy of a fixed $\omega \in \mathcal{P}(\Omega)$ is,

$$\mathcal{E}(\omega) = \frac{1}{2}\int_\Omega \omega(x) G[\omega](x) \mathrm{d}x, \quad \text{where} \quad G[\omega](x) = \int_\Omega G(x,y)\omega(y)\mathrm{d}y.$$

Let us denote by $\psi = G[\omega]$ the stream function, then $\psi \in W_0^{1,q}(\Omega)$, $q \in [1,2)$ is the unique solution ([72]) of

$$\begin{cases} -\Delta\psi = \omega & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Actually, since $\int_\Omega \omega \log(\omega)$ is bounded, it is well known ([73]) that $\psi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and the energy of the density ω is readily seen to be the Dirichlet energy of ψ:

$$\frac{1}{2}\int_\Omega |\nabla \psi|^2 \mathrm{d}x = \frac{1}{2}\int_\Omega (-\Delta\psi)\psi \mathrm{d}x = \frac{1}{2}\int_\Omega \omega G[\omega]\mathrm{d}x = \mathcal{E}(\omega).$$

Let us define

$$H(x) = e^{-4\pi\beta G_\Omega(x,0)} \prod_{j=1}^N e^{-4\pi\alpha_j G_\Omega(x,q_j)} = |x|^{2\beta} h_0(x),$$

where,

$$h_0(x) = e^{-4\pi\beta R_\Omega(x,0)} \prod_{j=1}^N e^{-4\pi\alpha_j G_\Omega(x,q_j)},$$

and we assume that,

$$x = 0 \in \Omega, \quad \beta \in (-1,0), \quad \alpha_j \in (0,+\infty),\, j \in \{1,\cdots,N\}. \tag{5.3}$$

Closely related to the mean field theory we have the so called Mean Field Equation,

$$\begin{cases} \Delta \mathrm{w} + \rho \dfrac{He^{\mathrm{w}}}{\int_\Omega He^{\mathrm{w}}} = 0 & \text{in } \Omega \\ \\ \mathrm{w} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4}$$

which is just the Euler-Lagrange equation relative to the natural variational principles for the vortex model, see [24], [25] for the regular case and the discussion below for the case with sinks. Here

$$\omega_\rho = \frac{He^{\mathrm{w}_\rho}}{\int_\Omega He^{\mathrm{w}_\rho}},$$

is the vorticity density corresponding to a solution $\mathrm{w}_\rho = \rho G_\Omega[\omega_\rho]$. The singularity at the origin describes a vortex of total vorticity $4\pi\rho^{-1}|\beta|$, whence co-rotating with the flow, i.e. of the same sign of ω_ρ. The remaining N singularities describe counter-rotating vortices of total vorticity $-4\pi\rho^{-1}\alpha_j$ each. See Remark 5.1 below for more details about this point.

A subtle problem in the analysis of (5.4) is the existence/non existence of solutions for $\rho = 8\pi(1+\beta)$, which is critical with respect to the singular ([1]) Moser-Trudinger ([65]) inequality, see Theorem 5.1 and Section 6 below. Interestingly enough, this problem is strictly related to that of the description of the thermodynamic equilibrium in the mean field theory of the vortex model and it has been solved in the regular case, see [24, 25], [28], [12], [15] showing that the geometry of the domain plays a crucial role. See [11] for the case of positive singular sources. In the regular case or either if the singular sources are positive, a crucial point for the characterization of the existence/non existence of solutions in terms of the geometry of Ω is the fact that the critical threshold is 8π and that solutions blowing-up as $\rho \to 8\pi$ make 1-point blow-up where the concentration point is a maximum point of the Robin function of Ω, see [24], [25],[28], [12], [11]. Both these features are drastically modified when adding negative singular sources. In particular, in the model case which we pursue here, where we have only one negative singular source with strength $\beta \in (-1,0)$, the critical threshold is $8\pi(1+\beta)$ and solutions blowing-up as $\rho \to 8\pi(1+\beta)$ make one point blow-up, but the concentration point is the singular source, which in principle need not be located at a maximum point of the Robin function. Therefore our results about mean field theory with sinks are two-fold. Indeed, on one side we extend the results about equivalence of the statistical ensembles and about the non concavity of the entropy ([25], [7]) on the other side we extend the characterization of the existence/non existence of solutions ([28], [12], [11]) of (5.4) to the case $\rho = 8\pi(1+\beta)$. Compared to [28], [12], [11], among other things we have to prove that entropy maximizers have to concentrate on the singular source in the limit of large energies. We will not go into details of those proof which can be worked out by a standard adaptation of known arguments in [25], [28] and [7]. On the other side, the characterization (Theorem 6.2) of the so called pairs (Ω, β) of first/second kind (see Definition 6.1) is more subtle. Another point which requires some care arises in the proof of the existence of negative specific heat states (Theorem 6.5) where we also use some recent results in [44].

5.1. **The Canonical Variational Principle.** To simplify the exposition, with an abuse of terminology (see Remark 5.1) we define the Free Energy functional as follows,

$$\mathcal{F}_\rho(\omega) = \int_\Omega \omega(x)(\log \omega(x) - \log H(x))\mathrm{d}x - \frac{\rho}{2}\int_\Omega \omega(x) G[\omega](x)\mathrm{d}x,$$

where $\omega \in \mathcal{P}(\Omega)$, and since we are interested in negative temperature states ([25, 67]), modulo some minor necessary exceptions, we will consider only the case $\rho > 0$. Indeed, in the statistical mechanics formulation ([24, 25]) we would have $-\frac{1}{\kappa T_{\mathrm{stat}}} = \rho$ where T_{stat} is the statistical Temperature and κ the Boltzmann constant.
Due to the inequality $ab \leq e^a + b(\log(b))$, $a \geq 0, b \geq 1$, it is readily seen that $\mathcal{F}_\rho(\omega)$ is well defined on $\mathcal{P}(\Omega)$. The thermodynamic equilibrium states are by definition those vorticity densities ω which solve the Canonical Variational Principle ((**CVP**) for short),

$$f(\rho) = \inf\{\mathcal{F}_\rho(\omega) \mid \omega \in \mathcal{P}(\Omega)\} \qquad (\mathbf{CVP})$$

Remark 5.1. *According to standard conventions* ([25]) *the physical free energy functional should be $-\rho^{-1}\mathcal{F}_\rho(\omega)$ and the free energy thus would take the form $-\rho^{-1}f(\rho)$. As mentioned above, $\rho = -\frac{1}{\kappa T_{\mathrm{stat}}}$ where κ is the Boltzmann constant and T_{stat} the statistical temperature.*

Actually, in this mean field model, the flow of positive vorticity ω is interacting with $N+1$ fixed vortices whose total vorticities are $4\pi|\beta|/\rho$, $-4\pi\alpha_j/\rho$ respectively. This particular formulation has the advantage that the Euler-Lagrange equations take the form (1.7)*, whose solutions are widely used to describe vortex equilibria of stationary Euler equations in vorticity form, see* [38, 81] *and references quoted therein.*

However, the more realistic model where the vorticities of the singular sources are fixed (i.e. they do not depend by ρ) should be defined via the standard entropy $-\int_{\Omega}\omega(x)\log\omega(x)\mathrm{d}x$, while the energy should contain the contribution due to the fixed vortices. In other words, we should use the same $\mathcal{F}_\rho(\omega)$ as above where $\int_{\Omega}\omega\log(\omega/H)$ should be replaced by $\int_{\Omega}\omega\log\omega$ while the energy part would be $\frac{\rho}{2}\int_{\Omega}\omega G[\omega]-\rho\int_{\Omega}\omega\log(H)$. This formulation in turn would yield an Euler-Lagrange equation as (1.7) *where β and α_j would be replaced by $\rho\beta$ and $\rho\alpha_j$. On the other side, as far as the blow-up at singular sources is concerned, some care is needed in the generalization to these equations of known results, such as concentration-compactness-quantization* ([16],[17]) *and in particular pointwise estimates* ([18]), ([90])*, local uniqueness* ([9],[19],[87]) *and asymptotic non degeneracy (see Theorem 1.4 above) which will be discussed elsewhere* ([4]).

We will often use the following consequences of recent results in [10] and [3].

Lemma 5.1. *Let w_0 be a solution of* (5.4) *for some $\rho=\rho_0$ and assume that the corresponding linearized equation admits only the trivial solution. For any $p\in(1,|\beta|^{-1})$ there exist open neighborhoods $I_0\subset\mathbb{R}$ of ρ_0 and $B_0\subset W_0^{2,p}(\Omega)$ of w_0 such that solutions of* (5.4) *in $I_0\times B_0$ describe a real analytic curve $\mathcal{G}_0$ of the form $I_0\ni\rho\mapsto\mathrm{w}_\rho\in B_0$.*

Proof. Based on the real analytic implicit function theorem ([23]), this is a straightforward generalization of Lemma 2.4 in [3]. □

Lemma 5.2.
(i) ([10]) *For any $\rho\in[0,8\pi(1+\beta))$ there exists a unique solution w_ρ of* (5.4)*;*
(ii)([10]) *For any $\rho\in[0,8\pi(1+\beta))$, the linearized operator of* (5.4) *evaluated at its unique solution has strictly positive first eigenvalue;*
(iii) *If a solution of* (5.4) *exists for $\rho=8\pi(1+\beta)$ then it is unique and the linearized operator evaluated at this solution has strictly positive first eigenvalue;*
(iv) *The map $[0,8\pi(1+\beta))\ni\rho\mapsto\mathrm{w}_\rho\in W_0^{2,p}(\Omega)$, $p\in(1,|\beta|^{-1})$ is real analytic;*
(v) *For any $\rho\in[0,8\pi(1+\beta))$ the map $\rho\mapsto E(\rho)$, where*

$$\rho\mapsto E(\rho)=\mathcal{E}(\omega_\rho),\quad \omega_\rho=\frac{He^{\mathrm{w}_\rho}}{\int_{\Omega}He^{\mathrm{w}_\rho}},$$

is real analytic and $\frac{dE(\rho)}{d\rho}>0$. In particular $\omega_0(x)=\frac{H(x)}{\int_{\Omega}H}$;
(vi) *If a solution of* (5.4) *exists for $\rho=8\pi(1+\beta)$ then the curve of unique solutions for $\rho\in[0,8\pi(1+\beta)]$ can be continued as a real analytic curve in a right neighborhood of $8\pi(1+\beta)$.*

Proof. By definition the map in (iv) is said to be real analytic at $\rho = 0$ if it can be extended to a real analytic map in a neighborhood of $\rho = 0$. The claim in (i), (ii) and (iii) have been recently proved in [10]. Based on (i), (ii), (iii) and Lemma 5.1, the proof of (iv), (v) and (vi) follow from a rather standard generalization of recent results in [3]. $\square$

Remark 5.2. *Actually, due to the strict convexity of the corresponding variational functional, the uniqueness of the solution of* (5.4) *holds as well for fixed $\rho < 0$, in which case the positivity of the first eigenvalue of the linearized problem relative to* (5.4) *is trivial. As a consequence, in view of Lemma 5.2, it is not too difficult to see that the map $\rho \mapsto E(\rho)$, is well defined and smooth and $\frac{dE(\rho)}{d\rho} > 0$ for any $\rho \in (-\infty, 8\pi(1+\beta))$.*

Before stating the first theorem of this section, we need the following lemma, which is well known in the regular case, see [37, 85]. Let us define,

$$J_\rho(\mathrm{w}) = \frac{1}{2\rho}\int_\Omega |\nabla \mathrm{w}|^2 \mathrm{d}x - \log\left(\int_\Omega He^{\mathrm{w}}\right), \quad \mathrm{w} \in W_0^{1,2}(\Omega),$$

for $\rho \in (0, +\infty)$.

Lemma 5.3. *For any $\rho > 0$ we have*

$$f(\rho) = \inf\{\mathcal{F}_\rho(\omega) \mid \omega \in \mathcal{P}(\Omega)\} = \inf\{J_\rho(\mathrm{w}) \mid \mathrm{w} \in W_0^{1,2}(\Omega)\}.$$

In particular we have:
(a) *For any $\omega \in \mathcal{P}(\Omega)$ let $\mathrm{w} = \rho G[\omega]$. Then $\mathrm{w} \in W_0^{1,2}(\Omega)$ and*

$$J_\rho(\mathrm{w}) \le \mathcal{F}_\rho(\omega),$$

where the equality holds if and only if w *is a solution of* (5.4)*;*
(b) *For any $\mathrm{w} \in W_0^{1,2}(\Omega)$ let $\omega = \frac{He^{\mathrm{w}}}{\int_\Omega He^{\mathrm{w}}}$. Then $\omega \in \mathcal{P}(\Omega)$ and*

$$J_\rho(\mathrm{w}) \ge \mathcal{F}_\rho(\omega),$$

where the equality holds if and only if w *is a solution of* (5.4).

Proof. We suppress the x dependence for the sake of simplicity. Obviously the first claim follows from (a) and (b).
(a) Let $\mathrm{w} = \rho G[\omega]$, we have ([73]) $\mathrm{w} \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and consequently

$$\int_\Omega He^{\mathrm{w}} < +\infty \quad \text{and} \quad \omega_{\mathrm{w}} = \frac{He^{\mathrm{w}}}{\int_\Omega He^{\mathrm{w}}} \in \mathcal{P}(\Omega).$$

Thus it is easy to check that,

$$J_\rho(\mathrm{w}) = \int_\Omega \omega\,(\log\omega_{\mathrm{w}} - \log H) - \frac{1}{2}\int_\Omega \omega \mathrm{w},$$

implying that,

$$\mathcal{F}_\rho(\omega) - J_\rho(\mathrm{w}) = \int_\Omega \omega\,(\log\omega - \log\omega_{\mathrm{w}}) \ge 0$$

where the inequality follows from the Jensen inequality

$$\int_{\Omega} s(f(x))\omega_{\mathrm{w}}(x)\mathrm{d}x \geq s\left(\int_{\Omega} f(x)\omega_{\mathrm{w}}(x)\mathrm{d}x\right),$$

where

$$s(t) = t\log(t),\ t \geq 0 \quad \text{and} \quad f(x) = \frac{\omega(x)}{\omega_{\mathrm{w}}(x)}.$$

Remark that, since $s(t)$ is strictly convex, the equality holds if and only if $f(x)$ is constant a.e., that is, since $\{\omega, \omega_{\mathrm{w}}\} \subset \mathcal{P}(\Omega)$, if and only if $\omega = \omega_{\mathrm{w}}$ a.e. in Ω. However this is the same as to say that $\mathrm{w} = \rho G[\omega_{\mathrm{w}}]$, that is, by standard elliptic estimates, that w is a solution of (5.4).
(b) Let $\omega = \frac{He^{\mathrm{w}}}{\int_{\Omega} He^{\mathrm{w}}}$, by the Moser-Trudinger inequality ([65]) and the Hölder inequality $He^{\mathrm{w}}\mathrm{d}x \in L^p(\Omega)$ for any $p \in [1, |\beta|^{-1})$ whence ω is well defined. Observe that, putting $c_{\mathrm{w}} = \log(\int_{\Omega} He^{\mathrm{w}})$, we have

$$\int_{\Omega} \omega \log \omega = -c_{\mathrm{w}} + \int_{\Omega} \omega(\mathrm{w} + \log H),$$

where by the Sobolev embedding $\mathrm{w} + \log H \in L^q(\Omega)$ for any $q \in [1, +\infty)$, implying that $\omega \in \mathcal{P}(\Omega)$. Let $\mathrm{w}_{\omega} = \rho G[\omega]$, then we have ([73])

$$\int_{\Omega} |\nabla \mathrm{w}_{\omega}|^2 = \rho \int_{\Omega} \omega \mathrm{w}_{\omega}, \quad \int_{\Omega} \nabla \mathrm{w}_{\omega} \cdot \nabla \mathrm{w} = \rho \int_{\Omega} \omega \mathrm{w}.$$

Therefore we deduce that,

$$J_{\rho}(\mathrm{w}) - \mathcal{F}_{\rho}(\omega) = \frac{1}{2\rho} \int_{\Omega} |\nabla(\mathrm{w} - \mathrm{w}_{\omega})|^2 \geq 0$$

where, since $\{\mathrm{w}, \mathrm{w}_{\omega}\} \in W_0^{1,2}(\Omega)$, the equality holds if and only if $\mathrm{w} = \mathrm{w}_{\omega}$ a.e., that is, by standard elliptic estimates, if and only if w is a solution of (5.4). $\square$

Now we are able to state the first result about the (**CVP**), which generalizes the corresponding results in [25] and [28].

Theorem 5.1. *For any $\rho \in (0, 8\pi(1+\beta))$ the* (**CVP**) *admits a unique minimizer ω_{ρ}. In particular $\mathrm{w}_{\rho} = \rho G[\omega_{\rho}]$ is the unique solution of* (5.4).

Proof. The uniqueness part is readily seen to be a straightforward consequence of Lemma 5.2-(i), whence we are just left to prove the existence part of the claim. By the Moser-Trudinger inequality ([65]) and the Hölder inequality, J_{ρ} is well defined. Furthermore, for any fixed $\rho \in [0, 8\pi(1+\beta))$, by the singular ([1]) Moser-Trudinger inequality, J_{ρ} is coercive and weakly lower semi-continuous on $W_0^{1,2}(\Omega)$. By the direct method we get that it admits a minimizer. The corresponding Euler-Lagrange equation is just (5.4) and the conclusion follows by Lemma 5.3. $\square$

To proceed further we restrict our attention to the physically meaningful situation where the singular source $x = 0 \in \Omega$ satisfies (1.16), that is, setting,

$$\gamma^*(x) = 8\pi(1+\beta)R_\Omega(x,0) + \log(h_0(x)), \quad x \in \Omega,$$

we assume that

$$\nabla\gamma^*(0) = 0. \tag{5.5}$$

With the same notations as in (2.29) let us set,

$$\Phi_{\Omega,0}(x) = 8\pi(1+\beta)(G_\Omega(x,0) - R_\Omega(0,0)) + \log(h_0(x)/h_0(0)) + 2\beta\log|x|,$$

which satisfies

$$e^{\Phi_{\Omega,0}(x)} = \frac{|x|^{2\beta}}{|x|^{4+4\beta}}e^{\gamma^*(x)-\gamma^*(0)}.$$

Then, in view of (5.5) the following quantity, is well defined,

$$D_\beta = \lim_{r\to 0} c_*\left(\int_{\Omega\setminus B_r} e^{\Phi_{\Omega,0}(x)} - \frac{\pi}{(1+\beta)}\frac{1}{r^{2(1+\beta)}}\right), \tag{5.6}$$

and we remark that D_β was already introduced in (2.28) above. Next, to simplify the evaluations, let us set,

$$c_\beta = \frac{h_0(0)}{8(1+\beta)^2}, \quad c_* = e^{\gamma^*(0)} = h_0(0)e^{8\pi(1+\beta)R_\Omega(0,0)},$$

and choose any $\tau > 0$ such that $B_\tau \Subset \Omega$. Then we define,

$$\mathrm{w}_\varepsilon(x) = -4(1+\beta)\log(\tau) + 2\log\left(\frac{\varepsilon^{2(1+\beta)} + c_\beta}{\varepsilon^{2(1+\beta)} + c_\beta(\tau^{-1}|x|)^{2(1+\beta)}}\right), \quad x \in \overline{B_\tau},$$

and

$$\widetilde{\mathrm{w}}_\varepsilon(x) = \begin{cases} -4(1+\beta)\log|x| + 8\pi(1+\beta)R_\Omega(x,0), & x \in \Omega\setminus B_\tau, \\ \\ \mathrm{w}_\varepsilon(x) + 8\pi(1+\beta)R_\Omega(x,0), & x \in \overline{B_\tau}, \end{cases}$$

so that in particular $\widetilde{\mathrm{w}}_\varepsilon \equiv 8\pi(1+\beta)G_\Omega(x,0)$ in $\Omega\setminus B_\tau$. At this point, we are ready to deduce a generalization of former results in [25], [28], [11], [12], which in particular provides an estimate for $f(8\pi(1+\beta))$ in terms of D_β.

Lemma 5.4. *Assume* (5.5) *and set*

$$\gamma_\Omega(x) = 4\pi(1+\beta)R_\Omega(x,x) + \log(h_0(x)),$$

then we have,

$$\begin{aligned} J_{8\pi(1+\beta)}(\widetilde{\mathrm{w}}_\varepsilon) = &-1 - \log\left(\frac{\pi}{1+\beta}\right) - \gamma_\Omega(0) \\ &- (\tau\varepsilon)^{2(1+\beta)}\frac{1+\beta}{\pi c_* c_\beta}\left(D_\beta + o_\varepsilon(1) + O(\tau^{-2\beta})\right), \end{aligned}$$

as $\varepsilon \to 0$, where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$, uniformly with respect to τ. Then we have

$$f(8\pi(1+\beta)) \leq -1 - \log\left(\frac{\pi}{1+\beta}\right) - \gamma_\Omega(0), \tag{5.7}$$

where the inequality is strict as far as $D_\beta > 0$.

Proof.

$$\begin{aligned}
\frac{1}{2}\int_\Omega |\nabla \widetilde{\mathrm{w}}_\varepsilon|^2 = & \frac{(1+\beta)^2}{2}\int_{\Omega\setminus B_\tau} \nabla\left(\log\frac{1}{|x|^4} + 8\pi R_\Omega(x,0)\right)\cdot\nabla\log\frac{1}{|x|^4} \\
& + 4\pi(1+\beta)^2\left(\int_{\Omega\setminus B_\tau}\nabla\log\frac{1}{|x|^4}\cdot\nabla R_\Omega(x,0) + 8\pi\int_\Omega|\nabla R_\Omega(x,0)|^2\right) \\
& + \frac{1}{2}\int_{B_\tau}\left|\frac{c_\beta\tau^{-2(1+\beta)}4(1+\beta)|x|^{1+2\beta}}{\varepsilon^{2(1+\beta)} + c_\beta(\tau^{-1}|x|)^{2(1+\beta)}}\right|^2 + 8\pi(1+\beta)\int_{B_\tau}\nabla \mathrm{w}_\varepsilon\cdot\nabla R_\Omega(x,0) \\
= & \frac{-(1+\beta)^2}{2}\int_{\partial B_\tau}\left(\log\frac{1}{|x|^4} + 8\pi R_\Omega(x,0)\right)\frac{\partial}{\partial\nu}\left(\log\frac{1}{|x|^4}\right)d\sigma \\
& - 4\pi(1+\beta)^2\int_{\partial B_\tau}\left(\log\frac{1}{|x|^4}\right)\frac{\partial R_\Omega}{\partial\nu}(x,0)d\sigma - 16\pi(1+\beta)^2\log(\varepsilon) \\
& - 8\pi(1+\beta)(1-\log(c_\beta)) + 16\pi(1+\beta)c_\beta^{-1}\varepsilon^{2(1+\beta)} + O(\varepsilon^{4(1+\beta)}) \\
= & -16\pi(1+\beta)^2\log(\tau) + 32\pi^2(1+\beta)^2R_\Omega(0,0) - 16\pi(1+\beta)^2\log(\varepsilon) \\
& - 8\pi(1+\beta)(1-\log(c_\beta)) + 16\pi(1+\beta)c_\beta^{-1}\varepsilon^{2(1+\beta)} + O(\varepsilon^{4(1+\beta)}),
\end{aligned}$$

as $\varepsilon \to 0$, where we used:

$$\left(\int_{\Omega\setminus B_\tau}\nabla\log\frac{1}{|x|}\cdot\nabla R_\Omega(x,0) + 2\pi\int_\Omega|\nabla R_\Omega(x,0)|^2\right) = \int_{\partial B_\tau}(\log|x|)\frac{\partial R_\Omega}{\partial\nu}(x,0)d\sigma = 0,$$

$$\begin{aligned}
\frac{1}{2}\int_{B_\tau}\left|\frac{c_\beta\tau^{-2(1+\beta)}4(1+\beta)|x|^{1+2\beta}}{\varepsilon^{2(1+\beta)} + c_\beta(\tau^{-1}|x|)^{2(1+\beta)}}\right|^2 = & -16\pi(1+\beta)^2\log(\varepsilon) - 8\pi(1+\beta)(1-\log(c_\beta)) \\
& + 16\pi(1+\beta)c_\beta^{-1}\varepsilon^{2(1+\beta)} + O(\varepsilon^{4(1+\beta)}),
\end{aligned}$$

as $\varepsilon \to 0$, and

$$\int_{B_\tau}\nabla \mathrm{w}_\varepsilon\cdot\nabla R_\Omega(x,0) = \int_{\partial B_\tau}\mathrm{w}_\varepsilon\frac{\partial}{\partial\nu}R_\Omega(x,0)d\sigma = 0;$$

$$\int_{\partial B_\tau}8\pi R_\Omega(x,0)\frac{\partial}{\partial\nu}\log\frac{1}{|x|^4} = -\frac{32\pi}{\tau}\int_{\partial B_\tau}R_\Omega(x,0)d\sigma = -64\pi^2R_\Omega(0,0).$$

Next observe that, recalling the definitions of c_* and $\Phi_{\Omega,0}$ above, we have

$$\int_{\Omega\setminus B_\tau}|x|^{2\beta}h_0(x)e^{\widetilde{\mathrm{w}}_\varepsilon} = c_*\int_{\Omega\setminus B_\tau}e^{\Phi_{\Omega,0}(x)} = c_*\int_{\Omega\setminus B_\varepsilon}e^{\Phi_{\Omega,0}(x)} - c_*\int_{B_\tau\setminus B_\varepsilon}e^{\Phi_{\Omega,0}(x)}$$

$$= c_* \int_{\Omega\setminus B_\varepsilon} e^{\Phi_{\Omega,0}(x)} + \frac{c_*\pi}{(1+\beta)}\left(\frac{1}{\tau^{2(1+\beta)}} - \frac{1}{\varepsilon^{2(1+\beta)}}\right) + O(\tau^{-2\beta})$$

where we used

$$\begin{aligned} c_* \int_{B_\tau\setminus B_\varepsilon} e^{\Phi_{\Omega,0}(x)} &= \int_{B_\tau\setminus B_\varepsilon} \frac{|x|^{2\beta}}{|x|^{4(1+\beta)}} h_0(x) e^{8\pi(1+\beta)R_\Omega(x,0)} \\ &= \frac{c_*\pi}{(1+\beta)}\left(\frac{1}{\varepsilon^{2(1+\beta)}} - \frac{1}{\tau^{2(1+\beta)}}\right) + O(\tau^{-2\beta}). \end{aligned}$$

Moreover we have,

$$\begin{aligned} \int_{B_\tau} |x|^{2\beta} h_0(x) e^{\widetilde{\mathrm{w}}_\varepsilon} = \int_{B_\tau} |x|^{2\beta} e^{\gamma^*(x)} e^{\mathrm{w}_\varepsilon} &= \int_{B_\tau\setminus B_\varepsilon} |x|^{2\beta} e^{\gamma^*(x)} e^{\mathrm{w}_\varepsilon} + \int_{B_\varepsilon} |x|^{2\beta} e^{\gamma^*(x)} e^{\mathrm{w}_\varepsilon} \\ &= \frac{\pi c_*}{(1+\beta)} \frac{1}{(\varepsilon\tau)^{2(1+\beta)}} \left(c_\beta + \varepsilon^{2(1+\beta)} + O(\tau^{-2\beta})\right), \end{aligned}$$

where, putting

$$\tau_\beta = (c_\beta)^{-\frac{1}{2(1+\beta)}}\tau,$$

we used

$$\begin{aligned} \int_{B_\varepsilon} |x|^{2\beta} e^{\gamma^*(x)} e^{\mathrm{w}_\varepsilon} &= \tau^{-4(1+\beta)} \int_{B_\varepsilon} |x|^{2\beta} e^{\gamma^*(x)} \left(\frac{\varepsilon^{2(1+\beta)} + c_\beta}{\varepsilon^{2(1+\beta)} + c_\beta(\tau^{-1}|x|)^{2(1+\beta)}}\right)^2 \\ &= \frac{\left(c_\beta + \varepsilon^{2(1+\beta)}\right)^2}{c_\beta(\varepsilon\tau)^{2(1+\beta)}} \int_{B_{\tau_\beta^{-1}}} \frac{|z|^{2\beta}}{(1+|z|^{2(1+\beta)})^2} e^{\gamma^*(\varepsilon\tau_\beta z)} \\ &= \frac{\left(c_\beta + \varepsilon^{2(1+\beta)}\right)^2}{c_\beta(\varepsilon\tau)^{2(1+\beta)}} \frac{\pi c_*}{(1+\beta)} \left(\frac{1}{1+\tau_\beta^{2(1+\beta)}} + O((\varepsilon\tau)^2)\right), \end{aligned}$$

and similarly

$$\begin{aligned} &\int_{B_\tau\setminus B_\varepsilon} |x|^{2\beta} e^{\gamma^*(x)} e^{\mathrm{w}_\varepsilon} \\ &= \frac{\left(c_\beta + \varepsilon^{2(1+\beta)}\right)^2}{c_\beta(\varepsilon\tau)^{2(1+\beta)}} \frac{\pi c_*}{(1+\beta)} \left(\frac{c_\beta}{c_\beta + \varepsilon^{2(1+\beta)}} - \frac{1}{1+\tau_\beta^{2(1+\beta)}} + O(\tau^2\varepsilon^{2(1+\beta)})\right). \end{aligned}$$

Therefore we have

$$\begin{aligned} &\int_\Omega |x|^{2\beta} h_0(x) e^{\widetilde{\mathrm{w}}_\varepsilon} = c_* \int_{\Omega\setminus B_\tau} e^{\Phi_{\Omega,0}(x)} + \int_{B_\tau} |x|^{2\beta} h_0(x) e^{\mathrm{w}_\varepsilon} \\ &= c_* \left(\int_{\Omega\setminus B_\varepsilon} e^{\Phi_{\Omega,0}(x)} - \frac{\pi}{(1+\beta)} \frac{1}{\varepsilon^{2(1+\beta)}}\right) + \frac{\pi c_*}{(1+\beta)} \frac{1}{\tau^{2(1+\beta)}} \\ &\quad + \frac{\pi c_*}{(1+\beta)} \frac{1}{(\tau\varepsilon)^{2(1+\beta)}} \left(c_\beta + \varepsilon^{2(1+\beta)} + O(\tau^{-2\beta})\right) + O(\tau^{-2\beta}) \end{aligned}$$

$$= \frac{\pi c_*}{(1+\beta)} \frac{1}{(\tau\varepsilon)^{2(1+\beta)}} \left(c_\beta + \left(\frac{(1+\beta)}{\pi c_*} \tau^{2(1+\beta)} (D_\beta + o_\varepsilon(1)) + 2 + O(\tau^{-2\beta}) \right) \varepsilon^{2(1+\beta)} \right),$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$ uniformly with respect to τ.

At this point, putting together these estimates, we see that,

$$\begin{aligned} J_{8\pi(1+\beta)(\tilde{w}_\varepsilon)} = & -2(1+\beta)\log(\tau) - 2(1+\beta)\log(\varepsilon) + 2c_\beta^{-1}\varepsilon^{2(1+\beta)} \\ & + 4\pi(1+\beta)R_\Omega(0,0) - 1 + \log(c_\beta) + O(\varepsilon^{4(1+\beta)}) \\ & - \log\left(\frac{\pi c_* c_\beta}{(1+\beta)} \frac{1}{(\tau\varepsilon)^{2(1+\beta)}} \right) \\ & - \log\left(1 + \left(\frac{(1+\beta)}{\pi c_* c_\beta} \tau^{2(1+\beta)} (D_\beta + o_\varepsilon(1)) + 2c_\beta^{-1} + O(\tau^{-2\beta}) \right) \varepsilon^{2(1+\beta)} \right) \\ = & -1 - 4\pi(1+\beta)R_\Omega(0,0) - \log\left(\frac{\pi h_0(0)}{1+\beta} \right) \\ & - (\tau\varepsilon)^{2(1+\beta)} \frac{1+\beta}{\pi c_* c_\beta} \left(D_\beta + o_\varepsilon(1) + O(\tau^{-2\beta}) \right), \quad \text{as } \varepsilon \to 0 \end{aligned}$$

as claimed. Passing to the limit we obtain the inequality for $f(8\pi(1+\beta))$, where if $D_0 > 0$ then obviously the inequality must be strict. $\square$

5.2. **Microcanonical Variational Principle.** With the definitions above, for $E > 0$ we define the Microcanonical Variational Principle ((**MVP**) for short),

$$S(E) = \sup\{\mathfrak{S}(\omega) \mid \omega \in \mathcal{P}(\Omega),\ \mathcal{E}(\omega) = E\} \qquad (\textbf{MVP})$$

where

$$\mathfrak{S}(\omega) = -\int_\Omega \omega(x)(\log\omega(x) - \log H(x))\mathrm{d}x.$$

As remarked above, due to the inequality $ab \leq e^a + b(\log(b))$, $a \geq 0, b \geq 1$, it is readily seen that $\mathfrak{S}(\omega)$ is well defined on $\mathcal{P}(\Omega)$. Concerning the (**MVP**) we have the following generalization of a result in [25].

Theorem 5.2. *(j) For any $E > 0$, $S(E) < +\infty$ and there exists ω_{E} which solves the* (**MVP**)*, $S(E) = \mathfrak{S}(\omega_{E})$. In particular there exists $\rho_{E} \in \mathbb{R}$ such that $\mathrm{w}_{E} = \rho_{E} G[\omega_{E}]$ is a solution of* (5.4) *for $\rho = \rho_{E}$ and $\omega_0(x) := \frac{H(x)}{\int_\Omega H}$ is the unique entropy maximizer for $E = E_0 := \mathcal{E}(\omega_0)$ and $\rho_{E_0} = 0$.*
(jj) There exists $E_c \in (E_0, +\infty]$ such that for any $E \in [E_0, E_c)$ the map $E \mapsto \rho_{E}$ is smooth and invertible with inverse $E = E(\rho)$, $\rho \in [0, 8\pi(1+\beta))$ and $E(\rho) \nearrow E_c$, as $\rho \nearrow 8\pi(1+\beta)$, $\frac{dE(\rho)}{d\rho} > 0$ in $[0, 8\pi(1+\beta))$.

Proof. (j) Observe that, putting $\mathrm{d}\mu_H = \frac{H(x)\mathrm{d}x}{\int_\Omega H(x)dx}$ and $f(x) = \frac{\omega(x)}{H(x)}$, then by a standard approximation argument and the Jensen inequality for $s(t) =$

$-t\log(t), t \geq 0$, we have,

$$\frac{\mathfrak{S}(\omega)}{\int_\Omega H(x)dx} = \int_\Omega s(f(x))\mathrm{d}\mu_H \leq s\left(\int_\Omega f(x)\mathrm{d}\mu_H\right) = \frac{\log\int_\Omega H(x)dx}{\int_\Omega H(x)dx},$$

where the equality holds if and only if f is constant a.e., that is, if and only if $\omega = \omega_0(x) = \frac{H(x)}{\int_\Omega H}$, which is therefore the unique maximizer of the entropy at $E = E_0 := \mathcal{E}(\omega_0)$. For $E \neq E_0$, let H_n be any sequence of smooth positive functions such that $H_n \to H$ in $L^p(\Omega)$, for some $p \in (1, |\beta|^{-1})$ and define

$$\mathfrak{S}_n(\omega) = -\int_\Omega \omega(x)(\log\omega(x) - \log H_n(x))\mathrm{d}x, \quad \omega \in \mathcal{P}(\Omega).$$

Modulo minor modifications due to the weight H_n, by the same argument adopted in [25] we see that for any $E > 0$, $S_n(E) < +\infty$ and there exists $\omega_{E,n} \in \mathcal{P}(\Omega)$ which maximizes $\mathfrak{S}_n(\omega)$ at fixed energy $\mathcal{E}(\omega_{E,n}) = E$, $S_n(E) = \mathfrak{S}(\omega_{E,n})$. In particular there exists $\rho_{E,n} \in \mathbb{R}$ such that $\mathrm{w}_{E,n} = \rho_{E,n}G[\omega_{E,n}]$ is a solution of (5.4) for $\rho = \rho_{E,n}$ and $H = H_n$. By standard elliptic regularity theory and the Sobolev embedding, for each fixed n, we have $\mathrm{w}_{E,n} \in W_0^{2,p}(\Omega) \cap C^{0,\gamma}(\overline{\Omega})$, for some $\gamma \in (0,1)$. Since Ω is simply connected and of class C^1, with the aid of a conformal map, by a well known trick based on the Pohozaev identity we can find $\overline{\rho}_\Omega$ such that $\rho_{E,n} \leq \overline{\rho}_\Omega$ for any n and E. Therefore, for a fixed E, by the equation (5.4) we readily deduce that $\|\nabla \mathrm{w}_{E,n}\|_2 \leq \overline{\rho}_\Omega E$, implying by the Moser-Trudinger inequality that $e^{\mathrm{w}_{E,n}}$ is uniformly bounded in $L^q(\Omega)$ for any $q \geq 1$ and in particular that $H_n e^{\mathrm{w}_{E,n}}$ is uniformly bounded in $L^t(\Omega)$ for some $t \in (1,p)$. Actually by the Sobolev embedding $\mathrm{w}_{E,n}$ is uniformly bounded in $L^q(\Omega)$ for any $q \geq 1$, implying by the Jensen inequality that $\int_\Omega H_n e^{\mathrm{w}_{E,n}}$ is bounded below away from 0. Therefore the right hand side of (5.4) with $\rho = \rho_{E,n}$, $H = H_n$ and $\mathrm{w} = \mathrm{w}_{E,n}$ is uniformly bounded in $L^t(\Omega)$ and by standard elliptic estimates and the Sobolev embedding $\mathrm{w}_{E,n}$ is uniformly bounded in $\mathrm{w}_{E,n} \in W_0^{2,t}(\Omega) \cap C^{0,\gamma}(\overline{\Omega})$, for some $\gamma \in (0,1)$. At this point it is readily seen that, possibly along a subsequence, as $n \to +\infty$, $\mathrm{w}_{E,n}$ converges uniformly and in $W_0^{1,2}(\Omega)$ to a solution w_E of (5.4) for some $\rho_E \in \mathbb{R}$, with $\mathcal{E}(\omega_E) = E$, where $\omega_E = \dfrac{He^{\mathrm{w}_E}}{\int_\Omega He^{\mathrm{w}_E}}$. Clearly ω_E is a maximizer of the entropy at fixed energy, $S(E) = \mathfrak{S}(\omega_E)$, which, excluding $\rho_{E_0} = 0$, concludes the proof of (j).

(jj) Remark that if $\rho_E < 0$ then J_ρ is strictly convex and consequently (5.4) admits a unique solution, which therefore is just w_E. By Lemma 5.2 and Remark 5.2 solutions of (5.4) are in fact unique and the first eigenvalue of the linearized problem relative to (5.4) is strictly positive as far as $\rho < 8\pi(1+\beta)$. In particular the inverse of $E \mapsto \rho_E$, say $E = E(\rho)$, is well defined and smooth and $\frac{dE(\rho)}{d\rho} > 0$ as far as $\rho \in (-\infty, 8\pi(1+\beta))$. As a consequence $\rho = \rho_E \leq 0$ if and only if $E(\rho) \leq E_0 = \mathcal{E}(\omega_0)$, where $\omega_0 = \frac{H(x)}{\int_\Omega H(x)}$. However

$\omega_{_E}=\dfrac{He^{\mathrm{w}_{_E}}}{\int_\Omega He^{\mathrm{w}_{_E}}}=\omega_0$ if and only if $\mathrm{w}_{_E}\equiv 0$ implying that $\rho_{E_0}=0$. By monotonicity the limit of $E(\rho)$ as $\rho\to 8\pi(1+\beta)^-$ is well defined which we take as defining E_c. $\square$

6. THE EXISTENCE/NON EXISTENCE OF SOLUTIONS FOR $\rho=8\pi(1+\beta)$. DOMAINS OF FIRST/SECOND KIND.

As mentioned above, the existence/non existence of solutions of (5.4) for $\rho=8\pi(1+\beta)$ is a subtle problem, since $J_{8\pi(1+\beta)}$ is bounded below ([1]) but not coercive, implying that in general minimizing sequences need not be compact. We discuss hereafter the case where a negative singular sink is described in the sense defined above.

First of all, as in [25], the critical value of the energy E_c defined in Theorem 5.2 plays a crucial in the rest of the argument.

Definition 6.1. *A pair (Ω,β), where Ω is a simply connected and bounded domain of class C^2, $\beta\in(-1,0)$ and $q=0\in\Omega$ satisfies* (5.5) *is said to be of* **first kind** *if $E_c=E_c(\Omega,\beta)=+\infty$, of* **second kind** *otherwise.*

Any disk centered at the origin, $\Omega=B_r(0)$, with $H(x)=e^{-4\pi\beta G_\Omega(x,0)}=\frac{|x|^{2\beta}}{r^{2\beta}}$ is of first kind for any $\beta\in(-1,0)$. In fact, by dilation invariance we are free to take $r=1$ and deduce that

$$D_0=-\frac{\pi}{1+\beta},$$

$$\mathrm{w}_\rho(x)=2\log\left(\frac{1+\gamma_{_\rho}}{1+\gamma_{_\rho}|x|^{2(1+\beta)}}\right),\quad \gamma_{_\rho}=\frac{\rho}{8\pi(1+\beta)-\rho},\quad |x|\leq 1.$$

Then

$$\omega_\rho(x)=\frac{|x|^{2\beta}e^{\mathrm{w}_\rho}}{\int_\Omega|x|^{2\beta}e^{\mathrm{w}_\rho}}=\frac{(1+\beta)}{\pi}\frac{(1+\gamma_{_\rho})|x|^{2\beta}}{\left(1+\gamma_{_\rho}|x|^{2(1+\beta)}\right)^2},$$

and

$$\mathcal{E}(\omega_\rho)=\frac{1}{\rho}\left(1+\left(1-\frac{1}{\gamma_\rho}\right)\log(1+\gamma_\rho)\right).$$

It is readily seen in particular that $E(\rho)=\mathcal{E}(\omega_\rho)\to+\infty$ as $\rho\nearrow 8\pi(1+\beta)$. Our first result concerning domains of first kind is a generalization of analogue results in the regular case, see Theorem 7.1 in [24] and Proposition 3.3 in [25].

Theorem 6.1. *Let (Ω,β) be of first kind, then we have:*
(i) $f(\rho)$ is smooth, strictly convex and increasing for any $\rho\in(0,8\pi(1+\beta))$. In particular

$$E(\rho)=-\frac{df(\rho)}{d\rho}=\frac{1}{2}\int_\Omega\omega_\rho G[\omega_\rho],\quad \rho\in(0,8\pi(1+\beta)),$$

where ω_ρ denotes the unique solution of the (**CVP**) *and* $\frac{dE(\rho)}{d\rho} > 0$;
(ii) *for* $E \in (E_0, +\infty)$ *we have that,*

$$S(E) = \inf_{\rho \in \mathbb{R}} \{-f(\rho) - \rho E\}$$

is smooth and concave;
(iii) [Equivalence of Statistical Ensembles] *If* $\omega(E)$ *solves the* (**MVP**) *then* $\omega(E(\rho)) = \omega_\rho$, *that is,* $\omega(E)$ *coincides with the unique solution of the* (**CVP**) *for* $\rho = \rho_E$, *where* $\rho_{_E}$ *is the inverse of* $E(\rho)$.
(iv) *As* $\rho \nearrow 8\pi(1+\beta)$,

$$f(\rho) \to f(8\pi(1+\beta)) = -1 - \log\left(\frac{\pi}{1+\beta}\right) - \gamma_\Omega(0),$$

and $\omega_\rho \rightharpoonup \delta_{q=0}$.

Proof. The proof of $(i) - (ii) - (iii)$ goes as in [25], with the exception that the smoothness of $f(\rho)$ as well as $E(\rho) = -\frac{df(\rho)}{d\rho}$ and $\frac{dE(\rho)}{d\rho} > 0$ both follow from Lemma 5.2-(v). Actually, in view of Lemma 5.2-(v), we also deduce that (see section 4 in [3] for details)

$$\frac{dS(E)}{dE} = -\rho_{_E}, \quad \frac{d^2S(E)}{dE^2} = -\frac{d\rho_{_E}}{dE} = -\left(\frac{dE(\rho)}{d\rho}\right)^{-1} < 0, \quad E \in (E_0, +\infty).$$

We are left with the proof of (iv). By definition of domain of first kind, $E(\rho) \to +\infty$ as $\rho \nearrow 8\pi(1+\beta)$. Consequently, according to well known blow-up results ([16]), we have that $\omega_\rho \rightharpoonup \delta_{q=0}$. It is not difficult to prove (see for example Proposition 7.3 in [24]) that $f(\rho)$ is continuous in $[0, 8\pi(1+\beta)]$. Let $\rho_n \nearrow 8\pi(1+\beta)$ and w_n be any sequence of minimizers of J_{ρ_n}, then we have from Lemma 5.3,

$$J_{\rho_n}(\mathrm{w}_n) = f(\rho_n) \to f(8\pi(1+\beta)), \quad \text{as} \quad \rho_n \nearrow 8\pi(1+\beta).$$

In particular, since w_n is a blow-up sequence, by using the pointwise estimates in [18] together with Proposition 2.1 and (2.27) we find that,

$$J_{\rho_n}(\mathrm{w}_n) \to f(8\pi(1+\beta)) = -1 - \log\left(\frac{\pi}{1+\beta}\right) - \gamma_\Omega(0).$$

The evaluation is very similar to that worked out in the proof of Lemma 5.4 where one also uses the mean field equation (5.4), and we omit it to here to avoid repetitions. The proof is completed. $\square$

As an immediate consequence of Theorem 6.1-(iv) and Lemma 5.4 we have,

Corollary 6.1. *Let* (Ω, β) *be of first kind, then* $D_\beta \leq 0$.

To prove the converse of this fact is more subtle, see Theorem 6.2 below.

Theorem 6.1 shows that for pairs (Ω,β) of first kind we have a complete description of the thermodynamic equilibrium, in the sense that for any $E>0$ there exists a unique ω_E which solves the **(MVP)** and there exists a unique $\rho_{\scriptscriptstyle E}$ such that there exists a unique solution of the **(CVP)** at $\rho=\rho_{\scriptscriptstyle E}$ which satisfies $\mathrm{w}_{\scriptscriptstyle E}=\rho_{\scriptscriptstyle E}G[\omega_E]$. In other words the two variational principles describe the same thermodynamic equilibrium state.

For pairs (Ω,β) of second kind the situation is much more involved, since entropy maximizer with $E>E_c$ correspond to values of $\rho=\rho_{\scriptscriptstyle E}>8\pi(1+\beta)$ where the **(CVP)** has no solution (since it can be shown by standard arguments that J_ρ is not bounded below) and moreover solutions of (5.4) are not unique, see Theorem 6.6 below. Concerning this point we generalize first some results in [25]. We refer to [3], [7], [21] for other recent progress in this direction in the regular case.

We will need hereafter a generalization of former results in [28] and [12]. We omit the proof which follows in part from known blow-up arguments ([22, 17, 16]) and otherwise from the invertibility of the linearized operator at $\rho=8\pi(1+\beta)$, see Lemma 5.2-(iii), (vi), and the implicit function theorem (see for example Proposition 6.1 in [28] for a proof in the regular case).

Lemma 6.1. *Let $\rho\in[0,8\pi(1+\beta))$ and w_ρ be the unique solution of* (5.4)*. Then, the following facts are equivalent:*
(-) (Ω,β) is of second kind;
(-) For $\rho=8\pi(1+\beta)$, equation (5.4) *admits a solution denoted by w_c;*
(-) $\mathrm{w}_\rho\to\mathrm{w}_c$ in $C^2(\Omega)\cap C^1(\overline{\Omega})$ as $\rho\nearrow 8\pi(1+\beta)$;
(-) a subsequence $\mathrm{w}_{\rho_n}\to\mathrm{w}_c$ in $C^2(\Omega)\cap C^1(\overline{\Omega})$ as $\rho_n\nearrow 8\pi(1+\beta)$;
(-) $J_{8\pi(1+\beta)}$ admits a minimizer.
Moreover, the following facts are equivalent:
(-) (Ω,β) is of first kind;
(-) $\|\mathrm{w}_\rho\|_\infty\to+\infty$, as $\rho\nearrow 8\pi(1+\beta)$;
(-) $\omega_\rho=\frac{He^{\mathrm{w}_\rho}}{\int_\Omega He^{\mathrm{w}_\rho}}\rightharpoonup\delta_{p=0}$ weakly in the sense of measures;
(-) a subsequence $\|\mathrm{w}_{\rho_n}\|_\infty\to+\infty$, as $\rho_n\nearrow 8\pi(1+\beta)$;
(-) Equation (5.4) *has no solution at $\rho=8\pi(1+\beta)$;*
(-) $J_{8\pi(1+\beta)}$ has no minimizer.

Remark 6.1. *Clearly if (Ω,β) is of second kind we deduce from Lemma 6.1 and Lemma 5.2-*(vi) *that as $\rho\nearrow 8\pi(1+\beta)$ the unique solutions of* (5.4) *converge to w_c in $C^2(\Omega)\cap C^1(\overline{\Omega})$ and then in particular $\mathcal{E}(\omega_\rho)=E(\rho)\to E_c=E(8\pi(1+\beta))=\mathcal{E}(\omega_c)$ where $\omega_c=\frac{He^{\mathrm{w}_c}}{\int_\Omega He^{\mathrm{w}_c}}$ and (recall Theorem 6.1) $\mathfrak{S}(\omega_\rho)=S(E)\to S(E_c)$, implying that $f(\rho)\to f(8\pi(1+\beta))=-S(E_c)-8\pi(1+\beta)E_c$.*

We will also need the following generalizations of results in [25] about domains of second kind.

Lemma 6.2. *Let (Ω, β) be of second kind and $E > E_c$. Then*

$$C_1 - 8\pi(1+\beta)E \leq S(E) \leq C_2 - 8\pi(1+\beta)E,$$

where

$$C_2 = S(E_c) + 8\pi(1+\beta)E_c = f(8\pi(1+\beta)).$$

Proof. In view of Remark 6.1 the same argument adopted in Proposition 6.1 in [25] works fine, in this case as well. □

Lemma 6.3. *Let (Ω, β) be of second kind and denote by ω_{E} any entropy maximizer defined by Theorem 5.2. Then, as $E \to +\infty$ we have $\rho_{\mathrm{E}} \to 8\pi(1+\beta)$ and*

$$\omega_{\mathrm{E}} \rightharpoonup \delta_{p=0}, \quad \textit{weakly in the sense of measures in } \Omega.$$

Proof. Let E_k, $\omega_k = \omega_{E_k}$ be any pair of sequences such that $E_k \to +\infty$ where ω_k is any entropy maximizer at energy E_k, put $\rho_k = \rho_{E_k}$, we prove that in fact $\omega_k \rightharpoonup \delta_{p=0}$ weakly in the sense of measures and $\rho_k \to 8\pi(1+\beta)$. We argue by contradiction and assume that along a subsequence any one of these properties is not satisfied.

As remarked in the proof of Theorem 5.2, by the Pohozaev identity we have that ρ_k is uniformly bounded, and since $E_k \to +\infty$, then it is not difficult to see that necessarily $\|\mathrm{w}_k\|_\infty \to +\infty$, where $\mathrm{w}_k = \rho_k G[\omega_k]$. Therefore w_k blows up as $k \to +\infty$ and then necessarily along a subsequence $\rho_k \to \rho_\infty$ where ([16, 17, 54])

$$\rho_\infty = 8\pi(1+\beta)n_0 + 8\pi \sum_{i=1}^{N} n_i(1+\alpha_i) + 8\pi n_r,$$

with

$$n_i \in \{0,1\}, i = 0,1,\cdots,N, \quad n_r \in \mathbb{N}, \quad n_0 + \sum_{j=1}^{N} n_j + n_r \geq 1.$$

Observe that if $\mathrm{supp}\{\omega_i\} = \Omega_i$, $i = 1,2$ and $\Omega_1 \cap \Omega_2 = \emptyset$, then $\mathfrak{S}(\omega_1 + \omega_2) = \mathfrak{S}(\omega_1) + \mathfrak{S}(\omega_2)$ while the energy $\mathcal{E}(\omega)$ is exactly the same as the one in [25]. Therefore it is readily verified that the argument adopted in Theorem 6.1 in [25] works fine, showing that, possibly along a subsequence,

$$\omega_k \rightharpoonup \delta_{x=p}, \quad \text{as} \quad k \to +\infty,$$

weakly in the sense of measures in Ω, for some $p \in \Omega$. Assume first that along a subsequence we reach a blow point p satisfying $p \notin \{0, q_j\}$. Since w_k is a regular 1-point blow-up sequence, then the subtle balance obtained in Lemma 5.4 breaks down and, by using the first order pointwise estimates in [54] together with equation (6.8) in [31], we would find that,

$$\begin{aligned} J_{8\pi(1+\beta)}(\mathrm{w}_k) &= \frac{1}{16\pi(1+\beta)} \int_\Omega |\nabla \mathrm{w}_k|^2 - \log\left(\int_\Omega H e^{\mathrm{w}_k}\right) \\ &= -\frac{2}{1+\beta}\log(\bar{\varepsilon}_k) + 2\log(\bar{\varepsilon}_k) + O(1) \to +\infty, \text{ as } k \to +\infty, \end{aligned}$$

for a suitably defined $\bar{\varepsilon}_k \to 0$. Assume next that $p \in \{q_j\}$, since again w_k is a 1-point blow-up sequence at a positive singular source, say of strength $\alpha_j > 0$, then by using the first order pointwise estimates in [5] together with Proposition 2.1, we would find that,

$$\begin{aligned} J_{8\pi(1+\beta)}(\mathrm{w}_k) &= \frac{1}{16\pi(1+\beta)} \int_\Omega |\nabla \mathrm{w}_k|^2 - \log\left(\int_\Omega H e^{\mathrm{w}_k}\right) \\ &= -\frac{2(1+\alpha_j)}{1+\beta}\log(\varepsilon_k) + 2\log(\varepsilon_k) + O(1) \to +\infty, \text{ as } k \to +\infty, \end{aligned}$$

for a suitably defined $\varepsilon_k \to 0$. On the other side, since $\omega_k = \omega_{E_k}$ is an entropy maximizer we have, from Lemma 6.2 and Lemma 5.3

$$J_{8\pi(1+\beta)}(\mathrm{w}_k) = \mathcal{F}_{8\pi(1+\beta)}(\omega_k) = -S(E_k) - 8\pi(1+\beta)E_k \le -C_1,$$

which is the desired contradiction. Therefore necessarily $p = 0$ in which case $\rho_\infty = 8\pi(1+\beta)$, as claimed. $\square$

Our first result concerning domains of second kind is a generalization of the analogue characterization in the regular case as first pursued in [24] and then completed in [28], [12], [15], see also [11]. Here D_β as defined in (5.6) plays a crucial role and in particular we will keep the notations of Lemma 5.4 whenever needed.

Theorem 6.2. *The following facts are equivalent:*
(a) (Ω, β) *is of second kind;*
(b) $D_\beta > 0$*;*
(c) $f(8\pi(1+\beta)) < -1 - \log\left(\frac{\pi}{1+\beta}\right) - \gamma_\Omega(0)$*;*
(d) (5.4) *has a solution for* $\rho = 8\pi(1+\beta)$.

Proof. If (b) holds, then, in view of Lemma 5.4, (c) holds as well.
Assume that (c) holds, we show that a solution exists for $\rho = 8\pi(1+\beta)$. If not, by Lemma 6.1, any sequence of solutions $\mathrm{w}_n = \mathrm{w}_{\rho_n}$ blows up as $\rho_n \nearrow 8\pi(1+\beta)$. However, as remarked above, it is not difficult to prove that (see for example Proposition 7.3 in [24]) $f(\rho)$ is continuous in $[0, 8\pi(1+\beta)]$, therefore

$$J_{\rho_n}(\mathrm{w}_n) = f(\rho_n) \to f(8\pi(1+\beta)), \quad \text{as} \quad \rho_n \nearrow 8\pi(1+\beta).$$

In particular, since w_n is a 1-point blow-up sequence with blow up point $q = 0$, by arguing as in the proof of Theorem 6.1-(iv) we would find that,

$$f(8\pi(1+\beta)) = -1 - \log\left(\frac{\pi}{1+\beta}\right) - \gamma_\Omega(0),$$

which is the desired contradiction to the strict inequality in (c).

Since by Lemma 6.1 we have that (a) and (d) are equivalent, we are just left to prove that if (a) and (d) hold then (b) holds as well, that is, $D_\beta > 0$.

By contradiction assume that a solution exists at $\rho = 8\pi(1+\beta)$, say w_c, that $E_c < +\infty$ but $D_\beta \le 0$. By Lemma 6.2, as $E_n \to +\infty$ and if $\mathrm{w}_n = \rho_n G_\Omega[\omega_n]$

is a sequence defined by the entropy maximizers of Theorem 5.2, then w_n makes 1-point blow-up and the blow-up point is in fact the singular point $q = 0$. Since from (2.27) we have that $\rho_n - 8\pi(1+\beta)$ has the same sign as D_β, then $D_\beta \geq 0$. In fact, if otherwise $D_\beta < 0$, then we would have $\rho_n < 8\pi(1+\beta)$ implying that the unique solutions for $\rho_n < 8\pi(1+\beta)$ blow-up, which contradicts Lemma 6.1. Therefore we have that necessarily $D_\beta = 0$. At this point observe that, in view of (5.5), and with the notations of Lemma 5.4, D_β can be written equivalently as follows

$$D^*_\beta := c_*^{-1} D_\beta = e^{-\gamma^*(0)} \int_\Omega \frac{|x|^{2\beta}}{|x|^{4(1+\beta)}} \left(e^{\gamma^*(x)} - e^{\gamma^*(0)}\right) - \int_{\Omega^c} \frac{|x|^{2\beta}}{|x|^{4(1+\beta)}} = \tag{6.1}$$

$$\int_\Omega \frac{|x|^{2\beta}}{|x|^{4(1+\beta)}} \left(e^{\Phi^*_\Omega(x)} - 1\right) - \int_{\Omega^c} \frac{|x|^{2\beta}}{|x|^{4(1+\beta)}},$$

where $c_* = e^{\gamma^*(0)}$ and

$$\Phi^*_\Omega(x) = \gamma^*(x) - \gamma^*(0).$$

For any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \int_{B_r} \frac{|x|^{2\beta}}{|x|^{4(1+\beta)}} \left(e^{\Phi^*_\Omega(x)} - 1\right) \right| < \epsilon, \ \forall r \leq \delta,$$

and we define

$$\begin{aligned} D^*_{\beta,\epsilon} &= D^*_\beta - \int_{B_\delta} \frac{|x|^{2\beta}}{|x|^{4(1+\beta)}} \left(e^{\Phi^*_\Omega(x)} - 1\right) \\ &= \int_{\Omega \setminus B_\delta} \frac{|x|^{2\beta}}{|x|^{4(1+\beta)}} \left(e^{\Phi^*_\Omega(x)} - 1\right) - \int_{\Omega^c} \frac{|x|^{2\beta}}{|x|^{4(1+\beta)}}. \end{aligned}$$

For any such δ fixed, let $0 < \delta < \tau < R$ such that $B_{2\tau} \Subset \Omega \Subset B_{2R}$ and $\mathcal{N} \in C^4([-1,1] \times \mathbb{R}^2; \mathbb{R}^2)$ be any vector field such that

$$\mathcal{N}(t,x) = \begin{cases} (0,0), & \text{if} \quad x \notin \overline{B_{2R} \setminus B_{2\tau}}, \\ -g(x)\nu(x), & \text{if} \quad x \in \partial\Omega, \end{cases}$$

where $\nu(x) = \nu_\Omega(x)$ is the unit outer normal at x and g a smooth function. Then it is well known ([49]) that the unique solution of $\frac{d}{dt} h(x,t) = \mathcal{N}(x,t)$, $x \in \Omega$, $t \in (-1,1)$, $h(x,t) = x$, defines for t near $t = 0$ a C^3 "curve of domains" $\Omega(t) := h(\Omega, t)$ such that $\Omega(0) = \Omega$ and in particular the map

$$t \mapsto D^*_{\beta,\epsilon}(t) := \int_{\Omega(t) \setminus B_\delta} \frac{|x|^{2\beta}}{|x|^{4(1+\beta)}} \left(e^{\Phi^*_{\Omega(t)}(x)} - 1\right) - \int_{\Omega^c(t)} \frac{|x|^{2\beta}}{|x|^{4(1+\beta)}}$$

is of class C^1 near $t = 0$ and

$$I_g(x; \partial\Omega) := \frac{d}{dt} \left((8\pi + 4\pi\beta) G_{\Omega(t)}(x,0) - 4\pi \sum_{j=1}^N \alpha_j G_{\Omega(t)}(x, q_j) \right) \Bigg|_{t=0}$$

$$= -(8\pi + 4\pi\beta)\int\limits_{\partial\Omega} \frac{\partial G_\Omega(x,y)}{\partial\nu(y)}\frac{\partial G_\Omega(y,0)}{\partial\nu(y)}\mathcal{N}(y,0)\cdot\nu(y)$$

$$+ 4\pi\sum_{j=1}^{N}\alpha_j\int\limits_{\partial\Omega} \frac{\partial G_\Omega(x,y)}{\partial\nu(y)}\frac{\partial G_\Omega(y,q_j)}{\partial\nu(y)}\mathcal{N}(y,0)\cdot\nu(y)$$

$$= \int\limits_{\partial\Omega} \frac{\partial G_\Omega(x,y)}{\partial\nu(y)}\left[8\pi(1+\beta)\frac{\partial G_\Omega(y,0)}{\partial\nu(y)} - 4\pi\sum_{j=1}^{N}\alpha_j\frac{\partial G_\Omega(y,q_j)}{\partial\nu(y)}\right]g(y),$$

where we used the following formula, see [49, Example 3.4]

$$\frac{d}{dt}G_{\Omega(t)}(x,p)\bigg|_{t=0} = -\int_\Omega \frac{\partial G_\Omega(x,y)}{\partial\nu(y)}\frac{\partial G_\Omega(y,p)}{\partial\nu(y)}\mathcal{N}(y,0)\cdot\nu(y)dy, \quad \forall x,\ p\in\Omega.$$

Then

(6.2)

$$\frac{d}{dt}D^*_{\beta,\epsilon}(t)\bigg|_{t=0}$$

$$= \int_{\partial\Omega}\frac{|x|^{2\beta}}{|x|^{4(1+\beta)}}\left(e^{\Phi^*_{\Omega(t)}}-1\right)\mathcal{N}(x,0)\cdot\nu(x)$$

$$- \int_{\partial\Omega^c}\frac{|x|^{2\beta}}{|x|^{4(1+\beta)}}\mathcal{N}(x,0)\cdot\nu_c(x)$$

$$+ \int_{\Omega\setminus B_\delta}\frac{|x|^{2\beta}e^{\Phi^*_{\Omega(t)}(x)}}{|x|^{4(1+\beta)}}\frac{d}{dt}\left((8\pi+4\pi\beta)G_{\Omega(t)}(x,0) - 4\pi\sum_{j=1}^{N}\alpha_j G_{\Omega(t)}(x,q_j)\right)\bigg|_{t=0}$$

$$= \int_{\partial\Omega}\frac{|x|^{2\beta}}{|x|^{4(1+\beta)}}e^{\Phi^*_{\Omega(t)}}\mathcal{N}(x,0)\cdot\nu(x) + \int_{\Omega\setminus B_\delta}\frac{|x|^{2\beta}}{|x|^{4(1+\beta)}}e^{\Phi^*_{\Omega(t)}}I_g(x;\partial\Omega)$$

$$= -\int_{\partial\Omega}\frac{|x|^{2\beta}}{|x|^{4(1+\beta)}}e^{\Phi^*_{\Omega(t)}}g(x) + \int_{\Omega\setminus B_\delta}\frac{|x|^{2\beta}}{|x|^{4(1+\beta)}}e^{\Phi^*_{\Omega(t)}}I_g(x;\partial\Omega)$$

$$= -\int_{\partial\Omega}g(x) + \int_{\Omega\setminus B_\delta}\frac{|x|^{2\beta}}{|x|^{4(1+\beta)}}e^{\Phi^*_{\Omega(t)}}I_g(x;\partial\Omega).$$

Let

$$D_\delta(t) := \int_{B_\delta}\frac{|x|^{2\beta}}{|x|^{4(1+\beta)}}\left(e^{\Phi^*_{\Omega(t)}}-1\right)$$

and remark that

$$D^*_\beta(t) := \int_{\Omega(t)}\frac{|x|^{2\beta}}{|x|^{4(1+\beta)}}\left(e^{\Phi^*_{\Omega(t)}}-1\right) - \int_{\Omega^c(t)}\frac{|x|^{2\beta}}{|x|^{4(1+\beta)}} = D^*_{\beta,\epsilon}(t) + D_\delta(t),$$

and that

$$|D_\delta(t)| < 2\epsilon,$$

for any t small enough.

To proceed our discussion, we need the following claim:

Claim: There exists function g such that $D^*_\beta(t) < 0$ for any t small enough.

We shall prove the claim by dividing our discussion in various cases. Let

$$G^*(y) = \left[(8\pi + 4\pi\beta)\frac{\partial G_\Omega(y,0)}{\partial \nu(y)} - 4\pi \sum_{j=1}^{N} \alpha_j \frac{\partial G_\Omega(y,q_j)}{\partial \nu(y)} \right], \, y \in \partial\Omega.$$

Assume at first that,

$$\Gamma_+ = \{y \in \partial\Omega : G^*(y) > 0\} \neq \emptyset,$$

then for any $\sigma > 0$ we can find $g \geq 0$ with compact support in Γ_+, such that $\int_{\Gamma_+} g(x) > \sigma$, which is always possible since Γ_+ is relatively open in $\partial\Omega$. Therefore, since by the strong maximum principle $\frac{\partial G_\Omega(y,0)}{\partial \nu(y)} < 0$ for $y \in \partial\Omega$, we deduce that $I_g(x,\partial\Omega) < 0$ and consequently that

$$\frac{d}{dt} D^*_{\beta,\epsilon}(t)\Big|_{t=0} < -\int_{\partial\Omega} g(x) < -\sigma.$$

Therefore, for any t small enough we have,

$$D^*_{\beta,\epsilon}(t) < D^*_{\beta,\epsilon}(0) - \sigma t,$$

and then we choose σ large enough such that,

$$D^*_\beta(t) < D^*_{\beta,\epsilon}(0) - \sigma t + 2\epsilon < 0.$$

Therefore we are left with the case where $G^*(y) \leq 0$ and we assume that

$$\Gamma_- = \{y \in \partial\Omega : G^*(y) < 0\} \neq \emptyset,$$

otherwise the proof is easier. Assume that $g \leq 0$ then $I_g(x,\partial\Omega) \leq 0$ and then, as far as $I_g(0,\partial\Omega) < 0$, we have,

$$\int_{\Omega\setminus B_\delta} \frac{|x|^{2\beta}}{|x|^{4(1+\beta)}} e^{\Phi^*_\Omega} I_g(x;\partial\Omega) \to -\infty \text{ as } \delta \to 0^+.$$

In this case we would deduce from (6.2) that, for any $\sigma > 0$, possibly taking a smaller δ,

$$\frac{d}{dt} D_{0,\epsilon}(t)\Big|_{t=0} < -\sigma,$$

implying as above that $D_0(t) < 0$ for any t small enough. Therefore we are left with showing that there exists $g \leq 0$ such that $I_g(0,\partial\Omega) \neq 0$. However this an easy consequence of

$$I_g(0,\partial\Omega) = \int_{\partial\Omega} \frac{\partial G_\Omega(0,y)}{\partial \nu(y)} G^*(y) g(y) = \int_{\Gamma_-} \frac{\partial G_\Omega(0,y)}{\partial \nu(y)} G^*(y) g(y)$$

and the fact that $\frac{\partial G_\Omega(0,y)}{\partial \nu(y)} > 0$ in Γ_-. This fact completes the proof of the Claim.

Therefore, since $\Omega(t)$ is simply connected and of class C^2 and since $D^*_\beta(t)$ is just proportional via a positive constant to the D_β relative to $\Omega(t)$, and for $t > 0$ small enough $D^*_\beta(t) < 0$, we see from Theorem 2.1 that the unique solutions on $\Omega(t)$ blow-up as $\rho \nearrow 8\pi(1+\beta)$. Let $t_j \searrow 0^+$ and $\mathrm{w}_{\rho,j}(x)$ be the unique solutions for $\rho < 8\pi(1+\beta)$ on $\Omega(t_j)$. By assumption there exists a solution w_c on Ω at $\rho = 8\pi(1+\beta)$ which is also unique by Lemma 5.2-(iii) and we define $M_c = \max\limits_{\Omega} \mathrm{w}_c$. Remark that for fixed j, $\mathrm{w}_{\rho,j}$ is a smooth function of ρ, see Lemma 5.2-(iv). Therefore for any $M > M_c$ and for any j there exists $\rho_{j,M} \leq 8\pi(1+\beta)$ such that $\max\limits_{\Omega(t_j)} \mathrm{w}_{\rho_{j,M},j} = M$. Thus, along a diagonal subsequence, $\rho_{j,M} \to \rho_M \leq 8\pi(1+\beta)$ and $\mathrm{w}_{\rho_{j,M},j} \to \mathrm{w}_{\rho_M}$ uniformly, where w_{ρ_M} is a solution of (5.4) for $\rho = \rho_M$ on Ω with $\max\limits_{\Omega} \mathrm{w}_{\rho_M} = M$. However, since w_c is the unique solution for $\rho = 8\pi(1+\beta)$, then $\rho_M \in (0, 8\pi(1+\beta))$. Since M is arbitrary we can take $M_k \to +\infty$ and construct a sequence of solutions of (5.4) which blows up with $\rho_{M_k} < 8\pi(1+\beta)$, implying necessarily ([16]) that $\rho_{M_k} \to 8\pi(1+\beta)$. This contradicts Lemma 6.1 which would imply that (Ω,β) is of first kind. $\square$

As an immediate consequence of Theorem 6.2, Proposition 2.1 and (2.27), we have the following generalization of a result in [11].

Corollary 6.2. *Let* w_k *be a sequence of solutions of* (5.4) *where* $q = 0$ *satisfies* (5.5)*, which makes one point blow-up at* $q = 0$ *as* $\rho_k \to 8\pi(1+\beta)$*.*
If $|\rho_k - 8\pi(1+\beta)| = o(e^{-\lambda_k})$ *then* $\rho_k < 8\pi(1+\beta)$*.*

Remark 6.2. *As far as* $\beta = 0$*, Corollary 6.2 is false on the flat two-torus, see* [60].

Even with Theorem 6.2 at hand is not easy to find explicit examples of domains of pairs (Ω,β) of second kind. It is an interesting open problem to check whether or not symmetric dumbbell type domains Ω with a negative pole sitting at the saddle point of the Robin function can be constructed to match the requirement $D_\beta > 0$. Another possibility could be that of dropping (5.5), which, although not very meaningful from the physical point of view, seems to be at hand for $\beta \in (-1, -\frac{1}{2})$. We will discuss these issues elsewhere. In any case, as an immediate consequence of Theorem 6.2 and Lemma 6.1, we have the carachterization of pairs (Ω,β) of first kind.

Theorem 6.3. *The following facts are equivalent:*
(a) (Ω,β) *is of first kind;*
(b) $D_\beta \leq 0$*;*
(c) $f(8\pi(1+\beta)) = -1 - \log\left(\frac{\pi}{1+\beta}\right) - \gamma_\Omega(0)$*;*
(d) (5.4) *has no solution for* $\rho = 8\pi(1+\beta)$*.*

Among other things, for (Ω,β) of second kind, the following Theorem shows that, locally near $(8\pi(1+\beta))^+$, large energy solutions are unique and define a smooth curve.

Theorem 6.4. *If (Ω,β) is of second kind, there exists $\varepsilon_*>0$ such that:*
(i) there exists one and only one solution w_ρ of (5.4) with $\rho\in(8\pi(1+\beta),8\pi(1+\beta)+\varepsilon_*)$ and $\mathcal{E}\left(\frac{He^{\mathrm{w}_\rho}}{\int_\Omega He^{\mathrm{w}_\rho}dx}\right)>(\varepsilon_*)^{-1}$;
(ii) Let $\mathcal{G}_\infty$ denote the set of solutions of (5.4) determined in (i). Then, for any fixed $p\in(1,|\beta|^{-1})$, $\mathcal{G}_\infty$ is a real analytic curve

$$(8\pi(1+\beta),8\pi(1+\beta)+\varepsilon_*)\ni\rho\mapsto \mathrm{w}_\rho\in W^{2,p}_0(\Omega);$$

(iii) along $\mathcal{G}_\infty$ we have that the map $E(\rho)=\mathcal{E}\left(\frac{He^{\mathrm{w}_\rho}}{\int_\Omega He^{\mathrm{w}_\rho}dx}\right)$ is real analytic and strictly decreasing with inverse $E\mapsto\rho_E$, $E\in((\varepsilon_*)^{-1},+\infty)$ as determined by Theorem 5.2. In particular $\frac{dE(\rho)}{d\rho}\leq 0$.
(iv) along $\mathcal{G}_\infty$ we have that the map $\mu^2(\rho)=\rho\left(\int_\Omega He^{\mathrm{w}_\rho}\right)^{-1}$ is real analytic and strictly increasing in $(8\pi(1+\beta),8\pi(1+\beta)+\varepsilon_*)$ with $\mu(\rho)\to 0^+$ as $\rho\to 8\pi(1+\beta)^+$. In particular the inverse map $\mu\mapsto\rho(\mu)$ is well defined and strictly increasing in $(0,\mu_*)$, where $\mu_*=\lim_{\rho\to 8\pi(1+\beta)+\varepsilon_*}\mu(\rho)$.

Proof. (i) In view of Theorem 6.2 and (2.27), if this was not the case we could find sequences $\mathrm{w}_{k,1}\neq\mathrm{w}_{k,2}$ solving (5.4) with the same value $\rho_k\to 8\pi(1+\beta)^+$ and

$$\mathcal{E}\left(\frac{He^{\mathrm{w}_{k,2}}}{\int_\Omega He^{\mathrm{w}_{k,2}}dx}\right)=E_{k,2}\geq E_{k,1}=\mathcal{E}\left(\frac{He^{\mathrm{w}_{k,1}}}{\int_\Omega He^{\mathrm{w}_{k,1}}dx}\right)\to+\infty.$$

It is readily seen that both $\mathrm{w}_{k,i}$, $i=1,2$ make 1-point blow-up at the negative singular source, which for k large contradicts the local uniqueness result in [19]. This observation proves (i).
(ii) The claim about $\mathcal{G}_\infty$ in (ii) follows immediately from Theorem 1.4, Lemma 5.1 and (i).
(iii) We first prove that, by taking a smaller ε_* if needed, the map $E\mapsto\rho_{E}$ as determined by Theorem 5.2 for solutions on $\mathcal{G}_\infty$ is injective. If not, there exist sequences $E_{k,1}\geq E_{k,2}\nearrow+\infty$ such that there exists $\rho_k\searrow 8\pi(1+\beta)$ such that $\rho_k=\rho_{E_{k,1}}=\rho_{E_{k,1}}$. Let $\mathrm{w}_{k,1},\mathrm{w}_{k,2}$ be the solutions of (5.4) with energies $E_{k,1}\geq E_{k,2}$, by the uniqueness result in [19] for k large we have that $\mathrm{w}_{k,1}\equiv\mathrm{w}_{k,2}$, implying that $E_{k,1}=E_{k,2}$, which is the desired contradiction.
Therefore, since the map $E\mapsto\rho_{E}$ is continuous and injective in $(\frac{1}{\varepsilon_*},+\infty)$ then it must be strictly monotone and since by Lemma 6.1 $\rho_{E}\to 8\pi(1+\beta)^+$ as $E\to+\infty$ then it must be strictly decreasing. Next observe that by (ii) the inverse map $\rho\mapsto E(\rho)$ is well defined and real analytic, whence we also have $\frac{dE(\rho)}{d\rho}\leq 0$.
(iv) The fact that $\mu(\rho)$ is well defined and real analytic along $\mathcal{G}_\infty$ obviously follows from (ii). Let $\mu\mapsto\rho(\mu)$ be defined as follows. Let u_μ be a solution

of

$$
\begin{cases} -\Delta u_\mu = \mu^2 H e^{u_\mu} & \text{in } \Omega, \\ \\ u_\mu = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.3}
$$

and $\mathcal{U}_\mu$ be the set of all solutions of (6.3). We define the multivalued function

$$
\Lambda(\mu) = \left\{ \mu^2 \int_\Omega H e^{u_\mu}, u_\mu \in \mathcal{U}_\mu \right\}.
$$

Let $(\rho, \mathrm{w}_\rho) \in \mathcal{G}_\infty$, then setting $\mu := \mu(\rho) = \left(\rho \left(\int_\Omega H e^{\mathrm{w}_\rho}\right)^{-1}\right)^{\frac{1}{2}}$ and $u_\mu := \mathrm{w}_\rho$, then u_μ solves (6.3) for this μ, whence in particular $\rho(\mu) = \mu^2 \int_\Omega H e^{u_\mu} \in \Lambda(\mu)$. We prove that, possibly taking a smaller ε_*, this argument defines an injective function $\mu \mapsto \rho(\mu)$, $\mu \in (0, \mu_*)$ with μ_* as defined in the claim. If not there exists a sequence $\mu_k \to 0^+$ such that we could find $\rho_{k,1} \neq \rho_{k,2}$ sharing the same value of μ_k such that $(\rho_{k,i}, \mathrm{w}_{\rho_{k,i}}) \in \mathcal{G}_\infty$, $i = 1, 2$ and $8\pi(1+\beta) < \rho_{k,1} < \rho_{k,2} \to 8\pi(1+\beta)^+$. Since both solutions belong to $\mathcal{G}_\infty$ they both have to blow-up at the negative singular source which for k large contradicts the local uniqueness result in [44]. This shows that $\mu \mapsto \rho(\mu)$ is a well defined function. However it is also injective because otherwise we would have that there exists $0 < \mu_{k,1} \leq \mu_{k,2} \to 0^+$ corresponding to the same $\rho_k := \rho(\mu_{k,1}) = \rho(\mu_{k,2})$, while the solutions $u_{k,1} \neq u_{k,2}$. This is again in contradiction with the uniqueness result in [19] which implies $\mathrm{w}_{k,1} := u_{k,1} \equiv u_{k,2} =: \mathrm{w}_{k,2}$. □

With the above Theorem, we are able to deduce a generalization of a recent result about the convexity of the entropy for domains of second kind in the regular case, see Theorem 1.4 in [7].

Theorem 6.5. [Negative Specific Heat States] *If (Ω, β) is of second kind, then there exists $E_* > E_c$ such that $S(E)$ is smooth and strictly convex in $(E_*, +\infty)$. In particular*

$$
\frac{dS(E)}{dE} = -\rho_E, \quad \frac{d^2 S(E)}{dE^2} = -\frac{d\rho_E}{dE} > 0,
$$

where ρ_E is smooth and strictly decreasing in $(E_*, +\infty)$ and $\rho_E \searrow 8\pi(1+\beta)$ as $E \nearrow +\infty$.

Proof. By Theorem 6.4 we can adapt an argument in [7]. Let $\mu \in (0, \mu_*)$, $\rho(\mu)$ as defined in Lemma 6.4-(iv) and $u_\mu \equiv \mathrm{w}_{\rho(\mu)}$ the corresponding unique solution of (6.3) in $\mathcal{G}_\infty$. By the non degeneracy of u_μ ([44]) we have that $v_\mu = \frac{du_\mu}{d\mu}$ is well defined and satisfies,

$$
\begin{cases} -\Delta v_\mu = 2\mu H e^{u_\mu} + \mu^2 H e^{u_\mu} v_\mu & \text{in } \Omega \\ \\ v_\mu = 0 & \text{on } \partial\Omega \end{cases} \tag{6.4}
$$

and since by definition $\rho(\mu)=\mu^2\int_\Omega He^{u_\mu}$, then

$$\frac{d\rho(\mu)}{d\mu}=2\mu\int_\Omega He^{u_\mu}+\mu^2\int_\Omega He^{u_\mu}v_\mu. \tag{6.5}$$

Remark that by definition $E(\rho)=\frac{1}{2\rho}\int_\Omega\frac{He^{\mathrm{w}_\rho}\mathrm{w}_\rho}{\int_\Omega He^{\mathrm{w}_\rho}}$, whence in particular

$$E_\mu:=E(\rho(\mu))=\frac{\mu^2}{2\rho^2(\mu)}\int_\Omega He^{u_\mu}u_\mu.$$

Multiplying the equation in (6.4) by u_μ, integrating by parts and using (6.3) we have

$$\mu^2\int_\Omega He^{u_\mu}v_\mu=2\mu\int_\Omega He^{u_\mu}u_\mu+\mu^2\int_\Omega He^{u_\mu}v_\mu u_\mu,$$

which readily implies that

$$\frac{d}{d\mu}\left(\mu^2\int_\Omega He^{u_\mu}u_\mu\right)=2\mu^2\int_\Omega He^{u_\mu}v_\mu. \tag{6.6}$$

We deduce from (6.5) and (6.6) that

$$\begin{aligned}\rho^4(\mu)\frac{dE_\mu}{d\mu}&=\frac{\rho^2(\mu)}{2}\frac{d}{d\mu}\left(\mu^2\int_\Omega He^{u_\mu}u_\mu\right)-\rho(\mu)\frac{d\rho(\mu)}{d\mu}\mu^2\int_\Omega He^{u_\mu}u_\mu\\&=\left(\mu^2\int_\Omega He^{u_\mu}v_\mu\right)\rho^2(\mu)-\rho(\mu)\frac{d\rho(\mu)}{d\mu}\mu^2\int_\Omega He^{u_\mu}u_\mu\\&=\left(\frac{d\rho(\mu)}{d\mu}-\frac{2\rho(\mu)}{\mu}\right)\rho^2(\mu)-\rho(\mu)\frac{d\rho(\mu)}{d\mu}\mu^2\int_\Omega He^{u_\mu}u_\mu\\&=\frac{d\rho(\mu)}{d\mu}\left(\rho^2(\mu)-\rho(\mu)\mu^2\int_\Omega He^{u_\mu}u_\mu\right)-2\frac{\rho^3(\mu)}{\mu}.\end{aligned}$$

At this point, by using the fact that $\frac{d\rho(\mu)}{d\mu}\geq 0$, $\mu^2\int_\Omega He^{u_\mu}u_\mu=2\rho^2(\mu)E_\mu\to+\infty$, $\rho(\mu)\to 8\pi(1+\beta)$ and $\mu\to 0^+$ we deduce that $\frac{dE_\mu}{d\mu}\to-\infty$. Observe moreover that (6.5) and (6.6) together imply that

$$\frac{d\rho(\mu)}{d\mu}=2\mu\int_\Omega He^{u_\mu}+\frac{1}{2}\frac{d}{d\mu}\left(\mu^2\int_\Omega He^{u_\mu}u_\mu\right)=2\mu\int_\Omega He^{u_\mu}+\frac{d}{d\mu}(\rho^2(\mu)E_\mu),$$

that is

$$\left(1-\frac{d}{d\rho}(\rho^2E(\rho))\right)\frac{d\rho(\mu)}{d\mu}=2\mu\int_\Omega He^{u_\mu},$$

which shows, since $\frac{d\rho(\mu)}{d\mu}\geq 0$ and since from (iii) $\frac{dE(\rho)}{d\rho}$ is well defined, that in fact $\frac{d\rho(\mu)}{d\mu}>0$. Remark also that, because of the non degeneracy of u_μ ([44]) we see from (6.5) that $\frac{d\rho(\mu)}{d\mu}$ is bounded, whence we have $\frac{dE(\rho)}{d\rho}=\frac{dE_\mu}{d\mu}\frac{d\mu(\rho)}{d\rho}<0$. At this point the conclusion follows from the fact that $\frac{dS(E)}{dE}=-\rho_{_E}$ which is done as in (4.1) in [3]. $\square$

Finally, by Lemma 5.2 and Theorem 6.4, we provide the exact counting of solutions in a right neighborhood of $8\pi(1+\beta)$ for domains of second kind.

Theorem 6.6. *If* (Ω, β) *is of second kind, there exists* $\varepsilon_* > 0$ *such that there exists exactly two solutions of of* (5.4) *with* $\rho \in (8\pi(1+\beta), 8\pi(1+\beta)+\varepsilon_*)$.

Proof. By Lemma 5.2-(vi) the branch of unique solutions for $\rho = 8\pi(1+\beta)$ can be continued in a small enough right neighborhood of $\rho = 8\pi(1+\beta)$, say $I_* = (8\pi(1+\beta), 8\pi(1+\beta)+\varepsilon_*)$. By Lemma 6.4-$(i)$-$(ii)$, possibly taking a smaller ε_*, there exists a smooth branch of solutions in I_* whose energy is larger than $(\varepsilon_*)^{-1}$. If the claim were false we could find a third distinct sequence of solutions, say w_k for some $\rho_k \to 8\pi(1+\beta)^+$. Possibly along a subsequence there are only two possibilities, either $\mathrm{w}_k \to \mathrm{w}_\infty$, where w_∞ is the unique solution at $8\pi(1+\beta)$ or $\|\mathrm{w}_k\|_\infty \to +\infty$. The former case is ruled out since then $\rho = 8\pi(1+\beta)$ would be a bifurcation point, which contradicts Lemma 5.1 and Lemma 5.2-(iii). The latter case is ruled out by the local uniqueness Theorem in [19]. $\square$

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