

COMPUTING THE ROOTS OF TWISTING SHEAVES OVER THE PROJECTIVE LINE ARISING FROM MONODROMY REPRESENTATIONS

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ABSTRACT. Given a monodromy representation ρ of the projective line minus m points, one can extend the resulting vector bundle with connection map, via the Deligne canonical extension, to a vector bundle with logarithmic connection map over all of the projective line. Now, since vector bundles split as twisting sheaves over the projective line, the focus of this work is to determine the exact decomposition; i.e. to computing the roots. In Particular, we compute the roots for all finite-dimensional ρ when $m = 2$ and for all ρ of dimension strictly less than 3 when $m = 3$.

CONTENTS

1. Introduction	1
1.1. Related Problems	1
1.2. Results	2
2. Preliminaries	5
3. Punctured Complex Line	6
4. Double Punctured Complex Line	8
5. The Monodromy Derivative	10
5.1. Auxiliary Connection	11
5.2. Monodromy Derivative	12
6. Example from the Modular Group	15
References	16

1. INTRODUCTION

1.1. Related Problems. Taking the m -punctured Riemann sphere $\mathbb{P}^1 - \{p_1, \dots, p_m\}$ as a starting point, it is a classical result that a representation of its fundamental group determines a holomorphic connection (see, e.g., [24]). A fundamental result of Deligne [2] ensures that given a holomorphic connection on $\mathbb{P}^1 - \{p_1, \dots, p_m\}$, there exists a unique extension to a locally free sheaf \mathcal{V} on \mathbb{P}^1 , known as the Deligne canonical extension, such that the connection map ∇ has logarithmic poles and the eigenvalues of the residues lie in a specified section of \mathbb{C}/\mathbb{Z} , which we refer to as the *principal branch*.

By the celebrated Birkhoff–Grothendieck theorem the underlying vector bundle of the logarithmic connection must decompose as a direct sum of twisting sheaves. The looming question becomes: given a representation of the fundamental group of $\mathbb{P}^1 - \{p_1, \dots, p_m\}$ what is the decomposition of the ensuing vector bundle on \mathbb{P}^1 ?

First, observe that in the case of a compact (no points removed) Riemann surface, the question chiefly translates to which vector bundles admit holomorphic connections. This was settled by Weil

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[25]: a holomorphic vector bundle on a Riemann surface admits a holomorphic connection if and only if each indecomposable summand has first Chern class 0.

Because the problem under scrutiny deals with monodromy representations and logarithmic connections, a natural consideration is to examine the moduli of such objects. While the moduli spaces in question have been studied for sufficiently nice varieties (see Simpson [22] and Nitsure [20]), and the interactions between monodromy, mapping class groups, and motives have been surveyed by Litt [12], these results are often too general to produce an answer to our specific question.

However, if we restrict the problem to only consider irreducible unitary monodromy representations of $\mathbb{P}^1 - \{p_1, \dots, p_m\}$, then by the work of Narasimhan, Mehta, and Seshadri in [17] and [19], this essentially amounts to the study of the moduli space of stable parabolic vector bundles on \mathbb{P}^1 ; particularly, Narasimhan, Mehta and Seshadri showed that there is a one-to-one correspondence between irreducible unitary monodromy representations and stable parabolic vector bundles with parabolic degree 0.

Regarding general properties of the moduli of stable parabolic vector bundles on \mathbb{P}^1 one may consult, for example, with the following papers [8, 9, 3, 4]. Of particular interest to us, are the result of [7], [16] and [18]. The authors of [7] generalize some of the results from [14, 13] and provide the decomposition of any rank 2 stable parabolic vector bundle of degree d over \mathbb{P}^1 with 4 or more marked points. In our setting, this means that the answer to our question is known when we remove 4 or more points and take ρ to be an irreducible unitary representation of dimension 2. In [16], Matsumoto finds the decomposition of any rank 3 stable parabolic vector bundle of degree -2 over \mathbb{P}^1 with 3 marked points. The degree -2 condition translates to the sum of the traces of the principal logarithms of the image of the generators of the fundamental group being equal to 2. The authors of [18] find a bound for all twisting parameters ξ_j in the decomposition of a rank ≥ 2 stable parabolic vector bundle with an extra set of conditions; they state that for each ξ_j , $-m < \xi_j < 0$. Again, in our setting, this means that the answer to our question is bounded when we remove m points and take ρ to be an irreducible unitary representation of dimension two or greater that satisfies the extra set of conditions.

In this paper, we are not necessarily interested in the overall description of certain moduli spaces but simply in the computation of the decomposition of the vector bundles over the projective line that arise from a monodromy representation, as a result we take a rather algebraic approach and consider generic representations. Particularly, we compute the decomposition for all finite-dimensional ρ when $m = 2$ and for all representations ρ of dimension strictly less than 3 when $m = 3$. Lastly, we remark that we use different terminology than that used in the setting of moduli of stable parabolic bundles on \mathbb{P}^1 .

1.2. Results. Let $\mathbf{Y}_{(p_1, \dots, p_m)} := \mathbb{P}_{\mathbb{C}}^1 - \{p_1, \dots, p_m\}$. The fundamental group of $\mathbf{Y}_{(p_1, \dots, p_m)}$ is known to be isomorphic to the free group on $m - 1$ generators, $\mathbb{Z}^{*(m-1)}$. Thus, a representation of the fundamental group of $\mathbf{Y}_{(p_1, \dots, p_m)}$ is determined by the images of the generators, each of which we call a *local monodromy*. A *monodromy representation* of $\mathbf{Y}_{(p_1, \dots, p_m)}$ is a representation of its fundamental group.

We denote a *holomorphic connection* on $\mathbf{Y}_{(p_1, \dots, p_m)}$ with a pair (\mathcal{V}, ∇) where \mathcal{V} is a locally free sheaf on $\mathbf{Y}_{(p_1, \dots, p_m)}$, and $\nabla : \mathcal{V} \rightarrow \mathcal{V} \otimes \Omega_{\mathbf{Y}_{(p_1, \dots, p_m)}}^1$ is a morphism of sheaves satisfying the Leibniz Rule. When we want to refer to ∇ alone we call it the *connection map*.

Let S be a finite number of points of \mathbb{P}^1 and $\Omega_{\mathbb{P}^1}^1(S)$ the sheaf of 1-forms with logarithmic poles along S . We denote a *connection with logarithmic poles along S* on \mathbb{P}^1 with a pair $(\bar{\mathcal{E}}, \bar{\nabla})$, where $\bar{\mathcal{E}}$ is a locally free sheaf on \mathbb{P}^1 , and $\bar{\nabla} : \bar{\mathcal{E}} \rightarrow \bar{\mathcal{E}} \otimes_{\mathcal{O}} \Omega_{\mathbb{P}^1}^1(S)$ is a \mathbb{C} -linear map satisfying the Leibniz Rule.

There is an equivalence of categories between finite-dimensional monodromy representations, ρ , of $\mathbf{Y}_{(p_1, \dots, p_m)}$, and holomorphic connections, (\mathcal{V}, ∇) , on $\mathbf{Y}_{(p_1, \dots, p_m)}$. Furthermore, we can extend a holomorphic connection (\mathcal{V}, ∇) on $\mathbf{Y}_{(p_1, \dots, p_m)}$, via the Deligne canonical extension, to a connection $(\bar{\mathcal{V}}, \bar{\nabla})$ on \mathbb{P}^1 with logarithmic poles along $\{p_1, \dots, p_m\}$ satisfying $(\bar{\mathcal{V}}, \bar{\nabla})|_{\mathbf{Y}_{(p_1, \dots, p_m)}} \cong (\mathcal{V}, \nabla)$ such that the residue at each pole is the principal logarithm of the local monodromies. If (\mathcal{V}, ∇) is the holomorphic connection associated to ρ , then we refer to $(\bar{\mathcal{V}}, \bar{\nabla})$ as the *associated extended logarithmic connection* and denote it as $(\mathcal{V}_{\text{Log}(\rho)}, \nabla_{\text{Log}(\rho)})$.

Thus, the question at hand is, given a monodromy representation of $\mathbf{Y}_{(p_1, \dots, p_m)}$, extending the resulting holomorphic connection, via the Deligne canonical extension, to a connection with logarithmic poles on \mathbb{P}^1 , and applying the Birkhoff–Grothendieck theorem, what is the explicit decomposition of the ensuing vector bundle as a direct sum of twisting sheaves? In this context, we refer to the twisting parameters of the twisting sheaves as the *roots*.

In the case of $\mathbf{Y}_{(0, \infty)}$ we can fully answer our question. When the monodromy representation is a character, the root is the first Chern class. Given a locally free sheaf \mathcal{E} on \mathbb{P}^1 , denote its first Chern class by $c_1(\mathcal{E})$.

Theorem A (Theorem 3.3). *Let $\rho : \mathbb{Z} \rightarrow \mathrm{GL}_n(\mathbb{C})$ be an n -dimensional monodromy representation of $\mathbf{Y}_{(0, \infty)}$ with $(\mathcal{V}_{\mathrm{Log}(\rho)}, \nabla_{\mathrm{Log}(\rho)})$ the associated extended logarithmic connection. Allow $\zeta = c_1(\mathcal{V}_{\mathrm{Log}(\rho)})$.*

- (1) *If ρ is an indecomposable representation determined by the Jordan block M with eigenvalue $\lambda = re^{2\pi i q}$ where $0 \leq q < 1$, then*

$$\mathcal{V}_{\mathrm{Log}(\rho)} \cong \begin{cases} \mathcal{O}(0)^{\oplus n} & \text{when } q = 0 \\ \mathcal{O}(-1)^{\oplus n} & \text{when } q \neq 0. \end{cases}$$

- (2) *If $\rho \cong \bigoplus_{i=1}^u \rho_i$ where each ρ_i is indecomposable, then*

$$\mathcal{V}_{\mathrm{Log}(\rho)} \cong \mathcal{O}(-1)^{\oplus -\zeta} \oplus \mathcal{O}(0)^{\oplus n + \zeta}.$$

Observe that the theorem above closes the case on $\mathbf{Y}_{(0, \infty)}$. As Theorem 2.11 shows, $c_1(\mathcal{V}_{\mathrm{Log}(\rho)})$ is determined by ρ . Hence, Theorem A(2) reduces finding the roots to computing the first Chern class.

With regard to $\mathbf{Y}_{(0, 1, \infty)}$, we restrict ourselves to only consider monodromy representations of dimension strictly less than 3. When the monodromy representation is a character, the root is the first Chern class. When ρ is a reducible 2-dimensional monodromy representation of $\mathbf{Y}_{(0, 1, \infty)}$ we obtain the following theorem:

Theorem B (Theorem 4.4). *Suppose that $0 \rightarrow \rho' \rightarrow \rho \rightarrow \rho'' \rightarrow 0$ is a short exact sequence of monodromy representations of $\mathbf{Y}_{(0, 1, \infty)}$ with ρ two-dimensional and ρ', ρ'' one-dimensional. Let $0 \rightarrow \mathcal{O}(\xi') \rightarrow \mathcal{V}_{\mathrm{Log}(\rho)} \rightarrow \mathcal{O}(\xi'') \rightarrow 0$ be the short exact sequence of the underlying vector bundles from the associated extended logarithmic connections.*

- (1) *If $\xi', \xi'' \neq -2, 0$ respectively, then $0 \rightarrow \mathcal{O}(\xi') \rightarrow \mathcal{V}_{\mathrm{Log}(\rho)} \rightarrow \mathcal{O}(\xi'') \rightarrow 0$ splits.*
(2) *If $\xi', \xi'' = -2, 0$ respectively, then $\mathcal{V}_{\mathrm{Log}(\rho)} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{V}_{\mathrm{Log}(\rho)} \cong \mathcal{O}(0) \oplus \mathcal{O}(-2)$.*

In contrast with $\mathbf{Y}_{(0, \infty)}$, the fundamental group of $\mathbf{Y}_{(0, 1, \infty)}$ admits irreducible representations of dimension greater than 1. Now, when ρ is an irreducible representation of $\mathbb{Z} * \mathbb{Z}$ the methods we have been using are not sufficient to find the roots as explained at the beginning of §5, so the problem necessitates a more refined algebraic toolset. Observing the use of the Modular derivative in [1, §3 and §4], we set out to construct a Modular derivative-like operator.

Take the character representation $\chi : \pi_1(\mathbf{Y}_{(0, 1, \infty)}) \rightarrow \mathbb{C}^*$ defined by mapping both generators to -1 , the resulting associated extended logarithmic connection is $(\mathcal{V}_{\mathrm{Log}(\chi)}, \nabla_{\mathrm{Log}(\chi)})$ over \mathbb{P}^1 with $\mathcal{V}_{\mathrm{Log}(\chi)} \cong \mathcal{O}(-1)$. Next, we dualize, and obtain $(\mathcal{O}(1), \nabla)$ with

$$(1) \quad \nabla : \mathcal{O}(1) \rightarrow \mathcal{O}(1) \otimes_{\mathcal{O}} \Omega_{\mathbb{P}^1}^1([0] + [1] + [\infty])$$

where

$$(2) \quad \Omega_{\mathbb{P}^1}^1([0] + [1] + [\infty]) \cong \mathcal{O}(1)$$

and so

$$(3) \quad \nabla : \mathcal{O}(1) \rightarrow \mathcal{O}(2).$$

Further, we take $\nabla^{\otimes 2}$ to be the connection map on $\mathcal{O}(1) \otimes \mathcal{O}(1)$ such that for all sections $s, t \in \mathcal{O}(1)$ the following is satisfied: $(\nabla \otimes \nabla)(s \otimes t) = \nabla(s) \otimes t + s \otimes \nabla(t)$. Observe that

$$(4) \quad \nabla^{\otimes 2} : \mathcal{O}(2) \rightarrow \mathcal{O}(2) \otimes \Omega_{\mathbb{P}^1}^1([0] + [1] + [\infty]),$$

and so, inductively, we define $\nabla_\xi := \nabla^{\otimes \xi} : \mathcal{O}(\xi) \rightarrow \mathcal{O}(\xi) \otimes_{\mathcal{O}} \Omega^1([0] + [1] + [\infty])$. Hence, using the fact that $\Omega_{\mathbb{P}^1}^1([0] + [1] + [\infty]) \cong \mathcal{O}(1)$, then

$$(5) \quad \nabla_\xi : \mathcal{O}(\xi) \rightarrow \mathcal{O}(\xi + 1).$$

We refer to ∇_ξ as the *auxiliary connection map of weight ξ arising from $\mathbf{Y}_{(0,1,\infty)}$* .

Thus, keeping in mind that $\Omega_{\mathbb{P}^1}^1([0] + [1] + [\infty]) \cong \mathcal{O}(1)$ we now have the extra tool

$$(6) \quad \nabla_{\text{Log}(\rho)} \otimes \nabla : \mathcal{V}_{\text{Log}(\rho)} \otimes_{\mathcal{O}} \mathcal{O}(1) \rightarrow \mathcal{V}_{\text{Log}(\rho)} \otimes_{\mathcal{O}} \mathcal{O}(2)$$

and more generally, considering an auxiliary connection map of arbitrary weight

$$(7) \quad \nabla_{\text{Log}(\rho)} \otimes \nabla_\xi : \mathcal{V}_{\text{Log}(\rho)} \otimes_{\mathcal{O}} \mathcal{O}(\xi) \rightarrow \mathcal{V}_{\text{Log}(\rho)} \otimes_{\mathcal{O}} \mathcal{O}(\xi + 1).$$

Let $\mathcal{N}_\xi(\rho) := H^0(\mathbb{P}^1, \mathcal{V}_{\text{Log}(\rho)} \otimes_{\mathcal{O}} \mathcal{O}(\xi))$, and furthermore define $\mathcal{N}(\rho) := \bigoplus_{\xi \in \mathbb{Z}} \mathcal{N}_\xi(\rho)$. We refer to $\mathcal{N}(\rho)$ as the *twisted module of ρ* .

Moreover, $\mathcal{N}(\rho)$ is a \mathbb{Z} -graded module of global sections over the ring $\mathbb{C}[x, y]$ where of course $\mathbb{P}^1 = \text{Proj } \mathbb{C}[x, y]$. On global sections, the new connection maps $\nabla_{\text{Log}(\rho)} \otimes \nabla_\xi$ give us a graded derivation of degree one. Indeed, let us denote the connection maps $\nabla_{\text{Log}(\rho)} \otimes \nabla_\xi$ as D_ξ when applied to global sections, so that $D_\xi : \mathcal{N}_\xi(\rho) \rightarrow \mathcal{N}_{\xi+1}(\rho)$. Observe that each D_ξ satisfies the Leibniz rule as each $\nabla_{\text{Log}(\rho)} \otimes \nabla_\xi$ is a connection map and so

$$(8) \quad D := \bigoplus_{\xi \in \mathbb{Z}} D_\xi$$

is a graded derivation of degree one on $\mathcal{N}(\rho)$. We refer to D as the *monodromy derivative of ρ* .

With the use of the monodromy derivative of ρ we are able to prove the following theorem:

Theorem C (Theorem 5.10). *Suppose that $\rho : \mathbb{Z} * \mathbb{Z} \rightarrow \text{GL}_2(\mathbb{C})$ is an irreducible two-dimensional monodromy representation of $\mathbf{Y}_{(0,1,\infty)}$ and let $(\mathcal{V}_{\text{Log}(\rho)}, \nabla_{\text{Log}(\rho)})$ denote the associated extended logarithmic connection. Allow $\zeta = c_1(\mathcal{V}_{\text{Log}(\rho)})$. Then*

$$\mathcal{V}_{\text{Log}(\rho)} \cong \begin{cases} \mathcal{O}(\frac{\zeta}{2})^{\oplus 2} & \text{when } \zeta \text{ is even} \\ \mathcal{O}(\frac{\zeta-1}{2}) \oplus \mathcal{O}(\frac{\zeta+1}{2}) & \text{when } \zeta \text{ is odd.} \end{cases}$$

Witness that the theorem above does not make any assumptions regarding ρ aside from the fact that it is an irreducible representation of dimension 2. Thus, this theorem along with Theorem B close the case on two-dimensional monodromy representations of $\mathbf{Y}_{(0,1,\infty)}$.

Remark 1.1. For a generalization of this result to very general curves of higher genus, see the work of Landesman and Litt [11]. Moreover, Szabó [23] demonstrates that by varying the choice of integer shifts of the residue eigenvalues, one may realize any splitting type permitted by the topology as the underlying bundle of a logarithmic connection.

Conventions and Assumptions.

- All representations ρ are assumed to be over \mathbb{C} .
- Throughout this paper, for any monodromy representation ρ , the associated extended bundle with connection $(\mathcal{V}_{\text{Log}(\rho)}, \nabla_{\text{Log}(\rho)})$ over \mathbb{P}^1 is constructed via the Deligne canonical extension. This means that we fix the section of the logarithm such that the exponents q_j of the eigenvalues $\lambda_j = r_j e^{2\pi i q_j}$ of the local monodromies satisfy $0 \leq q_j < 1$, we refer to this choice of logarithm as the *principal branch*. Unless otherwise stated, “the canonical extension” refers specifically to this Deligne construction.
- We denote the generators of the free group on two generators with γ_0 and γ_1 .

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2. PRELIMINARIES

The purpose of this section is to formally provide some background, introduce notation, and state a special case of the main result from [21], which is a fundamental tool in this work.

Let $\mathbf{Y}_{(p_1, \dots, p_m)} := \mathbb{P}^1 - \{p_1, \dots, p_m\}$. The fundamental group of $\mathbf{Y}_{(p_1, \dots, p_m)}$ is known to be isomorphic to the free group on $m - 1$ generators, $\mathbb{Z}^{*(m-1)}$. Thus, a representation of the fundamental group of $\mathbf{Y}_{(p_1, \dots, p_m)}$ is determined by the images of the generators, each of which we call a *local monodromy*.

Definition 2.1. A *monodromy representation* of $\mathbf{Y}_{(p_1, \dots, p_m)}$ is a representation of its fundamental group.

Definition 2.2. A *holomorphic connection* on $\mathbf{Y}_{(p_1, \dots, p_m)}$ is a pair (\mathcal{V}, ∇) where \mathcal{V} is a locally free sheaf on $\mathbf{Y}_{(p_1, \dots, p_m)}$, and $\nabla : \mathcal{V} \rightarrow \mathcal{V} \otimes \Omega_{\mathbf{Y}_{(p_1, \dots, p_m)}}^1$ is a morphism of sheaves satisfying the Leibniz Rule. When we want to refer to ∇ alone we call it the *connection map*. A morphism of connections $(\mathcal{V}, \nabla) \mapsto (\mathcal{V}', \nabla')$ is a morphism of \mathcal{O} -modules $\phi : \mathcal{V} \rightarrow \mathcal{V}'$ making the diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\nabla} & \mathcal{V} \otimes \Omega_{\mathbf{Y}_{(p_1, \dots, p_m)}}^1 \\ \downarrow \phi & & \downarrow \phi \otimes \text{id} \\ \mathcal{V}' & \xrightarrow{\nabla'} & \mathcal{V}' \otimes \Omega_{\mathbf{Y}_{(p_1, \dots, p_m)}}^1 \end{array}$$

commute.

Theorem 2.3. *Every locally free sheaf on $\mathbf{Y}_{(p_1, \dots, p_m)}$ is free.*

Proof. See [5, Theorem 30.4]. \square

Lemma 2.4. *The category of finite-dimensional monodromy representations of $\mathbf{Y}_{(p_1, \dots, p_m)}$ is equivalent to the category of holomorphic connections on $\mathbf{Y}_{(p_1, \dots, p_m)}$.*

Proof. Combine [24, Corollary 2.6.2] with [24, Proposition 2.7.5]. \square

Definition 2.5. By Lemma 2.4, given a monodromy representation of $\mathbf{Y}_{(p_1, \dots, p_m)}$, ρ , we obtain a holomorphic connection on $\mathbf{Y}_{(p_1, \dots, p_m)}$. We call such a connection the *associated complex connection* and denote it by $(\mathcal{Q}_\rho, \nabla_\rho)$.

Definition 2.6. Let S be a finite number of points of \mathbb{P}^1 and $\Omega_{\mathbb{P}^1}^1(S)$ the sheaf of 1-forms with logarithmic poles along S . A *connection with logarithmic poles along S on \mathbb{P}^1* is a pair $(\bar{\mathcal{E}}, \bar{\nabla})$, where $\bar{\mathcal{E}}$ is a locally free sheaf on \mathbb{P}^1 , and $\bar{\nabla} : \bar{\mathcal{E}} \rightarrow \bar{\mathcal{E}} \otimes_{\mathcal{O}} \Omega^1(S)$ is a \mathbb{C} -linear map satisfying the Leibniz Rule.

Lemma 2.7. *Given a holomorphic connection (\mathcal{V}, ∇) on $\mathbf{Y}_{(p_1, \dots, p_m)}$, there is a unique connection $(\bar{\mathcal{V}}, \bar{\nabla})$ on \mathbb{P}^1 with logarithmic poles along $\{p_1, \dots, p_m\}$ satisfying $(\bar{\mathcal{V}}, \bar{\nabla})|_{\mathbf{Y}_{(p_1, \dots, p_m)}} \cong (\mathcal{V}, \nabla)$ such that the residue at each pole is the principal logarithm of the local monodromies. Moreover, the extension $(\bar{\mathcal{V}}, \bar{\nabla})$ of (\mathcal{V}, ∇) is functorial and exact in (\mathcal{V}, ∇) . This extension is the Deligne canonical extension.*

Proof. See Deligne's work in [2, Proposition 5.4]. \square

Definition 2.8. As a result of applying Lemma 2.7 to Lemma 2.4 we can associate to each monodromy representation ρ of $\mathbf{Y}_{(p_1, \dots, p_m)}$ a unique logarithmic connection over \mathbb{P}^1 . We refer to such connections as the *associated extended logarithmic connection* and denote it as $(\mathcal{V}_{\text{Log}(\rho)}, \nabla_{\text{Log}(\rho)})$.

Definition 2.9. In the setting of Definition 2.8, as $\mathcal{V}_{\text{Log}(\rho)} \cong \bigoplus_{j=1}^{\dim(\rho)} \mathcal{O}(\xi_j)$ we call the twisting parameters ξ_j the *roots*.

Lemma 2.10. *The first Chern class of $\mathcal{V} \in \text{Vect}(\mathbb{P}^1)$, denoted $c_1(\mathcal{V})$, is the sum of the twisting parameters.*

Proof. As every vector bundle over \mathbb{P}^1 splits as a direct sum of twisting sheaves, it follows that $\mathcal{V} \cong \bigoplus_{j=1}^r \mathcal{O}(\xi_j)$, and so $\det(\mathcal{V}) = \det(\bigoplus_{j=1}^r \mathcal{O}(\xi_j)) = \bigotimes_{j=1}^r \mathcal{O}(\xi_j) = \mathcal{O}(\sum_{j=1}^r \xi_j)$. Thus, $c_1(\mathcal{V}) = \deg(\det(\mathcal{V})) = \deg(\mathcal{O}(\sum_{j=1}^r \xi_j)) = \sum_{j=1}^r \xi_j$. \square

Theorem 2.11. *Let (\mathcal{V}, ∇) be a connection on \mathbb{P}^1 with logarithmic poles along $\{p_1, \dots, p_m\}$, then the following holds:*

$$(9) \quad c_1(\mathcal{V}) = - \sum_{j=1}^m \text{Tr}(\text{Res}_{p_j} \nabla)$$

where $\text{Res}_{p_j} \nabla$ is the residue of ∇ at the pole p_j .

Proof. See [21]. \square

Remark 2.12. Observe that Theorem 2.11 provides for us an explicit formula for computing the first Chern class of any associated extended logarithmic connection. Moreover, observe that any higher Chern class vanishes as we are over \mathbb{P}^1 .

Definition 2.13. We refer to Equation (9) as *Ohtsuki's formula*.

Remark 2.14. We note that Theorem 2.3, Lemma 2.7 and Theorem 2.11 are special cases of the much more general statements proved in their respective sources.

3. PUNCTURED COMPLEX LINE

The goal of this section is to compute the roots arising from any monodromy representation of $\mathbf{Y}_{(0, \infty)}$.

Proposition 3.1. *Let $\chi : \mathbb{Z} \rightarrow \mathbb{C}^*$ be a monodromy representation of $\mathbf{Y}_{(0, \infty)}$ given by $1 \mapsto re^{2\pi iq}$ with $0 \leq q < 1$, and $(\mathcal{V}_{\text{Log}(\chi)}, \nabla_{\text{Log}(\chi)})$ the associated extended logarithmic connection. Then*

$$\mathcal{V}_{\text{Log}(\chi)} \cong \begin{cases} \mathcal{O}(0) & \text{when } q = 0 \\ \mathcal{O}(-1) & \text{when } q \neq 0. \end{cases}$$

Proof. Observe that the connection map has two poles, one at 0 and one at ∞ with respective residues $\log(re^{2\pi iq})$ and $\log(\frac{1}{r}e^{-2\pi iq})$.

By Ohtsuki's formula we have that $c_1(\mathcal{V}_{\text{Log}(\chi)}) = -\log(re^{2\pi iq}) - \log(\frac{1}{r}e^{-2\pi iq})$, choosing the branch of \log defined by $0 \leq q < 1$ we see that if $q = 0$, then $c_1(\mathcal{V}_{\text{Log}(\chi)}) = -\log(r) - 0 - \log(r^{-1}) + 0$. However, since $r \in \mathbb{R}$ then $\log(r^{-1}) = -\log(r)$ and so $c_1(\mathcal{V}_{\text{Log}(\chi)}) = -\log(r) + \log(r) = 0$. If $q \neq 0$, as $-q \notin [0, 1)$, we must use $-q + 1$, thus, $c_1(\mathcal{V}_{\text{Log}(\chi)}) = -\log(r) - q + \log(r) + q - 1 = -1$.

Hence,

$$c_1(\mathcal{V}_{\text{Log}(\chi)}) = \begin{cases} 0 & \text{when } q = 0 \\ -1 & \text{when } q \neq 0 \end{cases}$$

and the result follows as a consequence of Lemma 2.10. \square

Lemma 3.2. *Suppose that ρ is a monodromy representation of $\mathbf{Y}_{(0, \infty)}$ defined by $1 \mapsto M$ with Jordan decomposition by blocks M_j with eigenvalue $\lambda_j = re^{2\pi iq_j}$ where $0 \leq q_j < 1$. The first Chern class of the associated extended logarithmic connection $(\mathcal{V}_{\text{Log}(\rho)}, \nabla_{\text{Log}(\rho)})$ is*

$$c_1(\mathcal{V}_{\text{Log}(\rho)}) = \sum_{j=1}^{\alpha} \dim(M_j) c_1(\mathcal{V}_{\text{Log}(\chi_{q_j})})$$

where α is the number of Jordan blocks that make up M .

Proof. Recall the fact that if $S \in \mathrm{GL}_n(\mathbb{C})$ and $T \in \mathrm{Mat}(n \times n, \mathbb{C})$ then

$$(10) \quad \exp(S^{-1}TS) = S^{-1}\exp(T)S.$$

Observe that as a consequence we may suppose that M is in Jordan form. Now, take an arbitrary Jordan block M_j with eigenvalue $\lambda_j = re^{2\pi iq_j}$. Note that by assumption $q_j \in [0, 1)$. Then since $\log(M_j) = \log(\lambda_j)\mathbb{I} + N$ where N is a nilpotent matrix, we have $\mathrm{Tr}(\log(M_j)) = \sum_1^{\dim(M_j)} \log(\lambda_j)$. Next, since we are only interested in the trace, we can choose a basis where M_j^{-1} is a Jordan matrix to easily obtain the trace of $\log(M_j^{-1})$. It is clear that the eigenvalue of M_j^{-1} is λ_j^{-1} , and so

$$\mathrm{Tr}(\log(M_j^{-1})) = \sum_1^{\dim(M_j)} \log(\lambda_j^{-1}).$$

Observe that M_j determines a subrepresentation ρ_j , which by Lemma 2.4 and Lemma 2.7 we have $(\mathcal{V}_{\mathrm{Log}(\rho_j)}, \nabla_{\mathrm{Log}(\rho_j)})$. Thus, by Ohtsuki's formula we may write

$$\begin{aligned} c_1(\mathcal{V}_{\mathrm{Log}(\rho_j)}) &= -\mathrm{Tr}(\log(M_j)) - \mathrm{Tr}(\log(M_j^{-1})) \\ &= \sum_1^{\dim(M_j)} -\log(\lambda_j) - \sum_1^{\dim(M_j)} \log(\lambda_j^{-1}) \\ &= \sum_1^{\dim(M_j)} -\log(\lambda_j) - \log(\lambda_j^{-1}) \\ &= \sum_1^{\dim(M_j)} c_1(\mathcal{V}_{\mathrm{Log}(\chi_{q_j})}) \\ &= \dim(M_j)c_1(\mathcal{V}_{\mathrm{Log}(\chi_{q_j})}). \end{aligned}$$

If M is an indecomposable Jordan block then we are done. If not, then it is clear that $c_1(\mathcal{V}_{\mathrm{Log}(\rho)}) = \sum_{j=1}^{\alpha} \dim(M_j)c_1(\mathcal{V}_{\mathrm{Log}(\chi_{q_j})})$ where α is the number of Jordan blocks that make up M . \square

Theorem 3.3. *Let $\rho : \mathbb{Z} \rightarrow \mathrm{GL}_n(\mathbb{C})$ be an n -dimensional monodromy representation of $\mathbf{Y}_{(0, \infty)}$ with $(\mathcal{V}_{\mathrm{Log}(\rho)}, \nabla_{\mathrm{Log}(\rho)})$ the associated extended logarithmic connection. Allow $\zeta = c_1(\mathcal{V}_{\mathrm{Log}(\rho)})$.*

- (1) *If ρ is an indecomposable representation determined by the Jordan block M with eigenvalue $\lambda = re^{2\pi iq}$ where $0 \leq q < 1$, then*

$$\mathcal{V}_{\mathrm{Log}(\rho)} \cong \begin{cases} \mathcal{O}(0)^{\oplus n} & \text{when } q = 0 \\ \mathcal{O}(-1)^{\oplus n} & \text{when } q \neq 0. \end{cases}$$

- (2) *If $\rho \cong \bigoplus_{i=1}^u \rho_i$ where each ρ_i is indecomposable, then*

$$\mathcal{V}_{\mathrm{Log}(\rho)} \cong \mathcal{O}(-1)^{\oplus -\zeta} \oplus \mathcal{O}(0)^{\oplus n + \zeta}.$$

Proof. For part (1), we proceed by induction on the dimension of ρ .

Base Case. Beginning with a character $\chi : \mathbb{Z} \rightarrow \mathbb{C}^*$ determined by $1 \mapsto re^{2\pi iq}$, we know that by Proposition 3.1

$$\mathcal{V}_{\mathrm{Log}(\chi)} = \begin{cases} \mathcal{O}(0) & \text{when } q = 0 \\ \mathcal{O}(-1) & \text{when } q \neq 0. \end{cases}$$

Inductive Step. Consider a representation of dimension n , $\rho : \mathbb{Z} \rightarrow \mathrm{GL}_n(\mathbb{C})$ with ρ reducible but indecomposable. It follows that ρ is determined by $1 \mapsto M$, where M is a Jordan block with eigenvalue $\lambda = re^{2\pi iq}$. Without loss of generality we may write $0 \rightarrow \chi_q \rightarrow \rho \rightarrow \rho' \rightarrow 0$ which by Lemma 2.4 and

Lemma 2.7 we obtain $0 \rightarrow \mathcal{V}_{\text{Log}(\chi_q)} \rightarrow \mathcal{V}_{\text{Log}(\rho)} \rightarrow \mathcal{V}_{\text{Log}(\rho')} \rightarrow 0$. Moreover, $\mathcal{V}_{\text{Log}(\chi_q)} \cong \mathcal{O}(\xi_q)$ by the base case and $\mathcal{V}_{\text{Log}(\rho')} \cong \mathcal{O}(\xi_q)^{\oplus(n-1)}$ by the inductive assumption.

Hence, we have $0 \rightarrow \mathcal{O}(\xi_q) \rightarrow \mathcal{V}_{\text{Log}(\rho)} \rightarrow \mathcal{O}(\xi_q)^{\oplus(n-1)} \rightarrow 0$. Now, note that

$$\begin{aligned} H^1(\mathbb{P}^1, (\mathcal{O}(\xi_q)^{\oplus(n-1)})^* \otimes \mathcal{O}(\xi_q)) &= H^1(\mathbb{P}^1, \mathcal{O}(-\xi_q)^{\oplus(n-1)} \otimes \mathcal{O}(\xi_q)) \\ &= H^1(\mathbb{P}^1, \mathcal{O}(0)^{\oplus(n-1)}). \end{aligned}$$

Further, by Serre-Duality we have

$$\begin{aligned} H^1(\mathbb{P}^1, \mathcal{O}(0)^{\oplus(n-1)}) &= H^0(\mathbb{P}^1, (\mathcal{O}(0)^{\oplus(n-1)})^* \otimes \Omega_{\mathbb{P}^1}^1)^* \\ &= H^0(\mathbb{P}^1, \mathcal{O}(-2)^{\oplus(n-1)})^*. \end{aligned}$$

Of course $\dim H^0(\mathbb{P}^1, \mathcal{O}(-2)^{\oplus(n-1)})^* = 0$ and so $0 \rightarrow \mathcal{O}(\xi_q) \rightarrow \mathcal{V}_{\text{Log}(\rho)} \rightarrow \mathcal{O}(\xi_q)^{\oplus(n-1)} \rightarrow 0$ splits. Thus, we have the desired result which is

$$\mathcal{V}_{\text{Log}(\rho)} \cong \begin{cases} \mathcal{O}(0)^{\oplus n} & \text{when } q = 0 \\ \mathcal{O}(-1)^{\oplus n} & \text{when } q \neq 0. \end{cases}$$

For part (2), let $\rho : \mathbb{Z} \rightarrow \text{GL}_n(\mathbb{C})$ be a monodromy representation of $\mathbf{Y}_{(0,\infty)}$ with $(\mathcal{V}_{\text{Log}(\rho)}, \nabla_{\text{Log}(\rho)})$ the associated extended logarithmic connection where $\rho \cong \bigoplus_{i=1}^u \rho_i$ such that each ρ_i is indecomposable. As a result of Lemma 2.4 and Lemma 2.7, we may write $\mathcal{V}_{\text{Log}(\rho)} \cong \bigoplus_{i=1}^u \mathcal{V}_{\text{Log}(\rho_i)}$. Since each ρ_i is assumed to be indecomposable then we apply part (1) above to obtain that for each i , $\mathcal{V}_{\text{Log}(\rho_i)} \cong \mathcal{O}(-1)^{\oplus \dim(\rho_i)}$ or $\mathcal{V}_{\text{Log}(\rho_i)} \cong \mathcal{O}(0)^{\oplus \dim(\rho_i)}$ dependent on the argument of the eigenvalue associated with the image of the generator. Now, allow \mathcal{E} to be the direct sum of all the subbundles $\mathcal{V}_{\text{Log}(\rho_j)}$ whose root is -1 , then $\mathcal{E} = \bigoplus_{j=1}^\alpha \mathcal{O}(-1)^{\oplus \dim(\rho_j)} = \mathcal{O}(-1)^{\oplus \sum_{j=1}^\alpha \dim(\rho_j)} = \mathcal{O}(-1)^{\oplus -c_1(\mathcal{V}_{\text{Log}(\rho)})}$ where the last equality follows from Lemma 3.2 combined with Proposition 3.1. The remaining subbundles all have root 0 and the dimension of their direct sum must be $n + c_1(\mathcal{V}_{\text{Log}(\rho)})$. \square

Remark 3.4. Recall that \mathbb{Z} does not admit irreducible representations of dimension greater than one. As a consequence, Proposition 3.1 along with Theorem 3.3 close the case on monodromy representations of $\mathbf{Y}_{(0,\infty)}$.

4. DOUBLE PUNCTURED COMPLEX LINE

The goal of this section is to compute the roots arising from any two-dimensional reducible representation of $\mathbf{Y}_{(0,1,\infty)}$.

Proposition 4.1. *Let $\chi : \mathbb{Z} * \mathbb{Z} \rightarrow \mathbb{C}^*$ be a monodromy representation of $\mathbf{Y}_{(0,1,\infty)}$ given by $\gamma_0 \mapsto r_0 e^{2\pi i q_0}$ and $\gamma_1 \mapsto r_1 e^{2\pi i q_1}$, with $(\mathcal{V}_{\text{Log}(\chi)}, \nabla_{\text{Log}(\chi)})$ the associated extended logarithmic connection. Then*

$$\mathcal{V}_{\text{Log}(\chi)} \cong \begin{cases} \mathcal{O}(0) & \text{when } q_j = 0 \\ \mathcal{O}(-1) & \text{when } \sum q_j \leq 1 \text{ and not both } q_j = 0 \\ \mathcal{O}(-2) & \text{when } \sum q_j > 1. \end{cases}$$

Proof. Observe that the connection map has three poles, one at 0, one at 1, and one at ∞ with respective residues $\log(r_0 e^{2\pi i q_0})$, $\log(r_1 e^{2\pi i q_1})$, $\log((r_0 r_1)^{-1} e^{-2\pi i(q_0 + q_1)})$. By Ohtsuki's formula we have that $c_1(\mathcal{V}_{\text{Log}(\chi)}) = -\log(r_0 e^{2\pi i q_0}) - \log(r_1 e^{2\pi i q_1}) - \log(\frac{1}{r_0 r_1} e^{-2\pi i(q_0 + q_1)})$, consequently, choosing the branch of log defined by $0 \leq q_j < 1$ we have the following set of cases.

Case 1. If $q_j = 0$ it is clear that $c_1(\mathcal{V}_{\text{Log}(\chi)}) = 0$.

Case 2. Suppose that $\sum q_j \leq 1$ and either $q_0 \neq 0$ or $q_1 \neq 0$. Since by assumption $\sum q_j \leq 1$ and not both q_j are zero, then $-\sum q_j \notin [0, 1)$ and so we must use $-\sum q_j + 1$, thus, $c_1(\mathcal{V}_{\text{Log}(\chi)}) = -q_0 - q_1 - (-q_0 - q_1 + 1) = -1$.

Case 3. Suppose that $\sum q_j > 1$. Then, since $\sum q_j > 1$ it follows that $-\sum q_j \in (-2, -1)$, so we must use $-\sum q_j + 2 \in (0, 1)$. Thus, $c_1(\mathcal{V}_{\text{Log}(\chi)}) = -q_0 - q_1 - (-q_0 - q_1 + 2) = -2$.

Observe that the result follows from Lemma 2.10. \square

Remark 4.2. Before introducing the main result of this section, we first prove the following lemma regarding the first Chern class of associated extended logarithmic connections that arise from monodromy representations with some specific constraints.

Lemma 4.3. *Suppose that $0 \rightarrow \rho' \rightarrow \rho \rightarrow \rho'' \rightarrow 0$ is a short exact sequence of monodromy representations of $\mathbf{Y}_{(0,1,\infty)}$ with ρ two-dimensional. Allow ρ to be defined by $\gamma_0 \mapsto M_0$ and $\gamma_1 \mapsto M_1$ where $\lambda_0 = r_{\lambda,0}e^{2\pi i q_{\lambda,0}}$ and $\mu_0 = r_{\mu,0}e^{2\pi i q_{\mu,0}}$ are the eigenvalues of M_0 , while $\lambda_1 = r_{\lambda,1}e^{2\pi i q_{\lambda,1}}$ and $\mu_1 = r_{\mu,1}e^{2\pi i q_{\mu,1}}$ are the eigenvalues of M_1 with all q such that $0 \leq q < 1$. Further, suppose that the λ s determine ρ' and the μ s determine ρ'' . Then, if $\sum q_{\lambda,\iota} > 1$ and $q_{\mu,j} = 0$, the first Chern class of the associated extended logarithmic connection $(\mathcal{V}_{\text{Log}(\rho)}, \nabla_{\text{Log}(\rho)})$ is $c_1(\mathcal{V}_{\text{Log}(\rho)}) = -2$.*

Proof. Note that by the classification of 2-dimensional representations of the free group with two generators found in [10, Ch 2, Theorem 4.2.1], there are three conjugacy classes of ρ that fit the assumptions that ρ is reducible, $\sum q_{\lambda,\iota} > 1$, and $q_{\mu,j} = 0$ with $0 \leq q < 1$. The three conjugacy classes are given by

$$\begin{aligned} M_0 &\leftrightarrow \begin{pmatrix} \lambda_0 & 0 \\ 0 & \mu_0 \end{pmatrix}, \quad M_1 \leftrightarrow \begin{pmatrix} \lambda_1 & 0 \\ 0 & \mu_1 \end{pmatrix}; \\ M_0 &\leftrightarrow \begin{pmatrix} \lambda_0 & 0 \\ 0 & \mu_0 \end{pmatrix}, \quad M_1 \leftrightarrow \begin{pmatrix} \lambda_1 & 1 \\ 0 & \mu_1 \end{pmatrix}; \\ M_0 &\leftrightarrow \begin{pmatrix} \mu_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}, \quad M_1 \leftrightarrow \begin{pmatrix} \mu_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}. \end{aligned}$$

The connection map has three poles, one at 0, one at 1, and one at ∞ with respective residues $\log(M_0)$, $\log(M_1)$, $\log((M_0M_1)^{-1})$. By Ohtsuki's formula, $c_1(\mathcal{V}_{\text{Log}(\rho)}) = -\text{Tr}(\log(M_0)) - \text{Tr}(\log(M_1)) - \text{Tr}(\log((M_0M_1)^{-1}))$. In the case where both M_0 and M_1 are diagonalizable it is clear that

$$\begin{aligned} c_1(\mathcal{V}_{\text{Log}(\rho)}) &= -\log(r_{\lambda,0}e^{2\pi i q_{\lambda,0}}) - \log(r_{\mu,0}e^{2\pi i(0)}) - \log(r_{\lambda,1}e^{2\pi i q_{\lambda,1}}) - \log(r_{\mu,1}e^{2\pi i(0)}) \\ &\quad - \log\left(\frac{e^{2\pi i(2 - \sum q_{\lambda,\iota})}}{r_{\lambda,0}r_{\lambda,1}}\right) - \log\left(\frac{e^{2\pi i(0)}}{r_{\mu,0}r_{\mu,1}}\right) \end{aligned}$$

where $2 - \sum q_{\lambda,\iota}$ arises from the fact that $\sum q_{\lambda,\iota} > 1$ along with the choice of log as in the proof of Proposition 4.1. Moreover, note that all the r s cancel out. Thus,

$$\begin{aligned} c_1(\mathcal{V}_{\text{Log}(\rho)}) &= -\log(\lambda_0) - \log(\mu_0) - \log(\lambda_1) - \log(\mu_1) - \log((\lambda_0\lambda_1)^{-1}) - \log((\mu_0\mu_1)^{-1}) \\ &= -q_{\lambda,0} - q_{\lambda,1} - (-q_{\lambda,0} - q_{\lambda,1} + 2) \\ &= -2. \end{aligned}$$

For the second case, again by Ohtsuki's formula, $c_1(\mathcal{V}_{\text{Log}(\rho)}) = -\text{Tr}(\log(M_0)) - \text{Tr}(\log(M_1)) - \text{Tr}(\log((M_0M_1)^{-1}))$. The eigenvalues of $\log(M_0)$ are $\log(\lambda_0)$ and $\log(\mu_0)$. By assumption $\sum q_{\lambda,\iota} > 1$, $q_{\mu,j} = 0$, and by the choice of $0 \leq q < 1$ for all q . As a result $\lambda_1 \neq \mu_1$ and hence there exists a basis Υ' where M_1 is diagonal. In the basis Υ' , the trace of $\log(M_1)$ is $\log(\lambda_1) + \log(\mu_1)$. With regards to $(M_0M_1)^{-1}$, we know that M_0M_1 is upper triangular with diagonal entries $\lambda_0\lambda_1$ and $\mu_0\mu_1$, further, as a consequence of $\sum q_{\lambda,\iota} > 1$ and $q_{\mu,j} = 0$, with $0 \leq q < 1$, it follows that $\lambda_0\lambda_1 \neq \mu_0\mu_1$. Hence, there exists a basis, Υ'' , where M_0M_1 is diagonal. In the basis Υ'' , the trace of $\log((M_0M_1)^{-1})$ is $\log((\lambda_0\lambda_1)^{-1}) + \log((\mu_0\mu_1)^{-1})$. Thus, this case reduces to the diagonal case. The third case follows the exact same procedure as the second case. \square

Theorem 4.4. *Suppose that $0 \rightarrow \rho' \rightarrow \rho \rightarrow \rho'' \rightarrow 0$ is a short exact sequence of monodromy representations of $\mathbf{Y}_{(0,1,\infty)}$ with ρ two-dimensional and ρ', ρ'' one-dimensional. Now, let $0 \rightarrow \mathcal{O}(\xi') \rightarrow \mathcal{V}_{\text{Log}(\rho)} \rightarrow \mathcal{O}(\xi'') \rightarrow 0$ be the short exact sequence of the underlying vector bundles from the associated extended logarithmic connections.*

- (1) *If $\xi', \xi'' \neq -2, 0$ respectively, then $0 \rightarrow \mathcal{O}(\xi') \rightarrow \mathcal{V}_{\text{Log}(\rho)} \rightarrow \mathcal{O}(\xi'') \rightarrow 0$ splits.*
- (2) *If $\xi', \xi'' = -2, 0$ respectively, then $\mathcal{V}_{\text{Log}(\rho)} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{V}_{\text{Log}(\rho)} \cong \mathcal{O}(0) \oplus \mathcal{O}(-2)$.*

Proof. Let $\rho : \mathbb{Z} * \mathbb{Z} \rightarrow \mathrm{GL}_2(\mathbb{C})$ be a reducible 2-dimensional representation defined by $\gamma_0 \mapsto M_0$ and $\gamma_1 \mapsto M_1$, where λ_0, μ_0 and λ_1, μ_1 are the eigenvalues of M_0 and M_1 respectively. Further, let $\lambda_0 = r_{\lambda,0} e^{2\pi i q_{\lambda,0}}$, $\mu_0 = r_{\mu,0} e^{2\pi i q_{\mu,0}}$, $\lambda_1 = r_{\lambda,1} e^{2\pi i q_{\lambda,1}}$, and $\mu_1 = r_{\mu,1} e^{2\pi i q_{\mu,1}}$. As ρ is reducible, then by [10, Ch 2, Theorem 4.2.1] there exists a basis $\hat{\Upsilon}$ in which

$$M_0 \leftrightarrow \begin{pmatrix} \lambda_0 & \eta_0 \\ 0 & \mu_0 \end{pmatrix} \text{ and } M_1 \leftrightarrow \begin{pmatrix} \lambda_1 & \eta_1 \\ 0 & \mu_1 \end{pmatrix}$$

and moreover, we may write $0 \rightarrow \rho' \rightarrow \rho \rightarrow \rho'' \rightarrow 0$, where ρ' is determined by the λ s and ρ'' is determined by the μ s.

For (1), using cohomology,

$$\begin{aligned} H^1(\mathbb{P}^1, (\mathcal{O}(\xi'')^* \otimes \mathcal{O}(\xi'))) &= H^1(\mathbb{P}^1, \mathcal{O}(-\xi'') \otimes \mathcal{O}(\xi')) \\ &= H^1(\mathbb{P}^1, \mathcal{O}(-\xi'' + \xi')). \end{aligned}$$

Further by Serre-Duality we have

$$\begin{aligned} H^1(\mathbb{P}^1, \mathcal{O}(-\xi'' + \xi')) &= H^0(\mathbb{P}^1, (\mathcal{O}(-\xi'' + \xi')^* \otimes \Omega_{\mathbb{P}^1}^1))^* \\ &= H^0(\mathbb{P}^1, \mathcal{O}(\xi'' - \xi' - 2))^*. \end{aligned}$$

Hence, if $\xi'' - \xi' < 2$ then $\mathcal{V}_{\mathrm{Log}(\rho)} = \mathcal{O}(\xi') \oplus \mathcal{O}(\xi'')$. At this point observe that by Proposition 4.1 we know that

$$\xi' = \begin{cases} 0 & \text{when } q_{\lambda,i} = 0 \\ -1 & \text{when } \sum q_{\lambda,i} \leq 1 \text{ and not both } q_{\lambda,i} = 0 \\ -2 & \text{when } \sum q_{\lambda,i} > 1 \end{cases}$$

and similarly,

$$\xi'' = \begin{cases} 0 & \text{when } q_{\mu,i} = 0 \\ -1 & \text{when } \sum q_{\mu,i} \leq 1 \text{ and not both } q_{\mu,i} = 0 \\ -2 & \text{when } \sum q_{\mu,i} > 1. \end{cases}$$

It is clear that the only case that does not satisfy the inequality $\xi'' - \xi' < 2$, is when $\xi' = -2$ and $\xi'' = 0$.

For part (2), the event that $\xi' = -2$ and $\xi'' = 0$ gives rise to the following short exact sequence $0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{V}_{\mathrm{Log}(\rho)} \rightarrow \mathcal{O}(0) \rightarrow 0$. Looking at the long exact sequence in cohomology

$$\begin{aligned} 0 \rightarrow H^0(\mathbb{P}^1, \mathcal{O}(-2)) \rightarrow H^0(\mathbb{P}^1, \mathcal{V}_{\mathrm{Log}(\rho)}) \rightarrow H^0(\mathbb{P}^1, \mathcal{O}(0)) \xrightarrow{\delta} H^1(\mathbb{P}^1, \mathcal{O}(-2)) \\ \rightarrow H^1(\mathbb{P}^1, \mathcal{V}_{\mathrm{Log}(\rho)}) \rightarrow H^1(\mathbb{P}^1, \mathcal{O}(0)) \dots \end{aligned}$$

we note that $H^0(\mathbb{P}^1, \mathcal{O}(-2))$ and $H^1(\mathbb{P}^1, \mathcal{O}(0))$ are zero-dimensional complex vector spaces while $H^1(\mathbb{P}^1, \mathcal{O}(-2))$ is 1-dimensional. Hence, δ is either an isomorphism or the 0 map.

If δ is an isomorphism, then $\dim(H^0(\mathbb{P}^1, \mathcal{V}_{\mathrm{Log}(\rho)})) = 0$, meaning that $\mathcal{V}_{\mathrm{Log}(\rho)} \cong \mathcal{O}(\xi_1) \oplus \mathcal{O}(\xi_2)$ with $\xi_1, \xi_2 \in \mathbb{Z}^-$. If δ is the 0 map, then $H^0(\mathbb{P}^1, \mathcal{V}_{\mathrm{Log}(\rho)}) \cong \mathbb{C}$, meaning that $\mathcal{V}_{\mathrm{Log}(\rho)} \cong \mathcal{O}(0) \oplus \mathcal{O}(\xi)$ with $\xi \in \mathbb{Z}^-$.

However, given that we are working in the scenario where $\sum q_{\lambda,i} > 1$ and $q_{\mu,i} = 0$, then by Lemma 4.3, $c_1(\mathcal{V}_{\mathrm{Log}(\rho)}) = -2$. Thus, if δ is an isomorphism then $\mathcal{V}_{\mathrm{Log}(\rho)} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, whereas if δ is the 0 map, then $\mathcal{V}_{\mathrm{Log}(\rho)} \cong \mathcal{O}(0) \oplus \mathcal{O}(-2)$. \square

5. THE MONODROMY DERIVATIVE

Examining the case of $\mathbf{Y}_{(0,\infty)}$, we recall that \mathbb{Z} does not admit irreducible representations of dimension greater than one, as was highlighted by Remark 3.4. This meant that on every occasion we considered representations ρ with $\dim(\rho) > 1$ for which we were able to construct a short exact sequence $0 \rightarrow \rho' \rightarrow \rho \rightarrow \rho'' \rightarrow 0$. As a result of this short exact sequence we were allowed to apply sheaf cohomology on the resulting short exact sequence of the underlying vector bundles of the associated extended logarithmic connections.

When studying $\mathbf{Y}_{(0,1,\infty)}$, its fundamental group is $\mathbb{Z} * \mathbb{Z}$, which admits irreducible representations of dimension greater than 1, particularly of dimension 2 as seen in [10, Ch 2, Theorem 4.2.1]. Thus, the sheaf cohomology approach is not a fruitful approach when ρ is irreducible of dimension greater than one.

With regard to the case when ρ is irreducible, the idea is to construct a Modular derivative-like operator arising from $\mathbf{Y}_{(0,1,\infty)}$.

5.1. Auxiliary Connection. The construction of the Modular derivative-like operator will be two fold carried in this subsection and the next: concluding with Definition 5.5. The purpose of this subsection is to develop what we will call an *auxiliary connection* arising from $\mathbf{Y}_{(0,1,\infty)}$.

Take the character representation $\chi : \pi_1(\mathbf{Y}_{(0,1,\infty)}) \rightarrow \mathbb{C}^*$ defined by mapping both generators to -1 , applying Proposition 4.1, the resulting associated extended logarithmic connection is $(\mathcal{V}_{\text{Log}(\chi)}, \nabla_{\text{Log}(\chi)})$ over \mathbb{P}^1 with $\mathcal{V}_{\text{Log}(\chi)} \cong \mathcal{O}(-1)$. Moreover, we dualize and obtain $(\mathcal{O}(1), \nabla_{\text{Log}(\chi)}^*)$. At this point we make a simple change in notation and refer to $\nabla_{\text{Log}(\chi)}^*$ simply as ∇ . Now globally,

$$(11) \quad \nabla : \mathcal{O}(1) \rightarrow \mathcal{O}(1) \otimes_{\mathcal{O}} \Omega_{\mathbb{P}^1}^1([0] + [1] + [\infty])$$

where

$$\begin{aligned} \Omega_{\mathbb{P}^1}^1([0] + [1] + [\infty]) &= \Omega_{\mathbb{P}^1}^1 \otimes_{\mathcal{O}} \mathcal{O}([0] + [1] + [\infty]) \\ &\cong \mathcal{O}(-2) \otimes_{\mathcal{O}} \mathcal{O}(3) \\ &\cong \mathcal{O}(1) \end{aligned}$$

and so

$$(12) \quad \nabla : \mathcal{O}(1) \rightarrow \mathcal{O}(2).$$

Next, we take $\nabla^{\otimes 2}$ to be the connection map on $\mathcal{O}(1) \otimes \mathcal{O}(1)$ such that for all sections $s, t \in \mathcal{O}(1)$ the following is satisfied: $(\nabla \otimes \nabla)(s \otimes t) = \nabla(s) \otimes t + s \otimes \nabla(t)$. Observe that

$$(13) \quad \nabla^{\otimes 2} : \mathcal{O}(2) \rightarrow \mathcal{O}(2) \otimes_{\mathcal{O}} \Omega_{\mathbb{P}^1}^1([0] + [1] + [\infty]),$$

and so, inductively, we define $\nabla_{\xi} := \nabla^{\otimes \xi} : \mathcal{O}(\xi) \rightarrow \mathcal{O}(\xi) \otimes_{\mathcal{O}} \Omega_{\mathbb{P}^1}^1([0] + [1] + [\infty])$. Hence, using the fact that $\Omega_{\mathbb{P}^1}^1([0] + [1] + [\infty]) \cong \mathcal{O}(1)$, then

$$(14) \quad \nabla_{\xi} : \mathcal{O}(\xi) \rightarrow \mathcal{O}(\xi + 1).$$

Definition 5.1. In the construction above we refer to ∇_{ξ} as the *auxiliary connection map of weight ξ arising from $\mathbf{Y}_{(0,1,\infty)}$* .

Remark 5.2. The construction of the auxiliary connection map can be generalized as arising from the m th punctured projective line. However, observe that as the number of removed points increases so does the degree of the auxiliary connection arising from the space. To see this explicitly, let p_i be points of \mathbb{P}^1 , then

$$\begin{aligned} \Omega_{\mathbb{P}^1}^1\left(\sum_{i=1}^m [p_i]\right) &= \Omega_{\mathbb{P}^1}^1 \otimes_{\mathcal{O}} \mathcal{O}\left(\sum_{i=1}^m [p_i]\right) \\ &\cong \mathcal{O}(-2) \otimes_{\mathcal{O}} \mathcal{O}(m) \\ &\cong \mathcal{O}(m - 2) \end{aligned}$$

and so arising from \mathbb{P}^1 minus m points

$$(15) \quad \nabla : \mathcal{O}(1) \rightarrow \mathcal{O}(m - 1).$$

Hence, which is why when the auxiliary connection map arises from $\mathbf{Y}_{(0,1,\infty)}$ the co-domain is $\mathcal{O}(2)$ and whereas if it arises from $\mathbf{Y}_{(0,\infty)}$ the co-domain would be $\mathcal{O}(1)$.

Example 5.3. Consider $\mathbf{Y}_{(0,\infty)}$ whose fundamental group is \mathbb{Z} . Let χ be a character representation that maps the generator to -1 , then after dualizing, the associated extended vector bundle at hand is $\mathcal{O}(1)$. It follows that the auxiliary connection arising from $\mathbf{Y}_{(0,\infty)}$ is a map such that

$$(16) \quad \nabla : \mathcal{O}(1) \rightarrow \mathcal{O}(1) \otimes_{\mathcal{O}} \Omega^1([0] + [\infty]).$$

Observe that $\Omega^1([0] + [\infty]) \stackrel{\phi}{\cong} \mathcal{O}(0)$ by $\phi(\frac{dz}{z}) = 1$, which means that

$$(17) \quad \nabla : \mathcal{O}(1) \rightarrow \mathcal{O}(1).$$

Over global sections, ∇ gives rise to a degree preserving map $\delta : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]$, where x has a zero at 0, and y has a zero at ∞ . Now, since δ is a degree preserving map, we must have $\delta x = a_1 x + a_2 y$ and $\delta y = b_1 x + b_2 y$ with $a_j, b_i \in \mathbb{C}$.

Let $U_0 := \mathbb{P}^1 - \{0\}$ and $U_\infty := \mathbb{P}^1 - \{\infty\}$. Restricting over U_∞ gives, $\text{Res}(\delta x) = \text{Res}(a_1 x + a_2 y)$, and hence, $\delta z = a_1 z + a_2$ as δ commutes with restriction, where $z = \frac{x}{y}$ is the local coordinate on U_∞ . Moreover, as locally over $\mathbf{Y}_{(0,\infty)}$, $\nabla = d - \frac{1}{2} \frac{dz}{z}$, then

$$\begin{aligned} \delta z &= dz - \frac{dz}{2z}(z) \\ &= z \frac{dz}{z} - \frac{z}{2} \frac{dz}{z} \\ &= \left(\frac{1}{2}z\right) \frac{dz}{z} \end{aligned}$$

however, after applying ϕ we are left with $\delta z = \frac{1}{2}z$. At this point we compare coefficients, $a_1 = \frac{1}{2}$ and $a_2 = 0$. Similarly for U_0 , we arrive at $b_1 = 0$ and $b_2 = -\frac{1}{2}$. Thus, $\delta x = \frac{1}{2}x$ and $\delta y = -\frac{1}{2}y$.

5.2. Monodromy Derivative. We now return to the case at hand, that is, considering irreducible monodromy representations of $\mathbf{Y}_{(0,1,\infty)}$ of dimension 2. Justifying the naming in Definition 5.1 we have the following: given a monodromy representation ρ , the associated extended logarithmic connection $(\mathcal{V}_{\text{Log}(\rho)}, \nabla_{\text{Log}(\rho)})$ can be seen as a triple, $(\mathcal{V}_{\text{Log}(\rho)}, \nabla_{\text{Log}(\rho)}, \nabla)$, where ∇ is the auxiliary connection map arising from $\mathbf{Y}_{(0,1,\infty)}$.

Thus, keeping in mind that $\Omega_{\mathbb{P}^1}^1([0] + [1] + [\infty]) \cong \mathcal{O}(1)$, we now have the extra tool

$$(18) \quad \nabla_{\text{Log}(\rho)} \otimes \nabla : \mathcal{V}_{\text{Log}(\rho)} \otimes_{\mathcal{O}} \mathcal{O}(1) \rightarrow \mathcal{V}_{\text{Log}(\rho)} \otimes_{\mathcal{O}} \mathcal{O}(2)$$

and more generally, considering an auxiliary connection map of arbitrary weight

$$(19) \quad \nabla_{\text{Log}(\rho)} \otimes \nabla_\xi : \mathcal{V}_{\text{Log}(\rho)} \otimes_{\mathcal{O}} \mathcal{O}(\xi) \rightarrow \mathcal{V}_{\text{Log}(\rho)} \otimes_{\mathcal{O}} \mathcal{O}(\xi + 1).$$

Definition 5.4. Let $\mathcal{N}_\xi(\rho) := H^0(\mathbb{P}^1, \mathcal{V}_{\text{Log}(\rho)} \otimes_{\mathcal{O}} \mathcal{O}(\xi))$, and furthermore, define $\mathcal{N}(\rho) := \bigoplus_{\xi \in \mathbb{Z}} \mathcal{N}_\xi(\rho)$. We refer to $\mathcal{N}(\rho)$ as the *twisted module of ρ* .

Since $\mathcal{V}_{\text{Log}(\rho)}$ is a direct sum of twisting sheaves it is easy to see that there exists a ξ_{\min} such that for any $\xi_j < \xi_{\min}$ we have $\mathcal{N}_{\xi_j}(\rho) = 0$. We refer to ξ_{\min} as the *minimal weight* of $\mathcal{N}(\rho)$.

Moreover, $\mathcal{N}(\rho)$ is a \mathbb{Z} -graded module of global sections over the ring $R := \mathcal{N}(\mathbf{1}) = \mathbb{C}[x, y]$ where, of course, $\mathbb{P}^1 = \text{Proj } \mathbb{C}[x, y]$ and $\mathbf{1}$ is the trivial one-dimensional representation. By the Birkhoff-Grothendieck theorem along with the fact that H^0 and direct sums commute, this module is free of

rank equal to the dimension of the representation; indeed,

$$(20) \quad \mathcal{N}(\rho) = \bigoplus_{\xi \in \mathbb{Z}} H^0(\mathbb{P}^1, \mathcal{V}_{\text{Log}(\rho)} \otimes_{\mathcal{O}} \mathcal{O}(\xi))$$

$$(21) \quad \cong \bigoplus_{\xi \in \mathbb{Z}} H^0(\mathbb{P}^1, \bigoplus_{i=1}^{\dim(\rho)} \mathcal{O}(\xi_i + \xi))$$

$$(22) \quad \cong \bigoplus_{\xi \in \mathbb{Z}} \bigoplus_{i=1}^{\dim(\rho)} H^0(\mathbb{P}^1, \mathcal{O}(\xi_i + \xi))$$

$$(23) \quad \cong \bigoplus_{i=1}^{\dim(\rho)} R[-\xi_i]$$

thus, any choice of homogeneous generators gives an isomorphism

$$(24) \quad \mathcal{N}(\rho) \cong \bigoplus_{i=1}^{\dim(\rho)} R[-\xi_i]$$

where by $R[a]$ we denote the rank one graded module over R obtained by shifting the grading by a . On global sections, note that the new connection map $\nabla_{\text{Log}(\rho)} \otimes \nabla_{\xi}$ gives us a graded derivation of degree one. Indeed, let us denote the connection map $\nabla_{\text{Log}(\rho)} \otimes \nabla_{\xi}$ as D_{ξ} when applied to global sections, so that

$$(25) \quad D_{\xi} : \mathcal{N}_{\xi}(\rho) \rightarrow \mathcal{N}_{\xi+1}(\rho)$$

where of course D_{ξ} satisfies the Leibniz rule as $\nabla_{\text{Log}(\rho)} \otimes \nabla_{\xi}$ is a connection map and so

$$(26) \quad D := \bigoplus_{\xi \in \mathbb{Z}} D_{\xi}$$

is a graded derivation of degree one on $\mathcal{N}(\rho)$.

Definition 5.5. We refer to the the graded derivation D as the *monodromy derivative* of ρ .

Remark 5.6. Analogous to Remark 5.2, it is not hard to see how to generalize the monodromy derivative not only to any dimension of a monodromy representation of $\mathbf{Y}_{(0,1,\infty)}$, but to any monodromy representation of an m th punctured projective line. We provide a rough sketch of how to generalize. Let $\mathbf{Y}_{(m\text{th})}$ be the m th punctured projective line. Given a monodromy representation of $\mathbf{Y}_{(m\text{th})}$ we obtain the triplet $(\mathcal{V}_{\text{Log}(\rho)}, \nabla_{\text{Log}(\rho)}, \nabla)$ where ∇ is the auxiliary connection map arising from $\mathbf{Y}_{(m\text{th})}$. Next, we tensor the connection maps. However, as mentioned in Remark 5.2 the co-domain of the connection map $\nabla_{\text{Log}(\rho)} \otimes \nabla$ changes according to the number of removed points, hence,

$$(27) \quad \nabla_{\text{Log}(\rho)} \otimes \nabla : \mathcal{V}_{\text{Log}(\rho)} \otimes_{\mathcal{O}} \mathcal{O}(1) \rightarrow \mathcal{V}_{\text{Log}(\rho)} \otimes_{\mathcal{O}} \mathcal{O}(m-1).$$

We can take global sections as before and create $\mathcal{N}(\rho)$. Further, denoting the connection map $\nabla_{\text{Log}(\rho)} \otimes \nabla_{\xi}$ as D_{ξ} when applied to global sections and defining $D := \bigoplus_{\xi \in \mathbb{Z}} D_{\xi}$ we have a graded derivation on $\mathcal{N}(\rho)$.

Remark 5.7. Now that we have constructed the monodromy derivative we will use it to prove the main theorem of this section, but first we direct our attention to the following two lemmas which will be used in the proof of the main theorem.

Lemma 5.8. *Let ξ_{\min} be the minimal weight of the twisted module $\mathcal{N}(\rho)$ for some n -dimensional irreducible monodromy representation ρ of $\mathbf{Y}_{(0,1,\infty)}$ with $n \geq 2$. Let $\xi_{\min} \leq k$ and suppose that $f \in \mathcal{N}_k(\rho)$ with $f \neq 0$, then $D_k(f) \neq 0$.*

Proof. For the sake of contradiction let us assume that $D_k(f) = 0$.

Restricting to $\mathbf{Y}_{(0,1,\infty)}$, we see that $f|_{\mathbf{Y}_{(0,1,\infty)}}$ is a global section of the associated complex bundle \mathcal{Q}_ρ (see Definition 2.5) because $(\mathcal{V}_{\text{Log}(\rho)} \otimes \mathcal{O}(k))|_{\mathbf{Y}_{(0,1,\infty)}} \cong \mathcal{Q}_\rho$. Denote with $\mathcal{F} \subset \mathcal{Q}_\rho$ the subbundle generated by $f|_{\mathbf{Y}_{(0,1,\infty)}}$. It is clear that the following diagram commutes

$$\begin{array}{ccc} \mathcal{V}_{\text{Log}(\rho)} \otimes \mathcal{O}(k) & \xrightarrow{\nabla_{\text{Log}(\rho)} \otimes \nabla} & \mathcal{V}_{\text{Log}(\rho)} \otimes \mathcal{O}(k+1) \\ \downarrow \varphi & & \downarrow \varphi \\ \mathcal{Q}_\rho & \xrightarrow{(\nabla_{\text{Log}(\rho)} \otimes \nabla)|_{\mathbf{Y}_{(0,1,\infty)}}} & \mathcal{Q}_\rho \end{array}$$

where φ denotes the restriction map.

Since by assumption $D_k(f) = 0$, then after restricting $D_k|_{\mathbf{Y}_{(0,1,\infty)}}(f|_{\mathbf{Y}_{(0,1,\infty)}}) = 0$. Moreover, by Theorem 2.3, \mathcal{Q}_ρ is free over $\mathbf{Y}_{(0,1,\infty)} = \text{Spec}(A)$. Hence, by Theorem 2.3, \mathcal{F} can be viewed as a free module of rank one over A whose generator is stable under $D|_{\mathbf{Y}_{(0,1,\infty)}}$. Ergo, as \mathcal{F} is stable under $D|_{\mathbf{Y}_{(0,1,\infty)}} \cong \nabla_\rho$ then \mathcal{F} inherits a connection map $\hat{\nabla}$ with $(\mathcal{F}, \hat{\nabla}) \subset (\mathcal{Q}_\rho, \nabla_\rho)$ as connections. Now, since there is an equivalence of categories between holomorphic connections on $\mathbf{Y}_{(0,1,\infty)}$ and monodromy representations of $\mathbf{Y}_{(0,1,\infty)}$, it follows that $\rho' \subset \rho$ where ρ' corresponds to $(\mathcal{F}, \hat{\nabla})$. At this point notice that $(\mathcal{F}, \hat{\nabla})$ is nonempty which means that $\rho' \neq 0$, moreover, since $\mathcal{N}(\rho)$ is of rank n it follows that $(\mathcal{F}, \hat{\nabla}) \subsetneq (\mathcal{Q}_\rho, \nabla_\rho)$ which implies that $\rho' \subsetneq \rho$. Consequently, ρ' must be a proper subrepresentation of ρ , but this is impossible since ρ is irreducible. \square

Lemma 5.9. *Suppose that $\rho : \mathbb{Z} * \mathbb{Z} \rightarrow \text{GL}_2(\mathbb{C})$ is an irreducible two-dimensional monodromy representation of $\mathbf{Y}_{(0,1,\infty)}$. Denote the minimal weight of $\mathcal{N}(\rho)$ by ξ_{\min} and let $(\mathcal{V}_{\text{Log}(\rho)}, \nabla_{\text{Log}(\rho)})$ denote the associated extended logarithmic connection. Then either*

$$\begin{aligned} \mathcal{V}_{\text{Log}(\rho)} &\cong \mathcal{O}(-\xi_{\min})^{\oplus 2} \text{ or} \\ \mathcal{V}_{\text{Log}(\rho)} &\cong \mathcal{O}(-\xi_{\min}) \oplus \mathcal{O}(-\xi_{\min} - 1) \end{aligned}$$

Proof. Since ρ is two-dimensional $\mathcal{N}(\rho)$ has rank two, so let F and G be a pair of generators. Further, let ξ_{\min}, ξ_1 be the weights of F, G respectively, where by assumption ξ_{\min} is the minimal weight of $\mathcal{N}(\rho)$ and $\xi_{\min} \leq \xi_1$. Observe that by construction $D_{\xi_{\min}}(F) \in \mathcal{N}_{\xi_{\min}+1}(\rho)$ and so $D_{\xi_{\min}}(F)$ has weight $\xi_{\min} + 1$. Moreover since F and G generate $\mathcal{N}(\rho)$ we must have $D_{\xi_{\min}}(F) = \eta F + \tau G$ for some $\eta, \tau \in \mathbb{C}[x, y]$. As a result of Lemma 5.8 it follows that $D_{\xi_{\min}}(F) \neq 0$. At this point we remark that $\tau \neq 0$, as otherwise F is stable under D and an argument similar to that in the proof of Lemma 5.8 will lead to a contradiction. Ergo, τG must have weight $\xi_{\min} + 1$. Thus, $\xi_{\min} + 1 = a + \xi_1$, where $\deg(\tau) = a$, implying that

$$\xi_1 = \begin{cases} \xi_{\min} + 1 & \text{when } a = 0 \\ \xi_{\min} & \text{when } a = 1 \end{cases}$$

since $0 \leq a$ with $\xi_{\min} \leq \xi_1$. \square

Theorem 5.10. *Suppose that $\rho : \mathbb{Z} * \mathbb{Z} \rightarrow \text{GL}_2(\mathbb{C})$ is an irreducible two-dimensional monodromy representation of $\mathbf{Y}_{(0,1,\infty)}$ and let $(\mathcal{V}_{\text{Log}(\rho)}, \nabla_{\text{Log}(\rho)})$ denote the associated extended logarithmic connection. Allow $\zeta = c_1(\mathcal{V}_{\text{Log}(\rho)})$. Then*

$$\mathcal{V}_{\text{Log}(\rho)} \cong \begin{cases} \mathcal{O}(\frac{\zeta}{2})^{\oplus 2} & \text{when } \zeta \text{ is even} \\ \mathcal{O}(\frac{\zeta-1}{2}) \oplus \mathcal{O}(\frac{\zeta+1}{2}) & \text{when } \zeta \text{ is odd.} \end{cases}$$

Proof. Let $\zeta = 2\sigma$ for some $\sigma \in \mathbb{Z}$, and where by Lemma 5.9, for the sake of contradiction, we assume that $\mathcal{V}_{\text{Log}(\rho)} \cong \mathcal{O}(-\xi_{\min}) \oplus \mathcal{O}(-\xi_{\min} - 1)$. Then, by Lemma 2.10 it follows that $2\sigma = -2\xi_{\min} - 1$ which is impossible as $\sigma, \xi_{\min} \in \mathbb{Z}$. Hence, if ζ is even then $\mathcal{V}_{\text{Log}(\rho)} \cong \mathcal{O}(-\xi_{\min})^{\oplus 2}$. Moreover, by Lemma 2.10 the first Chern class is the sum of the roots, so $-\xi_{\min} = \frac{\zeta}{2}$. A similar argument shows the desired result for when ζ is odd. \square

Remark 5.11. Witness that the theorem above does not make any assumptions regarding ρ aside from the fact that it is an irreducible representation of dimension 2. Thus, this theorem along with Theorem 4.4 close the case on two-dimensional monodromy representations of $\mathbf{Y}_{(0,1,\infty)}$.

Remark 5.12. It is natural to inquire regarding alternative choices of the logarithm, i.e., those differing from the Deligne canonical extension. Shifting the eigenvalues of the residues by integers results in a corresponding shift in the first Chern class $c_1(\mathcal{V}_{\text{Log}(\rho)})$ of the associated extended bundle, which in turn shifts the degrees of the Grothendieck splitting.

As the character cases are determined by the first Chern class, and the auxiliary connection map ∇ is itself dependent on these character cases, the choice of logarithm directly impacts the definition of the connection map. This implies that both the graded module $\mathcal{N}(\rho)$ and the monodromy derivative D may exhibit fundamentally different behavior under a different branch choice. Consequently, the resulting constraints on the bundle decomposition could be quite drastic.

We note that Szabó [23] explores the flexibility of the correspondence when this constraint is relaxed, demonstrating that integer shifts of the residue eigenvalues allow for the realization of logarithmic connections on bundles of arbitrary splitting type.

Notably, the case $m = 3$ and $\text{rank}(\rho) = 2$ is closely related to the classical Gauss hypergeometric equation, where the residue eigenvalues of the monodromy representation correspond directly to the local exponents of the hypergeometric function. In this context, the choice of the principal branch aligns with the standard treatment of hypergeometric differential equations and their associated monodromy. This framework is utilized, for example, by Franc and Mason [6] to identify vector-valued modular forms as sections of holomorphic vector bundles and to study their Fourier coefficients.

6. EXAMPLE FROM THE MODULAR GROUP

In this section we will explicitly compute the roots of an irreducible two-dimensional representation of $\mathbb{Z} * \mathbb{Z}$ that factors through $\text{SL}_2(\mathbb{Z})$ using the main result from the monodromy derivative. To remind the reader of such representations we closely follow [15, §3].

We introduce standard matrices

$$R = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

of which any two of them generate $\text{SL}_2(\mathbb{Z})$; indeed

$$\Gamma := \text{SL}_2(\mathbb{Z}) = \langle R, S, T \mid RS = T, R^3 = S^4 = I \rangle.$$

Further, we assume that given $\rho : \Gamma \rightarrow \text{GL}_2(\mathbb{C})$ the matrix $\rho(T)$ is semisimple and ρ is irreducible. For such representations, as [15] shows, there exists a basis β in which the images of T, S and R are not scalar matrices and

$$\begin{aligned} \rho(T) &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\sigma \end{pmatrix}, \\ \rho(S) &= \begin{pmatrix} a & 1 \\ -a^2 - \sigma^3 & -a \end{pmatrix}, \\ \rho(R) &= \sigma^3 \begin{pmatrix} -a\lambda & -\lambda \\ \sigma\lambda^{-1}(a^2 + \sigma^3) & a\sigma\lambda^{-1} \end{pmatrix}, \end{aligned}$$

where σ is a sixth root of unity, $\sigma \neq \lambda^2$, $a = \frac{1}{\sigma(\lambda - \sigma\lambda^{-1})}$, and $a^2 \neq -\sigma^3$.

In the basis β above, we choose $\lambda = 1$, and $\sigma = -1$. Observe then that the restrictions are satisfied and

$$\rho(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \rho(S) = \begin{pmatrix} -\frac{1}{4} & 1 \\ \frac{3}{4} & \frac{1}{2} \end{pmatrix}, (\rho(T)\rho(S))^{-1} = \begin{pmatrix} -\frac{1}{4} & -1 \\ \frac{3}{4} & -\frac{1}{2} \end{pmatrix}.$$

Now, as stated above, $\rho(T)$ and $\rho(S)$ give us an irreducible representation of Γ and hence of $\mathbb{Z} * \mathbb{Z}$. In other words, ρ is an irreducible two-dimensional monodromy representation of $\mathbf{Y}_{(0,1,\infty)}$. Thus, the representation above gives us the associated extended logarithmic connection $(\mathcal{V}_{\text{Log}(\rho)}, \nabla_{\text{Log}(\rho)})$ over

\mathbb{P}^1 . The connection has three poles, one at 0, one at 1, and one at ∞ with respective residues $\log(\rho(T))$, $\log(\rho(S))$, $\log((\rho(T)\rho(S))^{-1})$.

By Ohtsuki's formula we have

$$c_1(\mathcal{V}_{\text{Log}(\rho)}) = -\text{Tr}(\log(\rho(T))) - \text{Tr}(\log(\rho(S))) - \text{Tr}(\log((\rho(T)\rho(S))^{-1}))$$

and as a result of Equation (10) combined with the fact that the trace is invariant under conjugation we may use the Jordan form of each image. Denote by B_J the Jordan form of the matrix B . First, we have that $\rho(S) = H\rho(S)_J H^{-1}$ where

$$(28) \quad \rho(S)_J = \begin{pmatrix} e^{2\pi i(\frac{1}{2})} & 0 \\ 0 & e^{2\pi i(0)} \end{pmatrix} \text{ with } H = \begin{pmatrix} -2 & \frac{2}{3} \\ 1 & 1 \end{pmatrix},$$

and further, we have $(\rho(T)\rho(S))^{-1} = K(\rho(T)\rho(S))_J^{-1} K^{-1}$ where

$$(29) \quad (\rho(T)\rho(S))_J^{-1} = \begin{pmatrix} e^{2\pi i(\frac{2}{3})} & 0 \\ 0 & e^{2\pi i(\frac{1}{3})} \end{pmatrix} \text{ with } K = \begin{pmatrix} -\frac{2i}{\sqrt{3}} & \frac{2i}{\sqrt{3}} \\ 1 & 1 \end{pmatrix}.$$

It follows then that

$$\begin{aligned} \text{Tr}(\log(\rho(T)_J)) &= \frac{1}{2} \\ \text{Tr}(\log(\rho(S)_J)) &= \frac{1}{2} \\ \text{Tr}(\log((\rho(T)\rho(S))_J^{-1})) &= 1 \end{aligned}$$

which implies that $c_1(\mathcal{V}_{\text{Log}(\rho)}) = -2$. Thus, by Theorem 5.10 we must be in the case where

$$(30) \quad \mathcal{V}_{\text{Log}(\rho)} \cong \mathcal{O}(-1)^{\oplus 2}.$$

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