POISSON SUMMATION FORMULA IN WEIGHTED LEBESGUE SPACES

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Abstract. We characterize the parameters (α, β, p, q) for which the condition $f|x|^\alpha \in L^p$ and $\widehat{f}|\xi|^\beta \in L^q$ implies the validity of the Poisson summation formula, thus completing the study of Kahane and Lemarié-Rieusset.

1. Introduction

The Fourier transform of an integrable function $f : \mathbb{R} \to \mathbb{C}$, is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx.$$

The classical Poisson summation formula (PSF) states that if

$$f|x|^\alpha, \widehat{f}|\xi|^\beta \in L^\infty, \quad \alpha, \beta > 1,$$

then f and $\widehat{f}$ are both continuous and the formula

$$\lim_N P_N(f) = \lim_N P_N(\widehat{f}) \tag{PSF}$$

is true, where

$$P_N(g) = \sum_{|k| \leq N} g(k).$$

By a classical example of Katznelson it is well known that there exist continuous functions with $f, \widehat{f} \in L^1$ for which the Poisson summation formula fails, in the sense that both sides of (PSF) converge to different values.

In [4], Kahane and Lemarié-Rieusset obtained the following almost complete characterization of the validity of (PSF) under the condition $f|x|^\alpha \in L^p$ and $\widehat{f}|\xi|^\beta \in L^q$:

2010 *Mathematics Subject Classification.* Primary 42B10, 42B35; Secondary 46E30.

Key words and phrases. Fourier transform, Poisson formula, weighted Lebesgue space.

This work is supported by AP23488596, the CERCA Programme of the Generalitat de Catalunya and the Severo Ochoa, and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M). M. Saucedo is supported by the Spanish Ministry of Universities through the FPU contract FPU21/04230. S. Tikhonov is supported by PID2023-150984NB-I00, 2021 SGR 00087.

Theorem 1.1. *[4] Let $1 \leq p, q \leq \infty$ and $\alpha - \frac{1}{p'}, \beta - \frac{1}{q'} > 0$. Let f be an integrable function which satisfies*

$$\|f|x|^\alpha\|_p + \left\|\widehat{f}|\xi|^\beta\right\|_q < \infty. \tag{1.1}$$

Then,

1. *if $(\alpha - \frac{1}{p'})(\beta - \frac{1}{q'}) > \frac{1}{pq}$, $\lim_N P_N(f) = \lim_N P_N(\widehat{f})$;*
2. *if $(\alpha - \frac{1}{p'})(\beta - \frac{1}{q'}) < \frac{1}{pq}$, there exists a function f for which $P_N(f)$ and $P_N(\widehat{f})$ converge to different values.*

Remark 1.2. *In fact, by analyzing the proof in [4], we see that if either $\alpha - \frac{1}{p'}$ or $\beta - \frac{1}{q'}$ is not strictly positive, then both sides of* (PSF) *can converge to different values.*

Kahane and Lemarié-Rieusset also considered the question of the absolute convergence of the series $P_N(f)$ and obtained the following theorem

Theorem 1.3. *[4] Let $1 \leq p, q \leq \infty$ and $\alpha - \frac{1}{p'}, \beta - \frac{1}{q'} > 0$. Let f be an integrable function which satisfies* (1.1). *Then,*

1. *if $(\alpha - \frac{1}{p'})(\beta - \frac{1}{q'}) > \max(\frac{1}{pq}, \frac{1}{2p})$, $\lim_N P_N(|f|) < \infty$;*
2. *if $q > 2$ and $(\alpha - \frac{1}{p'})(\beta - \frac{1}{q'}) < \frac{1}{2p}$, there exists a function f for which $\lim_N P_N(|f|) = \infty$.*

In this paper we complete the missing cases in Theorems 1.1 and 1.3. In particular, we show in Theorem 2.2 that for $(\alpha - \frac{1}{p'})(\beta - \frac{1}{q'}) = \frac{1}{pq}$, $(p, q) \neq (1, 1)$ if one side of (PSF) converges (to a finite or infinite value), the other side must also converge to the same value. However, it is possible for neither side to converge.

See also [2] and [3] for various conditions for (PSF) to be true.

We will use the notation $F \lesssim G$ to mean that $F \leq CG$ with a constant $C = C(\alpha, \beta, \gamma, p, q)$ that may change from line to line; $F \approx G$ means that both $F \lesssim G$ and $G \lesssim F$ hold. Besides, $\|F\|_p = \|F\|_{L^p(\mathbb{R})}$.

2. Main results

Before stating our main results, we give two simple properties of the space of functions which satisfy (1.1).

Proposition 2.1. *Let $\alpha - \frac{1}{p'}$ and $\beta - \frac{1}{q'} > 0$. Then the space of integrable functions f which satisfy* (1.1) *is a Banach space, and the Schwartz functions are dense in it. Moreover,*

$$\|f\|_1 + \left\|\widehat{f}\right\|_1 \lesssim \|f|x|^\alpha\|_p + \left\|\widehat{f}|\xi|^\beta\right\|_q.$$

Thus, $\|f|x|^\alpha\|_p + \left\|\widehat{f}|\xi|^\beta\right\|_q$ is equivalent to the canonical norm in the space of integrable functions which satisfy (1.1).

In particular, we have the equivalence

$$\|f|x|^{\alpha}\|_p + \left\|\widehat{f}|\xi|^{\beta}\right\|_q \approx \|f(1+|x|)^{\alpha}\|_p + \left\|\widehat{f}(1+|\xi|)^{\beta}\right\|_q .$$

As a consequence, for $\varepsilon_1, \varepsilon_2 \geq 0$,

$$\|f|x|^{\alpha}\|_p + \left\|\widehat{f}|\xi|^{\beta}\right\|_q \lesssim \left\|f|x|^{\alpha+\varepsilon_1}\right\|_p + \left\|\widehat{f}|\xi|^{\beta+\varepsilon_2}\right\|_q .$$

Proof. The first statement of the proposition is clear. We now prove the second one.

First, note that as a consequence of the Hölder inequality we have

$$\left\|\widehat{f}\mathbb{1}_{|\xi|\geq 1/2}\right\|_1 \lesssim \left\|\widehat{f}|\xi|^{\beta}\right\|_q .$$

Second, defining $\Delta^1(g)(x) = g(x) - g(x+1)$ and $\Delta^j = \Delta(\Delta^{j-1})$, we have

$$|\Delta^k(\widehat{f})(\xi)| \leq \left\|f(1-e^{2\pi i x})^k\right\|_1 \lesssim \|f|x|^{\alpha}\|_p ,$$

where $k = \lceil \alpha \rceil$. Thus,

$$\|f\|_\infty \leq \left\|\widehat{f}\right\|_1 \lesssim \left\|\widehat{f}\mathbb{1}_{|\xi|\geq 1/2}\right\|_1 + \left\|\Delta^k(\widehat{f})\mathbb{1}_{|\xi|\leq 1/2}\right\|_1 \lesssim \|f|x|^{\alpha}\|_p + \left\|\widehat{f}|\xi|^{\beta}\right\|_q .$$

Similarly, we obtain the same estimate for $\left\|\widehat{f}\right\|_\infty$, whence the result follows. □

Our main results are the following extensions of Theorems 1.1 and 1.3.

Theorem 2.2. *Let* $1 \leq p, q \leq \infty$ *and* $\alpha - \frac{1}{p'}, \beta - \frac{1}{q'} > 0$. *Let* f *be an integrable function which satisfies* (1.1). *Then,*

1. *if* $(\alpha - \frac{1}{p'})(\beta - \frac{1}{q'}) > \frac{1}{pq}$, $\lim_N P_N(f) = \lim_N P_N(\widehat{f})$;
2. *if* $(\alpha - \frac{1}{p'})(\beta - \frac{1}{q'}) = \frac{1}{pq}$ *and* $(p,q) = (1,1)$, $\lim_N P_N(f) = \lim_N P_N(\widehat{f})$;
3. *if* $(\alpha - \frac{1}{p'})(\beta - \frac{1}{q'}) = \frac{1}{pq}$ *and* $(p,q) \neq (1,1)$, *neither* $\lim_N P_N(\widehat{f})$ *nor* $\lim_N P_N(f)$ *need to exist, but if one does, then* $\lim_N P_N(f) = \lim_N P_N(\widehat{f})$. *Moreover, there exists* $\gamma > 0$ *depending on* (α, β, p, q) *such that*
$$\lim_{N\to\infty} (P_N(f) - P_{N^\gamma}(\widehat{f})) = 0;$$
4. *if* $(\alpha - \frac{1}{p'})(\beta - \frac{1}{q'}) < \frac{1}{pq}$, *there exists a function* f *for which* $P_N(f)$ *and* $P_N(\widehat{f})$ *converge to different values.*

Theorem 2.3. *Let* $1 \leq p, q \leq \infty$ *and* $\alpha - \frac{1}{p'}, \beta - \frac{1}{q'} > 0$. *Let* f *be an integrable function which satisfies* (1.1). *Then,*

1. *if* $(\alpha - \frac{1}{p'})(\beta - \frac{1}{q'}) > \max(\frac{1}{pq}, \frac{1}{2p})$, $\lim_N P_N(|f|) < \infty$;
2. *if* $(\alpha - \frac{1}{p'})(\beta - \frac{1}{q'}) = \frac{1}{pq}$ *and* $(p,q) = (1,1)$, $\lim_N P_N(|f|) < \infty$;
3. *if* $(\alpha - \frac{1}{p'})(\beta - \frac{1}{q'}) \leq \max(\frac{1}{pq}, \frac{1}{2p})$ *and* $(p,q) \neq (1,1)$, *there exists a function* f *for which* $\lim_N P_N(|f|) = \infty$.

Remark 2.4. *In order that*

$$\lim_N P_N(|f|)+\lim_N P_N(|\widehat{f}|)<\infty$$

for any integrable function f satisfying (1.1), *it is necessary and sufficient that*

$$(\alpha-\frac{1}{p'})(\beta-\frac{1}{q'})>\max(\frac{1}{pq},\frac{1}{2p},\frac{1}{2q}),\qquad (p,q)\neq(1,1)$$

and

$$\alpha\beta\geq 1,\qquad (p,q)=(1,1).$$

3. Auxiliary lemma

In this section we obtain a lemma which provides us with useful estimates for the norms of smooth step functions and their Fourier transforms.

For a sequence (c_j), we denote

$$S_k(\xi)=\sum_{j=0}^{k}c_je^{-2\pi ij\xi}\quad\text{ and }\quad\widetilde{S_k}(\xi)=\sum_{j=k}^{N}c_je^{-2\pi ij\xi}.$$

In what follows we denote, abusing notation, the Dirac delta measure at x_0 by $\delta(x-x_0)$, that is,

$$\int_{\mathbb{R}}\delta(x-x_0)f(x)dx=f(x_0),\quad f\text{ continuous.}$$

Observe that the Fourier transform of $\delta(x-x_0)$ is the function $e^{-2\pi i\xi x_0}$.

Lemma 3.1. *Let $1\leq p,q\leq\infty$. Let $\widehat{\phi}$ be a Schwartz, even, non-increasing function which is equal to 1 on $[-1/2,1/2]$ and supported on $(-1,1)$. Let $(\Delta_k)_{k=-1}^{\infty}$ be an increasing sequence with $\Delta_{-1}=0<1\leq\Delta_0$ and $\Delta_k\approx\Delta_{k+1}$ for $k\geq 0$. For any integer $N\geq 0$ and any sequence $(c_k)_{k=0}^{N}$, we set*

$$F(x)=\sum_{k=0}^{N}c_k\Delta_k\phi\big((x-k)\Delta_k\big)$$

and

$$G(x)=\sum_{k=0}^{N}c_k\Big(\delta(x-k)-\Delta_k\phi\big((x-k)\Delta_k\big)\Big).$$

Then, for any $\alpha,\beta\in\mathbb{R}$ and $\nu>\frac{1}{q'}$, we have

(1) $$\|F(|x|+1)^{\alpha}\|_p\lesssim\left(\sum_{k=0}^{N}|c_k|^p(k+1)^{p\alpha}\Delta_k^{p-1}\right)^{\frac{1}{p}};$$

(2) $$\left\|\widehat{F}(|\xi|+1)^{\beta}\right\|_q\lesssim\left(\sum_{k=0}^{N}\Delta_k^{\beta q}(\Delta_k-\Delta_{k-1})\left\|\widetilde{S_k}\right\|_{L^q(\mathbb{T})}^q\right)^{\frac{1}{q}};$$

(3) $$\left\|\widehat{G}(|\xi|+1)^{-\nu}\right\|_{q'} \lesssim \left(\sum_{k=0}^{N-1} \Delta_k^{-\nu q'}(\Delta_{k+1}-\Delta_k)\left\|S_k\right\|_{L^{q'}(\mathbb{T})}^{q'}\right)^{\frac{1}{q'}} + \Delta_N^{-\nu+\frac{1}{q'}}\left\|S_N\right\|_{L^{q'}(\mathbb{T})}.$$

(4) *for any $M \in \mathbb{N}$ there exists a constant K_M such that*

$$|F(n)-\phi(0)\Delta_n c_n| \leq K_M \left\|c\right\|_\infty (n^{-1}+\Delta_{\lfloor n/2\rfloor}^{-1})^M$$

for any integer n.

Proof. To prove the first item, observe that, by Hölder's inequality,

$$|F(x)| \leq \left(\sum_{k=0}^{N}|c_k|^p\Delta_k^p|\phi((x-k)\Delta_k)|\right)^{\frac{1}{p}}\left(\sum_{k=0}^{N}|\phi((x-k)\Delta_k)|\right)^{\frac{1}{p'}}.$$

The rapid decay of ϕ and the assumption $\Delta_k \geq 1$ imply that, for any x,

$$\sum_{k=0}^{\infty}|\phi((x-k)\Delta_k)| \lesssim 1.$$

Hence,

$$\left\|F(|x|+1)^\alpha\right\|_p \lesssim \left(\sum_{k=0}^{N}|c_k|^p\Delta_k^p\big\|(|x|+1)^\alpha\phi((x-k)\Delta_k)\big\|_p^p\right)^{\frac{1}{p}}.$$

Finally, once again because of the rapid decay of ϕ, we conclude the proof of the first item by noting that

$$\Delta_k^{\frac{1}{p}}\left\|(|x|+1)^\alpha\phi((x-k)\Delta_k)\right\|_p \lesssim (k+1)^\alpha.$$

To prove the second item, using the Abel transformation, we obtain that

$$\begin{aligned}
\widehat{F}(\xi) &= \sum_{k=0}^{N} c_k e^{-2\pi i k\xi}\widehat{\phi}(\xi/\Delta_k) \\
&= \widetilde{S_0}(\xi)\widehat{\phi}(\xi/\Delta_0) + \sum_{k=1}^{N}\widetilde{S_k}(\xi)(\widehat{\phi}(\xi/\Delta_k)-\widehat{\phi}(\xi/\Delta_{k-1})) \\
&\leq |\widetilde{S_0}(\xi)|\widehat{\phi}(\xi/\Delta_0) + \left(\sum_{k=1}^{N}|\widetilde{S_k}(\xi)|^q|\widehat{\phi}(\xi/\Delta_k)-\widehat{\phi}(\xi/\Delta_{k-1}))|\right)^{\frac{1}{q}},
\end{aligned}$$

where in the last inequality we used Hölder's inequality and the monotonicity of $\widehat{\phi}$. Next, because of the periodicity of $\widetilde{S_k}$, we have

$$\left\|\widehat{F}(|\xi|+1)^{\beta}\right\|_q \lesssim \Delta_0^{\beta+\frac{1}{q}} \left\|\widetilde{S_0}\right\|_{L^q(\mathbb{T})}$$
$$+\left(\sum_{k=1}^{N}\left\|\widetilde{S_k}\right\|_{L^q(\mathbb{T})}^q \sup_{0\le\xi\le1}\left(\sum_{j\in\mathbb{Z}}|\xi+j|^{\beta q}\left|\widehat{\phi}\left(\frac{\xi+j}{\Delta_k}\right)-\widehat{\phi}\left(\frac{\xi+j}{\Delta_{k-1}}\right)\right|\right)\right)^{\frac{1}{q}}.$$

Finally, by the support assumption, the inequality $|\widehat{\phi}(x)-\widehat{\phi}(y)| \lesssim |x-y|$ and the property $\Delta_k \approx \Delta_{k+1}$, we conclude that for $k \geq 1$

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}|\xi+j|^{\beta q}\left|\widehat{\phi}\left(\frac{\xi+j}{\Delta_k}\right)-\widehat{\phi}\left(\frac{\xi+j}{\Delta_{k-1}}\right)\right| \\
&\lesssim \sum_{\Delta_{k-1}/2\le j+\xi\le\Delta_k}(j+\xi)^{\beta q+1}(\Delta_{k-1}^{-1}-\Delta_k^{-1}) \\
&\lesssim \Delta_k^{q\beta+2}(\Delta_{k-1}^{-1}-\Delta_k^{-1}) \\
&\approx \Delta_k^{q\beta}(\Delta_k-\Delta_{k-1}).
\end{aligned}$$

The third item follows in a similar way. Indeed, by the Abel transformation,

$$\begin{aligned}
\widehat{G}(\xi) &= \sum_{k=0}^{N} c_k e^{-2\pi i k\xi}(1-\widehat{\phi}(\xi/\Delta_k)) \\
&= S_N(\xi)(1-\widehat{\phi}(\xi/\Delta_N)) + \sum_{k=0}^{N-1} S_k(\xi)(\widehat{\phi}(\xi/\Delta_{k+1})-\widehat{\phi}(\xi/\Delta_k)) \\
&\le |S_N(\xi)|(1-\widehat{\phi}(\xi/\Delta_N)) + \left(\sum_{k=0}^{N-1}|S_k(\xi)|^{q'}|\widehat{\phi}(\xi/\Delta_{k+1})-\widehat{\phi}(\xi/\Delta_k)|\right)^{\frac{1}{q'}}.
\end{aligned}$$

Using the periodicity of S_k,

$$\left\|\widehat{G}(|\xi|+1)^{-\nu}\right\|_{q'} \lesssim \Delta_N^{-\nu+\frac{1}{q'}} \|S_N\|_{L^{q'}(\mathbb{T})}$$
$$+\left(\sum_{k=0}^{N-1}\|S_k\|_{L^{q'}(\mathbb{T})}^{q'} \sup_{0\le\xi\le1}\left(\sum_{j\in\mathbb{Z}}|\xi+j|^{-\nu q'}\left|\widehat{\phi}\left(\frac{\xi+j}{\Delta_{k+1}}\right)-\widehat{\phi}\left(\frac{\xi+j}{\Delta_k}\right)\right|\right)\right)^{\frac{1}{q'}}.$$

Finally, repeating the previous arguments we conclude that

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} |\xi+j|^{-\nu q'} \left| \widehat{\phi}\left(\frac{\xi+j}{\Delta_{k+1}}\right) - \widehat{\phi}\left(\frac{\xi+j}{\Delta_k}\right) \right| \\
&\lesssim \sum_{\Delta_k/2\le\xi+j\le\Delta_{k+1}} (\xi+j)^{-\nu q'+1}(\Delta_k^{-1}-\Delta_{k+1}^{-1}) \\
&\lesssim \Delta_k^{-q'\nu+2}(\Delta_k^{-1}-\Delta_{k+1}^{-1}) \approx \Delta_k^{-q'\nu}(\Delta_{k+1}-\Delta_k).
\end{aligned}$$

To prove the fourth item, we use the estimate $|\phi(x)| \lesssim (1+|x|)^{-M}$ to deduce that

$$\begin{aligned}
|F(n)-c_n\Delta_n\phi(0)| &\le \sum_{k\neq n} \Delta_k |c_k\phi(\Delta_k(n-k))| \lesssim \|c\|_\infty \sum_{k\neq n} \frac{\Delta_k}{(\Delta_k|k-n|)^M} \\
&\lesssim \|c\|_\infty \left(\sum_{k=0}^{\frac{n}{2}} \frac{\Delta_k}{(\Delta_k n)^M} + \Delta_{\lfloor n/2\rfloor}^{1-M} \sum_{k\neq n} \frac{1}{|k-n|^M} + \sum_{k=2n}^{\infty} \frac{\Delta_k}{(\Delta_k k)^M} \right) \\
&\lesssim \|c\|_\infty \left(n^{1-M} + \Delta_{\lfloor n/2\rfloor}^{1-M} \right).
\end{aligned}$$

□

4. Proofs of main results

4.1. **Proof of Theorem 2.2.** It is a standard approach to establish the convergence of $\lim_N P_N(f)$ for any function satisfying $\|f|x|^\alpha\|_p + \left\|\widehat{f}|\xi|^\beta\right\|_q < \infty$ through the uniform boundedness of $P_N(f)$ with respect to N. The following lemma gives sufficient conditions under which P_N and its modifications are uniformly bounded:

Lemma 4.1. *Let $1 \le p, q < \infty$ and $\alpha - 1/p', \beta - 1/q' > 0$. For $\gamma \ge 0$, we set*

$$Q_N(f) = \frac{1}{N^\gamma} \sum_{k=0}^{N} c_k f(k)$$

with a positive non-decreasing sequence $(c_k)_{k=0}^\infty$ satisfying $c_k \lesssim k^\gamma$.

Then the inequality

$$|Q_N(f)| \lesssim \|f|x|^\alpha\|_p + \left\|\widehat{f}|\xi|^\beta\right\|_q \tag{4.1}$$

holds uniformly on N in the following cases:

1. *if $(\alpha-1/p')(\beta-1/q') = 1/(pq)$ and $\gamma > 0$;*
2. *if $p = q = 1$, $\alpha\beta = 1$ and $\gamma \ge 0$.*

Proof. Let $\widehat{\phi}$ be a smooth, non-increasing function supported on $(-1,1)$ and equal to 1 on $[-1/2,1/2]$. Set

$$\Delta_k = 1 + k^B, \qquad B = \frac{1/q}{\beta - 1/q'}.$$

In the notation of Lemma 3.1 and by Plancherel and Hölder's inequalities, we have

$$\begin{aligned} Q_N(f) &= N^{-\gamma}\left(\int_{\mathbb{R}} f\overline{F} + \int_{\mathbb{R}} f\overline{G}\right) \\ &= N^{-\gamma}\left(\int_{\mathbb{R}} f\overline{F} + \int_{\mathbb{R}} \widehat{f}\overline{\widehat{G}}\right) \\ &\lesssim N^{-\gamma}\left(\|f|x|^{\alpha}\|_p + \left\|\widehat{f}|\xi|^{\beta}\right\|_q\right)\left(\left\|F(1+|x|)^{-\alpha}\right\|_{p'} + \left\|\widehat{G}(1+|\xi|)^{-\beta}\right\|_{q'}\right), \end{aligned}$$

where in the last estimate we also used Proposition 2.1. Thus, the result follows if we prove that

$$\left\|F(1+|x|)^{-\alpha}\right\|_{p'} + \left\|\widehat{G}(1+|\xi|)^{-\beta}\right\|_{q'} \lesssim N^{\gamma}.$$

To estimate $\|F(1+|x|)^{-\alpha}\|_{p'}$ we use item (1) in Lemma 3.1. Since $c_k \lesssim k^{\gamma}$ and $(\alpha - 1/p')(\beta - 1/q') = 1/(pq)$, we have

$$\left\|F(1+|x|)^{-\alpha}\right\|_{p'} \lesssim N^{\gamma}.$$

Next we deal with the quantity $\left\|\widehat{G}(1+|\xi|)^{-\beta}\right\|_{q'}$ for $q > 1$. Since the sequence $(c_k)_k$ is non-decreasing, we can apply the Hardy-Littlewood inequality for monotone Fourier coefficients (see Chapter XII in [6]) to obtain that

$$\int_0^1 |S_k|^{q'} \lesssim \sum_{j=1}^{k} c_j^{q'}(k-j)^{q'-2} \lesssim k^{q'\gamma+q'-1}.$$

If $q = 1$, it is clear that $\|S_k\|_\infty \lesssim k^{\gamma+1}$. Therefore, for any $q \geq 1$ by using item (3) in Lemma 3.1 we deduce that

$$\left\|\widehat{G}(1+|\xi|)^{-\beta}\right\|_{q'} \lesssim N^{\gamma}.$$

The proof is now complete. □

Remark 4.2. *If $(\alpha - 1/p')(\beta - 1/q') > 1/(pq)$ and $\gamma = 0$, the same proof with $\frac{\frac{1}{q}}{\beta - \frac{1}{q'}} < B < \frac{\alpha - \frac{1}{p'}}{\frac{1}{p}}$ shows that (4.1) holds true. This yields an alternative proof of item (1) in Theorems 1.1 and 2.2.*

We are now in a position to prove our main results.

Proof of Theorem 2.2. The first and the forth statements are contained in Theorem 1.1, see also Remark 4.2 for an alternative proof of the first one. The second item follows from item (2) in Lemma 4.1 with $\gamma = 0$ by a standard density argument (see Proposition 2.1). We now prove the positive part of the third item.

Let $\widehat{\phi}$ be a smooth, even, non-increasing function supported on $(-1,1)$ such that $\widehat{\phi}(\xi)\equiv 1$ for $|\xi|\leq 1/2$. Set $M=\lceil N^{\frac{\beta-1/q'}{1/q}}\rceil=\lceil N^{\frac{1/p}{\alpha-1/p'}}\rceil$ and let

$$P_{\Phi,N}(\widehat{f}):=\sum_{k\in\mathbb{Z}}\widehat{f}(k)\widehat{\phi}(k/N).$$

We show that both

$$\left|P_{\Phi,N}(\widehat{f})-P_M(f)\right|\lesssim \left\|f|x|^{\alpha}\right\|_p+\left\|\widehat{f}|\xi|^{\beta}\right\|_q \tag{4.2}$$

and

$$\left|P_{\Phi,N}(\widehat{f})-P_N(\widehat{f})\right|\lesssim \left\|f|x|^{\alpha}\right\|_p+\left\|\widehat{f}|\xi|^{\beta}\right\|_q \tag{4.3}$$

hold independently of N. By the density of the Schwartz functions, for which (PSF) clearly holds, one has

$$\lim_N\left(P_N(\widehat{f})-P_M(f)\right)=0$$

for any f satisfying (1.1).

First, to prove inequality (4.2), we observe that by Parseval and Hölder's inequality,

$$\begin{aligned}\left|P_M(f)-\int_{\mathbb{R}}f(x)\sum_{|k|\leq M}N\phi\left((x-k)N\right)dx\right|&=\left|\int_{\mathbb{R}}\widehat{f}(\xi)D_M(\xi)\left(1-\widehat{\phi}(\xi/N)\right)d\xi\right|\\&\lesssim\left\|\widehat{f}\xi^{\beta}\right\|_q M^{\frac{1}{q}}N^{-\beta+\frac{1}{q'}}\approx\left\|\widehat{f}\xi^{\beta}\right\|_q.\end{aligned}$$

Here for the Dirichlet kernel $D_M(\xi)=\sum_{|k|\leq M}e^{2\pi ik\xi}$ we have used the estimate

$$\left(\int_{N/2}^{\infty}|D_M(\xi)|^{q'}|\xi|^{-q'\beta}d\xi\right)^{\frac{1}{q'}}\lesssim M^{\frac{1}{q}}N^{-\beta+\frac{1}{q'}}.$$

A further application of Hölder's inequality shows that

$$\left|\int_{\mathbb{R}}f(x)\sum_{|k|\geq M}N\phi((x-k)N)dx\right|\lesssim\left\|f(1+|x|)^{\alpha}\right\|_p N^{\frac{1}{p}}M^{-\alpha+\frac{1}{p'}}\approx\left\|f(1+|x|)^{\alpha}\right\|_p.$$

Hence, by the Poisson summation formula

$$\sum_{k\in\mathbb{Z}}\widehat{\phi}(k/N)e^{-2\pi ikx}=\sum_{k\in\mathbb{Z}}N\phi\left((x-k)N\right),$$

we deduce that

$$\left|P_M(f) - P_{\Phi,N}(\widehat{f})\right| = \left|P_M(f) - \int_{\mathbb{R}} f(x) \sum_{k\in\mathbb{Z}} N\phi\left((x-k)N\right) dx\right|$$
$$\lesssim \|f|x|^\alpha\|_p + \left\|\widehat{f}|\xi|^\beta\right\|_q .$$

This proves inequality (4.2).

Inequality (4.3) follows by applying item (1) in Lemma 4.1 with $\gamma = 1$ to

$$\sum_{|k|\le N} \widehat{f}(k)(1-\widehat{\phi}(k/N)),$$

because $0 \le 1 - \widehat{\phi}(k/N) \lesssim k/N$ and is $\widehat{\phi}$ is non-increasing.

We now prove the negative part of the third item. Without loss of generality, we can assume that $p \ge 1$ and $q > 1$. We show that there exists a function f such that

$$\limsup P_N(f) = -\liminf P_N(f) = +\infty. \tag{4.4}$$

Since we know that $\lim_N (P_N(f) - P_{N^\gamma}(\widehat{f})) = 0$ with some $\gamma(\alpha,\beta,p,q) > 0$, the same divergence result holds for $P_N(\widehat{f})$.

Let $N > 0$. We define

$$F_N(x) = \sum_{k=0}^{N} c_k \Delta_k \phi\left(\Delta_k(x-k)\right),$$

where

$$c_k = \frac{1}{(k+1)^{1+B}\log(k+2)^A}$$

and

$$\Delta_k = (k+1)^B \log(k+2)^{A-1},$$

with $A = \frac{\beta}{\beta - 1/q'} > 1$ and $B = \frac{\alpha - 1/p'}{1/p} = \frac{1/q}{\beta - 1/q'}$. By using item (4) in Lemma 3.1, noting that

$$\sum_n (n^{-1} + \Delta_{\lfloor n/2\rfloor}^{-1})^{M'} \approx 1$$

with large enough M', we then have, for M, N large enough,

$$P_M(F_N) \approx \sum_{n=0}^{\min(N,M)} c_n\Delta_n \approx \min(\log\log N, \log\log M). \tag{4.5}$$

Besides, Lemma 3.1 implies both

$$\|F_N|x|^\alpha\|_p \lesssim 1 \tag{4.6}$$

and

$$\left\|\widehat{F_N}|\xi|^\beta\right\|_q \lesssim \log^{\frac{1}{q}} \log N, \tag{4.7}$$

where, in the last inequality, we also used the Hardy-Littlewood estimate (see Chapter XII in [6])

$$\int_0^1 |\widetilde{S_k}|^q \approx \sum_{j=k}^{N} c_j^q (j-k+1)^{q-2} \lesssim k^{-Bq-1} \log^{-qA}(k+2).$$

Finally, for $(N_k)_{k=1}^{\infty}$ a rapidly increasing sequence, let

$$f(x) = \sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} \frac{F_{N_j}(x)}{\log^{\frac{1}{q}} \log N_j}.$$

Using the estimates of $\|F_N |x|^\alpha\|_p$ and $\left\|\widehat{F_N}|\xi|^\beta\right\|_q$, we arrive at $\|f|x|^\alpha\|_p + \left\|\widehat{f}|\xi|^\beta\right\|_q < \infty$.

We now show that

$$\lim_{k\to\infty} (-1)^k P_{N_k}(f) = +\infty \tag{4.8}$$

provided that N_k increases rapidly enough. Indeed,

$$P_{N_k}(f) = \left(\sum_{j=1}^{k} + \sum_{j=k+1}^{\infty}\right) \frac{(-1)^j}{j^2} \frac{P_{N_k}(F_{N_j})}{\log^{\frac{1}{q}} \log N_j} = I_1 + I_2.$$

Taking into account (4.5), we derive

$$\begin{aligned}(-1)^k I_1 &\gtrsim \frac{1}{k^2} \frac{P_{N_k}(F_{N_k})}{\log^{\frac{1}{q}} \log N_k} - \sum_{j=1}^{k-1} \frac{1}{j^2} \frac{P_{N_k}(F_{N_j})}{\log^{\frac{1}{q}} \log N_j} \\ &\gtrsim \frac{1}{k^2} \log^{1-\frac{1}{q}} \log N_k - \log^{1-\frac{1}{q}} \log N_{k-1} \to \infty.\end{aligned}$$

Using again (4.5),

$$\begin{aligned}|I_2| &\lesssim \sum_{j=k+1}^{\infty} \frac{1}{j^2} \frac{|P_{N_k}(F_{N_j})|}{\log^{\frac{1}{q}} \log N_j} \\ &\lesssim \sum_{j=k+1}^{\infty} \frac{1}{j^2} \frac{\log\log N_k}{\log^{\frac{1}{q}} \log N_j} \lesssim \frac{\log\log N_k}{\log^{\frac{1}{q}} \log N_{k+1}} \to 0.\end{aligned}$$

We have thus proved (4.8) and, therefore, (4.4).

□

4.2. **Proof of Theorem 2.3.** The following lemma is based on the classical Rademacher-Salem-Zygmund randomization technique:

Lemma 4.3. *For any sequences $(w_k)_{k=0}^N$ and $(c_k)_{k=0}^N$ with $w_k \geq 0$, there exists a choice of signs $(\varepsilon_k)_{k=0}^N$ such that, setting $S_n = \sum_{k=0}^n c_k \varepsilon_k e^{2\pi i k x}$,*

$$\left(\sum_{k=0}^{N} w_k \|S_k\|_{L^q(\mathbb{T})}^q\right)^{\frac{1}{q}} \lesssim \left(\sum_{k=0}^{N} w_k \|S_k\|_{L^2(\mathbb{T})}^q\right)^{\frac{1}{q}}, \quad q < \infty,$$

and, for $q=\infty$,

$$\|S_k\|_{L^\infty(\mathbb{T})} \lesssim (\log(k+1))^{\frac{1}{2}} \|S_k\|_{L^2(\mathbb{T})}, \quad 0 \le k \le N.$$

Proof. If $q<\infty$, by Khintchin's inequality (see, for instance, Theorem 6.2.3 in [1]), taking the average over all possible sign combinations we deduce that

$$\mathbb{E}\left(\sum_{k=0}^{N} w_k \|S_k\|^q_{L^q(\mathbb{T})}\right) \lesssim \left(\sum_{k=0}^{N} w_k \|S_k\|^q_{L^2(\mathbb{T})}\right),$$

whence the result follows. If $q=\infty$, it is shown in [5, p. 271] that there exists $c>0$ such that

$$\sum_{n=1}^{\infty} \exp\left(\frac{\log^{\frac{1}{2}}(n+1)\|S_n\|_\infty}{\|S_n\|_2} - c\log(n+1)\right) \lesssim 1$$

for some choice of signs, whence the result follows at once. □

Proof of Theorem 2.3. In view of Theorems 1.3 and 2.2, it remains to consider only three cases:

- $q \le 2$ and $(\alpha-\frac{1}{p'})(\beta-\frac{1}{q'}) \le \frac{1}{pq}$, $\quad (p,q) \neq (1,1)$,
- $p=q=1$, $(\alpha-\frac{1}{p'})(\beta-\frac{1}{q'}) = \frac{1}{pq}$,
- $q>2$, $(\alpha-\frac{1}{p'})(\beta-\frac{1}{q'}) = \frac{1}{2p}$.

First, let $q\le 2$ and $(p,q)\neq(1,1)$. By Proposition 2.1, it suffices to consider the case $(\alpha-\frac{1}{p'})(\beta-\frac{1}{q'}) = \frac{1}{pq}$. In this case, by Theorem 2.2 (3), there exists a function f such that $\lim_N P_N(f)$ does not exist, which implies that $\lim_N P_N(|f|)=\infty$.

Second, if $p=q=1$ and $a\beta=1$, set $\Delta_n=n^\alpha$. By the triangle inequality,

$$\begin{aligned}\sum_{n=1}^{\infty}|f(n)| \le \sum_{n=1}^{\infty}\left|f(n)-\int_{\mathbb{R}}\Delta_n\phi(\Delta_n(x-n))f(x)dx\right| \\ + \sum_{n=1}^{\infty}\left|\int_{\mathbb{R}}\Delta_n\phi(\Delta_n(x-n))f(x)dx\right|,\end{aligned}$$

where, as usual, $\widehat{\phi}$ is a smooth, even, non-increasing function supported on $(-1,1)$ such that $\widehat{\phi}(\xi)\equiv 1$ for $|\xi|\le 1/2$.

It is plain to see (for instance, by Lemma 3.1 item (4)) that one can estimate the second summand as follows:

$$\int_{\mathbb{R}}|f(x)|\sum_{n=1}^{\infty}|\Delta_n\phi(\Delta_n(x-n))|\,dx \lesssim \|f(x)(1+|x|)^\alpha\|_1.$$

For the first one, by Plancherel's formula,

$$\begin{aligned}\sum_{n=1}^{\infty}\left|f(n)-\int_{\mathbb{R}}\Delta_n\phi(\Delta_n(x-n))f(x)dx\right| &\le \int_{\mathbb{R}}|\widehat{f}(\xi)|\sum_n|1-\widehat{\phi}(\xi/\Delta_n)|d\xi \\ &\lesssim \left\|\widehat{f}|\xi|^\beta\right\|_1,\end{aligned}$$

where we used that, for M large enough,

$$\sum_{n=1}^{\infty} |1-\widehat{\phi}(\xi/\Delta_n)| \lesssim \sum_{n=1}^{\infty} \min\left(1, \left(\frac{\xi}{n^{\alpha}}\right)^{M}\right) \lesssim |\xi|^{1/\alpha} = |\xi|^{\beta}.$$

Let now $q > 2$. The counterexample is similar to the one used in Theorem 2.2. Assume first that $q < \infty$. Set

$$c_k = \varepsilon_k \frac{1}{(1+k)^{1+B}\log(k+2)^{A}},$$

and

$$\Delta_k = (1+k)^{B}\log(k+2)^{A-1},$$

with $A = \frac{\beta}{\beta-1/q'}$ and $B = \frac{\alpha-1/p'}{1/p} = \frac{1/2}{\beta-1/q'}$.

For $N \in \mathbb{N}$, define $F_N(x) = \sum_{k=0}^{N} c_k\Delta_k\phi\left(\Delta_k(x-k)\right)$. Then we obtain (compare with (4.5) and (4.6))

$$\lim_{M} P_M(|F_N|) \approx \sum_{n=0}^{N} |c_n|\Delta_n \approx \log\log N$$

and

$$\|F_N|x|^{\alpha}\|_p \lesssim 1.$$

To derive the estimate of $\left\|\widehat{F_N}|\xi|^{\beta}\right\|_q$, cf. (4.7), we observe that, in view of Lemma 4.3, there exists a choice of signs ε_k such that in item (2) of Lemma 3.1 the L^q norms can be replaced by L^2 norms, namely,

$$\left\|\widehat{F_N}|\xi|^{\beta}\right\|_q \lesssim \left(\sum_{k=0}^{N} \Delta_k^{\beta q}(\Delta_k-\Delta_{k-1})\left\|\widetilde{S_k}\right\|_{L^2(\mathbb{T})}^{q}\right)^{\frac{1}{q}}.$$

Since that

$$\left\|\widetilde{S_k}\right\|_{L^2(\mathbb{T})}^{q} = \left(\sum_{j=k}^{N}|c_j|^2\right)^{\frac{q}{2}} \lesssim k^{-q(1+B)+\frac{q}{2}}\log^{-qA}(k+2),$$

we deduce that $\left\|\widehat{F_N}|\xi|^{\beta}\right\|_q \lesssim \log^{\frac{1}{q}}\log N$.

Thus, in the case $q > 2$, the inequality

$$\sum_{n} |f(n)| \lesssim \|f|x|^{\alpha}\|_p + \left\|\widehat{f}|\xi|^{\beta}\right\|_q \tag{4.10}$$

does not hold. Therefore, there is a function f satisfying (1.1) for which $\lim_N P_N(|f|) = \infty$.

Finally, we deal with the case $q = \infty$, for which we recall that $\beta > 1$ and $(\alpha - 1/p')(\beta-1) = 1/2p$. Set

$$c_k = \begin{cases} \varepsilon_k \frac{1}{(1+k)^{1+B}}, & \text{if } \sqrt{N} \le k \le N \\ 0, & \text{otherwise} \end{cases}$$

and

$$\Delta_k = 1 + (1+k)^B (\log N)^{p^{\#} \frac{1}{4(\beta-1)}},$$

where $A = \frac{\beta}{\beta-1}$ and $B = \frac{\alpha-1/p'}{1/p}$, $p^{\#} = 0$ if $p > 1$ and $p^{\#} = 1$ if $p = 1$. It is easy to see that

$$\lim_M P_M(|F_N|) \approx \sum_{n=0}^{N} |c_n| \Delta_n \approx (\log N)^{p^{\#} \frac{1}{4(\beta-1)} + 1}.$$

Next, by item (1) in Lemma 3.1,

$$\| F_N |x|^\alpha \|_p \lesssim \log^{\frac{1}{p}} N.$$

Finally, in the same way as before, by Lemma 4.3 with $q = \infty$, we deduce that there exists a choice of signs for which for all k

$$\begin{aligned} \left\| \widetilde{S_k} \right\|_\infty &\lesssim \log^{\frac{1}{2}}(N+1) \left(\sum_{j=\max(k,\sqrt{N})}^{N} \frac{1}{j^{2+2B}} \right)^{\frac{1}{2}} \\ &\lesssim \log^{\frac{1}{2}}(N+1) \min\left(N^{-1/4-B/2}, k^{-1/2-B} \right). \end{aligned}$$

This implies that

$$\left\| |\xi|^\beta \widehat{F_N} \right\|_\infty \lesssim (\log N)^{p^{\#} \frac{\beta}{4(\beta-1)}} \log^{\frac{1}{2}}(N+1).$$

Combining the above estimates, we see that inequality (4.10) does not hold. The proof is now complete. ☐

5. Poisson formula for general weights

Lemma 3.1 can be easily generalized in the case of general weights.

Lemma 5.1. *Under all the conditions of Lemma 3.1, assuming that w is an even weight function, we have*

(1) $\| F w \|_p \lesssim \left(\sum_{k=0}^{N} |c_k|^p \, \| w(x) \phi((x-k)\Delta_k) \|_p^p \, \Delta_k^p \right)^{\frac{1}{p}}$;

(2) *if w non-decreasing,*

$$\begin{aligned} \left\| \widehat{F} w \right\|_q &\lesssim \left(\int_0^{\Delta_0+2} w^q \right)^{\frac{1}{q}} \left\| \widetilde{S_0} \right\|_{L^q(\mathbb{T})} \\ &\quad + \left(\sum_{k=1}^{N} w(\Delta_k)^q (\Delta_k - \Delta_{k-1}) \left\| \widetilde{S_k} \right\|_{L^q(\mathbb{T})}^q \right)^{\frac{1}{q}}; \end{aligned}$$

(3) *if w is non-increasing,*

$$\left\|\widehat{G}w\right\|_{q'} \lesssim \left(\sum_{k=0}^{N-1} w^{q'}(\Delta_k/2)(\Delta_{k+1}-\Delta_k)\left\|S_k\right\|_{L^{q'}(\mathbb{T})}^{q'}\right)^{\frac{1}{q'}} + \left(\int_{\frac{\Delta_N}{2}-1}^{\infty} w^{q'}\right)^{\frac{1}{q'}} \left\|S_N\right\|_{L^{q'}(\mathbb{T})}.$$

As a consequence, we obtain a sufficient condition for the validity of (PSF) in Lebesgue spaces with general weights. Recall that the Dirichlet kernel is given by $D_n(x) = \sum_{k=-n}^{n} e^{2\pi i k x}$.

Theorem 5.2. *Let u, v be even, non-decreasing weights and $1 \le p, q \le \infty$. Assume that the Schwartz functions are dense in the space with norm $\|fu\|_p + \left\|\widehat{f}v\right\|_q$ and that there exist two sequences Δ and $\widetilde{\Delta}$ as in Lemma 3.1 such that*

$$\sup_N \left(\sum_{k=0}^{N} \left\|u^{-1}(x)\phi((x-k)\Delta_k)\right\|_{p'}^{p'} \Delta_k^{p'}\right)^{\frac{1}{p'}} < \infty;$$

$$\sup_N \left(\sum_{k=0}^{N-1} v^{-q'}(\Delta_k/2)(\Delta_{k+1}-\Delta_k)\left\|D_k\right\|_{L^{q'}(\mathbb{T})}^{q'}\right)^{\frac{1}{q'}} + \left(\int_{\frac{\Delta_N}{2}-1}^{\infty} v^{-q'}\right)^{\frac{1}{q'}} \left\|D_N\right\|_{L^{q'}(\mathbb{T})} < \infty;$$

$$\sup_N \left(\sum_{k=0}^{N} \left\|v^{-1}(x)\phi((x-k)\widetilde{\Delta}_k)\right\|_{q'}^{q'} \widetilde{\Delta}_k^{q'}\right)^{\frac{1}{q'}} < \infty;$$

and

$$\sup_N \left(\sum_{k=0}^{N-1} u^{-p'}(\widetilde{\Delta}_k/2)(\widetilde{\Delta}_{k+1}-\widetilde{\Delta}_k)\left\|D_k\right\|_{L^{p'}(\mathbb{T})}^{p'}\right)^{\frac{1}{p'}} + \left(\int_{\frac{\widetilde{\Delta}_N}{2}-1}^{\infty} u^{-p'}\right)^{\frac{1}{p'}} \left\|D_N\right\|_{L^{p'}(\mathbb{T})} < \infty.$$

Then, for any f with $\|fu\|_p + \left\|\widehat{f}v\right\|_q < \infty$, the formula (PSF) *holds.*

Remark 5.3. *It is possible to obtain an analogue of Theorem 5.2 for the absolute convergence of $P_N(|f|)$ by using the same ideas as in the proof of Theorem 2.3, item* (2).

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