Dynamics of a Large System of Heavy Particles in a Newtonian Fluid

Marco Bravin[1] Eduard Feireisl [*] Arnab Roy[2,3] Arghir Zarnescu[2,3,4]

[1] Departmento de Matemática Aplicada y Ciencias de la Computacíon
E.T.S.I. Industriales y de Telecomunicacíon, Universidad de Cantabria, 39005 Santander, Spain.
[*] Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, CZ-115 67 Praha 1, Czech Republic
[2] BCAM, Basque Center for Applied Mathematics
Mazarredo 14, E48009 Bilbao, Bizkaia, Spain.
[3]IKERBASQUE, Basque Foundation for Science,
Plaza Euskadi 5, 48009 Bilbao, Bizkaia, Spain.
[4]'Simion Stoilow" Institute of the Romanian Academy,
21 Calea Griviţei, 010702 Bucharest, Romania.

Abstract

We consider the motion of a large number of heavy particles in a Newtonian fluid occupying a bounded spatial domain. When we say "heavy", we mean a particle with a mass density that approaches infinity at an appropriate rate as its radius vanishes. We show that the collective effect of heavy particles on the fluid motion is similar to the Brinkman perturbation of the Navier–Stokes system identified in the homogenization process.

Keywords: Rigid bodies in a Newtonian fluid, heavy body, Brinkman homogenization limit, fluid structure interaction

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Contents

1 Introduction

We study the behavior of a fluid in the presence of a large number of small bodies, aiming to understand the emerging behaviour as the number of bodies increase simultaneously with their size decreasing. It was Albert Einstein in his Ph.D. thesis and afterwards in [10] who first pointed out that in such a setting there can be an emerging effect of the bodies onto the fluid. Specifically in the case considered by Einstein one obtains a more viscous fluid, with the additional viscosity explicitly identified in terms of the parameters of the bodies involved.

Einsteins seminal work [10] on the effective viscosity of dilute suspensions laid the theoretical groundwork for a wide array of mathematical investigations into fluid-particle interactions. Building on this foundation, researchers have rigorously extended and refined his setting, particularly in the context of deriving, correcting, and justifying the effective viscosity formula for various suspension models. For instance, in [26], the authors provided a rigorous proof of Einsteins formula, while the papers [24, 23, 35] introduced corrections under mild assumptions and examined the influence of particle distribution on macroscopic flow properties. More recently, in [11, 12], the authors have contributed significantly by addressing the corrector equations and effective viscosity in colloidal suspensions, offering both analytical insights and homogenization frameworks that connect microscopic dynamics to macroscopic fluid behavior.

In parallel, there has been a rapidly growing body of literature devoted to understanding the homogenization effects that emerge when a fluid interacts with a large number of small particles or perforations. In [22] investigated suspensions beyond the dilute regime, identifying nonlinear viscosity effects, while the authors in [29] analyzed the hindered settling phenomenon in well-separated particle systems. The articles [34, 31] have been dedicated to developing methods to study convergence and filament interactions in Stokes flows, while [21] examined vortex dynamics in the zero-radius limit. The recent contribution [47] delves into complex suspension behaviors like orientation mixing, further enriching the theoretical landscape. Collectively, these works highlight the sophistication of modern homogenization theory in fluid dynamics and underscore its relevance in both applied and theoretical settings.

At the same time, considerable effort has been devoted to showing that a cloud of small particles can be effectively described by the VlasovNavierStokes model. Currently, only partial results are available. In [2], the VlasovNavierStokes system was derived from a coupled system of Boltzmann equations, one modeling the fluid and the other modeling the rigid bodies. In [18, 19], the authors started with an intermediate system in which the fluid occupies the entire domain, and the influence of the rigid bodies is represented by an external force resembling the Stokes drag. In the asymptotic limit, they obtained the VlasovNavierStokes system. Finally, in [36, 37], under the assumption of vanishing Reynolds number for the fluid, the authors derived, in the limit where the number of rigid bodies tends to infinity and the particle inertia tends to zero, a transportStokes type equation. They further demonstrated that the resulting model is well-approximated by an appropriate VlasovStokes system.

A common theme of most of the existing literature seems to be the focus on cases in which the fluid is restricted to be of Stokes type. Our current exploration takes the flow to be a general time-dependent Navier-Stokes. We highlight that in [14, 15] we examined the case of a compressible or nearly compressible fluid with a single immersed body. It was shown that the body has a negligible effect on the fluid, in the limit of vanishing size. This required the construction of a specific type of test function adapted to the geometry of the body within the fluid.

Furthermore, we consider in [16] the case of several bodies with the number of bodies increasing proportionally with their size decreasing. This involved a delicate management of interactions of the bodies in a manner largely independent of the relative velocities of the bodies. We were able to show that under suitable assumptions concerning the relationship between the size of the bodies and their number, the overall effect of the bodies is negligible, that is in the limit one recovers the standard fluid equations. Similar results hold also for a large class of non-Newtonian fluids [17].

The results of the present paper are complementary to those obtained in our previous work [16], where the motion of a large number of "light" particles was considered. Here "light" means the mass density of each particle is uniformly bounded. In the case of the planar 2d-motion, the result is in a sharp contrast with the homogenization problem, where the particles are fixed and produce an additional friction (Brinkman perturbation) of the limit Navier–Stokes system. Intuitively, the particles have less influence on the fluid motion if they are allowed to move together with the fluid. Previously, this phenomenon had also been observed in [39] for a single moving "light" particle by Lacave, Takahashi.

We consider the motion of a large number of heavy particles in a Newtonian fluid. By "heavy" we mean that the mass density of the particle is large approaching infinity at an appropriate rate when the radius of the particle vanishes. Furthermore, the bodies are assumed to have suitably small initial velocity. We show that the collective effect of the heavy particles on the fluid motion is similar to the Brinkman perturbation of the Navier–Stokes system identified in the homogenization process. We want to mention that in the case of a single "heavy" particle, Iftime and He demonstrate in [27, 28] that the solution of the fluid-rigid body system approaches a solution of the Navier-Stokes equations (without Brinkman perturbation) in whole space as the diameter of the rigid body approaches zero. The influence of a small rigid body moving in a compressible fluid has recently been analyzed in [14, 15, 3]. In comparing with the literature we note that the setting of "heavy" bodies provides a significant difference with the Einstein quasi-stationary setting, where the mass is irrelevant, and where the result is obtained not in the limit, but to a first order. The heaviness allows to slow the bodies movement and our assumptions in terms of the initial particle distribution are closer in spirit to those of [8] while also allowing to provide a full treatment of the inertial terms.

Let us emphasize that our results concern the genuine fluid–structure interaction problem of a system of rigid bodies immersed in a Newtonian fluid avoiding any simplifying assumptions leading to the Stokes–like quasistatic approximation of the fluid equations commonly used in the literature. Our approach is based on a new result in "dynamic homogenization", where the particles move with a prescribed velocity, which may be of independent interest. The main idea is conceptually simple, namely the particles will approach a static position in the asymptotic limit. However, its implementation generates complex issues, related fundamentally with the proper velocity decay in relationship with the size of the particle and their mutual distances.

First, we transform the problem to a fixed spatial domain by means of the change of variables introduced by Inoue and Wakimoto [38] and later elaborated by a number of authors, see e.g. Cumsille, Takahashi [7], and [44], [3] for the most recent treatment. The resulting problem is no longer of Navier–Stokes type containing a perturbation of the velocity in the time derivative. Next, time regularization is used to obtain a stationary homogenization problem at each fixed time level. At this stage, homogenization limit is performed for any fixed time. It is important that the limit is actually the same for any time depending solely on the spatial distribution of the particles and their radius, see Desvilletes, Golse, and Ricci [8], Marchenko and Khruslov [40], or a more recent treatment in [13]. Finally, we relax the time regularization by means of a modification of the Aubin–Lions argument.

The paper is organized as follows. In Section 2, we provide a precise formulation of the problems under consideration and state the main results. This includes the governing equations, specifying the assumptions on the fluid and the particles, and outlining the key mathematical statements that will be rigorously proved in later sections. Section 3 collects the necessary is dedicated to collecting essential background material on the homogenization of the stationary Stokes problem. We review relevant theoretical results and techniques that play a crucial role in our analysis, particularly in handling the presence of a large number of small bodies in the fluid. The change in variables is introduced in Section 4 and the problem is transformed to a fixed spatial domain in Section 5. Since the presence of many small moving bodies introduces a complex and evolving geometry, transforming the problem into a fixed reference domain allows for a more tractable mathematical approach. This step is crucial for establishing uniform estimates and passing to the limit in the subsequent sections. The core analytical work is carried out in Section 6, where we derive the asymptotic limit and rigorously prove the main results of the paper. This involves justifying the homogenization process, establishing convergence results, and demonstrating how the presence of the small bodies affects the overall fluid behavior in the limit. A priori estimates of the velocity of small rigid bodies are established in Section 7. The paper is concluded by a short discussion concerning the extension of the result to the space dimension $d = 2$.

2 Problem formulation, main result

We consider a system of N rigid balls $\{B_{n,\varepsilon} = B(\mathbf{h}_{n,\varepsilon}, r_\varepsilon)\}_{n=1}^N$, $N = N(\varepsilon)$, centered at points $\mathbf{h}_{n,\varepsilon} = \mathbf{h}_{n,\varepsilon}(t)$ and with the common radius r_ε immersed in an incompressible Newtonian fluid occupying a bounded domain $\Omega \subset R^3$. To avoid technicalities, we always assume Ω is sufficiently smooth, at least of class $C^{2+\nu}$. The balls move with the rigid velocities

$$\mathbf{w}_{n,\varepsilon}(t,x) = \mathbf{h}'_{n,\varepsilon}(t) + \boldsymbol{\omega}_{n,\varepsilon}(t) \wedge (x - \mathbf{h}_{n,\varepsilon}(t)),\ t \in [0,T],\ n = 1, \dots, N(\varepsilon). \tag{2.1}$$

The fluid occupies the domain

$$\Omega_{\varepsilon,t} = \Omega \setminus \cup_{n=1}^N B(\mathbf{h}_{n,\varepsilon}, r_\varepsilon) \tag{2.2}$$

at each time $t \in [0,T]$. The fluid velocity $\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon(t,x)$ satisfies the incompressibility constraint

$$\mathrm{div}_x \mathbf{u}_\varepsilon(t,\cdot) = 0 \text{ for any } t \in [0,T]. \tag{2.3}$$

The fluid motion is governed by the Navier–Stokes system

$$\partial_t \mathbf{u}_\varepsilon + \mathrm{div}_x(\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla_x \Pi_\varepsilon = \Delta_x \mathbf{u}_\varepsilon \text{ in } \cup_{t \in (0,T)} \Omega_{\varepsilon,t}. \tag{2.4}$$

For the sake of simplicity, we have set the fluid density as well the viscosity coefficient to be one.

We impose the no-slip boundary conditions at the fluid–body interface

$$\mathbf{u}_\varepsilon|_{\partial\Omega} = 0,\ \mathbf{u}_\varepsilon|_{\partial B_{n,\varepsilon}} = \mathbf{w}_{n,\varepsilon}|_{\partial B_{n,\varepsilon}},\ n = 1, \dots, N(\varepsilon). \tag{2.5}$$

Accordingly, we consider the initial velocity distribution $\mathbf{u}_{0,\varepsilon} \in L^2(\Omega; R^3)$ satisfying

$$\mathrm{div}_x \mathbf{u}_{0,\varepsilon} = 0, \quad \mathbf{u}_{0,\varepsilon} \cdot \mathbf{n}|_{\partial\Omega} = 0 \quad \text{and} \quad \mathbf{u}_{0,\varepsilon} \cdot \mathbf{n}|_{\partial B_{n,\varepsilon}} = \mathbf{w}_{n,\varepsilon}(0, .) \cdot \mathbf{n}|_{\partial B_{n,\varepsilon}},\ n = 1, \dots, N(\varepsilon). \tag{2.6}$$

Remark 2.1. We neglect the influence of gravitational force for two main reasons. First, the result should be understood as the counterpart of [FRZ] (where no gravitation was considered). Secondly, in the presence of gravitational forces, the rigid bodies accelerate toward the boundary under gravity at a constant rate in the limit when the radius of the rigid bodies tends to zero. As the rigid bodies approach the boundary, we lose control over their evolution, and analyzing this scenario presents significant difficulties.

2.1 Dynamic homogenization

We consider the initial configuration of the particles of *critical* type, where the total (Stokes) capacity of the whole ensemble of particles is bounded, while their mutual distance is large enough to keep the collective effect of drag forces balanced with the external forces driving the fluid motion.

Specifically, we assume

$$\frac{r_\varepsilon}{\varepsilon^3} \to 1 \text{ as } \varepsilon \to 0, \tag{2.7}$$

while the initial positions of the rigid bodies satisfy

$$|\mathbf{h}_{i,\varepsilon}(0) - \mathbf{h}_{j,\varepsilon}(0)| > 2\varepsilon,\ i \neq j,\ \mathrm{dist}[\mathbf{h}_{n,\varepsilon}(0); \partial\Omega] > \varepsilon;\ i, j, n \in \{1, \dots, N(\varepsilon)\} \tag{2.8}$$

In addition, we require the translational components of rigid velocities $\mathbf{w}_{n,\varepsilon}$ to be small,

$$\sup_{n=1,\dots N(\varepsilon)} \frac{\|\mathbf{h}'_{n,\varepsilon}\|_{L^\infty(0,T)}}{\varepsilon^2} \to 0 \text{ as } \varepsilon \to 0. \tag{2.9}$$

Note carefully that condition (2.9) prevents collisions of the bodies in a fixed time interval $[0, T]$ as well as collision of a rigid body with the boundary $\partial\Omega$.

We say that a velocity field $\mathbf{u}_\varepsilon$ is a weak solution of the problem (2.3), (2.4) with initial condition $\mathbf{u}_\varepsilon(0, \cdot) = \mathbf{u}_{0,\varepsilon}$ satisfying (2.6) if the following holds:

$$\mathbf{u}_\varepsilon \in L^\infty(0, T; L^2(\Omega; R^3)) \cap L^2(0, T; W_0^{1,2}(\Omega; R^3)), \tag{2.10}$$

$$\mathbf{u}_\varepsilon(t, \cdot)|_{B(\mathbf{h}_{n,\varepsilon}(t), r_\varepsilon)} = \mathbf{w}_{n,\varepsilon}(t, \cdot),\ n = 1, \dots, N,\ \text{for a.a. } t \in (0, T), \tag{2.11}$$

$$\mathrm{div}_x \mathbf{u}_\varepsilon = 0 \text{ in } (0, T) \times \Omega, \tag{2.12}$$

$$\int_0^T \int_\Omega \Big[\mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + [\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon] : \nabla_x \boldsymbol{\varphi} \Big] \ \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \nabla_x \mathbf{u}_\varepsilon : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t - \int_\Omega \mathbf{u}_{0,\varepsilon} \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}x \tag{2.13}$$

for any test function $\boldsymbol{\varphi} \in C^1_c([0,T) \times \Omega; R^3)$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$, satisfying

$$\boldsymbol{\varphi}(t,\cdot)|_{B(\mathbf{h}_{n,\varepsilon}(t), r_\varepsilon)} = 0, \ n = 1, \dots, N, \ t \in [0,T]. \tag{2.14}$$

Remark 2.2. Unlike the fluid–structure interaction problem considered in Section 2.2 below, the rigid velocities are given *a priori* in (2.10)–(2.11). Accordingly, the eligible test functions in the momentum equation (2.13) vanish on the rigid bodies.

We claim the following result.

Theorem 2.3 (Dynamic homogenization). *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$. Suppose the initial distribution of the rigid bodies satisfies*

$$\frac{r_\varepsilon}{\varepsilon^3} \to 1 \ \text{as } \varepsilon \to 0, \ |\mathbf{h}_{i,\varepsilon}(0) - \mathbf{h}_{j,\varepsilon}(0)| > 2\varepsilon, \ i \neq j, \ \mathrm{dist}[\mathbf{h}_{n,\varepsilon}(0); \partial\Omega] > \varepsilon; \ i,j,n \in \{1, \dots, N(\varepsilon)\}. \tag{2.15}$$

In addition, let the rigid body velocities satisfy

$$\sup_{n=1,\dots N(\varepsilon)} \frac{\|\mathbf{h}'_{n,\varepsilon}\|_{L^\infty(0,T)}}{\varepsilon^2} \to 0, \ \text{as } \varepsilon \to 0, \ \sup_{n=1,\dots N(\varepsilon)} r_\varepsilon \|\boldsymbol{\omega}_{n,\varepsilon}\|_{L^\infty(0,T)} \to 0 \ \text{as } \varepsilon \to 0. \tag{2.16}$$

Let $\mathbf{u}_\varepsilon$ be a weak solution to the moving boundary problem (2.11)–(2.13),

$$\mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon}$$

such that

$$\|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega;R^3))} + \|\mathbf{u}_\varepsilon\|_{L^2(0,T;W^{1,2}_0(\Omega;R^3))} \lesssim 1 \ \text{uniformly for } \varepsilon \to 0. \tag{2.17}$$

Then, up to a suitable subsequence

$$\varepsilon^3 \sum_{n=1}^{N(\varepsilon)} \delta_{\mathbf{h}_{n,\varepsilon}(0)} \to \mathfrak{R} \in L^\infty(\Omega) \ \text{weakly-(*) in } \mathfrak{M}(\overline{\Omega}) \tag{2.18}$$

and

$$\mathbf{u}_\varepsilon \to \mathbf{u} \ \text{in } L^2((0,T) \times \Omega; R^3) \ \text{and weakly in } L^2(0,T; W^{1,2}_0(\Omega; R^3)),$$

where the limit $\mathbf{u}$ is a weak solution of the Navier–Stokes problem with Brinkman friction,

$$\mathrm{div}_x \mathbf{u} = 0, \tag{2.19}$$

$$\int_0^T \int_\Omega \Big[\mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \boldsymbol{\varphi} \Big] \ \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \Big[\nabla_x \mathbf{u} : \nabla_x \boldsymbol{\varphi} + 6\pi \mathfrak{R} \mathbf{u} \cdot \boldsymbol{\varphi} \Big] \ \mathrm{d}x \, \mathrm{d}t - \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0,\cdot) \ \mathrm{d}x \tag{2.20}$$

for any $\boldsymbol{\varphi} \in C^1_c([0,T) \times \Omega; R^3)$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$.

Remark 2.4. Note that hypothesis (2.15) yields

$$\varepsilon^3 N(\varepsilon) \lesssim |\Omega| \ \Rightarrow \varepsilon^3 \ N(\varepsilon) \to \lambda \geq 0$$

at least for a suitable subsequence. Accordingly, (2.18) can be written in the form

$$\varepsilon^3 \sum_{n=1}^{N(\varepsilon)} \delta_{\mathbf{h}_{n,\varepsilon}(0)} = \varepsilon^3 N(\varepsilon) \left(\frac{1}{N(\varepsilon)} \sum_{n=1}^{N(\varepsilon)} \delta_{\mathbf{h}_{n,\varepsilon}(0)} \right),$$

where

$$\frac{1}{N(\varepsilon)} \sum_{n=1}^{N(\varepsilon)} \delta_{\mathbf{h}_{n,\varepsilon}(0)}$$

is the empirical measure introduced by Desvilletes, Golse, and Ricci [8].

Remark 2.5. A restrictive assumption we make concerns the initial velocities of the bodies. It is known that in related contexts (see [8, 30]) allowing for higher initial velocity can generate additional corrections. We cannot treat this case since we expect that in the case of heavy bodies, a large enough velocity will determine the body to move linearly (as shown in the case of a singled body in [4]).

Remark 2.6. We note that one could attempt to consider the case when the viscosity also goes to zero proportional to ε, as done for instance, in the case of fixed obstacles, in [32]. It is not immediately clear if one would obtain a similar result in our setting.

2.2 Fluid–structure interaction problem

In the fluid–structure interaction problem, the velocities of the particles are determined by the mutual interaction with the fluid. Specifically, in addition to the no–slip condition (2.5), continuity of the momenta is imposed.

For the sake of simplicity, we suppose all particles have the same mass density $\varrho_{\varepsilon,S} > 0$. The weak formulation of the fluid–structure interaction problem reads as follows, see e.g. [16, Section 2.1].

- **Regularity.**

$$\begin{aligned} \varrho_\varepsilon > 0,\ \varrho_\varepsilon \in L^\infty((0,T) \times R^3) \cap C([0,T]; L^1(\Omega)), \\ \mathbf{u}_\varepsilon \in L^\infty(0,T;\Omega) \cap L^2(0,T; W_0^{1,2}(\Omega)),\ \mathrm{div}_x \mathbf{u}_\varepsilon = 0, \\ \mathbf{h}_{n,\varepsilon} \in W^{1,\infty}([0,T]; R^3),\ \boldsymbol{\omega}_{n,\varepsilon} \in W^{1,\infty}([0,T]; R^3),\ n = 1, \dots, N. \end{aligned} \tag{2.21}$$

- **Compatibility.**

$$\begin{aligned} \varrho_\varepsilon(t,x) = \left\{ \begin{array}{l} \varrho_{\varepsilon,S} \text{ if } x \in B(\mathbf{h}_{n,\varepsilon}(t), r_\varepsilon),\ t \in [0,T] \\ 1 \text{ otherwise,} \end{array} \right. \\ \mathbf{u}_\varepsilon(t,\cdot)|_{B(\mathbf{h}_{n,\varepsilon}(t), r_\varepsilon)} = \mathbf{w}_{n,\varepsilon}(t),\ n = 1, \dots, N, \text{ for a.a. } t \in (0,T). \end{aligned} \tag{2.22}$$

- **Mass conservation.**

$$\int_0^T \int_{R^3} \Big[\varrho_\varepsilon \partial_t \varphi + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \Big] \,\mathrm{d}x \,\mathrm{d}t = - \int_{R^3} \varrho_{0,\varepsilon} \varphi(0,\cdot) \,\mathrm{d}x \tag{2.23}$$

for any $\varphi \in C^1_c([0,T) \times R^3)$.

- **Momentum balance.**

$$\int_0^T \int_\Omega \Big[\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + \varrho_\varepsilon [\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon] : \nabla_x \boldsymbol{\varphi} \Big] \ \mathrm{d}x \,\mathrm{d}t$$
$$= \int_0^T \int_\Omega \mathbb{D}_x \mathbf{u}_\varepsilon : \mathbb{D}_x \boldsymbol{\varphi} \ \mathrm{d}x \,\mathrm{d}t - \int_\Omega \varrho_{\varepsilon,0} \mathbf{u}_{\varepsilon,0} \boldsymbol{\varphi}(0,\cdot) \ \mathrm{d}x \tag{2.24}$$

for any test function $\boldsymbol{\varphi} \in C^1_c([0,T) \times \Omega)$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$,

$$\mathbb{D}_x \boldsymbol{\varphi}(t,\cdot) = 0 \text{ on an open neighbourhood of } B(\mathbf{h}_{n,\varepsilon}(t), r_\varepsilon),\ n = 1,\dots,N,\ t \in [0,T]. \tag{2.25}$$

The symbol $\mathbb{D}_x$ denotes the symmetric part of the gradient,

$$\mathbb{D}_x = \frac{1}{2} \left(\nabla_x + \nabla_x^t \right).$$

- **Energy dissipation.**

$$\frac{1}{2} \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 (\tau,\cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega |\mathbb{D}_x \mathbf{u}_\varepsilon|^2 \ \mathrm{d}x \,\mathrm{d}t \le \frac{1}{2} \int_\Omega \varrho_{\varepsilon,0} |\mathbf{u}_{\varepsilon,0}|^2 \ \mathrm{d}x \tag{2.26}$$

for a.a. $\tau \in (0,T)$.

The following result is a consequence of Theorem 2.3.

Theorem 2.7. (Fluid–structure interaction with heavy particles.) *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$. Suppose the initial distribution of the rigid bodies satisfies*

$$\frac{r_\varepsilon}{\varepsilon^3} \to 1 \ \textit{as}\ \varepsilon \to 0,$$
$$|\mathbf{h}_{i,\varepsilon}(0) - \mathbf{h}_{j,\varepsilon}(0)| > 2\varepsilon,\ i \ne j,\ \mathrm{dist}[\mathbf{h}_{n,\varepsilon}(0); \partial\Omega] > \varepsilon;\ i,j,n \in \{1,\dots,N(\varepsilon)\}.$$

Let the rigid densities and the initial velocity of the rigid body satisfy

$$\varrho_{\varepsilon,S} \varepsilon^{\frac{19}{2}} \to \infty \quad \textit{and} \quad \sup_{n \in N(\varepsilon)} \frac{|\mathbf{h}'_{n,\varepsilon}(0)|}{\varepsilon^2} \to 0 \ \textit{as}\ \varepsilon \to 0. \tag{2.27}$$

Let $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ be a weak solution to the fluid–structure interaction problem (2.21)–(2.26) *with bounded initial energy,*

$$\int_\Omega \varrho_{\varepsilon,0} |\mathbf{u}_{\varepsilon,0}|^2 \ \mathrm{d}x \lesssim 1 \ \textit{uniformly for}\ \varepsilon \to 0. \tag{2.28}$$

Then, up to a suitable subsequence

$$\varepsilon^3 \sum_{n=1}^{N(\varepsilon)} \delta_{\mathbf{h}_{n,\varepsilon}(0)} \to \mathfrak{R} \in L^\infty(\Omega) \ \textit{weakly-(*) in } \mathfrak{M}(\overline{\Omega}) \tag{2.29}$$

and

$$\mathbf{u}_\varepsilon \to \mathbf{u} \ \textit{in } L^2((0,T) \times \Omega; R^3) \ \textit{and weakly in } L^2(0,T; W_0^{1,2}(\Omega; R^3)),$$

where the limit $\mathbf{u}$ *is a weak solution of the Navier–Stokes problem with Brinkman friction,*

$$\mathrm{div}_x \mathbf{u} = 0, \tag{2.30}$$

$$\int_0^T \int_\Omega \Big[\mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \boldsymbol{\varphi} \Big] \ \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \Big[\nabla_x \mathbf{u} : \nabla_x \boldsymbol{\varphi} + 6\pi \mathfrak{R} \mathbf{u} \cdot \boldsymbol{\varphi} \Big] \ \mathrm{d}x \, \mathrm{d}t - \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0,.) \ \mathrm{d}x \tag{2.31}$$

for any $\boldsymbol{\varphi} \in C_c^1([0,T) \times \Omega; R^3)$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$.

Theorem 2.7 follows directly from Theorem 2.3 combined with the following result providing control of the velocity of a small rigid body based on the ideas of [4].

Proposition 2.8. *Let* $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ *be a weak solution to the fluid–structure interaction problem* (2.21)–(2.26) *with bounded initial energy* (2.28). *If*

$$\sup_{n \in N(\varepsilon)} \frac{|\mathbf{h}'_{n,\varepsilon}(0)|}{\varepsilon} \to 0 \quad \textit{and} \quad \sup_{n \in N(\varepsilon)} \frac{r_\varepsilon^{1/2}}{\varepsilon m_\varepsilon} \to 0 \quad \textit{as } \varepsilon \to 0, \tag{2.32}$$

then, there exists $\bar{\varepsilon} > 0$ *such that for any* $0 < \varepsilon < \bar{\varepsilon}$, *it holds*

$$\|\mathbf{h}'_{n,\varepsilon} - \mathbf{h}'_{n,\varepsilon}(0)\|_{L^\infty(0,T)} + r_\varepsilon \|\boldsymbol{\omega}_{n,\varepsilon} - \boldsymbol{\omega}_{n,\varepsilon}(0)\|_{L^\infty(0,T)} \leq C \frac{r_\varepsilon^{1/2}}{m_\varepsilon}, \tag{2.33}$$

where C *does not depends on* n *or* ε *and we denoted* $m_\varepsilon := r_\varepsilon^3 \varrho_{\varepsilon,S}$.

Proof of Theorem 2.7. The boundedness of the initial energy (2.28) implies that the velocity field $\mathbf{u}_\varepsilon$ satisfies the uniform bound (2.17). Moreover, assumption (2.27) together with $\frac{r_\varepsilon}{\varepsilon^3} \to 1$ as $\varepsilon \to 0$ imply (2.32). Thus, we have verified the hypothesis of Proposition 2.8. We deduce from (2.33) and (2.27) that

$$\sup_{n=1,\dots N(\varepsilon)} \frac{\|\mathbf{h}'_{n,\varepsilon}\|_{L^\infty(0,T)}}{\varepsilon^2} \to 0, \ \text{as } \varepsilon \to 0. \tag{2.34}$$

Furthermore, from the bound $\int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \ \mathrm{d}x \lesssim 1$, we have

$$\int_{B(\mathbf{h}_{n,\varepsilon}, r_\varepsilon)} \varrho_{\varepsilon,S} |\mathbf{w}_{n,\varepsilon}|^2 \, \mathrm{d}x \lesssim 1 \text{ for any } n = 1, \dots, N(\varepsilon). \tag{2.35}$$

Moreover, we have

$$\int_{B(\mathbf{h}_{n,\varepsilon},r_\varepsilon)} \varrho_{\varepsilon,S}|\mathbf{w}_{n,\varepsilon}|^2 = m_\varepsilon|\mathbf{h}'_{n,\varepsilon}|^2 + J_\varepsilon|\boldsymbol{\omega}_{n,\varepsilon}|^2 = m_\varepsilon|\mathbf{h}'_{n,\varepsilon}|^2 + \frac{2}{5}m_\varepsilon r_\varepsilon^2|\boldsymbol{\omega}_{n,\varepsilon}|^2 \geq \frac{2}{5}m_\varepsilon r_\varepsilon^2|\boldsymbol{\omega}_{n,\varepsilon}|^2,$$

where J_ε is the moment of inertia of $B(\mathbf{h}_{n,\varepsilon}(t), r_\varepsilon)$.

As $\frac{r_\varepsilon}{\varepsilon^3} \to 1$ as $\varepsilon \to 0$ and $\varrho_{\varepsilon,S} = \frac{m_\varepsilon}{r_\varepsilon^3}$, we get $m_\varepsilon \approx \varrho_{\varepsilon,S}\varepsilon^9$. Thus we obtain from (2.35) that

$$\varrho_{\varepsilon,S}\varepsilon^9 r_\varepsilon^2|\boldsymbol{\omega}_{n,\varepsilon}|^2 \lesssim 1 \implies r_\varepsilon^2|\boldsymbol{\omega}_{n,\varepsilon}|^2 \lesssim \frac{1}{\varrho_{\varepsilon,S}\varepsilon^9} = \frac{\varepsilon^{1/2}}{\varrho_{\varepsilon,S}\varepsilon^{19/2}}.$$

Hence,

$$\sup_{n=1,\dots N(\varepsilon)} r_\varepsilon\|\boldsymbol{\omega}_{n,\varepsilon}\|_{L^\infty(0,T)} \to 0 \text{ as } \varepsilon \to 0. \tag{2.36}$$

Thus the relations (2.36) and (2.34) give us (2.16). Now we can apply Theorem 2.3 in order to deduce Theorem 2.7.

□

3 Homogenization of the stationary Stokes problem

We collect some known results concerning homogenization of the *stationary* Stokes problem. The distributions of the particles is the same as in (2.15). To simplify the notation, we write $\mathbf{h}_{n,\varepsilon}$ instead of $\mathbf{h}_{n,\varepsilon}(0)$. Similarly, we write Ω_ε instead of $\Omega_{\varepsilon,0}$.

First, we claim that under the hypotheses of Theorem 2.3 on the initial distribution of the obstacles, there exists a subsequence (not relabelled) such that

$$\varepsilon^3 \sum_{n=1}^{N(\varepsilon)} \delta_{\mathbf{h}_{n,\varepsilon}} \to \mathfrak{R} \text{ weakly-(*) in } \mathfrak{M}(\overline{\Omega}), \tag{3.1}$$

where $\mathfrak{R} \in L^\infty(\Omega)$. Indeed, in view of hypothesis (2.8), we have $\varepsilon^3 N(\varepsilon) \stackrel{<}{\sim} |\Omega|$, in particular, the limit (3.1) exists for a suitable subsequence. Moreover, as any cube of the side ε can contain at most one point $\mathbf{h}_{n,\varepsilon}$, we conclude

$$\left\langle \varepsilon^3 \sum_{n=1}^{N(\varepsilon)} \delta_{\mathbf{h}_{n,\varepsilon}}, \phi \right\rangle \equiv \varepsilon^3 \sum_{n=1}^{N(\varepsilon)} \phi(\mathbf{h}_{n,\varepsilon}) \stackrel{<}{\sim} \int_\Omega \phi(x) \ \mathrm{d}x \text{ for any } \phi \in C(\overline{\Omega}),\ \phi \geq 0,$$

which yields $\mathfrak{R}$ is bounded absolutely continuous with respect to the Lebesgue measure. This provides the claimed implication (2.18) in Theorem 2.3

3.1 Stokes problem

Consider the Stokes problem

$$\Delta_x \mathbf{v}_\varepsilon + \nabla_x p_\varepsilon = \mathbf{f}_\varepsilon,\ \mathrm{div}_x \mathbf{v}_\varepsilon = 0 \text{ in } \Omega_\varepsilon$$
$$\mathbf{v}_\varepsilon|_{\partial\Omega_\varepsilon} = 0. \tag{3.2}$$

Proposition 3.1. (Homogenization of Stokes problem.)
Let $\Omega \subset R^3$ be a bounded domain of class at least $C^{2+\nu}$. Let the particles and their positions satisfy (2.7), (2.8). *Finally, let*

$$\mathbf{f}_\varepsilon \to \mathbf{f} \ \textit{in}\ W^{-1,2}(\Omega; R^3), \tag{3.3}$$

$$\varepsilon^3 \sum_{n=1}^{N(\varepsilon)} \delta_{\mathbf{h}_{n,\varepsilon}} \to \mathfrak{R} \ \textit{weakly-(*) in}\ \mathfrak{M}(\overline{\Omega}) \tag{3.4}$$

as $\varepsilon \to 0$, where the limit $\mathfrak{R}$ coincides with an L^∞ function on Ω. Then the Stokes problem (3.2) *admits a unique solution $(\mathbf{v}_\varepsilon, p_\varepsilon)$, and*

$$\mathbf{v}_\varepsilon \to \mathbf{v} \ \textit{weakly in}\ W_0^{1,2}(\Omega; R^3), \tag{3.5}$$

where $\mathbf{v}$ is the unique solution of the Stokes problem

$$\Delta_x \mathbf{v} + \nabla_x p = \mathbf{f} + 6\pi\mathfrak{R}\mathbf{v} \ \textit{in}\ \Omega,$$
$$\mathbf{v}|_{\partial\Omega} = 0. \tag{3.6}$$

For the proof see Desvillettes, Golse, and Ricci [8, Theorem 1]. A quantitative estimate in L^p norms is provided in [41]. One of the earliest results, in a periodic setting was obtained by G. Allaire in [1]. A more general setting with obstacles of different radii and mutual distances was treated in the monograph [40, Chapter 4, Theorem 4.7]. Obstacles of arbitrary shape were considered in [13, Proposition 5.1]. Various random settings were considered in [6, 20, 33].

3.2 Perturbation of the boundary velocity

We extend Proposition 3.1 to functions that do not necessarily vanish on $\partial\Omega_\varepsilon$.

Proposition 3.2 (Boundary velocity). *Under the hypotheses of Proposition 3.1, suppose that $(\mathbf{v}_\varepsilon, p_\varepsilon)$ is the unique solution of the Stokes problem*

$$\Delta_x \mathbf{v}_\varepsilon + \nabla_x p_\varepsilon = \mathbf{f}_\varepsilon,\ \mathrm{div}_x \mathbf{v}_\varepsilon = 0 \ \textit{in}\ \Omega_\varepsilon$$
$$\mathbf{v}_\varepsilon|_{\partial\Omega_\varepsilon} = \mathbf{w}_\varepsilon|_{\partial\Omega_\varepsilon}, \tag{3.7}$$

where

$$\mathrm{div}_x \mathbf{w}_\varepsilon = 0,\ \mathbf{w}_\varepsilon \to 0 \ \textit{in}\ W_0^{1,2}(\Omega; R^3).$$

Then

$$\mathbf{v}_\varepsilon \to \mathbf{v} \ \textit{weakly in} \ W_0^{1,2}(\Omega; R^3), \tag{3.8}$$

where $\mathbf{v}$ *is the unique solution of the Stokes problem*

$$\begin{aligned} \Delta_x \mathbf{v} + \nabla_x p &= \mathbf{f} + 6\pi\Re\mathbf{v} \ \textit{in} \ \Omega, \\ \mathbf{v}|_{\partial\Omega} &= 0. \end{aligned} \tag{3.9}$$

Proof. Apply Proposition 3.1 to the sequence $\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon$.

□

4 Change of variables

Let the assumptions (2.15)–(2.17) of Theorem 2.3 hold. Our goal is to introduce a suitable change of variables to transform the problem to a fixed domain. To this end, we introduce a suitable cut–off function

$$\chi \in C_c^\infty[0,1), \ \chi(z) = \begin{cases} 1 \text{ if } 0 \le z < \frac{1}{2}, \\ \in [0,1] \text{ if } \frac{1}{2} \le z \le \frac{3}{4}, \\ 0 \text{ if } z > \frac{3}{4}, \end{cases}$$

together with the vector field

$$\Lambda_\varepsilon(t,x) = \mathbf{curl}_x \left[\sum_{n=1}^{N(\varepsilon)} \chi\left(\frac{|x - \mathbf{h}_{n,\varepsilon}(0)|}{\varepsilon} \right) \mathbb{T}\Big(x - \mathbf{h}_{n,\varepsilon}(0) \Big) \cdot \mathbf{h}'_{n,\varepsilon}(t) \right], \tag{4.1}$$

where

$$\mathbb{T}(x) = \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}, \tag{4.2}$$

see Cumsille and Takahashi [7].

The following properties of Λ_ε can be verified by a direct calculation.

- $\Lambda_\varepsilon(t,\cdot) \in C^\infty(R^3; R^3)$ for a.a. $t \in [0,T]$.
- $\mathrm{div}_x \Lambda_\varepsilon(t,\cdot) = 0$ for a.a. $t \in [0,T]$.
- Under the hypotheses of Theorem 2.3, notably (2.16),

$$\frac{\|D_x^k \Lambda_\varepsilon\|_{L^\infty((0,T)\times R^3; R^{3(k+1)})}}{\varepsilon^{2-k}} \to 0 \text{ as } \varepsilon \to 0, \ k = 0,1,2. \tag{4.3}$$

In the definition of Λ_ε, we consider rigid velocity that have zero angular velocities. Actually any rigid velocities admit an extension onto Ω similar to (4.1). In the following lemma we present the extension. Moreover we show a bound in $W_0^{1,2}(\Omega)$ which we will use to apply Theorem 3.2.

Lemma 4.1. *Let a family of rigid velocities*

$$\mathbf{w}_{n,\varepsilon} = \mathbf{h}'_{n,\varepsilon} + \boldsymbol{\omega}_{n,\varepsilon} \wedge (y - \mathbf{h}_{n,\varepsilon}) \ \textit{on} \ B(\mathbf{h}_{n,\varepsilon}, r_\varepsilon),\ n = 1, \dots, N(\varepsilon)$$

be given. Then there exists a velocity field $\mathbf{w}_\varepsilon \in W_0^{1,2}(\Omega; R^3)$ *enjoying the following properties:*

$$\mathrm{div}_x \mathbf{w}_\varepsilon = 0 \ \textit{in} \ \Omega;$$

$$\mathbf{w}_\varepsilon = \mathbf{w}_{n,\varepsilon} \ \textit{on} \ B(\mathbf{h}_{n,\varepsilon}, r_\varepsilon),\ n = 1, \dots, N(\varepsilon);$$

$$\|\mathbf{w}_\varepsilon\|_{W_0^{1,2}(\Omega)} \lesssim \left(\max_{n=1,\dots,N(\varepsilon)} |\mathbf{h}'_{n,\varepsilon}| + r_\varepsilon \max_{n=1,\dots,N(\varepsilon)} |\boldsymbol{\omega}_{n,\varepsilon}| \right).$$

Proof. Consider the velocity field

$$\mathbf{w}_\varepsilon(y) = \sum_{n=1}^{N(\varepsilon)} \mathbf{curl}_x \left[\chi \left(2 \frac{|y - \mathbf{h}_{n,\varepsilon}|}{r_\varepsilon} \right) \mathbb{T}(y - \mathbf{h}_{n,\varepsilon}) \cdot \mathbf{h}'_{n,\varepsilon} - \chi \left(2 \frac{|y - \mathbf{h}_{n,\varepsilon}|}{r_\varepsilon} \right) \frac{|y - \mathbf{h}_{n,\varepsilon}|^2}{2} \boldsymbol{\omega}_{n,\varepsilon} \right]$$

Then $\mathrm{div}_x \mathbf{w}_\varepsilon = 0$, $\mathbf{w}_\varepsilon = \mathbf{w}_{n,\varepsilon}$ on $B(\mathbf{h}_{n,\varepsilon}, r_\varepsilon)$ for $n = 1, \dots, N(\varepsilon)$ and

$$\begin{aligned} \|\nabla \mathbf{w}_\varepsilon\|^2_{L^2(\Omega; R^{3\times 3})} &\lesssim \sum_{n=1}^{N(\varepsilon)} r_\varepsilon |\mathbf{h}'_{\varepsilon,n}|^2 + r_\varepsilon^3 |\boldsymbol{\omega}_{n,\varepsilon}|^2 \\ &\lesssim N(\varepsilon) r_\varepsilon \sup_{n=1,\dots,N(\varepsilon)} \left(|\mathbf{h}'_{\varepsilon,n}|^2 + r_\varepsilon^2 |\boldsymbol{\omega}_{n,\varepsilon}|^2 \right). \end{aligned}$$

□

4.1 Flow generated by Λ_ε

In view of (4.3), the vector field Λ_ε generates a flow $\mathbf{X}_\varepsilon$. Specifically, $\mathbf{X}_\varepsilon(t, y)$ is the unique solution of the non–linear ODE system

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{X}_\varepsilon(t, y) = \Lambda_\varepsilon(t, \mathbf{X}_\varepsilon(t, y)),\ \mathbf{X}_\varepsilon(0, y) = y,\ y \in \Omega. \tag{4.4}$$

As the velocities $\mathbf{h}'_{n,\varepsilon}$ are of order $o(\varepsilon^2)$, we have

$$\Lambda_\varepsilon(t, x) = \mathbf{h}'_{n,\varepsilon}(t) \text{ if } x \in B(\mathbf{h}_{n,\varepsilon}(t), r_\varepsilon),\ n = 1, \dots, N(\varepsilon). \tag{4.5}$$

Moreover,

$$\Lambda_\varepsilon(t, x) = 0 \text{ whenever dist} \left[x; \{ \mathbf{h}_{1,\varepsilon}(0), \dots, \mathbf{h}_{N,\varepsilon}(0) \} \right] > \frac{3}{4} \varepsilon. \tag{4.6}$$

4.2 Estimates on the flow mapping

Differentiating equation (4.4) in y, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}[D_y \mathbf{X}_\varepsilon] = \Big[D_x \Lambda_\varepsilon(t, \mathbf{X}_\varepsilon(t,y))\Big] \circ [D_y \mathbf{X}_\varepsilon],\ D_y \mathbf{X}_\varepsilon(0,\cdot) = \mathbb{I}. \tag{4.7}$$

In view of the bound (4.3), we may infer that

$$|D_y \mathbf{X}_\varepsilon(t,y) - \mathbb{I}| \stackrel{<}{\sim} \varepsilon \text{ uniformly for } t \in [0,T],\ y \in \Omega. \tag{4.8}$$

Next, differentiating (4.7), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[D^2_{y,y} \mathbf{X}_\varepsilon] = \Big[D_x \Lambda_\varepsilon(t, \mathbf{X}_\varepsilon(t,y))\Big] \circ [D^2_{y,y} \mathbf{X}_\varepsilon] + \Big[D^2_{x,x} \Lambda_\varepsilon(t, \mathbf{X}_\varepsilon(t,y))\Big] \circ [D_y \mathbf{X}_\varepsilon] \circ [D_y \mathbf{X}_\varepsilon],$$
$$D^2_{y,y} \mathbf{X}_\varepsilon(0,\cdot) = 0. \tag{4.9}$$

Consequently, in accordance with (4.3),

$$\Big| D^2_{y,y} \mathbf{X}_\varepsilon(t,y) \Big| \to 0 \text{ as } \varepsilon \to 0 \text{ uniformly in } t \in [0,T],\ y \in \Omega. \tag{4.10}$$

4.3 Inverse of the flow mapping

Together with the flow mapping $\mathbf{X}_\varepsilon = \mathbf{X}_\varepsilon(t,y)$ determined by (4.4), we introduce its inverse $\mathbf{Y}_\varepsilon(t,x)$ such that,

$$\mathbf{Y}_\varepsilon(t, \mathbf{X}_\varepsilon(t,y)) = y,\ y \in \Omega \quad \text{and} \quad \mathbf{X}_\varepsilon(t, \mathbf{Y}_\varepsilon(t,x)) = x,\ x \in \Omega. \tag{4.11}$$

In particular, differentiating (4.11) yields

$$D_x \mathbf{Y}_\varepsilon(t, \mathbf{X}_\varepsilon(t,y)) \circ D_y \mathbf{X}_\varepsilon(t,y) = \mathbb{I} \quad \text{and} \quad D_y \mathbf{X}_\varepsilon(t, \mathbf{Y}_\varepsilon(t,x)) \circ D_x \mathbf{Y}_\varepsilon(t,x) = \mathbb{I}. \tag{4.12}$$

Writing

$$\begin{aligned} D_x \mathbf{Y}_\varepsilon(t, \mathbf{X}_\varepsilon(t,y)) - \mathbb{I} &= D_x \mathbf{Y}_\varepsilon(t, \mathbf{X}_\varepsilon(t,y)) \circ (\mathbb{I} - D_y \mathbf{X}_\varepsilon(t,y)) \\ &= \Big[D_x \mathbf{Y}_\varepsilon(t, \mathbf{X}_\varepsilon(t,y)) - \mathbb{I}\Big] \circ \Big[\mathbb{I} - D_y \mathbf{X}_\varepsilon(t,y)\Big] + \Big[\mathbb{I} - D_y \mathbf{X}_\varepsilon(t,y)\Big]. \end{aligned}$$

In view of (4.8), we conclude

$$\Big| D_x \mathbf{Y}_\varepsilon(t,x) - \mathbb{I} \Big| \stackrel{<}{\sim} \varepsilon \text{ uniformly for } t \in [0,T],\ x \in \Omega. \tag{4.13}$$

Finally, differentiating once more (4.12) we obtain, by virtue of (4.10),

$$\Big| D^2_{x,x} \mathbf{Y}_\varepsilon(t,x) \Big| \to 0 \text{ as } \varepsilon \to 0 \text{ uniformly in } t \in [0,T],\ x \in \Omega. \tag{4.14}$$

4.4 Estimates of the time derivative

Finally, we estimate the time derivative of the flow mapping. To this end, differentiate equation (4.4) to obtain

$$\Big[\partial_t D_y \mathbf{X}_\varepsilon\Big](t,y) = D_x\Lambda_\varepsilon(t,\mathbf{X}_\varepsilon(t,y)) \circ D_y\mathbf{X}_\varepsilon(t,y) \stackrel{<}{\sim} \varepsilon \text{ uniformly for } t\in[0,T],\ y\in\Omega, \tag{4.15}$$

where we have used (4.3), (4.8).

Furthermore, differentiating (4.12) in time and using (4.15), (4.13) and (4.8) we conclude

$$\Big|\partial_t D_x \mathbf{Y}_\varepsilon(t,x)\Big| \stackrel{<}{\sim} \varepsilon \text{ uniformly for } t\in[0,T],\ x\in\Omega. \tag{4.16}$$

4.5 Change of variables

We consider again the change of variables

$$(t,x)\mapsto(t,y),\ y=\mathbf{Y}_\varepsilon(t,x),$$

where $\mathbf{Y}_\varepsilon$ is the inverse of the flow mas $\mathbf{X}_\varepsilon$ defined via (4.4). As $\mathrm{div}_x\Lambda_\varepsilon = 0$, the Jacobian of the transformation is equal to 1. Note that the transformation maps the time dependent domains $\Omega_{\varepsilon,t}$ on $\Omega_{\varepsilon,0}$, whereas the moving rigid particles $B(\mathbf{h}_{n,\varepsilon}(t), r_\varepsilon)$ are mapped onto static ones $B(\mathbf{h}_{n,\varepsilon}(0), r_\varepsilon)$.

Any function $f = f(t,x)$ is transformed to $\widetilde{f}(t,y)$,

$$\widetilde{f}(t,\mathbf{Y}_\varepsilon(t,x)) = f(t,x) \text{ or } \widetilde{f}(t,y) = f(t,\mathbf{X}_\varepsilon(t,y)) \tag{4.17}$$

$$\int_\Omega \widetilde{f}(\tau,y)\ \mathrm{d}y = \int_\Omega f(\tau,x)\ \ \mathrm{d}x \text{ for a.a. } \tau\in(0,T). \tag{4.18}$$

This type of change of variables "freezing" the moving objects is frequently used in the literature devoted to the fluid–structure interaction problems, see e.g. [3], Cumsille and Takahashi [7], Glass et al. [25], Roy and Takahashi [44] among others.

5 Navier–Stokes system in the new variables

Our next goal is to rewrite the Navier–Stokes system in the (t,y) variables introduced in Section 4.5.

5.1 Transformed velocity field

Revisiting the weak formulation of the problem (2.11)–(2.13) we introduce the transformed velocity

$$\widetilde{\mathbf{u}}_\varepsilon(t,\mathbf{Y}_\varepsilon(t,x)) = \mathbf{u}_\varepsilon(t,x).$$

Accordingly,

$$\nabla_y \widetilde{\mathbf{u}}_\varepsilon \Big(t, \mathbf{Y}_\varepsilon(t,x)\Big)[D_x \mathbf{Y}_\varepsilon](t,x) = \nabla_x \mathbf{u}_\varepsilon(t,x).$$

As the original velocity coincides with the rigid velocities on the particles (cf. (2.11)), we get

$$\widetilde{\mathbf{u}}_\varepsilon(t,y) = \mathbf{w}_{n,\varepsilon}(t, \mathbf{X}_\varepsilon(t,y)) \text{ for } y \in B(\mathbf{h}_{n,\varepsilon}(0), r_\varepsilon),\ n = 1,\dots,N(\varepsilon). \tag{5.1}$$

5.2 Weak formulation in the new variables

Applying the change of variables to a test function $\boldsymbol{\varphi}$ mentioned in (2.13), we get

$$\widetilde{\boldsymbol{\varphi}}(t, \mathbf{Y}_\varepsilon(t,x)) = \boldsymbol{\varphi}(t,x).$$

Consequently,

$$\nabla_y \widetilde{\boldsymbol{\varphi}} \Big(t, \mathbf{Y}_\varepsilon(t,x)\Big)[D_x \mathbf{Y}_\varepsilon](t,x) = \nabla_x \boldsymbol{\varphi}(t,x),$$

and

$$\partial_t \widetilde{\boldsymbol{\varphi}} \Big(t, \mathbf{Y}_\varepsilon(t,x)\Big) + [\nabla_y \widetilde{\boldsymbol{\varphi}}](t, \mathbf{Y}_\varepsilon(t,x))[\partial_t \mathbf{Y}_\varepsilon](t,x) = \partial_t \boldsymbol{\varphi}(t,x).$$

The transformed test functions are smooth in the spatial variable x and continuously differentiable in time. Moreover, in accordance with (2.14),

$$\widetilde{\boldsymbol{\varphi}} \in C^1_c([0,T) \times \overline{\Omega_{\varepsilon,0}}). \tag{5.2}$$

Rewriting the weak formulation (2.13) in the new variables (t,y) we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \Big[\widetilde{\mathbf{u}}_\varepsilon(t,y) \cdot \partial_t \widetilde{\boldsymbol{\varphi}}(t,y) + \widetilde{\mathbf{u}}_\varepsilon(t,y) \cdot \Big(\nabla_y \widetilde{\boldsymbol{\varphi}}(t,y)[\partial_t \mathbf{Y}_\varepsilon](t, \mathbf{X}_\varepsilon(t,y))\Big)\Big] \mathrm{d}y\, \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega [\widetilde{\mathbf{u}}_\varepsilon(t,y) \otimes \widetilde{\mathbf{u}}_\varepsilon(t,y)] : \Big[\nabla_y \widetilde{\boldsymbol{\varphi}}(t,y)[D_x \mathbf{Y}_\varepsilon](t, \mathbf{X}_\varepsilon(t,y))\Big] \mathrm{d}y\, \mathrm{d}t \\
&\quad = \int_0^T \int_\Omega \Big[\nabla_y \widetilde{\mathbf{u}}_\varepsilon(t,y)[D_x \mathbf{Y}_\varepsilon](t, \mathbf{X}_\varepsilon(t,y))\Big] : \Big[\nabla_y \widetilde{\boldsymbol{\varphi}}(t,y)[D_x \mathbf{Y}_\varepsilon](t, \mathbf{X}_\varepsilon(t,y))\Big] \mathrm{d}y\, \mathrm{d}t
\end{aligned} \tag{5.3}$$

provided $\boldsymbol{\varphi}(0,\cdot) = \widetilde{\boldsymbol{\varphi}}(0,\cdot) = 0$.

The apparent drawback of (5.3) is that neither $\widetilde{\mathbf{u}}_\varepsilon$ nor $\widetilde{\boldsymbol{\varphi}}$ are solenoidal. To overcome this problem, we introduce a new velocity field

$$\mathbf{v}_\varepsilon(t,y) = D_x \mathbf{Y}_\varepsilon(t, \mathbf{X}_\varepsilon(t,y)) \widetilde{\mathbf{u}}_\varepsilon(t,y). \tag{5.4}$$

Similarly, we consider a new class of test functions,

$$\boldsymbol{\psi}(t,y) = D_x \mathbf{Y}_\varepsilon(t, \mathbf{X}_\varepsilon(t,y)) \widetilde{\boldsymbol{\varphi}}(t,y). \tag{5.5}$$

It is easy to check that (see, for instance [45, Lemma 4.4])

$$\mathrm{div}_y \mathbf{v}_\varepsilon = 0, \ \mathrm{div}_y \boldsymbol{\psi} = 0. \tag{5.6}$$

Moreover, in accordance with (5.1),

$$\mathbf{v}_\varepsilon|_{\partial\Omega_{\varepsilon,0}} = \mathbf{w}_\varepsilon, \tag{5.7}$$

where $\mathbf{w}_\varepsilon$ is the extension of the rigid velocities constructed in Lemma 4.1. Finally, by virtue of (5.2),

$$\boldsymbol{\psi} \in C_c^1([0,T) \times \overline{\Omega_{\varepsilon,0}}). \tag{5.8}$$

The ultimate goal of this part is rewriting the weak formulation (5.3) in terms of the new variables $\mathbf{v}_\varepsilon$ and $\boldsymbol{\psi}$. This is a bit tedious but rather straightforward process, where we systematically use the identity

$$D_y \mathbf{X}_\varepsilon(t,y) \circ D_x \mathbf{Y}_\varepsilon(t, \mathbf{X}_\varepsilon(t,y)) = \mathbb{I}. \tag{5.9}$$

Moreover, for any vector $\mathbf{w}$ and any matrix M, we denote by $[D^2_{yy}\mathbf{X}_\varepsilon]\mathbf{w}$ the matrix with entries

$$([D^2_{yy}\mathbf{X}_\varepsilon]\mathbf{w})_{i,j} = \sum_{n=1}^{3} \partial_{y_j}\partial_{y_n}\mathbf{X}_{\varepsilon,i}\mathbf{w}_n. \tag{5.10}$$

and by $M \odot [D^2_{yy}\mathbf{X}_\varepsilon]$ the vector with entries

$$(M \odot [D^2_{yy}\mathbf{X}_\varepsilon])_i = \sum_{l,k=1}^{3} M_{k,l}\partial_{y_i}\partial_{y_l}\mathbf{X}_{\varepsilon,k}. \tag{5.11}$$

Now we want to write down several steps in order to calculate several terms of the weak formulation (5.3) in terms of the new variables $\mathbf{v}_\varepsilon$ and $\boldsymbol{\psi}$.

Step 1

The integral on the right–hand side of (5.3) reads

$$\begin{aligned}
&\int_0^T \int_\Omega \Big[\nabla_y \widetilde{\mathbf{u}}_\varepsilon(t,y)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big] : \Big[\nabla_y \widetilde{\boldsymbol{\varphi}}(t,y)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big] \mathrm{d}y\,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega \Big([D_y\mathbf{X}_\varepsilon](t,y)\nabla_y \mathbf{v}_\varepsilon(t,y)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y)) + ([D^2_{y,y}\mathbf{X}_\varepsilon](t,y)\mathbf{v}_\varepsilon(t,y))[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big) \\
&\qquad : \Big([D_y\mathbf{X}_\varepsilon](t,y)\nabla_y \boldsymbol{\psi}(t,y)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y)) + ([D^2_{y,y}\mathbf{X}_\varepsilon](t,y)\boldsymbol{\psi}(t,y))[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big) \ \mathrm{d}y\,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega \nabla_y \mathbf{v}_\varepsilon(t,y) : \nabla_y \boldsymbol{\psi}(t,y) \ \mathrm{d}y\,\mathrm{d}t \\
&\qquad - \int_0^T \int_\Omega \nabla_y \mathbf{v}_\varepsilon(t,y) : \Big(\nabla_y \boldsymbol{\psi}(t,y) - [D_y\mathbf{X}_\varepsilon](t,y)\nabla_y \boldsymbol{\psi}(t,y)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big) \ \mathrm{d}y\,\mathrm{d}t
\end{aligned}$$

$$+\int_0^T\int_\Omega \nabla_y \mathbf{v}_\varepsilon(t,y) : \Big(([D^2_{y,y}\mathbf{X}_\varepsilon](t,y)\boldsymbol{\psi}(t,y)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big)\, \mathrm{d}y\,\mathrm{d}t$$
$$-\int_0^T\int_\Omega \Big(\nabla_y \mathbf{v}_\varepsilon(t,y) - [D_y\mathbf{X}_\varepsilon](t,y)\nabla_y\mathbf{v}_\varepsilon(t,y)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big)$$
$$: \Big([D_y\mathbf{X}_\varepsilon](t,y)\nabla_y\boldsymbol{\psi}[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y)) + ([D^2_{y,y}\mathbf{X}_\varepsilon](t,y)\boldsymbol{\psi}(t,x))[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big)\, \mathrm{d}y\,\mathrm{d}t$$
$$+\int_0^T\int_\Omega \Big([D^2_{y,y}\mathbf{X}_\varepsilon](t,y)\mathbf{v}_\varepsilon(t,y)\Big)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))$$
$$: \Big([D_y\mathbf{X}_\varepsilon](t,y)\nabla_y\boldsymbol{\psi}[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y)) + ([D^2_{y,y}\mathbf{X}_\varepsilon](t,y)\boldsymbol{\psi}(t,y))[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big)\, \mathrm{d}y\,\mathrm{d}t. \tag{5.12}$$

Step 2

Similarly,

$$\int_0^T\int_\Omega [\widetilde{\mathbf{u}}_\varepsilon \otimes \widetilde{\mathbf{u}}_\varepsilon](t,y) : \Big[\nabla_y\widetilde{\boldsymbol{\varphi}}(t,y)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big]\mathrm{d}y\,\mathrm{d}t$$
$$= \int_0^T\int_\Omega [\widetilde{\mathbf{u}}_\varepsilon \otimes \widetilde{\mathbf{u}}_\varepsilon](t,y) : \nabla_y\boldsymbol{\psi}(t,y)\mathrm{d}y\,\mathrm{d}t$$
$$- \int_0^T\int_\Omega [\widetilde{\mathbf{u}}_\varepsilon \otimes \widetilde{\mathbf{u}}_\varepsilon](t,y) : \Big(\nabla_y\boldsymbol{\psi}(t,y) - [D_y\mathbf{X}_\varepsilon](t,y)\nabla_y\boldsymbol{\psi}(t,y)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big)\mathrm{d}y\,\mathrm{d}t$$
$$+ \int_0^T\int_\Omega [\widetilde{\mathbf{u}}_\varepsilon \otimes \widetilde{\mathbf{u}}_\varepsilon](t,y) : \Big(([D^2_{y,y}\mathbf{X}_\varepsilon](t,y)\boldsymbol{\psi}(t,y))[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big)\mathrm{d}y\,\mathrm{d}t. \tag{5.13}$$

Step 3

Finally, the remaining integral on the left-hand side of (5.3) can be handled as follows:

$$\int_0^T\int_\Omega \Big[\widetilde{\mathbf{u}}_\varepsilon(t,y)\cdot\partial_t\widetilde{\boldsymbol{\varphi}}(t,y) + \widetilde{\mathbf{u}}_\varepsilon(t,y)\cdot\Big(\nabla_y\widetilde{\boldsymbol{\varphi}}(t,y)[\partial_t\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big)\Big]\mathrm{d}y\,\mathrm{d}t$$
$$= \int_0^T\int_\Omega \Big[[D_y\mathbf{X}_\varepsilon](t,y)\cdot\Big([D_y\mathbf{X}_\varepsilon](t,y)\mathbf{v}_\varepsilon(t,y)\Big)\Big]\cdot\partial_t\boldsymbol{\psi}(t,y)\mathrm{d}y\,\mathrm{d}t$$
$$+ \int_0^T\int_\Omega \widetilde{\mathbf{u}}_\varepsilon(t,y)\cdot(\partial_t[D_y\mathbf{X}_\varepsilon](t,y)\boldsymbol{\psi}(t,y))\mathrm{d}y\,\mathrm{d}t$$
$$+ \int_0^T\int_\Omega \widetilde{\mathbf{u}}_\varepsilon(t,y)\cdot\nabla_y\Big[D_y\mathbf{X}_\varepsilon(t,y)\boldsymbol{\psi}(t,y)\Big][\partial_t\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\mathrm{d}y\,\mathrm{d}t. \tag{5.14}$$

Summing up the previous identities, we obtain the weak formulation in the new coordinates:

$$\int_0^T\int_\Omega \Big[[D_y\mathbf{X}_\varepsilon](t,y)\cdot\Big([D_y\mathbf{X}_\varepsilon](t,y)\mathbf{v}_\varepsilon(t,y)\Big)\Big]\cdot\partial_t\boldsymbol{\psi}(t,y)\mathrm{d}y\,\mathrm{d}t$$

$$+\int_0^T\int_\Omega [\widetilde{\mathbf{u}}_\varepsilon(t,y)\otimes\widetilde{\mathbf{u}}_\varepsilon(t,y)]:\nabla_y\boldsymbol{\psi}(t,y)\mathrm{d}y\,\mathrm{d}t=\int_0^T\int_\Omega \nabla_x\mathbf{v}_\varepsilon(t,y):\nabla_x\boldsymbol{\psi}(t,y)\ \mathrm{d}y\,\mathrm{d}t+\mathcal{R}_\varepsilon \tag{5.15}$$

for any $\boldsymbol{\psi}\in C^1_c((0,T)\times\Omega_{\varepsilon,0})$, $\mathrm{div}\boldsymbol{\psi}=0$. The remainder term $\mathcal{R}_\varepsilon$ reads

$$\begin{aligned}
\mathcal{R}_\varepsilon=&-\int_0^T\int_\Omega \widetilde{\mathbf{u}}_\varepsilon(t,y)\cdot(\partial_t[D_y\mathbf{X}_\varepsilon](t,y)\boldsymbol{\psi}(t,y))\mathrm{d}y\,\mathrm{d}t\\
&-\int_0^T\int_\Omega \widetilde{\mathbf{u}}_\varepsilon(t,y)\cdot\nabla_y\Big[D_y\mathbf{X}_\varepsilon(t,y)\boldsymbol{\psi}(t,y)\Big][\partial_t\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\mathrm{d}y\,\mathrm{d}t\\
&+\int_0^T\int_\Omega[\widetilde{\mathbf{u}}_\varepsilon\otimes\widetilde{\mathbf{u}}_\varepsilon](t,y):\Big(\nabla_y\boldsymbol{\psi}(t,y)-[D_y\mathbf{X}_\varepsilon](t,y)\nabla_y\boldsymbol{\psi}(t,y)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big)\mathrm{d}y\,\mathrm{d}t\\
&-\int_0^T\int_\Omega[\widetilde{\mathbf{u}}_\varepsilon\otimes\widetilde{\mathbf{u}}_\varepsilon](t,y):\Big(([D^2_{y,y}\mathbf{X}_\varepsilon](t,y)\boldsymbol{\psi}(t,y))[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big)\mathrm{d}y\,\mathrm{d}t\\
&-\int_0^T\int_\Omega\nabla_y\mathbf{v}_\varepsilon(t,y):\Big(\nabla_y\boldsymbol{\psi}(t,y)-[D_y\mathbf{X}_\varepsilon](t,y)\nabla_y\boldsymbol{\psi}(t,y)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big)\ \mathrm{d}y\,\mathrm{d}t\\
&+\int_0^T\int_\Omega\nabla_y\mathbf{v}_\varepsilon(t,y):\Big(([D^2_{y,y}\mathbf{X}_\varepsilon](t,y)\boldsymbol{\psi}(t,y)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big)\ \mathrm{d}y\,\mathrm{d}t\\
&-\int_0^T\int_\Omega\Big(\nabla_y\mathbf{v}_\varepsilon(t,y)-[D_y\mathbf{X}_\varepsilon](t,y)\nabla_y\mathbf{v}_\varepsilon(t,y)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big)\\
&\quad:\Big([D_y\mathbf{X}_\varepsilon](t,y)\nabla_y\boldsymbol{\psi}(t,y)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))+([D^2_{y,y}\mathbf{X}_\varepsilon](t,y)\boldsymbol{\psi}(t,y))[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big)\ \mathrm{d}y\,\mathrm{d}t\\
&+\int_0^T\int_\Omega\Big([D^2_{y,y}\mathbf{X}_\varepsilon](t,y)\mathbf{v}_\varepsilon(t,y)\Big)[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\\
&\quad:\Big([D_y\mathbf{X}_\varepsilon](t,y)\nabla_y\boldsymbol{\psi}[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))+([D^2_{y,y}\mathbf{X}_\varepsilon](t,y)\boldsymbol{\psi}(t,y))[D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y))\Big)\ \mathrm{d}y\,\mathrm{d}t.
\end{aligned}\tag{5.16}$$

6 Asymptotic limit

We complete the proof of Theorem 2.3 by performing the limit $\varepsilon\to 0$ in the transformed system (5.16). As the position of the particles are fixed, we shall write

$$\Omega_\varepsilon \text{ instead of } \Omega_{\varepsilon,0},\ \text{ and } \mathbf{h}_{n,\varepsilon} \text{ instead of } \mathbf{h}_{n,\varepsilon}(0).$$

6.1 Remainder term

Our first goal is to show that the remainder term $\mathcal{R}_\varepsilon$ tends to zero. To this end, first observe that both $\widetilde{\mathbf{u}}_\varepsilon$, $\mathbf{v}_\varepsilon$ inherit the uniform bounds from hypothesis 2.17, specifically, uniformly for $\varepsilon\to 0$,

$$\|\widetilde{\mathbf{u}}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega;R^3))}+\|\widetilde{\mathbf{u}}_\varepsilon\|_{L^2(0,T;W^{1,2}_0(\Omega;R^3))}\lesssim 1,$$

$$\|\mathbf{v}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega;R^3))} + \|\mathbf{v}_\varepsilon\|_{L^2(0,T;W_0^{1,2}(\Omega;R^3))} \lesssim 1. \tag{6.1}$$

Combining the uniform bounds established on the flow mapping established in Sections 4.2–4.4, notably (4.8), (4.10), (4.15) for $\mathbf{X}_\varepsilon$, (4.13), (4.14), (4.16) for $\mathbf{Y}_\varepsilon$, with (6.1) and the notation (5.10), (5.11), we easily deduce:

$$\begin{aligned}
\widetilde{\mathbf{u}}_\varepsilon \cdot \partial_t[D_y\mathbf{X}_\varepsilon] &\to 0 \text{ in } L^2(0,T;L^6(\Omega;R^3)),\\
\Big[\partial_t\mathbf{Y}_\varepsilon \otimes \widetilde{\mathbf{u}}_\varepsilon\Big][D_y\mathbf{X}_\varepsilon] &\to 0 \text{ in } L^2(0,T;L^6(\Omega;R^3)),\\
\widetilde{\mathbf{u}}_\varepsilon \cdot [D^2_{y,y}\mathbf{X}_\varepsilon][\partial_t\mathbf{Y}_\varepsilon] &\to 0 \text{ in } L^2(0,T;L^6(\Omega;R^3)),\\
\Big[\mathbb{I} - [D_y\mathbf{X}_\varepsilon]\Big] \cdot [\widetilde{\mathbf{u}}_\varepsilon \otimes \widetilde{\mathbf{u}}_\varepsilon] &\to 0 \text{ in } L^\infty(0,T;L^1(\Omega;R^3)) \cap L^1(0,T;L^3(\Omega;R^3))\\
\Big[\mathbb{I} - [D_y\mathbf{X}_\varepsilon]\Big] \cdot [\widetilde{\mathbf{u}}_\varepsilon \otimes \widetilde{\mathbf{u}}_\varepsilon]\Big[\mathbb{I} - [D_x^t\mathbf{Y}_\varepsilon]\Big] &\to 0 \text{ in } L^\infty(0,T;L^1(\Omega;R^3)) \cap L^1(0,T;L^3(\Omega;R^3))\\
\Big[[\widetilde{\mathbf{u}}_\varepsilon \otimes \widetilde{\mathbf{u}}_\varepsilon][D_x^t\mathbf{Y}_\varepsilon]\Big] \odot [D^2_{y,y}\mathbf{X}_\varepsilon] &\to 0 \text{ in } L^\infty(0,T;L^1(\Omega;R^3)) \cap L^1(0,T;L^2(\Omega;R^3))\\
\Big[\mathbb{I} - [D_y\mathbf{X}_\varepsilon]\Big] \cdot [\nabla_y\mathbf{v}_\varepsilon] &\to 0 \text{ in } L^2(0,T;L^2(\Omega;R^3))\\
\Big[\mathbb{I} - [D_y\mathbf{X}_\varepsilon]\Big] \cdot [\nabla_y\mathbf{v}_\varepsilon]\Big[\mathbb{I} - [D_x^t\mathbf{Y}_\varepsilon]\Big] &\to 0 \text{ in } L^2(0,T;L^2(\Omega;R^3))\\
\Big[[\nabla_y\mathbf{v}_\varepsilon][D_x^t\mathbf{Y}_\varepsilon]\Big] \odot [D^2_{y,y}\mathbf{X}_\varepsilon] &\to 0 \text{ in } L^2(0,T;L^2(\Omega;R^3))\\
\nabla_y\mathbf{v}_\varepsilon - [D_y\mathbf{X}_\varepsilon]\nabla_y\mathbf{v}_\varepsilon[D_x\mathbf{Y}_\varepsilon] &\to 0 \text{ in } L^2(0,T;L^2(\Omega;R^3))\\
\Big([D^2_{y,y}\mathbf{X}_\varepsilon]\mathbf{v}_\varepsilon\Big)[D_x\mathbf{Y}_\varepsilon] &\to 0 \text{ in } L^\infty(0,T;L^2(\Omega;R^3)) \cap L^2(0,T;L^6(\Omega;R^3))
\end{aligned} \tag{6.2}$$

as $\varepsilon \to 0$. Here, we have used the Sobolev embedding $W^{1,2} \hookrightarrow L^6$.

6.2 Strong convergence of the velocity field

Using the uniform bounds (2.17), (6.1) to obtain, up to a suitable subsequence,

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly-* in } L^\infty(0,T;L^2(\Omega;R^3)), \text{ and weakly in } L^2(0,T;W_0^{1,2}(\Omega;R^3)), \tag{6.3}$$

$$\widetilde{\mathbf{u}}_\varepsilon \to \widetilde{\mathbf{u}} \text{ weakly-* in } L^\infty(0,T;L^2(\Omega;R^3)), \text{ and weakly in } L^2(0,T;W_0^{1,2}(\Omega;R^3)), \tag{6.4}$$

$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ weakly-* in } L^\infty(0,T;L^2(\Omega;R^3)), \text{ and weakly in } L^2(0,T;W_0^{1,2}(\Omega;R^3)). \tag{6.5}$$

Moreover, it is easy to see that the three limits coincide, $\mathbf{u} = \widetilde{\mathbf{u}} = \mathbf{v}$. Indeed, in view of (5.4),

$$\mathbf{v}_\varepsilon(t,y) = \widetilde{\mathbf{u}}_\varepsilon(t,y) + \Big([D_x\mathbf{Y}_\varepsilon](t,\mathbf{X}_\varepsilon(t,y)) - \mathbb{I}\Big)\widetilde{\mathbf{u}}_\varepsilon(t,y)\cdot;$$

whence, by virtue of (4.13), $\tilde{\mathbf{u}} = \mathbf{v}$. To show $\mathbf{u} = \tilde{\mathbf{u}}$ first observe

$$\int_0^T \int_\Omega \widetilde{\mathbf{u}}_\varepsilon(t,y) \cdot \widetilde{\boldsymbol{\varphi}}(t,y) \ \mathrm{d}y\,\mathrm{d}t = \int_0^T \int_\Omega \mathbf{u}_\varepsilon(t,x) \cdot \boldsymbol{\varphi}(t,x) \ \mathrm{d}x\,\mathrm{d}t,$$

where

$$\widetilde{\boldsymbol{\varphi}}(t,y) = \boldsymbol{\varphi}(t, \mathbf{X}_\varepsilon(t,y)).$$

Consequently, fixing the test function $\boldsymbol{\varphi}$, we get

$$\begin{aligned}
\int_0^T \int_\Omega \widetilde{\mathbf{u}}(t,y) \cdot \boldsymbol{\varphi}(t,y) \ \mathrm{d}y \, \mathrm{d}t &= \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \widetilde{\mathbf{u}}_\varepsilon(t,y) \cdot \boldsymbol{\varphi}(t,y) \ \mathrm{d}y \, \mathrm{d}t \\
&= \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \widetilde{\mathbf{u}}_\varepsilon(t,y) \cdot \Big(\boldsymbol{\varphi}(t,y) - \boldsymbol{\varphi}(t, \mathbf{X}_\varepsilon(t,y)) \Big) \mathrm{d}y \, \mathrm{d}t + \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \mathbf{u}_\varepsilon(t,x) \cdot \boldsymbol{\varphi}(t,x) \ \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_\Omega \mathbf{u}(t,x) \cdot \boldsymbol{\varphi}(t,x) \ \mathrm{d}x \, \mathrm{d}t,
\end{aligned}$$

where we have used bounded of $\widetilde{\mathbf{u}}_\varepsilon$ in $L^\infty(0,T; L^2(\Omega; R^3))$, together with

$$\sup_{t \in [0,T], y \in \Omega} \Big| \boldsymbol{\varphi}(t,y) - \boldsymbol{\varphi}(t, \mathbf{X}_\varepsilon(t,y)) \Big| \to 0 \text{ as } \varepsilon \to 0$$

for any $\boldsymbol{\varphi} \in C^1_c((0,T) \times \Omega; R^3)$. As $\boldsymbol{\varphi}$ was arbitrary, we conclude $\tilde{\mathbf{u}} = \mathbf{u}$.

Our next goal is to establish the strong convergence, specifically,

$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ in } L^2((0,T) \times \Omega; R^3). \tag{6.6}$$

Repeating the above arguments, we can show easily that (6.6) yields the same for the original velocity,

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2((0,T) \times \Omega; R^3). \tag{6.7}$$

To this end, we need a *restriction* operator R_ε enjoying the following properties:

$$\begin{aligned}
R_\varepsilon : W^{1,2}_0(\Omega; R^3) &\to W^{1,2}_0(\Omega_\varepsilon; R^3), \\
\| R_\varepsilon[\boldsymbol{\varphi}] \|_{W^{1,2}(\Omega)} &\lesssim \| \boldsymbol{\varphi} \|_{W^{1,2}(\Omega; R^3)}, \\
\mathrm{div}_y \boldsymbol{\varphi} = 0 \ &\Rightarrow \ \mathrm{div}_y R_\varepsilon[\boldsymbol{\varphi}] = 0, \\
R_\varepsilon[\boldsymbol{\varphi}] \to \boldsymbol{\varphi} \text{ in } L^2&(\Omega; R^3) \text{ as } \varepsilon \to 0 \text{ for any } \boldsymbol{\varphi} \in W^{1,2}_0(\Omega).
\end{aligned} \tag{6.8}$$

There are several constructions of a restriction operator available, see e.g. Allaire [1], Diening et. al [9], Tartar [46]. Note that these constructions of the restriction operator adapt to the non–periodic distribution of obstacles in a straightforward manner.

Now, consider a function

$$\boldsymbol{\psi} = R_\varepsilon[\phi]\chi, \ \chi = \chi(t), \chi \in C^\infty_c(0,T), \ \phi \in C^\infty_c(\Omega; R^3), \ \mathrm{div}_y \phi = 0.$$

By virtue of the properties of the restriction operator, the function $\boldsymbol{\psi}$ is an eligible test function in the weak formulation (5.15). In view of the uniform bounds (6.1), (6.2), we deduce that the function

$$t \mapsto \int_\Omega \Big[[D_y \mathbf{X}_\varepsilon](t,y) \cdot \Big([D_y \mathbf{X}_\varepsilon](t,y) \mathbf{v}_\varepsilon(t,y) \Big) \Big] \cdot R_\varepsilon[\phi](t,y) \chi(t) \mathrm{d}y, \ t \in [0,T]$$

has a bounded time derivative in $L^q(0,T)$ for some $q>1$ uniformly for $\varepsilon \to 0$.

Next,

$$t \mapsto \int_\Omega \Big(\Big[[D_y \mathbf{X}_\varepsilon] \cdot \Big([D_y \mathbf{X}_\varepsilon] \mathbf{v}_\varepsilon \Big) \Big] \cdot R_\varepsilon[\phi] \chi \Big)(t,y) \mathrm{d}y = \int_\Omega \Big(\Big[[D_y \mathbf{X}_\varepsilon] \cdot \Big([D_y \mathbf{X}_\varepsilon] \mathbf{v}_\varepsilon \Big) \Big] \cdot \phi \chi \Big)(t,y) \mathrm{d}y$$
$$+ \int_\Omega \Big(\Big[[D_y \mathbf{X}_\varepsilon] \cdot \Big([D_y \mathbf{X}_\varepsilon] \mathbf{v}_\varepsilon \Big) \Big] \cdot (R_\varepsilon[\phi] - \phi) \chi \Big)(t,y) \mathrm{d}y,$$

where, by virtue of (6.8) and the bounds (6.1),

$$\int_\Omega \Big(\Big[[D_y \mathbf{X}_\varepsilon] \cdot \Big([D_y \mathbf{X}_\varepsilon] \mathbf{v}_\varepsilon \Big) \Big] \cdot (R_\varepsilon[\phi] - \phi) \chi \Big)(t,y) \mathrm{d}y \to 0 \text{ in } L^\infty(0,T).$$

Consequently, using Arzelá–Ascoli theorem and the bounds on $D_y \mathbf{X}_\varepsilon$ established in (4.8), we obtain

$$\left[t \mapsto \int_\Omega \Big(\Big[[D_y \mathbf{X}_\varepsilon] \cdot \Big([D_y \mathbf{X}_\varepsilon] \mathbf{v}_\varepsilon \Big) \Big] \cdot \phi \chi \Big)(t,y) \mathrm{d}y \right] \to \left[t \mapsto \int_\Omega [\mathbf{v} \cdot \phi \chi](t,y) \mathrm{d}y \right] \text{ in } L^\infty[0,T]$$

for any $\phi \in C_c^\infty(\Omega; R^3)$, $\mathrm{div}\phi = 0$ and $\chi \in C_c^\infty(0,T)$. Finally, using again (4.8), we may infer that

$$\left[t \mapsto \int_\Omega [\mathbf{v}_\varepsilon \cdot \phi \chi](t,y) \mathrm{d}y \right] \to \left[t \mapsto \int_\Omega [\mathbf{v} \cdot \phi \chi](t,y) \mathrm{d}y \right] \text{ in } L^\infty[0,T] \tag{6.9}$$

for any $\phi \in C_c^\infty(\Omega)$, $\mathrm{div}\phi = 0$. Consequently for any $q \in [1,\infty)$,

$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ in } L^q(0,T; L^2_{\text{weak}}(\Omega; R^3)) \;\Rightarrow\; \mathbf{v}_\varepsilon \to \mathbf{v} \text{ in } L^2(0,T; W^{-1,2}(\Omega; R^3)).$$

(the last implication is a straightforward modification of Lemma 6.4 in [42]). In the above expression $\mathbf{v}_\varepsilon \to \mathbf{v}$ in $L^q(0,T; L^2_{\text{weak}}(\Omega; R^3))$ if for any $\psi \in L^2(\Omega)$

$$\left[t \mapsto \int_\Omega [\mathbf{v}_\varepsilon \cdot \psi](t,y) \mathrm{d}y \right] \to \left[t \mapsto \int_\Omega [\mathbf{v} \cdot \psi](t,y) \mathrm{d}y \right] \text{ in } L^q[0,T].$$

The convergence of $\mathbf{v}_\varepsilon \to \mathbf{v}$ in $L^2(0,T; W^{-1,2}(\Omega; R^3))$ together with (6.5), yields

$$\int_0^T \int_\Omega |\mathbf{v}_\varepsilon|^2 \; \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega |\mathbf{v}|^2 \; \mathrm{d}x \, \mathrm{d}t$$

Thus we have proved (6.6).

Finally, observe that the same argument yields also the convergence of the initial values

$$\mathbf{u}_{0,\varepsilon} = \mathbf{v}_\varepsilon(0,\cdot) \to \mathbf{u}_0 \text{ weakly in } L^2(\Omega; R^3).$$

6.3 Time regularization and homogenization

Following the idea of [5] we regularize (5.15) by means of a convolution with a time dependent families of regularizing kernels $\{\theta_\delta\}_{\delta>0}$. This means we consider the functions

$$\theta_\delta(\tau - t)\phi(x),\ \phi \in W_0^{1,2}(\Omega_\varepsilon; R^3),\ \mathrm{div}_y\phi = 0,\ \tau \in (\delta, T-\delta),$$

as test functions in (5.15). We denote

$$[f]_\delta(\tau, y) = \theta_\delta * f(\tau, y) = \int_0^T \theta_\delta(\tau - t) f(t, y)\ \mathrm{d}t.$$

The weak formulation (5.15) gives rise to a family of *stationary* Stokes problems for any fixed δ and $\tau \in (0, T)$:

$$\int_\Omega \nabla_x [\mathbf{v}_\varepsilon]_\delta(\tau, \cdot) : \nabla_x \phi\ \mathrm{d}y = -\int_\Omega \partial_t \Big[[D_y \mathbf{X}_\varepsilon] \cdot \Big([D_y \mathbf{X}_\varepsilon] \mathbf{v}_\varepsilon \Big) \Big]_\delta (\tau, \cdot) \cdot \boldsymbol{\varphi} \mathrm{d}y$$
$$+ \int_\Omega [\widetilde{\mathbf{u}}_\varepsilon \otimes \widetilde{\mathbf{u}}_\varepsilon]_\delta : \nabla_y \phi\ \mathrm{d}y + \langle \mathbf{g}_{\varepsilon,\delta}(\tau, \cdot); \phi \rangle \tag{6.10}$$

for any

$$\phi \in W_0^{1,2}(\Omega_\varepsilon; R^3),\ \mathrm{div}_y \phi = 0.$$

Moreover, in view of the bounds established in (6.2),

$$\mathbf{g}_{\varepsilon,\delta}(\tau, \cdot) \to 0 \text{ in } W^{-1,2}(\Omega; R^3) \text{ as } \varepsilon \to 0 \text{ for any fixed } \delta > 0, \tau \in (0, T).$$

Seeing that the regularized velocity $[\mathbf{v}_\varepsilon]_\delta$ still satisfies the boundary conditions (5.7) (with regularized rigid boundary velocities), we may apply Proposition 3.2 for any fixed $\tau > 0$. Keeping $\delta > 0$ and $\tau \in (0, T)$ fixed, we let $\varepsilon \to 0$ in (6.10) obtaining

$$\int_\Omega \nabla_x [\mathbf{v}]_\delta(\tau, \cdot) : \nabla_x \phi\ \mathrm{d}y = -\int_\Omega \partial_t [\mathbf{v}]_\delta(\tau, \cdot) \cdot \phi\ \mathrm{d}y$$
$$+ \int_\Omega [\mathbf{v} \otimes \mathbf{v}]_\delta(\tau, \cdot) : \nabla_y \phi\ \mathrm{d}y + \int_\Omega 6\pi \mathfrak{R} [\mathbf{v}]_\delta(\tau, \cdot) \cdot \phi\ \mathrm{d}y\, \mathrm{d}t \tag{6.11}$$

for any

$$\phi \in W_0^{1,2}(\Omega; R^3),\ \mathrm{div}_y \phi = 0.$$

Here we have used the strong convergence of $\mathbf{v}_\varepsilon$ (hence also $\widetilde{\mathbf{u}}_\varepsilon$) established in (6.6) to pass to the limit in the quadratic term $[\widetilde{\mathbf{u}}_\varepsilon \otimes \widetilde{\mathbf{u}}_\varepsilon]_\delta$.

Finally, we let $\delta \to 0$ in (6.11), and use the identity of the limits $\mathbf{u} = \mathbf{v}$ established in Section 6.2 to conclude the proof of Theorem 2.3.

7 A priori estimate of the velocity of small rigid bodies

We prove Proposition 2.8, that implies a control of the velocity of a the rigid bodies using an idea from [4].

Proof of Proposition 2.8. Suppose that there exists $\bar{\varepsilon} > 0$ such that for any $0 < \varepsilon < \bar{\varepsilon}$

$$\sup_{t\in[0,T]} |\mathbf{h}_{i,\varepsilon}(t) - \mathbf{h}_{j,\varepsilon}(t)| > \varepsilon,\ i \neq j,\ \mathrm{dist}[\mathbf{h}_{n,\varepsilon}(0); \partial\Omega] > \frac{1}{2}\varepsilon. \tag{7.1}$$

Then for $0 < \varepsilon < \bar{\varepsilon}$, we start by recalling a cut–off function introduced in Section 4.

$$\chi \in C_c^\infty[0,1),\ \chi(z) = \begin{cases} 1 \text{ if } 0 \le z < \frac{1}{2}, \\ \in [0,1] \text{ if } \frac{1}{2} \le z \le \frac{3}{4}, \\ 0 \text{ if } z > \frac{3}{4}. \end{cases}$$

Moreover for $z \in R^3$, we define the velocity field

$$\Psi_{r_\varepsilon}[z](x) = \mathbf{curl}_x \left[\chi\left(2\frac{|x|}{r_\varepsilon}\right) \mathbb{T}(x) \cdot z \right], \tag{7.2}$$

where $\mathbb{T}$ is defined in (4.2).

Let $\gamma \in C_c^\infty[0,T)$ and testing (2.24) with

$$\psi_{n,\varepsilon} = \frac{1}{m_\varepsilon} \Psi_{r_\varepsilon} \left[\int_t^T \gamma(\tau)\ \mathrm{d}\tau \right] (x - \mathbf{h}_{n,\varepsilon}(t)), \tag{7.3}$$

we deduce

$$\int_0^T (\mathbf{h}'_{n,\varepsilon}(t) - \mathbf{h}'_{n,\varepsilon}(0)) \cdot \gamma(t)\ \mathrm{d}t = \int_0^T \int_{\Omega_{t,\varepsilon}} \mathbf{u}_\varepsilon \cdot \partial_t \psi_{n,\varepsilon} + [\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon] : \nabla_x \psi_{n,\varepsilon}\ \mathrm{d}x\mathrm{d}t$$
$$- \int_0^T \int_{\Omega_{\varepsilon,t}} \nabla_x \mathbf{u}_\varepsilon : \nabla_x \psi_{n,\varepsilon}\ \mathrm{d}x\mathrm{d}t + \int_{\Omega_{0,\varepsilon}} \mathbf{u}_{\varepsilon,0} \cdot \psi_{n,\varepsilon}(0,.)\ \mathrm{d}x.$$

We compute

$$\begin{aligned} \partial_t \psi_{n,\varepsilon}(t,x) = & - \frac{1}{m_\varepsilon} \Psi_{r_\varepsilon} [\gamma(t)] (x - \mathbf{h}_{n,\varepsilon}(t)) \\ & - \mathbf{h}'_{n,\varepsilon}(t) \cdot \nabla_x \left(\frac{1}{m_\varepsilon} \Psi_{r_\varepsilon} \left[\int_t^T \gamma(\tau)\ \mathrm{d}\tau \right] \right) (x - \mathbf{h}_{n,\varepsilon}(t)). \end{aligned}$$

We deduce, for a constant C independent of ε and n

$$\|\partial_t \psi_{n,\varepsilon}(t,x)\|_{L^1(0,T;L^2(R^3)))} \le C\frac{r_\varepsilon^{3/2}}{m_\varepsilon} \|\gamma\|_{L^1(0,T)} + C\frac{r_\varepsilon^{1/2}}{m_\varepsilon} \|\mathbf{h}'_{n,\varepsilon}\|_{L^1(0,T)} \|\gamma\|_{L^1(0,T)} \tag{7.4}$$

and similarly

$$\|\nabla_x \psi_{n,\varepsilon}(t,x)\|_{L^\infty(0,T;L^2(R^3)))} \leq C\frac{r_\varepsilon^{1/2}}{m_\varepsilon}\|\gamma\|_{L^1(0,T)}. \tag{7.5}$$

Korn inequality together with the Sobolev inequality imply that

$$\|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^6(\Omega))} \leq C\|D\mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq C, \tag{7.6}$$

by the energy estimates. The bounds (7.4)-(7.5)-(7.6) imply

$$\begin{aligned}\int_0^T (\mathbf{h}'_{n,\varepsilon}(t) - \mathbf{h}'_{n,\varepsilon}(0)) \cdot \gamma(t)\ \mathrm{d}t \leq C\Bigg(&\frac{r_\varepsilon^{3/2}}{m_\varepsilon}\|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \frac{r_\varepsilon^{1/2}}{m_\varepsilon}\|\mathbf{u}_\varepsilon\|^2_{L^2(0,T;L^6(\Omega))} \\ &+ \sqrt{T}\frac{r_\varepsilon^{1/2}}{m_\varepsilon}\|\nabla_x \mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \frac{r_\varepsilon^{3/2}}{m_\varepsilon}\|\mathbf{u}_{\varepsilon,0}\|_{L^2(\Omega)} \\ &+ \frac{r_\varepsilon^{1/2}}{m_\varepsilon}\|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega))} T|\mathbf{h}'_{n,\varepsilon}(0)|\Bigg)\|\gamma\|_{L^1(0,T)} \\ &+ C\frac{r_\varepsilon^{1/2}}{m_\varepsilon}\|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega))} T\|\mathbf{h}'_{n,\varepsilon} - \mathbf{h}'_{n,\varepsilon}(0)\|_{L^\infty(0,T)}\|\gamma\|_{L^1(0,T)}.\end{aligned}$$

Dividing by $\|\gamma\|_{L^1(0,T)}$ and taking the sup over all $\gamma \in C_c^\infty[0,T)$, we deduce

$$\|\mathbf{h}'_{n,\varepsilon} - \mathbf{h}'_{n,\varepsilon}(0)\|_{L^\infty(0,T)} \leq C\frac{r_\varepsilon^{1/2}}{m_\varepsilon} + C\frac{r_\varepsilon^{1/2}}{m_\varepsilon} T\|\mathbf{h}'_{n,\varepsilon} - \mathbf{h}'_{n,\varepsilon}(0)\|_{L^\infty(0,T)}.$$

From hypothesis (2.32), we can absorb the second term of the right hand side and deduce the desired result.

Similarly for the angular velocity, we consider, for $z \in R^3$, a auxiliary function

$$\Phi_{r_\varepsilon}[z](x) = \mathbf{curl}_x \left[-\chi\left(2\frac{|x|}{r_\varepsilon}\right)\frac{|x|^2}{2}z\right], \tag{7.7}$$

$$\phi_{n,\varepsilon} = \frac{1}{r_\varepsilon m_\varepsilon}\Phi_{r_\varepsilon}\left[\int_t^T \gamma(\tau)\ \mathrm{d}\tau\right](x - \mathbf{h}_{n,\varepsilon}(t)). \tag{7.8}$$

In the spirit of (7.4) and (7.5), we have

$$\|\partial_t \phi_{n,\varepsilon}(t,x)\|_{L^1(0,T;L^2(R^3)))} \leq C\frac{r_\varepsilon^{3/2}}{m_\varepsilon}\|\gamma\|_{L^1(0,T)} + C\frac{r_\varepsilon^{1/2}}{m_\varepsilon}\|\mathbf{h}'_{n,\varepsilon}\|_{L^1(0,T)}\|\gamma\|_{L^1(0,T)} \tag{7.9}$$

and similarly

$$\|\nabla_x \phi_{n,\varepsilon}(t,x)\|_{L^\infty(0,T;L^2(R^3)))} \leq C\frac{r_\varepsilon^{1/2}}{m_\varepsilon}\|\gamma\|_{L^1(0,T)}. \tag{7.10}$$

We deduce that

$$
\begin{aligned}
&\frac{2}{5} r_\varepsilon \int_0^T (\boldsymbol{\omega}'_{n,\varepsilon}(t) - \boldsymbol{\omega}'_{n,\varepsilon}(0)) \cdot \gamma(t) \, \mathrm{d}t \\
&\quad \le C\|\gamma\|_{L^1(0,T)} \left(\frac{r_\varepsilon^{1/2}}{m_\varepsilon} + C \frac{r_\varepsilon^{1/2}}{m_\varepsilon} \|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega))} T \|\mathbf{h}'_{n,\varepsilon}(t)\|_{L^\infty(0,T)} \right) \\
&\quad \le C\|\gamma\|_{L^1(0,T)} \frac{r_\varepsilon^{1/2}}{m_\varepsilon}.
\end{aligned}
$$

Dividing by $\|\gamma\|_{L^1(0,T)}$ and taking the sup over all possible γ, we deduce the result.

We conclude the proof by showing that hypothesis (7.1) holds by contradiction. Suppose by contradiction that (7.1) does not hold. There exists a sequence $\varepsilon_k \to 0$ such that (7.1) is not satisfied. By (2.32) there exists $\bar{k}$ such that for any $k > \bar{k}$ we have

$$
|\mathbf{h}'_{n,\varepsilon_k}(0)| < \frac{\varepsilon_k}{8T} \quad \text{and} \quad \frac{r_{\varepsilon_k}^{1/2}}{\varepsilon_k m_{\varepsilon_k}} < \frac{1}{4CT}. \tag{7.11}
$$

Let now t_k the infimum in $[0,T]$ such that (7.1) with $\varepsilon = \varepsilon_k$ does not hold. By continuity of $\mathbf{h}_{n,\varepsilon_k}$ and the fact that $N(\varepsilon_k) < \infty$ the infimum is a minimum. In $[0,t_k)$ the bound (2.33) holds. We deduce that

$$
\begin{aligned}
|\mathbf{h}_{i,\varepsilon_k}(t_k) - \mathbf{h}_{j,\varepsilon_k}(t_k)| &\ge |\mathbf{h}_{i,\varepsilon_k}(0) - \mathbf{h}_{j,\varepsilon_k}(0)| - \left| \int_0^{t_k} \mathbf{h}'_{i,\varepsilon_k}(\tau) \, \mathrm{d}\tau \right| - \left| \int_0^{t_k} \mathbf{h}'_{j,\varepsilon_k}(\tau) \, \mathrm{d}\tau \right| \\
&\ge 2\varepsilon_k - \frac{\varepsilon_k}{8} - \frac{\varepsilon_k}{8} - \frac{\varepsilon_k}{8} - \frac{\varepsilon_k}{8} = \frac{3}{2}\varepsilon_k.
\end{aligned} \tag{7.12}
$$

Similarly we have

$$
\mathrm{dist}[\mathbf{h}_{n,\varepsilon_k}(t); \partial\Omega] \ge \mathrm{dist}[\mathbf{h}_{n,\varepsilon_k}(0); \partial\Omega] - \left| \int_0^{t_k} \mathbf{h}'_{n,\varepsilon_k}(\tau) \, \mathrm{d}\tau \right| \ge \frac{3}{4}\varepsilon_k. \tag{7.13}
$$

Then (7.12) and (7.13), contradict that (7.1) does not holds. The proof is then completed.

□

8 Concluding remarks

The results were stated in the space dimension $d = 3$. Extension to $d = 2$ can be done with the appropriate scaling of the radius r_ε, specifically,

$$
-\varepsilon^2 \log(r_\varepsilon) \to C \in (0, \infty).
$$

The homogenization limit for the stationary Stokes problem stated in Proposition 3.2 can be extended to the periodic distribution of obstacles by the method of Allaire [1]. Rather general distribution of obstacles and their radii was considered by Marchenko and Khruslov [40, Chapter 4, Theorem 4.7], cf. also Namlyeyeva et al. [43]. We leave the details to the interested reader.

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