ON LIPSCHITZ SPACES IN THE DUNKL SETTING - SEMIGROUP APPROACH

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ABSTRACT. Let $\{P_t\}_{t>0}$ be the Dunkl-Poisson semigroup associated with a root system $R \subset \mathbb{R}^N$ and a multiplicity function $k \geq 0$. We say that a bounded measurable function f defined on $\mathbb{R}^N$ belongs to the inhomogeneous Lipschitz space Λ_k^β, $\beta > 0$, if

$$\sup_{t>0} t^{m-\beta} \Big\| \frac{d^m}{dt^m} P_t f \Big\|_{L^\infty} < \infty,$$

where $m = [\beta] + 1$. We prove that the spaces Λ_k^β coincide with the classical Lipschitz spaces. In order to prove the theorem, we provide other characterizations of the space and apply the K-interpolation method.

1. Introduction

The aim of this paper is to study inhomogeneous Lipschitz spaces in the Dunkl setting. On the Euclidean space $\mathbb{R}^N$ equipped with a root system R and a multiplicity function $k \geq 0$, we consider the Dunkl Laplace operator

$$\Delta_k = \sum_{j=1}^{N} D_j^2, \tag{1.1}$$

where

$$D_j f(\mathbf{x}) = \partial_j f(\mathbf{x}) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \langle \alpha, e_j \rangle \frac{f(\mathbf{x}) - f(\sigma_\alpha(\mathbf{x}))}{\langle \alpha, \mathbf{x} \rangle}$$

are the Dunkl operators. These operators were introduced by C. F. Dunkl in [5] in the late 1980s to study special functions and spherical harmonics with symmetries given by finite Coxeter groups. They were later applied in physics to investigate the quantum many-body Calogero-Moser-Sutherland model, particularly enabling the proof of its integrability. In recent years, they have been used in quantum physics problems of a symmetric nature, including anyons, supersymmetry, noncommutative geometry, and PT-symmetry (see e.g., [25] for more details about CMS model and the other applications). Beyond physics, these operators have applications in probability theory (e.g., Feller processes with jumps, see e.g., [18] for the probabilistic point of view) and algebra (Hecke algebras). Replacing classical derivative operators with Dunkl operators in differential equations allows for the study of various physics problems and operators with symmetries, e.g., Dunkl versions of the harmonic oscillator operator and Schrödinger operators. The function spaces which are frequently used in these types of problems are Sobolev spaces, as well as the related Lipschitz spaces discussed in the current article.

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For $0 < \beta < 1$, the classical inhomogeneous Lipschitz space $\Lambda^\beta(\mathbb{R}^N)$ on the Euclidean space $\mathbb{R}^N$ is defined as

$$\Lambda^\beta(\mathbb{R}^N) = \Big\{ f : \mathbb{R}^N \to \mathbb{C} \ : \ \|f\|_{L^\infty} + \sup_{\mathbf{x}\neq\mathbf{x}'} \frac{|f(\mathbf{x}) - f(\mathbf{x}')|}{\|\mathbf{x}-\mathbf{x}'\|^\beta} =: \|f\|_{\Lambda^\beta(\mathbb{R}^N)} < \infty \Big\}. \tag{1.2}$$

In Taibleson [21], the author used the Poisson integral (semigroup)

$$f(\mathbf{x},t) = c_N^{-1} \int_{\mathbb{R}^N} f(\mathbf{y}) \frac{t}{(t^2 + \|\mathbf{x}-\mathbf{y}\|^2)^{(N+1)/2}}\, d\mathbf{y} \tag{1.3}$$

to study properties of the Lipschitz spaces on $\Lambda^\beta(\mathbb{R}^N)$ (see also [20]). To be more precise, for any fixed positive integer m, the norm $\|f\|_{\Lambda^\beta(\mathbb{R}^N)}$ is equivalent to

$$\|f\|_{L^\infty} + \sup_{t>0} t^{m-\beta} \Big\| \frac{d^m}{dt^m} f(\mathbf{x},t) \Big\|_{L^\infty}, \tag{1.4}$$

see [21, Theorems 3 and 4] and [20, Chapter V, Proposition 7 and Lemma 5] .

The theorem motivates extending the notion of Lipschitz spaces for all positive parameters $\beta > 0$. So, for $\beta > 0$, let m be the smallest integer bigger than β. We say that $f \in \Lambda^\beta(\mathbb{R}^N)$, if (1.4) is finite (then (1.4) is taken as the norm in the space). Furthermore, one can consider the heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$ and take

$$\|f\|_{L^\infty} + \sup_{t>0} t^{m-\beta/2} \Big\| \frac{d^m}{dt^m} e^{t\Delta} f \Big\|_{L^\infty} \tag{1.5}$$

as an equivalent norm in the Lipschitz space $\Lambda^\beta(\mathbb{R}^N)$ (see [21, Theorem 7]).

It turns out (see [21, Theorem 4]) that if $0 < \beta < 2$, then the norm (1.4) (with any fixed $m \geq 2$) is equivalent to

$$\|f\|_{L^\infty} + \sup_{\mathbf{x}\in\mathbb{R}^N} \sup_{0\neq\mathbf{y}\in\mathbb{R}^N} \frac{\|f(\mathbf{x}+\mathbf{y}) + f(\mathbf{x}-\mathbf{y}) - 2f(\mathbf{x})\|}{\|\mathbf{y}\|^\beta}. \tag{1.6}$$

Such spaces can be thought as inhomogeneous generalized Zygmund classes.

Furthermore, if $\beta > 1$, then $f \in \Lambda^\beta(\mathbb{R}^N)$ if and only if $f \in L^\infty$ and $\partial_j f \in \Lambda^{\beta-1}(\mathbb{R}^N)$ for $j = 1, 2, ..., N$ (see e.g., [20, Chapter V, Proposition 9]). Moreover,

$$\|f\|_{\Lambda^\beta(\mathbb{R}^N)} \sim \|f\|_{L^\infty} + \sum_{j=1}^N \|\partial_j f\|_{\Lambda^{\beta-1}(\mathbb{R}^N)}. \tag{1.7}$$

Additionally, the Bessel potential $(I-\Delta)^{\gamma/2}$ is an isomorphism of the space $\Lambda^\beta(\mathbb{R}^N)$ onto $\Lambda^{\beta+\gamma}(\mathbb{R}^N)$ (see [21, Theorem 5]).

Let us now discuss the Dunkl counterparts of the theorems and objects described above. The operator Δ_k (see (1.1)) generates a contraction semigroup $H_t = e^{t\Delta_k}$ on $L^p(dw)$, $1 \leq p \leq \infty$ (strongly continuous for $1 \leq p < \infty$), where

$$dw(\mathbf{x}) = \prod_{\alpha\in R} |\langle \alpha, \mathbf{x}\rangle|^{k(\alpha)}\, d\mathbf{x}.$$

The semigroup has a unique extension to a uniformly bounded holomorphic semigroup on any sector $\boldsymbol{S}_\delta = \{z \in \mathbb{C} : |\arg z| < \delta\}$, $0 < \delta < \pi/2$. Let

$$P_t := \Gamma(1/2)^{-1} \int_0^\infty e^{-u} H_{t^2/(4u)} \frac{du}{\sqrt{u}} \tag{1.8}$$

be the subordinate (Dunkl-Poisson) semigroup. We say that a measurable function f defined on $\mathbb{R}^N$ belongs to the inhomogeneous Lipschitz space Λ_k^β, if

$$\|f\|_{\Lambda_k^\beta} := \|f\|_{L^\infty} + \sup_{t>0} t^{m-\beta} \Big\| \frac{d^m}{dt^m} P_t f \Big\|_{L^\infty} < \infty, \tag{1.9}$$

where m is the smallest positive integer greater than β. We are in a position to state our main result, which is obtained at the very end of the paper.

Theorem 1.1. *For any $\beta > 0$, the Lipschitz space Λ_k^β coincides with the classical Lipschitz space $\Lambda^\beta(\mathbb{R}^N)$ and the corresponding norms (1.9) and (1.4) are equivalent.*

Analyzing objects defining Lipschitz spaces in the Dunkl context, it is easy to observe that many of the properties of these objects differ from the classical ones. This was one of the reasons why we opted for a different approach than classical ones in some places. We utilize them to bypass difficulties arising from the specificity of the Dunkl context. Our proofs use elements of abstract theory of function spaces associated with generators of uniformly bounded holomorphic semigroups. We want to note that some of theorems in the theory of Lipschitz spaces (including those from the classical theory) can be obtained in this way. For example, the equivalence of the definitions of Lipschitz spaces by subordinate and original semigroups can be obtained by means of holomorphic functional calculi. We present this general approach in Part 1 of the paper.

The proof of Theorem 1.1 goes by a detailed examination of properties of the Λ_k^β spaces. Let us shortly describe its main steps.

The first milestone in proving Theorem 1.1 is to obtain the equality $\Lambda_k^\beta = \Lambda^\beta(\mathbb{R}^N)$ for $0 < \beta < 1$. Let us remark that the characterizations of the classical Lipschitz spaces are given by the behavior of convolutions which use Euclidean translations, natural to study Hölder's regularities (see e.g., [20, Chapter V, Proposition 7]). In the Dunkl setting the action of the Poisson semigroup is by means of generalized translations and full understanding of these operations is far from satisfactory. In particular, one of the well-known open problems is the question of the boundedness of the Dunkl translation on L^p. However, recent results concerning behavior and regularity of the integral kernels of the heat and Poisson Dunkl semigroups allow us to prove that for $0 < \beta < 1$ the space Λ_k^β and the classical Lipschitz space $\Lambda^\beta(\mathbb{R}^N)$ coincide (see Theorem 8.5).

Another important component of the proof is Theorem 8.3, which states that the Bessel-type potentials $((I - \Delta_k)^{-\gamma/2})^*$ and $((I + \sqrt{-\Delta_k})^{-\gamma})^*$ are isomorphisms of Λ_k^β onto $\Lambda_k^{\beta+\gamma}$, $\beta, \gamma > 0$, and, moreover, for $\beta > 1$, $f \in \Lambda_k^\beta$ if and only if $f \in L^\infty$ and $D_j^* f \in \Lambda_k^{\beta-1}$, $j = 1, 2, ..., N$. We emphasize that the results of Theorem 8.3 are consequences of the abstract approach mentioned above.

Since the L^∞-norms of the Dunkl derivatives are controlled by the L^∞-norms of the classical ones, the inclusions $\Lambda^\beta(\mathbb{R}^N) \subseteq \Lambda_k^\beta$ seem to be expected at this moment. However, the inverse inclusions $\Lambda_k^\beta \subseteq \Lambda^\beta(\mathbb{R}^N)$ are not at all obvious, because the Dunkl operators introduce non-local effects, and their behavior is influenced by actions of the reflections. To this end, we study the operators $\frac{\partial}{\partial x_j}((I + \sqrt{-\Delta_k})^{-\gamma})^*$. The results stated above, combined with properties of the Bessel potential kernels, allow us to verify that $\Lambda_k^\beta = \Lambda^\beta(\mathbb{R}^N)$ for all $\beta > 0$, $\beta \notin \mathbb{Z}$ (see Theorems 9.2 and 9.5).

Our final step is the equality $\Lambda_k^\beta = \Lambda^\beta(\mathbb{R}^N)$ for $\beta \in \mathbb{Z}$, $\beta \geq 1$. Let us recall that in the case of $\beta = 1$, the Zygmund condition (1.6) occurs and we face difficulties in handling it in the

Dunkl setting. In order to overcome these major obstacles, we use an interpolation argument to complete the proof of Theorem 1.1.

Such an approach to interpolation spaces associated with powers of generators of strongly continuous semigroups can be found e.g., in [2], [23], and references therein. For readers who are not familiar with the theory and for the sake of completeness, we present all abstract results used in the proofs of the theorems.

We remark that our approach to the interpolation differs a little from that described in [2, 23], because we interpolate between the spaces defined by means of actions of semigroups which are not strongly continuous on the Banach spaces under consideration. So, motivated by the approach to the classical Lipschitz spaces, which is based on the action of either the Poisson or the heat semigroup on L^∞-functions, which form the dual space of L^1, in Part 1 we consider an analytic strongly continuous semigroup $\mathcal{T}_t = e^{t\mathcal{A}}$ on a Banach space $(X, \|\cdot\|)$, which is uniformly bounded in a sector around the positive axis and the dual semigroup $\{\mathcal{T}_t^*\}_{t\geq 0}$ acting on the dual space $(X^*, \|\cdot\|_{X^*})$. We do not assume that X is reflexive, so the semigroup $\mathcal{T}_t^*$ is not necessarily strongly continuous in general. For such a semigroup the space $\Lambda^\beta_{\mathcal{A}}$ is defined as follows.

Definition 1.2. *For $\beta > 0$, let m be the smallest positive integer such that $m > \beta$. We say that $x^* \in X^*$ belongs to $\Lambda^\beta_{\mathcal{A}}$, if*

$$\Big\| \frac{d^m}{dt^m} \mathcal{T}_t^* x^* \Big\|_{X^*} \leq C t^{\beta - m}. \tag{1.10}$$

We equip the space $\Lambda^\beta_{\mathcal{A}}$ with the norm

$$\|x^*\|_{\Lambda^\beta_{\mathcal{A}}} := \|x^*\|_{X^*} + \sup_{t>0} t^{m-\beta} \Big\| \frac{d^m}{dt^m} \mathcal{T}_t^* x^* \Big\|_{X^*}. \tag{1.11}$$

It is easy to prove that $(\Lambda^\beta_{\mathcal{A}}, \|\cdot\|_{\Lambda^\beta_{\mathcal{A}}})$ is a Banach space. Furthermore, the subordinate semigroup is defined by $\{\mathcal{P}_t\}_{t\geq 0}$ according to the formula

$$\mathcal{P}_t := \Gamma(1/2)^{-1} \int_0^\infty e^{-u} \mathcal{T}_{t^2/(4u)} \frac{du}{\sqrt{u}} \tag{1.12}$$

acting on the Banach space $(X, \|\cdot\|)$. We denote by $-\sqrt{-\mathcal{A}}$ the infinitesimal generator of $\{\mathcal{P}_t\}_{t\geq 0}$. The semigroup $\{\mathcal{P}_t\}_{t\geq 0}$ is strongly continuous and has a unique extension to a uniformly bounded holomorphic semigroup in the same sector. Again, we consider the dual semigroup $\{\mathcal{P}_t^*\}_{t\geq 0}$ and define the related $\Lambda^\beta_{-\sqrt{-\mathcal{A}}}$ spaces, namely for $\beta > 0$ and m as above, we say that $x^* \in X^*$ belongs to $\Lambda^\beta_{-\sqrt{-\mathcal{A}}}$ if

$$\Big\| \frac{d^m}{dt^m} \mathcal{P}_t^* x^* \Big\|_{X^*} \leq C t^{\beta - m}. \tag{1.13}$$

Then, as in the previous case, we set

$$\|x^*\|_{\Lambda^\beta_{-\sqrt{-\mathcal{A}}}} = \|x^*\|_{X^*} + \sup_{t>0} t^{m-\beta} \Big\| \frac{d^m}{dt^m} \mathcal{P}_t^* x^* \Big\|_{X^*}.$$

We are now in a position to state the first result which is obtained by means of holomorphic functional calculi.

Theorem 1.3. *For $\beta > 0$, the spaces $\Lambda^\beta_{\mathcal{A}}$ and $\Lambda^{2\beta}_{-\sqrt{-\mathcal{A}}}$ coincide. Moreover, there is $C > 1$ such that for all $x^* \in X^*$, we have*

$$C^{-1}\|x^*\|_{\Lambda^\beta_{\mathcal{A}}} \leq \|x^*\|_{\Lambda^{2\beta}_{-\sqrt{-\mathcal{A}}}} \leq C\|x^*\|_{\Lambda^\beta_{\mathcal{A}}}. \tag{1.14}$$

Next, for $\gamma > 0$, we consider the Bessel type potentials

$$((I-\mathcal{A})^{-\gamma})^*, \quad ((I+\sqrt{-\mathcal{A}})^{-\gamma})^*$$

(see (2.17) and (2.18)).

Theorem 1.4. *Let $\beta, \gamma > 0$. The operator $((I-\mathcal{A})^{-\gamma})^*$ is an isomorphism of $\Lambda^\beta_{\mathcal{A}}$ onto $\Lambda^{\gamma+\beta}_{\mathcal{A}}$.*

The theorem applied to the semigroup $\{\mathcal{P}_t\}_{t>0}$ gives the following corollary.

Corollary 1.5. *Let $\beta, \gamma > 0$. The operator $((I+\sqrt{-\mathcal{A}})^{-\gamma})^*$ is an isomorphism of $\Lambda^\beta_{-\sqrt{-\mathcal{A}}}$ onto $\Lambda^{\beta+\gamma}_{-\sqrt{-\mathcal{A}}}$.*

In order to prove Theorem 1.4, we first establish the following relation between the spaces $\Lambda^{\beta+1}_{\mathcal{A}}$ and $\Lambda^\beta_{\mathcal{A}}$ and the action of $\mathcal{A}^*$.

Fix $x^* \in X^*$. We say that $y^* = \mathcal{A}^* x^* \in X^*$ in the mild sense, if for all $x \in X$ and all $t > 0$,

$$\langle y^*, \mathcal{T}_t x\rangle = \langle x^*, \mathcal{A}\mathcal{T}_t x\rangle.$$

Theorem 1.6. *Assume that $\beta > 0$, $x^* \in X^*$. Then $x^* \in \Lambda^{\beta+1}_{\mathcal{A}}$ if and only if $\mathcal{A}^* x^* \in \Lambda^\beta_{\mathcal{A}}$, where the action of $\mathcal{A}^*$ on x^* is understood in the mild sense. Moreover, the norms*

$$\|x^*\|_{\Lambda^{\beta+1}_{\mathcal{A}}} \quad \text{and} \quad \|x^*\|_{X^*} + \|\mathcal{A}^* x^*\|_{\Lambda^\beta_{\mathcal{A}}}$$

are equivalent.

We finish Part 1 of the paper with the following theorem, which will play a crucial role in the completion of the proof of Theorem 1.1. For the Banach spaces $\Lambda^{\beta_0}_{\mathcal{A}}$ and $\Lambda^{\beta_1}_{\mathcal{A}}$ and $0 < \theta < 1$, let $(\Lambda^{\beta_0}_{\mathcal{A}}, \Lambda^{\beta_1}_{\mathcal{A}})_\theta$ and $\|x^*\|_\theta$ denote the intermediate interpolation space and the interpolation norm obtained by the K-method of Peetre (see Section 6 for details).

Theorem 1.7 (cf. [23, Section 2.7 for the classical Lipschitz spaces])**.** *For $0 < \beta_0 < \beta_1$ and $0 < \theta < 1$, let $\beta = (1-\theta)\beta_0 + \theta\beta_1$. Then*

$$(\Lambda^{\beta_0}_{\mathcal{A}}, \Lambda^{\beta_1}_{\mathcal{A}})_\theta = \Lambda^\beta_{\mathcal{A}} \tag{1.15}$$

and there is a constant $C > 1$ such that for all $x^* \in X^*$ we have

$$C^{-1}\|x^*\|_{\Lambda^\beta_{\mathcal{A}}} \leq \|x^*\|_\theta \leq C\|x^*\|_{\Lambda^\beta_{\mathcal{A}}}. \tag{1.16}$$

In Part 2, we apply the results of Part 1 together with properties of the Dunkl heat and the Dunkl Poisson kernels to study inhomogeneous Lipschitz spaces associated with the Dunkl operators on the Euclidean space $\mathbb{R}^N$.

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Part 1 Semigroup approach to Λ^β-spaces

2. Analytic semigroups of linear operators

2.1. **Analytic semigroups.** In the present section we collect some facts concerning holomorphic (analytic) semigroups of operators on Banach spaces (see e.g., [3], [13]).

Let $\{\mathcal{T}_t\}_{t\geq 0}$ be a strongly continuous semigroup of linear operators on a Banach space $(X, \|\cdot\|)$, which, for certain $0<\delta<\pi/2$, has an extension to a uniformly bounded holomorphic semigroup $\{\mathcal{T}_z\}_{z\in \boldsymbol{S}_\delta}$ in the sector

$$\boldsymbol{S}_\delta = \{z\in\mathbb{C} : |\arg z|<\delta\}$$

about the positive axis.

Let $(X^*, \|\cdot\|_{X^*})$ be the dual space of the Banach space $(X,\|\cdot\|)$ and let $\mathcal{T}_t^*\in\mathcal{L}(X^*)$ denote the dual operator to $\mathcal{T}_t$. Then $\{\mathcal{T}_t^*\}_{t\geq 0}$ has a unique extension to a uniformly bounded holomorphic semigroup (which is in general not strongly continuous). It follows from the theory of analytic semigroups that if $(\mathcal{A},\mathfrak{D}(\mathcal{A}))$, where $\mathfrak{D}(\mathcal{A})$ denotes the domain of $\mathcal{A}$, is the infinitesimal generator of $\{\mathcal{T}_t\}_{t\geq 0}$, then

(1) $\mathcal{T}_t(X)\subseteq \bigcap_{n\in\mathbb{N}}\mathfrak{D}(\mathcal{A}^n)$ for all $t>0$,
(2) the functions $(0,\infty)\ni t\mapsto \mathcal{T}_t\in\mathcal{L}(X)$, $(0,\infty)\ni t\mapsto\mathcal{T}_t^*\in\mathcal{L}(X^*)$ are differentiable (even holomorphic in $\boldsymbol{S}_\delta$) and

$$\frac{d}{dt}\mathcal{T}_t=\mathcal{A}\mathcal{T}_t,\quad \frac{d}{dt}\mathcal{T}_t^*=(\mathcal{A}\mathcal{T}_t)^*, \tag{2.1}$$

$$\|\mathcal{A}\mathcal{T}_t\|=\|(\mathcal{A}\mathcal{T}_t)^*\|_{X^*}\leq Ct^{-1}, \tag{2.2}$$

$$\mathcal{A}\mathcal{T}_{t+s}=\mathcal{T}_t\mathcal{A}\mathcal{T}_s,\quad (\mathcal{A}\mathcal{T}_{t+s})^*=\mathcal{T}_t^*(\mathcal{A}\mathcal{T}_s)^*. \tag{2.3}$$

In particular, if $t=t_1+t_2+...+t_n$, $t_j>0$, then

$$\frac{d^n}{dt^n}\mathcal{T}_t=\mathcal{A}^n\mathcal{T}_t=\mathcal{A}\mathcal{T}_{t_1}\mathcal{A}\mathcal{T}_{t_2}...\mathcal{A}\mathcal{T}_{t_n},\quad \frac{d^n}{dt^n}\mathcal{T}_t^*=(\mathcal{A}^n\mathcal{T}_t)^*=(\mathcal{A}\mathcal{T}_{t_1})^*(\mathcal{A}\mathcal{T}_{t_2})^*...(\mathcal{A}\mathcal{T}_{t_n})^*. \tag{2.4}$$

Consequently, there is $C>0$ such that for all $n\in\mathbb{N}$ and $t>0$ we have

$$\Big\|\frac{d^n}{dt^n}\mathcal{T}_t\Big\|\leq C^nt^{-n},\quad \Big\|\frac{d^n}{dt^n}\mathcal{T}_t^*\Big\|_{X^*}\leq C^nt^{-n}. \tag{2.5}$$

By $\rho(\mathcal{A})$, we denote the resolvent set of $\mathcal{A}$, that is, the set of all $\lambda\in\mathbb{C}$ such that the operator $\lambda I-\mathcal{A}:\mathfrak{D}(\mathcal{A})\to X$ is one to one and onto, and its inverse, denoted by $R(\lambda:\mathcal{A})$, is bounded on X. The resolvent set $\rho(\mathcal{A})$ is open and the mapping $\rho(\mathcal{A})\ni\lambda\mapsto R(\lambda:\mathcal{A})\in\mathcal{L}(X)$ is holomorphic, so is $R(\lambda:\mathcal{A})^*$. Moreover, for all $\lambda,\mu\in\rho(\mathcal{A})$,

$$\begin{aligned}&R(\lambda:\mathcal{A})-R(\mu:\mathcal{A})=(\mu-\lambda)R(\lambda:\mathcal{A})R(\mu:\mathcal{A}),\quad \frac{d}{d\lambda}R(\lambda:\mathcal{A})=-R(\lambda:\mathcal{A})^2,\\ &R(\lambda:\mathcal{A})^*-R(\mu:\mathcal{A})^*=(\mu-\lambda)R(\lambda:\mathcal{A})^*R(\mu:\mathcal{A})^*,\quad \frac{d}{d\lambda}R(\lambda:\mathcal{A})^*=-R(\lambda:\mathcal{A})^{*2}.\end{aligned} \tag{2.6}$$

If $\{\mathcal{T}_t\}_{t\geq 0}$ is a holomorphic semigroup, uniformly bounded in $\boldsymbol{S}_\delta$, then

$$\Sigma_\delta:=\Big\{\lambda\in\mathbb{C}:|\arg\lambda|<\frac{\pi}{2}+\delta\Big\}\subseteq\rho(\mathcal{A}), \tag{2.7}$$

$$\|R(\lambda:\mathcal{A})\|\leq\frac{C'}{|\lambda|},\quad \text{for } \lambda\in\Sigma_\delta. \tag{2.8}$$

2.2. $\Lambda_{\mathcal{A}}^{\beta}$**-spaces - basic properties.** Let $\{\mathcal{T}_t\}_{t\geq 0}$ be a semigroup of linear operators generated by $\mathcal{A}$ which has an extension to a uniformly bounded holomorphic semigroup in a sector $\boldsymbol{S}_\delta$. In this section we state elementary properties of the space $\Lambda_{\mathcal{A}}^{\beta}$ (see Definition 1.2).

The following easily proved proposition asserts that, as in the classical case (cf. [20, Chapter V, Lemma 5]), the integer m in Definition 1.2 can be replaced by any integer $n > \beta$.

Proposition 2.1. *Let $x^* \in X^*$. Fix $\beta > 0$. If $n > m > \beta$ are integers, then the following two conditions*

$$\Big\|\frac{d^m}{dt^m}\mathcal{T}_t^* x^*\Big\|_{X^*} \leq C_m t^{\beta-m}, \quad \Big\|\frac{d^n}{dt^n}\mathcal{T}_t^* x^*\Big\|_{X^*} \leq C_n t^{\beta-n}, \tag{2.9}$$

are equivalent. Moreover, there is $C > 1$ (which depends on m and n and is independent of $x^* \in X^*$) such that the smallest constant C_m and C_n holding in the above inequalities satisfy:

$$C^{-1}C_m \leq C_n \leq CC_m.$$

Remark 2.2. It is worth emphasizing that, as in the classical case, the estimates (2.9) are of interest only for $0 < t \leq 1$, because, for $t > 1$, in virtue of (2.5), the following better estimates always hold, namely

$$\Big\|\frac{d^n}{dt^n}\mathcal{T}_t^* x^*\Big\|_{X^*} \leq C^n t^{-n} \|x^*\|_{X^*}.$$

Remark 2.2 together with Proposition 2.1 imply that

$$\Lambda_{\mathcal{A}}^{\beta_1} \subseteq \Lambda_{\mathcal{A}}^{\beta_2} \quad \text{and} \quad \|x^*\|_{\Lambda_{\mathcal{A}}^{\beta_2}} \leq C_{\beta_1,\beta_2}\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta_1}} \quad \text{for } \beta_1 \geq \beta_2 > 0. \tag{2.10}$$

Lemma 2.3. *Suppose $\beta > 0$. If $x^* \in \Lambda_{\mathcal{A}}^{\beta}$, then*

$$\Big\|\frac{d^n}{dt^n}\mathcal{T}_t^* x^*\Big\|_{X^*} \leq C_n \|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}} \quad \textit{for} \;\; n < \beta; \tag{2.11}$$

$$\lim_{t\to 0}\|x^* - \mathcal{T}_t^* x^*\|_{X^*} = 0. \tag{2.12}$$

Proof. The proof of (2.11) follows from integration of (1.10). To prove (2.12), there is no loss of generality if we assume that $0 < \beta < 1$. Let $0 < t_1 < t_2 < 1$. We write

$$\|\mathcal{T}_{t_2}x^* - \mathcal{T}_{t_1}^* x^*\|_{X^*} = \Big\|\int_{t_1}^{t_2}\frac{d}{ds}\mathcal{T}_s^* x^*\,ds\Big\|_{X^*} \leq C\int_{t_1}^{t_2} s^{\beta-1}\,ds = C'(t_2^\beta - t_1^\beta).$$

Hence, $\mathcal{T}_t^* x^*$ satisfies the Cauchy condition as $t \to 0$. Let $x_0^* = \lim_{t\to 0}\mathcal{T}_t^* x^*$ in the $\|\cdot\|_{X^*}$–norm. Then

$$\langle x_0^*, x\rangle = \lim_{t\to 0}\langle \mathcal{T}_t^* x^*, x\rangle = \lim_{t\to 0}\langle x^*, \mathcal{T}_t x\rangle = \langle x^*, x\rangle,$$

because $\{\mathcal{T}_t\}_{t\geq 0}$ is strongly continuous. So $x_0^* = x^*$. □

Proof of Theorem 1.6. Suppose that $x^* \in X^*$ is such that $y^* := \mathcal{A}^* x^* \in \Lambda_{\mathcal{A}}^{\beta}$. Let $m > \beta + 1$. Then $m - 1 > \beta$ and

$$\Big\|\frac{d^{m-1}}{dt^{m-1}}\mathcal{T}_t^* y^*\Big\|_{X^*} \leq t^{\beta-(m-1)}\|y^*\|_{\Lambda_{\mathcal{A}}^{\beta}}. \tag{2.13}$$

Since $\{\mathcal{T}_t\}_{t\geq 0}$ is a strongly continuous and uniformly bounded semigroup of linear operators, using (2.4) and (2.12), we get

$$\begin{aligned}
\Big\|\frac{d^m}{dt^m}\mathcal{T}_t^*x^*\Big\|_{X^*} &\leq \sup_{s>0,\,\|x\|\leq 1}\Big|\Big\langle\frac{d^m}{dt^m}\mathcal{T}_t^*x^*,\mathcal{T}_s x\Big\rangle\Big|\\
&= \sup_{s>0,\,\|x\|\leq 1}|\langle x^*,\mathcal{A}\mathcal{T}_{t/m}(\mathcal{A}\mathcal{T}_{t/m})^{m-1}\mathcal{T}_s x\rangle|\\
&= \sup_{s>0,\,\|x\|\leq 1}|\langle y^*,\mathcal{T}_{t/m}(\mathcal{A}\mathcal{T}_{t/m})^{m-1}\mathcal{T}_s x\rangle|\\
&= \sup_{s>0,\,\|x\|\leq 1}\Big|\Big\langle\frac{d^{m-1}}{dt^{m-1}}\mathcal{T}_t^*y^*,\mathcal{T}_s x\Big\rangle\Big| \leq C\|y^*\|_{\Lambda^\beta_\mathcal{A}}t^{(\beta+1)-m},
\end{aligned} \tag{2.14}$$

where in the last inequality we have used (2.13).

We now turn to prove the converse implication. Suppose that $x^*\in\Lambda^{\beta+1}_\mathcal{A}$. Let m be an integer, such that $1\leq m-1\leq\beta+1<m$. We start by proving that $\mathcal{A}^*x^*$ exists in the mild sense. Let $u(t):=\mathcal{T}_t^*x^*$, $v(t):=u'(t)=(\mathcal{A}\mathcal{T}_t)^*x^*$. Then, by our assumption,

$$\|v^{(m-1)}(t)\|_{X^*}=\|u^{(m)}(t)\|_{X^*}\leq\|x^*\|_{\Lambda^{\beta+1}_\mathcal{A}}t^{(\beta+1)-m}. \tag{2.15}$$

Consider the Taylor expansion of the function $v(t)$ around the point $t_0=1$:

$$v(t)=\sum_{\ell=0}^{m-2}\frac{1}{\ell!}v^{(\ell)}(1)(t-1)^\ell+\int_1^t\frac{(t-s)^{m-2}}{(m-2)!}v^{(m-1)}(s)\,ds. \tag{2.16}$$

It follows from (2.15) that $v(t)$ converges in the X^*-norm to a vector $y^*\in X^*$, as t tends to 0, and $\|y^*\|_{X^*}\leq C\|x^*\|_{\Lambda^{\beta+1}_\mathcal{A}}$. To see that $y^*=\mathcal{A}^*x^*$, we write

$$\langle y^*,\mathcal{T}_s x\rangle=\lim_{t\to 0}\langle v(t),\mathcal{T}_s x\rangle=\lim_{t\to 0}\langle(\mathcal{A}\mathcal{T}_t)^*x^*,\mathcal{T}_s x\rangle=\lim_{t\to 0}\langle x^*,\mathcal{A}\mathcal{T}_t\mathcal{T}_s x\rangle=\langle x^*,\mathcal{A}\mathcal{T}_s x\rangle,$$

where in the last equality we have used (2.3) and the strong continuity of $\{\mathcal{T}_t\}_{t\geq 0}$.

It suffices to verify that $y^*\in\Lambda^\beta_\mathcal{A}$. To this end, we recall that $\beta<m-1$ and, for $t>0$, applying (2.4), we obtain

$$\begin{aligned}
\Big\|\frac{d^{m-1}}{dt^{m-1}}\mathcal{T}_t^*y^*\Big\|_{X^*} &= \lim_{s\to 0}\|((\mathcal{A}\mathcal{T}_{t/(m-1)})^*)^{m-1}v(s)\|_{X^*}=\lim_{s\to 0}\|((\mathcal{A}\mathcal{T}_{t/(m-1)})^*)^{m-1}(\mathcal{A}\mathcal{T}_s)^*x^*\|_{X^*}\\
&= \lim_{s\to 0}\Big\|\frac{d^m}{d\tau^m}\mathcal{T}_\tau x^*_{\big|\tau=t+s}\Big\|_{X^*}\leq t^{(\beta+1)-m}\|x^*\|_{\Lambda^{\beta+1}_\mathcal{A}}=t^{\beta-(m-1)}\|x^*\|_{\Lambda^{\beta+1}_\mathcal{A}}.
\end{aligned}$$

□

2.3. **Bessel potentials.** For $\gamma>0$, the potential operator is defined by

$$(I-\mathcal{A})^{-\gamma}:=\Gamma(\gamma)^{-1}\int_0^\infty t^\gamma e^{-t}\mathcal{T}_t\,\frac{dt}{t} \tag{2.17}$$

and its conjugate operator

$$((I-\mathcal{A})^{-\gamma})^*:=\Gamma(\gamma)^{-1}\int_0^\infty t^\gamma e^{-t}\mathcal{T}_t^*\,\frac{dt}{t}. \tag{2.18}$$

The integrals in (2.17) and (2.18) converge in the operator norm topology and define bounded operators on X and X^* respectively. Moreover, for $\gamma_1,\gamma_2>0$,

$$(I-\mathcal{A})^{-\gamma_1}(I-\mathcal{A})^{-\gamma_2}=(I-\mathcal{A})^{-\gamma_1+\gamma_2},\quad ((I-\mathcal{A})^{-\gamma_1})^*((I-\mathcal{A})^{-\gamma_2})^*=((I-\mathcal{A})^{-\gamma_1+\gamma_2})^*. \tag{2.19}$$

Proof of Theorem 1.4. We start by proving that for any $\beta > 0$, the operator $((I-\mathcal{A})^{-\gamma})^*$ is bounded from $\Lambda^\beta_{\mathcal{A}}$ to $\Lambda^{\beta+\gamma}_{\mathcal{A}}$. Clearly, for $x^* \in \Lambda^\beta_{\mathcal{A}}$, we have

$$\|(I-\mathcal{A})^{-\gamma})^* x^*\|_{X^*} \leq C\|x^*\|_{X^*} \leq C'\|x^*\|_{\Lambda^\beta_{\mathcal{A}}}. \tag{2.20}$$

To verify that $y^* := ((I-\mathcal{A})^{-\gamma})^* x^*$ indeed belongs to $\Lambda^{\beta+\gamma}_{\mathcal{A}}$, and $\|y^*\|_{\Lambda^{\beta+\gamma}_{\mathcal{A}}} \leq C\|x^*\|_{\Lambda^\beta_{\mathcal{A}}}$, we fix $n > \beta+\gamma$ and $0 < t \leq 1$. Then, using (2.4) and our assumption, we obtain

$$\begin{aligned} \Big\|\frac{d^n}{dt^n}\mathcal{T}^*_t y^*\Big\|_{X^*} &= \Gamma(\gamma)^{-1}\Big\|\int_0^\infty s^\gamma e^{-s}((\mathcal{A}\mathcal{T}_{t/n})^*)^n \mathcal{T}^*_s x^* \frac{ds}{s}\Big\|_{X^*} \\ &= \Gamma(\gamma)^{-1}\Big\|\int_0^\infty s^\gamma e^{-s}((\mathcal{A}\mathcal{T}_{(t+s)/n})^*)^n x^* \frac{ds}{s}\Big\|_{X^*} \\ &\leq C\|x^*\|_{\Lambda^\beta_{\mathcal{A}}}\int_0^t s^\gamma e^{-s} t^{\beta-n}\frac{ds}{s} + C\|x^*\|_{\Lambda^\beta_{\mathcal{A}}}\int_t^\infty s^\gamma e^{-s} s^{\beta-n}\frac{ds}{s} \\ &\leq C'\|x^*\|_{\Lambda^\beta_{\mathcal{A}}} t^{\beta+\gamma-n}. \end{aligned} \tag{2.21}$$

Thus, we have shown that $((I-\mathcal{A})^{-\gamma})^*$ is a continuous linear transformation of $\Lambda^\beta_{\mathcal{A}}$ into $\Lambda^{\beta+\gamma}_{\mathcal{A}}$.

According to the Banach closed graph theorem, it remains to prove that the mapping $((I-\mathcal{A})^{-\gamma})^* : \Lambda^\beta_{\mathcal{A}} \to \Lambda^{\beta+\gamma}_{\mathcal{A}}$ is injective and onto. Using (2.19), by standard functional analysis arguments, it suffices to prove this for $\gamma = 1$, cf. [20, Chapter V, Section 4.4] for the proof in the case of the classical Lipschitz spaces. Since $R(1:\mathcal{A})(X) = \mathfrak{D}(\mathcal{A})$, which is a dense space in X, we conclude that $((I-\mathcal{A})^{-1})^* = R(1:\mathcal{A})^*$ is injective. To verify that $R(1:\mathcal{A})^*$ is onto, consider $x^* \in \Lambda^{\beta+1}_{\mathcal{A}}$. Then $y^* = x^* - \mathcal{A}^*x^*$ belongs to $\Lambda^\beta_{\mathcal{A}}$, where $\mathcal{A}^*x^*$ is understood in the mild sense, that is, $\langle y^*, \mathcal{T}_t x\rangle = \langle x^*, (I-\mathcal{A})\mathcal{T}_t x\rangle$. Now,

$$\begin{aligned} \langle R(1:\mathcal{A})^* y^*, \mathcal{T}_t x\rangle &= \langle y^*, R(1:\mathcal{A})\mathcal{T}_t x\rangle = \langle y^*, \mathcal{T}_t R(1:\mathcal{A})x\rangle \\ &= \langle x^*, (I-\mathcal{A})\mathcal{T}_t R(1:\mathcal{A})x\rangle = \langle x^*, \mathcal{T}_t x\rangle, \end{aligned} \tag{2.22}$$

and, consequently, $R(1:\mathcal{A})^* y^* = x^*$. □

3. Subordinate semigroup

In this section, for a holomorphic uniformly bounded semigroup $\{\mathcal{T}_t\}_{t\geq 0} = \{e^{t\mathcal{A}}\}_{t\geq 0}$ on a Banach space $(X, \|\cdot\|)$, we consider the subordinate semigroup $\{\mathcal{P}_t\}_{t\geq 0}$ defined by (1.12).

The integral in (1.12) is convergent in the operator norm topology $\mathcal{L}(X)$ and it defines a strongly continuous, uniformly bounded (in any sector $\boldsymbol{S}_\delta$, $0 < \delta < \pi/2$) holomorphic semigroup on X. Our aim is to prove Theorem 1.3. The relations

$$\Lambda^\beta_{\mathcal{A}} \subseteq \Lambda^{2\beta}_{-\sqrt{-\mathcal{A}}} \quad \text{with } \|x^*\|_{\Lambda^{2\beta}_{-\sqrt{-\mathcal{A}}}} \leq C\|x^*\|_{\Lambda^\beta_{\mathcal{A}}} \quad \text{for all } x^* \in \Lambda^\beta_{\mathcal{A}} \tag{3.1}$$

will be proved by utilizing (1.12). For the converse, namely for

$$\Lambda^{2\beta}_{-\sqrt{-\mathcal{A}}} \subseteq \Lambda^\beta_{\mathcal{A}} \quad \text{with } C^{-1}\|x^*\|_{\Lambda^\beta_{\mathcal{A}}} \leq \|x^*\|_{\Lambda^{2\beta}_{-\sqrt{-\mathcal{A}}}} \quad \text{for all } x^* \in \Lambda^{2\beta}_{-\sqrt{-\mathcal{A}}}, \tag{3.2}$$

we shall apply a holomorphic functional calculus. For the reader who is not familiar with these methods, we provide details in Section 4.

3.1. **Inclusion $\Lambda^\beta_{\mathcal A}\subseteq \Lambda^{2\beta}_{-\sqrt{-\mathcal A}}$.** We start with the following lemma.

Lemma 3.1. *Suppose that $(0,\infty)\ni t\mapsto \phi(t)$ is a bounded C^∞-function taking values in a Banach space such that for any $n\in\mathbb N$ there is $C_n>0$ such that for all $t>0$ we have*

$$\|\phi^{(n)}(t)\|\leq C_nt^{-n}. \tag{3.3}$$

Then the function

$$(0,\infty)\ni t\mapsto \psi(t):=\int_0^\infty e^{-u}\phi\Big(\frac{t^2}{4u}\Big)\frac{du}{\sqrt u}$$

is C^∞ and satisfies

$$\psi^{(2n+1)}(t)=(-1)^n\int_0^\infty e^{-u}\frac{t}{2u}\phi^{(n+1)}\Big(\frac{t^2}{4u}\Big)\frac{du}{\sqrt u},$$
$$\psi^{(2n)}(t)=(-1)^n\int_0^\infty e^{-u}\phi^{(n)}\Big(\frac{t^2}{4u}\Big)\frac{du}{\sqrt u}.$$

Proof. Clearly,

$$\psi'(t)=\int_0^\infty e^{-u}\frac{t}{2u}\phi'\Big(\frac{t^2}{4u}\Big)\frac{du}{\sqrt u}.$$

For any smooth function $(0,\infty)\ni t\mapsto g(t)$, we have

$$\frac{d}{dt}[g(t^2/(4u))]=-\frac{2u}{t}\frac{d}{du}[g(t^2/(4u))]=\frac{t}{2u}g'(t^2/(4u)). \tag{3.4}$$

Hence,

$$\begin{aligned}\psi''(t)&=\int_0^\infty e^{-u}\frac{1}{2u}\phi'\Big(\frac{t^2}{4u}\Big)\frac{du}{\sqrt u}+\int_0^\infty e^{-u}\frac{t}{2u}\frac{d}{dt}\Big[\phi'\Big(\frac{t^2}{4u}\Big)\Big]\frac{du}{\sqrt u}\\ &=\int_0^\infty e^{-u}\frac{1}{2u}\phi'\Big(\frac{t^2}{4u}\Big)\frac{du}{\sqrt u}-\int_0^\infty e^{-u}\frac{d}{du}\Big[\phi'\Big(\frac{t^2}{4u}\Big)\Big]\frac{du}{\sqrt u},\end{aligned}$$

where in the last identity we have used (3.4). Integrating by parts and using (3.3), we obtain

$$\psi''(t)=\int_0^\infty e^{-u}\frac{1}{2u}\phi'\Big(\frac{t^2}{4u}\Big)\frac{du}{\sqrt u}+\int_0^\infty\frac{d}{du}\Big[e^{-u}u^{-1/2}\Big]\phi'\Big(\frac{t^2}{4u}\Big)\,du=-\int_0^\infty e^{-u}\phi'\Big(\frac{t^2}{4u}\Big)\frac{du}{\sqrt u}.$$

The proof for higher order derivatives follows by iterating the above argument. □

Proof of the inclusion $\Lambda^\beta_{\mathcal A}\subseteq\Lambda^{2\beta}_{-\sqrt{-\mathcal A}}$. Assume that $x^*\in\Lambda^\beta_{\mathcal A}$. Consider the functions

$$\phi(t)=\Gamma(1/2)^{-1}\mathcal T_t^*x^*\quad\text{and}\quad\psi(t)=\mathcal P_t^*x^*=\int_0^\infty e^{-u}\phi\Big(\frac{t^2}{u}\Big)\frac{du}{\sqrt u},$$

which take values in X^*. If $n>\beta$ then, by Proposition 2.1, $\|\phi^{(n)}(s)\|\leq s^{\beta-n}\|x^*\|_{\Lambda^\beta_{\mathcal A}}$. Thus, applying Lemma 3.1, we get

$$\begin{aligned}\|\psi^{(2n)}(t)\|_{X^*}&\leq\int_0^\infty e^{-u}\|\phi^{(n)}(t^2/4u)\|_{X^*}\frac{du}{\sqrt u}\\ &\leq\int_0^\infty e^{-u}\|x^*\|_{\Lambda^\beta_{\mathcal A}}\Big(\frac{t^2}{4u}\Big)^{\beta-n}\frac{du}{\sqrt u}\leq Ct^{2\beta-2n}\|x^*\|_{\Lambda^\beta_{\mathcal A}},\end{aligned}$$

which proves the inclusion and the second inequality in (1.14). □

4. Holomorphic calculi

This section is devoted to the proof of the inclusion $\Lambda^{2\beta}_{-\sqrt{-\mathcal{A}}} \subseteq \Lambda^{\beta}_{\mathcal{A}}$. To this end, we utilize holomorphic functional calculi for generators of semigroups. For the reader who is not familiar with this topic, we provide full details. We refer to [11] and references therein for more results.

4.1. Admissible holomorphic functions in $\mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$.

Definition 4.1. We say that f belongs to the space of admissible holomorphic functions H^∞_a if it satisfies the following conditions:

(A) f is bounded and holomorphic on $\mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$;
(B) for all $x \leq 0$ and for all θ such that $\frac{\pi}{2} < \theta < \pi$, we have

$$\int_0^\infty |f(x + re^{\pm i\theta})|\, dr < \infty. \tag{4.1}$$

It is straightforward to verify that $f_1(z) = e^z$ and $f_2(z) = e^{-\sqrt{-z}}$ belong to H^∞_a. Here $\sqrt{z}$ denotes the branch of the square root defined on $\mathbb{C} \setminus \mathbb{R}_-$.

Let $\frac{\pi}{2} < \theta < \pi$. We define the parameterized path:

$$(-\infty, \infty) \ni t \mapsto \Gamma_\theta(t) = \begin{cases} |t|e^{-i\theta} & \text{if } t \leq 0, \\ |t|e^{i\theta} & \text{if } t > 0. \end{cases}$$

We then define the right and the left sides of Γ_θ:

$$\Gamma^+_\theta := \{\lambda + x \,:\, \lambda \in \Gamma_\theta,\ x > 0\}, \quad \Gamma^-_\theta := \{\lambda + x \,:\, \lambda \in \Gamma_\theta,\ x < 0\}. \tag{4.2}$$

For $\varepsilon > 0$ we define a modification $\Gamma_{\theta,\varepsilon}$ of the path Γ_θ by replacing its piece corresponding to the two intervals for parameters $0 \leq |t| \leq \varepsilon$ by the part of the circle $\varepsilon e^{i\omega}$, $\theta \leq |\omega| \leq \pi$, explicitly,

$$(-\infty, \infty) \ni t \mapsto \Gamma_{\theta,\varepsilon}(t) = \begin{cases} |t|e^{-i\theta} & \text{if } t \leq -\varepsilon, \\ \varepsilon e^{-it(\pi-\theta)/\varepsilon - i\pi} & \text{if } -\varepsilon < t \leq \varepsilon, \\ |t|e^{i\theta} & \text{if } t > \varepsilon. \end{cases}$$

Similarly,

$$\Gamma^+_{\theta,\varepsilon} = \{\lambda + x \,:\, \lambda \in \Gamma_{\theta,\varepsilon},\ \ x > 0,\}, \quad \Gamma^-_{\theta,\varepsilon} = \{\lambda + x \,:\, \lambda \in \Gamma_{\theta,\varepsilon},\ x < 0\}. \tag{4.3}$$

To unify our notation, we set $\Gamma_{\theta,\varepsilon} := \Gamma_\theta$ if $\varepsilon = 0$.

Lemma 4.2 (cf. [11, Lemma 8.2])**.** *For $f \in H^\infty_a$ and $\lambda \notin \Gamma_{\theta,\varepsilon}$, $\varepsilon \geq 0$, $\pi/2 < \theta < \pi$, let*

$$\widetilde{f}(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\varepsilon}} \frac{f(z)}{z - \lambda}\, dz. \tag{4.4}$$

Then $\widetilde{f}$ is holomorphic on $\mathbb{C} \setminus \Gamma_{\theta,\varepsilon}$ and

$$\widetilde{f}(\lambda) = \begin{cases} f(\lambda) & \text{if } \lambda \in \Gamma^-_{\theta,\varepsilon}, \\ 0 & \text{if } \lambda \in \Gamma^+_{\theta,\varepsilon}. \end{cases} \tag{4.5}$$

Proof. We present the proof for $\varepsilon > 0$. The proof for $\varepsilon = 0$ is obtained by letting $\varepsilon \to 0$.

Case $\lambda \in \Gamma^+_{\theta,\varepsilon}$. It follows from (4.4) that

$$\widetilde{f}'(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon}} \frac{f(z)}{(z-\lambda)^2}\,dz.$$

We will show that $\widetilde{f}'(\lambda) = 0$. To this end we fix $\lambda \in \Gamma^+_{\theta,\varepsilon}$. For $x < 0$ and $R > 2\varepsilon$, we define the closed path $\boldsymbol{\gamma}^{x,R} = \bigcup_{j=1}^4 \boldsymbol{\gamma}_j^{x,R}$, oriented counterclockwise, which consists of two curves:

$$\boldsymbol{\gamma}_1^{x,R} = \{\Gamma_{\theta,\varepsilon}(t) \,:\, t \in [-R,R]\}, \quad \boldsymbol{\gamma}_2^{x,R} = \boldsymbol{\gamma}_1{}^{x,R} + x,$$

and two line segments:

$$\boldsymbol{\gamma}_3^{x,R} = \{(Re^{i\theta} + x)(1-s) + Re^{i\theta}s : s \in [0,1]\},$$
$$\boldsymbol{\gamma}_4^{x,R} = \{(Re^{-i\theta} + x)(1-s) + Re^{-i\theta}s : s \in [0,1]\}.$$

Since the function $z \mapsto f(z)(z-\lambda)^{-2}$ is holomorphic in an open neighborhood of the compact region bounded by $\boldsymbol{\gamma}^{x,R}$, we have

$$\oint_{\boldsymbol{\gamma}^{x,R}} \frac{f(z)}{(z-\lambda)^2}\,dz = \sum_{j=1}^4 \int_{\boldsymbol{\gamma}_j^{x,R}} \frac{f(z)}{(z-\lambda)^2}\,dz = 0. \tag{4.6}$$

From the condition (A) we conclude that for each (fixed) $x < 0$, we have

$$\lim_{R\to\infty} \int_{\boldsymbol{\gamma}_j^{x,R}} \frac{f(z)}{(z-\lambda)^2}\,dz = 0 \quad \text{for } j = 3,4, \tag{4.7}$$

and, consequently, by (4.6), for each (fixed) $x < 0$, one has

$$\int_{\Gamma_{\theta,\varepsilon}} \frac{f(z)}{(z-\lambda)^2}\,dz - \int_{\Gamma_{\theta,\varepsilon}+x} \frac{f(z)}{(z-\lambda)^2}\,dz = 0. \tag{4.8}$$

Next, applying (A), we obtain

$$\lim_{x\to-\infty} \int_{\Gamma_{\theta,\varepsilon}+x} \frac{f(z)}{(z-\lambda)^2}\,dz = 0. \tag{4.9}$$

Hence, from (4.8) and (4.9) we conclude that $\widetilde{f}'(\lambda) = 0$. Therefore, $\widetilde{f}$ is constant on $\Gamma^+_{\theta,\varepsilon}$. To determine its value, applying the condition (B), we get

$$\lim_{\lambda\in\mathbb{R},\ \lambda\to\infty} \int_{\Gamma_{\theta,\varepsilon}} \frac{f(z)}{z-\lambda}\,dz = 0,$$

so $\widetilde{f}(\lambda) = 0$ on $\Gamma^+_{\theta,\varepsilon}$.

By the same argument we also conclude that

$$\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon}+x} \frac{f(z)}{z-\lambda}\,dz = 0 \quad \text{for all } x < 0 \quad \text{and} \quad \lambda \in (\Gamma_{\theta,\varepsilon}+x)^+. \tag{4.10}$$

Case $\lambda \in \Gamma^-_{\theta,\varepsilon}$. Let us consider $\boldsymbol{\gamma}^{x,R}$ as above such that λ stays inside the bounded region with the boundary $\boldsymbol{\gamma}^{x,R}$. The Cauchy integral formula asserts that

$$f(\lambda) = \frac{1}{2\pi i}\oint_{\boldsymbol{\gamma}^{x,R}} \frac{f(z)}{z-\lambda}\,dz = \sum_{j=1}^4 \frac{1}{2\pi i}\int_{\boldsymbol{\gamma}_j^{x,R}} \frac{f(z)}{z-\lambda}\,dz. \tag{4.11}$$

From the condition (A), we obtain

$$\lim_{R\to\infty}\int_{\gamma_j^{x,R}}\frac{f(z)}{z-\lambda}\,dz=0\quad\text{for}\quad j=3,4. \tag{4.12}$$

Hence, letting $R\to\infty$ in (4.11), we obtain

$$f(\lambda)=\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon}}\frac{f(z)}{z-\lambda}\,dz-\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon}+x}\frac{f(z)}{z-\lambda}\,dz. \tag{4.13}$$

But, for $x<0$, $|x|$ large enough (such that λ stays on the right side of the curve $\Gamma_{\theta,\varepsilon}+x$), (4.10) gives

$$\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon}+x}\frac{f(z)}{z-\lambda}\,dz=0. \tag{4.14}$$

Consequently, (4.13) and (4.14) yield the identity $\widetilde{f}(\lambda)=f(\lambda)$ for $\lambda\in\Gamma^-_{\theta,\varepsilon}$. □

4.2. **Properties of resolvent.** In this subsection, we assume that $\mathcal{A}$ is a closed operator on a Banach space such that

$$\rho(\mathcal{A})\supseteq\Sigma_\delta\cup\{0\}=\{z\in\mathbb{C}\;:\;|\arg(z)|<\frac{\pi}{2}+\delta\}\cup\{0\} \tag{4.15}$$

and there is $C>0$ such that for all $\lambda\in\Sigma_\delta$, $\lambda\neq 0$, we have

$$\|R(\lambda:\mathcal{A})\|\leq\frac{C}{|\lambda|}. \tag{4.16}$$

Since $0\in\rho(\mathcal{A})$, there is $r>0$ such that $B(0,r)=\{z\in\mathbb{C}:|z|<r\}\subseteq\rho(\mathcal{A})$ and

$$\sup_{z\in B(0,r)}\|R(z:\mathcal{A})\|<\infty. \tag{4.17}$$

Lemma 4.3. *Let $\pi<\theta<2\pi$ and $\varepsilon\geq 0$ be such that $\Gamma_{\theta,\varepsilon}\subseteq\Sigma_\delta\cup B(0,r)$. Suppose $z\in\Gamma^+_{\theta,\varepsilon}$. Then*

$$R(z:\mathcal{A})=\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon}}\frac{1}{z-\lambda}R(\lambda:\mathcal{A})\,d\lambda. \tag{4.18}$$

In addition, if $z_1,z_2\in\Gamma^+_{\theta,\varepsilon}$, then

$$R(z_1:\mathcal{A})R(z_2:\mathcal{A})=\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon}}\frac{1}{(z_1-\lambda)(z_2-\lambda)}R(\lambda:\mathcal{A})\,d\lambda. \tag{4.19}$$

Proof. We begin by proving (4.18). Let $z\in\Gamma^+_{\theta,\varepsilon}$. Let us consider the following parameterized (closed) contour,

$$(-R,2R)\ni t\mapsto\Gamma_{\theta,\varepsilon,R}(t)=\begin{cases}\Gamma_{\theta,\varepsilon}(t)=:\boldsymbol{\gamma}_{R,1}(t) & \text{if } -R<t\leq R,\\ Re^{-2it\theta/R+i3\theta}=:\boldsymbol{\gamma}_{R,2}(t) & \text{if } R\leq t<2R.\end{cases}$$

The contour $\Gamma_{\theta,\varepsilon,R}$ is oriented clockwise. Hence, taking R large enough and using the Cauchy formula, we get:

$$\begin{aligned}R(z:\mathcal{A})&=-\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon,R}}\frac{1}{\lambda-z}R(\lambda:\mathcal{A})\,d\lambda\\&=\frac{1}{2\pi i}\int_{\gamma_{R,1}}\frac{1}{z-\lambda}R(\lambda:\mathcal{A})\,d\lambda+\frac{1}{2\pi i}\int_{\gamma_{R,2}}\frac{1}{z-\lambda}R(\lambda:\mathcal{A})\,d\lambda.\end{aligned} \tag{4.20}$$

By virtue of (4.16) we have

$$\lim_{R\to\infty}\left|\frac{1}{2\pi i}\int_{\gamma_{R,2}}\frac{1}{z-\lambda}R(\lambda:\mathcal{A})\,d\lambda\right|\leq\lim_{R\to\infty}\frac{C}{R}=0.$$

Moreover, by the definition of $\Gamma_{\theta,\varepsilon}$, we have

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma_{R,1}}\frac{1}{z-\lambda}R(\lambda:\mathcal{A})\,d\lambda=\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon}}\frac{1}{z-\lambda}R(\lambda:\mathcal{A})\,d\lambda.$$

Therefore, (4.18) follows from (4.20) by letting $R\to\infty$.

In order to prove (4.19), we use (2.6) together with the identity

$$(z_1-\lambda)^{-1}(z_2-\lambda)^{-1}(z_2-z_1)=(z_1-\lambda)^{-1}-(z_2-\lambda)^{-1}$$

and apply the first part of the theorem. □

4.3. **Holomorphic calculi.** Let $\mathcal{A}$ be a closed linear operator satisfying (4.15) and (4.16). Let $r>0$ be such that (4.17) holds. For $f\in H^\infty_a$, let

$$f(\mathcal{A}):=\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon}}f(\lambda)R(\lambda:\mathcal{A})\,d\lambda, \tag{4.21}$$

where $\Gamma_{\theta,\varepsilon}\subseteq\Sigma_\delta\cup B(0,r)$, $\pi/2<\theta<\pi/2+\delta$. It follows from (B), (4.16), and (4.17) that the integral (4.21) converges absolutely and defines a bounded operator on X. The next proposition asserts that the integral does not depend on the path $\Gamma_{\theta,\varepsilon}$.

Proposition 4.4. *Suppose $\mathcal{A}$ satisfies (4.15) and (4.16). Let $\Gamma_{\theta_1,\varepsilon_1},\Gamma_{\theta_2,\varepsilon_2}\subseteq\Sigma_\delta\cup B(0,r)$, $\varepsilon_1,\varepsilon_2\geq 0$. Then*

$$\frac{1}{2\pi i}\int_{\Gamma_{\theta_1,\varepsilon_1}}f(\lambda)R(\lambda:\mathcal{A})\,d\lambda=\frac{1}{2\pi i}\int_{\Gamma_{\theta_2,\varepsilon_2}}f(\lambda)R(\lambda:\mathcal{A})\,d\lambda. \tag{4.22}$$

Proof. By holomorphy, for $\Gamma_{\theta,\varepsilon}$, $\Gamma_{\theta,\varepsilon'}\subseteq\Sigma_\delta\cup B(0,r)$, we have

$$\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon}}f(\lambda)R(\lambda:\mathcal{A})\,d\lambda=\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon'}}f(\lambda)R(\lambda:\mathcal{A})\,d\lambda. \tag{4.23}$$

Consider $\pi/2<\theta_1<\theta_2<\pi/2+\delta$. Let $0\leq\varepsilon_1<\varepsilon_2$ be such that $\Gamma_{\theta_1,\varepsilon_1},\Gamma_{\theta_2,\varepsilon_2}\subseteq\Sigma_\delta\cup B(0,r)$. Then, using (4.18), we write

$$\frac{1}{2\pi i}\int_{\Gamma_{\theta_1,\varepsilon_1}}f(\lambda)R(\lambda:\mathcal{A})\,d\lambda=\frac{1}{2\pi i}\int_{\Gamma_{\theta_1,\varepsilon_1}}f(\lambda)\,\frac{1}{2\pi i}\int_{\Gamma_{\theta_2,\varepsilon_2}}\frac{1}{\lambda-z}R(z:\mathcal{A})\,dz\,d\lambda. \tag{4.24}$$

Since the double integral on the right-side of (4.24) is absolutely convergent, we apply Fubini's theorem together with Lemma 4.2 and get the proposition. □

Proposition 4.5 (cf. [11, Theorems 8.3 and 9.6])**.** *Suppose $f,g\in H^\infty_a$. Then*

$$(f\cdot g)(\mathcal{A})=f(\mathcal{A})g(\mathcal{A}). \tag{4.25}$$

Proof. It is clear that if $f,g\in H_a^\infty$, then $f\cdot g\in H_a^\infty$. Fix $\pi/2<\theta_1<\theta_2<\theta<\pi$ and $0\leq \varepsilon_1<\varepsilon_2<\varepsilon$, such that $\Gamma_{\theta,\varepsilon}\subseteq \Sigma_\delta\cup B(0,r)$. Then, using Lemma 4.2, we get

$$(4.26)\qquad \begin{aligned}(f\cdot g)(\mathcal{A})&=\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon}} f(\lambda)g(\lambda)R(\lambda:\mathcal{A})\,d\lambda\\ &=\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon}}\Big(\frac{1}{2\pi i}\int_{\Gamma_{\theta_1,\varepsilon_1}}\frac{f(z_1)}{z_1-\lambda}\,dz_1\Big)\Big(\frac{1}{2\pi i}\int_{\Gamma_{\theta_2,\varepsilon_2}}\frac{g(z_2)}{z_2-\lambda}\,dz_2\Big)R(\lambda:\mathcal{A})\,d\lambda.\end{aligned}$$

Since the triple integral is absolutely convergent, we utilize Fubini's theorem together with (4.19) and obtain (4.25). □

Remark 4.6. Since the mapping $\mathcal{L}(X)\ni B\mapsto B^*\in\mathcal{L}(X^*)$ is an isometric injection and the integral (4.21) converges absolutely, we deduce that

$$(4.27)\qquad f(\mathcal{A})^*=\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon}} f(\lambda)R(\lambda:\mathcal{A})^*\,d\lambda,$$

provided $\Gamma_{\theta,\varepsilon}\subseteq\Sigma_\delta\cup B(0,r)$ and $f\in H_a^\infty$.

4.4. **Subordinate semigroups and holomorphic calculi.** Recall that for a uniformly bounded and strongly continuous semigroup $\{\mathcal{T}_t\}_{t\geq 0}$ its subordinate semigroup $\{\mathcal{P}_t\}_{t\geq0}$ is defined by (1.12). For the convenience of the reader, we provide a short proof that $\{\mathcal{P}_t\}_{t\geq0}$ is obtained by the holomorphic calculi.

Lemma 4.7. *Let $\{\mathcal{T}_t\}_{t\geq0}$ be an analytic, uniformly bounded semigroup on a Banach space $(X,\|\cdot\|)$. Let $\mathcal{A}$ denote its generator. Assume that $0\in\rho(\mathcal{A})$. There exists $\frac{\pi}{2}<\delta<\pi$ such that for all $\frac{\pi}{2}<\theta<\delta$ we have*

$$(4.28)\qquad \mathcal{P}_t=\frac{1}{2\pi i}\int_{\Gamma_\theta} e^{-t\sqrt{-\lambda}}R(\lambda:\mathcal{A})\,d\lambda.$$

Proof. Let $\delta,r>0$ be such that the conditions (4.15), (4.16), and (4.17) hold. Then [13, Theorem 2.5.2] asserts that

$$(4.29)\qquad \mathcal{T}_t=\frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}R(\lambda:\mathcal{A})\,d\lambda.$$

Substituting (4.29) into (1.12), we obtain

$$(4.30)\qquad \mathcal{P}_t=\frac{1}{\Gamma(1/2)2\pi i}\int_0^\infty\int_{\Gamma_\theta} e^{-u}e^{\frac{t^2}{4u}\lambda}R(\lambda:\mathcal{A})\frac{1}{\sqrt{u}}\,d\lambda\,du.$$

It is not difficult to check that the double integral (4.30) is absolutely convergent. Hence, by Fubini's theorem, we get

$$\mathcal{P}_t=\frac{1}{\Gamma(1/2)2\pi i}\int_{\Gamma_\theta}\int_0^\infty e^{-u}e^{-\frac{t^2}{4u}(-\lambda)}\frac{1}{\sqrt{u}}\,du\,R(\lambda:\mathcal{A})\,d\lambda=\frac{1}{2\pi i}\int_{\Gamma_\theta} e^{-t\sqrt{-\lambda}}R(\lambda:\mathcal{A})\,d\lambda.$$

□

Corollary 4.8. *Under the assumptions of Lemma 4.7, for all non-negative integers n and $0\leq\varepsilon<\varepsilon_0$, where ε_0 is sufficiently small, we have*

$$(4.31)\qquad \frac{d^n}{dt^n}\mathcal{P}_t=\frac{1}{2\pi i}\int_{\Gamma_{\theta,\varepsilon}}(-\sqrt{-\lambda})^n e^{-t\sqrt{-\lambda}}R(\lambda:\mathcal{A})\,d\lambda.$$

Remark 4.9. *The formulae* (4.28), (4.29), *and* (4.31) *hold for* $\mathcal{T}_t^*$ *and* $\mathcal{P}_t^*$ *by replacing* $R(\lambda:\mathcal{A})$ *by* $R(\lambda:\mathcal{A})^*$.

Proposition 4.10. *Assume that* $\mathcal{A}$ *is an infinitesimal generator of a uniformly bounded holomorphic semigroup* $\{\mathcal{T}_t\}_{t\geq 0}$ *in a sector satisfying* (4.15) *and* (4.16). *Fix a positive integer* n. *Then there is a constant* $C_n>0$, *which depends only on* n, δ, *and the constant* C *in* (4.16), *such that for all* $x^*\in X^*$, *if* $\eta(t)=\mathcal{T}_t^*x^*$ *and* $\zeta(t)=\mathcal{P}_t^*x^*$, *then*

$$\|\eta^{(n)}(t)\|_{X^*}\leq C_n t^{-n/2}\|\zeta^{(n)}(\sqrt{t})\|_{X^*}. \tag{4.32}$$

Proof. Using (4.29), the holomorphic calculus (Proposition 4.5), and Corollary 4.8, we have

$$\begin{aligned}\eta^{(n)}(t)&=\frac{1}{2\pi i}\int_{\Gamma_\theta}\lambda^n e^{t\lambda}R(\lambda:\mathcal{A})^*x^*\,d\lambda\\&=(-1)^n\frac{1}{2\pi i}\int_{\Gamma_\theta}(-\sqrt{-\lambda})^n e^{t\lambda+\sqrt{t}\sqrt{-\lambda}}(-\sqrt{-\lambda})^n e^{-t\sqrt{-\lambda}}R(\lambda:\mathcal{A})^*x^*\,d\lambda\\&=t^{-n/2}m(t\mathcal{A})^*\zeta^{(n)}(t),\end{aligned} \tag{4.33}$$

where $m(\lambda)=(\sqrt{-\lambda})^n e^{\lambda+\sqrt{-\lambda}}$. Observe that $m(t\cdot)\in H_a^\infty$ and $\|m(t\mathcal{A})\|\leq C_n$, where C_n depends on $n\geq 1$ and the constant C in (4.16), but is independent of $t>0$. □

4.5. **Proof of the inclusion** $\Lambda^{2\beta}_{-\sqrt{-\mathcal{A}}}\subseteq\Lambda^{\beta}_{\mathcal{A}}$**.** In this subsection we complete the proof of Theorem 1.3. We assume that $\mathcal{A}$ is an infinitesimal generator of a uniformly bounded holomorphic c_0-semigroup $\{\mathcal{T}_t\}_{t\geq 0}$ on a Banach space $(X,\|\cdot\|)$. Thus, $\lambda I-\mathcal{A}$ is an invertible operator for $\lambda\in\Sigma_\delta\setminus\{0\}$ and

$$\|R(\lambda:\mathcal{A})\|\leq\frac{C}{|\lambda|}\quad\text{for }\lambda\in\Sigma_\delta\setminus\{0\}.$$

Let us remark that we do not assume that $0\in\rho(\mathcal{A})$.

Consider the approximation semigroups $\mathcal{T}_t^\omega=e^{-\omega t}\mathcal{T}_t$, $0<\omega<1$, generated by $\mathcal{A}_\omega=\mathcal{A}-\omega I$ and the associated Poisson semigroups

$$\mathcal{P}_t^\omega=\Gamma(1/2)^{-1}\int_0^\infty e^{-u}\mathcal{T}^\omega_{t^2/4u}\frac{du}{\sqrt{u}}.$$

Clearly, $0\in\rho(\mathcal{A}_\omega)$, $R(\lambda:\mathcal{A}_\omega)=R(\omega+\lambda:\mathcal{A})$ and assuming that $0<\delta<\pi/2$, we have

$$\|R(\lambda:\mathcal{A}_\omega)\|\leq\frac{C'}{|\lambda|}\quad\text{for }\lambda\in\Sigma_\delta\setminus\{0\} \tag{4.34}$$

with C' independent of $\omega\in(0,1)$.

It is not difficult to prove that for each non-negative integer n,

$$\lim_{\omega\to 0}\frac{d^n}{dt^n}\mathcal{T}_t^\omega=\frac{d^n}{dt^n}\mathcal{T}_t,\quad\lim_{\omega\to 0}\frac{d^n}{dt^n}\mathcal{P}_t^\omega=\frac{d^n}{dt^n}\mathcal{P}_t \tag{4.35}$$

in the operator norm topology uniformly for t being in any compact subset of $(0,\infty)$. The same convergences hold in the operator norm topology of X^* for $\mathcal{T}_t^{\omega *}$, $\mathcal{T}_t^*$, $\mathcal{P}_t^{\omega *}$, and $\mathcal{P}_t^*$.

Proof of the inclusion $\Lambda^{2\beta}_{-\sqrt{-\mathcal{A}}}\subseteq\Lambda^{\beta}_{\mathcal{A}}$ *and the first inequality in* (1.14). Suppose $x^*\in\Lambda^{2\beta}_{-\sqrt{-\mathcal{A}}}$. Fix a positive integer $n>2\beta$. Then, for all $t>0$,

$$\Big\|\frac{d^n}{dt^n}\mathcal{P}_t^*x^*\Big\|_{X^*}\leq t^{2\beta-n}\|x^*\|_{\Lambda^{2\beta}_{-\sqrt{-\mathcal{A}}}}. \tag{4.36}$$

Set $\eta_\omega(t) = \mathcal{T}_t^{\omega *}x^*$, $\zeta_\omega(t) = \mathcal{P}_t^{\omega *}x^*$. Then

$$\lim_{\omega\to 0} \eta_\omega^{(n)}(t) = \frac{d^n}{dt^n}\mathcal{T}_t^* x^*, \tag{4.37}$$

$$\lim_{\omega\to 0} \zeta_\omega^{(n)}(t) = \frac{d^n}{dt^n}\mathcal{P}_t^* x^*, \tag{4.38}$$

where the convergences are in the norm in X^* and are uniform on any interval $[a, b] \subset (0, \infty)$. Applying Proposition 4.10, we have

$$\|\eta_\omega^{(n)}\|_{X^*} \le C_n \|\zeta_\omega^{(n)}(\sqrt{t})\|_{X^*}, \quad \omega \in (0, 1),\ t > 0. \tag{4.39}$$

Now, from (4.37) and (4.39), we conclude that

$$\begin{aligned}\Big\|\frac{d^n}{dt^n}\mathcal{T}_t^* x^*\Big\|_{X^*} &= \lim_{\omega\to 0}\Big\|\eta_\omega^{(n)}(t)\Big\|_{X^*} \le \lim_{\omega\to 0} CC_n t^{-n/2}\|\zeta_\omega^{(n)}(\sqrt{t})\|_{X^*}\\ &\le CC_n t^{-n/2}\sqrt{t}^{2\beta-n}\|x^*\|_{\Lambda^{2\beta}_{-\sqrt{-\mathcal{A}}}} = CC_n t^{\beta-n}\|x^*\|_{\Lambda^{2\beta}_{-\sqrt{-\mathcal{A}}}}\end{aligned}$$

where in the last inequality we have used (4.38) and (4.36). The required inclusion and the first inequality in (1.14) is established. □

5. $\Lambda^\beta_{\mathcal{A}}$-spaces associated with special forms of operators

Let $\{\mathcal{T}_t\}_{t\ge 0}$ be a uniformly bounded strongly continuous analytic semigroup on a Banach space $(X, \|\cdot\|)$. We introduce the space of test vectors $\mathfrak{D} = \{\mathcal{T}_t x : x \in X,\ t > 0\}$, which, by the strong continuity, is dense in X. We assume that its infinitesimal generator $\mathcal{A}$ has the following special form:

$$\mathcal{A}\mathcal{T}_t x = \sum_{j=1}^{n} \epsilon_j \mathcal{D}_j\mathcal{D}_j\mathcal{T}_t x, \quad t > 0,\ x \in X, \tag{5.1}$$

where $\epsilon_j \in \mathbb{C}$, $n \in \mathbb{N}$, and the linear operators $\mathcal{D}_j : \mathfrak{D} \to \mathfrak{D}$ satisfy

$$\mathcal{D}_j\mathcal{T}_{t+s}x = \mathcal{T}_t\mathcal{D}_j\mathcal{T}_s x, \quad x \in X; \tag{5.2}$$

$$\|\mathcal{D}_j\mathcal{T}_t x\| \le C_j t^{-1/2}\|x\|, \quad x \in X. \tag{5.3}$$

The conditions (5.2) and (5.3) imply that the mappings $(0, \infty) \ni t \mapsto \mathcal{D}_j\mathcal{T}_t \in \mathcal{L}(X)$ are C^∞-functions of t, and

$$\frac{d}{dt}\mathcal{D}_j\mathcal{T}_t = \mathcal{D}_j\mathcal{T}_{t_1}\mathcal{A}\mathcal{T}_{t_2} = \mathcal{A}\mathcal{T}_{t_2}\mathcal{D}_j\mathcal{T}_{t_1}, \quad t_1 + t_2 = t.$$

The same conclusions hold for $(\mathcal{D}_j\mathcal{T}_t)^*$, namely for all $1 \le j \le n$ and all $t, s > 0$, we have

$$(\mathcal{D}_j\mathcal{T}_{t+s})^* = \mathcal{T}_t^*(\mathcal{D}_j\mathcal{T}_s)^*, \tag{5.4}$$

$$\|(\mathcal{D}_j\mathcal{T}_t)^*\|_{X^*} \le C_j t^{-1/2}, \tag{5.5}$$

$$\frac{d}{dt}(\mathcal{D}_j\mathcal{T}_t)^* = (\mathcal{D}_j\mathcal{T}_{t_1})^*(\mathcal{A}\mathcal{T}_{t_2})^* = (\mathcal{A}\mathcal{T}_{t_2})^*(\mathcal{D}_j\mathcal{T}_{t_1})^*, \quad t_1 + t_2 = t. \tag{5.6}$$

For $x^* \in X^*$, we say that $\mathcal{D}_j^* x^*$ belongs to X^* in the mild sense, if there is $y_j^* \in X^*$ such that

$$\langle y_j^*, \mathcal{T}_t x\rangle = \langle x^*, \mathcal{D}_j\mathcal{T}_t x\rangle \tag{5.7}$$

for all $x \in X$ and $t > 0$. Then we write $y_j^* = \mathcal{D}_j^* x^*$.

Example 5.1. *On $\mathbb{R}^2$, consider the operator $\mathcal{A} = -(\partial_1)^4 + (\partial_2)^2$. It is a generator of an analytic semigroup $\{\mathcal{T}_t\}_{t\geq 0}$ on $L^2(d\mathbf{x})$ such that $\mathcal{T}_t f = f * k_t$, where $\hat{k}_t(\xi) = e^{-t\xi_1^4 + t\xi_2^2}$, $\xi = (\xi_1, \xi_2)$. One can prove that the operator $\mathcal{A}$ and the semigroup $\{\mathcal{T}_t\}_{t\geq 0}$ fit the framework described above. Moreover, for $0 < \beta < 1$, one can check that $f \in \Lambda_{\mathcal{A}}^{\beta}$ if and only if there exists a constant $C > 0$ such that for all $\mathbf{x}, h \in \mathbb{R}^2$, $h = (h_1, h_2)$ we have*

$$|f(\mathbf{x}) - f(\mathbf{x} + h)| \leq C(|h_1|^{\beta} + |h_2|^{\beta/2}).$$

Our aim is to prove the following theorem.

Theorem 5.1. *Assume that $\beta > 1/2$. Then $x^* \in \Lambda_{\mathcal{A}}^{\beta}$ if and only if $\mathcal{D}_j^* x^* \in \Lambda_{\mathcal{A}}^{\beta-1/2}$ for all $1 \leq j \leq n$, where the action of $\mathcal{D}_j^*$ on x^* is understood in the mild sense. Moreover, there is a constant $C > 1$ such that for all $x^* \in X^*$, we have*

$$C^{-1}\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}} \leq \|x^*\|_{X^*} + \sum_{j=1}^{n} \|\mathcal{D}_j^* x^*\|_{\Lambda_{\mathcal{A}}^{\beta-1/2}} \leq C\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}}.$$

Proof. The proof mimics that of Theorem 1.6. Assume that $x^* \in \Lambda_{\mathcal{A}}^{\beta}$. Fix a positive integer $m > \beta$ and $1 \leq j \leq n$. Consider the C^∞ function

$$(0, \infty) \ni t \mapsto v_j(t) = (\mathcal{D}_j \mathcal{T}_t)^* x^* \in X^*.$$

Then

$$\|v_j^{(m)}(t)\|_{X^*} \leq C\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}} t^{\beta - m - 1/2}. \tag{5.8}$$

We write the Taylor expansion of v_j at $t_0 = 1$:

$$v_j(t) = \sum_{\ell=0}^{m-1} \frac{1}{\ell!} v_j^{\ell}(1)(t-1)^{\ell} + \int_1^t \frac{(t-s)^{m-1}}{(m-1)!} v_j^{(m)}(s)\, ds.$$

It follows from (5.8) that $v_j(t)$ converges in the X^*-norm, as t tends to 0, to a vector which we denote by y_j^*. To prove that $y_j^* \in \Lambda_{\mathcal{A}}^{\beta-1/2}$, we write

$$\begin{aligned}
\Big\|\frac{d^m}{dt^m}\mathcal{T}_t^* y_j^*\Big\|_{X^*} &= \sup_{\|x\|=1} \Big|\Big\langle y_j^*, \frac{d^m}{dt^m}\mathcal{T}_t x\Big\rangle\Big| = \sup_{\|x\|=1} \lim_{s\to 0} \Big|\Big\langle (\mathcal{D}_j\mathcal{T}_s)^* x^*, \frac{d^m}{dt^m}\mathcal{T}_t x\Big\rangle\Big| \\
&= \sup_{\|x\|=1} \lim_{s\to 0} |\langle x^*, \mathcal{A}^m \mathcal{T}_{t/2}\mathcal{D}_j\mathcal{T}_{t/2}\mathcal{T}_s x\rangle| \\
&\leq \sup_{\|x\|=1} \limsup_{s\to 0} \|(\mathcal{A}^m\mathcal{T}_{t/2})^* x^*\|_{X^*} \|\mathcal{D}_j\mathcal{T}_{t/2}\mathcal{T}_s x\| \leq \|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}} \Big(\frac{t}{2}\Big)^{\beta-m} \Big(\frac{t}{2}\Big)^{-1/2},
\end{aligned}$$

where in the last inequality we have used (5.5). Hence, the claim is proved, since $m > \beta - 1/2$.

We now prove the converse implication. Suppose $y_j^* = \mathcal{D}_j^* x^* \in \Lambda_{\mathcal{A}}^{\beta-1/2}$, $j = 1, 2, ..., n$. Fix $m > \beta + 1/2$. Then,

$$\Big\|\frac{d^m}{dt^m}\mathcal{T}_t^* x^*\Big\|_{X^*} = \sup_{\|x\|=1} |\langle x^*, \mathcal{A}^m \mathcal{T}_t x\rangle| = \sup_{\|x\|=1} |\langle x^*, \sum_{j=1}^n \epsilon_j \mathcal{D}_j \mathcal{T}_{t/2} \mathcal{A}^{m-1} \mathcal{D}_j \mathcal{T}_{t/2} x\rangle|$$
$$\le \sum_{j=1}^n \sup_{\|x\|=1} |\langle y_j^*, \mathcal{A}^{m-1} \mathcal{T}_{t/2} \mathcal{D}_j \mathcal{T}_{t/2} x\rangle| \le \sum_{j=1}^n \sup_{\|x\|=1} |\langle (\mathcal{A}^{m-1}\mathcal{T}_{t/2})^* y_j^*, \mathcal{D}_j \mathcal{T}_{t/2} x\rangle|$$
$$\le \sum_{j=1}^n \|y_j^*\|_{\Lambda_{\mathcal{A}}^{\beta-1}} 2^m t^{\beta-1/2-(m-1)} C_j t^{-1/2} = \Big(2^m \sum_{j=1}^n C_j \|y_j^*\|_{\Lambda_{\mathcal{A}}^{\beta-1}}\Big) t^{\beta-m}.$$

□

6. Interpolation

For $0 < \beta_0 < \beta_1$, consider the interpolation couple $\{\Lambda_{\mathcal{A}}^{\beta_0}, \Lambda_{\mathcal{A}}^{\beta_1}\}$. Clearly, by (2.10), $\Lambda_{\mathcal{A}}^{\beta_0} + \Lambda_{\mathcal{A}}^{\beta_1} = \Lambda_{\mathcal{A}}^{\beta_0}$. For $0 < t < \infty$, set

$$K(t, x^*) := \inf_{x^* = x_0^* + x_1^*} (\|x_0^*\|_{\Lambda_{\mathcal{A}}^{\beta_0}} + t\|x_1^*\|_{\Lambda_{\mathcal{A}}^{\beta_1}}), \quad x_0^* \in \Lambda_{\mathcal{A}}^{\beta_0},\ x_1^* \in \Lambda_{\mathcal{A}}^{\beta_1}. \tag{6.1}$$

Since $\Lambda_{\mathcal{A}}^{\beta_1} \subseteq \Lambda_{\mathcal{A}}^{\beta_0}$, we conclude that there exists a constant $c \in (0, 1]$ such that

$$c\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta_0}} \le K(t, x^*) \le \|x^*\|_{\Lambda_{\mathcal{A}}^{\beta_0}} \quad \text{for all } t \ge 1 \text{ and } x^* \in \Lambda_{\mathcal{A}}^{\beta_0}. \tag{6.2}$$

Indeed, there is $0 < c \le 1$ such that if $x^* = x_0^* + x_1^*$, $x_0^* \in \Lambda_{\mathcal{A}}^{\beta_0}$, $x_1^* \in \Lambda_{\mathcal{A}}^{\beta_1}$, then

$$\|x_0^*\|_{\Lambda_{\mathcal{A}}^{\beta_0}} + t\|x_1^*\|_{\Lambda_{\mathcal{A}}^{\beta_1}} \ge \|x_0^*\|_{\Lambda_{\mathcal{A}}^{\beta_0}} + c\|x_1^*\|_{\Lambda_{\mathcal{A}}^{\beta_0}} \ge c\|x_0^* + x_1^*\|_{\Lambda_{\mathcal{A}}^{\beta_0}} = c\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta_0}},$$

which proves $K(t, x^*) \ge c\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta_0}}$ for $t \ge 1$. The second inequality in (6.2) is obvious, by taking $x_0^* = x^*$ and $x_1^* = 0$.

For $0 < \theta < 1$, the interpolation intermediate space defined by the K-method of Peetre is given by

$$(\Lambda_{\mathcal{A}}^{\beta_0}, \Lambda_{\mathcal{A}}^{\beta_1})_\theta = (\Lambda_{\mathcal{A}}^{\beta_0}, \Lambda_{\mathcal{A}}^{\beta_1})_{\theta,\infty} = \{x^* \in \Lambda_{\mathcal{A}}^{\beta_0} : \sup_{t>0} t^{-\theta} K(t, x^*) =: \|x^*\|_\theta < \infty\}, \tag{6.3}$$

see e.g., [2, Chapter III], [23, Section 1.3.3].

Proof of Theorem 1.7. We start by proving the relations

$$(\Lambda_{\mathcal{A}}^{\beta_0}, \Lambda_{\mathcal{A}}^{\beta_1})_\theta \subseteq \Lambda_{\mathcal{A}}^{\beta}, \quad \|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}} \le C\|x^*\|_\theta. \tag{6.4}$$

Let $x^* \in (\Lambda_{\mathcal{A}}^{\beta_0}, \Lambda_{\mathcal{A}}^{\beta_1})_\theta$. Then,

$$\|x^*\|_\theta \ge K(1, x^*) \ge C\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta_0}} \ge C'\|x^*\|_{X^*}. \tag{6.5}$$

Fix a positive integer $m > \beta_1$. It suffices to prove that there is $C > 0$ such that, for all $t > 0$, we have

$$\|(\mathcal{A}\mathcal{T}_{t/m})^{*m} x^*\|_{X^*} \le C t^{\beta-m} \|x^*\|_\theta. \tag{6.6}$$

To this end, we consider $K(t^{\beta_1-\beta_0}, x^*)$. By the definition of $K(t^{\beta_1-\beta_0}, x^*)$, there are $x_0^* \in \Lambda_{\mathcal{A}}^{\beta_0}$ and $x_1^* \in \Lambda_{\mathcal{A}}^{\beta_1}$ such that $x^* = x_0^* + x_1^*$ and

$$t^{-(\beta_1-\beta_0)\theta}(\|x_0^*\|_{\Lambda_{\mathcal{A}}^{\beta_0}} + t^{\beta_1-\beta_0}\|x_1^*\|_{\Lambda_{\mathcal{A}}^{\beta_1}}) \le 2\|x^*\|_\theta. \tag{6.7}$$

Consequently,

$$\begin{aligned}
\|(\mathcal{A}\mathcal{T}_{t/m})^{*m}x^*\|_{X^*} &\leq \|(\mathcal{A}\mathcal{T}_{t/m})^{*m}x_0^*\|_{X^*} + \|(\mathcal{A}\mathcal{T}_{t/m})^{*m}x_1^*\|_{X^*}\\
&\leq t^{\beta_0-m}\|x_0^*\|_{\Lambda_{\mathcal{A}}^{\beta_0}} + t^{\beta_1-m}\|x_1^*\|_{\Lambda_{\mathcal{A}}^{\beta_1}}\\
&= t^{\beta-m}(t^{-(\beta_1-\beta_0)\theta}\|x_0^*\|_{\Lambda_{\mathcal{A}}^{\beta_0}} + t^{(\beta_1-\beta_0)(1-\theta)}\|x_1^*\|_{\Lambda_{\mathcal{A}}^{\beta_1}})\\
&\leq 2t^{\beta-m}\|x^*\|_\theta,
\end{aligned} \tag{6.8}$$

where in the last inequality we have used (6.7). Thus, (6.4) follows from (6.5) and (6.8).

We now turn to prove the inverse relations to (6.4). Let $x^* \in \Lambda_{\mathcal{A}}^{\beta}$. Our first goal, for any fixed $0 < t < 1$, is to decompose

$$x^* = x_0^* + x_1^*, \quad t^{-\theta}(\|x_0^*\|_{\Lambda_{\mathcal{A}}^{\beta_0}} + t\|x_1^*\|_{\Lambda_{\mathcal{A}}^{\beta_1}}) \leq C\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}} \tag{6.9}$$

with a constant $C > 0$ independent of $x^* \in X^*$ and $0 < t < 1$. Set $v(s) := \mathcal{T}_s^* x^*$. Let m be the smallest integer satisfying $m \geq \beta$. Let $\tau = t^{1/(\beta_1-\beta_0)}$. Using the Taylor expansion of $v(s)$ at τ, we get

$$x^* = v(0) = \Big\{\sum_{\ell=0}^{m-1}\frac{1}{\ell!}v^{(\ell)}(\tau)(-\tau)^\ell\Big\} + \Big\{\int_\tau^0 \frac{(-s)^{m-1}}{(m-1)!}v^{(m)}(s)\,ds\Big\} =: x_1^* + x_0^*. \tag{6.10}$$

We will prove that

$$\|x_1^*\|_{\Lambda_{\mathcal{A}}^{\beta_1}} \leq Ct^{-(1-\theta)}\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}} \quad \text{for } 0 < t < 1. \tag{6.11}$$

For this purpose, it suffices to verify that for $0 \leq \ell \leq m-1$, one has

$$\|v^{(\ell)}(\tau)\tau^\ell\|_{\Lambda_{\mathcal{A}}^{\beta_1}} \leq Ct^{-(1-\theta)}\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}} \quad \text{for } 0 < t < 1. \tag{6.12}$$

First observe that by Lemma 2.3,

$$\|v^{(\ell)}(\tau)\|_{X^*} \leq C\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}}, \quad 0 \leq \ell \leq m-1. \tag{6.13}$$

Fix an integer $n > \beta_1$. Then, by Proposition 2.1, for $0 < s < 1$, we get

$$\begin{aligned}
\Big\|\frac{d^n}{ds^n}\mathcal{T}_s^*(v^{(\ell)}(\tau))\tau^\ell\Big\|_{X^*} &\leq \|v^{(\ell+n)}(\tau+s)\tau^\ell\|_{X^*}\\
&\leq (\tau+s)^{\beta-\ell-n}\tau^\ell\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}}\\
&= (\tau+s)^{-(1-\theta)(\beta_1-\beta_0)+(\beta_1-n)-\ell}\tau^\ell\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}}\\
&\leq s^{\beta_1-n}(\tau+s)^{-(1-\theta)(\beta_1-\beta_0)-\ell}\tau^\ell\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}}\\
&\leq s^{\beta_1-n}\tau^{-(1-\theta)(\beta_1-\beta_0)}\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}}\\
&= s^{\beta_1-n}t^{-(1-\theta)}\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}}.
\end{aligned} \tag{6.14}$$

Now (6.12) follows from (6.13) and (6.14). So, (6.11) is established.

We now turn to examine x_0^* defined in (6.10). Fix $0 \leq \varepsilon \leq \beta_0$ such that $\beta - \varepsilon \notin \mathbb{Z}$ and $0 < m - (\beta-\varepsilon) < 1$ (if $\beta \notin \mathbb{Z}$, then we take $\varepsilon = 0$). Then $x^* \in \Lambda_{\mathcal{A}}^{\beta-\varepsilon}$. Consequently,

$$\begin{aligned}
\|x_0^*\|_{X^*} &\leq C\Big(\int_0^\tau s^{m-1}s^{\beta-\varepsilon-m}\,ds\Big)\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta-\varepsilon}} \leq C\tau^{\beta-\varepsilon}\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}}\\
&= Ct^{(\beta-\varepsilon)/(\beta_1-\beta_0)}\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}} \leq Ct^\theta\|x^*\|_{\Lambda_{\mathcal{A}}^{\beta}},
\end{aligned} \tag{6.15}$$

provided ε is sufficiently small. Furthermore, let n be a fixed positive integer such that $n > \beta_0$. Then, by Proposition 2.1,

$$\begin{aligned}\Big\|\frac{d^n}{du^n}\mathcal{T}_u^* x_0^*\Big\|_{X^*} &\leq C\int_0^\tau s^{m-1}\|v^{(m+n)}(s+u)\|_{X^*}\,ds\\ &\leq C\Big(\int_0^\tau s^{m-1}(s+u)^{\beta-m-n}\,ds\Big)\|x^*\|_{\Lambda^\beta_{\mathcal{A}}}\\ &= C\Big(\int_0^\tau s^{m-1}(s+u)^{\theta(\beta_1-\beta_0)-m}(s+u)^{\beta_0-n}\,ds\Big)\|x^*\|_{\Lambda^\beta_{\mathcal{A}}}\\ &\leq C'u^{\beta_0-n}\Big(\int_0^\tau s^{m-1}s^{\theta(\beta_1-\beta_0)-m}\,ds\Big)\|x^*\|_{\Lambda^\beta_{\mathcal{A}}}\\ &\leq C'u^{\beta_0-n}\tau^{\theta(\beta_1-\beta_0)}\|x^*\|_{\Lambda^\beta_{\mathcal{A}}} = Cu^{\beta_0-n}t^\theta\|x^*\|_{\Lambda^\beta_{\mathcal{A}}}.\end{aligned} \tag{6.16}$$

From (6.15), (6.16), and Proposition 2.1, we conclude that

$$\|x_0^*\|_{\Lambda^{\beta_0}_{\mathcal{A}}} \leq Ct^\theta\|x^*\|_{\Lambda^\beta_{\mathcal{A}}}, \quad 0<t<1. \tag{6.17}$$

Combining (6.17) and (6.11), we obtain (6.9). Consequently, we have proved that

$$\sup_{0<t<1} t^{-\theta}K(t,x^*) \leq C\|x^*\|_{\Lambda^\beta_{\mathcal{A}}}.$$

If $t \geq 1$, then using (6.2), we get $\sup_{t\geq 1} t^{-\theta}K(t,x^*) \leq \|x^*\|_{\Lambda^{\beta_0}_{\mathcal{A}}} \leq C\|x^*\|_{\Lambda^\beta_{\mathcal{A}}}$. Consequently, $\|x^*\|_\theta = \sup_{t>0} t^{-\theta}K(t,x^*) \leq C\|x^*\|_{\Lambda^\beta_{\mathcal{A}}}$. □

Part 2 Lipschitz spaces in the Dunkl setting

7. Preliminaries

7.1. **Dunkl theory.** In this section, we present basic facts concerning the theory of the Dunkl operators. For more details, we refer the reader to [5], [15], [17], and [19].

We consider the Euclidean space $\mathbb{R}^N$ with the scalar product $\langle \mathbf{x},\mathbf{y}\rangle = \sum_{j=1}^N x_jy_j$, where $\mathbf{x}=(x_1,...,x_N)$, $\mathbf{y}=(y_1,...,y_N)$, and the norm $\|\mathbf{x}\|^2 = \langle\mathbf{x},\mathbf{x}\rangle$.

A *normalized root system* in $\mathbb{R}^N$ is a finite set $R\subset\mathbb{R}^N\setminus\{0\}$ such that $R\cap\alpha\mathbb{R}=\{\pm\alpha\}$, $\sigma_\alpha(R)=R$, and $\|\alpha\|=\sqrt{2}$ for all $\alpha\in R$, where σ_α is defined by

$$\sigma_\alpha(\mathbf{x}) = \mathbf{x} - 2\frac{\langle\mathbf{x},\alpha\rangle}{\|\alpha\|^2}\alpha. \tag{7.1}$$

The finite group G generated by the reflections σ_α, $\alpha\in R$, is called the *Coxeter group* (*reflection group*) of the root system.

A *multiplicity function* is a G-invariant function $k: R\to\mathbb{C}$, which will be fixed and non-negative throughout this paper.

The associated measure dw is defined by $dw(\mathbf{x}) = w(\mathbf{x})\,d\mathbf{x}$, where

$$w(\mathbf{x}) = \prod_{\alpha\in R}|\langle\mathbf{x},\alpha\rangle|^{k(\alpha)}. \tag{7.2}$$

Let $\mathbf{N} = N+\sum_{\alpha\in R}k(\alpha)$. Then,

$$w(B(t\mathbf{x},tr)) = t^{\mathbf{N}}w(B(\mathbf{x},r)) \ \text{ for all } \mathbf{x}\in\mathbb{R}^N,\ t,r>0. \tag{7.3}$$

Observe that there is a constant $C>1$ such that for all $\mathbf{x}\in\mathbb{R}^N$ and $r>0$, we have

$$C^{-1}w(B(\mathbf{x},r))\leq r^N\prod_{\alpha\in R}(|\langle\mathbf{x},\alpha\rangle|+r)^{k(\alpha)}\leq Cw(B(\mathbf{x},r)), \tag{7.4}$$

so $dw(\mathbf{x})$ is doubling.

For $\xi\in\mathbb{R}^N$, the *Dunkl operators* D_ξ are the following k-deformations of the directional derivatives ∂_ξ by difference operators:

$$D_\xi f(\mathbf{x})=\partial_\xi f(\mathbf{x})+\sum_{\alpha\in R}\frac{k(\alpha)}{2}\langle\alpha,\xi\rangle\frac{f(\mathbf{x})-f(\sigma_\alpha(\mathbf{x}))}{\langle\alpha,\mathbf{x}\rangle}. \tag{7.5}$$

We simply write D_j, if $\xi=e_j$, where $\{e_j\}_{j=1}^N$ stands for the canonical basis in $\mathbb{R}^N$.

The Dunkl operators D_ξ, which were introduced in [5], commute with each other and are skew-symmetric with respect to the G-invariant measure dw. Furthermore, if $f,g\in C^1(\mathbb{R}^N)$ and at least one of them is G–invariant, then

$$D_\xi(fg)=(D_\xi f)\cdot g+f\cdot(D_\xi g). \tag{7.6}$$

For multi-index $\boldsymbol{\beta}=(\beta_1,\beta_2,\dots,\beta_N)\in\mathbb{N}_0^N$, we denote

$$|\boldsymbol{\beta}|=\beta_1+\ldots+\beta_N,\ \partial^{\mathbf{0}}=I,\ \ \partial^{\boldsymbol{\beta}}=\partial_1^{\beta_1}\circ\ldots\circ\partial_N^{\beta_N},\ \ D^{\mathbf{0}}=I,\ \ D^{\boldsymbol{\beta}}=D_1^{\beta_1}\circ\ldots\circ D_N^{\beta_N}. \tag{7.7}$$

Let f be a bounded measurable function. Fix a multi-index $\boldsymbol{\beta}$. We say that $D^{\boldsymbol{\beta}}f$ belongs to L^∞ in the sense of distribution $\mathcal{S}'_{\text{Dunkl}}(\mathbb{R}^N)$, if there is a bounded function g such that

$$\int_{\mathbb{R}^N}g(\mathbf{x})\varphi(\mathbf{x})\,dw(\mathbf{x})=(-1)^{|\boldsymbol{\beta}|}\int_{\mathbb{R}^N}f(\mathbf{x})D^{\boldsymbol{\beta}}\varphi(\mathbf{x})\,dw(\mathbf{x})\quad\text{for all }\varphi\in\mathcal{S}(\mathbb{R}^N), \tag{7.8}$$

where $\mathcal{S}(\mathbb{R}^N)$ denote the class of Schwartz functions on $\mathbb{R}^N$.

7.2. **Dunkl kernel.** For any fixed $\mathbf{y}\in\mathbb{R}^N$, the *Dunkl kernel* $\mathbf{x}\longmapsto E(\mathbf{x},\mathbf{y})$ is a unique analytic solution to the system

$$D_\xi f=\langle\xi,\mathbf{y}\rangle f,\quad f(0)=1.$$

The function $E(\mathbf{x},\mathbf{y})$, which generalizes the exponential function $e^{\langle\mathbf{x},\mathbf{y}\rangle}$, has a unique extension to a holomorphic function $E(\mathbf{z},\mathbf{w})$ on $\mathbb{C}^N\times\mathbb{C}^N$.

7.3. **Dunkl transform.** Let $f\in L^1(dw)$. The *Dunkl transform* $\mathcal{F}f$ of f is defined by

$$\mathcal{F}f(\xi)=\mathbf{c}_k^{-1}\int_{\mathbb{R}^N}f(\mathbf{x})E(\mathbf{x},-i\xi)\,dw(\mathbf{x}),\ \text{where }\mathbf{c}_k=\int_{\mathbb{R}^N}e^{-\frac{\|\mathbf{x}\|^2}{2}}\,dw(\mathbf{x})>0. \tag{7.9}$$

The Dunkl transform is a generalization of the Fourier transform on $\mathbb{R}^N$. It was introduced in [6] for $k\geq 0$ and further studied in [4] in a more general setting. It possesses many properties analogous to those of the classical Fourier transform, for example,

$$\mathcal{F}(D_jf)(\xi)=i\xi_j\mathcal{F}f(\xi)\text{ for all }f\in\mathcal{S}(\mathbb{R}^{\mathbb{N}})\text{ and }j\in\{1,\dots,N\}. \tag{7.10}$$

Moreover, it was proved in [6, Corollary 2.7] (see also [4, Theorem 4.26]) that it extends uniquely to an isometry on $L^2(dw)$. Furthermore, the following inversion formula holds ([4, Theorem 4.20]): for all $f\in L^1(dw)$ such that $\mathcal{F}f\in L^1(dw)$ one has

$$f(\mathbf{x})=(\mathcal{F})^2f(-\mathbf{x})\text{ for almost all }\mathbf{x}\in\mathbb{R}^N.$$

7.4. **Dunkl translations.** Suppose that $f \in \mathcal{S}(\mathbb{R}^N)$. The *Dunkl translation* $\tau_{\mathbf{x}} f$ of f is defined by

$$\tau_{\mathbf{x}} f(-\mathbf{y}) = \mathbf{c}_k^{-1} \int_{\mathbb{R}^N} E(i\xi, \mathbf{x})\, E(-i\xi, \mathbf{y})\, \mathcal{F}f(\xi)\, dw(\xi) = \mathcal{F}^{-1}(E(i\cdot, \mathbf{x})\mathcal{F}f)(-\mathbf{y}). \tag{7.11}$$

The Dunkl translation was introduced in [14]. The definition can be extended to functions which are not necessarily in $\mathcal{S}(\mathbb{R}^N)$. For instance, using the Plancherel's theorem, one can define the Dunkl translation of $L^2(dw)$ function f by

$$\tau_{\mathbf{x}} f(-\mathbf{y}) = \mathcal{F}^{-1}(E(i\cdot, \mathbf{x})\mathcal{F}f(\cdot))(-\mathbf{y}) \tag{7.12}$$

(see [14] and [22, Definition 3.1]). In particular, the operators $f \mapsto \tau_{\mathbf{x}} f$ are contractions on $L^2(dw)$. Here and henceforth, for a reasonable function $g(\mathbf{x})$, we write $g(\mathbf{x}, \mathbf{y}) := \tau_{\mathbf{x}} g(-\mathbf{y})$. Moreover, it follows from (7.10) and (7.11) that for $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$D_{j,\mathbf{x}}\{\varphi(\mathbf{x}, \mathbf{y})\} = (D_j\varphi)(\mathbf{x}, \mathbf{y}) = -D_{j,\mathbf{y}}\varphi(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^N,\ j = 1, 2, \ldots, N. \tag{7.13}$$

The following specific formula for the Dunkl translations of (reasonable) radial functions $f(\mathbf{x}) = \tilde{f}(\|\mathbf{x}\|)$ was obtained by Rösler [16]:

$$\tau_{\mathbf{x}} f(-\mathbf{y}) = \int_{\mathbb{R}^N} (\tilde{f} \circ A)(\mathbf{x}, \mathbf{y}, \eta)\, d\mu_{\mathbf{x}}(\eta) \text{ for } \mathbf{x}, \mathbf{y} \in \mathbb{R}^N. \tag{7.14}$$

Here

$$A(\mathbf{x}, \mathbf{y}, \eta) = \sqrt{\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\langle \mathbf{y}, \eta\rangle} = \sqrt{\|\mathbf{x}\|^2 - \|\eta\|^2 + \|\mathbf{y} - \eta\|^2}$$

and $\mu_{\mathbf{x}}$ is a probability measure, which is supported in the set $\operatorname{conv} \mathcal{O}(\mathbf{x})$, where $\mathcal{O}(\mathbf{x}) = \{\sigma(\mathbf{x}) : \sigma \in G\}$ is the orbit of $\mathbf{x}$.

Formula (7.14) implies that for all radial $f \in L^1(dw)$ and $\mathbf{x} \in \mathbb{R}^N$, we have

$$\|\tau_{\mathbf{x}} f\|_{L^1(dw)} \leq \|f\|_{L^1(dw)}. \tag{7.15}$$

7.5. **Dunkl convolution.** Assume that $f, g \in L^2(dw)$. The *generalized convolution* (or the *Dunkl convolution*) $f * g$ is defined by the formula

$$f * g(\mathbf{x}) = \mathbf{c}_k \mathcal{F}^{-1}\big((\mathcal{F}f)(\mathcal{F}g)\big)(\mathbf{x}), \tag{7.16}$$

and equivalently, by

$$(f * g)(\mathbf{x}) = \int_{\mathbb{R}^N} f(\mathbf{y})\, \tau_{\mathbf{x}} g(-\mathbf{y})\, dw(\mathbf{y}) = \int_{\mathbb{R}^N} g(\mathbf{y})\, \tau_{\mathbf{x}} f(-\mathbf{y})\, dw(\mathbf{y}). \tag{7.17}$$

Generalized convolution of $f, g \in \mathcal{S}(\mathbb{R}^N)$ was considered in [14] and [24], the definition was extended to $f, g \in L^2(dw)$ in [22].

It follows from (7.15) that if $f \in L^1(dw)$ is radial, then for any $g \in L^p(dw)$, $1 \leq p \leq \infty$, we have

$$\|g * f\|_{L^p(dw)} \leq \|f\|_{L^1(dw)} \|g\|_{L^p(dw)}. \tag{7.18}$$

7.6. **Dunkl Laplacian, Dunkl heat semigroup, and Dunkl heat kernel.** The *Dunkl Laplacian* associated with R and k is the differential-difference operator

$$\Delta_k = \sum_{j=1}^{N} D_j^2. \tag{7.19}$$

It was introduced in [5], where it was also proved that Δ_k acts on $C^2(\mathbb{R}^N)$ functions by

$$\Delta_k f(\mathbf{x}) = \Delta f(\mathbf{x}) + \sum_{\alpha\in R} k(\alpha)\delta_\alpha f(\mathbf{x}), \text{ where } \delta_\alpha f(\mathbf{x}) = \frac{\partial_\alpha f(\mathbf{x})}{\langle \alpha, \mathbf{x}\rangle} - \frac{f(\mathbf{x}) - f(\sigma_\alpha(\mathbf{x}))}{\langle \alpha, \mathbf{x}\rangle^2}. \tag{7.20}$$

Here $\Delta = \sum_{j=1}^N \partial_j^2$. It follows from (7.10) that for all $\xi \in \mathbb{R}^N$ and $f \in \mathcal{S}(\mathbb{R}^N)$, we have

$$\mathcal{F}(\Delta_k f)(\xi) = -\|\xi\|^2 \mathcal{F}f(\xi). \tag{7.21}$$

The operator $(-\Delta_k, \mathcal{S}(\mathbb{R}^N))$ in $L^2(dw)$ is densely defined and closable. Its closure generates a strongly continuous and positivity-preserving contraction semigroup on $L^2(dw)$, which is given by

$$H_t f(\mathbf{x}) = f * h_t(\mathbf{x}) = \int_{\mathbb{R}^N} h_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, dw(\mathbf{y}), \tag{7.22}$$

where

$$h_t(\mathbf{x}, \mathbf{y}) = h_t(\mathbf{y}, \mathbf{x}) = \mathbf{c}_k^{-1}(2t)^{-\mathbf{N}/2} e^{-(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)/(4t)} E\left(\frac{\mathbf{x}}{\sqrt{2t}}, \frac{\mathbf{y}}{\sqrt{2t}}\right) \tag{7.23}$$

is so called the *generalized heat kernel* (or the *Dunkl heat kernel*), see [14] . The integral kernels $h_t(\mathbf{x}, \mathbf{y})$ are the generalized translations of the Schwartz-class functions:

$$h_t(\mathbf{x}, \mathbf{y}) = \tau_{\mathbf{x}} h_t(-\mathbf{y}), \quad h_t(\mathbf{x}) = \mathbf{c}_k^{-1}(2t)^{-\mathbf{N}/2} e^{-\|\mathbf{x}\|^2/4t}.$$

Moreover, for all $t > 0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, one has $h_t(\mathbf{x}, \mathbf{y}) > 0$ and

$$\int_{\mathbb{R}^N} h_t(\mathbf{x}, \mathbf{z})\, dw(\mathbf{z}) = 1.$$

The semigroup $\{H_t\}_{t>0}$ on $L^2(dw)$ can be expressed by means of the Dunkl transform, that is,

$$\mathcal{F}(H_t f)(\xi) = \mathcal{F}(h_t * f)(\xi) = \mathbf{c}_k \mathcal{F}(h_t)(\xi)\mathcal{F}f(\xi) = e^{-t\|\xi\|^2}\mathcal{F}f(\xi), \quad f \in L^2(dw). \tag{7.24}$$

Note that in the case $k \equiv 0$ the Dunkl heat kernel is the classical heat kernel.

Formula (7.22) defines contraction semigroups on the $L^p(dw)$-spaces, $1 \le p \le \infty$, which are strongly continuous for $1 \le p < \infty$.

7.7. **Estimates for generalized translations of some functions.** For the purpose of this work, we need bounds for the Dunkl heat kernel and its derivatives. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, let

$$d(\mathbf{x}, \mathbf{y}) = \min_{\sigma\in G} \|\sigma(\mathbf{x}) - \mathbf{y}\| \tag{7.25}$$

be the *distance of the orbit* of $\mathbf{x}$ to the orbit of $\mathbf{y}$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $t, r > 0$, we denote

$$V(\mathbf{x}, \mathbf{y}, r) := \max\{w(B(\mathbf{x}, r)), w(B(\mathbf{y}, r))\}, \quad \mathcal{G}_t(\mathbf{x}, \mathbf{y}) = \frac{1}{V(\mathbf{x}, \mathbf{y}, \sqrt{t})} e^{-\frac{d(\mathbf{x},\mathbf{y})^2}{t}}. \tag{7.26}$$

It follows by the standard arguments, using the G-invariance of w, that there is a constant $C>0$ such that for all $\mathbf{x}\in\mathbb{R}^N$ and $t>0$ we have

$$\int_{\mathbb{R}^N}\mathcal{G}_t(\mathbf{x},\mathbf{y})\,dw(\mathbf{y})\leq C. \tag{7.27}$$

The following theorem was proved in [7], see also [1, Theorem 4.1]. For more detailed upper and lower bounds for $h_t(\mathbf{x},\mathbf{y})$ we refer the reader to [8].

Theorem 7.1 (Theorem 4.1, [7])**.** *For every nonnegative integer m and for all multi-indices $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{N}_0^N$ there are constants $C_{m,\boldsymbol{\alpha},\boldsymbol{\beta}},c>0$ such that*

$$|\partial_t^m\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\partial_{\mathbf{y}}^{\boldsymbol{\beta}}h_t(\mathbf{x},\mathbf{y})|\leq C_{m,\boldsymbol{\alpha},\boldsymbol{\beta}}t^{-m-\frac{|\boldsymbol{\alpha}|}{2}-\frac{|\boldsymbol{\beta}|}{2}}\Big(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\Big)^{-2}\mathcal{G}_{t/c}(\mathbf{x},\mathbf{y}). \tag{7.28}$$

Moreover, if $\|\mathbf{y}-\mathbf{y}'\|\leq\sqrt{t}$, then

$$|\partial_t^m h_t(\mathbf{x},\mathbf{y})-\partial_t^m h_t(\mathbf{x},\mathbf{y}')|\leq C_m t^{-m}\frac{\|\mathbf{y}-\mathbf{y}'\|}{\sqrt{t}}\Big(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\Big)^{-2}\mathcal{G}_{t/c}(\mathbf{x},\mathbf{y}). \tag{7.29}$$

We have the following estimate for generalized translations of Schwartz class functions.

Theorem 7.2. [1, Theorem 4.1 and Remark 4.2] *Let $\varphi\in\mathcal{S}(\mathbb{R}^N)$ and $M>0$. There is a constant $C>0$ such that for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$, we have*

$$|\varphi_t(\mathbf{x},\mathbf{y})|\leq C\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{t}\right)^{-1}\Big(1+\frac{d(\mathbf{x},\mathbf{y})}{t}\Big)^{-M}\frac{1}{w(B(\mathbf{x},t))}, \tag{7.30}$$

where $\varphi_t(\mathbf{x})=t^{-\mathbf{N}}\varphi(\mathbf{x}/t)$.

The following corollary is a simple consequence of (7.13) and Theorem 7.2.

Corollary 7.3. *For any non-negative integer m and any multi-indices $\boldsymbol{\beta},\boldsymbol{\beta}'\in\mathbb{N}_0^N$, and any $M>0$ there is a constant $C>0$ such that*

$$|\partial_t^m D_{\mathbf{x}}^{\boldsymbol{\beta}}D_{\mathbf{y}}^{\boldsymbol{\beta}'}h_t(\mathbf{x},\mathbf{y})|\leq Ct^{-m-(|\boldsymbol{\beta}|+|\boldsymbol{\beta}'|)/2}V(\mathbf{x},\mathbf{y},\sqrt{t})^{-1}\Big(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\Big)^{-1}\Big(1+\frac{d(\mathbf{x},\mathbf{y})}{\sqrt{t}}\Big)^{-M}. \tag{7.31}$$

Estimates of the form (7.30) for the generalized translation allows us to extend the definition of the Dunkl convolution by using the formula

$$f*\varphi(\mathbf{x})=\int_{\mathbb{R}^N}f(y)\varphi(\mathbf{x},\mathbf{y})\,dw(\mathbf{y})$$

to broader classes of functions. In particular, it follows from (7.13) and Theorem 7.2 that if $f\in L^p(dw)$, $1\leq p\leq\infty$ and $\varphi\in\mathcal{S}(\mathbb{R}^N)$, then $f*\varphi_t\in C^\infty(\mathbb{R}^N)\cap L^p(dw)\cap L^\infty(\mathbb{R}^N)$,

$$D^I(f*\varphi_t)(\mathbf{x})=t^{-|I|}(f*(D^I\varphi)_t)(\mathbf{x})$$

and

$$\|D^I(f*\varphi_t)\|_{L^p(dw)}\leq C_{\varphi,I}t^{-|I|}\|f\|_{L^p(dw)}. \tag{7.32}$$

Especially,

$$D_jH_{t+s}f=D_j(f*h_{t+s})=f*(D_j(h_t*h_s))=f*h_t*(D_jh_s)=H_t(D_jH_s)f. \tag{7.33}$$

7.8. **k-Cauchy kernel and Dunkl Poisson semigroup.** Let $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$. The *k–Cauchy kernel* $p_t(\mathbf{x},\mathbf{y})$ is defined as the integral kernel of the operator $P_t=e^{-t\sqrt{-\Delta_k}}$. It is related to the Dunkl heat kernel by the subordination formula

$$p_t(\mathbf{x},\mathbf{y})=\Gamma(1/2)^{-1}\int_0^\infty e^{-u}h_{t^2/(4u)}(\mathbf{x},\mathbf{y})\frac{du}{\sqrt{u}}. \tag{7.34}$$

Clearly,

$$p_t(\mathbf{x},\mathbf{y})=p_t(\mathbf{y},\mathbf{x}) \text{ for all } \mathbf{x},\mathbf{y}\in\mathbb{R}^N \text{ and } t>0, \tag{7.35}$$

$$\int_{\mathbb{R}^N}p_t(\mathbf{x},\mathbf{y})\,dw(\mathbf{y})=1 \text{ for all } \mathbf{x}\in\mathbb{R}^N \text{ and } t>0. \tag{7.36}$$

The kernel $p_t(\mathbf{x},\mathbf{y})$ was introduced and studied in [18].

Theorem 7.4 ([18, Theorem 5.6])**.** *Let f be a bounded continuous function on $\mathbb{R}^N$. Then the function given by $u(\mathbf{x},t)=P_tf(\mathbf{x})$ is continuous and bounded. Moreover, it solves the Cauchy problem*

$$\begin{cases}\partial_t^2u(\mathbf{x},t)+\Delta_{k,\mathbf{x}}u(\mathbf{x},t)=0 \ \text{on } \mathbb{R}^N\times(0,\infty),\\ u(\mathbf{x},0)=f(\mathbf{x}) \ \text{for all } \mathbf{x}\in\mathbb{R}^N.\end{cases} \tag{7.37}$$

The k-Cauchy kernel is also called the *generalized Poisson kernel* (or the *Dunkl-Poisson kernel*) by the analogy with the classical Poisson semigroup. It follows from (7.24) and (7.34) that

$$\mathcal{F}(P_tf)(\xi)=e^{-t\|\xi\|}\mathcal{F}f(\xi),\quad f\in\mathcal{S}(\mathbb{R}^N),\quad \xi\in\mathbb{R}^N,\quad t>0. \tag{7.38}$$

We shall use the following bounds of the integral kernel $p_t(\mathbf{x},\mathbf{y})$ of the Dunkl-Poisson semigroup.

Proposition 7.5 ([1, Proposition 5.1])**.** *For any non-negative integer m and for any multi-index $\boldsymbol{\beta}\in\mathbb{N}_0^N$, there exists a constant $C>0$ such that, for all $t>0$ and for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$,*

$$\left|\partial_t^m\partial_{\mathbf{y}}^{\boldsymbol{\beta}}p_t(\mathbf{x},\mathbf{y})\right|\leq C\,p_t(\mathbf{x},\mathbf{y})\big(t+d(\mathbf{x},\mathbf{y})\big)^{-m-|\boldsymbol{\beta}|}\times\begin{cases}1 & \text{if } m=0,\\ 1+\frac{d(\mathbf{x},\mathbf{y})}{t} & \text{if } m>0.\end{cases} \tag{7.39}$$

Moreover, for any non-negative integer m and for all multi-indices $\boldsymbol{\alpha},\boldsymbol{\beta}$, there is a constant $C>0$ such that for all $t>0$ and for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$,

$$\left|\partial_t^m\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\partial_{\mathbf{y}}^{\boldsymbol{\beta}}p_t(\mathbf{x},\mathbf{y})\right|\leq C\,t^{-m-|\boldsymbol{\alpha}|-|\boldsymbol{\beta}|}\,p_t(\mathbf{x},\mathbf{y}). \tag{7.40}$$

Proposition 7.6 ([7, Proposition 3.6])**.** *There is a constant $C>0$ such that for all $t>0$, $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$, we have*

$$p_t(\mathbf{x},\mathbf{y})\leq C\frac{t}{V(\mathbf{x},\mathbf{y},d(\mathbf{x},\mathbf{y})+t)}\cdot\frac{d(\mathbf{x},\mathbf{y})+t}{\|\mathbf{x}-\mathbf{y}\|^2+t^2} \tag{7.41}$$

if $N\geq 2$. If $N=1$, then

$$p_t(\mathbf{x},\mathbf{y})\leq C\frac{t}{V(\mathbf{x},\mathbf{y},d(\mathbf{x},\mathbf{y})+t)}\cdot\frac{d(\mathbf{x},\mathbf{y})+t}{\|\mathbf{x}-\mathbf{y}\|^2+t^2}\cdot\ln\Big(1+\frac{\|\mathbf{x}-\mathbf{y}\|+t}{d(\mathbf{x},\mathbf{y})+t}\Big). \tag{7.42}$$

Proposition 7.6 implies that

$$p_t(\mathbf{x},\mathbf{y}) \leq CV(\mathbf{x},\mathbf{y},t+d(\mathbf{x},\mathbf{y}))^{-1}\Big(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{t}\Big)^{-1}, \quad \mathbf{x},\mathbf{y}\in\mathbb{R}^N,\ t>0. \tag{7.43}$$

Let us note that if $X = L^1(dw)$, then $p_t(\mathbf{x},\mathbf{y})$ is the integral kernel of the dual operator P_t^* acting on $X^* = L^\infty$ (see (7.35)). Therefore, we occasionally omit '$*$' and write $P_t f$ instead of $P_t^* f$ for $f \in L^\infty$.

The following corollary of (7.40) will be used later on.

Corollary 7.7. *Let $\boldsymbol{\alpha}$ be a multi-index and let m be a non-negative integer. There is a constant $C>0$ such that for all $f\in L^\infty$, $t>0$, and $\mathbf{x}\in\mathbb{R}^N$ we have:*
(a) the function

$$u(t,\mathbf{x}) := P_t f(\mathbf{x}) = \int_{\mathbb{R}^N} p_t(\mathbf{x},\mathbf{y}) f(\mathbf{y})\,dw(\mathbf{y})$$

is C^∞ as a function of the variables $(t,\mathbf{x})\in(0,\infty)\times\mathbb{R}^N$;
(b)

$$\partial_t^m\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} u(t,\mathbf{x}) = \int_{\mathbb{R}^N}\Big\{\partial_t^m\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} p_t(\mathbf{x},\mathbf{y})\Big\} f(\mathbf{y})\,dw(\mathbf{y}); \tag{7.44}$$

(c)

$$|\partial_t^m\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} u(t,\mathbf{x})| \leq C_{m,\boldsymbol{\alpha}} t^{-m-|\boldsymbol{\alpha}|}\|f\|_{L^\infty}. \tag{7.45}$$

8. Lipschitz spaces in the Dunkl setting

8.1. **Preface.** Consider $\mathcal{T}_t = H_t$, $X = L^1(dw)$, $X^* = L^\infty(\mathbb{R}^N)$. In the present subsection we apply the abstract approach presented in Part 1 to list basic properties of the Lipschitz spaces Λ_{Δ_k} and $\Lambda^\beta_{-\sqrt{-\Delta_k}}$.

From a general theory of semigroups (see e.g., Davies [3]) we conclude that the semigroup $\{H_t\}_{t\geq 0}$ is holomorphic and uniformly bounded in a sector $\boldsymbol{S}_\delta$ for some $\delta>0$.

Proposition 8.1. *Let $X = L^1(dw)$, $\mathcal{T}_t = H_t$, and $\mathcal{D}_j = D_j$. Then properties (5.1)–(5.3) hold.*

Proof. **Property (5.1).** It is a consequence of the explicit form of Δ_k (see (7.19)).
Property (5.2). It follows from (7.33).
Property (5.3). It is a consequence of the estimates (7.32). ☐

Definition 8.2. *Consider $X = L^1(dw)$ and its dual $X^* = L^\infty = L^\infty(\mathbb{R}^N)$. For $\beta>0$, we set $\Lambda_k^\beta = \Lambda^\beta_{-\sqrt{-\Delta_k}}$ and $\|f\|_{\Lambda_k^\beta} = \|f\|_{\Lambda^\beta_{-\sqrt{-\Delta_k}}}$.*

We are now in a position to state some results concerning Λ_k^β which follow from the general approach described in Part 1 of the paper (see Theorems 1.3, 1.4, 5.1, and Corollary 1.5).

Theorem 8.3. *(a) $\Lambda_k^{2\beta} = \Lambda^\beta_{\Delta_k}$, $\beta>0$.*
(b) For $\beta,\gamma>0$, the Dunkl type Bessel potentials (see (2.17))

$$f \mapsto ((I-\Delta_k)^{-\gamma/2})^* f = \Gamma(\gamma/2)^{-1}\int_0^\infty t^{\gamma/2}e^{-t}H_t^* f\,\frac{dt}{t},$$

$$f \mapsto ((I+\sqrt{-\Delta_k})^{-\gamma})^* f = \Gamma(\gamma)^{-1}\int_0^\infty t^\gamma e^{-t} P_t^* f \frac{dt}{t},$$

are isomorphisms of Λ_k^β *onto* $\Lambda_k^{\beta+\gamma}$.

(c) *Suppose* $\beta > 1$ *and* $f \in L^\infty$. *Then* $f \in \Lambda_k^\beta$ *if and only if* $D_j^* f \in \Lambda_k^{\beta-1}$. *Moreover, there is a constant* $C > 1$ *such that for all* $f \in \Lambda_k^\beta$, *we have*

$$C^{-1}\|f\|_{\Lambda_k^\beta} \leq \|f\|_{L^\infty} + \sum_{j=1}^N \|D_j^* f\|_{\Lambda_k^{\beta-1}} \leq C\|f\|_{\Lambda_k^\beta}.$$

Here $D_j^* f$ *is understood in the mild sense (see (5.7)).*

A remark is in order. The action of D_j^* on f in the mild sense used in the characterization stated in part (c) of the theorem can be equivalently understood in the distributional sense expressed in the proposition below.

Proposition 8.4. *Suppose* $f \in L^\infty(\mathbb{R}^N) = L^1(dw)^*$. *Then* $D_j^* f$ *exists in the mild sense and belongs to* $L^\infty(\mathbb{R}^N)$ *if and only if there is* $g \in L^\infty(\mathbb{R}^N)$ *such that for all* $\varphi \in C_c^\infty(\mathbb{R}^N)$ *one has*

$$\int_{\mathbb{R}^N} g(\mathbf{x})\varphi(\mathbf{x})\,dw(\mathbf{x}) = \int f(\mathbf{x})D_j\varphi(\mathbf{x})\,dw(\mathbf{x}). \tag{8.1}$$

Then $D_j^* f = g$. *In particular, if* f *is a bounded* C^1 *function such that* $D_j f$ *is bounded, then* $D_j^* f$ *exists in the mild sense and* $D_j^* f = -D_j f$.

Proof. Assume that $g = D_j^* f \in L^\infty(\mathbb{R}^N)$ in the mild sense. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$. Then, $H_t\varphi = h_t * \varphi \in \mathcal{S}(\mathbb{R}^N)$ and $D_j H_t\varphi = D_j(h_t * \varphi) = h_t * D_j\varphi = H_t D_j\varphi$. Since the Dunkl heat semigroup is strongly continuous on $L^1(dw)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} g(\mathbf{x})\varphi(\mathbf{x})\,dw(\mathbf{x}) &= \lim_{t\to 0}\int_{\mathbb{R}^N} g(\mathbf{x})H_t\varphi(\mathbf{x})\,dw(\mathbf{x}) = \lim_{t\to 0}\int_{\mathbb{R}^N} f(\mathbf{x})D_j(H_t\varphi)(\mathbf{x})\,dw(\mathbf{x})\\ &= \lim_{t\to 0}\int_{\mathbb{R}^N} f(\mathbf{x})(H_t D_j\varphi(\mathbf{x}))\,dw(\mathbf{x}) = \int_{\mathbb{R}^N} f(\mathbf{x})\,D_j\varphi(\mathbf{x})\,dw(\mathbf{x}),\end{aligned}$$

where in the second equality, we have used $g = D_j^* f$ in the mild sense.

Conversely, assume that for $f \in L^\infty(\mathbb{R}^N)$ and for a certain $j \in \{1, 2, \dots, N\}$, there is $g \in L^\infty(\mathbb{R}^N)$ such that (8.1) holds for all $\varphi \in C_c^\infty(\mathbb{R}^N)$. Fix a radial function $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $\psi(\mathbf{x}) = 1$ for $\|\mathbf{x}\| \leq 1$. Set $\psi_n(\mathbf{x}) = \psi(\mathbf{x}/n)$. Let $\phi \in L^1(dw)$. Then $H_t\phi = h_t * \phi \in C^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ (see Subsection 7.6). Consequently, $\psi_n H_t\phi \in C_c^\infty$ and, by the Leibniz rule (7.6),

$$D_j(\psi_n H_t\phi)(\mathbf{x}) = (D_j\psi_n(\mathbf{x}))H_t\phi(\mathbf{x}) + \psi_n(\mathbf{x})D_j(H_t\phi)(\mathbf{x}) = \frac{1}{n}(\partial_j\psi)(\mathbf{x}/n) + \psi_n(\mathbf{x})D_j(H_t\phi)(\mathbf{x}).$$

Applying the Lebesgue dominated convergence theorem two times, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} f(\mathbf{x})D_j(H_t\phi)(\mathbf{x})\,dw(\mathbf{x}) = \lim_{n\to\infty}\int_{\mathbb{R}^N} f(\mathbf{x})\psi_n(\mathbf{x})D_j(H_t\phi)(\mathbf{x})\,dw(\mathbf{x})\\ &= \lim_{n\to\infty}\Big(\int_{\mathbb{R}^N} f(\mathbf{x})D_j(\psi_n H_t\phi)(\mathbf{x})\,dw(\mathbf{x}) - \int_{\mathbb{R}^N} f(\mathbf{x})\frac{1}{n}(\partial_j\psi)(\mathbf{x}/n)(H_t\phi)(\mathbf{x})\,dw(\mathbf{x})\Big)\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N} f(\mathbf{x})D_j(\psi_n H_t\phi)(\mathbf{x})\,dw(\mathbf{x}).\end{aligned}$$

Now, using (8.1) and the Lebesgue dominated convergence theorem, we get

$$\int_{\mathbb{R}^N} f(\mathbf{x})D_j(H_t\phi)(\mathbf{x})\,dw(\mathbf{x}) = \lim_{n\to\infty}\int_{\mathbb{R}^N} g(\mathbf{x})\psi_n(\mathbf{x})H_t\phi(\mathbf{x})\,dw(\mathbf{x}) = \int_{\mathbb{R}^N} g(\mathbf{x})H_t\phi(\mathbf{x})\,dw(\mathbf{x}).$$

□

8.2. **Case** $0<\beta<1$**.**

Theorem 8.5. *For* $0<\beta<1$*, the spaces* $\Lambda^\beta(\mathbb{R}^N)$ *and* Λ_k^β *coincide and the corresponding norms* $\|\cdot\|_{\Lambda^\beta(\mathbb{R}^N)}$ *and* $\|\cdot\|_{\Lambda_k^\beta}$ *are equivalent.*

We start with the following lemma.

Lemma 8.6. *There is a constant* $C>0$ *such that for all* $f\in L^\infty$*,* $\mathbf{x},\mathbf{x}'\in\mathbb{R}^N$*, and* $t>0$ *we have*

$$|P_tf(\mathbf{x})-P_tf(\mathbf{x}')|\leq C\min\Big(1,\frac{\|\mathbf{x}-\mathbf{x}'\|}{t}\Big)\|f\|_{L^\infty}.$$

Proof. Using (7.45) it suffices to consider $\|\mathbf{x}-\mathbf{x}'\|<t$. Then

$$\begin{aligned}\int_{\mathbb{R}^N}|p_t(\mathbf{x},\mathbf{y})-p_t(\mathbf{x}',\mathbf{y})|\,dw(\mathbf{y}) &= \int_{\mathbb{R}^N}\Big|\int_0^1\partial_s\big\{p_t(\mathbf{x}'+s(\mathbf{x}-\mathbf{x}'),\mathbf{y})\big\}\,ds\Big|\,dw(\mathbf{y})\\ &\leq \int_{\mathbb{R}^N}\int_0^1\|\mathbf{x}-\mathbf{x}'\|\|\nabla_{\mathbf{x}}p_t(\mathbf{x}'+s(\mathbf{x}-\mathbf{x}'),\mathbf{y})\|\,ds\,dw(\mathbf{y})\\ &\leq Ct^{-1}\|\mathbf{x}-\mathbf{x}'\|,\end{aligned} \tag{8.2}$$

where in the last inequality we have used (7.40) (with $m=0$, $|\boldsymbol{\alpha}|=1$, and $\boldsymbol{\beta}=\mathbf{0}$) together with (7.43). Now the lemma follows from (8.2). □

Proof of Theorem 8.5. For $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$, we set

$$q_t(\mathbf{x}):=\frac{d}{dt}p_t(\mathbf{x}),\quad q_t(\mathbf{x},\mathbf{y}):=\frac{d}{dt}p_t(\mathbf{x},\mathbf{y}).$$

Then, by (7.35) and (7.36), $q_t(\mathbf{x},\mathbf{y})=q_t(\mathbf{y},\mathbf{x})$ and

$$Q_t^*f(\mathbf{x}):=\frac{d}{dt}P_t^*f(\mathbf{x})=\int_{\mathbb{R}^N}q_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\,dw(\mathbf{y}),$$

$$\int_{\mathbb{R}^N}q_t(\mathbf{x},\mathbf{y})\,dw(\mathbf{y})=0. \tag{8.3}$$

Let $0<\beta<1$. Suppose $f\in\Lambda^\beta(\mathbb{R}^N)$. Applying (8.3) together with (7.40) combined with Proposition 7.6 with $m=1$ and $\boldsymbol{\beta}=\mathbf{0}$, we have

$$\begin{aligned}\Big|\frac{d}{dt}P_t^*f(\mathbf{x})\Big| &= \Big|\int_{\mathbb{R}^N}q_t(\mathbf{x},\mathbf{y})(f(\mathbf{y})-f(\mathbf{x}))\,dw(\mathbf{y})\Big|\\ &\leq \|f\|_{\Lambda^\beta(\mathbb{R}^N)}\int_{\mathbb{R}^N}|q_t(\mathbf{x},\mathbf{y})|\|\mathbf{x}-\mathbf{y}\|^\beta\,dw(\mathbf{y})\\ &\leq C\|f\|_{\Lambda^\beta(\mathbb{R}^N)}\int_{\mathbb{R}^N}t^{\beta-1}V(\mathbf{x},\mathbf{y},t+d(\mathbf{x},\mathbf{y}))^{-1}\Big(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{t}\Big)^{-1}\frac{\|\mathbf{x}-\mathbf{y}\|^\beta}{t^\beta}\,dw(\mathbf{y})\\ &\leq C\|f\|_{\Lambda^\beta(\mathbb{R}^N)}t^{\beta-1}.\end{aligned}$$

Hence, $\|f\|_{\Lambda_k^\beta}\leq C\|f\|_{\Lambda^\beta(\mathbb{R}^N)}$.

The converse implication will be proven if we show that $\Lambda_k^\beta \subseteq \Lambda^\beta(\mathbb{R}^N)$ and there is a constant $C>0$ such that for all $f\in\Lambda_k^\beta$ we have

$$\|f\|_{\Lambda^\beta(\mathbb{R}^N)} \leq C\|f\|_{\Lambda_k^\beta}. \tag{8.4}$$

For this purpose, observe that, by virtue of Lemma 2.3, the functions P_t^*f (which are continuous on $\mathbb{R}^N$) converge uniformly to f, as t tends to 0. Hence, f is function on $\mathbb{R}^N$. We now turn to prove the Lipschitz regularity (1.2) of f and (8.4). Fix $\mathbf{x},\mathbf{x}'\in\mathbb{R}^N$ such that $0<\|\mathbf{x}-\mathbf{x}'\|\leq 1$. Set $\delta:=\|\mathbf{x}-\mathbf{x}'\|$. Then

$$\begin{aligned}|f(\mathbf{x}')-f(\mathbf{x})| &= \Big|\lim_{\varepsilon\to0}(P_\varepsilon^*f(\mathbf{x}')-P_\varepsilon^*f(\mathbf{x}))\Big|\\ &\leq \lim_{\varepsilon\to0}\Big|\int_\varepsilon^1\frac{d}{ds}\Big(P_s^*f(\mathbf{x}')-P_s^*f(\mathbf{x})\Big)\,ds\Big| + \Big|P_1^*f(\mathbf{x}')-P_1^*f(\mathbf{x})\Big|\\ &\leq \lim_{\varepsilon\to0}\int_\varepsilon^\delta\Big(\Big|\frac{d}{ds}P_s^*f(\mathbf{x})\Big|+\Big|\frac{d}{ds}P_s^*f(\mathbf{x}')\Big|\Big)\,ds\\ &\quad+\int_\delta^1\Big|\frac{d}{ds}\big(P_s^*f(\mathbf{x}')-P_s^*f(\mathbf{x})\big)\Big|\,ds + C\|\mathbf{x}-\mathbf{x}'\|\|f\|_{L^\infty},\end{aligned} \tag{8.5}$$

where in the last inequality we have used Lemma 8.6. Since $f\in\Lambda_k^\beta$, we have

$$\int_\varepsilon^\delta\Big(\Big|\frac{d}{ds}P_s^*f(\mathbf{x})\Big|+\Big|\frac{d}{ds}P_s^*f(\mathbf{x}')\Big|\Big)\,ds \leq 2\|f\|_{\Lambda_k^\beta}\int_\varepsilon^\delta s^{\beta-1}\,ds\leq C\|f\|_{\Lambda_k^\beta}\|\mathbf{x}-\mathbf{x}'\|^\beta. \tag{8.6}$$

Recall that, by the semigroup property, $\frac{d}{ds}P_s^*=Q_s=P_{s/2}^*Q_{s/2}^*$. Hence, using Lemma 8.6 and the assumption $f\in\Lambda_k^\beta$, we get

$$\begin{aligned}&\int_\delta^1\Big|\frac{d}{ds}(P_s^*f(\mathbf{x}')-P_s^*f(\mathbf{x}))\Big|\,ds=\int_\delta^1\Big|P_{s/2}^*(Q_{s/2}^*f)(\mathbf{x}')-P_{s/2}^*(Q_{s/2}^*f)(\mathbf{x})\Big|\,ds\\ &\leq C\int_\delta^1\frac{\|\mathbf{x}'-\mathbf{x}\|}{s}\|Q_{s/2}^*f\|_{L^\infty}\,ds\leq C'\int_\delta^1\frac{\|\mathbf{x}'-\mathbf{x}\|}{s}s^{\beta-1}\|f\|_{\Lambda_k^\beta}\,ds\leq C''\|\mathbf{x}'-\mathbf{x}\|^\beta\|f\|_{\Lambda_k^\beta}.\end{aligned}$$

If $\|\mathbf{x}-\mathbf{x}'\|>1$, then $|f(\mathbf{x})-f(\mathbf{x}')|\leq 2\|f\|_{L^\infty}\leq 2\|\mathbf{x}-\mathbf{x}'\|^\beta\|f\|_{L^\infty}$. Thus, the proof of the theorem is complete. □

Remark 8.7. For $0<\beta<1$ the homogeneous Lipschitz spaces

$$\dot\Lambda^\beta(\mathbb{R}^N)=\Big\{f:\mathbb{R}^N\to\mathbb{C}:\sup_{\mathbf{x}\neq\mathbf{y}}\frac{|f(\mathbf{x})-f(\mathbf{y})|}{\|\mathbf{x}-\mathbf{y}\|^\beta}<\infty\Big\}$$

associated with the Euclidean metric and the spaces

$$\dot\Lambda_d^\beta=\Big\{f:\mathbb{R}^N\to\mathbb{C}:\sup_{\mathbf{x}\neq\mathbf{y}}\frac{|f(\mathbf{x})-f(\mathbf{y})|}{d(\mathbf{x},\mathbf{y})^\beta}<\infty\Big\}$$

related to the orbit distance $d(\mathbf{x},\mathbf{y})$ (see (7.25)) where studied in [10]. One of the results of [10] asserts that the space $\dot\Lambda^\beta(\mathbb{R}^N)$ is characterized by the condition

$$\sup_B\frac{1}{\operatorname{diam}(B)^\beta}\Big(\frac{1}{w(B)}\int_B|f(\mathbf{x})-f_B|^q dw(\mathbf{x})\Big)^{1/q}<\infty$$

for any/all $1\leq q<\infty$, where the supremum is taken over all balls $B\subset\mathbb{R}^N$ and $f_B=w(B)^{-1}\int_B f\,dw$. We also refer the reader to [10] for results which relates the Triebel-Lizorkin spaces in the Dunkl setting and the commutators of Lipschitz functions b with the Dunkl-Riesz transforms or the Dunkl-Riesz potentials $(-\Delta_k)^{-\gamma}$.

It is worth emphasizing that in the Dunkl setting one can consider Lipschitz spaces in which regularity conditions are measured by means of the generalized translations, namely, by $\|\tau_{\mathbf{x}} f - f\|_{L^q(dw)}$ for $1 \leq q \leq \infty$. In dimension one, in which the translations are bounded operators on $L^p(dw)$, such spaces were studied in [12].

9. Proof of Theorem 1.1

The following theorem, which follows by an iteration of (1.7), will be used in the proof of Theorem 1.1.

Theorem 9.1 ([21, Theorem 10]). *Let $\beta > 0$, $n \in \mathbb{N}_0$, $0 \leq n < \beta$. Then $f \in \Lambda^\beta(\mathbb{R}^N)$ if and only if $f \in C^n(\mathbb{R}^N)$, $\partial^{\boldsymbol{\gamma}} f$ are bounded functions for $|\boldsymbol{\gamma}| \leq n$, and $\partial^{\boldsymbol{\gamma}} f \in \Lambda^{\beta-n}(\mathbb{R}^N)$ for all $\boldsymbol{\gamma} \in \mathbb{N}_0^N$ such that $|\boldsymbol{\gamma}| = n$. Moreover, there is a constant $C > 0$ such that for all $f \in \Lambda^\beta(\mathbb{R}^N)$ we have*

$$C^{-1}\|f\|_{\Lambda^\beta(\mathbb{R}^N)} \leq \sum_{|\boldsymbol{\gamma}|<n} \|\partial^{\boldsymbol{\gamma}} f\|_{L^\infty} + \sum_{|\boldsymbol{\gamma}|=n} \|\partial^{\boldsymbol{\gamma}} f\|_{\Lambda^{\beta-n}(\mathbb{R}^N)} \leq C\|f\|_{\Lambda^\beta(\mathbb{R}^N)}. \tag{9.1}$$

Theorem 9.2. *Let $\beta > 0$, $\beta \notin \mathbb{N}$. Then $\Lambda^\beta(\mathbb{R}^N) \subseteq \Lambda_k^\beta$ and there is a constant $C > 0$ such that for all $f \in \Lambda^\beta(\mathbb{R}^N)$ we have $\|f\|_{\Lambda_k^\beta} \leq C\|f\|_{\Lambda^\beta(\mathbb{R}^N)}$.*

Proof. Assume that $0 \leq n < \beta < n+1$. The proof of the inclusion proceeds by induction on n. If $n = 0$, the equality $\Lambda^\beta(\mathbb{R}^N) = \Lambda_k^\beta$ and the equivalence of the norms follow from Theorem 8.5. Suppose that $1 \leq n < \beta < n+1$ and our induction hypothesis holds for $\beta - 1 < n$. Let $f \in \Lambda^\beta(\mathbb{R}^N)$. Then $f \in C^n(\mathbb{R}^N)$ and $\partial_j f \in \Lambda^{\beta-1}(\mathbb{R}^N)$ for $j = 1, \dots, N$. According to part (c) of Theorem 8.3 combined with the induction hypothesis, it suffices to prove that $D_j^* f$ exists in the mild sense and belongs to $\Lambda^{\beta-1}(\mathbb{R}^N)$ for $j = 1, \dots, N$. Since $f \in C^n(\mathbb{R}^N)$ with $n \geq 1$, and $\partial_j f$ are bounded continuous functions for all $j = 1, 2, \dots, N$, and

$$\begin{aligned} D_j f(\mathbf{x}) &= \partial_j f(\mathbf{x}) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \alpha_j \frac{f(\mathbf{x}) - f(\sigma_\alpha(\mathbf{x}))}{\langle \alpha, \mathbf{x} \rangle} \\ &= \partial_j f(\mathbf{x}) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \alpha_j \int_0^1 \langle \alpha, \nabla f(\sigma_\alpha(\mathbf{x}) + s(\mathbf{x} - \sigma_\alpha(\mathbf{x}))) \rangle \, ds, \end{aligned} \tag{9.2}$$

we conclude that $D_j^* f$ exists in the mild sense and $D_j^* f = -D_j f$ (see Proposition 8.4). As $\partial_j f \in \Lambda^{\beta-1}(\mathbb{R}^N)$ and $\|\partial_j f\|_{\Lambda^{\beta-1}(\mathbb{R}^N)} \leq C\|f\|_{\Lambda^\beta(\mathbb{R}^N)}$, it remains (by (9.2)) to consider the functions

$$f_{\alpha,\ell}(\mathbf{x}) := \int_0^1 (\partial_\ell f)(\sigma_\alpha(\mathbf{x}) + s(\mathbf{x} - \sigma_\alpha(\mathbf{x}))) \, ds = \int_0^1 (\partial_\ell f)(A_{s,\alpha}(\mathbf{x}))) \, ds, \tag{9.3}$$

for $\ell = 1, 2, ..., N$, where $A_{s,\alpha}(\mathbf{x}) := \sigma_\alpha(\mathbf{x}) + s(\mathbf{x} - \sigma_\alpha(\mathbf{x}))$. Now the $C^{n-1}(\mathbb{R}^N)$-function $f_{\alpha,\ell}$ belongs to $\Lambda^{\beta-1}(\mathbb{R}^N)$ if and only if $\|f_{\alpha,\ell}\|_{\Lambda^{\beta-1}(\mathbb{R}^N)}$ is finite, i.e.,

$$\|f_{\alpha,\ell}\|_{\Lambda^{\beta-1}(\mathbb{R}^N)} \sim \sum_{|\boldsymbol{\gamma}|<n-1} \|\partial^{\boldsymbol{\gamma}} f_{\alpha,\ell}\|_{L^\infty} + \sum_{|\boldsymbol{\gamma}|=n-1} \|\partial^{\boldsymbol{\gamma}} f_{\alpha,\ell}\|_{\Lambda^{\beta-n}(\mathbb{R}^N)} < \infty. \tag{9.4}$$

Note that the linear mappings $A_{s,\alpha}$ on the Euclidean space $\mathbb{R}^N$ satisfy

$$\|A_{s,\alpha}\| = 1, \quad 0 \leq s \leq 1, \ \alpha \in R.$$

Thus, by the chain rule, we obtain

$$|\partial^{\gamma}(\{\partial_\ell f\}\circ A_{s,\alpha})(\mathbf{x})| \leq C_{\boldsymbol{\gamma}}\|f\|_{C^n(\mathbb{R}^N)} \leq C'_{\boldsymbol{\gamma}}\|f\|_{\Lambda^\beta(\mathbb{R}^N)} \quad \text{for } |\boldsymbol{\gamma}|\leq n-1$$

and

$$|\partial^{\gamma}(\{\partial_\ell f\}\circ A_{s,\alpha})(\mathbf{x}) - \partial^{\gamma}(\{\partial_\ell f\}\circ A_{s,\alpha})(\mathbf{x}')| \leq C\|f\|_{\Lambda^\beta(\mathbb{R}^N)}\|\mathbf{x}-\mathbf{x}'\|^{\beta-n} \quad \text{for } |\boldsymbol{\gamma}|=n-1.$$

Hence, from (9.3), we conclude (9.4). Consequently $D_jf\in\Lambda^{\beta-1}(\mathbb{R}^N)$ and $\|D_jf\|_{\Lambda^{\beta-1}(\mathbb{R}^N)}\leq C\|f\|_{\Lambda^\beta(\mathbb{R}^N)}$. Thus, by Theorem 8.3, $f\in\Lambda^\beta_k$ and $\|f\|_{\Lambda^\beta_k}\leq C\|f\|_{\Lambda^\beta(\mathbb{R}^N)}$. □

Lemma 9.3. *Let $\beta\in(0,1)$ and let m be a positive integer. There are positive constants $C_{m,\beta}$, $C'_{m,\beta}$ such that for all $\boldsymbol{\alpha}\in\mathbb{N}_0^N$ such that $|\boldsymbol{\alpha}|=m$, $f\in\Lambda^\beta_k$, $\mathbf{x},\mathbf{x}'\in\mathbb{R}^N$, and $t>0$ such that $\|\mathbf{x}-\mathbf{x}'\|<t$, we have*

$$\|\partial^{\boldsymbol{\alpha}}P_tf\|_{L^\infty}\leq C_{m,\beta}t^{\beta-m}\|f\|_{\Lambda^\beta_k}, \tag{9.5}$$

$$|\partial^{\boldsymbol{\alpha}}P_tf(\mathbf{x})-\partial^{\boldsymbol{\alpha}}P_tf(\mathbf{x}')|\leq C'_{m,\beta}\|\mathbf{x}-\mathbf{x}'\|t^{\beta-m-1}\|f\|_{\Lambda^\beta_k}. \tag{9.6}$$

Proof. By virtue of (7.45), it is enough to check (9.5) for $0<t<1$. For $f\in\Lambda^\beta_k$ and $0<t<1$ we have

$$P_tf=P_1f-\int_t^1\partial_sP_sf\,ds.$$

By (7.45), $\|\partial^{\boldsymbol{\alpha}}P_1f\|_{L^\infty}\leq C_{\boldsymbol{\alpha}}\|f\|_{L^\infty}$, so it remains to estimate

$$\partial^{\boldsymbol{\alpha}}\int_t^1\partial_sP_sf\,ds=\int_t^1\partial^{\boldsymbol{\alpha}}\partial_sP_sf\,ds. \tag{9.7}$$

Furthermore,

$$\partial^{\boldsymbol{\alpha}}\partial_sP_sf(\mathbf{x})=\int_{\mathbb{R}^N}\partial^{\boldsymbol{\alpha}}p_{s/2}(\mathbf{x},\mathbf{z})\int_{\mathbb{R}^N}q_{s/2}(\mathbf{z},\mathbf{y})f(\mathbf{y})\,dw(\mathbf{y})\,dw(\mathbf{z}). \tag{9.8}$$

Since $f\in\Lambda^\beta_k$, there is $C>0$ such that for all $s>0$ and $\mathbf{z}\in\mathbb{R}^N$, we have

$$\left|\int_{\mathbb{R}^N}q_{s/2}(\mathbf{z},\mathbf{y})f(\mathbf{y})\,dw(\mathbf{y})\right|\leq Cs^{\beta-1}\|f\|_{\Lambda^\beta_k}. \tag{9.9}$$

From (7.45) we conclude that there is $C_{\boldsymbol{\alpha}}>0$ such that for all $s>0$ and $\mathbf{x}\in\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N}|\partial^{\boldsymbol{\alpha}}p_{s/2}(\mathbf{x},\mathbf{z})|\,dw(\mathbf{z})\leq C_{\boldsymbol{\alpha}}s^{-m}. \tag{9.10}$$

Combining (9.8), (9.9), and (9.10) together with (9.7), we get (9.5). Finally, (9.6) is a consequence of (9.5) and the mean value theorem. □

Lemma 9.4. *Let $m\in\mathbb{N}$. There is a constant $C>0$ such that for all $\boldsymbol{\alpha}\in\mathbb{N}_0^N$ such that $|\boldsymbol{\alpha}|=m$, $f\in L^\infty$, $\mathbf{x},\mathbf{x}'\in\mathbb{R}^N$, and $t>0$ such that $\|\mathbf{x}-\mathbf{x}'\|<t$, we have*

$$|\partial^{\boldsymbol{\alpha}}P_tf(\mathbf{x})-\partial^{\boldsymbol{\alpha}}P_tf(\mathbf{x}')|\leq C\|\mathbf{x}-\mathbf{x}'\|t^{-m-1}\|f\|_{L^\infty}. \tag{9.11}$$

Proof. The lemma follows directly from the estimates (7.45). □

Theorem 9.5. *Let $\gamma>1$, $\gamma\notin\mathbb{N}$. We have $\Lambda^\gamma_k\subseteq\Lambda^\gamma(\mathbb{R}^N)$ and*

$$\|g\|_{\Lambda^\gamma(\mathbb{R}^N)}\leq C_\gamma\|g\|_{\Lambda^\gamma_k}. \tag{9.12}$$

Proof. Fix $\gamma > 1$, $\gamma \notin \mathbb{N}$. Let m be such that $m < \gamma < m+1$. We fix $\gamma_1, \gamma_2 > 0$ such that $\gamma_1 \in (0,1)$ and $m < \gamma_2 < m+1$, $\gamma_1 + \gamma_2 = \gamma$. Consider $g \in \Lambda_k^{\gamma}$. Our goal is to verify (9.12). For this purpose, by Theorem 9.1, it is enough to show that

$$\sum_{|\boldsymbol{\alpha}|<m} \|\partial^{\boldsymbol{\alpha}} g\|_{L^\infty} + \sum_{|\boldsymbol{\alpha}|=m} \|\partial^{\boldsymbol{\alpha}} g\|_{\Lambda^{\gamma-m}(\mathbb{R}^N)} \leq C\|g\|_{\Lambda_k^{\gamma}}. \tag{9.13}$$

Let us recall that $((I+\sqrt{-\Delta_k})^{-\gamma_2})^*$ is an isomorphism of the space $\Lambda_k^{\gamma_1}$ onto Λ_k^{γ} (see Theorem 8.3). Let $f \in \Lambda_k^{\gamma_1}$ be such that $((I+\sqrt{-\Delta_k})^{-\gamma_2})^* f = g$. Clearly, $\|g\|_{\Lambda_k^{\beta}} \sim \|f\|_{\Lambda_k^{\gamma_1}}$. Hence, instead of (9.13), it is enough to verify that

$$\sum_{|\boldsymbol{\alpha}|<m} \|\partial^{\boldsymbol{\alpha}}((I+\sqrt{-\Delta_k})^{-\gamma_2})^* f\|_{L^\infty} + \sum_{|\boldsymbol{\alpha}|=m} \|\partial^{\boldsymbol{\alpha}}((I+\sqrt{-\Delta_k})^{-\gamma_2})^* f\|_{\Lambda^{\gamma-m}(\mathbb{R}^N)} \leq C\|f\|_{\Lambda_k^{\gamma_1}}. \tag{9.14}$$

First, let us take $\boldsymbol{\alpha} \in \mathbb{N}_0^N$ such that $|\boldsymbol{\alpha}| \leq m$ and estimate $\|\partial^{\boldsymbol{\alpha}}((I+\sqrt{-\Delta_k})^{-\gamma_2})^* f\|_{L^\infty}$. By Lemma 9.3, we get

$$\begin{aligned} &\|\partial^{\boldsymbol{\alpha}}((I+\sqrt{-\Delta_k})^{-\gamma_2})^* f\|_{L^\infty} \leq \Gamma(\gamma_2)^{-1} \int_0^\infty e^{-t} t^{\gamma_2} \|\partial^{\boldsymbol{\alpha}} P_t^* f\|_{L^\infty} \frac{dt}{t} \\ &\leq C\|f\|_{\Lambda_k^{\gamma_1}} \int_0^\infty e^{-t} t^{\gamma_2} t^{\gamma_1 - |\boldsymbol{\alpha}|} \frac{dt}{t} \leq C\|f\|_{\Lambda_k^{\gamma_1}}. \end{aligned} \tag{9.15}$$

Then, let $\boldsymbol{\alpha} \in \mathbb{N}_0^N$ be such that $|\boldsymbol{\alpha}| = m$ and estimate $\|\partial^{\boldsymbol{\alpha}}((I+\sqrt{-\Delta_k})^{-\gamma_2})^* f\|_{\Lambda^{\gamma-m}(\mathbb{R}^N)}$. Since $\gamma - m \in (0,1)$, it is enough to verify the following Lipschitz condition:

$$\left|\partial^{\boldsymbol{\alpha}}((I+\sqrt{-\Delta_k})^{-\gamma_2})^* f(\mathbf{x}) - \partial^{\boldsymbol{\alpha}}((I+\sqrt{-\Delta_k})^{-\gamma_2})^* f(\mathbf{x}')\right| \leq C\|f\|_{\Lambda_k^{\gamma_1}} \|\mathbf{x}-\mathbf{x}'\|^{\gamma-m}. \tag{9.16}$$

For $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^N$ such that $\|\mathbf{x}-\mathbf{x}'\| \leq 1$, we write

$$\begin{aligned} &\left|\partial^{\boldsymbol{\alpha}}((I+\sqrt{-\Delta_k})^{-\gamma_2})^* f(\mathbf{x}) - \partial^{\boldsymbol{\alpha}}((I+\sqrt{-\Delta_k})^{-\gamma_2})^* f(\mathbf{x}')\right| \\ &\leq \Gamma(\gamma_2)^{-1} \int_0^\infty e^{-t} t^{\gamma_2} \left|\partial^{\boldsymbol{\alpha}} P_t^* f(\mathbf{x}) - \partial^{\boldsymbol{\alpha}} P_t^* f(\mathbf{x}')\right| \frac{dt}{t} \\ &\leq C \int_0^{\|\mathbf{x}-\mathbf{x}'\|} \cdots + C \int_{\|\mathbf{x}-\mathbf{x}'\|}^1 \cdots + C \int_1^\infty =: I_1 + I_2 + I_3. \end{aligned}$$

From Lemma 9.3 we get

$$I_1 \leq C \int_0^{\|\mathbf{x}-\mathbf{x}'\|} e^{-t} t^{\gamma_2} \|\partial^{\boldsymbol{\alpha}} P_t^* f\|_{L^\infty} \frac{dt}{t} \leq C\|f\|_{\Lambda_k^{\gamma_1}} \int_0^{\|\mathbf{x}-\mathbf{x}'\|} e^{-t} t^{\gamma_2} t^{\gamma_1 - m} \frac{dt}{t} \leq C\|f\|_{\Lambda_k^{\gamma_1}} \|\mathbf{x}-\mathbf{x}'\|^{\gamma-m}. \tag{9.17}$$

Furthermore, using (9.6), we obtain

$$\begin{aligned} I_2 &\leq C\|\mathbf{x}-\mathbf{x}'\| \|f\|_{\Lambda_k^{\gamma_1}} \int_{\|\mathbf{x}-\mathbf{x}'\|}^1 e^{-t} t^{\gamma_2} t^{\gamma_1 - m - 1} \frac{dt}{t} \\ &\leq C\|\mathbf{x}-\mathbf{x}'\| \|f\|_{\Lambda_k^{\gamma_1}} \|\mathbf{x}-\mathbf{x}'\|^{\gamma-m-1} = C\|\mathbf{x}-\mathbf{x}'\|^{\gamma-m} \|f\|_{\Lambda_k^{\gamma_1}}. \end{aligned} \tag{9.18}$$

Finally, utilizing Lemma 9.4, we have

$$I_3 \leq C\|\mathbf{x}-\mathbf{x}'\| \|f\|_{L^\infty} \int_1^\infty e^{-t} t^{\gamma_2 - m - 1} \frac{dt}{t} = C'\|\mathbf{x}-\mathbf{x}'\| \|f\|_{\Lambda_k^{\gamma_1}} \leq C\|\mathbf{x}-\mathbf{x}'\|^{\gamma-m} \|f\|_{\Lambda_k^{\gamma_1}}. \tag{9.19}$$

Collecting (9.17), (9.18), and (9.19), we get (9.16), which, together with (9.15), proves (9.14). □

Completing the proof of Theorem 1.1. By virtue of Theorems 9.2 and 9.5, it remains to prove the theorem for β being any positive integer. To this end, we consider the identity operator and apply Theorems 9.2 and 9.5 together with the interpolation Theorem 1.7.

□

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