PRYM LOCI OF BRANCHED DOUBLE COVERINGS AND GENERALIZED ANDREOTTI-MAYER LOCI

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Abstract. The Andreotti-Mayer locus is a subset of the moduli space of principally polarized abelian varieties, defined by a condition on the dimension of the singular locus of the theta divisor. It is known that the Jacobian locus in the moduli space is an irreducible component of the Andreotti-Mayer locus. In this paper, we generalize the Andreotti-Mayer locus to the case of the moduli space of abelian varieties with non-principal polarization and prove that the Prym locus of branched double coverings is an irreducible component of the generalized Andreotti-Mayer locus.

1. Introduction

Let C be a projective smooth curve of genus g over the complex numbers $\mathbb{C}$, and let $\phi : \tilde{C} \to C$ be a double covering of C branched at $2n$ points. When $n \geq 1$, the norm map $\mathrm{Nm} : \mathrm{Pic}\,(\tilde{C}) \to \mathrm{Pic}\,(C)$ associated with ϕ has the connected kernel $P = \mathrm{Ker}\,(\mathrm{Nm})$, which is an abelian subvariety of dimension $d = g + n - 1$ in the Jacobian variety $J_{\tilde{C}} = \mathrm{Pic}^0\,(\tilde{C})$. Let $\mathcal{L} \in \mathrm{Pic}\,(P)$ be the restriction of the invertible sheaf $\mathcal{O}_{J_{\tilde{C}}}(\Theta_{\tilde{C}})$ associated with the theta divisor $\Theta_{\tilde{C}}$ on $J_{\tilde{C}}$. Then the class $[\mathcal{L}] \in \mathrm{NS}\,(P)$ determines the polarization on P of type $\Delta = (\underbrace{1, \ldots, 1}_{n-1}, \underbrace{2, \ldots, 2}_{g})$, and the polarized abelian variety $(P, [\mathcal{L}])$ is called the Prym variety ([11]) of the branched covering ϕ. From this construction, we obtain the Prym map

$$\mathrm{Prym}_{g,2n} : \mathcal{R}_{g,2n} \longrightarrow \mathcal{A}_d^{\Delta};\ [\tilde{C} \stackrel{\phi}{\to} C] \longmapsto (P, [\mathcal{L}]),$$

where $\mathcal{R}_{g,2n}$ denotes the moduli space of double coverings of curves of genus g branched at $2n$ points, and $\mathcal{A}_d^{\Delta}$ denotes the moduli space of d-dimensional polarized abelian varieties of type Δ. When $g = 0$, the Prym variety $\mathrm{Prym}_{0,2n}(\phi) = (J_{\tilde{C}}, [\mathcal{O}_{J_{\tilde{C}}}(\Theta_{\tilde{C}})])$ is the hyperelliptic Jacobian, and the Prym map $\mathrm{Prym}_{0,2n}$ is injective by the Torelli theorem. When $g = 1$, it is proven in [8] that the Prym map $\mathrm{Prym}_{1,2n}$ is injective for $n \geq 3$. More generally, the following result holds:

Theorem 1.1 (Naranjo-Ortega [14]). *If $g \geq 1$ and $n \geq 3$, then the Prym map $\mathrm{Prym}_{g,2n}$ is injective.*

Let $\mathcal{P}_{g,2n} \subset \mathcal{A}_d^{\Delta}$ be the closure of the image of the Prym map $\mathrm{Prym}_{g,2n}$. We aim to understand which kinds of polarized abelian varieties are contained in the Prym locus $\mathcal{P}_{g,2n}$. In the case where the polarization is principal, the characterization of

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the Jacobian locus has been classically studied as the Schottky problem. Let $\mathcal{J}_d \subset \mathcal{A}_d^{(1,\dots,1)}$ (resp. $\mathcal{H}_d \subset \mathcal{A}_d^{(1,\dots,1)}$) denote the closure of the set of points representing the Jacobian varieties of projective smooth curves (resp. hyperelliptic curves) of genus d. In this paper, we characterize $\mathcal{P}_{g,2n}$ in $\mathcal{A}_d^{\Delta}$, following the approach of Andreotti and Mayer [1] to characterize $\mathcal{J}_d$ and $\mathcal{H}_d$ in $\mathcal{A}_d^{(1,\dots,1)}$.

For an ample invertible sheaf $\mathcal{L}$ on an abelian variety A, we define the higher base locus of $\mathcal{L}$ by

$$S(A,\mathcal{L}) = \bigcap_{\Theta \in |\mathcal{L}|} \Theta_{\mathrm{sing}} \subset A,$$

where Θ_{sing} denotes the singular locus of the hypersurface $\Theta \subset A$. If ample invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ are algebraically equivalent, then $S(A,\mathcal{L}_1) \simeq S(A,\mathcal{L}_2)$, because $t_{x_0}^*\mathcal{L}_1 \simeq \mathcal{L}_2$ by a translation

$$t_{x_0} : A \longrightarrow A;\ x \longmapsto x + x_0.$$

For an integer $m \geq 0$, a generalization of the Andreotti-Mayer locus is defined by

$$\mathcal{N}_{d,m}^{\Delta} = \{(A,[\mathcal{L}]) \in \mathcal{A}_d^{\Delta} \mid \dim S(A,\mathcal{L}) \geq m\}.$$

When $\Delta = (1,\dots,1)$, the subset $\mathcal{N}_{d,m}^{(1,\dots,1)} \subset \mathcal{A}_d^{(1,\dots,1)}$ is the original Andreotti-Mayer locus, and the following results are known:

Theorem 1.2 (Andreotti-Mayer [1]). (1) *If $d \geq 4$, then the Jacobian locus $\mathcal{J}_d$ is an irreducible component of $\mathcal{N}_{d,d-4}^{(1,\dots,1)}$.*

(2) *If $d \geq 3$, then the hyperelliptic Jacobian locus $\mathcal{H}_d$ is an irreducible component of $\mathcal{N}_{d,d-3}^{(1,\dots,1)}$.*

Theorem 1.3 (Debarre [4]). *If $d \geq 7$, then $\mathcal{P}_{d+1,0}$ is an irreducible component of $\mathcal{N}_{d,d-6}^{(1,\dots,1)}$, where $\mathcal{P}_{d+1,0} \subset \mathcal{A}_d^{(1,\dots,1)}$ denotes the Prym locus for unramified double coverings of curves of genus $d+1$.*

In this paper, we prove the following:

Theorem 1.4. *If $n \geq 4$ and $d = g+n-1$, then the Prym locus $\mathcal{P}_{g,2n} \subset \mathcal{A}_d^{\Delta}$ is an irreducible component of $\mathcal{N}_{d,n-4}^{\Delta}$.*

This is a generalization of Theorem 1.2 (2), because $\mathcal{P}_{0,2n} = \mathcal{H}_d \subset \mathcal{A}_d^{(1,\dots,1)}$ for $d = n-1$. According to the folk conjecture [7, Conjecture 3.15 (2)], it is expected that $\overline{\mathcal{N}_{d,d-3}^{(1,\dots,1)} \setminus \mathcal{N}_{d,d-2}^{(1,\dots,1)}} = \mathcal{H}_d$. More generally we expect that $\overline{\mathcal{N}_{d,n-4}^{\Delta} \setminus \mathcal{N}_{d,n-3}^{\Delta}} = \mathcal{P}_{g,2n}$ for $d = g+n-1$, but this remains an open problem.

This paper proceeds as follows: In Section 2, for the Prym variety $(P,[\mathcal{L}])$ of a branched double covering, we explicitly describe the higher base locus $S(P,\mathcal{L}) \subset P$, and prove that $\mathcal{P}_{g,2n} \subset \mathcal{N}_{d,n-4}^{\Delta}$. In the proof, we provide the equation of the tangent cone $\mathcal{T}_{\Theta,x} \subset T_{P,x} = T_{P,0}$ of $\Theta \in |\mathcal{L}|$ at $x \in S(P,\mathcal{L})$. In Section 3, we prove that the equations of the tangent cones $\mathcal{T}_{\Theta,x}$ generate a certain subspace in $\mathrm{Sym}^2 T_{P,0}^{\vee}$. In Section 4, following the approach in [1], we introduce a method to bound the dimension of an irreducible component $\mathcal{P}$ of $\mathcal{N}_{d,n-4}^{\Delta}$. When $\mathcal{P}_{g,2n} \subset \mathcal{P}$, by using independent equations of the tangent cones $\mathcal{T}_{\Theta,x}$, we can provide an upper bound for the dimension of $\mathcal{P}$. Combining this with the result from Section 3, we obtain $\dim \mathcal{P} \le \dim \mathcal{P}_{g,2n}$, which implies that $\mathcal{P} = \mathcal{P}_{g,2n}$.

2. Higher base loci and the tangent cone of theta divisors

We assume that $n \ge 1$ and $g \ge 1$. Let $(P, [\mathcal{L}])$ be the Prym variety of $[\tilde{C} \stackrel{\phi}{\to} C] \in \mathcal{R}_{g,2n}$. Following Mumford's paper [11], we describe the base locus of the linear system $|\mathcal{L}|$. By [11, p333], we may assume that

$$[-1]^*_{J_C}\mathcal{O}_{J_C}(\Theta_C) = \mathcal{O}_{J_C}(\Theta_C), \quad [-1]^*_{J_{\tilde{C}}}\mathcal{O}_{J_{\tilde{C}}}(\Theta_{\tilde{C}}) = \mathcal{O}_{J_{\tilde{C}}}(\Theta_{\tilde{C}}),$$

and

$$\psi^*\mathcal{O}_{J_{\tilde{C}}}(\Theta_{\tilde{C}}) = \mathcal{O}_{J_C}(2\Theta_C), \quad \mathcal{L} = \mathcal{O}_{\tilde{C}}(\Theta_{\tilde{C}})|_P,$$

where $\psi : J_C \to J_{\tilde{C}}$ is the homomorphism defined by

$$J_C = \mathrm{Pic}^0\,(C) \longrightarrow J_{\tilde{C}} = \mathrm{Pic}^0\,(\tilde{C});\ \xi \longmapsto \phi^*\xi.$$

We define the homomorphism

$$\pi :\ J_C \times P \longrightarrow J_{\tilde{C}};\ (\xi, \mathcal{F}) \longmapsto \mathcal{F} \otimes \phi^*\xi.$$

Then we have $\pi^*\mathcal{O}_{J_{\tilde{C}}}(\Theta_{\tilde{C}}) = \mathrm{pr}_1^*\mathcal{O}_{J_C}(2\Theta_C) \otimes \mathrm{pr}_2^*\mathcal{L}$, where pr_i denotes the i-th projection of $J_C \times P$. Let $s \in H^0(J_{\tilde{C}}, \mathcal{O}_{J_{\tilde{C}}}(\Theta_{\tilde{C}}))$ be a nontrivial section. Since $n \ge 1$, $\psi : J_C \to J_{\tilde{C}}$ is injective, and by [11, p335], we have

$$\pi^*s = \sum_{i=1}^{2^g} \mathrm{pr}_1^*\alpha_i \otimes \mathrm{pr}_2^*\beta_i \in H^0(J_C \times P, \mathrm{pr}_1^*\mathcal{O}_{J_C}(2\Theta_C) \otimes \mathrm{pr}_2^*\mathcal{L})$$

for some bases $\alpha_1, \ldots, \alpha_{2^g} \in H^0(J_C, \mathcal{O}_{J_C}(2\Theta_C))$ and $\beta_1, \ldots, \beta_{2^g} \in H^0(P, \mathcal{L})$. This implies the following proposition for the base locus of $|\mathcal{L}|$. For $\mathcal{F}_0 \in \mathrm{Pic}\,(\tilde{C})$, we denote the translation by

$$t_{\mathcal{F}_0} :\ \mathrm{Pic}\,(\tilde{C}) \longrightarrow \mathrm{Pic}\,(\tilde{C});\ \mathcal{F} \longmapsto \mathcal{F} \otimes \mathcal{F}_0.$$

Proposition 2.1 ([11] p334)**.** *The base locus of the linear system $|\mathcal{L}|$ is*

$$\mathrm{Bs}\,|\mathcal{L}| = \{\mathcal{F} \in P \mid \psi(J_C) \subset t_{\mathcal{F}}^{-1}(\Theta_{\tilde{C}})\} = \bigcap_{\xi \in \mathrm{Pic}^0\,(C)} t_{\phi^*\xi}^{-1}(\Theta_{\tilde{C}})|_P.$$

More precisely, $H^0(P, \mathcal{L})$ is generated by defining sections of the hypersurfaces $t_{\phi^*\xi}^{-1}(\Theta_{\tilde{C}})|_P \subset P$ for $\xi \in \mathrm{Pic}^0\,(C)$.

For $[\tilde{C} \stackrel{\phi}{\to} C] \in \mathcal{R}_{g,2n}$, there exists a unique invertible sheaf $\eta_\phi \in \mathrm{Pic}^n\,(C)$ such that $\Omega^1_{\tilde{C}} = \phi^*(\Omega^1_C \otimes \eta_\phi)$. Let

$$P_\phi = \mathrm{Nm}^{-1}(\Omega^1_C \otimes \eta_\phi) \subset \mathrm{Pic}^{\tilde{g}-1}\,(\tilde{C})$$

be the fiber of the norm map at $\Omega^1_C \otimes \eta_\phi \in \mathrm{Pic}\,(C)$, where $\tilde{g} = 2g + n - 1$ is the genus of $\tilde{C}$. We set

$$W_{\tilde{C}} = \{\mathcal{F} \in \mathrm{Pic}^{\tilde{g}-1}\,(\tilde{C}) \mid h^0(\tilde{C}, \mathcal{F}) \ge 1\}$$

and $\mathcal{L}_\phi = \mathcal{O}_{\mathrm{Pic}^{\tilde{g}-1}\,(\tilde{C})}(W_{\tilde{C}})|_{P_\phi}$. Then there exists $\mathcal{F}_0 \in P_\phi$ such that

$$\mathcal{O}_{J_{\tilde{C}}}(\Theta_{\tilde{C}}) = t^*_{\mathcal{F}_0}\mathcal{O}_{\mathrm{Pic}^{\tilde{g}-1}\,(\tilde{C})}(W_{\tilde{C}}),$$

hence the translation $t_{\mathcal{F}_0}$ induces isomorphisms $P = \mathrm{Ker}\,(\mathrm{Nm}) \xrightarrow{\simeq} P_\phi$ and $\mathrm{Bs}\,|\mathcal{L}| \xrightarrow{\simeq} \mathrm{Bs}\,|\mathcal{L}_\phi|$. In [13], Naranjo and Ortega gave an explicit description for $\mathrm{Bs}\,|\mathcal{L}_\phi|$. For $j \geq 0$, we denote by $\tilde{C}^{(j)}$ the j-th symmetric product of $\tilde{C}$, which parameterizes effective divisors of degree j on $\tilde{C}$. We define the subvariety in $\mathrm{Pic}^{\tilde{g}-1}\,(\tilde{C})$ by

$$B_i = \mathrm{Image}\,(\tilde{C}^{(n-2i)} \times \mathrm{Pic}^{g-1+i}\,(C) \longrightarrow \mathrm{Pic}^{\tilde{g}-1}\,(\tilde{C});\ (D, \mathcal{G}) \longmapsto \mathcal{O}_{\tilde{C}}(D) \otimes \phi^*\mathcal{G})$$

for $2i \leq n$. When $2i > n$, we set $B_i = \emptyset$.

Proposition 2.2 ([13] Proposition 1.6.)**.** $\mathrm{Bs}\,|\mathcal{L}_\phi| = B_1 \cap P_\phi$.

When we set $W_\xi = t^{-1}_{\phi^*\xi}(W_{\tilde{C}})|_{P_\phi} \in |\mathcal{L}_\phi|$ for $\xi \in \mathrm{Pic}^0\,(C)$, as in Proposition 2.1 we have

$$\mathrm{Bs}\,|\mathcal{L}_\phi| = \bigcap_{\xi \in \mathrm{Pic}^0\,(C)} W_\xi = B_1 \cap P_\phi,$$

and $H^0(P_\phi, \mathcal{L}_\phi)$ is generated by defining sections of W_ξ for $\xi \in \mathrm{Pic}^0\,(C)$. We define $S_i(P_\phi, \mathcal{L}_\phi) = \bigcap_{\Theta \in |\mathcal{L}_\phi|} \Theta_{\geq i}$, where $\Theta_{\geq i}$ denotes the higher multiplicity locus

$$\Theta_{\geq i} = \{\mathcal{F} \in \Theta \mid \mathrm{mult}_{\mathcal{F}}\,\Theta \geq i\} \subset P_\phi.$$

By the next proposition, we have

$$S(P, \mathcal{L}) \xrightarrow{\simeq} S_2(P_\phi, \mathcal{L}_\phi) = \bigcap_{\xi \in \mathrm{Pic}^0\,(C)} W_{\xi,\mathrm{sing}} = B_2 \cap P_\phi.$$

Proposition 2.3. *When* $i \geq 1$*,*

$$S_i(P_\phi, \mathcal{L}_\phi) = \bigcap_{\xi \in \mathrm{Pic}^0\,(C)} W_{\xi,\geq i} = B_i \cap P_\phi.$$

We prepare some lemmas to prove Proposition 2.3.

Lemma 2.4. *If* $\mathcal{F} \in B_i \cap P_\phi$*, then* $\mathrm{mult}_{\mathcal{F}}\,W_\xi \geq i$ *for any* $\xi \in \mathrm{Pic}^0\,(C)$*.*

Proof. There exist $D \in \tilde{C}^{(n-2i)}$ and $\mathcal{G} \in \mathrm{Pic}^{g-1+i}\,(\mathcal{G})$ such that $\mathcal{F} = \mathcal{O}_{\tilde{C}}(D) \otimes \phi^*\mathcal{G}$. Since

$$\mathcal{G} \otimes \xi \subset \phi_*\phi^*(\mathcal{G} \otimes \xi) \subset \phi_*(\mathcal{O}_{\tilde{C}}(D) \otimes \phi^*(\mathcal{G} \otimes \xi)) = \phi_*(\mathcal{F} \otimes \phi^*\xi),$$

by the Riemann's singularity theorem, we have

$$\mathrm{mult}_{\mathcal{F}}\,W_\xi \geq \mathrm{mult}_{\mathcal{F}}\,t^{-1}_{\phi^*\xi}(W_{\tilde{C}}) = h^0(\tilde{C}, \mathcal{F} \otimes \phi^*\xi) \geq h^0(C, \mathcal{G} \otimes \xi) \geq i.$$

□

The following Mumford's exact sequence is essential for the proof of Proposition 2.3.

Lemma 2.5 ([11] p338)**.** *Let* D *be an effective divisor on* $\tilde{C}$*. If* $\phi^*p \not\leq D$ *for any* $p \in C$*, then there exists an exact sequence*

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \phi_*\mathcal{O}_{\tilde{C}}(D) \longrightarrow \mathrm{Nm}\,(\mathcal{O}_{\tilde{C}}(D)) \otimes \eta_\phi^\vee \longrightarrow 0$$

of locally free sheaves on C*.*

In the following, we write $H^i(\mathcal{G}) = H^i(C, \mathcal{G})$ for any sheaf $\mathcal{G}$ on C. When $0 < 2i \le n$, for $E \in C^{(n-2i)}$ and $\eta \in \mathrm{Pic}^n(C)$, we define

$$K_i(E, \eta) = \{\mathcal{G} \in \mathrm{Pic}^{g-1+i}(C) \mid h^0(\mathcal{G}) = h^0(\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-E) \otimes \mathcal{G}^\vee) = i\}.$$

For $\mathcal{G} \in K_i(E, \eta)$, we define the hypersurface $\mathcal{V}_{E,\mathcal{G}} \subset H^0(\Omega^1_C \otimes \eta)^\vee$ of degree i by

$$\det \begin{pmatrix} \mu(s_1, t_1) & & \mu(s_1, t_i) \\ & \cdots & \\ \mu(s_i, t_1) & & \mu(s_i, t_i) \end{pmatrix} \in \mathrm{Sym}^i H^0(\Omega^1_C \otimes \eta),$$

where $s_1, \ldots, s_i \in H^0(\mathcal{G})$ and $t_1, \ldots, t_i \in H^0(\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-E) \otimes \mathcal{G}^\vee)$ are bases, and

$$\mu : H^0(\mathcal{G}) \times H^0(\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-E) \otimes \mathcal{G}^\vee) \longrightarrow H^0(\Omega^1_C \otimes \eta)$$

denotes the composition of the multiplication map and the injective homomorphism $H^0(\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-E)) \hookrightarrow H^0(\Omega^1_C \otimes \eta)$. We note that $\mathcal{V}_{E,\mathcal{G}}$ does not depend on the choice of bases $s_1, \ldots, s_i$ and $t_1, \ldots, t_i$.

Lemma 2.6. *Let $D \in \tilde{C}^{(n-2i)}$ and $\mathcal{G} \in \mathrm{Pic}^{g-1+i}(C)$ be such that $\mathcal{F} = \mathcal{O}_{\tilde{C}}(D) \otimes \phi^*\mathcal{G} \in P_\phi$. For $\xi \in \mathrm{Pic}^0(C)$, the following conditions are equivalent:*

(1) $\mathrm{mult}_{\mathcal{F}} W_\xi = i$.
(2) $\mathcal{G} \otimes \xi \in K_i(\phi(D), \eta_\phi)$, *and* $\phi^*p \not\le D$ *for any* $p \in C$.

If these conditions are satisfied, then the tangent cone $\mathcal{T}_{W_\xi,\mathcal{F}}$ of W_ξ at $\mathcal{F}$ is

$$\mathcal{V}_{\phi(D),\mathcal{G}\otimes\xi} \subset H^0(\Omega^1_C \otimes \eta_\phi)^\vee \simeq T_{P_\phi,\mathcal{F}}.$$

Proof. First, assume condition (1). If $\phi^*p \le D$ for some $p \in C$, then $\mathcal{F} \in B_{i+1} \cap P_\phi$, and by Lemma 2.4, we have $\mathrm{mult}_{\mathcal{F}} W_\xi \ge i+1$, which contradicts the assumption that $\mathrm{mult}_{\mathcal{F}} W_\xi = i$. Thus, $\phi^*p \not\le D$ for any $p \in C$. Since $\mathrm{Nm}(\mathcal{O}_{\tilde{C}}(D)) \otimes \mathcal{G}^{\otimes 2} = \Omega^1_C \otimes \eta_\phi$, by Lemma 2.5, we obtain the exact sequence

$$0 \longrightarrow \mathcal{G} \otimes \xi \longrightarrow \phi_*(\mathcal{F} \otimes \phi^*\xi) \longrightarrow \Omega^1_C \otimes \mathcal{G}^\vee \otimes \xi \longrightarrow 0.$$

Since

$$i = \mathrm{mult}_{\mathcal{F}} W_\xi \ge \mathrm{mult}_{\mathcal{F}} t^{-1}_{\phi^*\xi}(W_{\tilde{C}}) = h^0(\tilde{C}, \mathcal{F} \otimes \phi^*\xi) \ge h^0(\mathcal{G} \otimes \xi) \ge i,$$

it follows that $h^0(\mathcal{G} \otimes \xi) = i$ and $h^1(\mathcal{G} \otimes \xi) = 0$. Therefore, $\mathcal{G} \otimes \xi \in K_i(\phi(D), \eta_\phi)$, because

$$\begin{aligned} &h^0(\Omega^1_C \otimes \eta_\phi \otimes \mathrm{Nm}(\mathcal{O}_{\tilde{C}}(-D)) \otimes (\mathcal{G} \otimes \xi)^\vee) = h^0(\mathcal{G} \otimes \xi^\vee) \\ =&h^0(\Omega^1_C \otimes \mathcal{G}^\vee \otimes \xi) + i = h^0(\tilde{C}, \mathcal{F} \otimes \phi^*\xi) - h^0(\mathcal{G} \otimes \xi) + i = i. \end{aligned}$$

Next, assume condition (2). By using Lemma 2.5, we obtain

$$\mathrm{mult}_{\mathcal{F}} t^{-1}_{\phi^*\xi}(W_{\tilde{C}}) = h^0(\tilde{C}, \mathcal{F} \otimes \phi^*\xi) = h^0(\mathcal{G} \otimes \xi) = i.$$

Let $\sigma : \tilde{C} \to \tilde{C}$ be the covering involution of ϕ. Since $\mathcal{F} \in P_\phi$, we have

$$\Omega^1_{\tilde{C}} = \phi^*(\Omega^1_C \otimes \eta_\phi) = \mathcal{F} \otimes \sigma^*\mathcal{F} = \mathcal{F} \otimes \mathcal{O}_{\tilde{C}}(\sigma^*D) \otimes \phi^*\mathcal{G}.$$

The injective homomorphisms

$$\mathcal{G} \otimes \xi \hookrightarrow \phi_*(\mathcal{O}_{\tilde{C}}(D)) \otimes \mathcal{G} \otimes \xi = \phi_*(\mathcal{O}_{\tilde{C}}(D) \otimes \phi^*(\mathcal{G} \otimes \xi)) = \phi_*(\mathcal{F} \otimes \phi^*\xi)$$

and

$$\Omega^1_C \otimes \eta_\phi \otimes \mathrm{Nm}\,(\mathcal{O}_{\tilde{C}}(-D)) \otimes (\mathcal{G}\otimes\xi)^\vee = \mathcal{G}\otimes\xi^\vee$$
$$\hookrightarrow \phi_*(\mathcal{O}_{\tilde{C}}(\sigma^* D))\otimes\mathcal{G}\otimes\xi^\vee = \phi_*(\mathcal{O}_{\tilde{C}}(\sigma^*D)\otimes\phi^*(\mathcal{G}\otimes\xi^\vee)) = \phi_*(\Omega^1_{\tilde{C}}\otimes(\mathcal{F}\otimes\phi^*\xi)^\vee)$$

induce the isomorphisms

$$H^0(C,\mathcal{G}\otimes\xi) \xrightarrow{\simeq} H^0(\tilde{C},\mathcal{F}\otimes\phi^*\xi)$$

and

$$H^0(C,\Omega^1_C\otimes\eta_\phi\otimes\mathrm{Nm}\,(\mathcal{O}_{\tilde{C}}(-D))\otimes(\mathcal{G}\otimes\xi)^\vee) \xrightarrow{\simeq} H^0(\tilde{C},\Omega^1_{\tilde{C}}\otimes(\mathcal{F}\otimes\phi^*\xi)^\vee).$$

From the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{G}\otimes\xi)\times H^0(\Omega^1_C\otimes\eta_\phi\otimes\mathrm{Nm}\,(\mathcal{O}_{\tilde{C}}(-D))\otimes(\mathcal{G}\otimes\xi)^\vee) & \xrightarrow{\mu} & H^0(\Omega^1_C\otimes\eta_\phi) \\ \simeq\downarrow & \circlearrowleft & \downarrow j \\ H^0(\tilde{C},\mathcal{F}\otimes\phi^*\xi)\times H^0(\tilde{C},\Omega^1_{\tilde{C}}\otimes(\mathcal{F}\otimes\phi^*\xi)^\vee) & \underset{\cup}{\longrightarrow} & H^0(\tilde{C},\Omega^1_{\tilde{C}}), \end{array}$$

and by the Riemann-Kempf singularity theorem [9, Theorem 2], the tangent cone of $t^{-1}_{\phi^*\xi}(W_{\tilde{C}})$ at $\mathcal{F}$ is

$$(j^\vee)^{-1}(\mathcal{V}_{\phi(D),\mathcal{G}\otimes\xi}) \subsetneq H^0(\tilde{C},\Omega^1_{\tilde{C}})^\vee \simeq T_{\mathrm{Pic}^{\tilde{g}-1}(\tilde{C}),\mathcal{F}},$$

where $j^\vee : H^0(\tilde{C},\Omega^1_{\tilde{C}})^\vee \to H^0(\Omega^1_C\otimes\eta_\phi)^\vee$ is the dual of the injective homomorphism

$$j : H^0(\Omega^1_C\otimes\eta_\phi) \longrightarrow H^0(\tilde{C},\phi^*(\Omega^1_C\otimes\eta_\phi)) = H^0(\tilde{C},\Omega^1_{\tilde{C}}).$$

Since $T_{P_\phi,\mathcal{F}} \subset H^0(\tilde{C},\Omega^1_{\tilde{C}})^\vee$ is isomorphic to $H^0(\Omega^1_C\otimes\eta_\phi)^\vee$ via $j^\vee$, the tangent cone of W_ξ at $\mathcal{F}$ is

$$T_{P_\phi,\mathcal{F}}\cap(j^\vee)^{-1}(\mathcal{V}_{\phi(D),\mathcal{G}\otimes\xi}) \simeq \mathcal{V}_{\phi(D),\mathcal{G}\otimes\xi} \subsetneq H^0(\Omega^1_C\otimes\eta_\phi)^\vee,$$

and hence we conclude that $\mathrm{mult}_{\mathcal{F}} W_\xi = i$. □

Proof of Proposition 2.3. Let $\Theta \in |\mathcal{L}_\phi|$ be the hypersurface defined by a section $\beta \in H^0(P_\phi,\mathcal{L}_\phi)$. We have $\Theta_{\geq i} \supset \bigcap_{\xi\in\mathrm{Pic}^0(C)} W_{\xi,\geq i}$, because β is a linear combination of defining sections of W_ξ for $\xi \in \mathrm{Pic}^0(C)$. Hence, we have $S_i(P_\phi,\mathcal{L}_\phi) = \bigcap_{\xi\in\mathrm{Pic}^0(C)} W_{\xi,\geq i}$.

By Lemma 2.4, we have $B_i\cap P_\phi \subset \bigcap_{\xi\in\mathrm{Pic}^0(C)} W_{\xi,\geq i}$. By induction on i, we prove $B_i\cap P_\phi = \bigcap_{\xi\in\mathrm{Pic}^0(C)} W_{\xi,\geq i}$. When $i=1$, this is true by Proposition 2.2. We assume $i\geq 2$. By the induction assumption,

$$\bigcap_{\xi\in\mathrm{Pic}^0(C)} W_{\xi,\geq i} \subset \bigcap_{\xi\in\mathrm{Pic}^0(C)} W_{\xi,\geq i-1} = B_{i-1}\cap P_\phi.$$

When $n < 2(i-1)$, we have defined that $B_{i-1} = B_i = \emptyset$, and it follows that $\bigcap_{\xi\in\mathrm{Pic}^0(C)} W_{\xi,\geq i} = \emptyset$ and $B_i\cap P_\phi = \emptyset$. When $n \geq 2(i-1)$, for $\mathcal{F} \in \bigcap_{\xi\in\mathrm{Pic}^0(C)} W_{\xi,\geq i}$, there exist $D \in \tilde{C}^{(n-2(i-1))}$ and $\mathcal{G} \in \mathrm{Pic}^{g-1+(i-1)}(C)$ such that

$$\mathcal{F} = \mathcal{O}_{\tilde{C}}(D)\otimes\phi^*\mathcal{G}, \quad \Omega^1_C\otimes\eta_\phi = \mathrm{Nm}\,(\mathcal{O}_{\tilde{C}}(D))\otimes\mathcal{G}^{\otimes 2}.$$

Since $\deg\mathcal{G} = g+i-2$, we can find $\xi_0 \in \mathrm{Pic}^0(C)$ such that $h^0(\mathcal{G}\otimes\xi_0) = i-1$ and $h^0(\mathcal{G}\otimes\xi_0^\vee) = i-1$. Then, we have $\mathcal{G}\otimes\xi_0 \in K_{i-1}(\phi(D),\eta_\phi)$. If $\mathcal{F}\notin B_i$, then

$\phi^*p \nleq D$ for any $p \in D$, and by Lemma 2.6, we have $\operatorname{mult}_{\mathcal{F}} W_{\xi_0} = i-1$. This is a contradiction to $\mathcal{F} \in \bigcap_{\xi \in \operatorname{Pic}^0(C)} W_{\xi, \geq i}$, hence $\mathcal{F} \in B_i$. □

Proposition 2.7. $\dim(B_i \cap P_\phi) = n - 2i$ *for* $0 < 2i \leq n$*. If* $0 < 2i < n$*, then* $B_i \cap P_\phi$ *is irreducible. If* $0 < 2i = n$*, then* $\#(B_i \cap P_\phi) = 4^g$*.*

Proof. When $0 < 2i \leq n$, we define

$$Y_i = \{(D, \mathcal{G}) \in \tilde{C}^{(n-2i)} \times \operatorname{Pic}^{g-1+i}(C) \mid \operatorname{Nm}(\mathcal{O}_{\tilde{C}}(D)) \otimes \mathcal{G}^{\otimes 2} = \Omega^1_C \otimes \eta_\phi\}.$$

The projection

$$g_i : Y_i \longrightarrow \tilde{C}^{(n-2i)};\ (D, \mathcal{G}) \longmapsto D$$

is a finite étale covering of degree 4^g, hence Y_i is smooth. When $n = 2i$,

$$Y_i \longrightarrow B_i \cap P_\phi;\ (0, \mathcal{G}) \longmapsto \phi^*\mathcal{G}$$

is bijective, hence $\#(B_i \cap P_\phi) = \#Y_i = 4^g$. Assume $n \geq 2i+1$. Let $\gamma : [0,1] \to \tilde{C}$ be a loop on $\tilde{C}$, and let ι be the morphism defined by

$$\iota : C \longrightarrow J_C = \operatorname{Pic}^0(C);\ p \longmapsto \mathcal{O}_C(p - \phi(\gamma(0))).$$

Then, there is a unique lift $\delta : [0,1] \to H^0(\Omega^1_C)^\vee$ of $\iota \circ \phi \circ \gamma$ such that $\delta(0) = 0$:

$$\begin{array}{ccccc} [0,1] & \xrightarrow{\delta} & H^0(\Omega^1_C)^\vee & \longrightarrow & H^0(\Omega^1_C)^\vee / H_1(C,\mathbb{Z}) \\ \gamma\downarrow & & \circlearrowleft & & \downarrow\simeq \\ \tilde{C} & \xrightarrow[\phi]{} & C & \xrightarrow[\iota]{} & J_C. \end{array}$$

For $t \in [0,1]$, we denote by $\xi(t) \in \operatorname{Pic}^0(C)$ the invertible sheaf corresponding to $\left[\frac{1}{2}\delta(t)\right] \in H^0(\Omega^1_C)^\vee / H_1(C,\mathbb{Z}) = J_C$. If $(D, \mathcal{G}) = (\gamma(0) + D', \mathcal{G}) \in Y_i$, then the path

$$[0,1] \longrightarrow Y_i;\ t \longmapsto (\gamma(t) + D', \mathcal{G} \otimes \xi(t)^\vee)$$

on Y_i is the lift of the loop

$$[0,1] \longrightarrow \tilde{C}^{(n-2i)};\ t \longmapsto \gamma(t) + D'$$

on $\tilde{C}^{(n-2i)}$. Since $\phi_* : H_1(\tilde{C}, \mathbb{Z}) \to H_1(C, \mathbb{Z})$ is surjective,

$$H_1(\tilde{C}, \mathbb{Z}) \longrightarrow J_{C,2} = \operatorname{Ker}([2]_{J_C});\ [\gamma] \longmapsto \xi(1) = \left[\frac{1}{2}\delta(1)\right]$$

is surjective. This means that $\pi_1(\tilde{C}^{(n-2i)}, D)$ acts transitively on the fiber $g_i^{-1}(D)$, and it follows that Y_i is connected. Since Y_i is also smooth, it is irreducible. Let U_i be the subset of $\tilde{C}^{(n-2i)}$ defined by

$$U_i = \{D \in \tilde{C}^{(n-2i)} \mid h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D)) = 1,\ \phi^*p \nleq D \text{ for any } p \in C\}.$$

Since $0 < n - 2i \leq 2g + n - 1 = \tilde{g}$, it is a non-empty open subset of $\tilde{C}^{(n-2i)}$. For $(D_1, \mathcal{G}_1), (D_2, \mathcal{G}_2) \in g_i^{-1}(U_i)$, we have

$$h^0(\operatorname{Nm}(\mathcal{O}_{\tilde{C}}(D_1)) \otimes \eta_\phi^\vee \otimes \mathcal{G}_1 \otimes \mathcal{G}_2^\vee) = h^0(\Omega^1_C \otimes \mathcal{G}_1^\vee \otimes \mathcal{G}_2^\vee) = 0,$$

because $\deg(\Omega^1_C \otimes \mathcal{G}_1^\vee \otimes \mathcal{G}_2^\vee) = -2i < 0$. If $\mathcal{O}_{\tilde{C}}(D_1) \otimes \phi^*\mathcal{G}_1 = \mathcal{O}_{\tilde{C}}(D_2) \otimes \phi^*\mathcal{G}_2$, then by Lemma 2.5, we have

$$h^0(\mathcal{G}_1 \otimes \mathcal{G}_2^\vee) = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D_1) \otimes \phi^*(\mathcal{G}_1 \otimes \mathcal{G}_2^\vee)) = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D_2)) = 1,$$

hence $\mathcal{G}_1 \otimes \mathcal{G}_2^\vee = \mathcal{O}_C$ and $D_1 = D_2$. This means that the morphism

$$Y_i \longrightarrow B_i \cap P_\phi;\ (D, \mathcal{G}) \longmapsto \mathcal{O}_{\tilde{C}}(D) \otimes \phi^*\mathcal{G}$$

is birational. Hence, $B_i \cap P_\phi$ is irreducible, and

$$\dim (B_i \cap P_\phi) = \dim Y_i = n - 2i.$$

□

By Proposition 2.3 and Proposition 2.7, we have the following:

Corollary 2.8. *Let $(P, [\mathcal{L}])$ be the Prym variety of $[\tilde{C} \overset{\phi}{\to} C] \in \mathcal{R}_{g,2n}$ for $n \geq 1$ and $g \geq 1$.*

(1) (a) $\dim \operatorname{Bs}|\mathcal{L}| = n - 2$ *and* $\operatorname{Bs}|\mathcal{L}|$ *is irreducible for* $n \geq 3$,
 (b) $\dim \operatorname{Bs}|\mathcal{L}| = 0$ *and* $\#\operatorname{Bs}|\mathcal{L}| = 4^g$ *for* $n = 2$,
 (c) $\operatorname{Bs}|\mathcal{L}| = \emptyset$ *for* $n = 1$,
(2) (a) $\dim S(P, \mathcal{L}) = n - 4$ *and* $S(P, \mathcal{L})$ *is irreducible for* $n \geq 5$,
 (b) $\dim S(P, \mathcal{L}) = 0$ *and* $\#S(P, \mathcal{L}) = 4^g$ *for* $n = 4$,
 (c) $S(P, \mathcal{L}) = \emptyset$ *for* $1 \leq n \leq 3$.

3. Quadric hypersurfaces containing projective curves

Let C be a projective smooth curve of genus g, and let η be an invertible sheaf on C of degree $n \geq 3$. By [10, p55, Corollary], the invertible sheaf $\Omega_C^1 \otimes \eta$ is very ample, and the multiplication map

$$\rho_i : \operatorname{Sym}^i H^0(\Omega_C^1 \otimes \eta) \longrightarrow H^0((\Omega_C^1 \otimes \eta)^{\otimes i})$$

is surjective for $i \geq 1$. When $0 < 2i \leq n$, we have defined the hypersurface $\mathcal{V}_{E,\mathcal{G}} \subset H^0(\Omega_C^1 \otimes \eta)^\vee = \operatorname{Spec}(\operatorname{Sym} H^0(\Omega_C^1 \otimes \eta))$ for $E \in C^{(n-2i)}$ and $\mathcal{G} \in K_i(E, \eta)$. Let $Q_{E,\mathcal{G}} \subset \mathbb{P}(H^0(\Omega_C^1 \otimes \eta)) = \operatorname{Proj}(\operatorname{Sym} H^0(\Omega_C^1 \otimes \eta))$ be the projectivization of $\mathcal{V}_{E,\mathcal{G}}$. Then we have

$$Q_{E,\mathcal{G}} = \bigcup_{D \in |\mathcal{G}|} \mathbb{P}(H^0(\Omega_C^1 \otimes \eta)/V_{E+D}) = \bigcup_{D \in |\Omega_C^1 \otimes \eta \otimes \mathcal{O}_C(-E) \otimes \mathcal{G}^\vee|} \mathbb{P}(H^0(\Omega_C^1 \otimes \eta)/V_{E+D})$$

in $\mathbb{P}(H^0(\Omega_C^1 \otimes \eta))$, where

$$V_{E+D} = \operatorname{Image}(H^0(\Omega_C^1 \otimes \eta \otimes \mathcal{O}_C(-E - D)) \hookrightarrow H^0(\Omega_C^1 \otimes \eta)).$$

Let

$$\Phi = \Phi_{|\Omega_C^1 \otimes \eta|} : C \longrightarrow \mathbb{P}(H^0(\Omega_C^1 \otimes \eta)) = \mathbb{P}^{d-1}$$

be the closed immersion defined by the very ample linear system $|\Omega_C^1 \otimes \eta|$. When $i \geq 2$, the defining section of $Q_{E,\mathcal{G}} \subset \mathbb{P}^{d-1}$ is contained in

$$\operatorname{Ker}(\rho_i) \subset \operatorname{Sym}^i H^0(\Omega_C^1 \otimes \eta) = H^0(\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}}(i)),$$

hence, by the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\Phi_{|(\Omega_C^1 \otimes \eta)^{\otimes i}|}} & \mathbb{P}(H^0((\Omega_C^1 \otimes \eta)^{\otimes i})) \\ \Phi \downarrow & \circlearrowleft & \downarrow \rho_i^\vee \\ \mathbb{P}(H^0(\Omega_C^1 \otimes \eta)) = \mathbb{P}^{d-1} & \xrightarrow[|\mathcal{O}_{\mathbb{P}^{d-1}}(i)|]{} & \mathbb{P}(\operatorname{Sym}^i H^0(\Omega_C^1 \otimes \eta)) = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{d-1}}(i))), \end{array}$$

we have $\Phi(C) \subset Q_{E,\mathcal{G}}$.

Proposition 3.1. *Let $\Sigma \subset C$ be a finite subset. If $n \geq 4$ and C is an elliptic curve or a hyperelliptic curve, then* $\mathrm{Ker}\,(\rho_2)$ *is generated by the defining quadratics of $Q_{E,\mathcal{G}}$ for $E \in (C \setminus \Sigma)^{(n-4)}$ and $\mathcal{G} \in K_2(E,\eta)$.*

First we prove it in the case where $n = 4$.

Lemma 3.2. *If $n = 4$ and C is an elliptic curve or a hyperelliptic curve, then* $\mathrm{Ker}\,(\rho_2)$ *is generated by the defining quadratics of $Q_{0,\mathcal{G}}$ for $\mathcal{G} \in K_2(0,\eta)$.*

Proof. Since C is an elliptic curve or a hyperelliptic curve, there exists an invertible sheaf $\mathcal{H} \in \mathrm{Pic}^2\,(C)$ such that $h^0(\mathcal{H}) = 2$. Let $U \subset C^{g+1}$ be the set of $(p_1, \ldots, p_{g+1}) \in C^{g+1}$ satisfying the following conditions:

1. $p_i \neq p_j$ for $1 \leq i < j \leq g+1$,
2. $\mathcal{O}_C(p_1 + \cdots + p_{g+1}) \in K_2(0,\eta)$,
3. $\mathcal{H} \otimes \mathcal{O}_C(p_1 + \cdots + p_{g+1} - p_i - p_j) \in K_2(0,\eta)$ for $1 \leq i < j \leq g+1$,
4. $h^0(\mathcal{H}^\vee \otimes \mathcal{O}_C(p_1 + \cdots + p_{g+1})) = 0$,
5. $h^0(\Omega^1_C \otimes \eta \otimes \mathcal{H}^\vee \otimes \mathcal{O}_C(-p_1 - \cdots - p_{g+1})) = 0$.

For $\mathcal{G} \in \mathrm{Pic}^{g+1}\,(C)$, the condition $\mathcal{G} \in K(0,\eta)$ is equivalent to

$$h^0(\Omega^1_C \otimes \mathcal{G}^\vee) = h^0(\mathcal{G} \otimes \eta^\vee) = 0.$$

Since

$$\deg\,(\Omega^1_C \otimes \mathcal{G}^\vee) = \deg\,(\mathcal{G} \otimes \eta^\vee) = g - 3 < g,$$

$K_2(0,\eta) = \mathrm{Pic}^{g+1}\,(C)$ for $g \leq 2$, and $K_2(0,\eta)$ is a non-empty open subset of $\mathrm{Pic}^{g+1}\,(C)$ with $\dim\,(\mathrm{Pic}^{g+1}\,(C) \setminus K_2(0,\eta)) = g - 3$ for $g \geq 3$. Hence the set of $(p_1, \ldots, p_{g+1}) \in C^{g+1}$ satisfying (2) is a non-empty open subset of C^{g+1}. Since the image of the map

$$C^{g+1} \longrightarrow \mathrm{Pic}^{g+1}\,(C);\ (p_1, \ldots, p_{g+1}) \longmapsto \mathcal{H} \otimes \mathcal{O}_C(p_1 + \cdots + p_{g+1} - p_i - p_j)$$

has dimension $g - 1$ for $1 \leq i < j \leq g+1$, the set of $(p_1, \ldots, p_{g+1}) \in C^{g+1}$ satisfying (3) is a non-empty open subset of C^{g+1}. Since (1), (4) and (5) are also open conditions, U is a non-empty open subset of C^{g+1}. We fix $(p_1, \ldots, p_{g+1}) \in U$, and set $\mathcal{G}_0 = \mathcal{O}_C(p_1 + \cdots + p_{g+1})$ and $\mathcal{G}_{ij} = \mathcal{H} \otimes \mathcal{O}_C(p_1 + \cdots + p_{g+1} - p_i - p_j)$ for $1 \leq i < j \leq g+1$. We show that the defining quadratics of $Q_{0,\mathcal{G}}$ and $Q_{0,\mathcal{G}_{ij}}$ form a basis of $\mathrm{Ker}\,(\rho_2)$. For $E_0 \neq E_\infty \in |\mathcal{H}|$, there is a rational function f on C such that $\mathrm{Div}\,(f) = E_0 - E_\infty$. Let $\alpha_{ij} \in H^0(\mathcal{G}_{ij})$ be a section defining the divisor $E_\infty + p_1 + \cdots + p_{g+1} - p_i - p_j \in |\mathcal{G}_{ij}|$. Then $\alpha'_{ij} = f\alpha_{ij} \in H^0(\mathcal{G}_{ij})$ defines the divisor $E_0 + p_1 + \cdots + p_{g+1} - p_i - p_j \in |\mathcal{G}_{ij}|$, and by (3), $\alpha_{ij}, \alpha'_{ij}$ form a basis of $H^0(\mathcal{G}_{ij})$. We set

$$V = \mathrm{Image}\,(H^0(\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-E_\infty)) \hookrightarrow H^0(\Omega^1_C \otimes \eta))$$

and

$$V_k = \mathrm{Image}\,(H^0(\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-E_\infty - p_k)) \hookrightarrow H^0(\Omega^1_C \otimes \eta))$$

for $1 \leq k \leq g+1$. Since $\dim V = g+1$, by (1) and (5), we have $\bigcap_{k=1}^{g+1} V_k = \{0\}$, and $\dim \left(\bigcap_{k \neq i} V_k\right) = 1$ for $1 \leq i \leq g+1$. Thus, there is a unique divisor $D_i \in$

$|\Omega_C^1 \otimes \eta \otimes \mathcal{H}^\vee \otimes \mathcal{O}_C(-p_1 - \cdots - p_{g+1} + p_i)|$. By (1) and (5), we have

$$D_i + p_j \neq D_j + p_i \in |\Omega_C^1 \otimes \eta \otimes \mathcal{G}_{ij}^\vee|.$$

for $1 \leq i < j \leq g+1$. Let $\beta_{ij} \in H^0(\Omega_C^1 \otimes \eta \otimes \mathcal{G}_{ij}^\vee)$ define $D_i + p_j$, and $\beta'_{ij} \in H^0(\Omega_C^1 \otimes \eta \otimes \mathcal{G}_{ij}^\vee)$ define $D_j + p_i$. By (3), β_{ij}, β'_{ij} form a basis of $H^0(\Omega_C^1 \otimes \eta \otimes \mathcal{G}_{ij}^\vee)$. Let γ_i be a generator of $\bigcap_{k \neq i} V_k$ for $1 \leq i \leq g+1$. Then $\gamma_1, \ldots, \gamma_{g+1}$ form a basis of V. Since $\mu(\alpha_{ij}, \beta_{ij}) \in \bigcap_{k \neq i} V_k$ and $\mu(\alpha_{ij}, \beta'_{ij}) \in \bigcap_{k \neq j} V_k$, we have $\mu(\alpha_{ij}, \beta_{ij}) = c_{ij}\gamma_i$ and $\mu(\alpha_{ij}, \beta'_{ij}) = c'_{ij}\gamma_j$ for some $c_{ij}, c'_{ij} \in \mathbb{C}^*$. By using the rational function f, we define the injective homomorphism

$$\lambda : V \longrightarrow H^0(\Omega_C^1 \otimes \eta);\ \gamma \longmapsto f\gamma.$$

Then, by [1, p195, Corollary 2], the defining quadratics

$$\det \begin{pmatrix} \mu(\alpha_{ij}, \beta_{ij}) & \mu(\alpha_{ij}, \beta'_{ij}) \\ \mu(\alpha'_{ij}, \beta_{ij}) & \mu(\alpha'_{ij}, \beta'_{ij}) \end{pmatrix} = \det \begin{pmatrix} \mu(\alpha_{ij}, \beta_{ij}) & \mu(\alpha_{ij}, \beta'_{ij}) \\ \lambda(\mu(\alpha_{ij}, \beta_{ij})) & \lambda(\mu(\alpha_{ij}, \beta'_{ij})) \end{pmatrix}$$

$$= \det \begin{pmatrix} c_{ij}\gamma_i & c'_{ij}\gamma_j \\ \lambda(c_{ij}\gamma_i) & \lambda(c'_{ij}\gamma_j) \end{pmatrix} = c_{ij}c'_{ij} \det \begin{pmatrix} \gamma_i & \gamma_j \\ \lambda(\gamma_i) & \lambda(\gamma_j) \end{pmatrix} \in \operatorname{Sym}^2 H^0(\Omega_C^1 \otimes \eta)$$

of $Q_{0,\mathcal{G}_{ij}}$ for $1 \leq i < j \leq g+1$ are linearly independent over $\mathbb{C}$. Since

$$\dim \operatorname{Ker}(\rho_2) = \frac{(g+3)(g+4)}{2} - (3g+5) = \frac{g(g+1)}{2} + 1,$$

the defining quadratics of $Q_{0,\mathcal{G}_{ij}}$ for $1 \leq i < j \leq g+1$ generate a subspace of codimension 1 in $\operatorname{Ker}(\rho_2)$.

The quadric hypersurface $Q_{0,\mathcal{G}_{ij}} \subset \mathbb{P}(H^0(\Omega_C^1 \otimes \eta))$ contains the line

$$\ell_V = \mathbb{P}(H^0(\Omega_C^1 \otimes \eta)/V) \subset \mathbb{P}(H^0(\Omega_C^1 \otimes \eta))$$

for $1 \leq i < j \leq g+1$. It is enough to show that ℓ_V is not contained in

$$Q_{0,\mathcal{G}_0} = \bigcup_{D \in |\mathcal{G}_0|} \mathbb{P}(H^0(\Omega_C^1 \otimes \eta)/V_D) \subset \mathbb{P}(H^0(\Omega_C^1 \otimes \eta)).$$

If $D \in |\mathcal{G}_0|$ satisfies $\operatorname{supp}(E_\infty) \cap \operatorname{supp}(D) = \emptyset$, then, by (5), we have $V_D \cap V = \{0\}$ and $\ell_V \cap \mathbb{P}(H^0(\Omega_C^1 \otimes \eta)/V_D) = \emptyset$. By (2) and (4), since the pencil $|\mathcal{G}_0|$ has no base points, there exist at most two $D \in |\mathcal{G}_0|$ such that $\operatorname{supp}(E_\infty) \cap \operatorname{supp}(D) \neq \emptyset$. When $p \in \operatorname{supp}(E_\infty) \cap \operatorname{supp}(D)$ and $E_\infty = p + p'$, by (4), we have $p' \not\leq D$. If $V_D \subset V$, then

$$2 = h^0(\Omega_C^1 \otimes \eta \otimes \mathcal{O}_C(-D)) = h^0(\Omega_C^1 \otimes \eta \otimes \mathcal{O}_C(-D - p')),$$

and $h^0(\Omega_C^1 \otimes \eta \otimes \mathcal{O}_C(-D - p - p')) \geq 1$. But this contradicts (5), hence $V_D \not\subset V$ and $\ell_V \cap \mathbb{P}(H^0(\Omega_C^1 \otimes \eta)/V_D) = \{\Phi(p)\}$. This implies that $\ell_V \not\subset Q_{0,\mathcal{G}_0}$. □

Proof of Proposition 3.1. We prove it by induction on n. When $n = 4$, the result holds by Lemma 3.2. We assume $n \geq 5$. Let $p_1, p_2, p_3 \in C \setminus \Sigma$ be three distinct points. We denote by

$$\iota_i : \operatorname{Sym}^2 H^0(\Omega_C^1 \otimes \eta \otimes \mathcal{O}_C(-p_i)) \hookrightarrow \operatorname{Sym}^2 H^0(\Omega_C^1 \otimes \eta)$$

the natural injective homomorphism. Since the multiplication maps

$$\rho_{2,p_i} : \operatorname{Sym}^2 H^0(\Omega_C^1 \otimes \eta \otimes \mathcal{O}_C(-p_i)) \longrightarrow H^0((\Omega_C^1 \otimes \eta \otimes \mathcal{O}_C(-p_i))^{\otimes 2})$$

and

$$\rho_{2,p_i+p_j} : \operatorname{Sym}^2 H^0(\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-p_i - p_j)) \longrightarrow H^0((\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-p_i - p_j))^{\otimes 2})$$

are surjective, we have

$$\begin{aligned}&\dim(\iota_1(\operatorname{Ker}(\rho_{2,p_1})) + \iota_2(\operatorname{Ker}(\rho_{2,p_2})))\\ =&\dim \iota_1(\operatorname{Ker}(\rho_{2,p_1})) + \dim \iota_2(\operatorname{Ker}(\rho_{2,p_2})) - \dim(\iota_1(\operatorname{Ker}(\rho_{2,p_1})) \cap \iota_2(\operatorname{Ker}(\rho_{2,p_2})))\\ =&\dim \operatorname{Ker}(\rho_{2,p_1}) + \dim \operatorname{Ker}(\rho_{2,p_2}) - \dim \operatorname{Ker}(\rho_{2,p_1+p_2}) = \dim \operatorname{Ker}(\rho_2) - 1.\end{aligned}$$

Let

$$\pi_i : \mathbb{P}(H^0(\Omega^1_C \otimes \eta)) \setminus \{\Phi(p_i)\} \longrightarrow \mathbb{P}(H^0(\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-p_i)))$$

denote the projection from $\Phi(p_i)$, and let

$$\Phi_i = \Phi_{|\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-p_i)|} : C \longrightarrow \mathbb{P}(H^0(\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-p_i)))$$

be the closed immersion defined by the very ample linear system $|\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-p_i)|$. Let $\ell \subset \mathbb{P}(H^0(\Omega^1_C \otimes \eta))$ be the line through the points $\Phi(p_1)$ and $\Phi(p_2)$. Since

$$\pi_1(\ell) = \pi_1(\Phi(p_2)) = \Phi_1(p_2) \neq \Phi_1(p_3) = \pi_1(\Phi(p_3)),$$

we have $\Phi(p_3) \notin \ell$. Hence, $\pi_3(\ell)$ is a line in $\mathbb{P}(H^0(\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-p_3)))$, and we have

$$\pi_3(\ell) \nsubseteq \Phi_3(C) \subset \mathbb{P}(H^0(\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-p_3))).$$

By [15], we have $\Phi_3(C) = \bigcap_{g \in \operatorname{Ker}(\rho_{2,p_3})} Q_g$, where $Q_g \subset \mathbb{P}(H^0(\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-p_3)))$ denotes the quadric defined by g. Thus, there exists $g \in \operatorname{Ker}(\rho_{2,p_3})$ such that $\pi_3(\ell) \nsubseteq Q_g$. Since the quadric $\overline{\pi_3^{-1}(Q_g)} \subset \mathbb{P}(H^0(\Omega^1_C \otimes \eta))$ defined by $\iota_3(g) \in \operatorname{Ker}(\rho_2)$ does not contain the line ℓ, we have $\iota_3(g) \notin \iota_1(\operatorname{Ker}(\rho_{2,p_1})) + \iota_2(\operatorname{Ker}(\rho_{2,p_2}))$. Hence,

$$\operatorname{Ker}(\rho_2) = \iota_1(\operatorname{Ker}(\rho_{2,p_1})) + \iota_2(\operatorname{Ker}(\rho_{2,p_2})) + \iota_3(\operatorname{Ker}(\rho_{2,p_3})).$$

By the induction assumption, $\operatorname{Ker}(\rho_{2,p_i}) \subset \operatorname{Sym}^2 H^0(\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-p_i))$ is generated by the defining quadratics of $Q_{E,\mathcal{F}} \subset \mathbb{P}(H^0(\Omega^1_C \otimes \eta \otimes \mathcal{O}_C(-p_i)))$ for $E \in (C \setminus \Sigma)^{(n-5)}$ and $\mathcal{F} \in K_2(E, \eta \otimes \mathcal{O}_C(-p_i))$. Hence, $\iota_i(\operatorname{Ker}(\rho_{2,p_i}))$ is generated by the defining quadratics of $Q_{E+p_i,\mathcal{F}} = \overline{\pi_i^{-1}(Q_{E,\mathcal{F}})} \subset \mathbb{P}(H^0(\Omega^1_C \otimes \eta))$ for $E + p_i \in (C \setminus \Sigma)^{(n-4)}$ and $\mathcal{F} \in K_2(E, \eta \otimes \mathcal{O}_C(-p_i)) = K_2(E + p_i, \eta)$. □

Corollary 3.3. *Let $(P, [\mathcal{L}])$ be the Prym variety of $[\tilde{C} \stackrel{\phi}{\to} C] \in \mathcal{R}_{g,2n}$. If $n \geq 4$ and C is an elliptic curve or a hyperelliptic curve, then* $\operatorname{Ker}(\rho_2)$ *is generated by the defining quadratics of the tangent cones $\mathcal{T}_{\Theta,x}$ for $x \in S(P, \mathcal{L})$ and $\Theta \in |\mathcal{L}|$ satisfying* $\operatorname{mult}_x \Theta = 2$.

Proof. Since $S(P, \mathcal{L}) \stackrel{\simeq}{\to} S_2(P_\phi, \mathcal{L}_\phi)$, by Proposition 3.1, it is enough to show that for $E \in (C \setminus \operatorname{Branch}(\phi))^{(n-4)}$ and $\mathcal{G} \in K_2(E, \eta_\phi)$, the quadric $\mathcal{V}_{E,\mathcal{G}} \subset H^0(\Omega^1_C \otimes \eta_\phi)^\vee$ is the tangent cone $\mathcal{T}_{W_\xi,\mathcal{F}}$ for some $\mathcal{F} \in S_2(P_\phi, \mathcal{L}_\phi)$ and $\xi \in \operatorname{Pic}^0(C)$ satisfying $\operatorname{mult}_\mathcal{F} W_\xi = 2$. For $E \in (C \setminus \operatorname{Branch}(\phi))^{(n-4)}$, there exist $D \in \tilde{C}^{(n-4)}$ such that $\phi(D) = E$ and $\phi^* p \nleq D$ for any $p \in C$. Let $\delta \in \operatorname{Pic}^{g+1}(C)$ be an invertible sheaf such that $\delta^{\otimes 2} = \Omega^1_C \otimes \eta_\phi \otimes \mathcal{O}_C(-E)$. For $\mathcal{G} \in K_2(E, \eta_\phi)$, we set $\mathcal{F} = \mathcal{O}_{\tilde{C}}(D) \otimes \phi^*\delta$ and $\xi = \mathcal{G} \otimes \delta^\vee \in \operatorname{Pic}^0(C)$. Then, by Lemma 2.6, we have $\operatorname{mult}_\mathcal{F} W_\xi = 2$, and $\mathcal{V}_{E,\mathcal{G}}$ is the tangent cone of W_ξ at $\mathcal{F}$. □

4. Andreotti-Mayer loci

Let

$$\mathfrak{H}_d = \{\tau \in \mathrm{Mat}_d(\mathbb{C}) \mid \tau = {}^t\tau,\ \mathrm{Im}(\tau) > 0\}$$

be the Siegel upper half-space of degree $d = g + n - 1$. For the type of the polarization $\Delta = (\underbrace{1, \ldots, 1}_{n-1}, \underbrace{2, \ldots, 2}_{g})$, we denote the diagonal matrix $\mathrm{diag}\,\Delta \in \mathrm{Mat}_d(\mathbb{Z})$ by the same notation Δ. We define the action of $\mathbb{Z}^d \times \mathbb{Z}^d$ on $\mathbb{C}^d \times \mathfrak{H}_d$ by

$$(\mathbb{Z}^d \times \mathbb{Z}^d) \times (\mathbb{C}^d \times \mathfrak{H}_d) \longrightarrow \mathbb{C}^d \times \mathfrak{H}_d;\ ((\boldsymbol{m}', \boldsymbol{m}''), (\boldsymbol{z}, \tau)) \longmapsto (\boldsymbol{z} + \tau\boldsymbol{m}' + \Delta\boldsymbol{m}'', \tau),$$

and denote its quotient by $\mathcal{U}_d^{\Delta}$. Then

$$u : \mathcal{U}_d^{\Delta} \longrightarrow \mathfrak{H}_d;\ (\boldsymbol{z}, \tau) \longmapsto \tau$$

is a proper morphism of complex manifolds, and the fiber $A_\tau = u^{-1}(\tau)$ at $\tau \in \mathfrak{H}_d$ is an abelian variety of dimension d. Let $\mathcal{L}_\tau$ be the invertible sheaf on A_τ defined by the Hermitian form

$$H : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C};\ (\boldsymbol{z}, \boldsymbol{w}) \longmapsto {}^t\boldsymbol{z}\,(\mathrm{Im}(\tau))^{-1}\,\overline{\boldsymbol{w}}$$

and the semi-character

$$\chi_0 : \tau\mathbb{Z}^d + \Delta\mathbb{Z}^d \longrightarrow \mathbb{C}^*;\ \tau\boldsymbol{m}' + \Delta\boldsymbol{m}'' \longmapsto \exp\left(\pi\sqrt{-1}\ {}^t\boldsymbol{m}'\Delta\boldsymbol{m}''\right).$$

Then $(A_\tau, [\mathcal{L}_\tau])$ is a polarized abelian variety of type Δ. The moduli space $\mathcal{A}_d^{\Delta}$ is the quotient of $\mathfrak{H}_d$ by the action

$$\Gamma_\Delta \times \mathfrak{H}_d \longrightarrow \mathfrak{H}_d;\ (\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tau) \longmapsto (\alpha\tau + \beta)(\gamma\tau + \delta)^{-1},$$

where Γ_Δ is the discrete subgroup of $\mathrm{Sp}_{2d}(\mathbb{R})$ defined by

$$\Gamma_\Delta = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sp}_{2d}(\mathbb{R}) \;\middle|\; \begin{pmatrix} \alpha & \beta\Delta^{-1} \\ \Delta\gamma & \Delta\delta\Delta^{-1} \end{pmatrix} \in \mathrm{Mat}_{2d}(\mathbb{Z}) \right\}.$$

Following [12], for $\boldsymbol{c}', \boldsymbol{c}'' \in \frac{1}{2}\mathbb{Z}^d$ the theta function is defined by

$$\theta \begin{bmatrix} {}^t\boldsymbol{c}' \\ {}^t\boldsymbol{c}'' \end{bmatrix} (\boldsymbol{z}, \tau) = \sum_{\boldsymbol{m} \in \mathbb{Z}^d} \exp\left(\pi\sqrt{-1}\left({}^t(\boldsymbol{m} + \boldsymbol{c}')\tau(\boldsymbol{m} + \boldsymbol{c}') + 2\,{}^t(\boldsymbol{m} + \boldsymbol{c}')(\boldsymbol{z} + \boldsymbol{c}'')\right)\right).$$

Then, by [2, Theorem 3.2.7. and Remark 8.5.3.],

$$\theta_{\boldsymbol{i}}(\boldsymbol{z}, \tau) = \theta \begin{bmatrix} 0 & \cdots & 0 & i_1 & \cdots & i_g \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} (\boldsymbol{z}, \tau), \quad \left(\boldsymbol{i} = \begin{pmatrix} i_1 \\ \vdots \\ i_g \end{pmatrix} \in I = \left\{0, \frac{1}{2}\right\}^g\right)$$

form a basis of $H^0(A_\tau, \mathcal{L}_\tau)$. We consider the complex analytic space

$$\mathcal{S} = \left\{(\boldsymbol{z}, \tau) \in \mathcal{U}_d^{\Delta} \mid \theta_{\boldsymbol{i}}(\boldsymbol{z}, \tau) = 0,\ \frac{\partial\theta_{\boldsymbol{i}}}{\partial z_j}(\boldsymbol{z}, \tau) = 0 \quad (\boldsymbol{i} \in I,\ 1 \le j \le d)\right\},$$

where z_j denotes the coordinate of $\boldsymbol{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} \in \mathbb{C}^d$. Let $\upsilon = u|_{\mathcal{S}} : \mathcal{S} \to \mathfrak{H}_d$ be the restriction of $u : \mathcal{U}_d^{\Delta} \to \mathfrak{H}_d$. Then the fiber of υ at $\tau \in \mathfrak{H}_d$ is

$$\upsilon^{-1}(\tau) = S(A_\tau, \mathcal{L}_\tau) = \bigcap_{\Theta \in |\mathcal{L}_\tau|} \Theta_{\mathrm{sing}}.$$

Lemma 4.1. *If* $(\boldsymbol{a}, \tau) \in \mathcal{S}$, *then* $\left(\boldsymbol{a} + \tau \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{i} \end{pmatrix} + \Delta \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{j} \end{pmatrix}, \tau\right) \in \mathcal{S}$ *for* $\boldsymbol{i}, \boldsymbol{j} \in I$.

Proof. For $\boldsymbol{i}, \boldsymbol{i}' \in I$, there exists $\boldsymbol{i}'' \in I$ such that $\boldsymbol{i} + \boldsymbol{i}' - \boldsymbol{i}'' \in \mathbb{Z}^g$. We set $\boldsymbol{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{i} \end{pmatrix} \in \frac{1}{2}\mathbb{Z}^d$ and $\boldsymbol{d} = \Delta \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{j} \end{pmatrix} \in \mathbb{Z}^d$. Then we have

$$\theta_{\boldsymbol{i}'}(\boldsymbol{z} + \tau\boldsymbol{c} + \boldsymbol{d}, \tau) = \exp\left(\pi\sqrt{-1}(-{}^t\boldsymbol{c}\tau\boldsymbol{c} - 2\,{}^t\boldsymbol{c}\boldsymbol{z} + 4\,{}^t\boldsymbol{i}'\boldsymbol{j})\right)\theta_{\boldsymbol{i}''}(\boldsymbol{z}, \tau),$$

and

$$\begin{aligned} &\frac{\partial\theta_{\boldsymbol{i}'}}{\partial z_j}(\boldsymbol{z} + \tau\boldsymbol{c} + \boldsymbol{d}, \tau) \\ &\quad = \exp\left(\pi\sqrt{-1}(-{}^t\boldsymbol{c}\tau\boldsymbol{c} - 2\,{}^t\boldsymbol{c}\boldsymbol{z} + 4\,{}^t\boldsymbol{i}'\boldsymbol{j})\right)\left(-2\pi\sqrt{-1}\,c_j\theta_{\boldsymbol{i}''}(\boldsymbol{z}, \tau) + \frac{\partial\theta_{\boldsymbol{i}''}}{\partial z_j}(\boldsymbol{z}, \tau)\right) \end{aligned}$$

for $1 \le j \le d$. Since $\theta_{\boldsymbol{i}''}(\boldsymbol{a}, \tau) = 0$ and $\frac{\partial\theta_{\boldsymbol{i}''}}{\partial z_j}(\boldsymbol{a}, \tau) = 0$, we have $\theta_{\boldsymbol{i}'}(\boldsymbol{a} + \tau\boldsymbol{c} + \boldsymbol{d}, \tau) = 0$ and $\frac{\partial\theta_{\boldsymbol{i}'}}{\partial z_j}(\boldsymbol{a} + \tau\boldsymbol{c} + \boldsymbol{d}, \tau) = 0$. ☐

For $m \ge 0$, the degeneracy set ([5, 3.6.])

$$\mathcal{S}_m = \{(\boldsymbol{z}, \tau) \in \mathcal{S} \mid \dim_{(\boldsymbol{z},\tau)} \upsilon^{-1}(\tau) \ge m\}$$

is an analytic subset in $\mathcal{S}$, and by the proper mapping theorem ([6, 10.6.1.]), $\mathcal{N}_m = \upsilon(\mathcal{S}_m)$ is an analytic subset in $\mathfrak{H}_d$. Let $\mathcal{N}$ be a local irreducible component of $\mathcal{N}_m$ at $\tau_0 \in \mathcal{N}_m$. We denote by $\Lambda(\mathcal{N})$ the set of all irreducible components $\mathcal{M}$ of $\upsilon^{-1}(\mathcal{N}) \cap \mathcal{S}_m$ such that $\upsilon(\mathcal{M}) = \mathcal{N}$. Since υ is proper, $\Lambda(\mathcal{N})$ is a non-empty finite set. Following [4], we consider the subset $\mathcal{S}(\tau_0, \mathcal{N}) = \upsilon^{-1}(\tau_0) \cap \bigcup_{\mathcal{M} \in \Lambda(\mathcal{N})} \mathcal{M}$ in $\upsilon^{-1}(\tau_0)$.

Lemma 4.2. *If* $\mathcal{N}$ *is nonsingular, then*

$$T_{\mathcal{N},\tau_0} \subset \Big(\sum_{1 \le k \le l \le d} \frac{\partial\theta_{\boldsymbol{i}}}{\partial\tau_{kl}}(\boldsymbol{a}, \tau_0)\, d\tau_{kl}\Big)^{\perp} \subset T_{\mathfrak{H}_d,\tau_0}$$

for $\boldsymbol{i} \in I$ *and* $(\boldsymbol{a}, \tau_0) \in \mathcal{S}(\tau_0, \mathcal{N})$, *where* $\tau_{kl} = \tau_{lk}$ *denotes the coordinate of* $\tau = \begin{pmatrix} \tau_{11} & & \tau_{1d} \\ & \cdots & \\ \tau_{d1} & & \tau_{dd} \end{pmatrix} \in \mathfrak{H}_d$.

Proof. Since $(\boldsymbol{a}, \tau_0) \in \mathcal{S}(\tau_0, \mathcal{N})$, there exists $\mathcal{M} \in \Lambda(\mathcal{N})$ such that $(\boldsymbol{a}, \tau_0) \in \mathcal{M}$. Then,

$$U = \{(\boldsymbol{z}, \tau) \in \mathcal{M} \setminus \mathcal{M}_{\mathrm{sing}} \mid T_{\mathcal{M},(\boldsymbol{z},\tau)} \overset{d\upsilon}{\to} T_{\mathcal{N},\tau} \text{ is surjective}\}$$

is a non-empty open subset of $\mathcal{M}$. For $(\boldsymbol{z}, \tau) \in U$, we can take a local coordinate $\boldsymbol{t} = (t_1, \dots, t_{\dim \mathcal{N}})$ of $\mathcal{N}$ at τ and a local coordinate $(\boldsymbol{s}, \boldsymbol{t})$ of U at $(\boldsymbol{z}, \tau)$ such that $\upsilon|_U : U \to \mathcal{N}$ is given by $(\boldsymbol{s}, \boldsymbol{t}) \mapsto \boldsymbol{t}$. Let $(\boldsymbol{z}(\boldsymbol{s}, \boldsymbol{t}), \tau(\boldsymbol{t})) \in U \subset \mathcal{S}$ be the point corresponding to $(\boldsymbol{s}, \boldsymbol{t})$. Then we have

$$\theta_{\boldsymbol{i}}(\boldsymbol{z}(\boldsymbol{s}, \boldsymbol{t}), \tau(\boldsymbol{t})) = 0, \quad \frac{\partial \theta_{\boldsymbol{i}}}{\partial z_j}(\boldsymbol{z}(\boldsymbol{s}, \boldsymbol{t}), \tau(\boldsymbol{t})) = 0 \quad (\boldsymbol{i} \in I,\ 1 \le j \le d).$$

For $1 \le \nu \le \dim \mathcal{N}$, by the chain rule, we get

$$\begin{aligned}&\frac{\partial}{\partial t_\nu}\Big(\theta_{\boldsymbol{i}}(\boldsymbol{z}(\boldsymbol{s}, \boldsymbol{t}), \tau(\boldsymbol{t}))\Big) \\ &\qquad = \sum_{j=1}^{d} \frac{\partial \theta_{\boldsymbol{i}}}{\partial z_j}(\boldsymbol{z}(\boldsymbol{s}, \boldsymbol{t}), \tau(\boldsymbol{t})) \frac{\partial z_j}{\partial t_\nu}(\boldsymbol{s}, \boldsymbol{t}) + \sum_{1 \le k \le l \le d} \frac{\partial \theta_{\boldsymbol{i}}}{\partial \tau_{kl}}(\boldsymbol{z}(\boldsymbol{s}, \boldsymbol{t}), \tau(\boldsymbol{t})) \frac{\partial \tau_{kl}}{\partial t_\nu}(\boldsymbol{t}),\end{aligned}$$

which implies

$$\sum_{1 \le k \le l \le d} \frac{\partial \theta_{\boldsymbol{i}}}{\partial \tau_{kl}}(\boldsymbol{z}(\boldsymbol{s}, \boldsymbol{t}), \tau(\boldsymbol{t})) \frac{\partial \tau_{kl}}{\partial t_\nu}(\boldsymbol{t}) = 0.$$

This shows that

$$\frac{\partial}{\partial t_\nu} = \sum_{1 \le k \le l \le d} \frac{\partial \tau_{kl}}{\partial t_\nu}(\boldsymbol{t}) \frac{\partial}{\partial \tau_{kl}} \in \Big(\sum_{1 \le k \le l \le d} \frac{\partial \theta_{\boldsymbol{i}}}{\partial \tau_{kl}}(\boldsymbol{z}(\boldsymbol{s}, \boldsymbol{t}), \tau(\boldsymbol{t}))\, d\tau_{kl} \Big)^{\perp} \subset T_{\mathfrak{H}_d, \tau(\boldsymbol{t})}.$$

Since the open subset $U \subset \mathcal{M}$ is contained in the analytic subset

$$\mathcal{M}' = \left\{ (\boldsymbol{z}, \tau) \in \mathcal{M} \;\middle|\; T_{\mathcal{N},\tau} \subset \Big(\sum_{1 \le k \le l \le d} \frac{\partial \theta_{\boldsymbol{i}}}{\partial \tau_{kl}}(\boldsymbol{z}, \tau)\, d\tau_{kl} \Big)^{\perp} \right\},$$

we have $(\boldsymbol{a}, \tau_0) \in \mathcal{M} = \mathcal{M}'$. □

Proposition 4.3. *Assume that* $\dim \upsilon^{-1}(\tau_0) = m$. *If there exist* $c_{\boldsymbol{i},j} \in \mathbb{C}$ *and* $(\boldsymbol{a}_j, \tau_0) \in \mathcal{S}(\tau_0, \mathcal{N})$ *for* $\boldsymbol{i} \in I$ *and* $1 \le j \le r$ *such that cotangent vectors*

$$\sum_{1 \le k \le l \le d} \sum_{\boldsymbol{i} \in I} c_{\boldsymbol{i},j} \frac{\partial \theta_{\boldsymbol{i}}}{\partial \tau_{kl}}(\boldsymbol{a}_j, \tau_0)\, d\tau_{kl} \in T^{\vee}_{\mathfrak{H}_d, \tau_0} = \bigoplus_{1 \le k \le l \le d} \mathbb{C} d\tau_{kl} \quad (1 \le j \le r)$$

are linearly independent, then $\dim \mathcal{N} \le \dim \mathfrak{H}_d - r$.

Proof. For $1 \le j \le r$, let $\mathcal{M}_j \in \Lambda(\mathcal{N})$ satisfy $(\boldsymbol{a}_j, \tau_0) \in \mathcal{M}_j$. By the upper semi-continuity of the fiber dimension [5, 3.4.], there exist open neighborhoods $U_j \subset \mathcal{M}_j$ at $(\boldsymbol{a}_j, \tau_0)$ such that

$$m = \dim \upsilon^{-1}(\tau_0) \ge \dim_{(\boldsymbol{a}_j, \tau_0)} (\upsilon^{-1}(\tau_0) \cap \mathcal{M}) \ge \dim_{(\boldsymbol{b}, \tau)} (\upsilon^{-1}(\tau) \cap \mathcal{M}) \ge m$$

for any $(\boldsymbol{b},\tau) \in U_j$. Then, by [5, 3.10.], the restriction $\upsilon|_{U_j} : U_j \to \mathcal{N}$ is an open mapping. By replacing U_j with smaller open neighborhoods, we may assume that

$$\sum_{1\le k\le l\le d} \sum_{\boldsymbol{i}\in I} c_{\boldsymbol{i},j} \frac{\partial \theta_{\boldsymbol{i}}}{\partial \tau_{kl}}(\boldsymbol{b}_j,\tau_j)\, d\tau_{kl} \quad (1 \le j \le r)$$

are linearly independent for any $\{(\boldsymbol{b}_j,\tau_j)\}_{1\le j\le r} \in \prod_{j=1}^r U_j$. Since $\bigcap_{j=1}^r \upsilon(U_j)$ is an open neighborhood of $\mathcal{N}$ at τ_0, there exists a point $\tau_0' \in \bigcap_{j=1}^r \upsilon(U_j)$ such that $\mathcal{N}$ is nonsingular at τ_0'. Then, $\mathcal{N}' = \mathcal{N} \setminus \mathcal{N}_{\mathrm{sing}}$ is a local irreducible component of $\mathcal{N}_m$ at τ_0'. For $(\boldsymbol{a}_j',\tau_0') \in \upsilon^{-1}(\tau_0') \cap U_j$, we have $(\boldsymbol{a}_j',\tau_0') \in \mathcal{S}(\tau_0',\mathcal{N}')$ and

$$\sum_{1\le k\le l\le d} \sum_{\boldsymbol{i}\in I} c_{\boldsymbol{i},j} \frac{\partial \theta_{\boldsymbol{i}}}{\partial \tau_{kl}}(\boldsymbol{a}_j',\tau_0')\, d\tau_{kl} \quad (1 \le j \le r)$$

are linearly independent. By Lemma 4.2, we conclude that $\dim \mathcal{N} \le \dim \mathfrak{H}_d - r$. □

Let $\pi : \mathfrak{H}_d \to \mathcal{A}_d^{\Delta} = \Gamma_\Delta \backslash \mathfrak{H}_d$ be the quotient map. By Corollary 2.8, we have $\pi^{-1}(\mathrm{Prym}_{g,2n}(\mathcal{R}_{g,2n})) \subset \mathcal{N}_{n-4}$ for $n \ge 4$, and $\dim \upsilon^{-1}(\tau_0) = n-4$ for any $\tau_0 \in \pi^{-1}(\mathrm{Prym}_{g,2n}(\mathcal{R}_{g,2n}))$.

Lemma 4.4. *If $n \ge 4$ and $\tau_0 \in \pi^{-1}(\mathrm{Prym}_{g,2n}(\mathcal{R}_{g,2n}))$, then $\mathcal{S}(\tau_0,\mathcal{N}) = \upsilon^{-1}(\tau_0)$ for any local irreducible component $\mathcal{N}$ of $\mathcal{N}_{n-4}$ at τ_0.*

Proof. When $n \ge 5$, by Corollary 2.8, $\upsilon^{-1}(\tau_0)$ is irreducible of dimension $n-4$. Since

$$n - 4 = \dim \upsilon^{-1}(\tau_0) \ge \dim_{(\boldsymbol{z},\tau_0)} (\upsilon^{-1}(\tau_0) \cap \mathcal{M}) \ge n-4$$

for any irreducible component $\mathcal{M}$ of $\upsilon^{-1}(\mathcal{N}) \cap \mathcal{S}_m$ and $(\boldsymbol{z},\tau_0) \in \mathcal{M}$, we have $\upsilon^{-1}(\tau_0) \subset \mathcal{M}$ for any $\mathcal{M} \in \Lambda(\mathcal{N})$, hence $\upsilon^{-1}(\tau_0) \subset \mathcal{S}(\tau_0,\mathcal{N})$. When $n = 4$, by Lemma 4.1, for $\mathcal{M} \in \Lambda(\mathcal{N})$ and $\boldsymbol{i},\boldsymbol{j} \in I$, there exists $\mathcal{M}_{\boldsymbol{i},\boldsymbol{j}} \in \Lambda(\mathcal{N})$ such that

$$\mathcal{M}_{\boldsymbol{i},\boldsymbol{j}} = \left\{(\boldsymbol{z},\tau) \in \mathbb{C}^d \times \mathfrak{H}_d \mid (\boldsymbol{z} - \tau \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{i} \end{pmatrix} - \Delta \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{j} \end{pmatrix}, \tau) \in \mathcal{M}\right\}.$$

By Corollary 2.8, we have $\upsilon^{-1}(\tau_0) \subset \mathcal{S}(\tau_0,\mathcal{N})$. □

We define $\mathcal{R}_{1,2n}^h = \mathcal{R}_{1,2n}$, and for $g \ge 2$,

$$\mathcal{R}_{g,2n}^h = \{[\tilde{C} \xrightarrow{\phi} C] \in \mathcal{R}_{g,2h} \mid C \text{ is a hyperelliptic curve}\}.$$

Lemma 4.5. *Let $r = \frac{d(d+1)}{2} - 3g - 2n + 3$. For $\tau_0 \in \pi^{-1}(\mathrm{Prym}_{g,2n}(\mathcal{R}_{g,2n}^h))$, there exist*

$$(\boldsymbol{a}_1,\tau_0),\ldots,(\boldsymbol{a}_r,\tau_0) \in \upsilon^{-1}(\tau_0) = S(A_{\tau_0},\mathcal{L}_{\tau_0})$$

and

$$f_1(\boldsymbol{z}),\ldots,f_r(\boldsymbol{z}) \in \bigoplus_{\boldsymbol{i}\in I} \mathbb{C}\,\theta_{\boldsymbol{i}}(\boldsymbol{z},\tau_0) = H^0(A_{\tau_0},\mathcal{L}_{\tau_0})$$

such that quadratics

$$\sum_{k=1}^d \sum_{l=1}^d \frac{\partial^2 f_1}{\partial z_k \partial z_l}(\boldsymbol{a}_1) z_k z_l,\ \ldots,\ \sum_{k=1}^d \sum_{l=1}^d \frac{\partial^2 f_r}{\partial z_k \partial z_l}(\boldsymbol{a}_r) z_k z_l \in \mathbb{C}[z_1,\ldots,z_d]$$

are linearly independent.

Proof. Since $r = \dim \mathrm{Ker}\,(\rho_2)$, by Corollary 3.3, there exist

$$(\boldsymbol{a}_1, \tau_0), \ldots, (\boldsymbol{a}_r, \tau_0) \in \upsilon^{-1}(\tau_0) = S(A_{\tau_0}, \mathcal{L}_{\tau_0})$$

and

$$\Theta_1, \ldots, \Theta_r \in |\mathcal{L}_{\tau_0}|$$

such that $\mathrm{mult}_{(\boldsymbol{a}_j,\tau_0)}\,\Theta_j = 2$, and the defining quadratics of the tangent cones $\mathcal{T}_{\Theta_j,(\boldsymbol{a}_j,\tau_0)}$ for $1 \leq j \leq r$ form a basis of $\mathrm{Ker}\,(\rho_2)$. Let

$$f_j(\boldsymbol{z}) \in \bigoplus_{\boldsymbol{i}\in I} \mathbb{C}\,\theta_{\boldsymbol{i}}(\boldsymbol{z}, \tau_0) = H^0(A_{\tau_0}, \mathcal{L}_{\tau_0})$$

be a defining section of $\Theta_j \subset A_{\tau_0}$. Then the defining quadratic of the tangent cone $\mathcal{T}_{\Theta_j,(\boldsymbol{a}_j,\tau_0)}$ is given by $\sum_{k=1}^{d}\sum_{l=1}^{d} \frac{\partial^2 f_j}{\partial z_k \partial z_l}(\boldsymbol{a}_r) z_k z_l \in \mathbb{C}[z_1, \ldots, z_d]$. □

Lemma 4.6 (Heat equation, [2] Proposition 8.5.5.)**.** *For $\boldsymbol{i} \in I$ and $1 \leq k \leq l \leq d$,*

$$\frac{\partial^2 \theta_{\boldsymbol{i}}}{\partial z_k \partial z_l}(\boldsymbol{z}, \tau) = 2\pi\sqrt{-1}(1 + \delta_{kl})\frac{\partial \theta_{\boldsymbol{i}}}{\partial \tau_{kl}}(\boldsymbol{z}, \tau).$$

Proof of Theorem 1.4. Let $\mathcal{N}$ be a local irreducible component of $\mathcal{N}_{n-4}$ at $\tau_0 \in \pi^{-1}(\mathrm{Prym}_{g,2n}\,(\mathcal{R}^h_{g,2n}))$. By Lemma 4.5 and Lemma 4.6, there exist $(\boldsymbol{a}_j, \tau_0) \in \upsilon^{-1}(\tau_0)$ and $f_j(\boldsymbol{z}) \in \bigoplus_{\boldsymbol{i}\in I} \mathbb{C}\,\theta_{\boldsymbol{i}}(\boldsymbol{z}, \tau_0)$ for $1 \leq j \leq r$ such that the following quadratics are linearly independent:

$$\sum_{1\leq k\leq l\leq d} \frac{\partial f_1}{\partial \tau_{kl}}(\boldsymbol{a}_1) z_k z_l, \ldots, \sum_{1\leq k\leq l\leq d} \frac{\partial f_r}{\partial \tau_{kl}}(\boldsymbol{a}_r) z_k z_l \in \mathbb{C}[z_1, \ldots, z_d].$$

Hence, the cotangent vectors

$$\sum_{1\leq k\leq l\leq d} \frac{\partial f_1}{\partial \tau_{kl}}(\boldsymbol{a}_1) d\tau_{kl}, \ldots, \sum_{1\leq k\leq l\leq d} \frac{\partial f_r}{\partial \tau_{kl}}(\boldsymbol{a}_r) d\tau_{kl} \in \bigoplus_{1\leq k\leq l\leq d} \mathbb{C} d\tau_{kl}$$

are linearly independent. By Lemma 4.4 and Proposition 4.3, we have

$$\dim \mathcal{N} \leq \dim \mathfrak{H}_d - r = 3g + 2n - 3.$$

Since $\mathcal{R}_{g,2n}$ is irreducible by [3, p138], there exist an irreducible component $\mathcal{P}$ of the generalized Andreotti-Mayer locus $\mathcal{N}^{\Delta}_{d,n-4} \subset \mathcal{A}^{\Delta}_d$ such that $\mathrm{Prym}_{g,2n}\,(\mathcal{R}_{g,2n}) \subset \mathcal{P}$. Then, we have $\pi^{-1}(\mathcal{P}) \subset \mathcal{N}_{n-4}$, and

$$\dim \mathcal{P} = \dim_{\tau_0} \pi^{-1}(\mathcal{P}) \leq \dim_{\tau_0} \mathcal{N} \leq 3g + 2n - 3,$$

for some local irreducible component $\mathcal{N}$ of $\mathcal{N}_{n-4}$ at $\tau_0 \in \pi^{-1}(\mathrm{Prym}_{g,2n}\,(\mathcal{R}^h_{g,2n}))$. On the other hand, by Theorem 1.1, we have

$$\dim \mathcal{P} \geq \dim \mathrm{Prym}_{g,2n}\,(\mathcal{R}_{g,2n}) = 3g + 2n - 3.$$

Hence, we conclude that $\mathcal{P} = \overline{\mathrm{Prym}_{g,2n}\,(\mathcal{R}_{g,2n})}$. □

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