REGULARITY PROPERTIES OF THE α-WILTON FUNCTIONS

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ABSTRACT. The aim of this article is to study the regularity properties of the Wilton functions W_α associated with α-continued fractions. We prove that the Wilton function is BMO for $\alpha \in [1-g, g]$ (where $g := \frac{\sqrt{5}-1}{2}$ denotes the golden number), and we show that this result is optimal, since we find that on any left neighbourhood of $1 - g$ and on any right neighbourhood of g there are values α for which W_α is not BMO; the proof of this latter negative result exploits a special feature of the family of α-continued fractions called "matching". Our results complete those of Marmi–Moussa–Yoccoz (1997) and of Lee–Marmi–Petrykiewicz–Schindler (2024), where it is proven that Wilton function is BMO for, respectively, $\alpha = 1/2$ and $\alpha \in [\frac{1}{2}, g]$.

1. INTRODUCTION

For $0 \leq \alpha \leq 1$, let $\bar{\alpha} = \max(\alpha, 1 - \alpha)$; the α-*continued fraction expansion* of a real number $x \in (0, \bar{\alpha})$ is associated to the iteration of the map $A_\alpha : (0, \bar{\alpha}) \to [0, \bar{\alpha}]$ defined as follows:

$$(1.1) \qquad A_\alpha(x) = \left| \frac{1}{x} - \left[\frac{1}{x} + 1 - \alpha \right] \right|,$$

where $[\cdot]$ denotes the integer part.

The family of maps $\{A_\alpha\}_\alpha$ was introduced by Nakada in [17], and as special cases it includes the standard continued fraction map when $\alpha = 1$, the nearest-integer continued fraction map when $\alpha = \frac{1}{2}$ and the by-excess continued fractions map when $\alpha = 0$. For all $\alpha \in (0, 1]$, these maps are expanding and admit a unique absolutely continuous invariant probability measure $d_{\mu_\alpha} = \rho_\alpha(x) dx$ whose density is bounded from above and below by a constant dependent on α. In the case $\alpha = 0$, there is an indifferent fixed point and A_α does not have a finite invariant density but it preserves the infinite measure $d_{\mu_0}(x) = \frac{dx}{1-x}$.

The *Wilton function* associated with an α-continued fraction is defined as follows on $\mathbb{R} \setminus \mathbb{Q}$

$$(1.2) \qquad W_\alpha(x) = \sum_{j=0}^\infty (-1)^j \beta_{\alpha, j-1}(x) \log x_{\alpha, j}^{-1} = \sum_{j=0}^\infty (-1)^j \beta_{\alpha, j-1}(x) \log(1/A_\alpha^j(x_{\alpha, 0})),$$

where the sequence $x_{\alpha,n} = A_\alpha^n(x_{\alpha,0})$ with $x_{\alpha,0} = |x - \lfloor x + 1 - \bar{\alpha} \rfloor|$ and $\beta_{\alpha,n} = x_{\alpha,0}x_{\alpha,1} \cdots x_{\alpha,n}$ for $n \geq 0$, $\beta_{\alpha,-1} = 1$. When we consider $\alpha = 1$, then A_1 is simply the Gauss map; in this case we shall often omit the dependence on α, and write x_n, β_n rather than $x_{\alpha,n}, \beta_{\alpha,n}$.

Note that the formula (1.2) defines an $L^1(0,1)$ function which satisfies the functional equation
$$W_\alpha(x) = -\log(x) - xW_\alpha(A_\alpha(x)) \quad \text{for all } x \in (0, \bar{\alpha}) \setminus \mathbb{Q},$$
$$W_\alpha(x) = W_\alpha(1-x) \qquad\qquad \text{for all } x \in (0, \min\{\alpha, 1-\alpha\}) \setminus \mathbb{Q},$$

and more generally,

(1.3) $\qquad W_\alpha(x) = W_\alpha^{(K)}(x) + (-1)^{K+1}\beta_{\alpha,K}(x)W_\alpha(A_\alpha^{K+1}(x)) \quad (K \in \mathbb{N}, x \in (0, \bar{\alpha}) \setminus \mathbb{Q}),$

where $W^{(K)}$ denotes the partial sum

(1.4) $$W_\alpha^{(K)}(x) = \sum_{j=0}^{K}(-1)^j\beta_{\alpha,j-1}(x)\log(1/A_\alpha^j(x))$$

with respect to the α-continued fraction.

The series (1.2) was first introduced by Wilton [20] for $\alpha = 1$ in order the study of trigonometric series

(1.5) $$\phi_1(x) = -\frac{1}{\pi}\sum_{n \geq 1}\frac{\tau(n)}{n}\sin(2\pi nx),$$

where $\tau(n)$ is the number of divisors of the natural number n. Indeed, the author showed that the series (1.5) converges if and only if W_1 is convergent. When $\alpha = 1$, we will omit the subscript α in (1.2) and write as

(1.6) $$W_1(x) = \sum_{j=0}^{\infty}(-1)^j\beta_{j-1}(x)\log x_j^{-1}.$$

We will refer to the irrational real numbers x for which the series (1.2) converges as the *Wilton numbers*. It can be proved that the series (1.6) converges if and only if it fulfills the *Wilton condition*
$$\left|\sum_{j=0}^{\infty}(-1)^j\frac{\log(q_{j+1}(x))}{q_j(x)}\right| < \infty,$$
where q_j denotes the denominator of the jth convergent of x associated with the Gauss map A_1.

All Diophantine numbers, i.e. $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $q_{n+1} = O(q_n^{1+\tau})$ where $\tau \geq 0$, are Wilton numbers. Note that the Wilton function (1.6) is an alternating sign version of the Brjuno function, introduced by Yoccoz in 1988, which plays an important role in the theory of dynamical systems, more precisely in the study of iteration of a quadratic polynomials (for more details on the Brjuno function, see [12, 14, 15]).

Clearly, all Brjuno numbers are Wilton numbers but not vice versa. Whereas the Hausdorff dimension of the difference set is 0, i.e. $\dim_H(\mathcal{W} \setminus \mathcal{B}) = 0$, where $\mathcal{W}$ and $\mathcal{B}$ denote the set of Wilton and Brjuno numbers respectively. This follows from the fact that $(\mathcal{W} \setminus \mathcal{B}) \subset (\mathbb{R} \setminus \mathcal{B}) = \mathcal{B}^c$, and the Hausdorff dimension of the set $\mathcal{B}^c$ is 0 as it is properly contained in the union of the set of Liouville numbers and the set of rational numbers.

In recent years, Balazard–Martin [3] studied the Wilton function W_1 in terms of its convergence properties and in the context of the Nyman and Beurling criterion [1, 2]. For example, in [2], the authors reduced the study of the autocorrelation function to that of the Wilton function W_1 in order
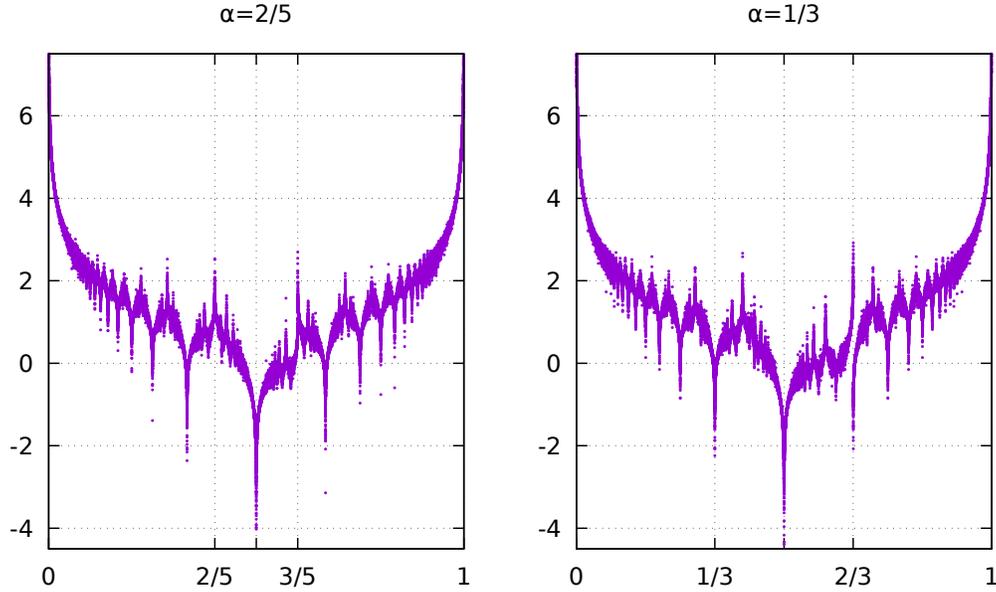
FIGURE 1. The graph of $W_{2/5}$ and $W_{1/3}$: the first is in BMO, the latter isn't (as one might guess observing asymmetricity of the logarithmic singularity at $x = 2/3$).

to show that the points of differentiability of the autocorrelation function $\mathcal{A}(\lambda) = \int_0^\infty \{t\}\{\lambda t\}\frac{dt}{t^2}$ are the positive irrational numbers such that the series $\sum_{j\geq 0}(-1)^{j+1}\frac{\log q_{j+1}}{q_j}$ converges.

The aim of this paper is to study BMO regularity properties of Wilton functions associated with α-continued fractions for $\alpha \in (0, 1)$. From [13, Thm 3.3], one can deduce that $W_{1/2}$ is in BMO (see also [10, p. 26]). Recently, the third author together with Marmi, Petrykiewicz and Schindler [10] improved this result by studying the regularity properties of Wilton function. In particular, they showed in [10] that $W_\alpha \in$ BMO for all $\alpha \in [1/2, g]$, where $g = \frac{\sqrt{5}-1}{2}$. The aim of this article is to further improve this result of Lee–Marmi–Petrykiewicz–Schindler by extending the interval of α.

Our first main result is as follows:

Theorem 1.1. *The Wilton function $W_\alpha \in$ BMO for all $\alpha \in [1-g, g]$.*

We will also show that this result is optimal:

Theorem 1.2.   (i) *If $\alpha \in (g, 1] \cap \mathbb{Q}$, then W_α is not in BMO.*
    (ii) *There exists a sequence $(u_m)_m$ of rational values, $u_m \uparrow 1 - g$ as $m \to +\infty$ such that W_{u_m} is not in BMO.*

It is interesting to point out that all these results (both in the positive and negative direction) are strictly linked with a remarkable feature of α-continued fractions called *matching*; the relevance of this property was first pointed out by [18] in relation to the study of the entropy of α-continued fractions, and it led to several results in this field (see [5]). In fact, the technique we use to prove our main result can be adapted to provide a new (and simple) proof that the entropy of α-continued fractions is constant on the interval $[1-g, g]$, a result which was already established through rather convoluted proofs (see appendix 5). The feature which is at the base of both the result on BMO regularity and the costancy of the entropy function on $[1-g, g]$ is that this interval is entirely tiled

by matching intervals of matching index zero, and yet it is surrounded by matching intervals of odd matching index.

2. NOTATIONS AND PRELIMINARY RESULTS

2.1. **Folded α-continued fractions.** Fix $\alpha \in (0,1]$, let $\bar{\alpha} = \max(\alpha, 1-\alpha)$ and consider the map $A_\alpha : [0,\bar{\alpha}] \to [0,\bar{\alpha}]$ be the transformation of α-continued fraction defined by $A_\alpha(0) = 0$ and

$$(2.1) \qquad A_\alpha(x) = \left| \frac{1}{x} - \left[\frac{1}{x}\right]_\alpha \right|,$$

for $x \in (0,\bar{\alpha}]$, where $[x]_\alpha = [x + 1 - \alpha]$ and $[\cdot]$ denotes the integer part. Put

$$(2.2) \qquad x_{\alpha,0} = |x - [x]_{\bar{\alpha}}|,$$

$a_{\alpha,0} = [x]_{\bar{\alpha}}$, $\epsilon_{\alpha,0}(x) = \mathrm{sgn}\,(x - [x]_{\bar{\alpha}})$ and define by recurrence for $n \geq 0$:

$$x_{\alpha,n+1} = A_\alpha(x_{\alpha,n}), \; a_{\alpha,n+1}(x) = \left[\frac{1}{x_{\alpha,n}}\right]_\alpha \; \text{ and } \; \epsilon_{\alpha,n+1} = \mathrm{sgn}\left(\frac{1}{x_{\alpha,n}} - \left[\frac{1}{x_{\alpha,n}}\right]_\alpha\right).$$

The α-continued fraction expansion of x is

$$x = a_{\alpha,0} + \cfrac{\epsilon_{\alpha,0}}{a_{\alpha,1} + \cfrac{\epsilon_{\alpha,1}}{\ddots + \cfrac{\epsilon_{\alpha,n-1}}{a_{\alpha,n} + \cfrac{\epsilon_{\alpha,n}}{\ddots}}}}.$$

Let $\frac{p_{\alpha,n}}{q_{\alpha,n}}$ be the nth finite truncation of this expansion, that is,

$$(2.3) \qquad \frac{p_{\alpha,n}}{q_{\alpha,n}} = a_{\alpha,0} + \cfrac{\epsilon_{\alpha,0}}{a_{\alpha,1} + \cfrac{\epsilon_{\alpha,1}}{\ddots + \cfrac{\epsilon_{\alpha,n-1}}{a_{\alpha,n}}}}.$$

It is called the nth *convergent* of x. Let $p_{\alpha,-1} = 1$, $q_{\alpha,-1} = 0$ for the convenience.

Thanks to the isomorphism between 2×2 matrices and fractional transformations the following notation will be useful

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}.$$

Then equation (2.3) induces, for $n \geq 1$,

$$(2.4) \qquad \begin{pmatrix} p_{\alpha,n-1} & p_{\alpha,n} \\ q_{\alpha,n-1} & q_{\alpha,n} \end{pmatrix} = \begin{pmatrix} 1 & a_{\alpha,0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \epsilon_{\alpha,0} \\ 1 & a_{\alpha,1} \end{pmatrix} \begin{pmatrix} 0 & \epsilon_{\alpha,1} \\ 1 & a_{\alpha,2} \end{pmatrix} \cdots \begin{pmatrix} 0 & \epsilon_{\alpha,n-2} \\ 1 & a_{\alpha,n-1} \end{pmatrix} \begin{pmatrix} 0 & \epsilon_{\alpha,n-1} \\ 1 & a_{\alpha,n} \end{pmatrix}.$$

By applying the above matrices on the point $\epsilon_{\alpha,n}x_{\alpha,n}$, we have

$$\frac{p_{\alpha,n} + \epsilon_{\alpha,n}p_{\alpha,n-1}x_{\alpha,n}}{q_{\alpha,n} + \epsilon_{\alpha,n}q_{\alpha,n-1}x_{\alpha,n}} = x.$$

By calculating the determinant of the matrices of (2.4), it is immediate that for $n \geq 1$

$$(2.5) \qquad p_{\alpha,n}q_{\alpha,n-1} - q_{\alpha,n}p_{\alpha,n-1} = (-1)^n \epsilon_{\alpha,0}\epsilon_{\alpha,1}\epsilon_{\alpha,2}\cdots\epsilon_{\alpha,n-1}.$$

Thus the convergents of x satisfy the following recursive relation:

$$(2.6) \qquad p_{\alpha,n} = a_{\alpha,n}p_{\alpha,n-1} + \epsilon_{\alpha,n-1}p_{\alpha,n-2}, \quad q_{\alpha,n} = a_{\alpha,n}q_{\alpha,n-1} + \epsilon_{\alpha,n-1}q_{\alpha,n-2}.$$

It follows

$$(2.7) \qquad x - \frac{p_{\alpha,n}}{q_{\alpha,n}} = \frac{p_{\alpha,n} + \epsilon_{\alpha,n}p_{\alpha,n-1}x_{\alpha,n}}{q_{\alpha,n} + \epsilon_{\alpha,n}q_{\alpha,n-1}x_{\alpha,n}} - \frac{p_{\alpha,n}}{q_{\alpha,n}} = \frac{(-1)^{n+1}\epsilon_{\alpha,0}\epsilon_{\alpha,1}\epsilon_{\alpha,2}\cdots\epsilon_{\alpha,n-1}\epsilon_{\alpha,n}x_{\alpha,n}}{q_{\alpha,n}(q_{\alpha,n} + \epsilon_{\alpha,n}q_{\alpha,n-1}x_{\alpha,n})}$$

and

$$\operatorname{sgn}\left(x - \frac{p_{\alpha,n}}{q_{\alpha,n}}\right) = \operatorname{sgn}(q_{\alpha,n}x - p_{\alpha,n}) = (-1)^{n+1}\epsilon_{\alpha,0}\epsilon_{\alpha,1}\epsilon_{\alpha,2}\cdots\epsilon_{\alpha,n}.$$

Define $\beta_{\alpha,n} := \prod_{i=0}^{n} x_{\alpha,i} = \prod_{i=0}^{n} A_\alpha^i(x_{\alpha,0})$ for $n \geq 0$ as the product of the iterates along the A_α-orbit with $\beta_{\alpha,-1} = 1$. From [11, Lemma 1], for all $n \geq 1$ we have $\beta_{\alpha,n} = |q_{\alpha,n}x - p_{\alpha,n}|$. By definition, $x_{\alpha,n} = \frac{\beta_{\alpha,n}}{\beta_{\alpha,n-1}}$. Combining with (2.7), we have

$$\beta_{\alpha,n} = \frac{\beta_{\alpha,n+1}}{x_{\alpha,n+1}} = \frac{1}{q_{\alpha,n+1} + q_{\alpha,n}\epsilon_{\alpha,n+1}x_{\alpha,n+1}}.$$

Since $q_{\alpha,n+1} > q_{\alpha,n} > 0$ ([16, Lemma 1]) and $\epsilon_{\alpha,n+1}x_{\alpha,n+1} = \frac{1}{x_{\alpha,n}} - [\frac{1}{x_{\alpha,n}}]_\alpha \in [\alpha - 1, \alpha)$, for $\alpha > 0$, we have

$$\frac{1}{1+\alpha} < \beta_{\alpha,n}q_{\alpha,n+1} < \frac{1}{\alpha}.$$

Proposition 2.1 ([16, Lemma 3]). *Let $\alpha > 0$ and $\bar{\alpha} = \max(\alpha, 1-\alpha)$. Then for all $n \geq 1$ one has*

$$\begin{array}{ccl}\beta_{\alpha,n} & \leq & \bar{\alpha}\rho_\alpha^n, \\ 1/q_{\alpha,n+1} & < & (1+\alpha)\bar{\alpha}\rho_\alpha^n,\end{array} \quad \text{where } \rho_\alpha = \begin{cases} g & g < \alpha \leq 1, \\ \sqrt{2}-1 & \sqrt{2}-1 \leq \alpha \leq g, \\ \sqrt{1-2\alpha} & 0 < \alpha < \sqrt{2}-1. \end{cases}$$

2.2. **Unfolded α-continued fractions and matching.** In this subsection, we recall another variant of α-continued fractions (called *unfolded α-continued fractions*), and we shall show that the two algorithms have the same features. In particular, the folded and unfolded algorithms lead essentially to the same Wilton function, and we shall use the unfolded version of the algorithm in order to directly use the results about matching (results which have been developed in the unfolded setting).

Following [18], consider the family of maps $(T_\alpha)_{\alpha\in[0,1]}$, $T_\alpha : [\alpha - 1, \alpha) \to [\alpha - 1, \alpha)$ defined by $T_\alpha(0) = 0$ and

$$T_\alpha(x) = \frac{\epsilon(x)}{x} - c_\alpha(x) \qquad \text{for } x \neq 0$$

with

$$\epsilon(x) := \operatorname{sgn}(x) \qquad c_\alpha(x) := \left[\frac{1}{|x|} + 1 - \alpha\right].$$

We also set

$$\tilde{\epsilon}_{\alpha,n} = \tilde{\epsilon}_{\alpha,n}(x) = \epsilon(T_\alpha^{n-1}(x)), \quad x_{\alpha,n} = T_\alpha^n(x) \quad \text{and } c_{\alpha,n} = c_{\alpha,n}(x) = c_\alpha(T_\alpha^{n-1}(x)).$$

With these notations, we have

$$(2.8) \qquad x = \cfrac{\tilde{\epsilon}_{\alpha,1}}{c_{\alpha,1} + \cfrac{\tilde{\epsilon}_{\alpha,2}}{\ddots + \cfrac{\tilde{\epsilon}_{\alpha,n}}{c_{\alpha,n} + x_{\alpha,n}}}} = \cfrac{\tilde{\epsilon}_{\alpha,1}}{c_{\alpha,1} + \cfrac{\tilde{\epsilon}_{\alpha,2}}{\ddots + \cfrac{\tilde{\epsilon}_{\alpha,n}}{c_{\alpha,n} + \ddots}}}.$$

The rightmost expression above is called the *infinite (unfolded) α-continued fraction expansion of* x.

As in the folded version, by setting

$$(2.9) \qquad M_{\alpha,x,n} := \begin{pmatrix} 0 & \tilde{\epsilon}_{\alpha,1} \\ 1 & c_{\alpha,1} \end{pmatrix} \begin{pmatrix} 0 & \tilde{\epsilon}_{\alpha,2} \\ 1 & c_{\alpha,2} \end{pmatrix} \cdots \begin{pmatrix} 0 & \tilde{\epsilon}_{\alpha,n} \\ 1 & c_{\alpha,n} \end{pmatrix},$$

we can rewrite equation (2.8) as $x = M_{\alpha,x,n} \cdot x_{\alpha,n}$ or, writing the entries of $M_{\alpha,x,n}$ explicitly,

$$x = \frac{\tilde{p}_{\alpha,n-1}x_n + \tilde{p}_{\alpha,n}}{\tilde{q}_{\alpha,n-1}x_n + \tilde{q}_{\alpha,n}}, \qquad \text{where } M_{\alpha,x,n} = \begin{pmatrix} \tilde{p}_{\alpha,n-1}(x) & \tilde{p}_{\alpha,n}(x) \\ \tilde{q}_{\alpha,n-1}(x) & \tilde{q}_{\alpha,n}(x) \end{pmatrix}$$

and $\frac{\tilde{p}_{\alpha,n}}{\tilde{q}_{\alpha,n}} := M_{\alpha,x,n} \cdot 0$, which corresponds to the truncated α-continued fraction (or *convergent*) of order n. In a similar way to obtain (2.7), the following approximation identity holds

$$\left| x - \frac{\tilde{p}_{\alpha,n}}{\tilde{q}_{\alpha,n}} \right| = \frac{|x_n|}{\tilde{q}_{\alpha,n}(\tilde{q}_{\alpha,n} + \tilde{q}_{\alpha,n-1}x_n)}.$$

2.2.1. *Folded vs. Unfolded algorithms.* The map A_α is just the folded version of T_α: the families $(T_\alpha)_\alpha$ and $(A_\alpha)_\alpha$ are semiconjugated by the map $x \mapsto |x|$, namely

$$(2.10) \qquad |T_\alpha^K(x)| = A_\alpha^K(|x|), \quad x \in [\alpha - 1, \alpha), \quad K \in \mathbb{N},$$

and they are associated to a pair of continued fraction expansion called respectively *unfolded* and *folded* α-continued fractions.

For $x \in [\alpha-1, \alpha)$, we can define $\tilde{\beta}_{\alpha,n}(x) := \prod_{i=0}^n |T_\alpha^i(x)| = \beta_{\alpha,n}(|x|)$, and also the Wilton function associated to the unfolded algorithm, which is the one periodic function $\tilde{W}_\alpha$ which satisfies

$$(2.11) \qquad \tilde{W}_\alpha(x) = \sum_{j=0}^\infty (-1)^j \tilde{\beta}_{\alpha,j-1}(x) \log(1/|T_\alpha^j(x)|), \quad x \in [\alpha - 1, \alpha).$$

It is immediate to check that

$$(2.12) \qquad \tilde{W}_\alpha(x) = W_\alpha(|x|) \quad \text{for } x \in [\alpha - 1, \alpha).$$

This means that $\tilde{W}_\alpha(x) = W_\alpha(x)$ when $\alpha \geq 1/2$ (the two periodic functions agree on $[0, \alpha)$ and by symmetry also on $[\alpha - 1, 0]$); on the other hand for $\alpha < 1/2$, one has that $\tilde{W}_\alpha(x) = W_\alpha(-x)$ (indeed, this identity holds on $[\alpha - 1, 0]$, and by symmetry also on $(0, \alpha)$). Obviously, the regularity properties of $\tilde{W}_\alpha$ and W_α are the same, and since all the results about matching are stated for the family T_α, in Section 3, we will prefer to work in the unfolded setting. As in the folded case, also the unfolded Wilton function satisfies a functional equation

$$(2.13) \qquad \tilde{W}_\alpha(x) = \tilde{W}_\alpha^{(K)}(x) + (-1)^{K+1} \tilde{\beta}_{\alpha,K}(x) \tilde{W}_\alpha(T_\alpha^{K+1}(x)) \quad (K \in \mathbb{N}, \ x \in ([\alpha - 1, \alpha) \setminus \mathbb{Q}),$$

where $\tilde{W}_\alpha^{(K)}$ is the partial sum $\sum_{j=0}^K (-1)^j \tilde{\beta}_{\alpha,K-1}(x) \log(1/|T_\alpha^j(x)|)$.)

2.2.2. *Matching property.* We now recall the *matching property* first discovered by [18] in connection with the study of the metric entropy of T_α[1]; in fact we will see that this matching property plays an important role also for the regularity properties of the Wilton function.

Definition 2.2. *The value $\alpha \in (0, 1]$ is said to satisfy an algebraic matching condition of order (n, m), denoted by $(n, m)_{\mathrm{alg}}$, when the following matrix identity holds:*

$$(2.14) \qquad (n, m)_{\mathrm{alg}} : \qquad M_{\alpha,\alpha,n} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} M_{\alpha,\alpha-1,m} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}.$$

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[1]Actually we shall follow the notation introduced in [5] (which is slightly different from the original in [18]).

To get some intuition of what this condition means from a dynamic point of view, one should note that $(n, m)_{\mathrm{alg}}$ implies
$$T_\alpha^{n+1}(\alpha) = T_\alpha^{m+1}(\alpha - 1)$$
(see [5, Appendix A1], and also the brief explanation on the next page). For this reason, the pair (n, m) satisfying $(n, m)_{\mathrm{alg}}$ is referred to as *matching exponents*; the difference $m - n$ is called *matching index*. Actually in [5] it is proved that the set
$$\mathcal{M}_{\mathrm{alg}} := \{\alpha \in (0, 1] : \exists\, n, m \in \mathbb{N} \text{ s.t. } \alpha \text{ satisfies } (n, m)_{\mathrm{alg}}\}$$
contains an open neighbourhood of $(0, 1] \cap \mathbb{Q}$ of full measure; the connected components of this open set are called *matching intervals*, and on any matching interval, both sides of (2.14) are constant (see [5, Lemma 3.7]). Any matching interval J contains a unique rational value p/q with a minimal denominator called the *pseudocenter* of J; moreover, the matching exponents (n, m) can be easily extracted from the even length continued fraction expansion of its pseudocenter: indeed if p/q is a pseudocenter, then by choosing the regular continued fraction expansion $[0; a_1, a_2, \cdots, a_\ell]$ with even length ℓ from its two possible expansions, the matching exponents (n, m) of J are
$$n := \sum_{j:\ \text{even}} a_j \qquad \text{and} \qquad m := \sum_{j:\ \text{odd}} a_j,$$
i.e. every $x \in J$ satisfies the matching condition $(n, m)_{\mathrm{alg}}$ (see [5, Theorem 3.1]).

In [4], a more explicit description of $\mathcal{M}_{\mathrm{alg}}$ is given in terms of the Gauss map T_1: indeed $[0, 1] \setminus \mathcal{M}_{\mathrm{alg}} = \mathcal{E}$, where

(2.15)
$$\mathcal{E} := \{x : \ T_1^k(x) \geq x \ \ \forall k \in \mathbb{N}\}.$$

Note that $\mathcal{E}$ is a zero measure set, but $\dim_H(\mathcal{E}) = 1$ since for all $N \in \mathbb{N}$ it contains the set
$$F_N := \{x \in [0, 1] : \ x = [0; N + 1, a_2, a_3, \ldots, a_n, \ldots], \quad a_j \leq N \ \ \forall j \geq 2\}$$
which is the image under the Lipschitz map $x \mapsto \frac{1}{N+1+x}$ of the set of numbers with all partial quotients bounded by N (see the proof of Theorem 1.2 in [5]).

The Gauss map T_1 can also be used to characterize those rational values p/q which are the pseudocenter of some matching interval J: indeed, this happens if and only if $T_1^k(p/q) \notin (0, p/q)$ for all $k \in \mathbb{N}$. Let us give a few examples of this phenomenon.

(1) The interval $(g, 1]$ is a matching interval of index -1.
(2) The interval $(1 - g, g)$ contains infinitely many matching intervals, all of index 0; the largest one is the rightmost one, namely $(\sqrt{2} - 1, g)$; however, $\dim_H(\mathcal{E} \cap [1 - g, g]) > 0$ (see [6]).
(3) Every left neighbourhood of $1 - g$ contains infinitely many matching intervals of index[2] $+1$: indeed any rational value of the type $u_m = [0; 2, 1^{2m-1}]$ (with a tail of $2m - 1$ ones) is the pseudocenter of a matching interval on the left of $1 - g$; these intervals accumulate on $1 - g$ as $m \to +\infty$.

We conclude this small subsection with a remark that will play an important role in the following discussion. It is known[3] that the condition (2.14) implies that
$$\frac{1}{T_\alpha^n(\alpha)} + \frac{1}{T_\alpha^m(\alpha - 1)} = -1.$$

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[2]In fact every left neighbourhood of $1 - g$ contains infinitely many matching intervals of any index (see [6]).
[3]See [5, Appendix A1].

This implies that the terms on the left side of the above sum have opposite signs, and if $T_\alpha^{n+1}(\alpha) = \frac{\epsilon}{T_\alpha^n(\alpha)} - c$, then

$$\frac{1}{|T_\alpha^m(\alpha-1)|} = -\frac{\epsilon}{T_\alpha^m(\alpha-1)} = \epsilon + \frac{\epsilon}{T_\alpha^n(\alpha)} = \epsilon + c + T_\alpha^{n+1}(\alpha)$$

and this last equality implies that

$$T_\alpha^{m+1}(\alpha-1) = \frac{1}{|T_\alpha^m(\alpha-1)|} - \epsilon - c = T_\alpha^{n+1}(\alpha).$$

This last equality corresponds to the following matrix identity

$$M_{\alpha,\alpha-1,m+1}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = M_{\alpha,\alpha,n+1}^{-1}.$$

From the above identity, we also get that

(2.16) $$\tilde{p}_{\alpha,n}(\alpha) = \tilde{p}_{\alpha,m}(\alpha-1) + \tilde{q}_{\alpha,m}(\alpha-1), \qquad \tilde{q}_{\alpha,n}(\alpha) = \tilde{q}_{\alpha,m}(\alpha-1).$$

Moreover, if α is rational, then one has that, for sufficiently small $\varepsilon > 0$,

(2.17) $$\begin{aligned} M_{\alpha,x-1,m+1} = M_{\alpha,\alpha-1,m+1} \quad & \text{for } \alpha < x < \alpha+\varepsilon, \\ M_{\alpha,x,n+1} = M_{\alpha,\alpha,n+1} \quad & \text{for } \alpha-\varepsilon < x < \alpha, \end{aligned}$$

and this implies that, setting $\phi(x) := M_{\alpha,\alpha,n+1}^{-1} \cdot x$, for sufficiently small $\varepsilon > 0$, we have

(2.18) $$\begin{aligned} T_\alpha^{m+1}(x-1) = \phi(x) \quad & \text{for } \alpha < x < \alpha+\varepsilon, \\ T_\alpha^{n+1}(x) = \phi(x), \quad & \text{for } \alpha-\varepsilon < x < \alpha. \end{aligned}$$

From (2.16) and (2.17), we also get that there is an analytic function $b(x)$ such that

(2.19) $$\begin{aligned} \tilde{\beta}_{\alpha,m}(x-1) = b(x) \quad & \text{for } \alpha < x < \alpha+\varepsilon, \\ \tilde{\beta}_{\alpha,n}(x) = b(x) \quad & \text{for } \alpha-\varepsilon < x < \alpha. \end{aligned}$$

Lemma 2.3. *Let J be a matching interval with matching exponents (m,n) and let $\alpha \in J \cap \mathbb{Q}$. Then there exist a neighbourhood U of α, functions β, ϕ which are smooth on U and $h \in L^\infty(U)$ such that*

(2.20) $$\tilde{W}_\alpha(x) = h(x) + \text{sgn}(\alpha-x)^{n-m}\beta(x)\tilde{W}_\alpha(\phi(x)) \qquad \text{for } x \in U.$$

Proof. We will use the functional equation (2.13). If x is in a left neighbourhood of α, then we can write

$$\tilde{W}_\alpha(x) = \tilde{W}_\alpha^{(n)}(x) + (-1)^{n+1}\tilde{\beta}_{\alpha,n}(x)\tilde{W}_\alpha(T_\alpha^{n+1}(x)).$$

On the other hand, if x belongs to a right neighbourhood of α, then we can write

$$\tilde{W}_\alpha(x) = \tilde{W}_\alpha(x-1) = \tilde{W}_\alpha^{(m)}(x-1) + (-1)^{m+1}\tilde{\beta}_{\alpha,m}(x-1)\tilde{W}_\alpha(T_\alpha^{m+1}(x-1)).$$

By (2.19) we have that the analytic expression $b(x)$ coincides with $\tilde{W}_\alpha^{(n)}(x)$ on a left neighbourhood of α and with $\tilde{W}_\alpha^{(m)}(x-1)$ on a right neighbourhood of α; similarly by (2.18) the fractional transtormation ϕ coincides with $T_\alpha^{n+1}(x)$ on a left neighbourhood of α and with $T_\alpha^{m+1}(x-1)$ on a right neighbourhood of α. Therefore, due to the alternating term in the Wilton sum, the singular part of $\tilde{W}_\alpha$ at α is either symmetric (if $n-m \equiv 0 \mod 2$) or antisymmetric (if $n-m \equiv 1 \mod 2$). Thus, on a neighbourhood of α we can write the following decomposition:

$$\tilde{W}_\alpha(x) = h(x) + \text{sgn}(\alpha-x)^{n-m}\beta(x)\tilde{W}_\alpha(\phi(x)),$$

where

$$h(x) = \tilde{W}_\alpha^{(n)}(x)\chi_{(\alpha-\varepsilon,\alpha)}(x) + \tilde{W}_\alpha^{(m)}(x-1)\chi_{(\alpha,\alpha+\varepsilon)}(x) \text{ and } \quad \beta(x) = \pm b(x).$$

$\square$

3. Behaviour of $\tilde{W}_\alpha$ near rational points

Lemma 3.1. *Let $\alpha \in (0,1]$, then*

$$\int_0^x \tilde{W}_\alpha(t)dt = -x\log x + x + o(x) \quad as \ x \to 0^+.$$

Proof. The proof of this follows directly integrating the functional equation $\tilde{W}_\alpha(x) = -\log x - x\tilde{W}_\alpha(T_\alpha(x))$: the term $-x\log x + x$ comes from the integration of $-\log x$, while the integration of $x\tilde{W}_\alpha(T_\alpha(x))$ leads to a term which is $o(x)$ as $x \to 0$ because the function $\tilde{W}_\alpha \circ T_\alpha$ is in L^1. Indeed, this last property follows directly from the fact that the invariant measure has a BV density $d\mu_\alpha(x) = \rho_\alpha(x)dx$ such that $0 < m \le \rho_\alpha(x) \le M$, and since

$$\int_{\alpha-1}^\alpha |\tilde{W}_\alpha| \circ T_\alpha(x)\rho_\alpha(x)dx = \int_{\alpha-1}^\alpha |\tilde{W}_\alpha(x)|\rho_\alpha(x)dx,$$

we easily get that $\|\tilde{W}_\alpha \circ T_\alpha\|_1 \le \frac{M}{m}\|\tilde{W}_\alpha\|_1$, where $\|\cdot\|_1$ is the L^1-norm with respect to the Lebesgue measure. The fact $\tilde{W}_\alpha \in L^1$ is derived from $B_\alpha = \sum_j \beta_{\alpha,j-1}\log\frac{1}{x_{\alpha,j}} \in L^1$ proven in [16, Corollary 13]. $\square$

Remark 3.2. *From the above lemma, one easily deduces the behaviour of $\tilde{W}_\alpha$ on a symmetric neighbourhood of the origin:*

- *If $\alpha < 1$, then $\tilde{W}_\alpha$ is even on a neighbourhood of 0, and the above expansion holds also for negative values:*

$$\int_0^x \tilde{W}_\alpha(t)dt = -x\log|x| + x + o(x) \quad as \ x \to 0.$$

- *However, this is not the case for $\alpha = 1$; indeed, in this case,*

$$\int_0^x \tilde{W}_1(t)dt = -|x|\log|x| + O(x) \ as \ x \to 0.$$

In order to prove the second claim of the above remark, let us observe that if $x < 0$, then one can use the functional equation $\tilde{W}_1(x) = \tilde{W}_1(x+1) = -\log(1+x) - (1+x)\tilde{W}_1(|x|/(1+x))$ and change of variable to get

$$
\begin{aligned}
\int_0^x \tilde{W}_1(t)dt = -\int_x^0 \tilde{W}_1(t)dt &= O(x^2) + \int_x^0 (1+t)\tilde{W}_1(-t/(1+t))dt \\
&= O(x^2) + \int_0^{\frac{-x}{1+x}} \frac{1}{(1+y)^3}W_1(y)dy \\
&= x\log|x| + O(x) \text{ for } x \to 0^-.
\end{aligned}
$$

Let us anticipate that the behaviour of $\tilde{W}_1$ near the origin leads to the failure of BMO property (as we shall soon see in Lemma 3.5).

In order to discuss the different asymptotic properties of $\tilde{W}_\alpha$ at rational points, we give a couple of definitions as follows.

Definition 3.3. *Let $w \in L^1(a,b)$ and $\xi \in (a,b)$; we say that ξ is a singularity of* type A *if one of the following conditions holds*

(A_+) $\lim_{h\to 0} \frac{1}{|h|}\int_\xi^{\xi+h} w(t)dt = +\infty,$

(A_-) $\lim_{h\to 0} \frac{1}{|h|}\int_\xi^{\xi+h} w(t)dt = -\infty.$

We say that ξ is a singularity of type B *if one of the following conditions holds*

(B_+) $\lim_{h \to 0} \frac{1}{h} \int_{\xi}^{\xi+h} w(t)dt = +\infty$,

(B_-) $\lim_{h \to 0} \frac{1}{h} \int_{\xi}^{\xi+h} w(t)dt = -\infty$.

By Remark 3.2, the point $\xi = 0$ is a type A singularity for $\tilde{W}_1$, while for $\alpha \in (0,1)$, $\xi = 0$ is the prototype of type B singularity.

Type A and B are mutually exclusive conditions at a point ξ, and the multiplication by the function $\sigma(x) := \text{sign}(\xi - x)$ produces a switch between type A and B. Even if in general, types A and B do not cover all possible singularities of an L^1 function, this definition will well describe the behaviour of $\tilde{W}_\alpha$ at rational values.

Theorem 3.4. *Let $\alpha \in [0,1]$ and $\xi \in [\alpha - 1, \alpha) \cap \mathbb{Q}$.*

    (i) *If $\{\alpha, \alpha - 1\} \cap \{T_\alpha^k(\xi), k \in \mathbb{N}\} = \emptyset$, then ξ is a type B singularity for $\tilde{W}_\alpha$.*

    (ii) *Otherwise, $\alpha \in \mathbb{Q} \cap [0,1]$; and in this latter case, ξ is a type B singularity for $\tilde{W}_\alpha$ if and only if α belongs to a matching interval of even index.*

When the condition of Theorem 3.4-(i) holds, we will say that ξ is α-*regular*. Note that if $\alpha \in (0,1] \setminus \mathbb{Q}$, then every $\xi \in \mathbb{Q}$ is α-regular. On the other hand, if some $\xi \in \mathbb{Q}$ is not α-regular, then $\alpha \in \mathbb{Q}$, and it belongs to some matching interval J; in this latter case, ξ is a type A (resp. B) singularity for $\tilde{W}_\alpha$ if the matching index of J is odd (resp. even). In particular, if $\alpha \in [0,1] \cap \mathbb{Q}$ is a rational parameter belonging to a matching interval of odd index, then α is a type A singularity of $\tilde{W}_\alpha$.

The above consideration, together with the following general principle, will be the main tool to show that BMO condition fails for some parameters.

Lemma 3.5. *Let $w \in L^1(a, b)$, and let $\xi \in (a, b)$ be a type A singularity for w. Then,*

    (i) *for every $\varepsilon > 0$, there exist $x^+ \in (\xi, \xi + \varepsilon)$ and $x^- \in (\xi - \varepsilon, \xi)$ such that*

$$\int_{x^-}^{x^+} w(t)dt = 0,$$

    and

    (ii) *$w \notin \text{BMO}$.*

We shall also need another lemma, which will be very useful in combination with the functional equation.

Lemma 3.6. *Let $w \in L^1(a, b)$, let $\xi \in (a, b)$, and let β, ϕ be two smooth functions such that*

- *$\beta(\xi) \neq 0$,*
- *$\phi'(\xi) \neq 0$ (hence ϕ is locally invertible near ξ).*

If $\phi(\xi) \in (a, b)$ is a singularity of type B (resp. type A) for w, then the function $g(x) := \beta(x)w(\phi(x))$ has a singularity of type B (resp. type A) at ξ.

Before proving our claims, let us show what happens for $\alpha \in (g, (5 - \sqrt{13})/2)$. We have $1 - \alpha \in (\frac{1}{4-\alpha}, \frac{1}{2+\alpha})$ and $\frac{1}{\alpha} - 1 \in (\frac{1}{3-\alpha}, \frac{1}{1+\alpha})$. If ϵ is sufficiently small and $\alpha + \epsilon > x > \alpha$, then, by the functional equation (2.13) for $K = 0$, we get

$$\tilde{W}_\alpha(x) = \tilde{W}_\alpha(1 - x) = -\log(1 - x) - (1 - x)\tilde{W}_\alpha\left(3 - \frac{1}{1-x}\right).$$

Let us set $\phi(x) := 3 - \frac{1}{1-x} = \frac{2-3x}{1-x}$, using the functional equation (2.13) for $K = 1$, we get, for $\alpha - \epsilon < x < \alpha$,

$$\tilde{W}_\alpha(x) = -\log x - x \log\left(\frac{x}{1-x}\right) + (1-x)\tilde{W}_\alpha(\phi(x)).$$

Therefore, in a neighbourhood $(\alpha - \epsilon, \alpha + \epsilon)$, we have

$$\tilde{W}_\alpha(x) = h(x) + \text{sgn}(\alpha - x)(1-x)\tilde{W}_\alpha(\phi(x)) \quad \text{with } h(x) = \begin{cases} -\log(1-x), & x > \alpha, \\ -\log x - x\log(\frac{x}{1-x}), & x < \alpha. \end{cases}$$

We note that

- $h(x) = O(1)$ for $x \to \alpha$, hence it does not change the kind of singularity of $\tilde{W}_\alpha$ at α;
- by Lemma 3.6, the singularity of the function $g(x) := (1-x)\tilde{W}_\alpha(\phi(x))$ at α is of the same kind as the singularity of $\tilde{W}_\alpha$ at $\phi(\alpha)$;
- by Lemma 3.6, the function $\tilde{W}_\alpha$ has a type (B) singularity at $\xi = \phi(\alpha)$. This is trivial if $\alpha = 2/3$ (since $\phi(\alpha) = 0$), and in the other cases, it can be easily seen using the functional equation (2.13) with the smallest integer K such that $T_\alpha^K(\xi) = 0$ (indeed, $T_\alpha^K(x)$ will be a smooth fractional transformation for x in a neighbourhood of $\xi = \phi(\alpha)$);
- the function $\text{sgn}(\alpha - x)g(x)$ has a type (A) singularity at α.

Therefore we can conclude that $\tilde{W}_\alpha$ has a type (A) singularity at α, and is not BMO by Lemma 3.5. The same argument applies to the other values in the interval $(g, 1]$, the only thing that can change is the analytic form of the fractional transformation ϕ.

Proof of Theorem 1.2. The claim (i) is an immediate consequence of the above discussion, together with Lemma 3.5. As for the claim (ii), we already observed that each of the rational values $u_m = [0; 2, 1^{2m-1}]$ is the pseudocenter of a matching interval of index $+1$, and $u_m \to 1 - g$ as $m \to +\infty$, and by Theorem 3.4-(ii) and Lemma 3.5, we get that $\tilde{W}_{u_m} \notin$ BMO for all $m \in \mathbb{N}$, and this concludes the proof of Theorem 1.2. $\qquad \square$

3.1. **Technical proofs.**
In this section, we present technical proofs of Lemma 3.5, Lemma 3.6 and Theorem 3.4.

Proof of Lemma 3.5. WLOG, we may assume that w satisfies Condition (A_+). Moreover, we may also assume that $\varepsilon > 0$ is such that $(\xi - \varepsilon, \xi + \varepsilon) \subset (a, b)$ and that $\int_{\xi-\varepsilon}^{\xi} w(t)dt < 0$ and $\int_{\xi}^{\xi+\varepsilon} w(t)dt > 0$.

Let us consider $I(\varepsilon) := \int_{\xi-\varepsilon}^{\xi+\varepsilon} w(t)dt$, if $I(\epsilon) = 0$, then assertion (i) holds; if not, let us assume $I(\varepsilon) > 0$, and consider the function $g(x) := \int_{\xi-\epsilon}^{\xi+x} w(t)dt$: g is continuous on $[0, \epsilon]$, $g(0) < 0$ and $g(\varepsilon) > 0$, so by the intermediate value theorem, there is $x^+ \in (0, \varepsilon)$ such that $g(x^+) = \int_{\xi-\varepsilon}^{x^+} w(t)dt = 0$. An analogous argument works if $I(\varepsilon) < 0$, and this concludes the proof of (i).

To prove that $w \notin$ BMO, let M be any constant; we can choose $\varepsilon > 0$ so that

$$\frac{1}{|x|}\int_{\xi}^{\xi+x} w(t)dt \geq M, \quad \forall |x| < \varepsilon.$$

Let us also fix $x^+ \in (\xi, \xi + \varepsilon)$ and $x^- \in (\xi - \varepsilon, \xi)$ such that $\int_{x^-}^{x^+} w(t)dt = 0$, and let us estimate the quantity $\frac{1}{|I|} \int_I |w(t) - w_I| dt$, where $I = [x^-, x^+]$ and $w_I = \frac{1}{|I|} \int_I w(t)dt = 0$:

$$\frac{1}{x^+ - x^-} \int_{x^-}^{x^+} |w(t)| dt \geq \frac{1}{x^+ - x^-} [M(\xi - x^-) + M(x^+ - \xi)] = M.$$

This ends the proof of (ii). □

Proof of Lemma 3.6. Let us first consider the case $\phi(x) = x$; let us set $W(x) := \int_{\xi}^{\xi+x} w(t)dt$, and note that since $w \in L^1$, the function W is continuous. Hence, integrating by parts, we get

$$\int_{\xi}^{\xi+x} \beta(t)w(t)dt = W(\xi + x)\beta(\xi + x) - \int_{\xi}^{\xi+x} W(t)\beta'(t)dt.$$

Note that the second term in the right-hand side is $O(x)$ as $x \to 0$. Thus, if w has a type A singularity at ξ, we get that

$$\frac{1}{|x|} \int_{\xi}^{\xi+x} \beta(t)w(t)dt = \frac{1}{|x|} W(\xi + x)\beta(\xi + x) + O(1) \quad \text{for } x \to 0;$$

hence, the product βw also has a type A singularity at ξ. The case of type B singularity goes through in the very same way.

To handle the general case, it is enough to use a change of coordinates; setting $s = \phi(t)$ and $\psi := \phi^{-1}$, we have

$$\int_{\xi}^{\xi+x} \beta(t)w(\phi(t))dt = \int_{\phi(\xi)}^{\phi(\xi+x)} \beta(\psi(s))w(s) \frac{1}{\phi'(\psi(s))} ds,$$

and we get our claim using the previous point, with $\tilde{\beta}(s) := \frac{\beta(\psi(s))}{\phi'(\psi(s))}$ instead of β. □

Proof of Theorem 3.4. Let us first prove claim (i); we shall consider $0 < \alpha < 1$ (otherwise, the hypotheses are not met). Since $\xi \in \mathbb{Q}$, there is a smallest $k_0 \in \mathbb{N}$ such that $T_\alpha^{k_0+1}(\xi) = 0$; so, by the functional equation (2.13), we get

$$\tilde{W}_\alpha(x) = \tilde{W}_\alpha^{(k_0)}(x) + (-1)^{k_0+1}\tilde{\beta}_{\alpha,k_0}(x)\tilde{W}_\alpha(T_\alpha^{k_0+1}(x)) \quad \text{with} \quad \tilde{W}_\alpha^{(k_0)}(x) = \sum_{k=0}^{k_0} (-1)^k \tilde{\beta}_{\alpha,k-1}(x) \log \frac{1}{|T_\alpha^k(x)|}.$$

Since $T_\alpha^k(\xi) \notin \{\alpha, \alpha-1\}$ for $k \leq k_0$, we get that, in a neighbourhood of ξ, $T_\alpha^{k_0+1}$ coincides with a fractional transformation ϕ; for the same reason the term $\tilde{W}_\alpha^{(k_0)}$ is smooth in a neighbourhood of ξ (hence it's irrelevant for the type of singularity); therefore by Lemma 3.6 the singularity in ξ is of the same type as in $\phi(\xi) = 0$, namely it is type B.

For the proof of claim (ii), let us first point out that the hypotheses of (i) are met whenever $\alpha \notin \mathbb{Q}$ or when $\alpha \in \mathbb{Q}$ but $\text{den}(\alpha) > \text{den}(\xi)$, where "den" denotes the denominator of a rational number. Thus if we are in case (ii), we have that $\alpha \in \mathbb{Q}$ and $\text{den}(\alpha) \leq \text{den}(\xi)$.

If $\xi \notin \{\alpha - 1, \alpha\}$, then there is $k_0 \geq 1$ such that $T_\alpha^{k_0}(\xi) = \alpha - 1$ and we can write

$$\tilde{W}_\alpha(x) = \tilde{W}_\alpha^{(k_0-1)}(x) + (-1)^{k_0}\beta_{k_0-1}(x)\tilde{W}_\alpha(T_\alpha^{k_0}(x)).$$

However, since $T_\alpha^k(\xi) \notin \{\alpha, \alpha-1\}$ for $k < k_0$, $T_\alpha^{k_0}$ coincides with a fractional transformation ϕ on a neighbourhood of ξ, hence by Lemma 3.6, ξ and $\alpha - 1$ are singularities of the same type for T_α.

Therefore to prove claim (ii), it is enough to show that, for $\alpha \in \mathbb{Q}$, α is of type A if and only if α belongs to a matching interval J of odd index. To prove this, we use equation (2.20) to express

W_α near α:
$$W_\alpha(x) = h(x) + \text{sgn}(\alpha - x)^{n-m} \beta(x) W_\alpha(\phi(x)),$$
where $h \in L^\infty$ on a neighbourhood of α and β, ϕ are smooth near α. We first point out that $\xi := \phi(\alpha)$ is a type B singularity, since $\text{den}(\xi) < \text{den}(\alpha)$, and hence falls in case (i) we just treated above. Then, by Lemma 3.6, the expression $\beta(x) \tilde{W}_\alpha(\phi(x))$ has a type B singularity in α, hence we deduce that if $n - m$ is even, then the singularity of $\tilde{W}_\alpha$ at α is of type B as well, while if $n - m$ is odd, then it becomes of type A. Since h is bounded, this ends the proof of claim (ii). $\qquad \square$

4. BMO PROPERTY FOR $\alpha \in [1 - g, g]$

The BMO property of W_α for $\alpha \in [\frac{1}{2}, g]$ was proven in [10, Theorem 2.3]. In this section, we focus on W_α for $\alpha \in [1 - g, \frac{1}{2}]$. Since BMO property is preserved when summing with an L^∞ function, it suffices to prove the following theorem to establish Theorem 1.1.

Theorem 4.1. *For $\alpha \in [1 - g, \frac{1}{2}]$, $W_\alpha - W_{1/2}$ is uniformly bounded.*

The proof of the theorem follows the proof of [10, Proposition 2.9]. We provide a self-contained proof of the theorem to ensure readability. To deduce that $B_\alpha(x) - \sum_{n=0}^\infty \frac{\log q_{\alpha,n+1}}{q_{\alpha,n}}$ is uniformly bounded in [16, Theorem 8], they proved $\sum_{n=0}^\infty \left| \beta_{\alpha,n-1} \log x_{\alpha,n}^{-1} - \frac{\log q_{\alpha,n+1}}{q_{\alpha,n}} \right|$ is uniformly bounded. From this, we derive the following proposition which allows us to use the series $\sum_{n=0}^\infty (-1)^n \frac{\log q_{\alpha,n+1}}{q_{\alpha,n}}$ in place of W_α.

Proposition 4.2. *For $\alpha \in (0, 1]$, $\left| W_\alpha(x) - \sum_{n=0}^\infty (-1)^n \frac{\log q_{\alpha,n+1}}{q_{\alpha,n}} \right|$ is uniformly bounded.*

In this section, we compare the orbits under A_α and $A_{1/2}$ associated with the same point $x \in \mathbb{R}$. We first consider $x_{\alpha,0}$ and $x_{1/2,0}$, defined as in (2.2) for the same $x \in \mathbb{R}$. Then we compare the orbits $\{A_\alpha^k(x_{\alpha,0})\}$ and $\{A_{1/2}^k(x_{1/2,0})\}$.

For the sake of brevity, let us denote the A_α-orbits and the $A_{1/2}$-orbits by $x_k = A_\alpha^k(x_{\alpha,0})$ and $x_k' = A_{1/2}^k(x_{1/2,0})$. Accordingly, we denote the convergents and the partial quotients associated with A_α by $\frac{p_k}{q_k}$ and (a_k, ϵ_k), and those associated with $A_{1/2}$ by $\frac{p_k'}{q_k'}$ and (a_k', ϵ_k'). We recall some fundamental properties of α-continued fractions.

Remark 4.3. *(1) For $x, x' \in (0, 1]$ such that $A_1(x) = A_1(x')$,*

(4.1)
$$\begin{cases} A_\alpha(x) \leq 1/2 & \text{if and only if } A_\alpha(x) = A_{1/2}(x'), \\ A_\alpha(x) > 1/2 & \text{if and only if } A_\alpha(x) + A_{1/2}(x') = 1. \end{cases}$$

(2) For $\alpha \in (0, \frac{1}{2})$, we have

(4.2)
$$\begin{cases} (a_k, \epsilon_k) = (a_k', \epsilon_k') & \text{if } x_k = x_k' \text{ and } x_{k-1} = x_{k-1}', \\ (a_k, \epsilon_k) = (a_k' + 1, \epsilon_k') & \text{if } x_k = x_k' \text{ and } \frac{1}{x_{k-1}} - 1 = \frac{1}{x_{k-1}'}, \\ a_k = a_k' + 1, \ \epsilon_k = -1, \ \epsilon_k' = 1 & \text{if } x_k = 1 - x_k' \text{ and } x_{k-1} = x_{k-1}', \\ a_k = a_k' + 2, \ \epsilon_k = -1, \ \epsilon_k' = 1 & \text{if } x_k = 1 - x_k' \text{ and } \frac{1}{x_{k-1}} - 1 = \frac{1}{x_{k-1}'}. \end{cases}$$

(3) For $\alpha \in (0, 1]$, we have $q_n > \frac{1}{C \cdot \lambda^n}$, where $C > 0$ and $0 < \lambda < 1$. Combining with $x^{-1/2} \log x \leq 2/e$ for $x > 0$, the series

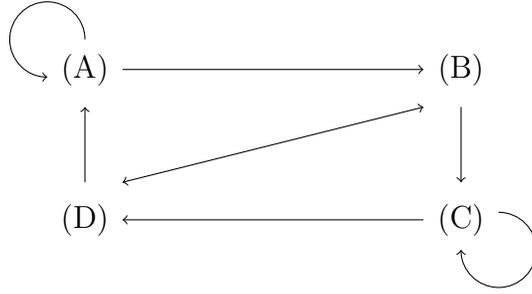$$\sum_{n=0}^\infty \frac{1}{q_n} \quad \text{and} \quad \sum_{n=0}^\infty \frac{\log q_n}{q_n}$$

FIGURE 2. Diagram of the four states in Proposition 4.4.

are uniformly bounded, see [16, Eq. (3.9) and Proof of Theorem 8] *and also* [13, Remark 1.7] *for details.*

Proposition 4.4. *For $x \in \mathbb{R}$, a tuple $S_i := (x_i, x'_i, q_i, q'_i)$ is in one of the following four states:*

(A) $x_i = x'_i$ and $q_i = q'_i$,
(B) $x_i = 1 - x'_i$, $x'_i \in (\alpha, \frac{1}{2}]$ and $q_i - q'_i = q_{i-1}$,
(C) $\frac{1}{x_i} - 1 = \frac{1}{1-x'_i}$ and $q_i - q'_i = -q'_{i-1}$,
(D) $\frac{1}{x_i} - 1 = \frac{1}{x'_i}$ and $q_i = q'_i$.

Moreover, the states change according to the diagram in Figure 2.

Remark 4.5. *If S_i is in state* (B), *then $x_i \geq 1/2$. If it is in state* (C), *then $x_i \geq 1/3$.*

Proof of Proposition 4.4. We will prove our claim inductively. Let us point out that S_0 can only be in state (A) or (B), so the statement is true at the start of the induction chain. We then assume that the statement holds for $0 \leq i \leq k$ and deduce that the statement holds also for $k + 1$; this will be accomplished proving that the quadruple evolves accordingly to the arrows in the diagram 4.4.

Let us point out that (C) and (D) occur only in the chain of (B)-(C)-(C)-$\cdots$-(C)-(D), where the case (B)–(D) (with no occurrence of (C)) is also included. Thus, it is not necessary to consider (C) and (D) independently.

Case (A). If $S_k \in$ (A), then $S_{k+1} \in \{(A), (B)\}$.

Suppose that $S_k \in$ (A). From the induction hypothesis, $S_{k-1} \in$ (A) or $S_{k-1} \in$ (D). Thus we have

$$x_k = x'_k, \quad q_k = q'_k, \quad q_{k-1} = q'_{k-1} \text{ and, by (4.2), } \epsilon_k = \epsilon'_k.$$

We divide into two cases according to whether $x_{k+1} \leq 1/2$ or $x_{k+1} > 1/2$. If $x_{k+1} \leq 1/2$, then $x_{k+1} = x'_{k+1}$ by (4.1). From (4.2), $a_{k+1} = a'_{k+1}$, which implies $q_{k+1} = q'_{k+1}$. Thus $S_{k+1} \in$ (A). For the case $x_{k+1} > 1/2$, with a similar argument, we have $x_{k+1} = 1 - x'_{k+1}$, $x_{k+1} \in (\alpha, \frac{1}{2}]$ and $a_{k+1} = a'_{k+1} + 1$. Thus, we have $q_{k+1} = a_{k+1}q_k + \epsilon_k q_{k-1} = q'_{k+1} + q_k$. Therefore, $S_{k+1} \in$ (B).

Case (B). If $S_k \in$ (B), then there exists $m \geq 0$ such that $S_{k+i} \in$ (C) for $i = 1, \ldots, m-1$, $S_{k+m} \in$ (D) and $S_{k+m+1} \in \{(A), (B)\}$.

Suppose that $S_k \in$ (B). Then $S_{k-1} \in$ (A) or $S_{k-1} \in$ (D). Thus we have $x_k = 1 - x'_k$, $x'_k \in (\alpha, \frac{1}{2}]$ and

(4.3) $$q_k - q'_k = q_{k-1}, \quad q_{k-1} = q'_{k-1}, \quad \epsilon_k = -1, \quad \epsilon'_k = 1.$$

Let $u_m = [0; 2, 1^{2m-1}]$ and $t_m = [0; 2, 1^{2m}]$, where $u_0 = 0$. Since $u_m \uparrow 1 - g$ and $t_m \downarrow 1 - g$, we have

$$(0, 1 - g) = \sqcup_{m=1}^{\infty}(u_{m-1}, u_m] \quad \text{and} \quad \left(1 - g, \frac{1}{2}\right] = \sqcup_{m=1}^{\infty}(t_m, t_{m-1}].$$

Since $x_k' \in (\alpha, \frac{1}{2}]$, there exists m such that $x_k' \in (t_m, t_{m-1}]$.

If $m = 1$, i.e. $x_k' \in \left(\max\{\frac{2}{5}, \alpha\}, \frac{1}{2}\right]$ and $x_k \in \left[\frac{1}{2}, \min\{\frac{3}{5}, 1 - \alpha\}\right)$, then

$$x_{k+1}' = \frac{1}{x_k'} - 2 \text{ with } (a_{k+1}', \epsilon_{k+1}') = (2, 1), \quad \text{and } x_{k+1} = 2 - \frac{1}{x_k} \text{ with } (a_{k+1}, \epsilon_{k+1}) = (2, -1).$$

Thus $\frac{1}{x_{k+1}} - 1 = \frac{1}{x_{k+1}'}$. By (4.3), $q_{k+1} = 2q_k - q_{k-1} = 2q_k' + q_{k-1}' = q_{k+1}'$. It means that $S_{k+1} \in$ (D).

For $m > 1$, we shall show that

(4.4) $\qquad S_{k+j} \in$ (C) for $j = 1, \cdots, m - 1$, and $S_{k+m} \in$ (D).

For brevity, we define functions f and g by

$$f(x) := 3 - \frac{1}{x} \quad \text{and} \quad g(x) := \frac{2x - 1}{x - 1}.$$

First, we have $f(t_i) = t_{i-1}$, $f(u_i) = u_{i-1}$ and $u_i = 2 - \frac{1}{1-t_i}$ for all i. These equations imply that

(4.5) $\qquad \begin{cases} x_{k+j}' = f(x_{k+j-1}') & \text{and } (a_{k+j}', \epsilon_{k+j}') = (3, -1) \quad \text{for } j = 1, \cdots, m - 1, \\ x_{k+m}' = \frac{1}{x_{k+m-1}'} - 2 & \text{and } (a_{k+m}', \epsilon_{k+m}') = (2, 1), \\ x_{k+1} = 2 - \frac{1}{x_k}, & \text{and } (a_{k+1}, \epsilon_{k+1}) = (2, -1), \\ x_{k+j} = f(x_{k+j-1}) & \text{and } (a_{k+j}, \epsilon_{k+j}) = (3, -1) \quad \text{for } j = 2, \cdots, m. \end{cases}$

Note that the relation of x_i and x_i' in (C) is equivalent to $x_i' = g(x_i)$. We have

$$x_{k+1}' = f(x_k') = f\left(1 - \frac{1}{2 - x_{k+1}}\right) = \frac{2x_{k+1} - 1}{x_{k+1} - 1} = g(x_{k+1}).$$

Since $f \circ g(x) = g \circ f(x)$, the relation $x_{k+j-1}' = g(x_{k+j-1})$ implies that

$$x_{k+j}' = f(x_{k+j-1}') = f \circ g(x_{k+j-1}) = g \circ f(x_{k+j-1}) = g(x_{k+j}) \quad \text{for } j = 2, \cdots, m - 1.$$

From above, for $j = m$, we have

$$x_{k+m}' = \frac{1}{x_{k+m-1}'} - 2 = \frac{1}{g(x_{k+m-1})} - 2 = 1 - f(g(x_{k+m-1})) = \frac{x_{k+m}}{1 - x_{k+m}}.$$

On the other hand, combining (4.5) and (4.3) with a recurrence relation of q_i, we have

(4.6) $\qquad \begin{cases} q_{k+1}' - q_{k+1} & = 3q_k' - 2q_k + 2q_{k-1} = q_k', \\ q_{k+j}' - q_{k+j} & = 3(q_{k+j-1}' - q_{k+j-1}) - (q_{k+j-2}' - q_{k+j-2}) = q_{k+j-1}' \text{ for } 2 \le j \le m - 1, \\ q_{k+m}' - q_{k+m} & = 3(q_{k+m-1}' - q_{k+m-1}) - (q_{k+m-2}' - q_{k+m-2}) - q_{k+m-1}' = 0, \end{cases}$

inductively. Thus (4.4) holds.

We now only need to prove that $S_{k+m+1} \in \{(A), (B)\}$. First, from $a_{k+m}' = 2$, $\epsilon_{k+m-1}' = -1$ as in (4.5), $q_{k+m}' - q_{k+m-1}' = q_{k+m-1}' - q_{k+m-2}'$. By (4.6), we have

(4.7) $\qquad q_{k+m}' - q_{k+m-1}' - q_{k+m-1} = 0.$

Since $S_{k+m} \in$ (D), we have the relation between x_{k+m} and x_{k+m}' in (D), which implies $A_1(x_{k+m}) = A_1(x_{k+m}')$. We divide into two cases according to whether $A_\alpha(x_{k+m}') \le 1/2$ or $A_\alpha(x_{k+m}') > 1/2$.

If $A_\alpha(x'_{k+m}) \leq 1/2$, then $A_{1/2}(x'_{k+m}) = A_\alpha(x_{k+m})$ by (4.1). By (4.2), $a_{k+m+1} = a'_{k+m+1} + 1$. Recall that $q_{k+m} = q'_{k+m}$ in (4.6) and $\epsilon_{k+m} = -1$ and $\epsilon'_{k+m} = 1$ in (4.5). Combining with (4.7), we have $q_{k+m+1} - q'_{k+m+1} = q_{k+m} - q_{k+m-1} - q'_{k+m-1} = 0$. Therefore, $S_{k+m+1} \in (A)$.

If $A_\alpha(x'_{k+m}) > 1/2$, then $A_{1/2}(x'_{k+m}) = 1 - A_\alpha(x_{k+m}) \in (\alpha, \frac{1}{2})$ and $a_{k+m+1} = a'_{k+m+1} + 2$. Thus, we have $q_{k+m+1} - q'_{k+m+1} = 2q_{k+m} - q_{k+m-1} - q'_{k+m-1} = q'_{k+m}$ with a similar argument above. Therefore, $S_{k+m+1} \in (B)$. $\qquad\square$

Proof of Theorem 4.1. The difference $W_\alpha - W_{1/2}$ is $\mathbb{Z}$-periodic and symmetric on $(0, \alpha)$. By Proposition 4.2, it is enough to show that $\sum_{n=0}^{\infty} \left| \frac{\log q_{n+1}}{q_n} - \frac{\log q'_{n+1}}{q'_n} \right|$ is uniformly bounded for $x \in [0, 1-\alpha]$. We have

$$\frac{\log q_{n+1}}{q_n} - \frac{\log q'_{n+1}}{q'_n} = \frac{1}{q_n} \log \frac{q_{n+1}}{q'_{n+1}} + \left(\frac{1}{q_n} - \frac{1}{q'_n} \right) \log(q'_{n+1}).$$

By using the recurrence relation of q'_i, Hurwitz proved that $\frac{q'_i}{q'_{i+1}} \leq g$ for all i in [7, §3], see also [19, Satz 5.18 (B) in §43] and [8, p. 421]. By Proposition 4.4, $|q_i - q'_i| \leq q_{i-1}$ for all i. Thus we have

$$1 - g \leq \frac{-q'_n + q'_{n+1}}{q'_{n+1}} \leq \frac{q_{n+1}}{q'_{n+1}} \leq \frac{q'_n + q'_{n+1}}{q'_{n+1}} \leq 2,$$

which implies that

$$(4.8) \qquad \left| \log \frac{q_{n+1}}{q'_{n+1}} \right| \leq \max \left\{ \log 2, \log \frac{1}{1-g} \right\} = \log(g+2).$$

In the proof of Proposition 4.4, we saw that if $|q'_n - q_n| = q'_{n-1}$, then $a'_{n+1} = 2$ or 3, see (4.5). Thus $q'_{n+1} \leq 3q'_n + q'_{n-1} \leq 4q'_n$. Then we have

$$(4.9) \qquad \left| \frac{1}{q_n} - \frac{1}{q'_n} \right| = \frac{|q'_n - q_n|}{q_n q'_n} \leq \frac{q'_{n-1}}{(q'_n - q'_{n-1})q'_n} = \frac{1}{\left(\frac{q'_n}{q'_{n-1}} - 1 \right) q'_n} \leq \frac{1}{\left(\frac{1}{g} - 1 \right) q'_n} \leq \frac{4}{g q'_{n+1}}.$$

By (4.8), (4.9) and Remark 4.3-(3),

$$\sum_{n=0}^{\infty} \left| \frac{\log q_{n+1}}{q_n} - \frac{\log q'_{n+1}}{q'_n} \right| \leq \sum_{n=0}^{\infty} \frac{\log(g+2)}{q_n} + \sum_{n=0}^{\infty} \frac{4}{g} \cdot \frac{\log q'_{n+1}}{q'_{n+1}}$$

is uniformly bounded. $\qquad\square$

Let us remark that for α near $1/2$, we can provide a much more precise estimate for the difference $W_\alpha - W_{1/2}$ exploiting the matching phenomenon: we give here an explicit example of the difference $W_{2/5} - W_{1/2}$, which has the following graphs.

To interpret these pictures, let us first focus on some $x \in (1/2, 3/5)$ (see Figure 3, right), so that we can use the classical regular continued fraction and write $x = [0; 1, 1, a+y]$ with $a \in \mathbb{N}$, $y \in (0, 1)$ and $a + y > 2$. We first observe that, letting $x_0 = x_{0, 2/5}$ and $x' = x_{0, 1/2}$, we get

$$\begin{aligned} x_0 &= x = [0; 1, 1, a+y], & x_1 &= A_{2/5}(x) = 1 - A_1(x) = [0; a+1+y], & A_1(x_1) &= y; \\ x'_0 &= 1 - x = [0; 2, a+y], & x'_1 &= A_{1/2}(1-x) = A_1(1-x) = [0; a+y], & A_1(x'_1) &= y. \end{aligned}$$

Here we see that the orbits follow the diagram in Figure 2. More precisely, we start from state (B), and pass directly to state (D); after that we can either end up in state (A) or (B). Using the
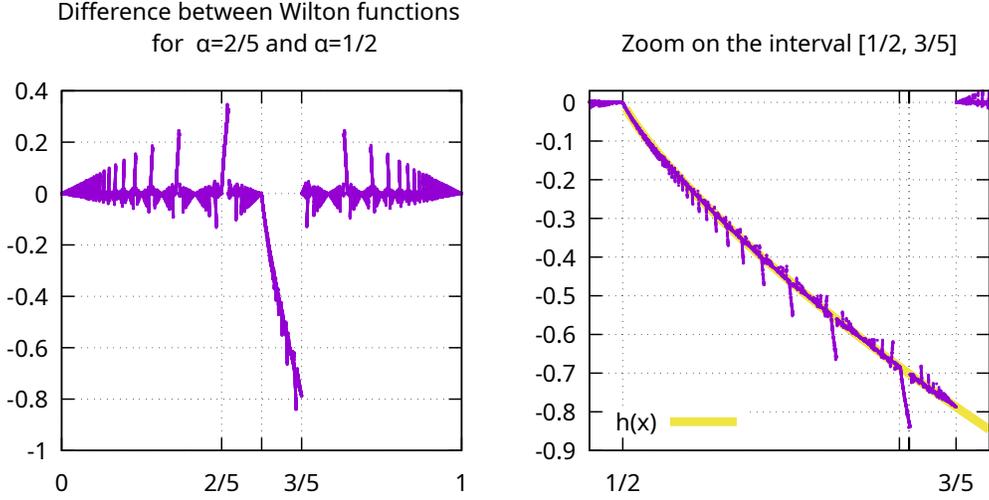
FIGURE 3. $W_\alpha - W_{1/2}$ for $\alpha = 2/5$

functional equation (2.13), we get

$$W_{2/5}(x) = -\log x - x \log \frac{x}{2x-1} + (2x-1)W_{2/5}(x_2),$$

$$W_{1/2}(x) = W_{1/2}(1-x) = -\log(1-x) - (1-x)\log\frac{1-x}{2x-1} + (2x-1)W_{1/2}(x_2').$$

Therefore the difference can be written as

$$(4.10) \quad W_{2/5}(x) - W_{1/2}(x) = h(x) + (2x-1)[W_{2/5}(x_2) - W_{1/2}(x_2')] \quad \text{with}$$

$$(4.11) \qquad h(x) = -\log x - x \log \frac{x}{2x-1} + \log(1-x) + (1-x)\log\frac{1-x}{2x-1},$$

where either $x_2 = x_2'$ (state (A)) or $x_2 = 1 - x_2'$ (state (B)); in either case, one has that $\beta_1 = \beta_1'$.

Note that h is the function plotted with a yellow thick line in Figure 3, which follows the graph of the difference quite closely. Indeed, this can be explained easily: if we denote by $\tilde{B} = \{k \in \mathbb{N} : (x_k, x_k')$ is in state (B)$\}$, then we easily realize that

$$(4.12) \qquad W_{2/5}(x) - W_{1/2}(x) = \sum_{k \in \tilde{B}} \beta_{k-1} h(x_k).$$

This formula holds in general, and explains the structure of the graph well; for instance, the part of the graph where $W_{2/5}(x) - W_{1/2}(x)$ closely shadows $h(x)$ corresponds to a point for which (x_k, x_k') stays in state (A) for quite a few iterations, while intervals where the graph of the difference parts from that of h corresponds to quick returns to state (B) (the largest "hair" shooting off the graph of h for $x \in (7/12, 18/31)$ corresponds to the case $x_2 = 1 - x_2' = y$ with $y \in (1/2, 3/5)$): namely a transition (D) to (B) without passing through state (A). One can use (4.12) together with the estimate of Proposition 2.1 to prove rigorously that $\|W_{2/5} - W_{1/2}\|_\infty < 1$ (which can be guessed from Figure 3).

5. Appendix

In Section 3, we saw how the matching condition, whose relevance was first understood in connection with the study of the entropy of α-continued fractions, plays a key role in the mechanism leading to the failure of the BMO property. Actually, the techniques used in Section 4 have the same flavour, even if the matching property is never explicitly mentioned. In fact, we can also use the intermediate results in Section 4 to recover a very simple proof of the following non trivial fact:

Proposition 5.1. *The metric entropy of A_α is constant for $\alpha \in [1-g, g]$.*

For $\alpha \in [1/2, g]$, this result is known since the eighties ([17]), but extending it to $[1-g, 1/2]$ is much harder, because the range $[1-g, 1/2]$ is split into countably many matching intervals. Indeed the proofs of this result that can be found in the literature are rather convoluted (see[4] for instance Section 10 of [9] or Proposition 6.1 of [6]).

However, the results in the previous section provide a straightforward proof of the "hard" case $\alpha \in [1-g, 1/2]$.

Proof of Proposition 5.1. It is well known that for $\alpha > 0$ the map A_α has an (unique) ergodic absolute continuous invariant probability measure μ_α, and for a.e. $x \in [0, \bar{\alpha}]$ the invariant measure μ_α and the metric entropy can be computed as follows:

$$\mu_\alpha([a,b]) = \lim_{n \to +\infty} \frac{\sum_{k=0}^{n-1} \chi_{[a,b]}(A_\alpha^k(x))}{n}, \qquad h_{\mu_\alpha}(A_\alpha) = 2 \lim_{k \to +\infty} \frac{1}{k} \log q_{\alpha,k}(x),$$

see [18, Proposition 1]. We shall call *typical* a value for which both above formulas hold.

Let $\alpha \in [1-g, 1/2]$ be fixed, and let us pick $x_0 \in (0, 1/3)$ which is typical both for $A_{1/2}$ and A_α; resuming the notation of the previous section, we set $x_k = A_\alpha^k(x_0)$, $x_k' = A_{1/2}^k(x_0)$ and define $\frac{p_k}{q_k}$, $\frac{p_k'}{q_k'}$ as the convergents of x_0 associated with $A_{1/2}$ and A_α, respectively.

Since x_0 is typical then there is an infinite set J of indices k such that $x_k \in (0, 1/3)$ for $k \in J$, and thus by Remark 4.5 the pair (x_k, x_k') is either in state (A) or (D) for all $k \in J$ and $q_k = q_k'$ for all $k \in J$. Therefore,

$$h_{\mu_\alpha}(A_\alpha) = 2 \lim_{k \to +\infty} \frac{1}{k} \log q_k = 2 \lim_{\substack{k \to +\infty, \\ k \in J}} \frac{1}{k} \log q_k = 2 \lim_{\substack{k \to +\infty, \\ k \in J}} \frac{1}{k} \log q_k' = 2 \lim_{k \to +\infty} \frac{1}{k} \log q_k' = h_{\mu_\alpha}(A_{1/2}).$$

$\square$

References

[1] M. Balazard and B. Martin. *Comportement local moyen de la fonction de Brjuno.* Fund. Math. **218** (2012), 193–224.

[2] M. Balazard and B. Martin. *Sur l'autocorrélation multiplicative de la fonction "partie fractionnaire" et une fonction définie par J. R. Wilton. https://hal.archives-ouvertes.fr/hal-00823899v1* (2013).

[3] M. Balazard and B. Martin. *Sur certaines équations fonctionnelles approchées, liées à la transformation de Gauss.* Aequationes Math. **93** (2019), 563–585.

[4] C. Bonanno, C. Carminati, S. Isola, and G. Tiozzo. *Dynamics of continued fractions and kneading sequences of unimodal maps.* Discrete Contin. Dyn. Syst. **33** (2013), 1313–1332.

[5] C. Carminati and G. Tiozzo. *A canonical thickening of $\mathbb{Q}$ and the entropy of α-continued fraction transformations.* Ergodic Theory Dyn. Syst. **32** (2012), 1249–1269.

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[4]Actually the results of [9, 6] are established for the unfolded algorithm T_α, but it is not difficult to see that the entropy of A_α and T_α is the same for all $\alpha \in [0, 1]$.

[6] C. Carminati and G. Tiozzo. *Tuning and plateaux for the entropy of α-continued fractions.* Nonlinearity **26** (2013), 1049–1070.

[7] A. Hurwitz. *Über eine besondere art der kettenbruch-entwicklung reeller grössen.* Acta math **12** (1889) 367–405.

[8] H. Jager. *Metrical results for the nearest integer continued fraction.* Indagationes Mathematicae, **88** (1985), 417–427.

[9] C. Kraaikamp, T. A. Schmidt, and W. Steiner, Natural extensions and entropy of α-continued fractions, *Nonlinearity*, 25 (2012), no. 8, 2207–2243,

[10] S. B. Lee, S. Marmi, I. Petrykiewicz, and T. I. Schindler. *Regularity properties of k-Brjuno and Wilton functions.* Aequat. Math. **98** (2024), 13–85.

[11] L. Luzzi, S. Marmi, H. Nakada, and R. Natsui. *Generalized Brjuno functions associated to α-continued fractions.* J. Approx. Theory **162** (2010).

[12] S. Marmi. *Critical functions for complex analytic maps.* J. Phys. A **23** (1990), 3447–3474.

[13] S. Marmi, P. Moussa, and J.-C. Yoccoz. *The Brjuno functions and their regularity properties.* Comm. Math. Phys. **186** (1997), 265–293.

[14] S. Marmi, P. Moussa, and J.-C. Yoccoz. *Complex Brjuno functions.* J. Amer. Math. Soc. **14** (2001), 783–841.

[15] S. Marmi, P. Moussa, and J.-C. Yoccoz. *Some properties of real and complex Brjuno functions.* In "Frontiers in number theory, physics, and geometry. I", Springer, Berlin, 2006, 601–623.

[16] P. Moussa, A. Cassa, and S. Marmi. *Continued fractions and Brjuno functions.* J. Comput. Appl. Math. **105** (1999), 403–415.

[17] H. Nakada. *Metrical theory for a class of continued fraction transformations and their natural extensions.* Tokyo J. Math. **4** (1981) 399–426.

[18] H. Nakada and R. Natsui. *The non-monotonicity of the entropy of α-continued fraction transformations.* Nonlinearity **21** (2008), 1207–1225.

[19] O. Perron. "Die Lehre von den Kettenbrüchen Band I: Elementare Kettenbrüche." Teubner, 1954.

[20] J. R. Wilton. *An approximate functional equation with applications to a problem of Diophantine approximation.* J. Reine Angew. Math. **169** (1933), 219–237.

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