$\mathbf{S^1}$-index theory for the Lorentz force equation

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Abstract. In this paper we prove that the S^1-invariance of the Poincaré action functional associated to the Lorentz force equation gives the existence of multiple critical points which are periodic solutions with a fixed period. To do this, we prove an abstract multiplicity result which is based upon the Lusternik-Schnirelman method with the S^1-index. The corresponding result in the context of the Fadell-Rabinowitz index is proved in Ekeland and Lasry (Ann. Math., 112 (1980)). The main feature of our abstract result is that it allows us to consider nonsmooth functionals satisfying only a weak compactness condition well adapted to the Poincaré functional.

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1 Introduction

In what follows $\mathbb{R}^3$ is endowed with the Euclidean scalar product "$\cdot$" and with the Euclidean norm "$|\cdot|$". Let $T > 0$ be a fixed period. Let $V : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ and $W : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ be two C^1-functions. In this paper we consider the Lorentz force equation (LFE) with the electric potential V and the magnetic potential W, namely

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' = E(t,q) + q' \times B(t,q),$$

where

$$E = -\nabla_q V - \frac{\partial W}{\partial t}, \qquad B = \operatorname{curl}_q W$$

are the electric and magnetic fields respectively, and $E + q' \times B$ is the well known Lorentz force. By a **solution** q of the LFE we mean a function $q = (q_1, q_2, q_3) : [0,T] \to \mathbb{R}^3$ of class C^2 such that $|q'(t)| < 1$ for all $t \in [0,T]$ and which verifies the equation. In what follows we consider T**-periodic solutions**, that is, solutions q such that

$$q(0) = q(T), \quad q'(0) = q'(T).$$

Following Feynman [14] (see also [17]), the above equation is the relativistically correct equation of motion for a single charge in the fields E and B. The Lorentz force equation is first introduced by Lorentz [18] and Poincaré [22]. For more details about the Lorentz force equation, see for example Lorentz's paper [19]. On the other hand, in the Conclusions Section from Damour's paper [10] we can read that "one of the most important advances made by Poincaré [22] is the relativistic electron Lagrangian

$$L_{\text{electron}} = -m_{\text{electron}} c^2 \sqrt{1 - \frac{\mathbf{v^2}}{c^2}},$$

where

$$m_{\text{electron}} = \frac{4}{3}\frac{E_{\text{em}}}{c^2}."$$

Concerning the Lorentz force equation and the above Lagrangian introduced by Poincaré we can read in [14] that: "The formula in this case of relativity is the following:

$$S = -m_0 c^2 \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2}} dt - q \int_{t_1}^{t_2} [\phi(x,y,z,t) - \mathbf{v} \cdot A(x,y,z,t)] dt$$

[...] This action function gives the complete theory of relativistic motion of a single particle in an electromagnetic field."

Next, it is proved in Theorem 6 [3] that a function $q : [0,T] \to \mathbb{R}^3$ is a T-periodic solution of the Lorentz force equation with the electric potential V and the magnetic potential W if and only if q is a T-periodic Lipschitz function with $||q'||_\infty \leq 1$ and q is a solution of the variational inequality

$$\begin{aligned} \int_0^T [\sqrt{1-|q'|^2} - \sqrt{1-|\varphi'|^2}] dt \quad &+ \quad \int_0^T [\mathcal{E}(t,q,q') - \nabla_q V(t,q)] \cdot (\varphi - q) dt \\ &+ \quad \int_0^T W(t,q) \cdot (\varphi' - q') dt \geq 0 \end{aligned}$$

for all T-periodic Lipschitz functions $\varphi : [0,T] \to \mathbb{R}^3$ such that $||\varphi'||_\infty \leq 1$, where $\mathcal{E} : [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\mathcal{E}(t,q,p) = (p \cdot D_{q_1} W(t,q), p \cdot D_{q_2} W(t,q), p \cdot D_{q_3} W(t,q)).$$

This means more or less that $0 \in \partial S(q)$, or q is a critical point of the Poincaré action functional $\mathcal{I}_*$ associated to the Lorentz force equation with T-periodic boundary conditions (see Theorem 4).

In this paper we consider autonomous potentials V and W, i.e. $V : \mathbb{R}^3 \to \mathbb{R}$ and $W : \mathbb{R}^3 \to \mathbb{R}^3$, so both potentials are independent of the time variable. The main remark is that in this case the action functional $\mathcal{I}_*$ is S^1-invariant. Our main result about the Lorentz force equation is roughly speaking as follows (see Theorem 5). First, we assume that V is of class C^2, $V(0) = 0$, $V > 0$ and $V' \neq 0$ on $\mathbb{R}^3 \setminus \{0\}$, and at infinity one has that

$$\lim_{|q| \to \infty} V(q) = l^* > 0.$$

Regarding the behaviour of the electric potential V around the origin, we assume that for some positive integer m and for some $\lambda > 0$ one has

$$V(q) \geq \lambda |q|^2$$

for all q around the origin. Moreover, we only consider electric potentials V such that V'' is bounded. For the magnetic potential W, we assume that W is of class C^2 with W, W', W'' bounded on $\mathbb{R}^3$. Then, we prove that there exists $\Lambda_m > 0$ such that if λ introduced above is such that $\lambda \geq \Lambda_m$, then $\mathcal{I}_*$ has at least $3m$ critical orbits at negative levels which are 2π-periodic solutions of the Lorentz force equation. The constant Λ_m quantifies the interaction of the electric potential V with the magnetic potential W.

To prove that the Poincaré action functional has multiple critical orbits at negative levels, we develop a Lusternik-Schnirelman theory with the S^1-index for nonsmooth functionals. First, we prove our abstract result for smooth functionals satisfying only a weak Palais-Smale compactness condition (Theorem 2). In this smooth context our result is a generalization of the classical Lusternik-Schnirelman method with the S^1-index or with the Fadell-Rabinowitz index (see, for example, Theorem 1 [12], Theorem 6.1 [21], Proposition 10 [11]). The main tool in the proof of our Theorem 2 is Ghoussoub's location theorem [15, 16, 11]. To pass from the smooth case (Theorem 2) to the nonsmooth case (Theorem 3), we use the Ekeland-Lasry regularization procedure (Lemma 7 in [12]). For other results about periodic solutions of the Lorentz force equation via the Lusternik-Schnirelman method applied to the Poincaré action functional, see [4, 6, 8, 9]. In the newtonian situation, for results concerning periodic solutions of nonlinear bounded perturbations of the operator $u \mapsto u''$ see for example [1]. For a good introduction to the Lusternik-Schnirelman theory, see the monographs [2]-Chapter 9, [11]-Chapter 5, [21]-Chapter 6, [16]-Chapter 7.

The paper is organized as follows. In Section 2 we prove our abstract result (Theorem 2) . In Section 3 we introduce the Poincaré action functional on the Hilbert space H^1_T and prove the suitable compactness condition in the context of H^1_T, that is, Proposition 1. In Section 4 we prove the main result concerning periodic solutions with a fixed period of the Lorentz force equation (Theorem 5) that was described above.

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2 S^1-invariant nonsmooth functionals

In this section $(X, \langle \cdot, \cdot \rangle)$ is a real Hilbert space and $(Y, \|\cdot\|_Y)$ is a real Banach space such that $X \subset Y$ and the canonical injection $i : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is a compact operator. Next, we consider the functional $\mathcal{I} : X \to (-\infty, \infty]$ satisfying the following hypothesis:

(H) $\mathcal{I} = \Psi + \mathcal{F}$, where $\mathcal{F} \in C^1(X, \mathbb{R})$ and $\Psi : X \to (-\infty, +\infty]$ is a convex

lower semicontinous functional with a nonempty domain, i.e.
$$D(\Psi) = \{q \in X : \Psi(q) < +\infty\} \neq \emptyset.$$

The dual of X is denoted by X^*, and for any $q \in X$, using the Riesz Representation Theorem, there exists a unique element $\nabla \mathcal{F}(q) \in X$ such that

$$\mathcal{F}'(q)[\varphi] = \langle \nabla \mathcal{F}(q), \varphi \rangle \quad \text{for all } \varphi \in X.$$

2.1 Critical points and compactness conditions

For any $q \in D(\Psi)$, we consider **the subdifferential** of Ψ at q (see [13]) given by

$$\partial \Psi(q) = \{f^* \in X^* : \Psi(\varphi) - \Psi(q) \geq f^*[\varphi - q] \text{ for all } \varphi \in D(\Psi)\}.$$

It follows that for any $q \in D(\Psi)$, one has

$$\partial \mathcal{I}(q) = \mathcal{F}'(q) + \partial \Psi(q) = \{\mathcal{F}'(q) + f^* : f^* \in \partial \Psi(q)\}.$$

Notice that, by the Riesz Representation Theorem, $\partial \Psi(q)$ and $\partial \mathcal{I}(q)$ are subsets of X for every $q \in D(\Psi)$. A point $q \in X$ is **a critical point** of $\mathcal{I}$ if $q \in D(\Psi)$ and

$$0 \in \partial \mathcal{I}(q),$$

or, equivalently,

$$-\mathcal{F}'(q) \in \partial \Psi(q),$$

or, equivalently, the following variational inequality holds:

$$\Psi(\varphi) - \Psi(q) + \mathcal{F}'(q)[\varphi - q] \geq 0 \text{ for all } \varphi \in D(\Psi).$$

A (PS)-sequence at the level $c \in \mathbb{R}$ for $\mathcal{I}$ (see [23]) is a sequence $(q_n) \subset D(\Psi)$ such that $\mathcal{I}(q_n) \to c$ and there exists a sequence $(\varepsilon_n) \subset [0, \infty)$ having the property that $\varepsilon_n \to 0$ and, for every $n \in \mathbb{N}$, one has

$$\Psi(\varphi) - \Psi(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq -\varepsilon_n \|\varphi - q_n\| \text{ for all } \varphi \in D(\Psi).$$

A $(PS)^*$-sequence at the level $c \in \mathbb{R}$ for $\mathcal{I}$ (see [23]) is a sequence $(q_n) \subset D(\Psi)$ such that $\mathcal{I}(q_n) \to c$ and there exists a sequence $(f_n^*) \subset X^*$ having the property that $\|f_n^*\| \to 0$ and, for every $n \in \mathbb{N}$, one has

$$f_n^* \in \partial \mathcal{I}(q_n),$$

or, equivalently,

$$\Psi(\varphi) - \Psi(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq f_n^*[\varphi - q_n] \text{ for all } \varphi \in D(\Psi).$$

We need the following result (see Lemma 1.3 in [23]).

Lemma 1 *Consider a convex lower semicontinous function* $\chi : X \to (-\infty, +\infty]$ *with* $\chi(0) = 0$. *Assume that*

$$\chi(q) \geq -\|q\| \text{ for all } q \in X.$$

Then, there exists $f^* \in X^*$ *with* $\|f^*\| \leq 1$ *and*

$$\chi(q) \geq f^*[q] \text{ for all } q \in X.$$

Lemma 2 *Consider a sequence* $(q_n) \subset D(\Psi)$ *and* $c \in \mathbb{R}$. *We have that* (q_n) *is a* (PS)*-sequence at the level* c *for* $\mathcal{I}$ *if and only if* (q_n) *is a* $(PS)^*$*-sequence at the level* c *for* $\mathcal{I}$.

Proof. Assume that (q_n) is a $(PS)^*$-sequence at the level c for $\mathcal{I}$, i.e. $\mathcal{I}(q_n) \to c$ and there exists a sequence $(f_n^*) \subset X^*$ having the property that $\|f_n^*\| \to 0$ and, for every $n \in \mathbb{N}$, one has

$$\Psi(\varphi) - \Psi(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq f_n^*[\varphi - q_n] \text{ for all } \varphi \in D(\Psi).$$

For any $n \in \mathbb{N}$, one has that

$$|f_n^*[x]| \leq \|f_n^*\|\|x\| \text{ for all } x \in X.$$

In particular,

$$f_n^*[\varphi - q_n] \geq -\|f_n^*\|\|\varphi - q_n\| \text{ for all } \varphi \in D(\Psi).$$

It follows that

$$\Psi(\varphi) - \Psi(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq -\|f_n^*\|\|\varphi - q_n\| \text{ for all } \varphi \in D(\Psi),$$

and (q_n) is a (PS)-sequence at the level c for $\mathcal{I}$ by just taking $\varepsilon_n = \|f_n^*\|$ $(n \in \mathbb{N})$.

Next, assume that (q_n) is a (PS)-sequence at the level c for $\mathcal{I}$, i.e. $\mathcal{I}(q_n) \to c$ and there exists a sequence $(\varepsilon_n) \subset [0, \infty)$ having the property that $\varepsilon_n \to 0$ and, for every $n \in \mathbb{N}$, one has

$$\Psi(\varphi) - \Psi(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq -\varepsilon_n\|\varphi - q_n\| \text{ for all } \varphi \in D(\Psi).$$

If $\varepsilon_n = 0$, we take $f_n^* = 0$. Now consider $\varepsilon_n > 0$ and $\chi : X \to (-\infty, +\infty]$ given by

$$\chi(q) = \varepsilon_n^{-1}(\Psi(q + q_n) - \Psi(q_n) + \mathcal{F}'(q_n)[q]) \quad (q \in X).$$

It is clear that χ is convex, lower semicontinuous, $\chi(0) = 0$, and

$$\chi(q) \geq -\|q\| \text{ for all } q \in X.$$

Hence, using the above Lemma 1, there exists $g_n^* \in X^*$ with $\|g_n^*\| \leq 1$ and

$$\chi(q) \geq g_n^*[q] \text{ for all } q \in X.$$

If we take $f_n^* = \varepsilon_n g_n^*$, we have $\|f_n^*\| \to 0$ and, using that

$$\varepsilon_n \chi(\varphi - q_n) \geq f_n^*[\varphi - q_n] \text{ for all } \varphi \in D(\Psi),$$

the conclusion is now clear. ■

2.2 Ekeland-Lasry regularization procedure

The following result, which is one of the main tools in this paper, is due to Ekeland and Lasry, Lemma 7 [12].

Lemma 3 *Assume that $\mathcal{I}$ is bounded from below and the following assumption holds true:*

(*) There exists $\alpha > 0$ such that the function
$D(\Psi) \ni q \mapsto \mathcal{I}(q) + \alpha\|q\|^2 \in (-\infty, \infty)$ is convex.

Consider $0 < \varepsilon < \alpha^{-1}$, and let $\mathcal{I}_\varepsilon : X \to \mathbb{R}$ be given by

$$\mathcal{I}_\varepsilon(q) = \inf_{\varphi \in X}(\varepsilon^{-1}\|\varphi - q\|^2 + \mathcal{I}(\varphi)).$$

The functional $\mathcal{I}_\varepsilon$ satisifes the following properties:
$(EL1)$ $\mathcal{I}_\varepsilon \in C^1(X, \mathbb{R})$;
$(EL2)$ *for all $q \in X$, one has that*

$$\inf_X \mathcal{I} \le \mathcal{I}_\varepsilon(q) \le \mathcal{I}(q);$$

$(EL3)$ *for a fixed $q \in X$, one has that*

$$\mathcal{I}'_\varepsilon(q) = 0 \Leftrightarrow [q \in D(\Psi), 0 \in \partial\mathcal{I}(q)] \Leftrightarrow \mathcal{I}_\varepsilon(q) = \mathcal{I}(q).$$

Moreover, consider the function $\gamma : X \to X$ given by

$$\gamma(q) = q - \frac{\varepsilon}{2}\nabla\mathcal{I}_\varepsilon(q) \quad (q \in X).$$

Then, for any $q \in X$, one has that $\gamma(q) \in D(\Psi)$, and
$(EL4)$ $\mathcal{I}(\gamma(q)) = \mathcal{I}_\varepsilon(q) - \varepsilon^{-1}\|q - \gamma(q)\|^2$;
$(EL5)$ $2\varepsilon^{-1}(q - \gamma(q)) \in \partial\mathcal{I}(\gamma(q))$.

Proof. $(EL2)$ is relation (41) in Lemma 7 [12]. $(EL3)$ is relation (42) in Lemma 7 [12]. Our function γ is the function ψ introduced in Step 1 of the proof of Lemma 7 [12]. The formula for γ given above is relation (63) in Step 2 of the proof of Lemma 7 [12]. $(EL4)$ is relation (47) in Step 1 of the proof of Lemma 7 [12]. $(EL5)$ is relation (48) in Step 1 of the proof of Lemma 7 [12]. ∎

Given $c \in \mathbb{R}$, we say that $\mathcal{I}$ satisfies **the weak Palais-Smale condition at the level** c (for short, $(wPS)_c$**-condition**) (see [3, 4]) if for any sequence (q_n) from $D(\Psi)$ which is a (PS)-sequence at the level c, there exists a subsequence (q_{n_k}) converging in Y to a critical point q^* of $\mathcal{I}$ such that $\mathcal{I}(q^*) = c$, that is q^* is a critical point of $\mathcal{I}$ at the level c and $\|q_{n_k} - q^*\|_Y \to 0$.

Given $c \in \mathbb{R}$, we say that $\mathcal{I}_\varepsilon$ satisfies **the weak Palais-Smale condition at the level** c if for any sequence (q_n) from X such that $\mathcal{I}_\varepsilon(q_n) \to c$ and $\mathcal{I}'_\varepsilon(q_n) \to 0$, there exists a subsequence (q_{n_k}) converging in Y to $q^* \in X$ such that $\mathcal{I}_\varepsilon(q^*) = c$ and $\mathcal{I}'_\varepsilon(q^*) = 0$.

Lemma 4 *Consider $c \in \mathbb{R}$ such that $\mathcal{I}$ satisfies $(wPS)_c$-condition. If $0 < \varepsilon < \alpha^{-1}$, with α given in $(*)$, then $\mathcal{I}_\varepsilon$ satisfies $(wPS)_c$-condition.*

Proof. Consider a sequence $(q_n) \subset X$ such that

$$\mathcal{I}_\varepsilon(q_n) \to c \text{ and } \mathcal{I}'_\varepsilon(q_n) \to 0.$$

For any positive integer n, we consider

$$\varphi_n = \gamma(q_n), \quad u_n = 2\varepsilon^{-1}(q_n - \varphi_n).$$

Using Lemma 3 - $(EL4)$, we have that

$$\begin{aligned}
\mathcal{I}(\varphi_n) &= \mathcal{I}(\gamma(q_n)) \\
&= \mathcal{I}_\varepsilon(q_n) - \varepsilon^{-1}\|q_n - \gamma(q_n)\|^2 \\
&= \mathcal{I}_\varepsilon(q_n) - \varepsilon^{-1}\|q_n - \varphi_n\|^2 \\
&= \mathcal{I}_\varepsilon(q_n) - \frac{\varepsilon}{4}\|u_n\|^2.
\end{aligned}$$

Using Lemma 3 - $(EL5)$, we have that

$$2\varepsilon^{-1}(q_n - \gamma(q_n)) \in \partial\mathcal{I}(\gamma(q_n)),$$

that is

$$u_n \in \partial\mathcal{I}(\varphi_n).$$

But, from the definition of γ, one has that

$$\nabla\mathcal{I}_\varepsilon(q_n) = 2\varepsilon^{-1}(q_n - \gamma(q_n)) = u_n,$$

so $u_n \to 0$ and $\mathcal{I}(\varphi_n) \to c$. We deduce that (φ_n) is a $(PS)^*$-sequence of $\mathcal{I}$ at the level c, and using Lemma 2, it follows that (φ_n) is a (PS)-sequence of $\mathcal{I}$ at the level c. But $\mathcal{I}$ satisfies $(wPS)_c$-condition, so there exists a subsequence (φ_{n_k}) and $q^* \in D(\Psi)$ such that $\mathcal{I}(q^*) = c$, $0 \in \partial\mathcal{I}(q^*)$, and $\|\varphi_{n_k} - q^*\|_Y \to 0$. Then, using Lemma 3 - $(EL3)$, we deduce that $\mathcal{I}_\varepsilon(q^*) = c$, $\mathcal{I}'_\varepsilon(q^*) = 0$, and $\mathcal{I}_\varepsilon$ satisfies $(wPS)_c$-condition. ∎

2.3 Benci's S^1-index and Ghoussoub's location theorem

In this subsection we recall some known results.

First, we introduce Benci's S^1-index (see [5, 21]). Consider

$$S^1 = \{z \in \mathbb{C} : |z| = 1\},$$

which is a compact topological group with the multiplication of complex numbers. We identify the reals mod 2π, that is $\mathbb{R}/2\pi$, with S^1 by $\theta \leftrightarrow e^{i\theta}$. Thus, the group $(\mathbb{R}/2\pi, +)$ is identified with $(S^1, \cdot)$. We consider a real Banach space $\mathcal{X}$, and we denote by $B(\mathcal{X})$ the Banach space of all bounded linear operators acting on $\mathcal{X}$ endowed with the usual operator norm. We recall that an operator $A \in B(\mathcal{X})$ is called an **isometry** if $\|Ax\| = \|x\|$ for all $x \in \mathcal{X}$. Next, a **representation** of S^1 over the Banach space $\mathcal{X}$ is a function

$$L : S^1 \to B(\mathcal{X}) : \quad \theta \mapsto L(\theta)$$

having the following properties:

$$L(0) = \text{id},$$
$$L(\theta_1 + \theta_2) = L(\theta_1)L(\theta_2) \text{ for all } \theta_1, \theta_2 \in S^1,$$
$$S^1 \times \mathcal{X} \ni (\theta, x) \mapsto L(\theta)x \in \mathcal{X} \text{ is continuous.}$$

A subset A of $\mathcal{X}$ is **invariant** under the representation L if $L(\theta)A = A$ for all $\theta \in S^1$. A representation L of S^1 over $\mathcal{X}$ is **isometric** if $L(\theta)$ is an isometry for all $\theta \in S^1$. A mapping R between two invariant subsets of $\mathcal{X}$ under the representation L of S^1 is **equivariant** if

$$R \circ L(\theta) = L(\theta) \circ R \text{ for all } \theta \in S^1.$$

Consider

$$\mathcal{C} = \{A \subset \mathcal{X} : A \text{ is closed and invariant}\}.$$

The **S^1-index** associated to the isometric representation L is the function

$$\text{ind} : \mathcal{C} \to \mathbb{N} \cup \{+\infty\} : \ A \mapsto \text{ind}(A)$$

defined as follows: for $A \in \mathcal{C}$, the S^1-index of A is the smallest integer k such that there exists $n \in \mathbb{N} \setminus \{0\}$ and $\Phi \in C(A, \mathbb{C}^k \setminus \{0\})$ with

$$\Phi(L(\theta)x) = e^{in\theta}\Phi(x) \text{ for all } \theta \in S^1, x \in A.$$

If such a mapping does not exist, then we define

$$\text{ind}(A) = +\infty.$$

Finally, we define

$$\text{ind}(\emptyset) = 0.$$

The mapping "ind" defined above is an index for the representation L, that is it has the following properties:

(i) $\text{ind}(A) = 0$ if and only if $A = \emptyset$;

(ii) if $R : A_1 \to A_2$ is equivariant and continuous, then

$$\text{ind}(A_1) \leq \text{ind}(A_2);$$

(iii) if $A \in \mathcal{C}$ is compact, then there exists $N \in \mathcal{C}$ with $A \subset \text{int}(N)$ and

$$\text{ind}(N) = \text{ind}(A);$$

(iv) for all $A_1, A_2 \in \mathcal{C}$, one has that

$$\text{ind}(A_1 \cup A_2) \le \text{ind}(A_1) + \text{ind}(A_2).$$

Next, we introduce Ghoussoub's location theorem (see [15, 16, 11]). A functional $\mathcal{J} : \mathcal{X} \to (-\infty, \infty]$ is **invariant** for the representation L of S^1 over $\mathcal{X}$ if

$$\mathcal{J} \circ L(\theta) = \mathcal{J} \text{ for all } \theta \in S^1.$$

From Lemma 7 in [12], one has the following result.

Lemma 5 *Under the assumptions of Lemma 3, if $\mathcal{I}$ is invariant, then $\mathcal{I}_\varepsilon$ is invariant for any $0 < \varepsilon < \alpha^{-1}$.*

A class $\mathcal{G}$ of compact invariant subsets of $\mathcal{X}$ is **stable by equivariant homotopies** if for any $A \in \mathcal{G}$ and $h \in C([0,1] \times \mathcal{X}, \mathcal{X})$ such that $h(s, \cdot)$ is equivariant for all $s \in [0,1]$ and $h(0, \cdot) = \text{id}$, one has $h(1, A) \in \mathcal{G}$.

Theorem 1 *Consider a class $\mathcal{G}$ of compact invariant subsets of $\mathcal{X}$ which is stable by equivariant homotopies, and let $\mathcal{J} : \mathcal{X} \to \mathbb{R}$ be a C^1 invariant functional such that*

$$c := \inf_{A \in \mathcal{G}} \max_A \mathcal{J} > -\infty.$$

Assume that F is a closed invariant subset of $\mathcal{X}$ such that for all $A \in \mathcal{G}$, one has that

- $A \cap F \neq \emptyset$,
- $\sup\limits_{A \cap F} \mathcal{J} \ge c$.

Then, there exists a sequence (x_n) in $\mathcal{X}$ such that

$$\mathcal{J}(x_n) \to c, \quad \mathcal{J}'(x_n) \to 0,$$

and one has the following localization:

$$\text{dist}(x_n, F) \to 0.$$

2.4 Lusternik-Schnirelman method with the S^1-index and weak compactness

Computations with the S^1-index

In this subsection we work in the abstract setting we have introduced. Let L be an isometric representation of S^1 over the Banach space Y such that $L(\theta)X = X$ for all $\theta \in S^1$, and $L|_X = \{L(\theta)|_X\}_{\theta \in S^1}$ is an isometric representation of S^1 over the Hilbert space X with the associated (to $L|_X$) S^1-index also denoted as above by "ind". Consider

$$\text{Fix}(S^1) = \{y \in Y : L(\theta)y = y \text{ for all } \theta \in S^1\},$$

and assume that

$$\mathrm{Fix}(S^1) \subset X.$$

For any $y \in Y$, the **orbit** of y is given by

$$\mathcal{O}(y) = \{L(\theta)y : \theta \in S^1\}.$$

For every $y \in Y$ and $\delta > 0$, we consider

$$\mathcal{O}^\delta(y) = \{z \in Y : \mathrm{dist}_Y(z, \mathcal{O}(y)) \leq \delta\}.$$

Notice that $\mathcal{O}(y)$ is compact invariant in Y, and $\mathcal{O}^\delta(y)$ is closed invariant in Y. For any $q \in X$, one has that $\mathcal{O}(q) \subset X$, but $\mathcal{O}^\delta(q) \subset Y$ ($\delta > 0$). It is clear that

$$\mathrm{ind}(\mathcal{O}(q)) = +\infty \text{ for all } q \in \mathrm{Fix}(S^1).$$

Moreover, if $q \in X \setminus \mathrm{Fix}(S^1)$, then the 2π-periodic continuous function

$$S^1 \ni \theta \mapsto L(\theta)q \in X$$

is not constant, so it has a minimal period $T^* > 0$, and there exists a positive integer n such that $2\pi = nT^*$. Thus, for any positive integer j, the function

$$\mathcal{O}(q) \ni L(\theta)q \mapsto e^{ijn\theta} \in \mathbb{C} \setminus \{0\}, \ \theta \in S^1,$$

is well-defined and continuous. Hence, in particular, it is clear that

$$\mathrm{ind}(\mathcal{O}(q)) = 1.$$

The next result is a generalization of this remark.

Lemma 6 *If $q_1, ..., q_m \in X \setminus \mathrm{Fix}(S^1)$, then there exists $\delta > 0$ such that*

$$\bigcup_{j=1}^{m} \mathcal{O}^\delta(q_j) \cap \mathrm{Fix}(S^1) = \emptyset, \quad \mathrm{ind}\left(\bigcup_{j=1}^{m} \mathcal{O}^\delta(q_j) \cap X\right) = 1.$$

Proof. First, it is clear that there exists $\delta' > 0$ such that $\mathcal{O}^{\delta'}(q_j) \cap \mathcal{O}^{\delta'}(q_k) = \emptyset$ for all $1 \leq j < k \leq m$. For $j = 1, ..., m$ fixed, we consider the 2π-periodic continuous function

$$S^1 \ni \theta \mapsto L(\theta)q_j \in X \subset Y.$$

Using that $q_j \notin \mathrm{Fix}(S^1)$, the above function is not constant, so it has a minimal period $T_j > 0$, and there exists a positive integer n_j such that $2\pi = n_j T_j$. Consider $n = n_1 n_2 \cdots n_m$ and the $\|\cdot\|_Y$-continuous function

$$\Phi : \bigcup_{j=1}^{m} \mathcal{O}(q_j) \to \mathbb{C} \setminus \{0\} \ : \quad L(\theta)q_j \mapsto e^{in\theta}, \ 1 \leq j \leq m, \ \theta \in S^1.$$

Using Tietze's extension theorem, there exists a $\|\cdot\|_Y$-continuous function $\tilde{\Phi}: Y \to \mathbb{C}$ such that $\tilde{\Phi}|_{\cup_{j=1}^m \mathcal{O}(q_j)} = \Phi$. Now we define the continuous function $\Lambda: Y \to \mathbb{C}$ by

$$\Lambda(y) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\tilde{\Phi}(L(\theta)y)d\theta \text{ for all } y \in Y.$$

For any $\tau \in [0, 2\pi]$ and $y \in Y$, one has that

$$\begin{aligned} \Lambda(L(\tau)y) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\tilde{\Phi}(L(\theta)L(\tau)y)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\tilde{\Phi}(L(\theta+\tau)y)d\theta \\ &= \frac{1}{2\pi}\int_\tau^{2\pi+\tau} e^{-in(t-\tau)}\tilde{\Phi}(L(t)y)dt \\ &= e^{in\tau}\Lambda(y). \end{aligned}$$

In particular, for $y = q_j$ with $1 \le j \le m$, one has that

$$\Lambda(L(\tau)q_j) = e^{in\tau}\Lambda(q_j) = e^{in\tau} = \Phi(L(\tau)q_j),$$

that is

$$\Lambda(q) = \Phi(q) \quad \text{for all } q \in \bigcup_{j=1}^m \mathcal{O}(q_j).$$

We claim that there exists $\delta \le \delta'$ such that $\Lambda(q) \neq 0$ for all $q \in \cup_{j=1}^m \mathcal{O}^\delta(q_j) \cap X$. Assume by contradiction that our claim is not true. Then, for all positive integers $k \ge \delta'^{-1}$, there exists $v_k \in \cup_{j=1}^m \mathcal{O}^{1/k}(q_j) \cap X$ with $\Lambda(v_k) = 0$. Let $u_k \in \cup_{j=1}^m \mathcal{O}(q_j)$ be such that $\|v_k - u_k\|_Y \le 1/k$. Passing to a subsequence, there exists $u \in \cup_{j=1}^m \mathcal{O}(q_j)$ such that $\|u_k - u\|_X \to 0$, and so $\|v_k - u\|_Y \to 0$. Hence, $\Lambda(v_k) \to \Lambda(u)$ and $\Lambda(u) = \Phi(u) = 0$, contradicting the definition of Φ. Thus, $\Lambda : \cup_{j=1}^m \mathcal{O}^\delta(q_j) \cap X \to \mathbb{C}\setminus\{0\}$ is a $\|\cdot\|_X$-continuous function such that $\Lambda(L(\theta)q) = e^{in\theta}\Lambda(q)$ for all $q \in \cup_{j=1}^m \mathcal{O}^\delta(q_j) \cap X$, $\theta \in S^1$, which implies that $\mathrm{ind}(\cup_{j=1}^m \mathcal{O}^\delta(q_j) \cap X) = 1$. Now the other conclusion is clear. ■

The smooth case

Next, consider an invariant functional $\mathcal{J} \in C^1(X, \mathbb{R})$, and for any positive integer j we define

$$\mathcal{G}_j = \{A \subset X : A \text{ compact, invariant with } \mathrm{ind}(A) \ge j\}$$

and the associated Lusternik-Schnirelman levels,

$$c_j = \inf_{A \in \mathcal{G}_j} \max_A \mathcal{J}.$$

Using property (ii) of the S^1-index, it follows that the class $\mathcal{G}_j$ is stable by equivariant homotopies for any positive integer j. Notice that $\operatorname{ind}(\mathcal{O}(0)) = +\infty$, so $\mathcal{O}(0) \in \mathcal{G}_j$ and $c_j \leq \mathcal{J}(0)$ for all positive integers j. Hence, it follows that

$$-\infty \leq c_1 \leq c_2 \leq \cdots \leq \mathcal{J}(0).$$

For $c \in \mathbb{R}$, we consider the set of critical points of $\mathcal{J}$ at the level c,

$$K_c = \{q \in X : \mathcal{J}(q) = c, \mathcal{J}'(q) = 0\}.$$

Notice that since $\mathcal{J}$ is invariant, it follows that if $q \in K_c$, then $\mathcal{O}(q) \subset K_c$. We say that $\mathcal{O}(q)$ is a **critical orbit** of $\mathcal{J}$ at the level c.

Theorem 2 *Let $\mathcal{J} \in C^1(X, \mathbb{R})$ be an invariant functional.*

- *If for some positive integer j one has that $c_j > -\infty$ and $\mathcal{J}$ satisfies $(wPS)_{c_j}$-condition, then $K_{c_j} \neq \emptyset$.*
- *Moreover, if for some positive integers $j < k$ one has that $c_j = c_k = c > -\infty$, $\mathcal{J}$ satisfies $(wPS)_c$-condition, and $K_c \cap \operatorname{Fix}(S^1) = \emptyset$, then there exists a sequence $(q_n) \subset X$ such that $\cup_n \mathcal{O}(q_n) \subset K_c$ and $\mathcal{O}(q_m) \cap \mathcal{O}(q_n) = \emptyset$ for all positve integers m, n such that $m \neq n$, that is, there exists a sequence of distinct critical orbits of $\mathcal{J}$ at the level c.*

Proof. We consider a positive integer j such that $c_j > -\infty$ and $\mathcal{J}$ satisfies $(wPS)_{c_j}$-condition. Using Ghoussoub's location theorem, that is Theorem 1, with $F = X$ and $\mathcal{G} = \mathcal{G}_j$, it follows that there exists a sequence $(q_n) \subset X$ such that

$$\mathcal{J}(q_n) \to c_j, \quad \mathcal{J}'(q_n) \to 0.$$

Using that $\mathcal{J}$ satisfies $(wPS)_{c_j}$-condition, passing to a subsequence it follows that there exists $q \in X$ such that $\|q_n - q\|_Y \to 0$, $\mathcal{J}(q) = c_j$, and $\mathcal{J}'(q) = 0$. In particular, $q \in K_{c_j}$, so $K_{c_j} \neq \emptyset$.

For the second assertion, assume by contradiction that K_c contains only a finite number of orbits $\mathcal{O}(q_1), \cdots, \mathcal{O}(q_l)$. Now let $\delta > 0$ be given by Lemma 6, so one has that

$$\bigcup_{m=1}^{l} \mathcal{O}^\delta(q_m) \cap \operatorname{Fix}(S^1) = \emptyset, \quad \operatorname{ind}\left(\bigcup_{m=1}^{l} \mathcal{O}^\delta(q_m) \cap X\right) = 1.$$

Consider in X the invariant set F given by

$$F = \{q \in X : \ \operatorname{dist}_Y(q, \cup_{m=1}^{l} \mathcal{O}(q_m)) \geq \delta\}.$$

Using that the canonical injection from X into Y is a compact operator and

$$F = X \cap \ (\operatorname{dist}_Y(\cdot, \cup_{m=1}^{l} \mathcal{O}(q_m)))^{-1}[\delta, \infty),$$

it follows that F is closed in X. Notice that by our assumption,

$$F \cap K_c = \emptyset.$$

For any fixed $A \in \mathcal{G}_k$, it is clear that

$$A \subset (A \cap F) \cup (\cup_{m=1}^{l} \mathcal{O}^{\delta}(q_m) \cap X),$$

and using property (iv) of the S^1-index we deduce that

$$j < k \leq \mathrm{ind}(A) \leq \mathrm{ind}(A \cap F) + \mathrm{ind}(\cup_{m=1}^{l} \mathcal{O}^{\delta}(q_m) \cap X) = \mathrm{ind}(A \cap F) + 1,$$

which implies that

$$\mathrm{ind}(A \cap F) \geq j,$$

so $A \cap F \neq \emptyset$, $A \cap F \in \mathcal{G}_j$, and $\max_{A \cap F} \mathcal{J} \geq c_j = c$. Using Ghoussoub's location theorem, that is Theorem 1, with F defined above and $\mathcal{G} = \mathcal{G}_k$, it follows that there exists a sequence $(u_n) \subset X$ such that

$$\mathcal{J}(u_n) \to c, \quad \mathcal{J}'(u_n) \to 0, \text{ and } \mathrm{dist}_X(u_n, F) \to 0.$$

Using that $\mathcal{J}$ satisfies $(wPS)_c$-condition, passing to a subsequence it follows that there exists $u \in X$ such that $\|u_n - u\|_Y \to 0$, $\mathcal{J}(u) = c$, and $\mathcal{J}'(u) = 0$. Let $(v_n) \subset F$ be a sequence in F such that $\|u_n - v_n\|_X \to 0$, hence $\|v_n - u\|_Y \to 0$. Using that $\mathrm{dist}_Y(\cdot, \cup_{m=1}^{l} \mathcal{O}(q_m))$ is a continuous function on the Banach space $(Y, \|\cdot\|_Y)$ and $u \in X$, it follows that $u \in F$ and $\mathcal{O}(u) \subset F \cap K_c$, contradicting $F \cap K_c = \emptyset$. Thus, the second assertion is proved. ∎

Remark 1 The above result, that is Theorem 2, holds true without any change in the proof if X is a Banach space. We need X to be a Hilbert space in order to use the Ekeland-Lasry result (Lemma 3).

The nonsmooth case

Now let us return to our nonsmooth functional $\mathcal{I} = \Psi + \mathcal{F}$ which satisfies the main hypothesis (H), and assume that the functionals Ψ and $\mathcal{F}$ are invariant (under the isometric action $L|_X$ of the compact group S^1 over the Hilbert space X). For a fixed real number $c \in \mathbb{R}$, we consider, as in the smooth case,

$$K_c(\mathcal{I}) = \{q \in X : \mathcal{I}(q) = c, q \text{ is a critical point of } \mathcal{I}\}.$$

Moreover, we consider

$$K(\mathcal{I}) = \{q \in X : q \text{ is a critical point of } \mathcal{I}\}.$$

Using that Ψ and $\mathcal{F}$ are invariant, it is easy to prove that if $q \in K_c(\mathcal{I})$, then $\mathcal{O}(q) \subset K_c(\mathcal{I})$. As in the smooth case, for any positive integer j, we consider the Lusternik-Schnirelman levels associated to the nonsmooth functional $\mathcal{I}$,

$$c_j(\mathcal{I}) = \inf_{A \in \mathcal{G}_j} \sup_A \mathcal{I}.$$

One has that

$$-\infty \leq c_1(\mathcal{I}) \leq c_2(\mathcal{I}) \leq \cdots \leq \mathcal{I}(0).$$

Moreover, following Ekeland-Lasry [12], we consider the set

$$\Omega = \{q \in X : \mathcal{I}(q) < 0\}.$$

Our main abstract result, which will be applied to the Poincaré action functional, goes as follows.

Theorem 3 *Assume that the invariant nonsmooth functional* $\mathcal{I}$ *is bounded from below, satisfies* (H)*, the Ekeland-Lasry condition* $(*)$*,* $\mathcal{I}(0) = 0$*, there exists some* $\omega < 0$ *such that* $\mathcal{I}$ *satisfies* $(wPS)_d$*-condition for all* $d < \omega$*, and*

$$\mathrm{Fix}(S^1) \cap \Omega \cap K(\mathcal{I}) = \emptyset.$$

Consider $0 < \varepsilon < \alpha^{-1}$.

- *One has that* $K_{c_j(\mathcal{I}_\varepsilon)}(\mathcal{I}_\varepsilon) = K_{c_j(\mathcal{I}_\varepsilon)}(\mathcal{I})$ *is a nonempty set for all positive integers* j *such that* $c_j(\mathcal{I}_\varepsilon) < \omega$.
- *Moreover, if there exist positive integers* j, k *such that*

  $$j < k \text{ and } c_j(\mathcal{I}_\varepsilon) = c_k(\mathcal{I}_\varepsilon) < \omega,$$

  then $K_{c_j(\mathcal{I}_\varepsilon)}(\mathcal{I})$ *contains infinitely many orbits.*
- *In particular, if for some positive integer* k *one has that* $c_k(\mathcal{I}_\varepsilon) < \omega$*, then* $K(\mathcal{I})$ *contains at least* k *orbits at negative levels.*

Proof. From Lemma 3 - $(EL1)$ it follows that $\mathcal{I}_\varepsilon \in C^1(X, \mathbb{R})$, and using Lemma 7 in [12], we have that $\mathcal{I}_\varepsilon$ is invariant. Moreover, using Lemma 4 and Lemma 3 - $(EL2)$ we deduce that $\mathcal{I}_\varepsilon$ satisfies $(wPS)_d$-condition for all $d < \omega$. Now the first and second parts of the proof follow immediately from Theorem 2 and Lemma 3 - $(EL3)$. The last part follows from the two previous parts. ∎

We recall the following result (see Proposition 5.3 in [21]).

Lemma 7 *Let* Z *be a finite-dimensional invariant subspace of* X *and let* D *be an open bounded invariant neighbourhood of* 0 *in* Z*. If* $Z \cap \mathrm{Fix}(S^1) = \{0\}$*, then*

$$\mathrm{ind}(\partial D) = \frac{1}{2} dim(Z).$$

From the above Lemma 7 and our main abtract result, Theorem 3, we deduce the following important result.

Corollary 1 *Assume that the invariant nonsmooth functional* $\mathcal{I}$ *is bounded from below and satisfies* (H)*, the Ekeland-Lasry condition* $(*)$*,* $\mathcal{I}(0) = 0$*, there exists* $\omega < 0$ *such that* $\mathcal{I}$ *satisfies* $(wPS)_d$ *for all* $d < \omega$*, and* $\mathrm{Fix}(S^1) \cap \Omega \cap K(\mathcal{I}) = \emptyset$*. Let* Z *be a (2k)-dimensional invariant subspace of* X *such that* $Z \cap \mathrm{Fix}(S^1) = \{0\}$*, and let* D *be an open bounded invariant neighborhood of* 0 *in* Z *with* $\sup_{\partial D} \mathcal{I} < \omega$*. Then,* $K(\mathcal{I})$ *contains at least* k *orbits at negative levels.*

Proof. Using Lemma 7, we deduce that $\text{ind}(\partial D) = k$. But $\sup_{\partial D} \mathcal{I} < \omega$, so using Lemma 3 - $(EL2)$, one has that $\max_{\partial D} \mathcal{I}_\varepsilon < \omega$, which together with $\partial D \in \mathcal{G}_k$ implies that $c_k(\mathcal{I}_\varepsilon) < \omega$. Now the result follows from the last part of Theorem 3. ■

3 Poincaré action functional on H^1_T

3.1 Function spaces

We denote by C_T the Banach space of continuous functions $q : [0, T] \to \mathbb{R}^3$ with $q(0) = q(T)$, endowed with the usual norm

$$||q||_\infty = \max_{[0,T]} |q| \quad \text{for all } q \in C_T.$$

The norm in $L^\infty(0, T)$ will also be denoted by $||\cdot||_\infty$. If $W^{1,\infty}(0, T)$ denotes the space of all real valued Lipschitz functions in $[0, T]$ (or, equivalently, the absolutely continuous functions on $[0, T]$ with bounded derivatives a.e.), we consider the Banach space

$$W^{1,\infty}_T = \{q \in [W^{1,\infty}(0, T)]^3 : q(0) = q(T)\}$$

endowed with the usual norm $\| \cdot \|_{1,\infty}$ given by

$$\|q\|_{1,\infty} = \|q\|_\infty + \|q'\|_\infty \quad (q \in W^{1,\infty}_T).$$

The Sobolev space H^1_T is the space of functions $q \in L^2(0, T; \mathbb{R}^3)$ having a weak derivative $q' \in L^2(0, T; \mathbb{R}^3)$ and which are T-periodic, that is $q(0) = q(T)$, or, equivalently, the space of absolutely continuous functions $q : [0, T] \to \mathbb{R}^3$ with $q' \in L^2(0, T; \mathbb{R}^3)$ and which are T-periodic. The Sobolev space H^1_T is a Hilbert space with the inner product

$$(q|\varphi)_{1,2} = \int_0^T [q(t) \cdot \varphi(t) + q'(t) \cdot \varphi'(t)]dt,$$

and we shall denote the corresponding norm by $\| \cdot \|_{1,2}$. Notice that $W^{1,\infty}_T \subset H^1_T$ continuously and the Arzelà-Ascoli theorem implies that $H^1_T \subset C_T$ compactly.

3.2 The nonsmooth part of the action functional

Consider

$$D(\Psi_*) = \{q \in W^{1,\infty}_T : \|q'\|_\infty \leq 1\},$$

and $\Psi_* : H^1_T \to (-\infty, +\infty]$ given by

$$\Psi_*(q) = \begin{cases} \displaystyle\int_0^T [1 - \sqrt{1 - |q'(t)|^2}]dt, & \text{if } q \in D(\Psi_*), \\ +\infty, & \text{if } q \in H^1_T \setminus D(\Psi_*). \end{cases}$$

Following [20, 7] one has the following result:

Lemma 8 *(i) The set $D(\Psi_*)$ is convex and closed in C_T. Moreover, if (q_n) is a sequence in $D(\Psi_*)$ converging pointwise in $[0,T]$ to a continuous function $q : [0,T] \to \mathbb{R}^3$, then $q \in D(\Psi_*)$ and $q_n' \to q'$ in the w^*-topology $\sigma(L^\infty, L^1)$.*

(ii) If (q_n) is a sequence in $D(\Psi_*)$ converging in C_T to q, then $q \in \mathcal{D}(\Psi_*)$ and

$$\Psi_*(q) \leq \liminf_{n\to\infty} \Psi_*(q_n).$$

In particular, the functional Ψ_* is weakly lower semicontinuous and convex on H_T^1.

3.3 The smooth part of the action functional

Let $\mathcal{F}_* : H_T^1 \to \mathbb{R}$ be given by

$$\mathcal{F}_*(q) := \int_0^T [q'(t) \cdot W(t, q(t)) - V(t, q(t))]dt \qquad \text{for all } q \in H_T^1.$$

It is standard to prove (see [21]) that if $V : [0,T]\times\mathbb{R}^3 \to \mathbb{R}$ and $W : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ are C^1 functions with $W(0,\cdot) = W(T,\cdot)$, then $\mathcal{F}_* \in C^1(H_T^1, \mathbb{R})$, with

$$\begin{aligned} \mathcal{F}_*'(q)[\varphi] &= \int_0^T (\mathcal{E}(t, q(t), q'(t)) - \nabla_q V(t, q(t))) \cdot \varphi(t)dt \\ &+ \int_0^T W(t, q(t)) \cdot \varphi'(t)dt \end{aligned}$$

for every $q, \varphi \in H_T^1$, where $\mathcal{E} : [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\mathcal{E}(t, q, p) = (p \cdot D_{q_1} W(t, q), p \cdot D_{q_2} W(t, q), p \cdot D_{q_3} W(t, q)).$$

3.4 The action functional

The action functional on H_T^1 (see [3] for $W_T^{1,\infty}$ case) associated to the LFE with the electric potential V, the magnetic potential W, and periodic boundary conditions on $[0,T]$ is given by

$$\mathcal{I}_* : H_T^1 \to (-\infty, +\infty], \quad \mathcal{I}_* = \Psi_* + \mathcal{F}_*.$$

One has that $\mathcal{I}_*$ satisfies (H), hence a point $q \in H_T^1$ is **a critical point** of $\mathcal{I}_*$ if $q \in D(\Psi_*)$ and

$$\Psi_*(\varphi) - \Psi_*(q) + \mathcal{F}_*'(q)[\varphi - q] \geq 0 \quad \text{for all } \varphi \in D(\Psi_*),$$

or, equivalently,

$$\int_0^T [\sqrt{1-|q'|^2} - \sqrt{1-|\varphi'|^2}]dt + \int_0^T [\mathcal{E}(t,q,q') - \nabla_q V(t,q)] \cdot (\varphi - q)dt$$
$$+ \int_0^T W(t,q) \cdot (\varphi' - q')dt \geq 0 \quad \text{for all } \varphi \in D(\Psi_*).$$

We need the following important result (see Lemma 14 in [20]). For $q \in L^1(0,T;\mathbb{R}^3)$, we denote

$$\overline{q} = \frac{1}{T}\int_0^T q(t)dt.$$

Lemma 9 *For every $f \in L^1(0,T;\mathbb{R}^3)$, there exists a unique $q_f \in W^{2,1}(0,T;\mathbb{R}^3)$ such that $\|q_f'\|_\infty < 1$ and*

$$\left(\frac{q_f'}{\sqrt{1-|q_f'|^2}}\right)' = \overline{q_f} + f, \quad q_f(0) = q_f(T),\ q_f'(0) = q_f'(T).$$

Moreover, q_f is the unique solution $q \in D(\Psi_*)$ of the variational inequality

$$\int_0^T [\sqrt{1-|q'|^2} - \sqrt{1-|\varphi'|^2}]dt + T\overline{q}\cdot(\overline{\varphi} - \overline{q}) + \int_0^T f\cdot(\varphi - q)dt \geq 0$$
$$\textit{for all } \varphi \in D(\Psi_*).$$

Using Lemma 9 and the same strategy as in the proof of Proposition 1 in [7] (see also Proposition 1 in [20] and Theorem 6 in [3]), we have the following key result.

Theorem 4 *A function $q \in H_T^1$ is a T-periodic solution of the LFE with the electric potential V and the magnetic potential W if and only if q is a critical point of the action functional $\mathcal{I}_*$.*

In the following result we introduce the weak compactness property of the Poincaré action functional on H_T^1. For the $W_T^{1,\infty}$ situation see Lemma 6 in [3].

Proposition 1 *Assume that any (PS)-sequence (q_n) of $\mathcal{I}_*$ at the level $c \in \mathbb{R}$ is such that $(\overline{q_n})$ is bounded. Then, $\mathcal{I}_*$ satisfies $(wPS)_c$-condition with respect to C_T.*

Proof. Let $(q_n) \subset D(\Psi_*)$ be a (PS)-sequence of $\mathcal{I}_*$ at the level $c \in \mathbb{R}$. Using that $(\overline{q_n})$ is bounded, it follows that (q_n) is bounded in $W_T^{1,\infty}$ and passing to a subsequence we can assume that $\|q - q_n\|_\infty \to 0$ as $n \to \infty$ for some $q \in C_T$. From Lemma 8 we deduce that $q \in D(\Psi_*)$, $\Psi_*(q) \leq \liminf_{n\to\infty} \Psi_*(q_n)$, and $q_n' \to q'$ in the w^*-topology $\sigma(L^\infty, L^1)$. It follows that

$$\lim_{n\to\infty} \int_0^T V(t,q_n)dt = \int_0^T V(t,q)dt.$$

Moreover, for any positive integer n, we have

$$\begin{aligned}\left|\int_0^T q_n' \cdot (W(t,q_n) - W(t,q))dt\right| &\leq \int_0^T |q_n'|\, |W(t,q_n) - W(t,q)|dt \\ &\leq \int_0^T |W(t,q_n) - W(t,q)|dt,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \int_0^T q_n' \cdot (W(t,q_n) - W(t,q))dt = 0.$$

Next, using that $W(\cdot,q) \in L^\infty(0,T;\mathbb{R}^3)$ we deduce that

$$\lim_{n\to\infty} \int_0^T q_n' \cdot W(t,q)dt = \int_0^T q' \cdot W(t,q)dt,$$

hence

$$\lim_{n\to\infty} \int_0^T q_n' \cdot W(t,q_n)dt = \int_0^T q' \cdot W(t,q)dt$$

and

$$\lim_{n\to\infty} \mathcal{F}_*(q_n) = \mathcal{F}_*(q).$$

Analogously, one has

$$\lim_{n\to\infty} \int_0^T (\mathcal{E}(t,q_n,q_n') - \nabla_q V(t,q_n)) \cdot (\varphi - q_n)dt =$$
$$\int_0^T (\mathcal{E}(t,q,q') - \nabla_q V(t,q)) \cdot (\varphi - q)dt \quad \text{for all } \varphi \in D(\Psi_*),$$

and we deduce that

$$\lim_{n\to\infty} \mathcal{F}_*'(q_n)[\varphi - q_n] = \mathcal{F}_*'(q)[\varphi - q] \quad \text{for all } \varphi \in D(\Psi_*).$$

We recall that $\mathcal{I}_*(q_n) \to c$ and we consider a sequence $(\varepsilon_n) \subset [0,\infty)$ having the property that $\varepsilon_n \to 0$ and, for every positive integer n, one has

$$\Psi_*(\varphi) - \Psi_*(q_n) + \mathcal{F}_*'(q_n)[\varphi - q_n] \geq -\varepsilon_n \|\varphi - q_n\|_{1,2} \text{ for all } \varphi \in D(\Psi_*).$$

If we take $n \to \infty$, we deduce that q is a critical point of $\mathcal{I}_*$. Moreover, taking $\varphi = q$ it follows that

$$\Psi_*(q) - \Psi_*(q_n) + \mathcal{F}_*'(q_n)[q - q_n] \geq -\varepsilon_n \|q - q_n\|_{1,2} \text{ for all positive integers } n,$$

and taking $n \to \infty$ it follows that $\Psi_*(q) = \lim\limits_{n\to\infty} \Psi_*(q_n)$, so $c = \mathcal{I}_*(q)$. The proof is completed. ■

4 Main result

For convenience, we take our fixed period to be $T = 2\pi$. Let $L : S^1 \to B(C_{2\pi})$ be given by

$$(L(\theta)q)(t) = q(t+\theta) \quad \text{for all } \theta \in S^1, t \in \mathbb{R}, q \in C_{2\pi}.$$

One has that L is an isometric representation of S^1 over the Banach space $C_{2\pi}$. Notice that

$$\text{Fix}(S^1) = \mathbb{R}^3 \subset H^1_{2\pi}.$$

Moreover, it is clear that $(L(\theta)|_{H^1_{2\pi}})_{\theta \in S^1}$ is an isometric representation of S^1 over the Hilbert space $H^1_{2\pi}$. The nonsmooth part of the Poincaré action $\mathcal{I}_*$, that is Ψ_*, is invariant under the representation L. In this section we consider the Lorentz force equation with autonomous electric and magnetic potentials, so

$$V : \mathbb{R}^3 \to \mathbb{R}, \quad W : \mathbb{R}^3 \to \mathbb{R}^3.$$

This implies that the smooth part of the Poincaré action $\mathcal{I}_*$, that is $\mathcal{F}_*$, is invariant under the representation L.

Lemma 10 *Assume that the conditions*

- (V_1) *V is of class C^2 on $\mathbb{R}^3$, $V(0) = 0$, $V > 0$ on $\mathbb{R}^3 \setminus \{0\}$, $V' \neq 0$ on $\mathbb{R}^3 \setminus \{0\}$, V'' is bounded on $\mathbb{R}^3$, and there exists $l^* > 0$ such that $\lim_{|q|\to\infty} V(q) = l^*$.*
- (W_1) *W is of class C^2 on $\mathbb{R}^3$ and W, W', W'' are bounded on $\mathbb{R}^3$.*

are satisfied. Then, the action $\mathcal{I}_*$ satisfies the Ekeland-Lasry convexity condition $(*)$, that is, there exists $\alpha > 0$ such that the function

$$D(\Psi_*) \ni q \mapsto \mathcal{I}_*(q) + \alpha \|q\|^2_{1,2} \in \mathbb{R}$$

is convex.

Proof. Using (V_1), let $\alpha_1 > 0$ be such that the function

$$\mathbb{R}^3 \ni q \mapsto -V(q) + \alpha_1 |q|^2 \in \mathbb{R}$$

is convex. But, for any $q \in H^1_{2\pi}$, one has that

$$-\int_0^{2\pi} V(q(t))dt + \alpha_1 \|q\|^2_{1,2} = \int_0^{2\pi} (-V(q(t)))dt$$
$$+\alpha_1 \int_0^{2\pi} (|q(t)|^2 + |q'(t)|^2)dt = \int_0^{2\pi} (-V(q(t)) + \alpha_1 |q(t)|^2)dt$$
$$+\alpha_1 \int_0^{2\pi} |q'(t)|^2 dt,$$

and then we deduce that

$$H^1_{2\pi} \ni q \mapsto -\int_0^{2\pi} V(q(t))dt + \alpha_1 \|q\|^2_{1,2} \in \mathbb{R}$$

is convex. Next, for a fixed constant $\alpha_2 > 0$, we consider the function $\mathcal{H}_* : W^{1,\infty}_{2\pi} \to \mathbb{R}$ given by

$$\mathcal{H}_*(q) = \int_0^{2\pi} q'(t) \cdot W(q(t))dt + \alpha_2 \|q\|^2_{1,2} \quad \text{for all } q \in W^{1,\infty}_{2\pi}.$$

One has that $\mathcal{H}_* \in C^2(W^{1,\infty}_{2\pi}, \mathbb{R})$, with the second order derivative given by

$$\begin{aligned}
\mathcal{H}''_*(q)[\varphi, \psi] &= \int_0^{2\pi} q'(t) \cdot W''(q(t))[\varphi(t), \psi(t)]dt \\
&+ \int_0^{2\pi} \psi'(t) \cdot W'(q(t))[\varphi(t)]dt \\
&+ \int_0^{2\pi} \varphi'(t) \cdot W'(q(t))[\psi(t)]dt \\
&+ 2\alpha_2(\varphi|\psi)_{1,2} \text{ for all } q, \varphi, \psi \in W^{1,\infty}_{2\pi},
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathcal{H}''_*(q)[\varphi - q, \varphi - q] &= \int_0^{2\pi} q'(t) \cdot W''(q(t))[\varphi(t) - q(t), \varphi(t) - q(t)]dt \\
&+ 2\int_0^{2\pi} (\varphi'(t) - q'(t)) \cdot W'(q(t))[\varphi(t) - q(t)]dt \\
&+ 2\alpha_2\|\varphi - q\|^2_{1,2} \text{ for all } q, \varphi \in W^{1,\infty}_{2\pi}.
\end{aligned}$$

Using (W_1), we consider two positive constants c_1 and c_2 such that $||W'(q)|| \leq c_1$ and $||W''(q)|| \leq c_2$ for all $q \in \mathbb{R}^3$. Then, for any $q, \varphi \in D(\Psi_*)$, one has

$$\left| \int_0^{2\pi} q'(t) \cdot W''(q(t))[\varphi(t) - q(t), \varphi(t) - q(t)]dt \right| \leq c_2\|\varphi - q\|^2_{L^2} \leq c_2\|\varphi - q\|^2_{1,2},$$

and, on the other hand

$$\begin{aligned}
\left| \int_0^{2\pi} (\varphi'(t) - q'(t)) \cdot W'(q(t))[\varphi(t) - q(t)]dt \right| &\leq c_1\|\varphi' - q'\|_{L^2}\|\varphi - q\|_{L^2} \\
&\leq c_1\|\varphi - q\|^2_{1.2}.
\end{aligned}$$

It follows that for $\alpha_2 > 0$ sufficiently large one has

$$\mathcal{H}''_*(q)[\varphi - q, \varphi - q] \geq 0 \quad \text{for all } q, \varphi \in D(\Psi_*),$$

which implies that $\mathcal{H}_* : \text{int}(D(\Psi_*)) \to \mathbb{R}$ is convex. This together with the continuity of $\mathcal{H}_*$ implies the convexity of $\mathcal{H}_*$ on $D(\Psi_*)$.

Thus, if α is sufficiently large, then the function

$$D(\Psi_*) \ni q \mapsto \mathcal{F}_*(q) + \alpha\|q\|^2_{1,2} \in \mathbb{R}$$

is convex, which together with Ψ_* being convex on $D(\Psi_*)$ implies the conclusion. ∎

Lemma 11 *If the conditions* (V_1) *and* (W_1) *are satisfied, then the action* $\mathcal{I}_* : H^1_{2\pi} \to (-\infty, +\infty]$ *is bounded from below and satisfies* $(wPS)_c$*-condition with respect to* $C_{2\pi}$ *for any* $c < -2\pi(l^* + \sup_{\mathbb{R}^3} |W|)$.

Proof. First of all, notice that from (V_1) it follows that V is bounded. Moreover, from (W_1) it follows that for any $q \in D(\Psi_*)$, we have

$$\left| \int_0^{2\pi} q'(t) \cdot W(q(t))dt \right| \leq 2\pi \sup_{\mathbb{R}^3} |W| < \infty,$$

so $\mathcal{I}_*$ is bounded from below on $H^1_{2\pi}$. Next, consider $c < -2\pi(l^* + \sup_{\mathbb{R}^3} |W|)$, and let $(q_n) \subset D(\Psi_*)$ be a (PS)-sequence at the level c. Assume that $(\overline{q_n})$ is not bounded, and passing to a subsequence assume that $|\overline{q_n}| \to \infty$. Consider the decomposition $q_n = \overline{q_n} + \tilde{q}_n$ and notice that $\|\tilde{q}_n\|_\infty \leq 2\pi$ for any $n \in \mathbb{N}$. Thus, using (V_1) we deduce that

$$\int_0^{2\pi} V(q_n(t))dt \to 2\pi l^* \text{ as } n \to \infty.$$

On the other hand, for all $n \in \mathbb{N}$, we have that

$$\begin{aligned} \mathcal{I}_*(q_n) &= \int_0^{2\pi} [1 - \sqrt{1 - |q_n'|^2}]dt - \int_0^{2\pi} V(q_n)dt + \int_0^{2\pi} q_n' \cdot W(q_n)dt \\ &\geq -2\pi \sup_{\mathbb{R}^3} |W| - \int_0^{2\pi} V(q_n)dt \end{aligned}$$

Letting $n \to \infty$ it follows that $c \geq -2\pi(l^* + \sup_{\mathbb{R}^3} |W|)$, a contradiction. Hence, $(\overline{q_n})$ is bounded and the conclusion follows using Proposition 1. ∎

Lemma 12 *Assume additionally that the following condition holds true.*

- (V_2) *There exist* $\lambda > 0$, *and* $r_0 > 0$ *such that* $V(q) \geq \lambda |q|^2$ *for all* $q \in \mathbb{R}^3$ *with* $|q| \leq r_0$.

Consider $m \in \mathbb{N}$ *and*

$$Z_m = \left\{ \sum_{j=1}^m [\cos(jt)a_j + \sin(jt)b_j] : a_j, b_j \in \mathbb{R}^3, 1 \leq j \leq m \right\},$$

and

$$D = \{q \in Z_m : \|q\|_{1,\infty} < r\},$$

where $r < \min(r_0, 1)$. *Then,* Z_m *is a* $(6m)$*-dimensional invariant subspace of* $H^1_{2\pi}$ *such that* $Z_m \cap \text{Fix}(S^1) = \{0\}$, *and* D *is an open bounded invariant neighbourhood of* 0 *in* Z_m. *Moreover, there exists* $\Lambda_m(l^*, W) > 0$ *such that if* $\lambda \geq \Lambda_m$, *then*

$$\sup_{\partial D} \mathcal{I}_* < -2\pi(l^* + \sup_{\mathbb{R}^3} |W|).$$

Proof. One has that Z_m is invariant due to the trigonometric formulas

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$
$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

Thus, all we have to prove is that $\mathcal{I}_*(q) < -2\pi(l^* + \sup_{\mathbb{R}^3}|W|)$ for all $q \in \partial D$ and for all λ large enough. Using that $r \leq 1$, one has $\overline{D} \subset D(\Psi_*)$, and using that

$$1 - \sqrt{1-s^2} \leq s^2 \quad \text{for all } s \in [0,1],$$

and

$$m^2 \int_0^{2\pi} |q(t)|^2 dt \geq \int_0^{2\pi} |q'(t)|^2 dt \quad \text{for all } q \in Z_m,$$

from (V_2) and (W_1) we deduce that for all $q \in \overline{D}$,

$$\begin{aligned} \mathcal{I}_*(q) &= \int_0^{2\pi} [1 - \sqrt{1 - |q'(t)|^2}]dt - \int_0^{2\pi} V(q(t))dt + \int_0^{2\pi} q'(t) \cdot W(q(t))dt \\ &\leq (m^2 - \lambda) \int_0^{2\pi} |q(t)|^2 dt + 2\pi \sup_{\mathbb{R}^3} |W|. \end{aligned}$$

Let $\gamma_m > 0$ be such that

$$||q||_{L^2} \geq \gamma_m ||q||_{1,\infty} \quad \text{for all } q \in Z_m.$$

It follows that for all $q \in \partial D$, one has that

$$\mathcal{I}_*(q) \leq (m^2 - \lambda) 2\pi \gamma_m r^2 + 2\pi \sup_{\mathbb{R}^3} |W|,$$

and the conclusion follows. ■

Remark 2 The constant Λ_m quantifies the link between the behaviour of the electric potential V around the origin and the behaviour of the electric potential V at infinity together with the magnetic potential W.

Lemma 13 *Consider the set*

$$\Omega_* = \{q \in H^1_{2\pi} : \mathcal{I}_*(q) < 0\}.$$

Then, we have that

$$\mathrm{Fix}(S^1) \cap \Omega_* \cap K(\mathcal{I}_*) = \emptyset.$$

Proof. Assume that there exists $q \in \mathbb{R}^3 = \mathrm{Fix}(S^1)$ such that q is a critical point of $\mathcal{I}_*$ and $\mathcal{I}_*(q) < 0$. From Theorem 4 it follows that q is a solution of the Lorentz force equation, that is $V'(q) = 0$, a contradiction. The proof is completed. ■

Our main application goes as follows.

Theorem 5 *Suppose that the assumptions* $(V_{1,2})$ *and* (W_1) *hold true. For any* $m \in \mathbb{N}$ *there exists* $\Lambda_m > 0$ *such that if* λ *in* (V_2) *satisfies* $\lambda \geq \Lambda_m$, *then* $\mathcal{I}_*$ *has at least* $3m$ *critical orbits at negative levels, which are* 2π*-periodic solutions of the Lorentz force equation.*

Proof. The result follows from Theorem 4, Corollary 1, and Lemmas 10, 11, 12, 13. ■

Example 1 For any $\lambda > 0$ one has that

$$\lim_{x \to 0} x^{-2} \arctan(\lambda x^2) = \lambda, \qquad \lim_{|x| \to \infty} \arctan(\lambda x^2) = \frac{\pi}{2}.$$

Consider the electric potential V given by

$$V : \mathbb{R}^3 \to \mathbb{R}, \qquad V(q) = \arctan(\lambda |q|^2).$$

One has that V satisfies $(V_{1,2})$ with $l^* = \frac{\pi}{2}$ and $V(q) \geq \frac{\lambda}{2}|q|^2$ around zero. Assume also that the magnetic potential W satisfies (W_1). Let $m \in \mathbb{N}$ be fixed. From the above theorem it follows that there exists $\Lambda_m > 0$ such that the Lorentz force equation has at least $3m$ 2π-periodic solutions for any λ such that $\frac{\lambda}{2} \geq \Lambda_m$. In particular, if $\lambda \to \infty$, then the number of 2π-periodic solutions of the Lorentz force equation goes to infinity.

Remark 3 It is well known that without the normalization we used in this paper the relativistically correct equation of the motion of a charged particle is

$$\left(\frac{q'}{\sqrt{1 - |q'|^2/c^2}} \right)' = \frac{\beta}{m_0} (E(t,q) + q' \times B(t,q)),$$

where c is the speed of light, m_0 is the rest mass, and $\beta \in \mathbb{R}$ is the charge of the particle. Notice that m_0 and β are prescribed. Thus, to apply our result, if we want a positively charged particle, our electric potential V must be attractive, i.e. $V > 0$, and if we want a negatively charged particle, our electric potential V must be repulsive, i.e. $V < 0$.

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