

NORMAL HOLONOMY OF COMPLEX HYPERBOLIC SUBMANIFOLDS

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ABSTRACT. We prove that the restricted normal holonomy group of a Kähler submanifold of the complex hyperbolic space $\mathbb{C}H^n$ is always transitive, provided the index of relative nullity is zero. This contrasts with the case of $\mathbb{C}P^n$, where a Berger type result was proved by Console, Di Scala, and the second author. The proof is based on lifting the submanifold to the pseudo-Riemannian space $\mathbb{C}^{n,1}$ and developing new tools to handle the difficulties arising from possible degeneracies in holonomy tubes and associated distributions. In particular, we introduce the notion of weakly polar actions and a framework for dealing with degenerate submanifolds. These techniques could contribute to a broader understanding of submanifold geometry in spaces with indefinite signature, offering new insight into submanifolds in the dual setting of complex projective geometry.

1. INTRODUCCION

For submanifolds of spaces of constant curvature, a fundamental result is the so-called normal holonomy theorem [8]. It states that the representation of the restricted normal holonomy group on the normal space is, up to a trivial factor, equivalent to an s -representation (i.e., the isotropy representation of a semisimple symmetric space). This result is an important tool for studying submanifold geometry, particularly for submanifolds with simple geometric invariants, such as isoparametric and homogeneous submanifolds. Moreover, there is a subtle interplay between Riemannian and normal holonomy which has led to a geometric proof of the Berger holonomy theorem [9] (for a general reference on this topic, see [3]). The normal holonomy theorem was extended to Kähler submanifolds of the complex space forms $\mathbb{C}P^n$ and $\mathbb{C}H^n$ by Alekseevsky and Di Scala [2]. They proved that if the normal holonomy representation is irreducible, then it is a Hermitian s -representation. In the reducible case, up to multiplication by complex numbers of unit modulus, it is still a Hermitian s -representation. Moreover, they showed that the normal holonomy representation is always irreducible when the index of relative nullity is zero. In this context, one has a Berger type holonomy theorem [4]: *a complete, full complex submanifold of $\mathbb{C}P^n$ with a*

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non-transitive normal holonomy group is the complex orbit, in the projective space, of an irreducible Hermitian s -representation (see [5] for a generalization). In fact, the assumption of completeness is used only to guarantee, by a result of Abe and Magid [1], that the index of relative nullity is zero. The main techniques consisted of taking the canonical lift of the submanifold to \mathbb{C}^{n+1} and using methods from submanifold geometry.

The main purpose of this article is to address the natural question of whether the aforementioned results can be extended to complex submanifolds of complex hyperbolic space $\mathbb{C}H^n$. To this end, we lift the complex submanifold to $\mathbb{C}_-^{n,1}$, the open subset of $\mathbb{C}^{n,1} \simeq \mathbb{C}^{n+1}$ consisting of vectors v satisfying $\langle v, v \rangle < 0$, where $\langle \cdot, \cdot \rangle$ denotes the Hermitian inner product of complex signature $(n, 1)$. The main challenge stems from the fact that the geometry of submanifolds in pseudo-Riemannian spaces is significantly more intricate, primarily due to the possible degeneracy — in the sense that the ambient metric may restrict degenerately — of holonomy tubes and of the equivalence classes defined by certain distributions. To tackle this issue, we first introduce the concept of weakly polar actions and develop a geometric framework for dealing with degenerate submanifolds. Although the normal connection is not well-defined for such submanifolds, the notion of a parallel normal field remains meaningful.

Theorem 1.1. *Let \bar{N}^n be a full Kähler submanifold of the complex hyperbolic space $\mathbb{C}H^{n+k}$ with zero index of relative nullity. Then the restricted normal holonomy group Φ is transitive (or equivalently, $\Phi \simeq U_k$, since it acts as a Hermitian s -representation).*

Let us note that when the index of relative nullity is non-zero, the normal holonomy group representation may be reducible. For example, if M is a complex submanifold of $\mathbb{C}H^m$ and N is a complex submanifold of $\mathbb{C}P^n$, then the open subset \mathcal{O} of negative points of the abstract join $J(M, N)$ forms a complex submanifold of $\mathbb{C}H^{m+n+1}$ whose normal holonomy group is reducible.

We hope that the techniques developed in this paper will be useful for studying submanifolds in spaces with indefinite signature, with a focus on normal holonomy.

The paper is organized as follows. Section 2 contains the preliminaries and basic facts necessary for our purposes. In this section, we develop general tools that may also be useful in a broader context. We begin with standard results on the adapted normal curvature tensor in $\mathbb{R}^{r,s}$, reviewing in §2.1 known facts about isoparametric submanifolds in Lorentz space. In §2.2, we define the concept of an *essentially Riemannian submanifold*, and in §2.3, we prove a normal holonomy theorem for such submanifolds. In §2.4, we define *weakly polar actions*, without requiring that the maximal dimensional orbits be non-degenerate. The main general result is Proposition 2.13, which is applied in Theorem 2.14 to the study of

normal holonomy. This, in turn, is used to extend normal vectors to parallel normal fields, possibly in a degenerate context. In §2.5 it is extended the theory of holonomy tubes of Euclidean submanifolds to $\mathbb{R}^{r,s}$, even in degenerate cases. In §2.6, we define the *horosphere embedding*, which will play a crucial role in the focalization of the 0-eigendistributions associated with parallel normal fields. This may be regarded as a focalization at infinity.

Section 3 is concerned with the lift of complex submanifolds of $\mathbb{C}H^n$ to $\mathbb{C}^{n,1}$, relating the respective normal holonomy groups and relative nullity distributions.

Section 4 is concerned with generalized holonomy tubes and their relation with the so-called *canonical foliation*, extending arguments in [4]. The delicate point is the proof of Main Lemma 4.3.

In Section 5, we study the geometry of the equivalence classes of the distribution perpendicular to the nullity. Coxeter groups are defined, inspired by Terng's construction of such groups for isoparametric submanifolds. This section includes the proof of Theorem 1.1.

2. PRELIMINARIES AND BASIC FACTS

Let \mathbb{V} be a real vector space of dimension n and let $\langle \cdot, \cdot \rangle$ be an inner product of signature (r, s) , where $n = r + s$ with s being the dimension of a maximal negatively definite subspace of \mathbb{V} . We will often refer to s as the signature of \mathbb{V} , when the inner product is clear from the context. As usual, $\mathfrak{so}(\mathbb{V})$ denotes the Lie algebra of the skew-symmetric (i.e. anti self-adjoint) endomorphisms of $(\mathbb{V}, \langle \cdot, \cdot \rangle)$. The inner product induces an inner product, also denoted by $\langle \cdot, \cdot \rangle$, on tensors of a fixed type. In particular, $\langle x \otimes y, w \otimes z \rangle = \langle x, w \rangle \langle y, z \rangle$.

We focus on the inner product induced on $\Lambda^2(\mathbb{V})$. One has that

$$\langle x \wedge y, w \wedge z \rangle = 2(\langle x, w \rangle \langle y, z \rangle - \langle x, z \rangle \langle y, w \rangle), \quad (2.1)$$

where $u \wedge v = u \otimes v - v \otimes u$.

If e_1, \dots, e_n is an orthonormal basis of \mathbb{V} , then $\frac{1}{\sqrt{2}} e_i \wedge e_j = \frac{1}{\sqrt{2}} (e_i \otimes e_j - e_j \otimes e_i)$, is an orthonormal basis of $\Lambda^2(\mathbb{V})$, $i, j = 1, \dots, n$, $i < j$.

The vector space $\Lambda^2(\mathbb{V})$ is naturally identified with $\mathfrak{so}(\mathbb{V})$ by means of

$$\ell : \Lambda^2(\mathbb{V}) \rightarrow \mathfrak{so}(\mathbb{V}), \quad (2.2)$$

where ℓ is determined by

$$\langle \ell(x \wedge y)w, z \rangle = \langle x, w \rangle \langle y, z \rangle - \langle x, z \rangle \langle y, w \rangle. \quad (2.3)$$

Observe that

$$\langle \ell(x \wedge y)w, z \rangle = \frac{1}{2} \langle x \wedge y, w \wedge z \rangle \quad (2.4)$$

Endow $\mathfrak{so}(\mathbb{V})$ with the usual inner product

$$\langle B, C \rangle = -\text{trace}(B \circ C).$$

A straightforward calculation shows that

$$\langle \ell(x \wedge y), \ell(w \wedge z) \rangle = \langle x \wedge y, w \wedge z \rangle$$

which implies that ℓ is a linear isometry.

One has, from (2.4), that

$$\langle \ell^{-1}(B), w \wedge z \rangle = 2 \langle Bw, z \rangle \quad (2.5)$$

and hence

$$\langle \ell^{-1}(B), \frac{1}{\sqrt{2}} e_i \wedge e_j \rangle = \sqrt{2} \langle Be_i, e_j \rangle, \quad (2.6)$$

and hence

$$\ell^{-1}(B) = \sum_{i < j} \epsilon_i \epsilon_j \langle Be_i, e_j \rangle e_i \wedge e_j, \quad (2.7)$$

where $\epsilon_k = \langle e_k, e_k \rangle = \pm 1$.

In order to fix notation, since the word *degenerate* is ambiguous, we explicit the following definition:

Definition 2.1. A (regular) submanifold of a pseudo-Riemannian manifold is called *degenerate* if the induced metric is degenerate.

Let $M^{k,l} \subset \mathbb{R}^{r,s}$ be a non-degenerate local submanifold of the flat space form of signature s and dimension $n = r + s$. Here, *local* means that we work in a neighborhood of a point, without any global assumptions. Let us consider the normal curvature tensor R^\perp at some arbitrary $q \in M$. Recall the Ricci identity $\langle R_{x,y}^\perp \xi, \eta \rangle = \langle [A_\xi, A_\eta]x, y \rangle$, where A is the shape operator of M .

Just for the sake of saving notation, we use the same letter ℓ for the isometry $\ell : \Lambda^2(\mathbb{V}) \rightarrow \mathfrak{so}(\mathbb{V})$, where \mathbb{V} is either $T_q M$ or $\nu_q M$. Let $x, y \in T_q M$ and $\xi, \eta \in \nu_q M$ be arbitrary. Since $\ell^{-1}(R_{x,y}^\perp)$ is skew-symmetric in x, y it extends to a linear map $\tilde{R}^\perp : \Lambda^2(T_q M) \rightarrow \Lambda^2(\nu_q M)$, by defining

$$\tilde{R}^\perp(x \wedge y) = \ell^{-1}(R_{x,y}^\perp). \quad (2.8)$$

We will refer to \tilde{R}^\perp as the normal curvature operator.

$$\begin{aligned} \frac{1}{2} \langle \tilde{R}^\perp(x \wedge y), \xi \wedge \eta \rangle &= \langle R_{x,y}^\perp \xi, \eta \rangle = \langle [A_\xi, A_\eta]x, y \rangle \\ &= \frac{1}{2} \langle \ell^{-1}([A_\xi, A_\eta]), x \wedge y \rangle = \frac{1}{2} \langle \tilde{A}(\xi \wedge \eta), x \wedge y \rangle \end{aligned} \quad (2.9)$$

where $\tilde{A} : \Lambda^2(\nu_q M) \rightarrow \Lambda^2(T_q M)$ is the linear map defined by $\tilde{A}(\xi \wedge \eta) = \ell^{-1}([A_\xi, A_\eta])$. Then

$$\langle \tilde{R}^\perp(x \wedge y), \xi \wedge \eta \rangle = \langle x \wedge y, \tilde{A}(\xi \wedge \eta) \rangle \quad (2.10)$$

This implies that \tilde{A} is the transpose morphism of the normal curvature operator \tilde{R}^\perp (or, equivalently, \tilde{R}^\perp is the transpose of \tilde{A}).

Remark 2.1. Let us define the so-called *adapted* normal curvature tensor:

$$\begin{aligned} \langle \mathcal{R}_{\xi_1, \xi_2, \xi_3, \xi_4} \rangle &:= \langle \tilde{R} \circ \tilde{A}(\xi_1 \wedge \xi_2), \xi_3 \wedge \xi_4 \rangle = \langle \tilde{A}(\xi_1 \wedge \xi_2), \tilde{A}(\xi_3 \wedge \xi_4) \rangle \\ &= \langle \ell^{-1}([A_{\xi_1}, A_{\xi_2}]), \ell^{-1}([A_{\xi_3}, A_{\xi_4}]) \rangle \\ &= \langle [A_{\xi_1}, A_{\xi_2}], [A_{\xi_3}, A_{\xi_4}] \rangle = -\text{trace}([A_{\xi_1}, A_{\xi_2}] \circ [A_{\xi_3}, A_{\xi_4}]) \end{aligned}$$

Then, by the same arguments in [8], \mathcal{R} satisfy the identities of a pseudo-Riemannian curvature tensor on the normal space $\nu_q(M)$.

Lemma 2.2. *Let \mathbb{V} be a vector space with a positive definite inner product and let \mathbb{W} be a vector space with an inner product. We denote both inner products by $\langle \cdot, \cdot \rangle$. Let $L : \mathbb{V} \rightarrow \mathbb{W}$ be a linear map and let $L^t : \mathbb{W} \rightarrow \mathbb{V}$ be its transpose. Then the image of L coincides with the image of $L \circ L^t$.*

Proof. The inclusion $L \circ L^t(\mathbb{W}) \subset L(\mathbb{V})$ is clear. If $\mathbb{V}' = L^t(\mathbb{W})^\perp$, then

$$\{0\} = \langle \mathbb{V}', L^t(\mathbb{W}) \rangle = \langle L(\mathbb{V}'), \mathbb{W} \rangle.$$

Thus, $L(\mathbb{V}') = \{0\}$. Since $\mathbb{V} = \mathbb{V}' \oplus L^t(\mathbb{W})$ the lemma follows. \square

Remark 2.3. In the notation and assumptions of the Lemma 2.2, let $C := L \circ L^t$. Then $\langle C(w), w \rangle = \langle L^t(w), L^t(w) \rangle \geq 0$ with equality if and only if $L^t(w) = 0$.

2.1. Riemannian isoparametric submanifolds of the Lorentz space. The object of this section is to point out some local results that in the bibliography are only proved for complete submanifolds (see [13],[12], [3, Section 4.2.6]).

Let M^n be a local isoparametric Riemannian submanifold of Lorentz space $\mathbb{R}^{m,1}$. Namely, M is a local Riemannian submanifold with (globally) flat normal bundle and, the shape operator A_ξ has constant eigenvalues for any parallel normal section ξ . As in the Euclidean ambient case we have an orthogonal decomposition $TM = E_0 \oplus \cdots \oplus E_g$, perhaps where $E_0 = 0$, and different parallel normal fields, known as curvature normals, $0 = \eta_0, \cdots, \eta_g$ such that any of the so-called eigendistributions E_i is invariant under all the shape operators of M and

$$A_{\xi|_{E_i}} = \langle \eta_i, \xi \rangle Id|_{E_i}.$$

One has, due to Codazzi identity, that any eigendistribution is autoparallel. Moreover, the integral manifold $S_i(x)$ of E_i by x is an umbilical submanifold of the ambient space, which is contained in the affine subspace

$$L_i(x) = x + E_i(x) \oplus \mathbb{R}\eta_i(x) \subset \mathbb{R}^{m,1}.$$

It turns out that $L_i(x) = L_i(y)$ if $S_i(x) = S_i(y)$, $i = 0, \dots, g$. Let $k_i = \dim E_i$. One has, for $i = 0$, that $S_0(x)$ is an open part of $L_0(x) = x + E_0(x)$. If $i > 1$, then $S_i(x)$ is an umbilical hypersurface of $L_i(x)$ that belongs to one of the following types:

(1) If η_i is spacelike, then $S_i(x)$ is an open subset of the round k_i -sphere of $L_i(x)$ of center c and radius ρ given by

$$c = x + \frac{1}{\langle \tilde{\eta}_i(x), \tilde{\eta}_i(x) \rangle} \tilde{\eta}_i, \quad \rho^2 = \frac{1}{\langle \tilde{\eta}_i(x), \tilde{\eta}_i(x) \rangle}. \quad (2.11)$$

In this case the geodesics of $S_i(x)$ are circles

(2) If η_i is timelike then $S_i(x)$ is an open subset of the hyperbolic space of $L_i(x)$ defined by

$$H_r^{k_i} = \{X \in x + E_i(x) \oplus \mathbb{R}\eta_i(x) : \langle X - c, X - c \rangle = -r^2\}^o$$

, where $-r^2 = \langle x - c, x - c \rangle$, c has the same expression as in (1) and $()^o$ denotes the connected component by x . In this case the geodesics of $S_i(x)$, are of the form

$$- \cosh(t)\eta_i(x) + \sinh(t)w, \quad (2.12)$$

where $w \in T_x S_i(x)$ and $\langle w, w \rangle = r^2 = -\langle \eta_i(x), \eta_i(x) \rangle$

(3) If η_i is lightlike, then $S_i(x)$ is a horosphere of an appropriate real hyperbolic space. In fact, there always exist a timelike $z \in \nu_x M$ such that $\langle \eta_i(y), z \rangle = 1$. Extend z to a parallel normal field \tilde{z} of $S_i(x)$. Then the shape operator $A_{\tilde{z}}$ is the identity, since $\langle \eta_i, \tilde{z} \rangle = 1$. Then the image of the parallel map $y \mapsto y + \tilde{z}_y$, from $S_i(x)$ into $\mathbb{R}^{m,1}$, is a constant $c = x + z$, since its differential is zero. Then $\langle y - c, y - c \rangle = \langle \tilde{z}, \tilde{z} \rangle := -r^2$, for all $y \in S_i(x)$. Let

$$H_r^{k_i+1} = \{X \in x + E_i(x) \oplus \mathbb{R}\eta_i(x) \oplus \mathbb{R}z : \langle X - c, X - c \rangle = -r^2\}^o.$$

Then $S_i(x)$ is an open subset of the horosphere defined by

$$\left(x + E_i(x) \oplus \mathbb{R}\eta_i(x) \right) \cap H_r^{k_i+1}.$$

Any component of a geodesic $\gamma(t)$ of $S_i(x)$ is quadratic, i.e. of the form $a_1 t^2 + a_2 t + a_t$ (see Section 2.6, and [12] for an explicit expression).

Proposition 2.4. *Let M be a Riemannian isoparametric submanifold of the Lorentz space $\mathbb{R}^{m,1}$. Then any non-space like curvature normal η_i is perpendicular to any other curvature normal.*

Proof. We may assume that $\eta_i \neq 0$. Let $S_i(p)$ be an integral manifold of E_i , let $M_i := (M)_{\xi_i}$ be a parallel focal manifold such that $\ker(I - A_{\xi_i}) = E_i$, and let π be the projection from $M \rightarrow M_i$, i.e. $\pi(q) = q + \xi_i(q)$. Since $\langle \eta_i, \eta_i \rangle \leq 0$, then $S_i(p)$ is a open subset of an unbounded complete Riemannian umbilical submanifold $\tilde{S}_i(p)$ of the Lorentzian affine normal space $\pi(p) + \nu_{\pi(p)}M_i \subset \mathbb{R}^{m,1}$ (see the discussions preceding this proposition). By the tube formula, the eigenvalues of the shape operator $A_{q-\pi(p)}^i$ of M_i do not depend on $q \in S_i(p)$ and hence, $d := \|A_{q-\pi(p)}^i\|$ does not depend on $q \in S_i(p)$. Let $\gamma(t)$, $|t| < \varepsilon$ be a geodesic of $S_i(p)$ and let $\tilde{\gamma}(t)$ be its extension to a complete geodesic of $\tilde{S}_i(p)$, $t \in \mathbb{R}$. We have that $\|A_{\tilde{\gamma}(t)-\pi(p)}^i\| = d$, for $|t| < \varepsilon$. By the explicit form of the geodesics, and by making use standard arguments relying on the (real) analyticity of $\tilde{\gamma}(t)$, we obtain that $\|A_{\tilde{\gamma}(t)-\pi(p)}^i\| = d$, for all $t \in \mathbb{R}$. Then the image of $\tilde{\gamma}(t)$ under the affine map $u \mapsto A_{u-\pi(p)}^i$, from $\pi(p) + \nu_{\pi(p)}M_i$ into the symmetric endomorphisms of $T_{\pi(p)}M_i$, is bounded. This is a contradiction, from the explicit expression of $\tilde{\gamma}(t)$, unless $A_{\tilde{\gamma}(t)-\pi(p)}^i$ is constant and thus, $A_{q-\pi(p)}^i$ does not depend on $q \in S_i(p)$ (cf. [12, Lemma 4] or [3, Lemma 4.2.19]). Then, by the proof of Lemma 4.2.20 of [3], we obtain that E_i is a parallel distribution of M . Let v_i and v_j of unit length and tangent to $E_i(p)$ and $E_j(p)$, respectively ($i \neq j$). By making use of the Gauss equation, taking into account that E_i is a parallel distribution and that $\alpha(E_i, E_j) = 0$, we obtain that $\langle R(v_1, v_2)v_1, v_2 \rangle = \langle \alpha(v_1, v_2), \alpha(v_1, v_2) \rangle - \langle \alpha(v_1, v_1), \alpha(v_2, v_2) \rangle = -\langle \eta_i(p), \eta_j(p) \rangle = 0$. \square

2.2. Essentially Riemannian submanifolds.

Definition 2.5. A non-degenerate (immersed) submanifold $M^{k,l}$ of $\mathbb{R}^{r,s}$ is called *essentially Riemannian* if there exists a distribution \mathcal{D} on M , where $\langle \cdot, \cdot \rangle$ is positive definite, such that \mathcal{D} is invariant under all shape operators A_{ξ} , and the family of the shape operators, restricted to \mathcal{D}^\perp , is a commuting family.

Let $M^{k,l}$ be an essentially Riemannian submanifold of $\mathbb{R}^{r,s}$ with associated Riemannian distribution \mathcal{D} . Since we will work locally, we assume that $M \subset \mathbb{R}^{r,s}$ is an embedded submanifold. Let, for $q \in M$,

$$C_q = \{R_{x_q, y_q}^\perp : x_q, y_q \in T_q M\} \quad (2.13)$$

By the Ricci identity, and the fact that the family of shape operators restricted to \mathcal{D}^\perp is a commuting family one has that

$$C_q = \{R_{x_q, y_q}^\perp : x_q, y_q \in \mathcal{D}_q\} \quad (2.14)$$

Since the bracket of any two shape operators $[A_\xi, A_\eta]$ is zero when restricted to \mathcal{D}^\perp , and the restriction of $\langle \cdot, \cdot \rangle$ to \mathcal{D} is positive definite, one obtains, from Remark 2.1, Lemma 2.2 and Remark 2.3, the following:

Lemma 2.6. *Let M be an essentially Riemannian submanifold of $\mathbb{R}^{r,s}$ and let \mathcal{R} be its adapted normal curvature tensor. Then*

- (1) \mathcal{R} has non-positive sectional curvatures, i.e. $\langle \mathcal{R}_{\xi_1, \xi_2} \xi_2, \xi_1 \rangle \leq 0$, for all $\xi_1, \xi_2 \in \nu_q M, q \in M$.
- (2) $\langle \mathcal{R}_{\xi_1, \xi_2} \xi_2, \xi_1 \rangle = 0$ if and only if $\mathcal{R}_{\xi_1, \xi_2} = 0$.
- (3) $\langle \mathcal{R}_{\xi_1, \xi_2} \xi_2, \xi_1 \rangle = 0$ if and only if $[A_{\xi_1}, A_{\xi_2}] = 0$.
- (4) The linear span of $\{R_{x,y}^\perp : x, y \in T_q M\}$ coincides with the linear span of $\{\mathcal{R}_{\xi, \eta} : \xi, \eta \in \nu_q M\}$.
- (5) Let $\bar{\mathcal{R}} = \bar{R}^\perp \circ \tilde{A}$ be the curvature operator on $\nu_q M$ associated to \mathcal{R} . Then $\langle \bar{\mathcal{R}}(u), u \rangle \geq 0$ for all $u \in \Lambda^2(\nu_q M)$. Moreover, the equality holds if and only if $\bar{\mathcal{R}}(u) = 0$.

2.3. Normal holonomy of essentially Riemannian submanifolds.

Let M be an essentially Riemannian submanifold of $\mathbb{R}^{r,s}$ with adapted normal curvature tensor \mathcal{R} . Let τ_c^\perp denote the ∇^\perp -parallel transport along a (piecewise differentiable) curve c from p to q , and let $\tau_c(\mathcal{R})$ be the algebraic curvature tensor of $\nu_q M$ defined by

$$\tau_c(\mathcal{R})_{\xi_1, \xi_2} \xi_3 := \tau_c \mathcal{R}_{\tau_c^{-1} \xi_1, \tau_c^{-1} \xi_2} \tau_c^{-1} \xi_3.$$

Let $\mathfrak{hol}(q)$ denote the normal holonomy algebra at q , i.e., the Lie algebra of the normal holonomy group $\Phi(q)$ of M at q . Then, by the Ambrose-Singer theorem and Lemma 2.6(4), one has that

$$\mathfrak{hol}(q) = \text{linear span of } \{R_{\xi, \eta} : R \in F(q), \xi, \eta \in \nu_q M\} \quad (2.15)$$

where

$$F(q) := \{\tau_c(\mathcal{R}_x) : c \text{ is an arbitrary curve from } x \text{ to } q, x \in M\}. \quad (2.16)$$

Observe that any $R \in F(q)$ is an algebraic curvature tensor of $\nu_q(M)$. Moreover, it is positive semi-definite when regarded as a symmetric endomorphism of $\Lambda^2(\nu_q M)$. That is, if $u \in \Lambda^2(\nu_q M)$, then $\langle R(u), u \rangle \geq 0$ with equality if and only if $R(u) = 0$,

Lemma 2.7. *We are under the previous notation and assumptions. There exists $R \in F(q)$ such that $\mathfrak{hol}(q) = \{R_{\xi, \eta} : \xi, \eta \in \nu_q M\}$.*

Proof. Any $R \in F(q)$ will be regarded as a symmetric endomorphism of $\Lambda^2(\nu_q M) \simeq \mathfrak{so}(\nu_q M)$. By means of this identification (2.15) is equivalent to

$$\ell(\mathfrak{hol}(q)) = \text{linear span of } \{\text{Im}(R) : R \in F(q)\}, \quad (2.17)$$

where Im denotes the image.

One has that $\text{Im}(R)^\perp = \ker(R)$, for all $R \in F(q)$. If $R, R' \in F(q)$, then

$$(\text{Im}(R) + \text{Im}(R'))^\perp = \ker(R) \cap \ker(R').$$

Let $u \in \Lambda^2(\nu_q M)$ that belongs to $\ker(R + R')$. Then

$$0 = \langle (R + R')(u), u \rangle = \langle R(u), u \rangle + \langle R'(u), u \rangle.$$

Since both $\langle R(u), u \rangle$ and $\langle R'(u), u \rangle$ are non-positive, then $\langle R(u), u \rangle = \langle R'(u), u \rangle = 0$. Then, by Lemma 2.6 (5), $R(u) = R'(u) = 0$. Then $\ker(R + R') \subset \ker(R) \cap \ker(R')$. Since the other inclusion is trivial we obtain the equality. Hence,

$$\text{Im}(R + R') = \ker(R + R')^\perp = (\ker(R) \cap \ker(R'))^\perp = \text{Im}(R) + \text{Im}(R').$$

Since $\ell(\mathfrak{hol}(q))$ is the sum of the images of a finite number of elements of $F(q)$, by making use of the previous argument, we conclude the proof. \square

Let us recall the concept of weak irreducibility. Let \mathbb{V} be a vector space endowed with an inner product $\langle \cdot, \cdot \rangle$, with signature s , and let G be a Lie subgroup of $SO(\mathbb{V}, \langle \cdot, \cdot \rangle)$. We say that G acts on \mathbb{V} *weakly irreducibly* if any G -invariant proper subspace of \mathbb{V} is degenerate (i.e., $\langle \cdot, \cdot \rangle$ is degenerate on \mathbb{V}).

With the same proof as in [8] (see also Section 3.3 of [3]) we have the following:

Proposition 2.8. *Let $\Phi(q)$ be the restricted normal holonomy group at q of an essentially Riemannian submanifold M of $\mathbb{R}^{r,s}$. Then the normal space decomposes as $\nu_q M = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_k$, orthogonal direct sum of non-degenerate $\Phi(q)$ -invariant subspaces and $\Phi = \Phi_0 \times \cdots \times \Phi_k$, where $\Phi_0 = \{Id\}$ and Φ_i acts trivially on \mathbb{V}_j if $i \neq j$ and weakly irreducible on \mathbb{V}_i for $i \geq 1$. (Note that \mathbb{V}_0 is contained, possibly properly, in the set of fixed vectors of $\Phi(q)$.)*

Note that the existence of a splitting into weakly irreducible invariant subspaces of $\Phi(q)$, plus a factor contained in the fixed vectors, is a standard fact.

With the same proof of the normal holonomy theorem in [8] (see also [3], Theorem 3.2.1) one obtains:

Theorem 2.9. *Let $M^{n,s}$ be an essentially Riemannian submanifold of $\mathbb{R}^{r,s}$ of the same signature as the ambient space. Then the restricted normal holonomy $\Phi(q)$ of M at q acts on the orthogonal complement of its fixed set as the isotropy representation of a semisimple Riemannian symmetric space.*

2.4. Weakly polar actions.

Let G act by isometries on a pseudo-Riemannian manifold $M^{r,s}$, and let \mathfrak{g} be its Lie algebra. Let Ω be the open and dense subset of M such that the dimension of the G -orbits is locally constant. Let \mathcal{V} be the distribution on Ω given by the tangent spaces to the G -orbits, and let $\mathcal{H} := \mathcal{V}^\perp$ be the distribution of normal spaces to the G -orbits. If $q \in \Omega$, then $\dim \mathcal{V}_q + \dim \mathcal{H}_q = \dim M = r + s$. However $\mathcal{V}_q \cap \mathcal{H}_q$ could be non-trivial if \mathcal{V}_q is a degenerate subspace.

The proof of the following lemma is standard.

Lemma 2.10. *Let G be a Lie group acting on a manifold M and let $G \cdot p$ be a locally maximal dimensional orbit. Then*

$$\mathfrak{g}_p \cdot T_p M \subset T_p(G \cdot p) \quad (2.18)$$

where \mathfrak{g}_p denotes the isotropy algebra at p .

□

Lemma 2.11. *Let G be a Lie group of isometries of a pseudo-Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$. Let $G \cdot p$ be a (locally) maximal dimensional, possible degenerate, orbit. Then the identity component G_p^o of the isotropy group at p acts trivially on the normal space $\nu_p(G \cdot p)$.*

Proof. Let \mathfrak{g} and \mathfrak{g}_p be the Lie algebras of G and G_p , respectively. Then, by Lemma 2.10,

$$0 = \langle \mathfrak{g}_p \cdot T_p M, \nu_p(G \cdot p) \rangle = \langle T_p M, \mathfrak{g}_p \cdot \nu_p(G \cdot p) \rangle$$

□

The following lemma is well-known in the Riemannian case. The same arguments apply to the pseudo-Riemannian case.

Lemma 2.12. *We are under the previous notation and assumptions. The distribution \mathcal{H} is integrable if and only if it is autoparallel.*

Proof. Let ξ, η be local fields on Ω that lie in \mathcal{H} and let X be an arbitrary Killing field induced by G .

Since $\langle \xi, X \rangle = 0$, then $0 = \eta \langle \xi, X \rangle = \langle \nabla_\eta \xi, X \rangle + \langle \xi, \nabla_\eta X \rangle$ (and the same is true by interchanging ξ and η). Then $\langle \nabla_\eta \xi, X \rangle = -\langle \xi, \nabla_\eta X \rangle = \langle \eta, \nabla_\xi X \rangle$, where the last equality is due to the Killing equation. This implies that $\langle \nabla_\eta \xi, X \rangle$ is skew-symmetric in ξ, η and hence, $\langle [\xi, \eta], X \rangle = \langle \nabla_\xi \eta - \nabla_\eta \xi, X \rangle = 2\langle \nabla_\xi \eta, X \rangle$. □

Definition 2.2. The group G acts *weakly polarly* on M if the distribution \mathcal{H} of Ω is integrable.

Proposition 2.13. *Let \mathbb{V} be a vector space with a non-degenerate inner product of signature s , and let G be a Lie subgroup of $SO(\mathbb{V})$. Assume that there exists an algebraic pseudo-Riemannian curvature tensor R of \mathbb{V} such that the curvature endomorphisms linearly span the Lie algebra \mathfrak{g} of G . Furthermore, assume that R , regarded as a symmetric endomorphism of $\Lambda^2(\mathbb{V})$, is positive semi-definite (i.e., if $u \in \Lambda^2(\mathbb{V})$ satisfies $\langle R(u), u \rangle = 0$, then $R(u) = 0$). Let N be a non-degenerate submanifold of \mathbb{V} which is locally invariant under the action of G .*

- (i) G acts weakly polarly on N .
- (ii) $T_q(G \cdot q)$ is invariant under any shape operator of N at q , for all $q \in \Omega$, where Ω is the open and dense subset of N where the dimensions of the G orbits are locally constant.

Proof. Let $u, v \in \nu_q N$ and consider the Killing field X of \mathbb{V} given by $X_x = R_{u,v}x$. Then X is a linear Killing field, so $\nabla_w X = R_{u,v}w$, where ∇ is the usual Levi-Civita flat connection of \mathbb{V} .

Let \mathcal{V} be the distribution of Ω tangent to the G -orbits, and let $\mathcal{H} = \mathcal{V}^\perp$. If $x \in N$, then $\xi \in \mathcal{H}_x$ if and only if $0 = \langle R_{u,v}x, \xi \rangle = \langle R_{x,\xi}u, v \rangle$ for all $u, v \in \mathbb{V}$. Thus, $\mathcal{H}_x = \{\xi \in T_x N : R_{x,\xi} = 0\}$. Let $\xi, \eta \in \mathcal{H}_x$. By making use of the Bianchi identity we have that $R_{\xi,\eta}x = 0$ and thus, $R_{\xi,\eta} \in \mathfrak{g}_x$. Then, from Lemma 2.10, $\langle R_{\xi,\eta}\mathcal{H}_x, \mathcal{H}_x \rangle = \{0\}$, and therefore $\langle R_{\xi,\eta}\xi, \eta \rangle = 0$. Since R is positive semi-definite, we conclude that $R_{\xi,\eta} = 0$, for all $\xi, \eta \in \mathcal{H}_x$.

Let $\tilde{\xi}, \tilde{\eta}$ be fields of N that lie in \mathcal{H} , and let X be the field of N given by $X_x = R_{u,v}x$, where $u, v \in \mathbb{V}$ are arbitrary. Then $\langle \tilde{\xi}, X \rangle = 0$, and hence, differentiating in the direction of η one obtains

$$\langle \nabla_{\tilde{\eta}} \tilde{\xi}, X \rangle = -\langle \tilde{\xi}, \nabla_{\tilde{\eta}} X \rangle = -\langle \tilde{\xi}, R_{u,v} \tilde{\eta} \rangle = \langle R_{\tilde{\xi}, \tilde{\eta}} u, v \rangle = 0$$

and hence, since u, v are arbitrary, \mathcal{H} is autoparallel. This proves (i).

Let $q \in \Omega$ and let us consider the orbit $G \cdot q$. Let $\xi \in \mathcal{H}_q$ and let $\eta \in \nu_q N$ be arbitrary. Note that ξ, η are orthogonal to $T_q(G \cdot q) = \mathcal{V}_q$. Then, as in the proof of part (i), $R_{\xi,\eta}q = 0$, and so $R_{\eta,\xi}$ belongs to the isotropy algebra \mathfrak{g}_q . Then, by Lemma 2.10, $R_{\eta,\xi}\xi$ belongs to \mathcal{V}_q . Thus, $\langle R_{\eta,\xi}\xi, \eta \rangle = 0$, which implies that $R_{\eta,\xi} = 0$. Let, for $u, v \in \mathbb{V}$,

$$\phi_t := \text{Exp}(tR_{u,v}) = e^{tR_{u,v}}.$$

Then $\phi_t \xi$ is a field along $c(t) := \phi_t q$ that lies in $\mathcal{H}_{c(t)}$. Differentiating at $t = 0$ one obtains $R_{u,v} \xi = \frac{D}{dt}|_0 \phi_t \xi$, where $\frac{D}{dt}$ is the ambient covariant derivative along the curve $c(t)$. Denote the second fundamental form and the shape operator of N as α and A , respectively. Then

$$\begin{aligned} 0 &= \langle R_{\xi, \eta} u, v \rangle = \langle R_{u, v} \xi, \eta \rangle = \langle \alpha(c'(0), \xi), \eta \rangle \\ &= \langle A_\eta c'(0), \xi \rangle = \langle A_\eta R_{u, v} q, \xi \rangle. \end{aligned}$$

Since the vectors $R_{u, v} q$, $u, v \in \mathbb{V} \text{ span } T_q(G \cdot q) = \mathcal{V}_q$ and $\xi \in \mathcal{H}_q$ is arbitrary, we conclude that $A_\eta(T_q(G \cdot q)) \subset T_q(G \cdot q)$

□

Theorem 2.14. *Let $\Phi(q)$ be the restricted normal holonomy group at q of an essentially Riemannian submanifold M of $\mathbb{R}^{r,s}$. Let N be a non-degenerate submanifold of the normal space $\nu_q M$ which is locally invariant by $\Phi(q)$. Then $\Phi(q)$ acts weakly polarly on N .*

Proof. The proof follows immediately from Lemma 2.7 and Proposition 2.13

□

Remark 2.15. *In a degenerate submanifold S of a pseudo-Riemannian manifold, the normal connection is not defined. Nevertheless, a section $\tilde{\xi}$ of the normal bundle $\nu S = (T_p S)^\perp$ is called a parallel normal field, if for any tangent field X of S , $\nabla_X \tilde{\xi}$ is a tangent field of S , where ∇ is the Levi-Civita connection of the ambient space. Thus, the shape operator $A_{\tilde{\xi}}$ is defined by $A_{\tilde{\xi}} X := -\nabla_X \tilde{\xi}$. The same proof of the Gauss formula, since ∇ is torsion-free and the bracket between tangent fields of S is tangent to S , proves that $\langle A_{\tilde{\xi}} X, Y \rangle = \langle A_{\tilde{\xi}} Y, X \rangle$. Now, assume that X_p is a degenerate vector, and let Y_p be arbitrary. Then the above equality shows that $A_{\tilde{\xi}_p} X_p$ is degenerate. Then $A_{\tilde{\xi}_p}$ leaves invariant the degeneracy subspace of $T_p S$. Analogously, a normal field $\eta(t)$ of S along a curve $c(t)$ is called parallel if $\frac{d}{dt} \eta(t) \in T_{c(t)} S$.*

Corollary 2.16. *Let G be a Lie group of isometries of a pseudo-Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ which acts weakly polarly on M . Let $N = G \cdot p$ be a maximal dimensional orbit, with a (possible) degenerate induced metric. Then any $\xi \in \nu_p N$ extends, in a neighborhood U of p in N , to a parallel normal field $\tilde{\xi}$.*

Proof. From Lemma 2.11 it follows that ξ extends to a G -invariant section $\tilde{\xi}$ of νN in a neighbourhood U of N . Since the arguments are local, we may assume that $U = N$ is an embedded submanifold of M . Let \mathcal{H} be the autoparallel distribution given by the normal spaces to the G -orbits, defined in a neighbourhood Ω of p in M (see Lemma 2.12). Without loss of generality we may assume that $N \subset \Omega$. Since $\tilde{\xi}$ is tangent to \mathcal{H} , this normal field extends to a field of Ω that lies in \mathcal{H} (perhaps by making Ω smaller). We denote such an

extension also by $\tilde{\xi}$. Let X be a Killing field of M induced by G , and let ϕ_t be its associated flow. Since the normal field $\tilde{\xi}$ is G -invariant, then $d\phi_t(\tilde{\xi}_p) = \tilde{\xi}_{\phi_t(p)}$. Hence,

$$[X, \tilde{\xi}]_p = 0 \quad (2.19)$$

Let $\tilde{\eta}$ be a field of Ω that lies in \mathcal{H} . Since X is tangent to the G -orbits, then $\langle X, \tilde{\eta} \rangle = 0$. By differentiating this equality in the direction of $\tilde{\xi}$ we obtain that

$$\tilde{\xi}_p \langle X, \tilde{\eta} \rangle = \langle X_p, \nabla_{\tilde{\xi}_p} \tilde{\eta} \rangle + \langle \nabla_{\tilde{\xi}_p} X, \tilde{\eta}_p \rangle = 0.$$

Since \mathcal{H} is autoparallel, and by making use of (2.19), we obtain that $\langle \nabla_{X_p} \tilde{\xi}, \tilde{\eta}_p \rangle = 0$. Since X and $\tilde{\eta}$ are arbitrary, we conclude that $\nabla_{T_p N} \tilde{\xi} \subset T_p N$. \square

The proof of the following result is standard. In the case of a non-degenerate submanifold it is a special case of Ricci identity.

Lemma 2.17. *Let M be a possible degenerate submanifold of $\mathbb{R}^{r,s}$ and let ξ, η be parallel normal fields of M (see Remark 2.15). Then $\langle [A_\xi, A_\eta]X, Y \rangle = 0$, for all fields X, Y tangent to M .*

\square

2.5. Holonomy tubes around a focal manifold. Let $M \subset \mathbb{R}^{r,s}$ be a local submanifold with a non-degenerate induced metric. Let $\tilde{\xi}$ be a parallel normal field of M and assume that $0 < \dim \ker(Id - A_{\tilde{\xi}(x)}) < \dim M$, and that $\dim \ker(Id - A_{\tilde{\xi}(x)})$ is independent of $x \in M$, where A is the shape operator of M .

Assumptions: The *vertical* distribution $\ker(Id - A_{\tilde{\xi}(x)})$ of M is pseudo-Riemannian and the *horizontal* distribution $\mathcal{H}^{\tilde{\xi}} := (\ker(Id - A_{\tilde{\xi}(x)}))^\perp$ is Riemannian.

By the Codazzi equation, $\ker(Id - A_{\tilde{\xi}(x)})$ defines an autoparallel distribution of M . Let us consider, locally, the Riemannian parallel focal manifold $M_{\tilde{\xi}} = \{x + \tilde{\xi}(x) : x \in M\}$. If $\pi : M \rightarrow M_{\tilde{\xi}}$ is the projection, i.e. $\pi(x) = x + \tilde{\xi}(x)$, then $\ker \mathbf{d}\pi = \ker(Id - A_{\tilde{\xi}})$. Observe that $T_{\pi(x)} M_{\tilde{\xi}} = (\ker \mathbf{d}_x \pi)^\perp \subset T_x M$, as subspaces of the ambient space. Moreover, any fiber $\pi^{-1}(\{\pi(x)\})$ is contained in the (affine) normal space $\pi(x) + \nu_{\pi(x)} M_{\tilde{\xi}}$, and the ∇^\perp -parallel transport τ_c^\perp along an arbitrary curve c of $M_{\tilde{\xi}}$ from $\pi(x)$ to $\pi(y)$ maps (locally) $\pi^{-1}(\{\pi(x)\})$ into $\pi^{-1}(\{\pi(y)\})$ (see [3, Lemma 3.4.10]). In particular,

$$\pi(x) + \Phi(\pi(x)) \cdot (x - \pi(x)) \subset \pi^{-1}(\{\pi(x)\}) \quad (\text{locally}), \quad (2.20)$$

where Φ denotes the local normal holonomy group of $M_{\tilde{\xi}}$. We regard, in the obvious way, this parallel transport as a map from the affine normal spaces, i.e. $\tau_c^\perp : \pi(x) + \nu_{\pi(x)} M_{\tilde{\xi}} \rightarrow \pi(y) + \nu_{\pi(y)} M_{\tilde{\xi}} \subset \mathbb{R}^{r,s}$. If $v \in T_x \pi^{-1}(\{\pi(x)\})$, then $d\tau_c^\perp(v)$ is naturally identified with the

linear parallel transport $\tau_c^\perp(v)$. Any of these possible interpretations of the normal parallel transport will be clear from the context.

One has that M is (locally) foliated by the holonomy tubes (see [3, p. 220])

$$H^{\tilde{\xi}}(x) := (M_{\tilde{\xi}})_{x-\pi(x)} = (M_{\tilde{\xi}})_{-\tilde{\xi}(x)} \quad (2.21)$$

By considering a smaller neighborhood of a nearby generic point, we may assume that all holonomy tubes have the same dimension, or equivalently, that $\dim(\Phi(\pi(x)) \cdot (-\tilde{\xi}(x)))$ does not depend on $x \in M$. The induced metric on $(M_{\tilde{\xi}})_{x-\pi(x)}$ may be degenerate at x . This occurs if and only if $\Phi(\pi(x)) \cdot (-\tilde{\xi}(x))$ is a degenerate orbit. Additionally, we may assume that the dimension of the degeneracy of the induced metric on $H^{\tilde{\xi}}(x)$ is constant.

Let $\tilde{\nu}$ be the distribution of M perpendicular to the distribution \mathcal{T} defined by the tangent spaces of the holonomy tubes. When the holonomy tubes are degenerate, then $\tilde{\nu}$ and \mathcal{T} have a non-trivial intersection. Let us consider the distribution $\mathcal{H}^{\tilde{\xi}} = (\ker \mathbf{d}\pi)^\perp$ and observe that $(\mathcal{H}^{\tilde{\xi}})_x = T_{\pi(x)}M_{\tilde{\xi}}$ (as linear subspaces of the ambient space). Moreover,

$$\mathcal{T}_x := T_x H^{\tilde{\xi}}(x) = T_x \left(\pi(x) + \Phi(\pi(x)) \cdot (x - \pi(x)) \right) \oplus (\mathcal{H}^{\tilde{\xi}})_x \quad (2.22)$$

Since $\tilde{\xi}$ is a parallel normal field of M , by the Ricci identity, the shape operator $A_{\tilde{\xi}}$ commutes with any other shape operator of M . Thus, $\ker \mathbf{d}\pi$ and \mathcal{H} are distributions which are invariant under all shape operators of M . From the Codazzi identity, it follows that the distribution $\ker \mathbf{d}\pi$ is autoparallel. Furthermore, from the construction of the holonomy tubes inside M , and by making use of Theorem 2.14, the distribution $\tilde{\nu}$ is autoparallel, and contained in $\ker \mathbf{d}\pi$.

Observe that the normal space $\nu_x M$ of M at x coincides with the normal space at x of $\pi^{-1}(\{\pi(x)\})$, regarded as a submanifold of the affine normal space $\pi(x) + \nu_{\pi(x)}M_{\tilde{\xi}}$. Then, taking into account that $\ker \mathbf{d}\pi$ is invariant under all the shape operators of M and Proposition 2.13, we obtain the following results (keeping the assumptions and notation of this section).

Lemma 2.18. *The distributions $\ker \mathbf{d}\pi$, $\mathcal{H}^{\tilde{\xi}}$, \mathcal{T} , and $\tilde{\nu}$ are invariant under all shape operators of M . Moreover, $\ker \mathbf{d}\pi$ and $\tilde{\nu}$ are autoparallel. \square*

Corollary 2.19. *Let $\tilde{\eta}$ be a parallel normal field of M . Then $\tilde{\eta}|_{H^{\tilde{\xi}}(x)}$ is a parallel normal field of $H^{\tilde{\xi}}(x)$, for all $x \in M$. In particular, $\tilde{\xi}|_{H^{\tilde{\xi}}(x)}$ is a parallel normal field of $H^{\tilde{\xi}}(x)$. \square*

(The definition of a parallel normal field, if $H^{\tilde{\xi}}$ is degenerate, is given by Remark 2.15).

Let $c(t)$ be a horizontal curve in $H^{\tilde{\xi}}(x)$ and let $\eta(t)$ be a normal field of $H^{\tilde{\xi}}(x)$ along $c(t)$. Then it is standard to show, and well-known in a Euclidean ambient space by an

argument that goes back to [7], that $\eta(t)$ is a parallel normal field along $c(t)$ if and only if $\eta(t)$ is a parallel normal field of $M_{\tilde{\xi}}$ along the curve $\pi(c(t))$.

Remark 2.20. *The distribution of M tangent to the normal holonomy orbits of the focal manifold is given by $\mathcal{T} \cap \ker \mathbf{d}\pi$; see (2.22).*

Lemma 2.21. *Let $\psi \in \tilde{\nu}_x$. Then ψ extends (locally) to a section $\tilde{\psi}$ of $\tilde{\nu}|_{H^{\tilde{\xi}}(x)}$, which is a parallel normal field of $H^{\tilde{\xi}}(x)$. Moreover, the shape operator $\hat{A}_{\tilde{\psi}}$ of $H^{\tilde{\xi}}(x)$ leaves invariant the horizontal distribution $\mathcal{H}^{\tilde{\xi}}$.*

Proof. If $H^{\tilde{\xi}}(x)$ is non-degenerate, the proof follows analogous arguments to those in part (iii) of Proposition 7.1.1 of [3]. In the degenerate case the arguments are similar, after applying Corollary 2.16 to construct a parallel normal field of the degenerate normal holonomy orbit $\pi(x) + \Phi(\pi(x)) \cdot (x - \pi(x))$ of the focal manifold $M_{\tilde{\xi}}$. In fact, let $y \in H^{\tilde{\xi}}(x)$, and let $c : [0, 1] \rightarrow H^{\tilde{\xi}}(x)$ be a horizontal curve with $c(0) = x$, $c(1) = y$. Let $\bar{\psi}(t)$ be the parallel normal field along the curve $\pi(c(t))$ of the Riemannian manifold $M_{\tilde{\xi}}$ with $\bar{\psi}(0) = \psi$. Then $\bar{\psi}(t)$ is a parallel normal field of $H^{\tilde{\xi}}(x)$ along the curve $c(t)$ (see Remark 2.15). We define $\tilde{\psi}(y) = \bar{\psi}(1)$. From Corollary 2.16 one obtains that $\tilde{\psi}$ is well defined (near x), defines a parallel normal field of $H^{\tilde{\xi}}(x)$. The last assertion follows from the construction of $\tilde{\psi}$ \square

Remark 2.22. *Let us define on M the following equivalence class: $x \underset{\tilde{\xi}}{\sim} y$ if there is a curve in M from x to y which lies in the horizontal distribution \mathcal{H} . Let $[x]$ denote the equivalence class of x . Then, locally,*

$$H^{\tilde{\xi}}(x) = [x]$$

(see last paragraph of [3, p. 224]).

2.6. The horosphere embedding. Let $\mathbb{R}^{r,s}$ be the pseudo-Euclidean space \mathbb{R}^{r+s} with signature s where the inner product is given by $\langle v, v \rangle = -v_1^2 - \dots - v_s^2 + v_{s+1}^2 + \dots + v_{s+r}^2$.

The *horosphere embedding* is the isometric map $f : \mathbb{R}^{r,s} \rightarrow \mathbb{R}^{r+1,s+1} \simeq \mathbb{R}^{1,1} \times \mathbb{R}^{r,s}$ given by

$$f(x) = \left(\frac{1}{2} \langle x, x \rangle + 1, \frac{1}{2} \langle x, x \rangle, x \right) \quad (2.23)$$

Then $Q^{r,s} := f(\mathbb{R}^{r,s})$ is called the *pseudo-horosphere* of the pseudo-hyperbolic space

$$H^{r+1,s} = \{v \in \mathbb{R}^{r+1,s+1} : \langle v, v \rangle = -1\}^o \quad (2.24)$$

where $e_{-1}, e_0, \dots, e_{r+s}$ is the canonical basis of $\mathbb{R}^{1,1} \times \mathbb{R}^{r,s}$ and $\{\}^o$ denotes the connected component by e_{-1} (we will frequently write Q instead of $Q^{r,s}$). Namely,

$$Q = H^{r+1,s} \cap E \quad (2.25)$$

where E is the degenerate affine subspace of $\mathbb{R}^{1,1} \times \mathbb{R}^{r,s}$ given by the equation $x_{-1} - x_0 = 1$. One has that $f : \mathbb{R}^{r,s} \rightarrow Q$ is an isometric diffeomorphism and the map $f : \mathbb{R}^{r,s} \rightarrow \mathbb{R}^{r+1,s+1}$ is an isometric ρ -equivariant embedding, is a Lie group morphism from the isometry group of $\mathbb{R}^{r,s}$ into the orthogonal group $O(r+1, s+1)$. In fact, let $g \in O(r, s)$ and let τ_v be the translation by $v \in \mathbb{R}^{r,s}$ in $\mathbb{R}^{r,s}$. Then $\rho(g)$ is given by the natural inclusion of $O(r, s) \subset O(r+1, s+1)$, where $\mathbb{R}^{r,s} \subset \mathbb{R}^{1,1} \times \mathbb{R}^{r,s} = \mathbb{R}^{r+1,s+1}$. Moreover,

$$\begin{aligned} \rho(\tau_v)(x_{-1}, x_0, x) = & \quad (2.26) \\ (x_{-1} + \langle x, v \rangle + \frac{1}{2}(x_{-1} - x_0)\langle v, v \rangle, x_0 + \langle x, v \rangle + \frac{1}{2}(x_{-1} - x_0)\langle v, v \rangle, x + (x_{-1} - x_0)v) \end{aligned}$$

One has that $Q^{r,s} \simeq \mathbb{R}^{r,s}$ is a pseudo-Riemannian flat manifold of signature s .

If $c(t) = (c_{-1}(t), c_0(t), \dots, c_{r+s}(t))$ is a curve in Q , then $c_{-1}(t) - c_0(t) = 1$ and hence, differentiating, $0 = c'_{-1}(t) - c'_0(t) = \langle -e_{-1} + e_0, c'(t) \rangle$. Then $\xi^0 = -e_{-1} + e_0$ is a constant ∇^\perp -parallel normal vector field to Q . Moreover, if A is the shape operator of $Q \hookrightarrow \mathbb{R}^{r+1,s+1}$, then $A_{\xi^0} = 0$.

The position vector field ξ^1 of $H^{r+1,s} \subset \mathbb{R}^{r+1,s+1}$ is an umbilical parallel normal field. Namely, $A'_{\xi^1} = -\text{Id}$, where A' is the shape operator of $H^{r+1,s}$. Thus, the restriction of ξ^1 to Q is also a parallel normal field and $A_{\xi^1} = -\text{Id}$, where A is the shape operator of the horosphere. Then the normal space νQ of Q in $\mathbb{R}^{r+1,s+1}$ is generated by the parallel independent normal fields ξ^0, ξ^1 , which are umbilical. Let $i : M \rightarrow Q \simeq \mathbb{R}^{r,s}$ be an isometric immersion and let νM be the normal bundle of M . Then the normal bundle of M , regarded as a submanifold of $\mathbb{R}^{r+1,s+1}$, decomposes orthogonally as

$$\bar{\nu}M = i^*(\nu Q) \oplus \nu M \quad (2.27)$$

where $i^*(\nu Q)$ is the pull-back bundle, which is a parallel, flat and umbilical sub-bundle of $\bar{\nu}M$.

Remark 2.23. *Since the pseudo-horosphere $Q^{r,s}$ is umbilical, it follows that, via the horosphere embedding, it is an essentially Riemannian submanifold of $\mathbb{R}^{r+1,s+1}$.*

The proof of the following lemma is the same as that for Euclidean submanifolds when dealing with the zero distribution associated to the kernel of the shape operator of a parallel normal field (see [3, Section 7.1]).

Lemma 2.24. *Let M be a local pseudo-Riemannian submanifold and let $\tilde{\eta}$ be a parallel normal field such that the kernel of the shape operator $A_{\tilde{\eta}}$ has constant dimension. We identify, by means of the horosphere embedding, M with its image \tilde{M} under the horosphere embedding and $\tilde{\eta}$ with a parallel normal field of \tilde{M} (tangent to the horosphere). Let $\bar{\nu}$ be*

the position (parallel normal) field of \tilde{M} . Then $\ker A_{\tilde{\eta}} = \ker(\text{Id} - \tilde{A}_{\tilde{\eta}-\tilde{v}})$, where \tilde{A} is the shape operator of \tilde{M} . Thus, $\ker A_{\tilde{\eta}}$ is the vertical distribution associated to the projection $pr : \tilde{M} \rightarrow \tilde{M}_{\tilde{\eta}-\tilde{v}}$, $pr(x) = x + \tilde{\eta}(x) - \tilde{v}(x) = \tilde{\eta}(x)$.

3. THE LIFT OF A KÄHLER SUBMANIFOLD OF $\mathbb{C}H^n$ TO $\mathbb{C}^{n,1}$

Let $\mathbb{C}^{n,1}$ be the complex space \mathbb{C}^{n+1} endowed with the pseudo-Hermitian inner product $\langle \cdot, \cdot \rangle^H$ given by

$$\langle (z_0, z_1, \dots, z_n), (z'_0, z'_1, \dots, z'_n) \rangle^H = -z_0 \bar{z}'_0 + z_1 \bar{z}'_1 + \dots + z_n \bar{z}'_n. \quad (3.1)$$

The induced (real) inner product, i.e. the real part of the pseudo-Hermitian inner product will be denoted by $\langle \cdot, \cdot \rangle$. Let $\langle \langle \cdot, \cdot \rangle \rangle^H$ be the canonical Hermitian inner product of \mathbb{C}^{n+1} . This Hermitian inner product induces the canonical inner product of $\mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$ which will be denoted by $\langle \langle \cdot, \cdot \rangle \rangle$.

Observe that $\langle \cdot, \cdot \rangle$ and $\langle \langle \cdot, \cdot \rangle \rangle$ naturally induce on \mathbb{C}^{n+1} flat pseudo-Riemannian and Riemannian metrics, respectively. Such metric tensors will be also denoted $\langle \cdot, \cdot \rangle$ and $\langle \langle \cdot, \cdot \rangle \rangle$, respectively. Nevertheless, the associated Levi-Civita connections coincide. In fact, it is the usual connection ∇ of a vector space. The Kähler structure J of \mathbb{C}^{n+1} is also a pseudo-Kähler structure of $\mathbb{C}^{n,1}$.

Observe that $\mathbb{C}^{n,1}$, regarded as a real pseudo-Euclidean space, has signature 2 and thus $\mathbb{C}^{n,1} \simeq \mathbb{R}^{2n,2}$. Let

$$\mathbb{C}_-^{n,1} = \{z \in \mathbb{C}^{n,1} : \langle z, z \rangle < 0\} \quad (3.2)$$

which is an open subset of $\mathbb{C}^{n,1} \simeq \mathbb{C}^{n+1}$. Observe that $\lambda \mathbb{C}_-^{n,1} = \mathbb{C}_-^{n,1}$, for any $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$.

The complex hyperbolic space $\mathbb{C}H^n$ is the projectivized space of $\mathbb{C}_-^{n,1}$. Moreover, it is the symmetric dual space of the complex projective space $\mathbb{C}P^n$. The symmetric presentation is

$$\mathbb{C}H^n = \text{SU}_{n,1}/\text{S}(\text{U}_1 \text{U}_n),$$

where the group $\text{SU}_{n,1}$ is the group of complex linear transformations of \mathbb{C}^{n+1} that preserve $\langle \cdot, \cdot \rangle$. The Riemannian metric on $\mathbb{C}H^n = \text{SU}_{n,1}/\text{S}(\text{U}_1 \text{U}_n)$, up to a scaling, is unique and has constant and negative holomorphic curvature. We choose such a Riemannian metric to have holomorphic curvature equal to -4 . Observe that $\mathbb{C}H^n$ may be regarded as an open subset of $\mathbb{C}P^n$; see (3.2). But the symmetric Riemannian metric is different.

Let $\pi : \mathbb{C}_-^{n,1} \rightarrow \mathbb{C}H^n$ be the projection. Then π is a submersion and

$$\ker(d\pi)_q = T_q(\mathbb{C}^*q) \simeq \mathbb{C}q \quad (3.3)$$

Definition 3.1. *The lift N of a submanifold \bar{N} of $\mathbb{C}H^n$, to a submanifold of $\mathbb{C}^{n,1}$, is $N = h(\pi^{-1}(\bar{N}))$, where $h : \mathbb{C}_-^{n,1} \rightarrow \mathbb{C}^{n,1}$ is the inclusion.*

Such a lift, when restricted to the anti-de Sitter space, has been considered in [6] with the aim of classifying isoparametric hypersurfaces in the complex projective space.

One has, from (3.3), that the lift of a submanifold of $\mathbb{C}H^n$ is a non-degenerate submanifold of $\mathbb{C}^{n,1}$ with signature 2. Moreover, $\pi : \mathbb{C}_-^{n,1} \rightarrow \mathbb{C}H^n$ is a fibration with fibers $\pi^{-1}(\{\pi(q)\}) = \mathbb{C}^*q$, where $\mathbb{C}^* = \mathbb{C} - \{0\}$. Let \mathcal{V} be the vertical distribution of $\mathbb{C}_-^{n,1}$; i.e tangent to the fibers of π . One has that $\mathcal{V}_q = \mathbb{C}q$, regarded as a subspace of $T_q\mathbb{C}_-^{n,1}$. Observe that, for all $q \in \mathbb{C}_-^{n,1}$, \mathcal{V}_q is a negative 2-dimensional (real) subspace of $T_q\mathbb{C}_-^{n,1}$. Then the perpendicular distribution, the so-called horizontal distribution, $\mathcal{H} := \mathcal{V}^\perp$ is a Riemannian distribution.

Let us consider the (real) pseudo-hyperbolic space

$$H_r^{2n,1} = \{v \in \mathbb{C}_-^{n,1} : \langle v, v \rangle = -r^2\} \subset \mathbb{C}_-^{n,1} \quad (3.4)$$

of constant negative curvature $-1/r^2$, $r > 0$; cf. (2.24). Observe that for any $r > 0$, $\pi : H_r^{2n,1} \rightarrow \mathbb{C}H^n$ is a submersion. Moreover, it is a fibration with non-degenerate negative definite fibers $S^1 \cdot q$, where S^1 here denotes the unit complex numbers. The vertical distribution at q is given by $\mathcal{V}_q \cap T_q H_r^{2n,1} = Jq$. The horizontal distribution is just the restriction of \mathcal{H} to $H_r^{2n,1}$. It is well-known that $\pi|_{H_r^{2n,1}}$ is a pseudo-Riemannian submersion of factor $1/r$, i.e. $d_q\pi : \mathcal{H}_q \rightarrow T_{\pi(q)}\mathbb{C}H^n$ is a homothety of factor $1/r$. Namely,

$$\langle d_q\pi(u), d_q\pi(u) \rangle = r^{-2}\langle u, u \rangle.$$

The distributions \mathcal{V} and \mathcal{H} of $\mathbb{C}_-^{n,1}$ are both J -invariant. Moreover, if \bar{J} is the Kähler structure of $\mathbb{C}H^n$, one has that $d\pi(Jv) = \bar{J}d\pi(v)$, for all $v \in T\mathbb{C}_-^{n,1}$. This implies that $N = \pi^{-1}(\bar{N})$ is a pseudo-Kähler submanifold of $\mathbb{C}^{n,1}$ if and only if \bar{N} is a Kähler submanifold of $\mathbb{C}H^n$.

Recall that a submanifold of a Riemannian manifold is called *full* if it is not contained in a proper totally geodesic submanifold of the ambient space.

Remark 3.2. *If X is either $\mathbb{C}P^n$ or $\mathbb{C}H^n$, then any totally geodesic submanifold of X is complex or totally real. Assume that a Kähler submanifold \bar{N} is contained in a totally geodesic submanifold $\bar{\Sigma}$ of X . Then $\bar{\Sigma}$ is Kähler.*

If N is a submanifold of a real vector space \mathbb{V} and $q \in N$, then the affine subspace generated by the set N coincides with $q + \mathbb{W}$, where \mathbb{W} is the (real) linear subspace generated by all the tangent spaces of N . If \mathbb{V} is complex and N is Kähler then any tangent space

is complex and so \mathbb{W} is complex. Let now $N \subset \mathbb{C}_-^{n,1} \subset \mathbb{V} := \mathbb{C}^{n,1}$ be the lift of a Kähler submanifold of $\mathbb{C}H^n$. Then, for any given $q \in N$, $\mathbb{C}^*q \subset q + \mathbb{W}$ and so the limit point 0 belongs to the affine subspace generated by N . Then $q + \mathbb{W} = \mathbb{W}$ and \mathbb{W} is a complex subspace of $\mathbb{C}^{n,1}$. Since $\mathbb{C}q \subset T_qN$ and $\mathbb{C}q$ is a negative definite complex line, we obtain that the signature of \mathbb{W} is 2.

It is well-known, and standard to prove, that $\bar{\Sigma}$ is a totally geodesic Kähler submanifold of $\mathbb{C}H^n$ if and only if its lift Σ is the intersection of a complex subspace \mathbb{W} of signature 2 with $\mathbb{C}_-^{n,1}$. Then, the previous discussion and Remark 3.2 imply:

Proposition 3.3. *A submanifold \bar{N} of $\mathbb{C}H^n$ is full if and only if its lift N is a full submanifold of $\mathbb{C}^{n,1}$.*

One has the following result:

Lemma 3.4. *Let \bar{N} be a Kähler submanifold of $\mathbb{C}H^n$, and let N be its lift to $\mathbb{C}^{n,1}$. Let \mathcal{N}_q be the nullity of the second fundamental form of N at q , and let $\bar{\mathcal{N}}_{\pi(q)}$ be the nullity of the second fundamental form of \bar{N} at $\pi(q)$. Then, for all $q \in N$,*

- (i) $\mathcal{V}_q \subset \mathcal{N}_q$
- (ii) $\mathcal{N}_q = (d_q\pi)^{-1}(\bar{\mathcal{N}}_{\pi(q)})$.

In particular, if $\bar{\mathcal{N}}_{\pi(q)} = \{0\}$, then $\mathcal{N}_q = \mathcal{V}_q$.

Proof. Recall that the lift of a Kähler submanifold of $\mathbb{C}H^n$ is a pseudo-Kähler submanifold of $\mathbb{C}^{n,1}$ and observe that $\mathbb{C}^*N = N$. Let, for $\lambda \in \mathbb{C}^*$, $\mu_\lambda : \mathbb{C}^{n,1} \rightarrow \mathbb{C}^{n,1}$ denote the multiplication by λ . If $q \in N$, then $T_{\mu_\lambda(q)}N = d\mu_\lambda(T_qN) = \lambda(T_qN) = T_qN$. This means that the tangent spaces of N are constant along any fiber. This implies that $\mathcal{V}|_N \subset \mathcal{N}$. This shows (i).

Let $-r^2 = \langle q, q \rangle$, let \bar{X}, \bar{Y} be fields of \bar{N} around $\pi(q)$ and let X, Y be their horizontal lifts to $H_r^{2n,1}$. Since the normal space $\mathbb{R}q$ of $H_r^{2n,1}$ at q is included in \mathcal{N}_q , one obtains that $v \in T_qH_r^{2n,1}$ belongs to \mathcal{N}_q if and only if v is in the nullity of the second fundamental form $\hat{\alpha}$ of $\hat{N} := N \cap H_r^{2n,1}$ as a submanifold of $H_r^{2n,1}$. If $\hat{\nabla}$ is the Levi-Civita connection of $H_r^{2n,1}$ we obtain, from O'Neill formulas that

$$d\pi(\hat{\nabla}_{X_q}Y) = \bar{\nabla}_{\bar{X}_q}\bar{Y},$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\mathbb{C}H^n$. Since the normal space of \hat{N} in $H_r^{2n,1}$ is included in \mathcal{H}_q , we obtain, by taking normal components, that

$$d\pi\hat{\alpha}(X_q, Y_q) = \bar{\alpha}(\bar{X}_q, \bar{Y}_q),$$

where $\bar{\alpha}$ is the second fundamental form of \bar{N} . From this it follows (ii). \square

It is clear that the normal holonomy of pseudo-Kähler submanifolds of pseudo-Kähler spaces acts by complex endomorphisms.

Remark 3.5. *Since the restriction of vertical distribution \mathcal{V} to N is tangent to N , for any $q \in N$, $d_q\pi : \nu_q N \rightarrow \nu_{\pi(q)}\bar{N}$ is a homothety of factor r^{-2} , where $r^2 = -\langle q, q \rangle$.*

We have the following result:

Lemma 3.6. *Let \bar{N} be a Kähler submanifold of $\mathbb{C}H^n$ and let N be its lift to $\mathbb{C}^{n,1}$. Let $q \in N$ be arbitrary and let $\bar{q} = \pi(q)$. Then*

$$S^1\bar{\Phi}(\bar{q}) = d_q\pi(S^1\Phi(q)) := d_q\pi \circ (S^1\Phi(q)) \circ (d_q\pi|_{\nu_q N})^{-1}$$

where Φ and $\bar{\Phi}$ are the local normal holonomy groups of N and \bar{N} , respectively and S^1 is the group of unit complex numbers acting on the normal spaces.

Proof. The arguments are the same as those inside the proof of Lemma 7.5.4 of [3] for proving formula (7.7) there. \square

Let us recall that the *index of relative nullity*, of a non-degenerate submanifold of a pseudo-Riemannian manifold, is the dimension of \mathcal{N}_q where \mathcal{N}_q is the nullity subspace of the second fundamental form at q . The set of points where the index of relative nullity attain its minimum is open. If the submanifold is connected and analytic, then this set is also dense.

We recall a result from Alekseevsky and Di Scala (see Theorem 1, Theorem 2 and Corollary 1 of [2]):

Theorem 3.7 ([2]). *Let \bar{N} be a Kähler submanifold of a space of constant holomorphic curvature. If the index of relative nullity at \bar{q} is zero, then the restricted normal holonomy group $\bar{\Phi}(\bar{q})$ acts on the normal space as the isotropy representation of an irreducible Hermitian symmetric space. In particular, $\bar{\Phi}(\bar{q})$ contains the group of multiplications by unit complex numbers.*

Remark 3.8. *If one replace in Theorem 3.7 the restricted normal holonomy group by the local holonomy group, then the conclusion is the same. In fact, the local normal holonomy group at p is the normal holonomy group at p of a small simply connected neighbourhood of p .*

Observe that the lift of a Kähler submanifold of $\mathbb{C}H^n$ is an essentially Riemannian submanifold $\mathbb{C}^{n,1}$ with the same signature. Then Lemma 3.6, Theorem 2.9, and Theorem 3.7 imply:

Corollary 3.9. *Let \bar{N} be a Kähler submanifold of $\mathbb{C}H^n$ with index of relative nullity $\nu_{\bar{q}} = 0$ at $\bar{q} \in \bar{N}$. Let N be the lift of \bar{N} to $\mathbb{C}^{n,1}$, let $q \in \tilde{N}$ be such that $\pi(q) = \bar{q}$. Then*

$$\bar{\Phi}(\bar{q}) = d_q \pi(\Phi(q))$$

where $\bar{\Phi}$ and Φ are the local normal holonomy groups of \bar{N} and N , respectively. Moreover, $\bar{\Phi}(\bar{q})$ and $\Phi(q)$ act irreducibly as the isotropy representation of a Hermitian symmetric space.

Lemma 3.10. *Let \bar{N} be a Kähler submanifold of $\mathbb{C}H^n$ and let N be the lift of \bar{N} to $\mathbb{C}_-^{n,1}$. Let \mathcal{V}' be the restriction to N of the vertical distribution \mathcal{V} of $\mathbb{C}_-^{n,1}$ and let $\mathcal{H}' = \mathcal{V}'^\perp = \mathcal{H} \cap TN$. Then \mathcal{H}' has no integral manifolds.*

Proof. Assume that N' is an integral manifold of \mathcal{H}' . Since \mathcal{V}' is J -invariant, then N' is a pseudo-Kähler (Riemannian) submanifold of $\mathbb{C}_-^{n,1}$. Observe, since N' is always perpendicular to the position vector, that $N' \subset H_r^{2n,1}$, where $-r^2 = \langle q, q \rangle$ is independent of $q \in N'$. Let ξ be the restriction to N' of the position vector field, which is an umbilical parallel normal vector field of N' , i.e. $A_\xi = -Id$ where A is the shape operator of N' . Observe that $J\xi$, as well as ξ , is a parallel normal field and $A_{J\xi} = JA_\xi = -JId$. The left hand side of this equality is a symmetric $(1, 1)$ tensor on N' while the right hand side is skew-symmetric and non-null. A contradiction. \square

4. HOLONOMY TUBES AND THE CANONICAL FOLIATION

The general arguments for this section are based on [4], [3, Section 7], but our notation is slightly different for the restricted normal holonomy groups. We will adapt the arguments in these references to the pseudo-Riemannian case. Moreover, we will simplify some crucial proofs there. The main difficulty is to deal with degenerate orbits of normal holonomy groups associated to focalization at infinity of the leaves of nullity foliations.

We keep the general notation of previous sections.

General assumption: \bar{N} is a Kähler (local) submanifold of $\mathbb{C}H^n$ with zero index of relative nullity at any point.

Let $N := \pi^{-1}(\bar{N})$ be the lift of \bar{N} to $\mathbb{C}^{n,1}$. Then, by Lemma 3.4, the nullity distribution \mathcal{N} of N coincides with the restriction to N of the 2-dimensional π -vertical distribution \mathcal{V} , i.e., $\mathcal{V}_q = \mathbb{C}q$.

Since the signature of N is the same as that of the ambient space, the normal space νN is Riemannian. Moreover, by Corollary 3.9, the normal holonomy group $\Phi(q)$ of N at q acts as an irreducible Hermitian s -representation.

Let $\zeta_q \in \nu_q N$ be a small principal vector for the normal holonomy action and let $(N)_{\zeta_q}$ be its associated holonomy tube (possibly, by making N smaller around q). Observe that $(N)_{\zeta_q} = (N)_{\zeta'}$, where ζ' is the normal parallel transport of ζ along any curve starting at q .

Let η_1, \dots, η_g be the curvature normals, with associated eigendistributions E_1, \dots, E_g , of the commuting family of shape operators of the isoparametric homogeneous submanifold $\Phi(q) \cdot \zeta_q$ of $\nu_q N$ (see [10] or chapter 4 of [3]). Moreover, the curvature normals are parallel in the normal connection of the orbit $\Phi(q) \cdot \zeta_q$. We regard such an orbit as a submanifold of the affine normal space $q + \nu_q N$. That is, we identify

$$\nu_q N \simeq q + \nu_q N \quad \text{and} \quad \Phi(q) \cdot \zeta_q \simeq q + \Phi(q) \cdot \zeta_q.$$

Since $\Phi(q)$ acts irreducibly, $\Phi(q) \cdot \zeta_q$ is full in the normal space $\nu_q N$ and thus, the curvature normals span the normal space of $\Phi(q) \cdot \zeta_q$ at any point of the orbit. Observe that the normal space to such an orbit coincides with the normal space of the holonomy tube $(N)_{\zeta_q}$ (see [3, p.130]). The integral manifold $S_i(x)$ of E_i by $x \in \Phi(q) \cdot \zeta_q$ is an extrinsic sphere, a so-called curvature sphere, of $\nu_q N$. One has that

$$S_i(x) \subset E_i(x) \oplus \mathbb{R} \eta_i(x). \quad (4.1)$$

As in the case of Riemannian submanifolds of Euclidean space, every curvature normal of the holonomy orbit $\Phi(q) \cdot \zeta_q$ extends to a parallel normal field of $(N)_{\zeta_q}$ and its associated autoparallel eigendistribution extends in a natural way to $(N)_{\zeta_q}$. We denote such extensions by $\tilde{\eta}_i$ and \tilde{E}_i , $i = 1, \dots, g$. If $\text{pr} : (N)_{\zeta_q} \rightarrow N$ denotes the projection, then the fibers, which are totally geodesic and invariant under all shape operators of $(N)_{\zeta_q}$, are given by

$$\text{pr}^{-1}(\{\text{pr}(x)\}) = \text{pr}(x) + \Phi(\text{pr}(x)) \cdot (x - \text{pr}(x)) \subset \text{pr}(x) + \nu_{\text{pr}(x)} N. \quad (4.2)$$

Moreover, $\tilde{\eta}_i(x)$ is a curvature normal at x of the orbit $\text{pr}(x) + \Phi(\text{pr}(x)) \cdot (x - \text{pr}(x)) \subset \text{pr}(x) + \nu_{\text{pr}(x)} N$. Furthermore, $\tilde{E}_i(x)$ is the eigenspace associated with $\tilde{\eta}_i(x)$ (see [3, Remark 7.3.1]). Observe that $\Phi(\text{pr}(x)) \cdot (x - \text{pr}(x))$ is identified with $\Phi(q) \cdot \zeta_q$ by means of the normal parallel transport in N along any curve from q to $\text{pr}(x)$.

Remark 4.1. *The normal space $\nu_x(N)_{\zeta_q}$ coincides with the normal space of $\Phi(\text{pr}(x)) \cdot (x - \text{pr}(x)) \subset \nu_{\text{pr}(x)} N$. Hence, $\tilde{\eta}_1(x), \dots, \tilde{\eta}_g(x)$ span $\nu_x(N)_{\zeta_q}$ for all $x \in (N)_{\zeta_q}$. Then the principal holonomy tube $(N)_{\zeta_q}$ has a flat normal bundle (see [3, Thm. 4.4.12]). In particular, the normal field $\tilde{\zeta}$ of $(N)_{\zeta_q}$ defined by $\tilde{\zeta}(x) = \text{pr}(x) - x$ is parallel and hence N is a parallel focal manifold of the holonomy tube. Namely,*

$$N = ((N)_{\zeta_q})_{\tilde{\zeta}} \quad (4.3)$$

Note that $\langle \tilde{\zeta}, \tilde{\eta}_i \rangle = 1$ for $i = 1, \dots, g$. In particular, $\tilde{\eta}_i \neq 0$.

Recall that the nullity distribution \mathcal{N} of N coincides with the vertical distribution \mathcal{V} . Since the perpendicular distribution to \mathcal{V} in N is Riemannian, then the commuting family of shape operators of $(N)_{\zeta_q}$ can be simultaneously diagonalized, with real eigenvalues functions. In fact, this follows from the *tube formula* [3, Lemma 3.4.7]. Namely, at any point $x \in (N)_{\zeta_q}$ there exist different curvature normals $0 = \tilde{\eta}_0(x), \tilde{\eta}_1(x), \dots, \tilde{\eta}_{d(x)} \in \nu_x(N)_{\zeta_q}$, $d(x) > g$, and orthogonal decomposition $T_x(N)_{\zeta_q} = \tilde{E}_0(x) \oplus \dots \oplus \tilde{E}_{d(x)}(x)$ such that

$$\tilde{A}_{\psi|\tilde{E}_i(x)} = \langle \psi, \tilde{\eta}_i(x) \rangle Id_{\tilde{E}_i(x)}$$

for all $\psi \in \nu_x(N)_{\zeta_q}$, where \tilde{A} denotes the shape operator of $(N)_{\zeta_q}$. As for Euclidean submanifolds, in an open and dense subset Ω of $(N)_{\zeta_q}$, $d(x)$ is locally constant. Moreover, \tilde{E}_i is an integrable distribution and $\tilde{\eta}_i$ is a smooth normal field. Since we are working locally, we may assume that $\Omega = (N)_{\zeta_q}$ and that $d = d(x)$ does not depend on x . In our notation $\tilde{\eta}_1, \dots, \tilde{\eta}_g$ are the above mentioned extensions of the curvature normals of the holonomy orbit, being $\tilde{E}_1, \dots, \tilde{E}_g$ their associated (autoparallel) eigendistributions. Namely, $\tilde{E}_1, \dots, \tilde{E}_g$ are the *vertical* eigendistributions of $\nu_x(N)_{\zeta_q}$, with respect to the projection $\text{pr} : \nu_x(N)_{\zeta_q} \rightarrow N$.

In general, the curvature normals $\tilde{\eta}_{g+1}, \dots, \tilde{\eta}_d$ are not ∇^\perp -parallel and the eigendistributions $\tilde{E}_{g+1}, \dots, \tilde{E}_d$ are not autoparallel. One has that $T(N)_{\zeta_q}$ decompose orthogonally as

$$T(N)_{\zeta_q} = \hat{\mathcal{V}} \oplus \mathcal{H} \quad (4.4)$$

where $\mathcal{H} = (\ker(d \text{ pr}))^\perp$ is the pr-horizontal distribution of the holonomy tube $(N)_{\zeta_q}$ and $\hat{\mathcal{V}} = \tilde{E}_1 \oplus \dots \oplus \tilde{E}_g$ is the vertical distribution.

Remark 4.2. From the tube formula [3, Lemma 3.4.7] (see (4.6)) one obtains that

$$\tilde{\mathcal{V}} \subset \tilde{E}_0 \quad (4.5)$$

where \tilde{E}_0 is the nullity distribution of $(N)_{\zeta_q}$ and $\tilde{\mathcal{V}}$ is the pr-horizontal lift of the distribution \mathcal{V} of N (in particular, $\tilde{A}_{\tilde{\zeta}}(\tilde{\mathcal{V}}) = 0$). Then

$$\mathcal{V}_{\text{pr}(x)} = d_x \text{pr}(\tilde{\mathcal{V}}_x) = (Id - \tilde{A}_{\tilde{\zeta}(x)})(\tilde{\mathcal{V}}_x) = \tilde{\mathcal{V}}_x \quad (4.6)$$

This implies that the distribution $\tilde{\mathcal{V}}$ is constant, in the ambient space $\mathbb{C}^{n,1}$, along any fibre $S(x) := \text{pr}^{-1}(\{\text{pr}(x)\})$.

Let $\psi \in \nu_{\tilde{q}}(N)_{\zeta_q}$ be generic in the sense that it is not perpendicular to some $\tilde{\eta}_i(\tilde{q}) - \tilde{\eta}_j(\tilde{q})$, $i, j \in \{0, 1, \dots, d\}$, $i \neq j$. Then ψ extends to a parallel normal field $\tilde{\psi}$ around \tilde{q} that distinguishes all the eigenvalues functions $\lambda_i(\cdot) := \langle \cdot, \tilde{\eta}_i \rangle$. However, we will be interested in some parallel normal fields $\tilde{\xi}$ that do not distinguish such eigenvalue functions. In

particular, in the case that $\ker \tilde{A}_{\tilde{\xi}}$ is bigger than \tilde{E}_0 , around a generic point where $\dim \ker \tilde{A}_{\tilde{\xi}}$ is constant and hence a distribution.

Let now $\tilde{\xi}$ be a parallel normal field of $(N)_{\zeta_q}$. Since we work locally, we may assume that $\ker \tilde{A}_{\tilde{\xi}}$ has constant dimension and so, by Codazzi identity, $\ker \tilde{A}_{\tilde{\xi}}$ is an autoparallel distribution that is invariant, due to Ricci identity, by all the shape operators of $(N)_{\zeta_q}$. Since $\tilde{\mathcal{V}} \subset \ker \tilde{A}_{\tilde{\xi}}$, this distribution is pseudo-Riemannian and the orthogonally complementary distribution $\mathcal{H}^{\tilde{\xi}} := (\ker \tilde{A}_{\tilde{\xi}})^\perp$ is Riemannian. Let us consider the equivalence relation on $(N)_{\zeta_q}$ given by $x \sim_{\tilde{\xi}} y$ if there exists a $\mathcal{H}^{\tilde{\xi}}$ -horizontal curve that connects x with y (see [3, p.224]). About a generic point the equivalence classes have all the same dimension. By means of the horosphere embedding f (see Section 2.6) every equivalence class $H^{\tilde{\xi}}(x) := [x]$ may be locally viewed as a (possible degenerate) holonomy tube around a focal Riemannian manifold. Thus, we can apply the results of Section 2.5, after replacing N by $M = f((N)_{\zeta_q})$. Observe that under these identifications $\ker \tilde{A}_{\tilde{\xi}} = \ker(Id - A_{\tilde{\xi}-\tilde{v}})$, where A is the shape operator of M and \tilde{v} is the (umbilical) position vector field. In particular, by Lemma 2.18, the tangent space to any equivalence class $\mathcal{T}_x := T_x H^{\tilde{\xi}}(x)$ is invariant under all shape operators of $(N)_{\zeta_q}$.

Before stating the next crucial result, we introduce some notation: let \mathbf{F} be the foliation of N given by the hypersurfaces obtained by the intersection of N with the family of pseudo-hyperbolic spaces $H_r^{2n,1}$ (see (3.4)). The element of \mathbf{F} that contains $q \in N$ is denoted by $F(q)$. Let $\tilde{\mathbf{F}} := \text{pr}^{-1}(\mathbf{F})$ which is a foliation of $(N)_{\zeta_q}$ by hypersurfaces. The element of $\tilde{\mathbf{F}}$ that contains x is denoted by $\tilde{F}(x)$. Then:

Main Lemma 4.3. *We are under the assumptions and notation of this section. Let U be an open subset of $(N)_{\zeta_q}$ such that for all $x \in U$ the equivalence classes $H^{\tilde{\xi}}(x)$ have the same dimension. Then, for all $x \in U$,*

- (1) $S(x) := \text{pr}^{-1}(\{\text{pr}(x)\}) \subset H^{\tilde{\xi}}(x)$ (locally).
- (2) $H^{\tilde{\xi}}(x)$ is non-degenerate.
- (3) $H^{\tilde{\xi}}(x) = \tilde{F}(x)$ (locally). In particular, the foliation $\tilde{\mathbf{F}}$ does not depend on $\tilde{\xi}$ (and its is called the canonical foliation of the holonomy tube).

Proof. Part (1) follows with exactly the same arguments, relying on the Homogeneous Slice Theorem, as those used for Euclidean submanifolds in [3, Section 7.3, p. 217]. (see also [4, p. 202]). In order to prove part (2), we will first prove that the induced metric on $H^{\tilde{\xi}}(x)$, if degenerate, is positive semi-definite with a one-dimensional degeneracy. Let us consider the foliation \mathbf{F} of N . Note that any leaf $F(p)$ of this foliation is a pseudo-Riemannian hypersurface of N with signature 1. This foliation is perpendicular to the

position vector field \vec{v} . Observe that \vec{v} lies in the vertical distribution \mathcal{V} and hence in the nullity distribution of N . Let us consider the foliation $\tilde{\mathbf{F}} = \text{pr}^{-1}(\mathbf{F})$ by pseudo-Riemannian hypersurfaces of signature 1 of $(N)_{\zeta_q}$. Let \tilde{v} be the pr-horizontal lift of \vec{v} . Then, by the tube formula [3, Lemma 3.4.7], \tilde{v} lies in the nullity distribution of $(N)_{\zeta_q}$ and hence in $\ker \tilde{A}_{\tilde{\zeta}}$. Note that $T_x \tilde{F}(x) = \tilde{v}_x^\perp$. Then $H^{\tilde{\zeta}}(x)$ lies in the leaf $\tilde{F}(x)$ of $\tilde{\mathbf{F}}$ through x , which implies our assertion, since a linear subspace of a space of signature 1 is either Lorentzian or positive semi-definite.

Let $\tilde{\nu}$ be the (autoparallel) distribution of U which is perpendicular to the distribution \mathcal{T} of tangent spaces of the equivalence classes $H^{\tilde{\zeta}}(x)$ (see Section 2.5).

Assume that $H^{\tilde{\zeta}}(x)$ is degenerate at x . Then, the intersection of $T_x H^{\tilde{\zeta}}(x) \cap \tilde{\nu}_x$ is one-dimensional. Let $\psi \neq 0$ belong to this intersection. Note that ψ is an isotropic vector, i.e. $\langle \psi, \psi \rangle = 0$. From Lemma 2.18 the distribution \mathcal{T} is invariant under all shape operators of $(N)_{\zeta_q}$ and in particular by $\tilde{A}_{\tilde{\zeta}}$. Hence, $\tilde{A}_{\tilde{\zeta}}|_{\mathcal{T}_x} = \hat{A}_{\tilde{\zeta}_x}$ where \hat{A} is the shape operator of $H^{\tilde{\zeta}}(x)$ (see Remark 2.15 for the definition of a parallel normal field to a degenerate submanifold, and its associated shape operator). By the first part of this section, $\tilde{A}_{\tilde{\zeta}}$ is diagonalizable, with real eigenvalues $\langle \tilde{\zeta}_x, (\tilde{\eta}_i)_x \rangle$, $i = 0, \dots, d$ (see the paragraph below Remark 4.1). Then, $\hat{A}_{\tilde{\zeta}_x}$ is diagonalizable with real eigenvalues. Since, from part (i), the distribution tangent to the pr-fibres, is contained in \mathcal{T} , then the 1-eigenspace of $\hat{A}_{\tilde{\zeta}_x}$ coincides with the 1-eigenspace $E_1^{\tilde{\zeta}}(x)$ of $\tilde{A}_{\tilde{\zeta}_x}$. By the last part of Remark 2.15, $\hat{A}_{\tilde{\zeta}_x} \mathbb{R}\psi \subset \mathbb{R}\psi$ and hence ψ is an eigenvector. The only non-positive definite eigenspace of $\tilde{A}_{\tilde{\zeta}_x}$ is $\ker \tilde{A}_{\tilde{\zeta}_x}$. Since ψ is isotropic, we conclude that ψ is a 0-eigenvector, i.e. $\hat{A}_{\tilde{\zeta}_x} \psi = 0$. We regard now the isotropic vector ψ as a vector perpendicular to $H^{\tilde{\zeta}}$ at x , and hence it extends to a parallel normal field $\tilde{\psi}$ of $H^{\tilde{\zeta}}$ (see Lemma 2.21). Let v belong to

$$\tilde{E}_1^{\tilde{\zeta}}(x) = \ker(Id - \tilde{A}_{\tilde{\zeta}_x}) = \ker(Id - \hat{A}_{\tilde{\zeta}_x})$$

and let $w = \hat{A}_{\tilde{\psi}_x}(v)$. Then, from Lemma 2.17, one has that

$$\hat{A}_{\tilde{\zeta}_x} w = w + \lambda(v) \tilde{\psi}_x \tag{4.7}$$

for some scalar $\lambda(v)$ (we have used that the degeneracy of the metric of $H^{\tilde{\zeta}}(x)$ has dimension 1). Observe that the subspace \mathcal{T}_x is invariant under the shape operator $\tilde{A}_{\tilde{\zeta}_x}$ of $(N)_{\zeta_q}$. Thus, $\hat{A}_{\tilde{\zeta}_x} = (\tilde{A}_{\tilde{\zeta}_x})|_{\mathcal{T}}$ diagonalizes with real different eigenvalues $\lambda_0 = 0, \lambda_1 = 1, \lambda_2, \dots, \lambda_m$. Decompose $w = w_0 + w_1 + \dots + w_m$, where w_i is an eigenvector associated with λ_i , $i = 1, \dots, m$.

Then, by equation (4.7), since $\tilde{\psi}_x$ is a 0-eigenvector, we conclude that $w = \hat{A}_{\tilde{\psi}_x}(v)$ is an 1-eigenvector of $\hat{A}_{\tilde{\zeta}_x}$. Then

$$\hat{A}_{\tilde{\psi}_x} \tilde{E}_1^{\tilde{\zeta}}(x) \subset \tilde{E}_1^{\tilde{\zeta}}(x) \quad (4.8)$$

and the same is true if one replaces x for any arbitrary nearby $y \in H^{\tilde{\zeta}}(x)$. This implies that $\tilde{\psi}$ is a parallel normal field of $S(x) = \text{pr}^{-1}(\{\text{pr}(x)\})$.

Since $S(x)$ is a totally geodesic submanifold of $(N)_{\zeta_q}$ which is invariant under all shape operators, it is contained in the affine subspace

$$y + T_y S(x) \oplus \nu_y(N)_{\zeta_q} \supset S(x) \quad (4.9)$$

for all $y \in S(x)$. The affine subspace $y + T_y S(y) \oplus \nu_y(N)_{\zeta_q}$ does not depend on $y \in S(x)$ (observe that $S(x) = S(y)$). Observe that $\tilde{\psi}$ is both perpendicular and tangent to the (degenerate) equivalence class $H^{\tilde{\zeta}}$. While the latter condition implies that it is perpendicular to $\nu(N)_{\zeta_q}$, the first condition implies that it is perpendicular to the pr-fibers $S(x)$ (see part (i)). Then $\tilde{\psi}$ is a constant field when restricted to $S(x)$, since it is a parallel normal field which is perpendicular to an affine subspace that contains $S(x)$.

Recall that $\text{pr}(y) = y + \tilde{\zeta}(y)$ (see equality (4.3)). Then

$$d_y(\text{pr})(\tilde{\psi}(y)) = (Id - \tilde{A}_{\tilde{\zeta}(y)})\tilde{\psi}(y) = \tilde{\psi}(y). \quad (4.10)$$

Since $\tilde{\psi}$ is constant along the fiber $S(x)$ we obtain that the constant field $\tilde{\psi}|_{S(x)}$ projects down to the vector $\tilde{\psi}(x) \in T_{\text{pr}(x)}N$. Observe that the union of the normal spaces of $(N)_{\zeta_q}$ at different points of $S(x)$ generates $\nu_{\text{pr}(x)}N$. Then, taking into account that $\psi = \tilde{\psi}(y)$ belongs to the nullity $\tilde{E}_0(y)$ of $(N)_{\zeta_q}$ for any $y \in S(x)$, we obtain from the tube formula that $\tilde{\psi}(y) = \psi$ belongs to the nullity subspace $\mathcal{N}_{\text{pr}(x)}$ of N . Then the vector $\psi \in T_y(N)_{\zeta_q}$ is time-like. A contradiction since ψ is isotropic. This proves (2).

(3) The inclusion $H^{\tilde{\zeta}}(x) \subset \tilde{F}(x)$ was proved inside the demonstration of part (2). The following arguments are similar to those in [4, section 2] (see also [3, chap. 7]). Let us consider the distribution $\tilde{\nu}$ perpendicular to the equivalence classes $H^{\tilde{\zeta}}(y)$ and let $\Sigma(p)$ be the totally geodesic integral manifold of $\tilde{\nu}$ by p . Since the equivalence classes are non-degenerate by part (2), the same argument used in the proof of Proposition 2 (iv) of [4] shows that the equivalence classes are parallel manifolds of the ambient space. Namely,

$$H^{\tilde{\zeta}}(x) = (H^{\tilde{\zeta}}(p))_{\mu_{(p,x)}}$$

where $\mu_{(p,x)}$ is the parallel normal field of $H^{\tilde{\zeta}}(p)$ with $\mu_{(p,x)}(p) = x - p$ ($x \in \Sigma(p)$, near p). Let $\tilde{E}_1, \dots, \tilde{E}_g$ be the autoparallel eigendistributions of $(N)_{\zeta_q}$, with associated parallel curvature normals $\tilde{\eta}_1, \dots, \tilde{\eta}_g$, determined by the isoparametric full submanifolds $S(x)$ of

the affine normal space $\text{pr}(x) + \nu_{\text{pr}(x)}N$. By part (i), the restriction of \tilde{E}_i to any $H^{\tilde{\xi}}(x)$ is tangent to this equivalence class. By the tube formula

$$\tilde{\eta}_i(x) = \frac{1}{1 - \langle (x - p), \tilde{\eta}_i(p) \rangle} \tilde{\eta}_i(p), \quad (4.11)$$

$x \in \Sigma(p)$ near p , $i = 1, \dots, g$. Since $\tilde{\eta}_i$ has constant length, we conclude that

$$\langle (x - p), \tilde{\eta}_i(p) \rangle = 0$$

or, more generally,

$$\langle \mu_{(p,x)}, \tilde{\eta}_i|_{H^{\tilde{\xi}}(p)} \rangle = 0. \quad (4.12)$$

Since the curvature normals $\tilde{\eta}_1, \dots, \tilde{\eta}_g$ associated to the pr -fibres generate, at any point, the normal space of $S(p)$, regarded as a submanifold of $\nu_{\text{pr}(p)}N$. Then, from (4.12), $\mu_{(p,x)|S(p)}$ is a constant normal field along $S(p)$, regarded as a submanifold of the full ambient space $\mathbb{C}^{n,1}$. Then, $x - p$ projects trivially to $\nu_p(N)_{\zeta_q}$, since it is spanned by $\tilde{\eta}_1(p), \dots, \tilde{\eta}_g(p)$. Observe that $x - p$ is perpendicular to $H^{\tilde{\xi}}(p)$ at p , since it is the initial condition at p of the normal field $\mu_{(p,x)}$. Since x is arbitrary in $\Sigma(p)$, we obtain that $\Sigma(p)$ is contained in the affine subspace $p + \tilde{\nu}_q = p + T_p\Sigma(p)$ and so it locally coincides with this subspace near p . This implies that $\tilde{\alpha}(\tilde{\nu}_p, \tilde{\nu}_p) = 0$, where $\tilde{\alpha}$ is the second fundamental form of $(N)_{\zeta_q}$. Since $\tilde{\nu}_p$ is invariant under all shapes operators of $(N)_{\zeta_q}$ at p , we obtain that $\tilde{\nu}_p$ is contained in the nullity of $\tilde{\alpha}$. Taking into account that the parallel normal field $\mu_{(p,x)}$ of $H^{\tilde{\xi}}(p)$ is constant along $S(p)$, one obtains that $\Sigma(p)$ is a parallel affine subspace to $\Sigma(r)$ in the full ambient space, for all $r \in S(p)$ (locally). Then $\tilde{\nu}_p = \tilde{\nu}_r$ for all $r \in S(p)$, as linear subspaces. This implies, by the tube formula and the fact that the normal spaces of $\nu_r S(p)$, $r \in S(p) \subset \text{pr}(p) + \nu_{\text{pr}(p)}N$ span $\nu_{\text{pr}(p)}N$, that $\tilde{\nu}_p$ belongs to the nullity of the second fundamental form α of N at $\text{pr}(p)$ (see [3, chap. 7.3.2]). Since the nullity of α is the distribution $y \rightarrow \mathbb{C}y$, we obtain that $\dim \tilde{\nu}_p \leq 2$ and thus, the codimension of $H^{\tilde{\xi}}(x)$ in $(N)_{\zeta_q}$ is at most 2. Then the equivalence classes $H^{\tilde{\xi}}(x)$, since $H^{\tilde{\xi}}(x) \subset \tilde{F}(x)$, locally coincide with $\tilde{F}(x)$ or have codimension 1 in $\tilde{F}(x)$. In the first case we are done. In the second case, since $S(x) \subset H^{\tilde{\xi}}(x)$, the integrable foliation $\tilde{\mathcal{T}}$, given by the tangent spaces of the equivalence classes $H^{\tilde{\xi}}(x)$, projects down to an integrable distribution $\mathcal{T} := d\text{pr}(\tilde{\mathcal{T}})$ which (locally) coincides with the distribution perpendicular to the vertical foliation $q \mapsto \mathbb{C}^*q$ of N . This contradicts Lemma 3.10. Thus, $H^{\tilde{\xi}}(x)$ coincides locally with $\tilde{F}(x)$. \square

Remark 4.4. *We keep the notation and assumptions of this section. It was proved, inside the proof of Lemma 3.4, that $v \in T_z H_r^{2n,1}$ belongs to the nullity space of N if and only if v belongs to the nullity of the second fundamental form of $F(z) = N \cap H_r^{2n,1}$ as a*

submanifold of $H_r^{2n,1}$. Since the nullity of N coincides with the distribution $y \mapsto \mathbb{C}y$, we obtain that $\mathbb{R}Jz$ coincides with the nullity of $F(z)$ as a submanifold of the umbilical submanifold $H_r^{2n,1} \subset \mathbb{C}^{n,1}$, $r = \|z\|$. If we regard $F(z)$ as a submanifold of $\mathbb{C}^{n,1}$, then we can decompose orthogonally the normal bundle into two parallel sub-bundles. Namely, $\nu F(z) = \nu_1 \oplus \nu_2$ where ν_1 is one-dimensional, spanned by the position vector field, and ν_2 is the normal bundle of $F(z)$ as a submanifold of $H_r^{2n,1}$. Then the distribution of $F(z)$ given by $x \rightarrow Jx$ is the common kernel of the family of shape operators $\{A_\mu : \nu \text{ is a section of } \nu_2\}$. Observe that the shape operator of the position vector field of $H_r^{2n,1}$ is minus the identity.

Let \tilde{v} be the pr -horizontal lift of $J\vec{v}$, where \vec{v} is the position vector field of N . Observe that \tilde{v} is time-like, and it is tangent to any $\tilde{F}(x) := \text{pr}^{-1}(F(\text{pr}(x)))$. Moreover, by making use of the tube formula, we obtain that the one-dimensional distribution $\mathbb{R}\tilde{v}|_{\tilde{F}(x)}$ is invariant under all the shape operators of $\tilde{F}(x)$. Taking into account that the one-dimensional bundle ν_1 is time-like, we obtain that the curvature normal $\tilde{\eta}_{g+1}$, associated with the distribution $\mathbb{R}\tilde{v}|_{\tilde{F}(x)}$ of $\tilde{F}(x)$, is

$$\tilde{\eta}_{g+1} = \frac{1}{r^2} \vec{u}|_{\tilde{F}(x)} \quad (4.13)$$

where \vec{u} is the horizontal lift to $(N)_{\zeta_q}$ of the position (tangent) vector field \vec{v} of N and $r^2 = -\langle \vec{v}_{\text{pr}(x)}, \vec{v}_{\text{pr}(x)} \rangle$. Note, from the definition, that $F(z) = F(z')$, if $z' \in F(z)$ and $\langle z, z \rangle = \langle z', z' \rangle$. Moreover, $\tilde{\eta}_{g+1}$ is parallel in the normal connection of $\tilde{F}(x)$, as a Lorentzian submanifold of the ambient space $\mathbb{C}^{n,1}$.

The labeling index of $\tilde{\eta}_{g+1}$ is due to the fact that in our notation $\tilde{\eta}_1, \dots, \tilde{\eta}_g$ are the parallel curvature normals associated to the vertical autoparallel distribution of $\tilde{F}(x)$ whose integral manifolds are isoparametric submanifolds of the ambient space (see part (1) of Lemma 4.3). The eigendistribution \tilde{E}_{g+1} , associated with $\tilde{\eta}_{g+1}$, could be bigger than $\mathbb{R}\tilde{v}$. In fact, it coincides with the restriction to $\tilde{F}(x)$ of the orthogonal complement of \vec{u} in \tilde{E}_0 , where \tilde{E}_0 is the nullity distribution of $(N)_{\zeta_q}$.

Lemma 4.5. *The local normal holonomy at $\text{pr}(x)$ of $F(\text{pr}(x))$, restricted to the orthogonal complement of the position vector \vec{v} , coincides with the local normal holonomy group of N at $\text{pr}(x)$.*

Proof. The proof is the same as that of Lemma 7.3.5 (i). □

5. THE GEOMETRY OF THE EQUIVALENCE CLASSES

To ensure clarity, we begin by summarizing the main results of the previous section, explaining them in some detail. Recall that we work locally, and our results—though not

always stated explicitly—hold in a neighborhood of a generic point of the submanifolds involved.

If $\tilde{\xi}$ is a parallel normal field of $(N)_{\zeta_q}$, then $H^{\tilde{\xi}}(x) \subset (N)_{\zeta_q}$ is the equivalence class of x , where $x \sim y$ if there exists a curve perpendicular to $\ker \tilde{A}_{\tilde{\xi}}$ connecting x with y . Then $H^{\tilde{\xi}}(x)$ is a hypersurface of $(N)_{\zeta_q}$ that (locally) coincides with $\tilde{F}(x) = \text{pr}^{-1}(F(\text{pr}(x)))$, where $F(\text{pr}(x)) = N \cap H_r^{2n,1}$ and $r^2 = -\langle \text{pr}(x), \text{pr}(x) \rangle$ (see Lemma 4.3). One has that $\tilde{F}(x)$ is invariant under all shape operators of $(N)_{\zeta_q}$ (see Lemma 2.18 and the paragraphs previous to Lemma 4.3). Moreover, the normal bundle of $\tilde{F}(x) \subset \mathbb{C}^{n,1}$ splits as the orthogonal sum of the following parallel and flat subbundles

$$\nu\tilde{F}(x) = \tilde{\nu}_1 \oplus \tilde{\nu}_2, \quad (5.1)$$

where $\tilde{\nu}_1 = \mathbb{R}\tilde{u}|_{\tilde{F}(x)}$, $\tilde{\nu}_2 = (\nu(N)_{\zeta_q})|_{\tilde{F}(x)}$ and \tilde{u} is the pr-horizontal lift of the position vector field \tilde{v} of N . One has that both $\tilde{F}(x)$ and its normal bundle are Lorentzian. Moreover, the commuting symmetric family of shape operators $\{\tilde{A}_\mu\}$ of $\tilde{F}(x)$ diagonalize simultaneously with real eigenvalues. In fact, the eigendistribution \tilde{E}_{g+1} associated to the parallel section $\tilde{\eta}_{g+1}$ is non-zero and contains the timelike vector \tilde{v} (see last part of Remark 4.4). Since $\tilde{F}(x)$ is Lorentzian, \tilde{E}_0 is non-degenerate and its orthogonal complement is a Riemannian distribution. This implies our assertion.

Let us finally recall that $\tilde{F}(x)$ contains the fibre $\text{pr}^{-1}(\{\text{pr}(x)\})$ (see Lemma 4.3).

Remark 5.1. *The nullity of $\tilde{F}(x)$ is trivial, as a submanifold of $\mathbb{C}^{n,1}$. In fact, let $\tilde{\zeta}$ be the parallel normal vector field of $(N)_{\zeta_q}$ such that $\text{pr}(y) = y + \tilde{\zeta}(y)$. Then, by the tube formula, see (4.10), $d\text{pr}(\tilde{u}_y) = (Id - \tilde{A}_{\tilde{\zeta}})\tilde{u}_y = \tilde{u}_y = \tilde{v}_{\text{pr}(y)}$, as vectors of the ambient space $\mathbb{C}^{n,1}$. Taking into account that the position (normal) vector field of $F(\text{pr}(y))$ is umbilical, we obtain*

$$-Id = A_{\tilde{v}(\text{pr}(y))} = \tilde{A}_{\tilde{u}_y}((Id - \tilde{A}_{\tilde{\zeta}})|_{\mathcal{H}})^{-1},$$

where \mathcal{H} is the pr-horizontal distribution of $\tilde{F}(x)$. This shows that $\tilde{A}_{\tilde{u}_y}|_{\mathcal{H}}$ has no kernel. Since the pr-fibres are irreducible isoparametric submanifolds, the family of shape operators \tilde{A}_ψ , $\psi \in (\tilde{\nu}_2)_y$, restricted to the pr-vertical distribution \mathcal{H}^\perp , have no common kernel. The previous observations imply our assertion.

Let, keeping the notation of Section 4, and Remark 5.1, $\tilde{\eta}_1, \dots, \tilde{\eta}_d$ ($d \geq g+1$) be the curvature normals of $\tilde{F}(x)$ with associated eigendistribution $\tilde{E}_1, \dots, \tilde{E}_d$, which are integrable due to the Codazzi identity (perhaps in a neighbourhood of a point close to x). Recall that $\tilde{\eta}_1, \dots, \tilde{\eta}_g$ are parallel, and $\tilde{\eta}_{g+1}$ is also parallel due to Remark 4.4. Moreover, all eigendistributions are Riemannian with the exception of \tilde{E}_{g+1} which is Lorentzian.

Asume that $\tilde{\eta}_i$ is a parallel curvature normal, then any integral manifold $S(y)$ of \tilde{E}_i is totally geodesic in $\tilde{F}(x)$. Moreover, $S_i(y)$ is an umbilical submanifold of the ambient space $\mathbb{C}^{n,1}$ which is contained in the affine space

$$y + \tilde{E}_i(y) + \mathbb{R}\tilde{\eta}_i(y).$$

(a) If $\tilde{\eta}_i(y)$ is spacelike, and so $i \neq g + 1$, then $S_i(y)$ is an open subset of the sphere of the Euclidean space $y + \tilde{E}_i(y) + \mathbb{R}\tilde{\eta}_i(y)$ with center c and radius ρ given by

$$c = y + \frac{1}{\langle \tilde{\eta}_i(y), \tilde{\eta}_i(y) \rangle} \tilde{\eta}_i, \quad \rho = \frac{1}{\sqrt{\langle \tilde{\eta}_i(y), \tilde{\eta}_i(y) \rangle}}. \quad (5.2)$$

In this case $S_i(y)$ is called a *curvature sphere*

(b) If $\tilde{\eta}_i$ is lightlike, and so $i > g + 1$, then $S_i(y)$ is a horosphere of an appropriate real hyperbolic space. In fact, there always exist a timelike $z \in \nu_y \tilde{F}(y)$ such that $\langle \tilde{\eta}_i, z \rangle = 1$. Let $-r^2 = \langle z, z \rangle$ and let

$$H_r^k = \{w \in y + \tilde{E}_i(y) \oplus \mathbb{R}\tilde{\eta}_i(y) \oplus \mathbb{R}z : \langle w - (y + z), w - (y + z) \rangle = -r^2\}^\circ$$

where $k = \dim \tilde{E}_i(y) + 1$ and $(\)^\circ$ denotes the connected component by y . Then $S_i(x)$ is an open subset of the horosphere defined by

$$\left(y + \tilde{E}_i(y) \oplus \mathbb{R}\tilde{\eta}_i(y) \right) \cap H_r^k.$$

(c) If η_i is timelike, $i \neq g + 1$, then $S_i(x)$ is an open subset of the hyperbolic space of $L_i(x)$ defined by

$$H_r^{k_i} = \{X \in x + E_i(x) \oplus \mathbb{R}\eta_i(x) : \langle X - c, X - c \rangle = -r^2\}^\circ$$

, where $-r^2 = \langle x - c, x - c \rangle$, and c has the same expression as in (a).

(d) If $i = g + 1$, there are two cases:

- $\dim \tilde{E}_{g+1} = 1$. Then, by Remark 4.4, $\tilde{E}_{g+1} = \mathbb{R}\tilde{v}$. In this case $S_i(y)$ is an open subset of the (compact) anti-circle of the negative definite affine plane $y + \tilde{E}_{g+1}(y) \oplus \mathbb{R}\tilde{\eta}_{g+1}(y)$ of center

$$c = y + \frac{1}{\langle \tilde{\eta}_{g+1}(y), \tilde{\eta}_{g+1}(y) \rangle} \tilde{\eta}_{g+1}$$

and given by the equation

$$\langle w - c, w - c \rangle = \frac{1}{\langle \tilde{\eta}_{g+1}(y), \tilde{\eta}_{g+1}(y) \rangle}.$$

- $\dim \tilde{E}_{g+1} > 1$. One gets the same formulas as in the previous case. But, instead of an anti-circle one obtains a pseudo-hyperbolic space.

5.1. The parallel focal set.

We keep the notation and assumptions of this section. In order to simplify the exposition we introduce the following notation:

$$\tilde{F} := \tilde{F}(x), \quad F := F(\text{pr}(x)).$$

We next discuss some standard facts, or definitions, that are well-known in a Euclidean ambient space and extend to our setting with straightforward modifications.

The *affine normal space* of \tilde{F} at y is the affine subspace

$$y + \nu_y \tilde{F} \subset \mathbb{C}^{n,1} \simeq \mathbb{R}^{2n,2}.$$

The *affine focal hyperplane* $\tilde{\Sigma}_j(y) \subset y + \nu_y \tilde{F}$ associated to $\tilde{\eta}_j(y)$ is

$$\tilde{\Sigma}_j(y) := y + \tilde{H}_j(y),$$

where $\tilde{H}_j(y)$ is the linear hyperplane of $\nu_y \tilde{F}$ defined by the equation $\langle \tilde{\eta}_j(y), \cdot \rangle = 1$ ($j = 1, \dots, d$). The *focal set* at y is defined by

$$\cup_{j=1}^d \tilde{\Sigma}_j(y)$$

and the *parallel focal set* at y is defined by

$$\cup_{i \in I} \tilde{\Sigma}_i(y)$$

where

$$I = \{i : \tilde{\eta}_i \text{ is parallel}, 1 \leq i \leq d\}. \quad (5.3)$$

Let ξ be a parallel normal field of \tilde{F} such that the parallel manifold \tilde{F}_ξ is not singular, i.e., $I - \tilde{A}_\xi$ is never singular (perhaps after shrinking ξ and F). Equivalently, $\langle \xi_y, \tilde{\eta}_j(y) \rangle \neq 1$, for all $y \in \tilde{F}$, $j = 1, \dots, d$. Let f be the parallel map $y \xrightarrow{f} y + \xi(y)$. Then affine normal spaces $y + \nu_y \tilde{F}$ and $f(y) + \nu_{f(y)} \tilde{F}_\xi$ do coincide. Moreover, the focal set of \tilde{F} at y coincides with the focal set of \tilde{F}_ξ at $f(y)$. In fact, this is a consequence of the tube formula that relates the shape operators of parallel manifolds (see Lemma 3.4.7 and Proposition 4.4.11 of [3]). One has that f maps the eigendistribution \tilde{E}_j of \tilde{F} into an eigendistribution $\tilde{E}_j^\xi := f_*(\tilde{E}_j)$ of \tilde{F}_ξ . Moreover, from the tube formula, the curvature normal $\tilde{\eta}_j^\xi$ associated to \tilde{E}_j^ξ is given by

$$\tilde{\eta}_j^\xi(f(p)) = \frac{1}{1 - \langle \tilde{\eta}_j(y), \xi(y) \rangle} \tilde{\eta}_j(y).$$

Observe that \tilde{E}_j is Riemannian if and only if \tilde{E}_j^ξ is Riemannian. Moreover, $\tilde{\eta}_j$ is parallel if and only if $\tilde{\eta}_j^\xi$ is parallel. Then the parallel focal set of \tilde{F} at y coincides with that of \tilde{F}_ξ at $f(y)$. Observe that f maps a curvature sphere $S_i(y)$ into a curvature sphere $S_i^\xi(f(y))$; see (5.2).

5.2. The Coxeter group.

We keep the notation and assumptions of this section.

We keep the notation and assumptions of Sections 3 and 4. In particular, we assume that the index of relative nullity of \bar{N} is zero. This implies that the normal holonomy of N acts irreducibly. Let us further assume that the normal holonomy is not transitive on the unit sphere of the normal space. This implies, in particular, that the dimension of the normal space of \tilde{F} is at least 3.

The next main tools are inspired by Terng's construction of the Coxeter group of an isoparametric submanifold [10, Section 6.3] (see also [3, Section 4.2]). We may assume, since we work locally, that \tilde{F} is simply connected and so $\nu\tilde{F}$ is globally flat. Let $y, y' \in \tilde{F}$ and let $\tau_{y,y'} : \nu_y\tilde{F} \rightarrow \nu_{y'}\tilde{F}$ be the parallel transport with respect to the normal connection. Let $\tilde{\tau}_{y,y'} : y + \nu_y\tilde{F} \rightarrow y' + \nu_{y'}\tilde{F}$ be the so-called *affine parallel transport*. Namely, $\tilde{\tau}_{y,y'}(y) = y'$ and $d_y\tilde{\tau}_{y,y'} = \tau_{y,y'}$. The affine parallel transport maps parallel focal hyperplanes into parallel focal hyperplanes. That is, for any $i \in I = \{i : \tilde{\eta}_i \text{ is parallel, } 1 \leq i \leq d\}$,

$$\tilde{\tau}_{y,y'}(\tilde{\Sigma}_i(y)) = \tilde{\Sigma}_i(y')$$

and hence the affine parallel transport maps parallel focal sets into parallel focal sets:

$$\tilde{\tau}_{y,y'}(\cup_{i \in I} \tilde{\Sigma}_i(y)) = \cup_{i \in I} \tilde{\Sigma}_i(y') \quad (5.4)$$

Let

$$\begin{aligned} I_0 &:= \{i \in I : \text{the integral manifolds of } \tilde{E}_i \text{ are curvature spheres}\} \\ &= \{i \in I : \tilde{E}_i \text{ is Riemannian and } \tilde{\eta}_i \text{ is spacelike}\} \end{aligned} \quad (5.5)$$

Equivalently, \tilde{E}_i is Riemannian, and $\tilde{\eta}_i$ is parallel and spacelike. Observe that $\{1, \dots, g\} \subset I_0$. Then

$$\tilde{\tau}_{y,y'}(\tilde{\Sigma}_{i_0}(y)) = \tilde{\Sigma}_{i_0}(y'), \text{ for any } i_0 \in I_0. \quad (5.6)$$

Let $i_0 \in I_0$ and assume that the curvature spheres $S_{i_0}(y)$, $y \in \tilde{F}$, are complete. Let, for $y \in \tilde{F}$, $\tilde{y} \in S_{i_0}(y)$ be the antipodal point of y . Then ξ_{i_0} , defined by $\xi_{i_0}(y) = \tilde{y} - y$ is a parallel normal field and $\tilde{F}_{\xi_{i_0}} = \tilde{F}$. In fact, $\xi_{i_0} = \frac{2}{\langle \tilde{\eta}_{i_0}, \tilde{\eta}_{i_0} \rangle} \tilde{\eta}_{i_0}$.

The affine parallel transport $\tilde{\tau}_{y,\tilde{y}}$ may be achieved by parallel transporting along a curve in S_{i_0} from y to \tilde{y} . It turns out that this parallel transport coincides with the reflection R_{i_0} in the focal affine hyperplane $\tilde{\Sigma}_{i_0}(y)$ (see [3, Section 4.2.2]). The affine normal spaces of $\tilde{F} = \tilde{F}_{\xi_{i_0}}$ at y and \tilde{y} coincide. Moreover,

$$\cup_{j \in J} \tilde{\Sigma}_j(y) = \cup_{j \in J} \tilde{\Sigma}_j(\tilde{y}), \quad (5.7)$$

where J is any of the following sets: $\{1, \dots, d\}$, I , I_0 . In fact, for $J = \{1, \dots, d\}$ the equality (5.7) is true since \tilde{F} is a parallel manifold to itself. For $J = I$, I_0 it is a consequence of equalities (5.4) and (5.6).

By the previous discussions we obtain that

$$R_{i_0}(\cup_{j \in J} \tilde{\Sigma}_j(y)) = \cup_{j \in J} \tilde{\Sigma}_j(y) \quad (5.8)$$

In particular,

$$R_{i_0}(\cup_{i \in I_0} \tilde{\Sigma}_i(y)) = \cup_{i \in I_0} \tilde{\Sigma}_i(y) \quad (5.9)$$

If the curvature sphere $S_{i_0}(y)$ is not complete, we can use an argument, used by Terng in the proof of Theorem 3.4 in [11], in order to extend locally \tilde{F} and so that the curvature sphere $S_{i_0}(y)$ is complete. We now sketch this argument. Let us consider the parallel focal manifold F_ξ , where $\xi = \frac{1}{\langle \tilde{\eta}_{i_0}, \tilde{\eta}_{i_0} \rangle} \tilde{\eta}_{i_0}$. Perhaps, by passing before to a nearby generic parallel manifold to \tilde{F} so that $\langle \tilde{\eta}_{i_0}, \tilde{\eta}_j \rangle = 1$ if and only if $j = i_0$ ($j = 1 \dots d$). Then consider the normal (parallel) subbundle B over \tilde{F}_ξ given by $B_z = \tilde{E}_{i_0}(y) \oplus \mathbb{R}\tilde{\eta}_{i_0}(y)$, where $\pi(y) := y + \xi(y) = z$. This subspace does not depend on y such $\pi(y) = z$. In contrast to the framework in Terng's proof, not all curvature normals are necessarily parallel. Consequently, in our context, we have to consider the complete sphere bundle SB of B of radius $\beta \langle \xi, \xi \rangle^{1/2}$, $\beta > 0$ small. The image of B under the normal exponential map of \tilde{F}_ξ is the desired extension of \tilde{B} .

By equation (4.2), $S(y) = \text{pr}^{-1}(\{\text{pr}(y)\}) \subset \text{pr}(y) + \nu_{\text{pr}(y)}N$ is an irreducible isoparametric submanifold. Then

$$\{R_{i|y+\nu_y(N)_{\zeta_q}} : 1 \leq i \leq g\}$$

generates a (finite) Coxeter group W which acts irreducibly on the affine normal space $y + \nu_y(N)_{\zeta_q}$. Moreover, $\text{pr}(y)$ is the only fixed point in such a space. Taking into account that $y + \nu_y(N)_{\zeta_q}$ is an affine hyperplane of $y + \nu_y\tilde{F}$, we obtain that

$$\cap_{i \in I_0} \tilde{\Sigma}(y) = \text{pr}(y) + \mathbb{R}\vec{u}_y$$

and hence the line

$$\text{pr}(y) + \mathbb{R}\vec{u}_y \quad (5.10)$$

is the fixed set of W acting on $y + \nu_y\tilde{F}(y)$.

Let \tilde{W} be the group of affine transformations of $y + \nu_y\tilde{F}$ generated by the set $\{R_i : i \in I_0\}$. As for the case of Euclidean reflections, \tilde{W} is a finite group. In fact, observe first that $R_i = R_i^{-1}$. If we now write $g \in \tilde{R}$ as a word of minimal length $g = R_{i_1} \dots R_{i_m}$ then all factors are different due to the commutation law $R_i R_j R_i = R_k$ for some k . Since \tilde{W} is finite, it has a fixed point p . Such a point must belong to the line $\text{pr}(y) + \mathbb{R}\vec{u}_y$, since

$W \subset \tilde{W}$ (see (5.10)). We will identify \tilde{W} with a linear group with center p . Assume that \tilde{W} acts irreducibly on $y + \nu_y \tilde{F}$. Then, since \tilde{W} is finite there exists a positive definite scalar product (\cdot, \cdot) that is invariant by \tilde{W} . By the Schur lemma, the Lorentz inner product of $y + \nu_y \tilde{F}$ is a multiple of (\cdot, \cdot) . A contradiction since $\dim \nu_y \tilde{F} \geq 2$. Thus, the action of \tilde{W} is reducible. From the fact that $W \subset \tilde{W}$ acts irreducibly on the hyperplane $y + \nu_y(N)_{\zeta_q}$ of $y + \nu_y \tilde{F}$ it follows that the unique non-trivial irreducible subspaces of \tilde{W} are $y + \nu_y(N)_{\zeta_q}$ and $\text{pr}(y) + \mathbb{R}\vec{u}_y$. Moreover, since the finite group $\tilde{W}|_{y + \nu_y(N)_{\zeta_q}}$ must have a fixed point, and the only fixed point of W is such a space is $\text{pr}(y)$, we conclude that $\text{pr}(y)$ is a fixed point of \tilde{W} . Then the parallel normal vector field $y \xrightarrow{\tilde{\zeta}} y - \text{pr}(y)$ satisfies that

$$\ker(Id - \tilde{A}_{\tilde{\zeta}(y)}) \supset \oplus_{i \in I_0} \tilde{E}_i(y).$$

On the other hand, since $\tilde{F}_{\tilde{\zeta}} = F$, it follows that

$$\ker(Id - \tilde{A}_{\tilde{\zeta}(y)}) = \oplus_{i=1}^g \tilde{E}_i(y)$$

(see (4.3)). Then $\tilde{W} = W$ and

$$I_0 = \{1, \dots, g\}. \quad (5.11)$$

Corollary 5.2. *The only curvature spheres of \tilde{F} are the vertical ones associated to the isoparametric fibers of $\tilde{F} \xrightarrow{\text{pr}} F$. \square*

The above corollary together with the following lemma will be crucial for our purposes.

Lemma 5.3. *Not all curvature normals of \tilde{F} are parallel.*

Proof. Assume that all curvature normals are parallel. Observe, keeping the notation of previous sections, that there should exist $i > g + 1$ such that $\tilde{\eta}_i$ is not a scalar multiple of $\tilde{\eta}_{g+1}$. If not, $\ker \tilde{A}_{\tilde{\zeta}}$ and $\ker(Id - \tilde{A}_{\tilde{\zeta}})$ would be two (orthogonally) complementary non-degenerate totally geodesic distributions which are invariant by all shape operators. The affine subspace generated by any integral manifold of $\ker(Id - \tilde{A}_{\tilde{\zeta}})$ is Euclidean and hence non degenerate. Then, by Moore's lemma (see e.g. [12, Lemma 2] and [3, Corollary 1.7.4]), \tilde{F} locally splits and hence F locally splits. The flat part $\nu_0 F$ of normal bundle νF has dimension 1, and locally $N = \cup_{\xi} F_{\xi}$, where ξ is a small parallel section of $\nu_0 F$. Then N locally splits which is a contradiction, since the normal holonomy of N acts irreducibly (see the beginning of this section).

Let $\tilde{\eta}_i$, $i > g$ be such that it is not a scalar multiple of \vec{u} . Then $i \neq g + 1$ and thus, \tilde{E}_i is Riemannian (see the paragraph below Remark 5.1). If $\tilde{\eta}_i$ is spacelike then the integral manifolds of \tilde{E}_i are curvature spheres (see (5.5)). This contradicts Corollary 5.2. Then $\tilde{\eta}_i = \lambda \vec{u} + \tilde{\mu}$, where $\lambda \neq 0$ and $\tilde{\mu} \neq 0$ is the projection of $\tilde{\eta}_i$ to the parallel normal

subbundle $\tilde{\nu}_2 = \tilde{u}^\perp = \nu(N)_{\zeta_q|_{\tilde{F}}}$ (see (5.1)). Let $j \in \{1, \dots, g\}$ and let $\tilde{\psi}$ be a parallel section of $\tilde{\nu}_2$ such that $\langle \tilde{\psi}, \tilde{\eta}_j \rangle = 1 = \langle \tilde{\psi}, \tilde{\mu} \rangle$. In fact define $\tilde{\psi}$ as an appropriate scalar multiple of a non-zero parallel section of the bundle $\mathbb{R}\tilde{\eta}_j \oplus \mathbb{R}\tilde{\mu}$ which is perpendicular to $\tilde{\eta}_j - \tilde{\mu}$. Then $\tilde{E} := \ker(Id - \tilde{A}_{\tilde{\psi}})$, which is a direct sum of eigendistributions, contains $\tilde{E}_i \oplus \tilde{E}_j$ and does not contain \tilde{E}_{g+1} . Then any integral manifold $\tilde{S}(y)$ of the autoparallel distribution \tilde{E} is a Riemannian isoparametric submanifold of the affine Lorentzian subspace $y + \tilde{E}(y) \oplus \nu_y \tilde{F}$. Since $\tilde{\eta}_j$ is spacelike and η_i is not so, then $\langle \tilde{\eta}_i, \tilde{\eta}_j \rangle = 0$ by Proposition 2.4. Since $\{\eta_j : 1 \leq j \leq g\}$ generates $\tilde{\nu}_2$, then η_i lies in $\tilde{\nu}_1 = \mathbb{R}\tilde{u}$. A contradiction that implies the lemma. □

Proof of Theorem 1.1. By Corollary 3.9, the normal holonomy group of N acts irreducibly on the normal space. Let N be the lift of \bar{N} to $\mathbb{C}^{n,1}$, and let $\tilde{\xi}$ be a parallel normal field to N such that $\ker A_{\tilde{\xi}}$ has constant dimension. Then, according to Lemma 4.3 (3), $H^{\tilde{\xi}}(x) = \tilde{F}(x)$, and so it does not depend on $\tilde{\xi}$. By Lemma 2.18, $T_y \tilde{F}(x)$ is invariant under all shape operators of $(N)_{\zeta_q}$. Moreover, $\ker A_{\tilde{\xi}|_{TF(x)}}$ is invariant under all the shape operators of $(N)_{\zeta_q}$. Let $S^{\tilde{\xi}}(y)$ be the total geodesic and non-degenerate integral manifold of $\ker A_{\tilde{\xi}|_{TF(x)}}$ by $y \in \tilde{F}(x)$. Then $S^{\tilde{\xi}}(y)$ is (locally) the orbit of a weakly polar action. Namely, via the horosphere embedding, it coincides with the normal holonomy orbit of an appropriate focal manifold. Since $\tilde{F}(x)$ has flat normal bundle, and its family of shape operators are simultaneously diagonalizable, with real eigenvalues, the same is true for $S^{\tilde{\xi}}(y)$. Moreover, since $S^{\tilde{\xi}}(y)$ is the orbit of a weakly polar action, it follows that the curvature normals of $S^{\tilde{\xi}}(y)$ are parallel in the normal connection. As for the curvature normals associated to the fibers of (N_{ζ_q}) , any curvature normal of $S^{\tilde{\xi}}(y)$ extends to a parallel curvature normal of $\tilde{F}(x)$ (see [3, sec.7.1]). If the normal holonomy group of N is not transitive we can find, as in [3, sect. 7.4], parallel normal fields $\tilde{\xi}, \tilde{\xi}'$ of $\tilde{F}(x)$ such that $\ker A_{\tilde{\xi}} + \ker A_{\tilde{\xi}'}$ contains the horizontal distribution of $\tilde{F}(x)$. Then, Proposition 7.36 of [3] applies with the same proof to show that any curvature normal of $\tilde{F}(x)$ is parallel in the normal connection. This contradicts Lemma 5.3, proving that the normal holonomy must be transitive. □

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