UNEXPECTED PHENOMENA IN A ONE-DIMENSIONAL ELLIPTIC EQUATION WITH A SINGULAR FIRST ORDER DIVERGENCE TERM

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Abstract. In this paper we study the possible solutions u of the one-dimensional non-linear singular problem which formally reads as

$$\text{(S)} \qquad \begin{cases} -\dfrac{d}{dx}\left(a(x)\dfrac{du}{dx}\right) = -\dfrac{d\phi(u)}{dx} - \dfrac{dg(x)}{dx} & \text{in } (0,L), \\ u(0) = u(L) = 0\,, \end{cases}$$

where $L > 0$, and where the data $(a,\, g,\, \phi)$ are as follow: a is a function of $L^\infty(0,L)$ which is bounded between two positive constants, g is a function of $L^2(0,L)$, and the singular function $\phi : \mathbb{R} \mapsto \mathbb{R}\cup\{+\infty\}$ is continuous as a function with values in $\mathbb{R}\cup\{+\infty\}$, and satisfies $\phi(0) = +\infty$ and $\phi(s) < +\infty$ for every $s \in \mathbb{R}$, $s \neq 0$; the model example for the singular function ϕ is $\phi_\gamma(s) = |s|^{-\gamma}$ with $\gamma > 0$.

We first study the behaviour of the solutions of approximating problems (S_n) involving non-singular functions ϕ_n which converge to ϕ in a sense that we specify, and we prove that these solutions have subsequences which either converge to weak solutions of (S) (for a definition of weak solutions that we specify), or converge to zero. We then prove that for a large class of data $(a,\, g,\, \phi)$ it does not exist any weak solution of (S), while for another large class of data $(a,\, g,\, \phi)$ it does exist at least one weak solution of (S).

Thanks to the study of an associated singular ODE (this study is of independent interest), we prove that under additional assumptions which are satisfied by the model example $\phi_\gamma(s) = |s|^{-\gamma}$ when $0 < \gamma < 1$, if for some data $(a,\, g,\, \phi)$ there exists one weak solution of (S), then for the same data it also exist infinitely many weak solutions of (S) which are parametrized by $c \in (-\infty, c^*]$ for some finite c^*.

We finally prove that for any given data $(a,\, g,\, \phi)$ and for any weak solution u of (S) corresponding to these data, there exist sequences of data $(a,\, g_n,\, \phi_n)$, with non-singular functions ϕ_n which converge to $(a,\, g,\, \phi)$, for which the solutions converge to u, while there also exist other sequences of data $(a,\, g_n,\, \phi_n)$, with non-singular functions ϕ_n which converge to $(a,\, g,\, \phi)$, for which the solutions converge to zero.

Most of these results are unexpected.

Contents

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1. Introduction

Setting of the problem

In the present paper we deal with a singular one-dimensional problem. Our main aim consists in trying to find a function u which formally satisfies

$$\begin{cases} -\dfrac{d}{dx}\left(a(x)\dfrac{du}{dx}\right) = -\dfrac{d\phi(u)}{dx} - \dfrac{dg}{dx} \quad \text{in } (0,L), \\ u(0) = u(L) = 0\,. \end{cases} \tag{1.1}$$

We will soon give a mathematically correct and natural definition of solutions of this problem, but let us first give the assumptions on its data.

We will assume that $L > 0$ and that the data $(a,\, g,\, \phi)$ satisfy

$$a \in L^\infty(0,L)\,, \quad \exists\ \alpha > 0:\ a(x) \geq \alpha\,, \quad \text{a.e. in} \quad (0,L)\,; \tag{1.2}$$

$$g \in L^2(0,L)\,; \tag{1.3}$$

$$\begin{cases} \phi : \mathbb{R} \mapsto \mathbb{R}\cup\{+\infty\},\ \phi \text{ is continuous with values in } \mathbb{R}\cup\{+\infty\}, \\ \phi(s) < +\infty \quad \forall s \in \mathbb{R}, \quad s \neq 0\,. \end{cases} \tag{1.4}$$

Our main purpose is to study the case where

$$\phi(0) = +\infty \tag{1.5}$$

even though some results will be true (and new) also in the case where $\phi(0) < +\infty$.

The model case for the function ϕ is

$$\phi_\gamma(s) = \frac{c}{|s|^\gamma} + \varphi(s), \text{ with } c > 0,\ \gamma > 0,\ \varphi \in C^0(\mathbb{R})\,.$$

Assuming $\phi(0) = -\infty$ in place of $\phi(0) = +\infty$ is just a variant of problem (1.1) by a simple change of variable (see Remark 2.3 below) and we will not treat that case.

Definition of a weak solution

We introduce the following definition of a weak solution of problem (1.1) (for more details see Definition 2.6 and Subsection 2.2 below).

Definition 1.1. We say that u is a *weak solution* of problem (1.1) if u satisfies

$$\begin{cases} u \in H_0^1(0,L)\,, \qquad \phi(u) \in L^2(0,L), \\ \\ -\dfrac{d}{dx}\left(a(x)\dfrac{du}{dx}\right) = -\dfrac{d\phi(u)}{dx} - \dfrac{dg}{dx} \quad \text{in } \mathcal{D}'(0,L)\,. \end{cases} \tag{1.6}$$

□

Note that (1.6) holds true if and only if u solves (see Proposition 2.13 below)

$$\begin{cases} u \in H_0^1(0,L) \quad \phi(u) \in L^2(0,L), \\ a(x)\dfrac{du}{dx} = \phi(u) + g + c \ \text{ in } \ \mathcal{D}'(0,L), \end{cases} \tag{1.7}$$

for

$$c = -\frac{\displaystyle\int_0^L \frac{\phi(u)}{a(x)}dx + \int_0^L \frac{g(x)}{a(x)}dx}{\displaystyle\int_0^L \frac{1}{a(x)}dx}. \tag{1.8}$$

Three other classes of singular problems and some bibliographical references

A first class of singular problems is as follows:

If one has $\phi \in C^0(\mathbb{R})$ and $N \geq 1$, it is proved in [7] and [8] that there exists a *renormalized solution* of the N-dimensional problem which formally reads as

$$\begin{cases} -\text{div}(A(x)\nabla u) = -\text{div}(\phi(u)) - \text{div}g & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where Ω is a bounded domain of $\mathbb{R}^N$ and $A(x)$ is a bounded matrix satisfying, for some $\alpha > 0$,

$$A(x)\xi \cdot \xi \geq \alpha|\xi|^2, \quad \text{a.e. } \ x \in \Omega, \quad \forall\ \xi \in \mathbb{R}^N.$$

These solutions turn out to be also weak solutions if the growth at infinity of $\phi(s)$ is sufficiently low.

In the special case $N = 1$ with $\phi \in C^0(\mathbb{R})$ this notion coincides with the notion of classical weak solution:

$$\begin{cases} u \in H_0^1(0,L), \\ \displaystyle\int_0^L a(x)\frac{du}{dx}\frac{dz}{dx}\,dx = \int_0^L \phi(u)\frac{dz}{dx}\,dx + \int_0^L g(x)\frac{dz}{dx}\,dx \quad \forall z \in H_0^1(0,L), \end{cases}$$

and there exists a classical weak solution of this problem (see Proposition 2.15 below).

A second class of singular problems is the class of semilinear problems involving a zeroth-order term $h(x,u) \geq 0$ which is singular in $u = 0$, for which one looks for a nonnegative solution u of

$$\begin{cases} -\text{div}(A(x)\nabla u) = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

For this class, let us quote several important papers. First, the cases $h(x,s) = f(x)e^{1/s}$ or $h(x,s) = \dfrac{f(x)}{s^\gamma}$ for a regular function $f(x)$ are treated in [15] where the authors prove the existence of a classical solution when $A(x)$ is the identity matrix. In [13, 30] similar results are proved for a regular matrix $A(x)$ and a regular function $h(x,s)$ uniformly bounded for $s > 1$ with $\lim_{s\to 0} h(x,s) = +\infty$ uniformly for $x \in \overline{\Omega}$. Moreover, continuity properties of the solution are proved in [13] if $h(x,s)$ does not depend on x.

The case where the nonlinearity is of the form $h(x,s) = \dfrac{f(x)}{s^\gamma}$ with $f(x)$ a positive Hölder continuous function in $\overline{\Omega}$ is studied in [25] where it is proved that problem (1.10) has a classical non-negative solution which does not always belong to $H_0^1(\Omega)$. More precisely the authors prove in [25] that the solution belongs to $H_0^1(\Omega)$ if and only if $\gamma < 3$. Furthermore, they demonstrate that for $\gamma > 1$ the solution is not in $C^1(\overline{\Omega})$. In the case $h(x,s) = f(x)\tilde{h}(s)$, some extensions may be found, among others, in [23, 24] for $\Omega = \mathbb{R}^N$ and in [31] for bounded domains. In the latest case $f(x)$ may also be singular at the boundary of Ω.

Let us highlight the paper [9], in which the authors extensively study the semi-linear problem in the case $h(x,s) = \dfrac{f(x)}{s^\gamma}$ with $f \geq 0$, $f \in L^m(\Omega)$ for $m \geq 1$, and prove existence results depending on γ and on m. For $\gamma = 1$ and $f \in L^1(\Omega)$, they prove the existence of a solution belonging to $H_0^1(\Omega)$. They also prove a similar result when $f \in L^m(\Omega)$ with $m \geq C(N,\gamma) > 1$. Finally, for the case $\gamma > 1$ and $f \in L^1(\Omega)$ they prove the existence of a solution u belonging to $H^1_{\text{loc}}(\Omega)$ satisfying $u^{\frac{\gamma+1}{2}} \in H_0^1(\Omega)$.

In [28] (see also [1]), the authors prove the existence of a solution in $H_0^1(\Omega)$ if $f \in L^m(\Omega)$, f positive, $1 < \gamma < 3 - \frac{2}{m}$. These results are optimal for $f \in L^\infty(\Omega)$, and they fit with the result of [25], i.e. $u \in H_0^1(\Omega)$ for all $\gamma < 3$ if one formally takes $m = +\infty$.

In [16, 17, 18, 19, 20] the authors introduce, in the case of strong singularities (which in the model case corresponds to $\gamma > 1$), a new definition of the solution, with a space for the solution which is unconventional, and test functions which are reminiscent of the notion of solution defined by transposition. In this framework they prove results of existence, stability and uniqueness. They also prove [17, 19] results of homogenization in this framework. Another step in this direction is [12] where the right-hand side of the equation can change sign. Other homogenization results can be found in [6] and [14].

Let us also point out that the cases $h(x,s) = \dfrac{f(x)}{s^\gamma} + \mu$ and $h(x,s) = \mu\tilde{h}(s)$ with μ a non-negative Radon measure have been studied in [27]. Moreover, the case of a variable exponent , i.e. $h(x,s) = \dfrac{f(x)}{s^{\gamma(x)}}$ is considered in [11]. For more details and a gentle introduction to singular elliptic problems we refer to the recent survey [29].

The third class of singular problems concerns the case of singularities which appear in first order terms with natural growth in the gradient, which has also been extensively studied. For a far to be complete account on these problems see [5, 26, 2, 3, 4], and references therein.

Originality of problem (1.1)

Let us emphasize some features which are specific to problem (1.1). Both in the singular case (i.e. $\phi(0) = +\infty$) and in the non-singular case (i.e. $\phi \in C^0(\mathbb{R})$), for any (possible) weak solution u of (1.1) one has $\displaystyle\int_0^L \phi(u)\frac{du}{dx}\,dx = 0$, since

$$\text{if } z \in H_0^1(0,L) \text{ with } \phi(z) \in L^2(0,L), \text{ then } \int_0^L \phi(z)\frac{dz}{dx}\,dx = 0,$$

(see Lemma 2.12 below); here the hypothesis $\phi(z) \in L^2(0,L)$ is essential. This implies that every weak solution u of (1.1) defined above by (1.6) satisfies the following important a priori estimate

$$\left\|\frac{du}{dx}\right\|_{L^2(0,L)} \le \frac{1}{\alpha}\|g\|_{L^2(0,L)}, \tag{1.11}$$

and, by Morrey's embedding (here $N = 1$ is crucial)

$$\|u\|_{L^\infty(0,L)} \le \frac{\sqrt{L}}{\alpha}\|g\|_{L^2(0,L)}\,. \tag{1.12}$$

In order to (try to) prove the existence of a weak solution of problem (1.1), we proceed as usual by approximation.

We consider sequences of a_n, g_n and ϕ_n which satisfy

$$\begin{cases} a_n \in L^\infty(0,L),\ \alpha \le a_n \le \beta \text{ for some fixed constant } \beta, \\ a_n(x) \to a(x) \text{ a.e. } x \in (0,L), \end{cases} \tag{1.13}$$

$$g_n \in L^2(0,L),\ g_n \rightharpoonup g \text{ weakly in } L^2(0,L), \tag{1.14}$$

$$\phi_n \in C^0(\mathbb{R}) \quad \text{for every} \quad n \in \mathbb{N}, \tag{1.15}$$

$$\text{if } \ s_n \to s \text{ in } \mathbb{R} \ \text{ then } \phi_n(s_n) \to \phi(s) \text{ in } \mathbb{R}\cup\{+\infty\}; \tag{1.16}$$

the latest property (1.16) is equivalent (see Proposition 2.18 and Remark 2.19 below) to say that the sequence ϕ_n locally uniformly converges to ϕ, even in the case in which $\phi(0) = +\infty$.

Examples of such *reasonable approximations* of ϕ are the truncations (i.e. $\phi_n(s) = T_n(\phi(s))$) and the homographic approximations (i.e. $\phi_n(s) = \dfrac{\phi(s)}{1 + \frac{1}{n}\phi(s)}$).

For (a_n, g_n, ϕ_n) satisfying (1.13)–(1.16) it is quite easy to show that, for every $n \in \mathbb{N}$, there exists at least one function u_n which satisfies

$$(1.17)\qquad \begin{cases} u_n \in H_0^1(0,L), \\ \displaystyle\int_0^L a_n(x)\frac{du_n}{dx}\frac{dz}{dx}\,dx = \int_0^L \phi_n(u_n)\frac{dz}{dx}\,dx + \int_0^L g_n\frac{dz}{dx}\,dx \quad \forall z \in H_0^1(0,L). \end{cases}$$

Therefore, due to estimate (1.11), there exists $u \in H_0^1(0,L)$ such that, for a subsequence

$$(1.18)\qquad \begin{cases} u_n \rightharpoonup u \ \text{ weakly in } \ H_0^1(0,L), \\ u_n(x) \to u(x) \ \text{ a.e. } \ x \in (0,L). \end{cases}$$

An alternative

Our first main result (see Theorem 3.4 below) consists in the following alternative for the weak limit u: if $\phi(0) = +\infty$, then

- either $u \equiv 0$,
- or u is a weak solution of problem (1.1) in the sense of Definition 1.1.

In the proof of this alternative we use a new estimate on the sequence $\phi_n(u_n)$ in $L^2(0,L)$. This estimate, which is specific to the one-dimensional setting, can only be obtained when the sequence u_n weakly converges to a function u which is not identically zero.

Note that the result is a true alternative, in the sense that the two possible situations are mutually exclusive since 0 is not a weak solution of (1.1) in the sense of Definition 1.1 when $\phi(0) = +\infty$.

Moreover the two cases of the alternative effectively happen depending on the data g and ϕ. Indeed on the first hand there exists a large class of functions $g \in L^2(0,L)$ (see Theorem 4.1 below), and a large class of functions ϕ which satisfy (1.4) and (1.5) (see Theorem 4.4 below), such that the limit of the sequence u_n is always $u \equiv 0$. On the other hand there exists (see Section 7 below) a large class of data g and ϕ such that (1.1) has a weak solution, and for every weak solution, there exists (see Proposition 8.1 in Section 8 below) sequences of approximations of the data which have solutions which converge to this weak solution.

Let us emphasize that hypothesis $\phi(0) = +\infty$ is essential in order to get the alternative. When $\phi(0) < +\infty$, then for any sequence ϕ_n of approximations of ϕ, one has $\phi_n(u_n)$ bounded in $L^\infty(0,L)$ and $\phi_n(u_n)$ converges strongly in $L^2(0,L)$ to $\phi(u)$. Moreover, in this setting, the case $u \equiv 0$ occurs if and only if $g \equiv c$ for some $c \in \mathbb{R}$.

The results of Theorem 4.1 and Theorem 4.4 proved in Section 4 are in fact results of non-existence of weak solutions of problem (1.1) in the sense of Definition 1.1.

The first one, Theorem 4.1, states that when the datum g is bounded from below, there exists no weak solution of problem (1.1) in the sense of Definition 1.1. This is essentially due to the fact that in this case the datum g cannot compensate the singular behaviour of the function $\phi(u(x))$ when $u(x)$ tends to zero.

The second one, Theorem 4.4, states that when the function $\phi(s)$ is not integrable in $s = 0^+$ and in $s = 0^-$, there exists no weak solution of problem (1.1) in the sense of Definition 1.1. This is proved by showing that the strong behaviour of the singular function $\phi(s)$ near $s = 0$ implies that the class of functions $u \in H_0^1(0,L)$ with $\phi(u) \in L^2(0,L)$ is empty.

An associated ODE

If we look at (1.7)-(1.8), which is an equivalent formulation of the definition of a weak solution of problem (1.1) in the sense of Definition 1.1, one is naturally led to consider, for $h \in L^2(0,L)$, the ODE

$$(1.19)\qquad \begin{cases} v \in H^1(0,L), \quad \phi(v) \in L^2(0,L), \\ a(x)\dfrac{dv}{dx} = \phi(v) + h \ \text{ in } \ \mathcal{D}'(0,L), \\ v(0) = 0, \end{cases}$$

which is a singular ODE when $\phi(0) = +\infty$.

This problem is clearly related to our problem (1.1). Indeed, if $h = g + c$, any solution u of this ODE is a weak solution of (1.1) when c is given by (1.8). Conversely, any solution v of the ODE (1.19) is a weak solution of problem (1.6) when v satisfies $v(L) = 0$.

In Section 5 below we study the ODE (1.19). This is original when $\phi(0) = +\infty$. We obtain existence, non-negativity, comparison, and uniqueness results for the solution v of (1.19), under further assumptions on ϕ and on g; note that these assumptions on ϕ are satisfied in the model case $\phi_\gamma(s) = \frac{c}{|s|^\gamma} + \varphi(s)$ when $c > 0$, $0 < \gamma < 1$, and $\varphi \in C_b^0(\mathbb{R})$ with $\phi_\gamma(s)$ non-increasing for $s > 0$.

In order to get existence of solutions v to (1.19), we use the integrability of the function ϕ at $s = 0$ and its boundedness at infinity (see Theorem 5.5 below), while for the comparison and uniqueness results (see Proposition 5.16 below) we further require that $\phi(s)$ is monotone non-increasing for $s > 0$.

A synthesis of the results that we prove for the ODE (1.19) is given in Subsection 5.3 below. These results are new and, in our opinion, of independent interest, because of the singular behaviour of the function $\phi(s)$ in $s = 0$.

These results will then be strongly used for proving the existence of weak solutions of problem (1.1) (see Section 7 below) as well as the multiplicity result that we describe now.

A multiplicity result for the solutions of the singular problem (1.1)

Another interesting consequence of these results is indeed a multiplicity result for the weak solutions of problem (1.1) in the sense of Definition 1.1 (see Section 6 below). This multiplicity result, which is stated in Theorem 6.1 below, is quite unexpected: it says that, whenever a solution of problem (1.1) in the sense of Definition 1.1 exists for some given data $(a,\, g,\, \phi)$, then infinitely many solutions exist for the same data. These solutions are indexed by a real parameter c which varies in an interval $(-\infty, c^*]$ where c^* is finite. These infinitely many solutions are strictly ordered with respect to c and any possible solution of problem (1.1) in the sense of Definition 1.1 for these data correspond to some $c \in (-\infty, c^*]$.

Stability and instability of the approximations

We also show, in Section 8 below, two quite remarkable results concerning the stability of the weak solutions of (1.1) for the data $(a,\, g_n,\, \phi_n)$ when g_n converges to g in $L^2(0, L)$ and when ϕ_n is a reasonable sequence of approximations of ϕ. We prove in particular that for any weak solution u of problem (1.1) in the sense of Definition 1.1, approximations $(a,\, g_n,\, \phi_n)$ can be built for which the solutions u_n of the approximating problems converge to u. In other terms this result asserts that a weak solution of problem (1.1) in the sense of Definition 1.1 is never isolated. But in contrast with this result, we also show that for any weak solution u of problem (1.1) in the sense of Definition 1.1 one can build approximations $(a,\, g_n,\, \phi_n)$ for which the solutions u_n of the approximating problems converge to 0. This can be viewed as a strong instability result. Let us stress that in both results, the approximating sequence ϕ_n can be any reasonable sequence of approximations of ϕ, while the sequence g_n should be chosen accordingly.

Existence of solutions of problem (1.1)

Finally, Section 7 below is devoted to produce explicit large classes of data for which solutions of (1.1) in the sense of Definition 1.1 do exist.

These results are essentially straightforward consequences of the fact that for any given ϕ satisfying (1.4) and (1.5) as well as the fact that ϕ is integrable in 0, one can construct a large class of functions u such that $u \in H_0^1(0, L)$ with $\phi(u) \in L^2(0, L)$. Defining then $g \in L^2(0, L)$ by $g(x) = a(x)\dfrac{du}{dx} - \phi(u)$, one has built data such that problem (1.1) admits u as a weak solution.

Another result, Theorem 7.5 below, is in some sense a "density result": it asserts that for any ϕ satisfying (1.4) and (1.5) as well as the fact that ϕ is integrable in 0, for any $g \in L^2(0, L)$, and for any $\delta > 0$, one can construct a function $\hat{g}_\delta \in L^2(0, L)$ such that $\hat{g}_\delta \equiv g$ on $(0, L - \delta)$ for which the problem (1.1) has a weak solution in the sense of Definition 1.1 for the data $(a,\, \hat{g}_\delta,\, \phi)$.

Let us note that most of the results that we obtain and prove in this article, even if simple and obtained through elementary proofs, are new.

In any case, they are unexpected.

Concluding remarks and comments

Much to our regrets, our results are confined to the one-dimensional setting, since, as said before, in this case we are able to find, when $u \neq 0$, a new estimate on $\phi_n(u_n)$ in $L^2(0,L)$, which is no more the case in the N-dimensional setting for $N > 1$. The only N-dimensional situations that we were able to face are special forms of equations which can be solved by separation of variables, and the case of radial solutions under special (radial) assumptions on the data. We will publish in [21] these partial results along with some variants of problem (1.1), namely

- the case where a zeroth-order term $+b(x)u$ with $b(x) \geq 0$ (or more generally $+b(x,u)$ with $b(x,s)s \geq 0$) is added to the left hand side of (1.1);
- the case where in (1.1) the linear operator $-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)$ is replaced by a nonlinear monotone operator $-\frac{d}{dx}a(x,\frac{du}{dx})$ (or even by a nonlinear pseudo-monotone operator $-\frac{d}{dx}a(x,u,\frac{du}{dx})$) where $a : (x,\xi) \in (0,L) \times \mathbb{R} \to a(x,\xi) \in \mathbb{R}$ is a Carathéodory function that satisfies, for some p, α, β, $b(x)$ with $1 < p < +\infty$, $\alpha > 0$, $\beta \geq \alpha$, $b \in L^p(0,L)$, the classical monotonicity properties

$$\begin{cases} a(x,\xi)\xi \geq \alpha|\xi|^p, \quad |a(x,\xi)| \leq \beta(|\xi|^{p-1} + |b(x)|^{p-1}), \\ (a(x,\xi) - a(x,\eta))(\xi - \eta) > 0, \quad \text{a.e. } x \in (0,L),\ \forall \xi \in \mathbb{R},\ \forall \eta \in \mathbb{R},\ \xi \neq \eta. \end{cases}$$

In a second paper ([22]) we will treat the problem where the model singular function $\phi_\gamma(s) = \frac{1}{|s|^\gamma}$ is replaced by a function that is singular in $s = m$ with $m \neq 0$ whose model is

$$\phi_\gamma^m(s) = \frac{1}{|s-m|^\gamma}, \quad \text{with } \gamma > 0,\ m \neq 0.$$

Notation

In the present paper we will use classical notations. Here we just recall and precise some of them.

We denote by $C^0(\mathbb{R})$ the space of functions $\phi : \mathbb{R} \mapsto \mathbb{R}$ which are continuous at each point of $\mathbb{R}$. Observe that a function $\phi : \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\}$ which is continuous (see an example in (2.10) below) does not belong to $C^0(\mathbb{R})$ when $\phi(s_0) = +\infty$ for some $s_0 \in \mathbb{R}$.

We denote by $C_b^0(\mathbb{R})$ the space of functions of $C^0(\mathbb{R})$ which are bounded on $\mathbb{R}$, namely $C_b^0(\mathbb{R}) = C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

We denote by $\mathcal{D}'(0,L)$ the space of distributions on the open interval $(0,L)$, namely the dual of the space $C_c^\infty(0,L)$ of functions which have derivatives of any order and which have a compact support in $(0,L)$.

We denote by $\mathrm{Lip}(\mathbb{R})$ the space of the Lipschitz continuous functions on $\mathbb{R}$, namely the space of those functions $\psi \in C^0(\mathbb{R})$ such that

$$\|\psi\|_{\mathrm{Lip}(\mathbb{R})} = \sup_{s,t\in\mathbb{R},\, s\neq t} \frac{|\psi(s) - \psi(t)|}{|s-t|} < +\infty.$$

We denote by $H_0^1(0,L)$ the Sobolev space of those functions $z \in L^2(0,L)$ whose distributional first derivative $\dfrac{dz}{dx}$ belongs to $L^2(0,L)$ and which satisfy $z(0) = z(L) = 0$. The space $H_0^1(0,L)$ will be equipped with the norm

$$\|z\|_{H_0^1(0,L)} = \left\|\frac{dz}{dx}\right\|_{L^2(0,L)}, \quad \forall\, z \in H_0^1(0,L), \tag{1.20}$$

since the Poincaré inequality asserts that

$$\|z\|_{L^2(0,L)} \leq L\left\|\frac{dz}{dx}\right\|_{L^2(0,L)}, \quad \forall z \in H_0^1(0,L). \tag{1.21}$$

Recall that the Morrey's embedding and estimate, which are specific to the one-dimensional case, assert that

$$H_0^1(0,L) \subset L^\infty(0,L) \quad \text{with} \quad \|z\|_{L^\infty(0,L)} \leq \sqrt{L}\left\|\frac{dz}{dx}\right\|_{L^2(0,L)}, \ \forall z \in H_0^1(0,L), \tag{1.22}$$

and also that

$$\begin{cases} H^1(0,L) \subset C^{0,\frac{1}{2}}([0,L]), \ \text{with} \\ \|z\|_{C^{0,\frac{1}{2}}([0,L])} = \sup\limits_{\substack{x,y\in[0,L]\\ x\neq y}} \dfrac{|z(x)-z(y)|}{\sqrt{|x-y|}} \leq \left\|\dfrac{dz}{dx}\right\|_{L^2(0,L)}, \quad \forall z \in H^1(0,L). \end{cases} \tag{1.23}$$

Finally, for $k \in \mathbb{R}^+$, let us denote by $T_k : \mathbb{R} \mapsto \mathbb{R}$ the truncation function at height k, i.e. the function given by

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ k\dfrac{s}{|s|} & \text{if } |s| \ge k, \end{cases} \quad \forall s \in \mathbb{R}. \tag{1.24}$$

2. Assumptions and definitions

As mentioned in the Introduction, in this paper we will study a one-dimensional singular problem that we formally write as

$$\begin{cases} -\dfrac{d}{dx}\left(a(x)\dfrac{du}{dx}\right) = -\dfrac{d\phi(u)}{dx} - \dfrac{dg(x)}{dx} & \text{in } (0, L), \\ u(0) = u(L) = 0\,, \end{cases} \tag{2.1}$$

where u is the unknown and where $(a,\, g,\, \phi)$ are data which will be specified below (see Subsection 2.1).

One of the main difficulties of the problem is to give a correct mathematical meaning to problem (2.1) (see Subsection 2.2 below).

2.1. **Assumptions.**

We will always assume that

$$N = 1 \quad \text{and} \quad L > 0. \tag{2.2}$$

As far as the data (a, g, ϕ) are concerned, we will assume that they satisfy

$$a \in L^\infty(0, L),\ \exists\ \alpha, \beta,\ 0 < \alpha < \beta,\ \alpha \le a(x) \le \beta \quad \text{a.e. } x \in (0, L), \tag{2.3}$$

$$g \in L^2(0, L), \tag{2.4}$$

$$\begin{cases} \phi : \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\},\ \phi \ \text{ is continuous with values in } \mathbb{R} \cup \{+\infty\}, \\ \phi(s) < +\infty, \quad \forall s \in \mathbb{R}, \quad s \neq 0\,. \end{cases} \tag{2.5}$$

We are mainly interested in the case where ϕ is singular at $s = 0$, i.e. in the case where

$$\phi(0) = +\infty\,; \tag{2.6}$$

this will be the originality and the difficulty of the problem.

Note however that we will also consider functions ϕ which do not satisfy (2.6), for instance when approximating a singular function ϕ which satisfies (2.5) and (2.6) by a sequence of functions ϕ_n which belong to $C^0(\mathbb{R})$, and therefore satisfy (2.5) but not (2.6).

Remark 2.1. When ϕ satisfies (2.5), condition (2.6) is equivalent to

$$\phi(s) \to +\infty \quad \text{as} \quad s \to 0, \tag{2.7}$$

which implies that

$$\phi(s) \ge 0 \quad \text{for } s \text{ sufficiently small.} \tag{2.8}$$

Note that conditions (2.5)-(2.6) exclude oscillatory singularities of the type $\phi(s) = \dfrac{\sin(1/s)}{|s|^\gamma}$, and $\phi(s) = \dfrac{1 + \sin(1/s)}{|s|^\gamma}$.

Observe also that when ϕ satisfies (2.5), then ϕ satisfies

$$\phi \in C_b^0([-R, -\delta]) \cap C_b^0([+\delta, +R]), \quad \forall (\delta, R),\ 0 < \delta < R < +\infty. \tag{2.9}$$

Finally note that (2.5) does not impose any behaviour of ϕ as s tends to $+\infty$ and $-\infty$.

□

Remark 2.2. The **model case** for the function ϕ is the case of the function ϕ_γ given by

$$\phi_\gamma(s) = \frac{c}{|s|^\gamma} + \varphi(s),\ \text{with } c > 0,\ \gamma > 0,\ \varphi \in C^0(\mathbb{R}), \tag{2.10}$$

which satisfies (2.5)-(2.6) for every $\gamma > 0$.

□

Remark 2.3. Assuming $\phi(0) = -\infty$ in place of $\phi(0) = +\infty$ in (2.6) is just a variant of problem (2.1): indeed the problem

$$\begin{cases} -\dfrac{d}{dx}\left(a(x)\dfrac{d\hat{u}}{dx}\right) = -\dfrac{d\hat{\phi}(\hat{u})}{dx} - \dfrac{d\hat{g}}{dx} \text{ in } \mathcal{D}'(0,L), \\ \hat{u}(0) = \hat{u}(L) = 0\,, \end{cases} \tag{2.11}$$

with

$$\begin{cases} \hat{\phi} : \mathbb{R} \mapsto \mathbb{R} \cup \{-\infty\},\ \hat{\phi} \text{ is continuous with values in } \mathbb{R} \cup \{-\infty\}, \\ \hat{\phi}(s) > -\infty,\ \forall s \in \mathbb{R},\ s \neq 0, \end{cases} \tag{2.12}$$

reduces to problem (2.1) with assumptions (2.5)-(2.6) on ϕ by setting

$$u = -\hat{u}, \quad g = -\hat{g}, \quad \phi(s) = -\hat{\phi}(-s), \quad \forall s \in \mathbb{R}. \tag{2.13}$$

□

Remark 2.4. Observe that when u is a solution of (2.1), then u is still a solution of (2.1) when ϕ is changed in $\phi + c_1$ and g in $g + c_2$, where c_1 and c_2 are arbitrary constants.

To avoid confusion, we then emphasize the fact that in the whole of the present paper, *the data g and ϕ are fixed once and for all; in other terms the data g and ϕ are not defined up to additive constants, but fixed.*

□

Remark 2.5. When the data (a, g, ϕ) are given and when they satisfy the hypotheses (2.3)-(2.5) above, we claim that *it is always possible to assume without any loss of generality that ϕ also enjoys the following property*

$$\phi \in C_b^0(\mathbb{R}\backslash(-\delta, +\delta)), \quad \forall\, \delta > 0^1, \tag{2.14}$$

(compare with (2.9)), if one considers weak solutions of problem (2.1) in the sense of Definition 2.6 below for the data (a, g, ϕ) and if g satisfies for some $M > 0$

$$\|g\|_{L^2(0,L)} \leq \frac{\alpha}{\sqrt{L}} M. \tag{2.15}$$

Note that *condition* (2.15) *is not a restriction on g*, since M is an arbitrary constant; on the contrary, this condition allows g to vary into a given ball of $L^2(0, L)$ in the proof below.

Let us prove this claim.

Consider data (a, g, ϕ) which satisfy (2.3)-(2.5) and (2.15), and let u be any weak solution of problem (2.1) in the sense of Definition 2.6 below for these data (a, g, ϕ), i.e. a function u which satisfies

$$\begin{cases} u \in H_0^1(0, L),\ \phi(u) \in L^2(0, L), \\ -\dfrac{d}{dx}\left(a(x)\dfrac{du}{dx}\right) = -\dfrac{d\phi(u)}{dx} - \dfrac{dg(x)}{dx} \text{ in } \mathcal{D}'(0, L). \end{cases} \tag{2.16}$$

Then, in view of (2.25) (see Proposition 2.10 below) and of condition (2.15) on g and M, the function u satisfies

$$\|u\|_{L^\infty(0,L)} \leq \frac{\sqrt{L}}{\alpha}\|g\|_{L^2(0,L)} \leq M. \tag{2.17}$$

Define the function $\hat{\phi}_M : \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\}$ by the formula

$$\hat{\phi}_M(s) = \begin{cases} \phi(-M) & \text{if } s \leq -M, \\ \phi(s) & \text{if } -M \leq s \leq +M, \\ \phi(+M) & \text{if } s \geq +M. \end{cases} \tag{2.18}$$

Then the function $\hat{\phi}_M$ satisfies both (2.5) and (2.14).

On the other hand, in view of (2.17) one has

$$\hat{\phi}_M(u) = \phi(u) \in L^2(0, L) \quad \text{and} \quad \frac{d\hat{\phi}_M(u)}{dx} = \frac{d\phi(u)}{dx} \text{ in } \mathcal{D}'(0, L). \tag{2.19}$$

[1]or, equivalently, $\phi \in C_b^0(\mathbb{R}\backslash(-\delta, +\delta))$ for some given $\delta > 0$.

When g satisfies both (2.4) and (2.15), this implies that, any weak solution u of problem (2.1) in the sense of Definition 2.6 below for the data (a, g, ϕ) also satisfies

$$\begin{cases} u \in H_0^1(0,L), \hat{\phi}_M(u) \in L^2(0,L), \\ -\dfrac{d}{dx}\left(a(x)\dfrac{du}{dx}\right) = -\dfrac{d\hat{\phi}_M(u)}{dx} - \dfrac{dg(x)}{dx} \quad \text{in} \quad \mathcal{D}'(0,L), \end{cases} \tag{2.20}$$

or, in other terms, u is a weak solution of problem (2.1) in the sense of Definition 2.6 below for the data $(a, g, \hat{\phi}_M)$.

Conversely, when g satisfies both (2.4) and (2.15), any weak solution u of problem (2.1) in the sense of Definition 2.6 below for the data $(a, g, \hat{\phi}_M)$, or in other terms, any solution u of (2.20), is also a solution of (2.16), or in other terms a weak solution of problem (2.1) in the sense of Definition 2.6 below for the data (a, g, ϕ).

In brief, when g satisfies both (2.4) and (2.15) for some $M > 0$, one can always replace the function ϕ, which only satisfy (2.5), by the function $\hat{\phi}_M$, which satisfies both (2.5) and (2.14), if one considers weak solutions of problem (2.1) in the sense of Definition 2.6 below. This proves the claim.

□

2.2. **Definition of a weak solution of problem** (2.1)**.**

We introduce the following notion of solution:

Definition 2.6. Assume that (2.2) holds true, and that the data (a, g, ϕ) satisfy hypotheses (2.3)-(2.5). We will say that u is a *weak solution* of problem (2.1) if u satisfies

$$\begin{cases} u \in H_0^1(0,L), \phi(u) \in L^2(0,L), \\ -\dfrac{d}{dx}\left(a(x)\dfrac{du}{dx}\right) = -\dfrac{d\phi(u)}{dx} - \dfrac{dg(x)}{dx} \quad \text{in} \quad \mathcal{D}'(0,L). \end{cases} \tag{2.21}$$

□

Remark 2.7. The above Definition 2.6 will be justified by the result presented below in Theorem 3.4 (Alternative).

Let us emphasize that there are cases where problem (2.1) does not have any solution in the sense of Definition 2.6.

This is for example the case if the nonlinearity ϕ satisfies $\phi(0) = +\infty$ (hypothesis (2.6)) and if the source term g is bounded from below (see Theorem 4.1 below).

This is also the case if g is arbitrary in $L^2(0,L)$ and if

$$\int_0^{+\delta} \phi(t)\, dt = +\infty \quad \text{and} \quad \int_{-\delta}^{0} \phi(t)\, dt = +\infty$$

(see Theorem 4.4 below).

For this reason we will state most of the result of the present paper assuming that there exists at least a solution of problem (2.1) in the sense of Definition 2.6. We will face the problem of existence of such a solution in Section 7.

□

Remark 2.8. Problem (2.21) has a precise mathematical meaning, in contrast with problem (2.1) which has no mathematical meaning, since in (2.1) the spaces to which u and $\phi(u)$ have to belong to are not specified, and since the mathematical meanings of the two equations in (2.1) are not specified neither.

□

Remark 2.9. Since the first line in (2.21) asserts that $u \in H_0^1(\Omega)$ and $\phi(u) \in L^2(0,L)$ while a and g satisfy (2.3)-(2.4), the second line of (2.21) is equivalent to the *variational formulation*

$$\int_0^L a(x)\frac{du}{dx}\frac{dz}{dx} = \int_0^L \phi(u)\frac{dz}{dx} + \int_0^L g(x)\frac{dz}{dx}, \quad \forall z \in H_0^1(\Omega). \tag{2.22}$$

□

Proposition 2.10. *Assume that* (2.2) *holds true, and that the data* (a, g, ϕ) *satisfy* (2.3)-(2.5). *Then every possible weak solution* u *of problem* (2.1) *in the sense of Definition 2.6 satisfies*

$$\int_0^L a(x)\frac{du}{dx}\frac{du}{dx}\,dx = \int_0^L g(x)\frac{du}{dx}\,dx. \tag{2.23}$$

This energy equality in particular implies that

$$\left\|\frac{du}{dx}\right\|_{L^2(0,L)} \le \frac{1}{\alpha}\|g\|_{L^2(0,L)}, \tag{2.24}$$

which in turn implies that

$$\|u\|_{L^\infty(0,L)} \le \frac{\sqrt{L}}{\alpha}\|g\|_{L^2(0,L)}. \tag{2.25}$$

Remark 2.11. Proposition 2.10 asserts that every possible solution of (2.21) satisfies the a priori estimates (2.24)-(2.25), i.e. $H_0^1(0,L)$ and $L^\infty(0,L)$ bounds which depend only on L, on the coercivity constant α of a, and on $\|g\|_{L^2(0,L)}$, but not on the function ϕ, which is only assumed to satisfy (2.5).
□

The three results of Proposition 2.10 immediately follow from (2.22), from the coercivity (2.3), from Morrey's embedding (1.22) which is specific to the dimension one, and from Lemma 2.12 below.

Lemma 2.12. *Assume that* ϕ *satisfies hypothesis* (2.5)*, and let* z *be such that*

$$z \in H_0^1(0,L) \quad \textit{with} \quad \phi(z) \in L^2(0,L). \tag{2.26}$$

Then one has:

$$\int_0^L \phi(z)\frac{dz}{dx}\,dx = 0. \tag{2.27}$$

Proof. For $n > 0$, let $T_n : \mathbb{R} \mapsto \mathbb{R}$ be the truncation at height n defined by (1.24), and let $\psi_n : \mathbb{R} \mapsto \mathbb{R}$ be the function defined by

$$\psi_n(s) = \int_0^s T_n(\phi(t))\,dt.$$

Since $T_n(\phi) \in C_b^0(\mathbb{R})$, one has $\psi_n \in C^0(\mathbb{R})$ with $\dfrac{d\psi_n}{ds} \in C_b^0(\mathbb{R})$, so that

$$\forall z \in H_0^1(0,L),\ \psi_n(z) \in H_0^1(0,L) \ \text{ with } \ \frac{d\psi_n(z)}{dx} = (T_n(\phi(z)))\frac{dz}{dx},$$

and therefore

$$\int_0^L T_n(\phi(z))\frac{dz}{dx}\,dx = \int_0^L \frac{d\psi_n(z)}{dx}\,dx = \psi_n(z(L)) - \psi_n(z(0)) = 0 - 0 = 0. \tag{2.28}$$

Since $|T_n(\phi(z(x)))| \le |\phi(z(x))|$ a.e. $x \in [0,L]$, and since by hypothesis (2.26) $\phi(z)$ belongs to $L^2(0,L)$, while

$$T_n(\phi(z(x))) \to \phi(z(x)) \ \text{ a.e. } \ x \in (0,L) \ \text{ as } \ n \to +\infty,$$

passing to the limit in (2.28) thanks to Lebesgue's dominated convergence theorem implies that

$$\int_0^L \phi(z)\frac{dz}{dx}\,dx = 0, \tag{2.29}$$

which proves (2.27).

□

The proof of the following proposition is straightforward (the last line of (2.31) below is just dividing the second line by $a(x)$, integrating on $(0,L)$, and using that $u(0) = u(L)$).

Proposition 2.13 (**Equivalence**). *Assume that* (2.2) *holds true, and that the data* (a, g, ϕ) *satisfy* (2.3)-(2.5). *Then* u *is a weak solution of* (2.1) *in the sense of Definition 2.6, (i.e.* u *satisfies* (2.21)*) if and only if* u *satisfies*

$$\begin{cases} u \in H_0^1(0,L), \ \ \phi(u) \in L^2(0,L), \\ \exists\, c \in \mathbb{R}, \ a(x)\dfrac{du}{dx} = \phi(u) + g + c \ \ in \ \ \mathcal{D}'(0,L)\,, \end{cases} \tag{2.30}$$

or, equivalently, if and only if

$$\begin{cases} u \in H_0^1(0,L), \ \ \phi(u) \in L^2(0,L), \\ a(x)\dfrac{du}{dx} = \phi(u) + g + c \ \ in \ \ \mathcal{D}'(0,L), \\ with \ \ c = -\dfrac{\displaystyle\int_0^L \frac{\phi(u)}{a(x)}\,dx + \int_0^L \frac{g(x)}{a(x)}\,dx}{\displaystyle\int_0^L \frac{1}{a(x)}\,dx}. \end{cases} \tag{2.31}$$

Remark 2.14. Consider a function $\phi : \mathbb{R} \mapsto \mathbb{R}$ which satisfies

$$\phi \in C^0(\mathbb{R}). \tag{2.32}$$

Then the function ϕ satisfies (2.5) and the notion of weak solution in the sense of Definition 2.6 of problem (2.1) for this function ϕ is defined.

On the other hand, when ϕ satisfies (2.32), it is standard to define a *classical weak solution of problem* of (2.1) for such a function ϕ as a function u which satisfies

$$\begin{cases} u \in H_0^1(0,L), \\ -\dfrac{d}{dx}\left(a(x)\dfrac{du}{dx}\right) = -\dfrac{d\phi(u)}{dx} - \dfrac{dg(x)}{dx} \ \ \text{in} \ \ \mathcal{D}'(0,L)\,, \end{cases} \tag{2.33}$$

where $\phi(u)$ "automatically" belongs to $L^2(0,L)$, since by Morrey's embedding (see (1.22)) one has $H_0^1(0,L) \subset L^\infty(0,L)$, which implies when $\phi \in C^0(\mathbb{R})$ that

$$\phi(z) \in L^\infty(0,L) \subset L^2(0,L), \ \forall z \in H_0^1(0,L).$$

When ϕ satisfies (2.32), the definition of "weak solution of problem (2.1) in the sense of Definition 2.6" coincides with the definition of "classical weak solution of problem (2.1)" given by (2.33).

Definition 2.6 is actually an extension of definition (2.33) of a "classical weak" solution of the singular case where $\phi(0) = +\infty$.

□

2.3. **Existence of a weak solution when $\phi \in C^0(\mathbb{R})$.**

Proposition 2.15. *Assume that* (2.2) *holds true, and that the data* (a, g, ϕ) *satisfy* (2.3)-(2.5). *Assume also that*

$$\phi \in C^0(\mathbb{R}). \tag{2.34}$$

Then there exists at least one classical weak solution u of problem (2.1) *in the sense of* (2.33), *or equivalently a function* u *which satisfies*

$$\begin{cases} u \in H_0^1(0,L), \\ \displaystyle\int_0^L a(x)\frac{du}{dx}\frac{dz}{dx}\,dx = \int_0^L \phi(u)\frac{dz}{dx}\,dx + \int_0^L g(x)\frac{dz}{dx}\,dx, \quad \forall z \in H_0^1(0,L). \end{cases} \tag{2.35}$$

This classical weak solution *is also* a weak solution of problem (2.1) in the sense of Definition 2.6. *Moreover any solution* u *of* (2.35) *satisfies*

$$\left\| \frac{du}{dx} \right\|_{L^2(0,L)} \leq \frac{1}{\alpha} \|g\|_{L^2(0,L)}\,, \tag{2.36}$$

which in turn implies that

$$\|u\|_{L^\infty(0,L)} \leq \frac{\sqrt{L}}{\alpha}\|g\|_{L^2(0,L)}\,. \tag{2.37}$$

Proof of Proposition 2.15. **Step 1.** Further to (2.34), let us assume in this first step that

$$\phi \in C_b^0(\mathbb{R}) = C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}). \tag{2.38}$$

For every $\overline{u} \in L^2(0,L)$, define u as the unique solution of the linear problem

$$\begin{cases} u \in H_0^1(0,L), \\ -\dfrac{d}{dx}\left(a(x)\dfrac{du}{dx}\right) = -\dfrac{d\phi(\overline{u})}{dx} - \dfrac{dg}{dx} \ \text{ in } \ \mathcal{D}'(0,L). \end{cases} \tag{2.39}$$

Using u as test function in the variational formulation of (2.39), one has

$$\int_0^L a(x)\left|\frac{du}{dx}\right|^2 dx = \int_0^L \phi(\overline{u})\frac{du}{dx}\,dx + \int_0^L g(x)\frac{du}{dx}\,dx,$$

which implies that

$$\alpha\left\|\frac{du}{dx}\right\|_{L^2(0,L)} \leq \sqrt{L}\|\phi\|_{L^\infty(\mathbb{R})} + \|g\|_{L^2(0,L)}.$$

Recalling Poincaré's inequality (1.21), we get that

$$\|u\|_{L^2(0,L)} \leq L\left\|\frac{du}{dx}\right\|_{L^2(0,L)} \leq \frac{L}{\alpha}\left(\sqrt{L}\|\phi\|_{L^\infty(\mathbb{R})} + \|g\|_{L^2(0,L)}\right).$$

Then Leray-Schauder's fixed point theorem applied to the map W defined by

$$W : \overline{u} \in L^2(0,L) \to W(\overline{u}) = u \in L^2(0,L)$$

and to the ball B of $L^2(0,L)$ defined by

$$B = \left\{ z \in L^2(0,L) : \|z\|_{L^2(0,L)} \leq \frac{L}{\alpha}\left(\sqrt{L}\|\phi\|_{L^\infty(\mathbb{R})} + \|g\|_{L^2(0,L)}\right)\right\}$$

implies, using also the Rellich-Kondrachov's compactness theorem, that W has at least a fixed point in B.

When ϕ satisfies (2.38), this proves that there exists a classical weak solution of problem (2.1), i.e. a solution of (2.35) (or of (2.33), which is equivalent to (2.35)).

Using then $z = u$ as test function in (2.35) and Lemma 2.12 implies that

$$\int_0^L a(x)\frac{du}{dx}\frac{du}{dx}\,dx = \int_0^L g(x)\frac{du}{dx}\,dx,$$

which immediately implies (2.36), which in turns implies (2.37) using Morrey's inequality (1.22) (where the latest inequality is specific to the one-dimensional case).

Observe that when ϕ satisfies (2.38) and not only (2.34) in Proposition 2.15, the existence of (at least) one classical weak solution of (2.35) and estimate (2.36), as well as their proofs, continue to hold true in dimension $N > 1$; in contrast estimate (2.37) is a result specific to the case $N = 1$, since it follows from Morrey's inequality (1.22).

Step 2. Let us now consider the case where the sole hypothesis (2.34) holds true, i.e. the case where ϕ belongs to $C^0(\mathbb{R})$ and not necessarily to $C_b^0(\mathbb{R})$.

In this case, fix m which satisfies

$$m \geq \frac{\sqrt{L}}{\alpha}\|g\|_{L^2(0,L)}, \tag{2.40}$$

and consider the function $\phi_m : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$\phi_m(s) = \phi(T_m(s)), \quad \forall s \in \mathbb{R},$$

where T_m is the truncation at height m defined by (1.24). Then ϕ_m belongs to $C_b^0(\mathbb{R})$, and Step 1 implies that there exists at least one solution u_m of

$$\begin{cases} u_m \in H_0^1(0,L), \\ \displaystyle\int_0^L a(x)\frac{du_m}{dx}\frac{dz}{dx}\,dx = \int_0^L \phi_m(u_m)\frac{dz}{dx}\,dx + \int_0^L g(x)\frac{dz}{dx}\,dx, \ \ \forall z \in H_0^1(0,L). \end{cases} \tag{2.41}$$

Moreover every solution u_m of (2.41) satisfies (2.36) and (2.37), and therefore in view of (2.40)

$$\|u_m\|_{L^\infty(0,L)} \le \frac{\sqrt{L}}{\alpha}\|g\|_{L^2(0,L)} \le m.$$

This implies that

$$T_m(u_m) = u_m, \quad \text{and} \quad \phi_m(u_m) = \phi(T_m(u_m)) = \phi(u_m),$$

which in turn implies that u_m is also a solution of (2.35) for the function ϕ.

Proposition 2.15 is then proved in full generality.

□

2.4. **Definitions of reasonable sequence of approximations and good sequence of approximations of ϕ.** Let us conclude this section by introducing the following definition.

Definition 2.16. Let ϕ be a function which satisfies (2.5). We will say that a sequence ϕ_n of functions is a *reasonable sequence of approximations* (or simply a *reasonable approximation*) of ϕ if the sequence ϕ_n satisfies

$$\phi_n \text{ satisfies assumption (2.5) for every given } n, \tag{2.42}$$

$$\begin{cases} \text{for every sequence } s_n \in \mathbb{R} \text{ and every } s \in \mathbb{R} \text{ such that } s_n \to s \text{ in } \mathbb{R}, \\ \text{then } \phi_n(s_n) \to \phi(s) \text{ in } \mathbb{R} \cup \{+\infty\}. \end{cases} \tag{2.43}$$

Moreover, we will say that a sequence ϕ_n of functions is a *good sequence of approximations* (or simply a *good approximation*) of ϕ if the sequence ϕ_n is a reasonable sequence of approximations of ϕ for which every ϕ_n belongs to $C^0(\mathbb{R})$. □

Remark 2.17 (**Examples**)**.** Let us give, for functions ϕ which satisfy (2.5) (and possibly (2.6)), five examples of reasonable approximations of ϕ. The first, the second and the fifth examples are actually good approximations, while, when $\phi(0) = +\infty$, the third and the fourth examples are not good approximations of ϕ.

The **first example** is the **approximation by truncation**, which consists in taking, for every function ϕ which satisfies (2.5), the sequence of functions ϕ_n defined by

$$\phi_n(s) = T_n(\phi(s)), \quad \forall s \in \mathbb{R},\ \forall n \in \mathbb{N}, \tag{2.44}$$

where T_n is the truncation at height n defined by (1.24).

It is easy to prove that $\phi_n \in C_b^0(\mathbb{R})$, which proves (2.42). It is also easy to prove (2.43) when $s \neq 0$, as well as when $\phi(0) < +\infty$.

The case where $\phi(0) = +\infty$ and where $s_n \to 0$ requires special attention. In this case one has indeed to prove that

$$\text{if} \quad s_n \to 0, \quad \text{then} \quad \phi_n(s_n) = T_n(\phi(s_n)) \to \phi(0) = +\infty.$$

This can be done using the facts that $T_n(r) \ge T_m(r)$ for every $n \ge m > 0$ and for every $r > 0$. Indeed, as $s_n \to 0$, one has $\phi(s_n) > 0$ for every n sufficiently large. Therefore for every $m > 0$ and n sufficiently large, one has

$$\phi_n(s_n) = T_n(\phi(s_n)) \ge T_m(\phi(s_n)) \to T_m(+\infty) = m \quad \text{for every } m > 0 \text{ fixed.}$$

This completes the proof of the fact that the sequence ϕ_n of approximations by truncation defined by (2.44) is a good sequence of approximations of any function ϕ which satisfies (2.5).

Note that the approximations by truncation (2.44) satisfy

$$\phi_n \in C_b^0(\mathbb{R}), \quad |\phi_n(s)| \le |\phi(s)|, \quad |\phi_n(s)| \le n, \quad \forall s \in \mathbb{R},\ \forall n \in \mathbb{N}. \tag{2.45}$$

The **second example** is the **homographic approximation**, which consists in taking, for every function ϕ which satisfies (2.5), the sequence of functions ϕ_n defined by

$$\begin{cases} \phi_n(s) = \dfrac{\phi(s)}{1 + \frac{1}{n}|\phi(s)|}, \quad \forall s \in \mathbb{R},\ s \neq 0, \\ \phi_n(0) = \begin{cases} \dfrac{\phi(0)}{1 + \frac{1}{n}|\phi(0)|}, & \text{when } \phi(0) < +\infty, \\ n, & \text{when } \phi(0) = +\infty. \end{cases} \end{cases} \tag{2.46}$$

It is easy to prove that $\phi_n \in C^0_b(\mathbb{R})$, which implies (2.42): indeed, when $\phi(0) = +\infty$, one has, for every fixed n,

$$\phi_n(s) \sim \frac{\phi(s)}{\frac{1}{n}|\phi(s)|} = n, \quad \text{if } s \to 0,\ s \neq 0.$$

Here again the only (small) difficulty in proving (2.43) is the case where $\phi(0) = +\infty$ and where $s_n \to 0$. Since $\phi(s_n) > 0$ for n sufficiently large, this is done by considering first subsequences $\{n'\} \subset \{n\}$ for which

$$\frac{1}{n'}|\phi(s_{n'})| = \frac{1}{n'}\phi(s_{n'}) \to c, \quad \text{with} \quad 0 \leq c < +\infty, \ \text{ as } n' \to +\infty,$$

and then subsequences $\{n'\} \subset \{n\}$ for which

$$\frac{1}{n'}|\phi(s_{n'})| = \frac{1}{n'}\phi(s_{n'}) \to +\infty, \quad \text{ as } n' \to +\infty;$$

in the latest case, one has either $\phi_{n'}(s_{n'}) = n'$ if $s_{n'} = 0$, or if $s_{n'} \neq 0$,

$$\phi_{n'}(s_{n'}) = \frac{\phi(s_{n'})}{1 + \frac{1}{n'}|\phi(s_{n'})|} \sim \frac{\phi(s_{n'})}{\frac{1}{n'}|\phi(s_{n'})|} = \frac{\phi(s_{n'})}{\frac{1}{n'}\phi(s_{n'})} = n'.$$

In any cases one has

$$\phi_{n'}(s_{n'}) \to +\infty = \phi(0), \quad \text{ as } \ n' \to +\infty,$$

which proves (2.43).

This completes the proof of the fact that the sequence ϕ_n of homographic approximations defined by (2.46) is a good approximation of any function ϕ which satisfies (2.5).

Note that here again the homographic approximations satisfy (2.45).

The **third example** is the **trivial approximation** of the function ϕ by itself, which consists in taking, for every function ϕ which satisfies (2.5) the sequence of approximations defined by

$$\phi_n(s) = \phi(s), \quad \forall s \in \mathbb{R},\ \forall n \in \mathbb{N}. \tag{2.47}$$

In this trivial example, it is clear that (2.42)-(2.43) hold true, and that the trivial sequence of approximations defined by (2.47) is a reasonable sequence of approximations of any function ϕ which satisfies (2.5), but not a good approximation if ϕ satisfies (2.6).

The **fourth example** consists in **approximating** $\dfrac{1}{|s|^\gamma}$ **by** $\dfrac{1}{|s|^{\gamma_n}}$, in the specific case where ϕ is the model example (2.10) given by

$$\phi_\gamma(s) = \frac{c}{|s|^\gamma} + \varphi(s), \text{ with } c > 0,\ \gamma > 0,\ \varphi \in C^0(\mathbb{R}). \tag{2.48}$$

In this case one can approximate the function ϕ_γ by the sequence of functions ϕ_n given by

$$\phi_n(s) = \frac{c_n}{|s|^{\gamma_n}} + \varphi_n(s), \text{ with } c_n > 0,\ \gamma_n > 0,\ \varphi_n \in C^0(\mathbb{R}), \tag{2.49}$$

where

$$c_n \to c, \quad \gamma_n \to \gamma, \quad \varphi_n \to \varphi \quad \text{ uniformly in } C^0([-R, +R]) \text{ for every fixed } R. \tag{2.50}$$

Using in particular the fact that for every ε, with $0 < \varepsilon < \min\{\gamma, c\}$, one has

$$\phi_n(s_n) = \frac{c_n}{|s_n|^{\gamma_n}} + \varphi_n(s_n) \geq \frac{c - \varepsilon}{|s_n|^{\gamma - \varepsilon}} + \varphi_n(s_n) \to +\infty = \phi(0), \quad \text{ as } \ s_n \to 0,\ s_n \neq 0,$$

allows one to prove that the sequence ϕ_n defined by (2.49)-(2.50) is a reasonable sequence of (but not a good sequence of) approximations of the function ϕ defined by (2.48).

The **fifth example** concerns the case where $\phi \in C^0(\mathbb{R})$ and consists in this case in the classical **approximation by convolution**, namely

$$\phi_\varepsilon = \phi * \rho_\varepsilon\,,$$

where ρ_ε is a standard sequence of mollifiers, i.e. $\rho_\varepsilon = \varepsilon\rho(\varepsilon x)$ with $\rho \in C^\infty_c(\mathbb{R})$, $\rho \geq 0$, such that $\int_{\mathbb{R}} \rho = 1$. Then ϕ_ε is a good sequence of approximations of ϕ since ϕ_ε converges locally uniformly to ϕ. Here one has $\phi_\varepsilon \in C^\infty(\mathbb{R})$.

Moreover, in the case where $\phi \in C^0(\mathbb{R})$ is constant at infinity, namely when there exists $R > 0$ such that

$$\phi(s) = \begin{cases} \phi(+R) & \text{if } s > +R, \\ \phi(-R) & \text{if } s < -R, \end{cases}$$

then for every ε the function $\phi_\varepsilon = \phi * \rho_\varepsilon$ is Lipschitz continuous on $\mathbb{R}$.

□

To conclude this section, let us state and prove a characterization of a reasonable sequence of approximations defined by Definition 2.16.

Proposition 2.18. *Assume that ϕ satisfies hypotheses* (2.5)-(2.6). *Then Definition 2.16 is equivalent to assert that the sequence ϕ_n satisfies* (2.42) *as well the following two properties*

$$\begin{cases} \text{for every } \eta \text{ and } R,\ 0 < \eta < R, \\ \phi_n \to \phi \text{ uniformly in } C^0([+\eta, +R]) \text{ and in } C^0([-R, -\eta]), \end{cases} \tag{2.51}$$

$$\liminf_{n\to+\infty} \left(\inf_{t\in[-\eta,+\eta]} \phi_n(t) \right) \to +\infty, \quad \text{as } \eta \to 0^+. \tag{2.52}$$

Remark 2.19. The meaning of the two properties (2.51)-(2.52) is that, *in some sense*, the sequence of functions $\phi_n : \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\}$ (which are assumed to satisfy (2.42)) *locally uniformly converges* to the function $\phi : \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\}$ which satisfies (2.5)-(2.6). Note however that the local uniform convergence is usually defined only for functions from $\mathbb{R}$ into $\mathbb{R}$, while here one has $\phi(0) = +\infty$.

Property (2.51) is indeed nothing but the classical local uniform convergence of the sequence ϕ_n to ϕ in $\mathbb{R} - \{0\}$, while property (2.52) asserts that ϕ_n *uniformly converges to* $+\infty$ *around* $s = 0$.

When $\phi(0)$ is finite, a variant of the proof below shows that it is equivalent for an approximation ϕ_n to satisfy (2.43) or to converge locally uniformly on the whole of $\mathbb{R}$, i.e. in $C^0([-R, +R])$ for every $R \in \mathbb{R}$. Note that when $\phi(0)$ is finite, then necessarily $\phi_n(0)$ is finite for n sufficiently large (take $s_n = 0$ in (2.43)).

□

Proof of Proposition 2.18. **Step 1.** Let us first prove that if ϕ satisfies (2.5)-(2.6), and if the sequence ϕ_n satifies (2.42), (2.51)-(2.52), then the sequence ϕ_n satisfies (2.43).

Consider indeed on the first hand a sequence s_n which satisfies

$$s_n \to s \ \text{ as } \ n \to +\infty,\ s \neq 0,$$

and write

$$\phi_n(s_n) - \phi(s) = (\phi_n(s_n) - \phi(s_n)) + (\phi(s_n) - \phi(s)).$$

Then using (2.51) and (2.9) proves (2.43).

Consider on the other hand a sequence s_n which satisfies

$$s_n \to 0 \ \text{ as } \ n \to +\infty.$$

Then for every $\eta > 0$, one has $|s_n| \leq \eta$ for n sufficiently large, and the inequality $\phi_n(s_n) \geq \inf_{t\in[-\eta,+\eta]} \phi_n(t)$ for n sufficiently large implies that

$$\liminf_{n\to+\infty} \phi_n(s_n) \geq \liminf_{n\to+\infty} \inf_{t\in[-\eta,+\eta]} \phi_n(t),$$

which combined with (2.52) proves that

$$\phi_n(s_n) \to +\infty = \phi(0), \quad \text{as } \ n \to +\infty.$$

This completes the proof of (2.43).

Step 2. Conversely let us prove that if ϕ satisfies (2.5)-(2.6), and if the sequence ϕ_n satisfies (2.42)-(2.43), then the sequence ϕ_n satisfies (2.51)-(2.52).

As far as (2.51) **is concerned**, fix η and R with $0 < \eta < R$ and choose any $s_n \in [\eta, R]$ such that

$$|\phi_n(s_n) - \phi(s_n)| = \max_{t\in[\eta,R)} |\phi_n(t) - \phi(t)| = \sup_{t\in[\eta,R]} |\phi_n(t) - \phi(t)| = \|\phi_n - \phi\|_{C^0([\eta,R])}; \tag{2.53}$$

observe indeed that in (2.53) the supremum on $[\eta, R]$ is actually a maximum. From any subsequence denoted by $\{n'\}$ of $\{n\}$, extract from n' a subsequence $\{n''\} \subset \{n'\}$ such that the subsequence $s''_n \in [\eta, R]$ converges to some $s \in [\eta, R]$, and apply (2.43) to s''_n; then

$$\phi_{n''}(s_{n''}) \to \phi(s), \quad \text{as} \quad n'' \to +\infty.$$

Since

$$\|\phi_{n''} - \phi\|_{C^0([\eta,R])} = |\phi_{n''}(s_{n''}) - \phi(s_{n''})| \leq |\phi_{n''}(s_{n''}) - \phi(s)| + |\phi(s) - \phi(s_{n''})|,$$

using (2.43) and (2.5) implies that

$$\phi_{n''} \to \phi \ \text{ uniformly in } \ C^0([\eta, R]), \quad \text{as } n'' \to +\infty. \tag{2.54}$$

The fact that the limit in (2.54) does not depend on the subsequence n' implies that the convergence (2.54) takes place for the whole sequence $\{n\} = \mathbb{N}$.

A similar proof implies the similar result in $C^0([-R, -\eta])$, and (2.51) is proved.

As far as (2.52) **is concerned**, fix $\eta > 0$ and choose any $s_n \in [-\eta, +\eta]$ such that

$$\phi_n(s_n) = \min_{t\in[-\eta,+\eta]} \phi_n(t) = \inf_{t\in[-\eta,+\eta]} \phi_n(t); \tag{2.55}$$

observe indeed that in (2.55) the infimum on $[-\eta, +\eta]$ is actually a minimum, and that $\phi_n(s_n)$ is finite for every n since, for every $\overline{s} \in [-\eta, +\eta]$, one has

$$\phi_n(s_n) = \inf_{|t|\leq\eta} \phi_n(t) \leq \phi_n(\overline{s}).$$

Since

$$\phi_n(\overline{s}) \to \phi(\overline{s}), \quad \text{as} \quad n \to +\infty,$$

this implies that for some constant $\overline{C}$, one has

$$\phi_n(s_n) \leq \overline{C}, \quad \forall n \in \mathbb{N}, \tag{2.56}$$

as well as

$$\liminf_n \phi_n(s_n) \leq \lim_n \phi_n(\overline{s}) = \phi(\overline{s}), \quad \forall \overline{s} \in [-\eta, +\eta],$$

so that

$$\liminf_n \phi_n(s_n) \leq \inf_{s\in[-\eta,+\eta]} \phi(s). \tag{2.57}$$

Let us now prove that for some constant $\underline{C}$, one has

$$\phi_n(s_n) \geq \underline{C}, \quad \forall n \in \mathbb{N}. \tag{2.58}$$

Indeed, if (2.58) does not hold true, there exists a subsequence $\{n'\} \subset \{n\} = \mathbb{N}$ such that

$$\phi_{n'}(s_{n'}) \to -\infty, \quad \text{as} \quad n' \to +\infty. \tag{2.59}$$

Extract from $\{n'\}$ a subsequence $\{n''\} \subset \{n'\}$ such that the subsequence $s''_n \in [-\eta, +\eta]$ converges to some $s \in [-\eta, +\eta]$, and apply (2.43) to s''_n; then

$$\phi_{n''}(s_{n''}) \to \phi(s), \quad \text{as} \quad n'' \to +\infty,$$

in contradiction with (2.59), since $\phi : \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\}$.

We use the fact that from any sequence ρ_n with $\underline{c} \leq \rho_n \leq \overline{c}$ for some $\underline{c}, \overline{c} \in \mathbb{R}$, one can extract a subsequence $\{n'\} \subset \{n\}$ such that

$$\lim_{n'} \rho_{n'} = \liminf_n \rho_n.$$

Using this result with $\rho_n = \phi_n(s_n)$, which satisfies $\underline{C} \leq \phi_n(s_n) \leq \overline{C}$ in view of (2.58) and (2.56), there exists a subsequence $\{n'\} \subset \{n\}$ such that

$$\lim_{n'} \phi_{n'}(s_{n'}) = \liminf_n \phi_n(s_n).$$

Extract from $\{n'\}$ a subsequence $\{n''\} \subset \{n'\}$ such that the subsequence $s''_n \in [-\eta, +\eta]$ converges to some $s \in [-\eta, +\eta]$, and apply (2.43) to s''_n; then

$$\liminf_n \phi_n(s_n) = \lim_{n''} \phi_{n''}(s_{n''}) = \phi(s) \geq \inf_{t\in[-\eta,+\eta]} \phi(t).$$

Combining this result with (2.57), we have proved that

$$\forall \eta > 0, \quad \liminf_n \left(\inf_{t\in[-\eta,+\eta]} \phi_n(t) \right) = \liminf_n \phi_n(s_n) = \inf_{t\in[-\eta,+\eta]} \phi(t). \tag{2.60}$$

Since in view of (2.5)-(2.6) one has

$$\inf_{t\in[-\eta,+\eta]} \phi(t) \to \phi(0) = +\infty, \quad \text{as } \eta \to 0^+, \tag{2.61}$$

we have proved (2.52).

This completes the proof of Proposition 2.18.

□

3. Approximation of problem (2.1), a priori estimates, and an alternative

As we already said, in order to (try to) prove the existence of a weak solution of problem (2.1) in the sense of Definition 2.6, one proceed as usual by approximation, finding suitable priori estimates, and finally passing to the limit.

3.1. **Approximation of problem (2.1) and the main difficulty.** We assume that hypothesis (2.2) holds true (so that we are dealing with a one-dimensional problem), and that the data (a, g, ϕ) satisfy hypotheses (2.3)-(2.5) and (2.6), and we consider sequences (a_n, g_n, ϕ_n) of "approximated data" which satisfy, for some $\beta > \alpha > 0$, and some $c_0 > 0$

$$a_n \in L^\infty(0,L),\ \exists\, \alpha,\beta,\ 0<\alpha<\beta,\ \alpha \le a_n(x) \le \beta,\ a_n(x) \to a(x) \ \text{ a.e. } x \in (0,L), \tag{3.1}$$

$$g_n \in L^2(0,L), \quad \|g_n\|_{L^2(0,L)} \le c_0,\ g_n \rightharpoonup g \ \text{ weakly in } \ L^2(0,L), \tag{3.2}$$

$$\phi_n \ \text{ is a good sequence of approximations of } \phi, \tag{3.3}$$

(recall Definition 2.16 in Section 2), so that ϕ_n satisfies

$$\phi_n \in C^0(\mathbb{R}) \ \text{ for every given } \ n. \tag{3.4}$$

Proposition 2.15 then ensures that for every n there exists at least one classical weak solution of problem (2.1) for every (a_n, g_n, ϕ_n), namely at least one function u_n which satisfies (see (2.23)) the energy equality

$$\int_0^L a_n(x)\frac{du_n}{dx}\frac{du_n}{dx}\,dx = \int_0^L g_n(x)\frac{du_n}{dx}\,dx, \tag{3.5}$$

which implies

$$\begin{cases} u_n \in H_0^1(0,L), \\ \displaystyle\int_0^L a_n(x)\frac{du_n}{dx}\frac{dz}{dx}\,dx = \int_0^L \phi_n(u_n)\frac{dz}{dx}\,dx + \int_0^L g_n\frac{dz}{dx}\,dx, \quad \forall z \in H_0^1(0,L). \end{cases} \tag{3.6}$$

Moreover the function u_n satisfies (see (2.36))

$$\left\|\frac{du_n}{dx}\right\|_{L^2(0,L)} \le \frac{1}{\alpha}\|g_n\|_{L^2(0,L)} \le \frac{c_0}{\alpha}. \tag{3.7}$$

One can therefore extract a subsequence, denoted by n', and there exists some $u \in H_0^1(0,L)$ such that

$$\begin{cases} u_{n'} \rightharpoonup u \ \text{ weakly in } \ H_0^1(0,L), \\ u_{n'}(x) \to u(x) \ \text{ a.e. } \ x \in (0,L), \end{cases} \tag{3.8}$$

thanks to Rellich-Kondrashov's theorem. One easily passes to the limit in the first and last terms of (3.6) thanks to the strong convergence of $a_n \dfrac{dz}{dx}$ and the weak convergence of g_n in $L^2(0,L)$, which result from (3.1) and (3.2), obtaining

$$\begin{cases} \displaystyle\int_0^L a_{n'}(x)\frac{du_{n'}}{dx}\frac{dz}{dx}\,dx \to \int_0^L a(x)\frac{du}{dx}\frac{dz}{dx}\,dx, \\ \displaystyle\int_0^L g_{n'}\frac{dz}{dx}\,dx \to \int_0^L g\frac{dz}{dx}\,dx, \end{cases} \quad \text{as } \ n' \to +\infty. \tag{3.9}$$

As far as the second term of (3.6) is concerned, the almost everywhere convergence in $(0,L)$ of $u_{n'}$ stated in (3.8) and the fact that ϕ_n is a reasonable sequence of approximations of ϕ immediately imply (see (2.43)) that

$$\phi_{n'}(u_{n'}(x)) \to \phi(u(x)) \text{ a.e. } x \in (0,L), \quad \text{as } n' \to +\infty. \tag{3.10}$$

Observe however that this almost everywhere convergence does not allow one to pass to the limit in the term

$$\int_0^L \phi_{n'}(u_{n'}) \frac{dz}{dx}\, dx,$$

since the a.e. convergence of $\phi_{n'}(u_{n'})$ is not sufficient to imply the convergence of the integrals.

Observe that up to now, we could have obtained results similar to (3.6)-(3.10) in an N-dimensional setting.

3.2. **A new a priori estimate due to the one-dimensional setting.** We will now prove a new a priori estimate which is specific to the one-dimensional case, see assumption (2.2).

Lemma 3.1. *Assume that* (2.2) *holds true, and that the data* (a,g,ϕ) *and* (a_n,g_n,ϕ_n) *satisfy hypotheses* (2.3)-(2.5) *and* (2.6), *and* (3.1)-(3.3) *and* (3.4). *If a subsequence, denoted by* $u_{n'}$, *satisfies* (3.6) *and* (3.8) *for some* $u \in H_0^1(0,L)$, *and if*

$$u \neq 0, \tag{3.11}$$

then

$$\phi_{n'}(u_{n'}) \text{ is bounded in } L^2(0,L), \tag{3.12}$$

and

$$\phi_{n'}(u_{n'}) \rightharpoonup \phi(u) \text{ weakly in } L^2(0,L). \tag{3.13}$$

Proof. Step 1. We will strongly use in the present proof the assumption that $N=1$ in two ways.

First by using Morrey's embedding theorem (see (1.23)) which asserts that, when $N=1$, then $H_0^1(0,L) \subset C^{0,\frac{1}{2}}([0,L])$, so that, in view of (3.8), that

$$u_{n'} \to u \text{ uniformly in } C^0([0,L]), \quad \text{as } n' \to +\infty. \tag{3.14}$$

And, second, by using the characterization of a weak solution of problem (2.1) in the sense of Definition 2.6 given in Proposition 2.13, see (2.30); applied to $u_{n'}$, (2.31) implies that

$$\begin{cases} u_{n'} \in H_0^1(0,L),\ \phi_{n'}(u_{n'}) \in L^2(0,L), \\ \exists\, c_{n'} \in \mathbb{R},\ a_{n'}(x)\dfrac{du_{n'}}{dx} = \phi_{n'}(u_{n'}) + g_{n'}(x) + c_{n'} \text{ in } \mathcal{D}'(0,L)\,, \\ \text{with } c_{n'} = -\dfrac{\displaystyle\int_0^L \frac{\phi_{n'}(u_{n'})}{a_{n'}(x)}\,dx + \int_0^L \frac{g_{n'}(x)}{a_{n'}(x)}\,dx}{\displaystyle\int_0^L \frac{1}{a_{n'}(x)}\,dx}. \end{cases} \tag{3.15}$$

Observe that here $\phi_{n'} \in C^0(\mathbb{R})$ (see (3.4)), and that $u_{n'} \in H_0^1(0,L) \subset L^\infty(0,L)$, so that $\phi_n(u_{n'})$ "automatically" belongs to $L^\infty(0,L) \subset L^2(0,L)$ for each n'.

Step 2. If we assume that $u \neq 0$ (hypothesis (3.11)), there exists at least one $x_0 \in (0,L)$, such that

$$u(x_0) \neq 0. \tag{3.16}$$

We claim that

$$c_{n'} \text{ is bounded in } \mathbb{R}. \tag{3.17}$$

Let us assume for a moment that

$$u(x_0) > 0. \tag{3.18}$$

(the proof will be similar in the case where $u(x_0) < 0$).

Since $u \in H_0^1(0,L) \subset C^0([0,L])$, (3.18) implies that there exists some $\delta > 0$ and $\eta > 0$ such that

$$0 < x_0 - \delta < x_0 < x_0 + \delta < L, \text{ with } u(x) \geq \eta, \quad \forall x \in [x_0-\delta, x_0+\delta], \tag{3.19}$$

and the uniform convergence (3.14) implies that

$$\text{for } n' \text{ sufficiently large, } \forall x \in [x_0 - \delta, x_0 + \delta] \text{ one has } u_{n'}(x) \geq \frac{\eta}{2}. \tag{3.20}$$

On the other hand, in view of the Morrey's inequality (1.22), and of (3.7) and (3.2), we have

$$\|u_{n'}\|_{L^\infty(0,L)} \leq \sqrt{L}\left\|\frac{du_{n'}}{dx}\right\|_{L^2(0,L)} \leq \frac{\sqrt{L}}{\alpha}\|g_{n'}\|_{L^2(0,L)} \leq \frac{\sqrt{L}}{\alpha}c_0, \tag{3.21}$$

therefore one has in view of (3.20)

$$\text{for } n' \text{ sufficiently large, } \forall x \in [x_0 - \delta, x_0 + \delta] \quad \frac{\eta}{2} \leq u_{n'}(x) \leq \frac{\sqrt{L}}{\alpha}c_0. \tag{3.22}$$

Since $\phi_{n'}$, which is a reasonable approximation of ϕ, satisfies

$$\phi_{n'} \to \phi \text{ uniformly in } C^0\left(\left[\frac{\eta}{2}, \frac{\sqrt{L}}{\alpha}c_0\right]\right), \tag{3.23}$$

see (2.51); we deduce that

$$\text{for } n' \text{ sufficiently large, } \phi_{n'} \text{ is bounded in } L^\infty(x_0 - \delta, x_0 + \delta). \tag{3.24}$$

From (3.15), (3.1), (3.7), (3.2), and (3.24), we deduce that

$$\text{for } n' \text{ sufficiently large, } c_{n'} \text{ is bounded in } L^2(x_0 - \delta, x_0 + \delta), \tag{3.25}$$

which implies that

$$c_{n'} \text{ is bounded in } \mathbb{R}, \tag{3.26}$$

which proves the claim (3.17).

Turning back to (3.15), the estimates (3.1), (3.7), (3.2), and (3.26) together imply that

$$\phi_{n'}(u_{n'}) \text{ is bounded in } L^2(0, L). \tag{3.27}$$

We have proved that (3.12) holds true.

The weak convergence (3.13) of $\phi_{n'}(u_{n'})$ to $\phi(u)$ then results from (3.27) and (3.10), since a bounded sequence $z_{n'}$ in $L^p(\Omega)$ which converges a.e. in Ω to some z also converges to z weakly in $L^p(\Omega)$ when $1 < p < +\infty$ (this results from Vitali's theorem since the sequence $z_{n'}$ is equi-integrable in $L^1(\Omega)$, and therefore it converges strongly in $L^1(\Omega)$).

Lemma 3.1 is proved.

□

Remark 3.2. As we said at the beginning of its Step 1, the proof of Lemma 3.1 strongly uses the assumption $N = 1$.

On the other hand, the proof given in the second step is very surprising, since it consists to transform the local estimate (3.24), which is only valid in $(x_0 - \delta, x_0 + \delta)$, into the global estimate (3.27), which is valid in $(0, L)$. This passing from local to global is also specific to the dimension $N = 1$.

□

Remark 3.3. It is assumed in hypothesis (3.2) that g_n converges weakly to g in $L^2(0, L)$. If this hypothesis is reinforced in

$$g_n \to g \text{ strongly in } L^2(0, L), \tag{3.28}$$

then the weak convergence (3.8) is reinforced in

$$u_{n'} \to u \text{ strongly in } H_0^1(0, L). \tag{3.29}$$

Indeed once a subsequence n' has been extracted for which one has (3.8) for some $u \in H_0^1(0, L)$, one has

$$\int_0^L g_{n'}\frac{du_{n'}}{dx}\,dx \to \int_0^L g\frac{du}{dx}\,dx, \quad \text{as } n' \to +\infty. \tag{3.30}$$

when the strong convergence (3.28) holds true. Then either $u \equiv 0$, in which case (3.5) implies that

$$u_{n'} \to 0 \text{ strongly in } H_0^1(0, L),$$

or $u \not\equiv 0$, in which case, in view of Theorem 3.4 (Alternative) below, u is a weak solution of problem (2.1) in the sense of Definition 2.6, which therefore satisfies the energy equality (2.23) in view of Proposition 2.10. Passing to the limit in (3.5) and using (3.30) proves that

$$\int_0^L a_{n'}(x)\frac{du_{n'}}{dx}\frac{du_{n'}}{dx}\,dx \to \int_0^L a(x)\frac{du}{dx}\frac{du}{dx}\,dx, \quad \text{as} \quad n' \to +\infty.$$

which with the weak convergence (3.8) implies the strong convergence (3.29) by passing to the limit in

$$\begin{aligned}\alpha\int_0^L \left|\frac{du_{n'}}{dx}-\frac{du}{dx}\right|^2 dx \le \int_0^L a_{n'}(x)\left(\frac{du_{n'}}{dx}-\frac{du}{dx}\right)\left(\frac{du_{n'}}{dx}-\frac{du}{dx}\right)dx =\\ = \int_0^L a_{n'}(x)\frac{du_{n'}}{dx}\frac{du_{n'}}{dx}\,dx - 2\int_0^L a_{n'}(x)\frac{du}{dx}\frac{du_{n'}}{dx}\,dx+\\ + \int_0^L a_{n'}(x)\frac{du}{dx}\frac{du}{dx}\,dx.\end{aligned}$$

□

3.3. **An alternative.** From the results obtained in Subsection 3.1, and from Lemma 3.1 of Subsection 3.2, we deduce that we are in front of *an alternative*:

Theorem 3.4 (**Alternative**). *Assume that hypothesis* (2.2) *holds true, and that the data* (a, g, ϕ) *satisfy hypotheses* (2.3)-(2.5) *and* (2.6)*. Consider approximations* (a_n, g_n, ϕ_n) *which satisfy* (3.1)-(3.3) *and* (3.4)*.*

Then for every $n \in \mathbb{N}$ *there exists at least one function* u_n *which satisfies* (3.6) *and* (3.7)*. If one extract a subsequence, denoted by* $u_{n'}$*, such that* (3.8) *holds for some* $u \in H_0^1(0, L)$*, then one has the alternative:*

- *either* $u \equiv 0$*,*
- *or* u *is a weak solution of problem* (2.1) *in the sense of Definition 2.6.*

Remark 3.5. Let us emphasize that in the whole of the present section, and in particular in Theorem 3.4, we have assumed hypothesis (2.6), namely $\phi(0) = +\infty$. If we do not make this hypothesis, but $\phi(0) < +\infty$, we have $\phi \in C^0(\mathbb{R})$ and we are in the hypotheses of Proposition 2.15 with good approximations ϕ_n which converge uniformly on $C^0([-R, +R])$ for every $R < +\infty$. In this classical setting, $u \equiv 0$ can be a solution[2], and there is no alternative: all the converging subsequences $u_{n'}$ converge to a classical weak solution, see the proof of Proposition 2.15.

In Theorem 3.4 the alternative is indeed due to the fact that ϕ is singular in $s = 0$.

□

Let us complete Theorem 3.4 by a result which characterizes the behaviour of the constant c_n which appears in (3.15), and also the behaviour of $\phi_n(u_n)$.

To this aim observe that, since $c_n \in \mathbb{R}$ for any given n, but without any bound on $|c_n|$, we are at liberty to extract from n' a further subsequence denoted by n'' such that

$$\exists\, \tilde{c} \in \mathbb{R}\cup\{+\infty\}\cup\{-\infty\} \text{ such that } c_{n''} \to \tilde{c} \text{ in } \mathbb{R}\cup\{+\infty\}\cup\{-\infty\}, \text{ as } n'' \to +\infty. \tag{3.31}$$

We then have the following result, which describes the links between the possible limits of u_n, $\phi_n(u_n)$, and c_n.

Proposition 3.6. *Assume that hypothesis* (2.2) *holds true, and that the data* (a, g, ϕ) *satisfy hypotheses* (2.3)-(2.5) *and* (2.6)*. Consider approximations* (a_n, g_n, ϕ_n) *which satisfy* (3.1)-(3.3) *and* (3.4)*. For* $\tilde{c}$ *and* n'' *defined by* (3.31)*, we have the following equivalences:*

$$\begin{cases} u = 0 \Leftrightarrow \tilde{c} = -\infty \Leftrightarrow \\ \Leftrightarrow \forall M \in \mathbb{R},\ \phi_{n''}(u_{n''}(x)) \ge M,\ \forall x \in (0, L), \quad \textit{for } n'' \textit{ sufficiently large} \Leftrightarrow \\ \Leftrightarrow \phi_{n''}(u_{n''}(x)) \to +\infty \quad \textit{uniformly in } [0, L], \quad \textit{as } n'' \to +\infty. \end{cases} \tag{3.32}$$

$$\begin{cases} u \ne 0 \Leftrightarrow -\infty < \tilde{c} < +\infty \Leftrightarrow \\ \Leftrightarrow \phi_{n''}(u_{n''}) \rightharpoonup \phi(u) \quad \textit{weakly in } L^2(0, L), \quad \textit{as} \quad n'' \to +\infty. \end{cases} \tag{3.33}$$

[2] As a side remark, note that when $\phi \in C^0(\mathbb{R})$ (or in other terms when $\phi(0) < +\infty$), $u \equiv 0$ is a classical weak solution of problem (2.1) if and only if g is constant (see (2.33)).

Remark 3.7. Since the limit u of a subsequence $u_{n'}$ can only be equal to $u = 0$, or to $u \neq 0$, taking into account the equivalences in (3.32) and in (3.33), one sees that the limit $\tilde{c}$ of a subsequence $c_{n''}$ can never be equal to $+\infty$ (but only be either finite or equal to $-\infty$), and that the limit $\phi(u)$ of a subsequence $\phi_{n''}(u_{n''})$ can only be equal to $+\infty$, or to $\phi(u)$ for u a weak solution of problem (2.1) in the sense of Definition 2.6.

□

Proof of Proposition 3.6. **Step 1: The case $u \neq 0$.** In this case, we have proved in Step 2 of the proof of Lemma 3.1 that (see (3.26)-(3.27))

$$u \neq 0 \Rightarrow c_{n''} \text{ is bounded in } \mathbb{R} \Rightarrow \phi_{n''}(u_{n''}) \text{ is bounded in } L^2(0, L),$$

which implies that (see the last paragraph of the proof of Lemma 3.1)

$$\begin{cases} u \neq 0 \Rightarrow -\infty < \tilde{c} < +\infty \Rightarrow \\ \Rightarrow \phi_{n''}(u_{n''}) \rightharpoonup \phi(u) \text{ weakly in } L^2(0, L), \quad \text{as } n'' \to +\infty. \end{cases} \tag{3.34}$$

Step 2: The case $u = 0$. In view of (3.14) $u_{n''}$ converges to 0 uniformly in $C^0([0, L])$, and therefore

$$\forall \eta > 0, \quad |u_{n''}(x)| \leq \eta, \quad \forall x \in [0, L] \quad \text{for } n'' \text{ sufficiently large},$$

which implies that

$$\forall \eta > 0, \quad \phi_{n''}(u_{n''}(x)) \geq \inf_{t \in [-\eta, +\eta]} \phi_{n''}(t) \quad \forall x \in [0, L] \quad \text{for } n'' \text{ sufficiently large}. \tag{3.35}$$

On the other hand, in the second part of Step 2 of the proof of Proposition 2.18, we have proved, see (2.60), that

$$\forall \eta > 0, \quad \liminf_{n''} \left\{ \inf_{t \in [-\eta, +\eta]} \phi_{n''}(t) \right\} = \inf_{t \in [-\eta, +\eta]} \phi(t),$$

which implies that for every $k < \inf_{t \in [-\eta, +\eta]} \phi(t)$, one has

$$\inf_{t \in [-\eta, +\eta]} \phi_{n''}(t) \geq k, \quad \text{for } n'' \text{ sufficiently large}.$$

Since it results from (2.5)-(2.6) that

$$\inf_{t \in [-\eta, +\eta]} \phi(t) \to \phi(0) = +\infty, \quad \text{as } \eta \to 0,$$

we have proved that

$$\forall M \in \mathbb{R}, \quad \inf_{t \in [-\eta, +\eta]} \phi_{n''}(t) \geq M, \quad \text{for } n'' \text{ sufficiently large}. \tag{3.36}$$

Combining (3.35) and (3.36) we have proved that

$$u = 0 \Rightarrow \forall M \in \mathbb{R}, \ \phi_{n''}(u_{n''}(x)) \geq M, \ \forall x \in (0, L), \quad \text{for } n'' \text{ sufficiently large}. \tag{3.37}$$

On the other hand, in view of (3.15) one has

$$c_{n''} + \phi_{n''}(u_{n''}) = a_{n''}(x) \frac{du_{n''}}{dx} - g_{n''}(x) \quad \text{in } L^2(0, L),$$

which combined with (3.37) implies that

$$\forall M \in \mathbb{R}, \quad c_{n''} + M \leq a_{n''}(x) \frac{du_{n''}}{dx} - g_{n''}(x) \quad \text{in } L^2(0, L), \quad \text{for } n'' \text{ sufficiently large}.$$

Since the right-hand side of this inequality is bounded in $L^2(0, L)$ in view of (3.1)-(3.2) and (3.7), integrating on $(0, L)$ and dividing by L implies that there exists a constant $c_0 < +\infty$ such that

$$\forall M \in \mathbb{R}, \quad c_{n''} + M \leq c_0, \quad \text{for } n'' \text{ sufficiently large},$$

which implies that $c_{n''} \to -\infty$ as $n'' \to +\infty$, or in other terms that $\tilde{c} = -\infty$.

We have proved that

$$\begin{cases} u = 0 \Rightarrow \\ \Rightarrow \forall M \in \mathbb{R}, \ \phi_{n''}(u_{n''}(x)) \geq M, \ \forall x \in (0, L), \quad \text{for } n'' \text{ sufficiently large} \Rightarrow \\ \Rightarrow \tilde{c} = -\infty. \end{cases} \tag{3.38}$$

Step 3: Proof of the two equivalences (3.32) and (3.33). We will deduce (3.32) and (3.33) from the two results (3.34) and (3.38), and from the fact that when a subsequence $u_{n''}$ converges weakly to u in $H^1_0(0,L)$ (see (3.8)), then one has the dicotomy "either $u \neq 0$ or $u = 0$".

Indeed, when considering $\tilde{c}$, one deduces from (3.32) and (3.33) and from the dicotomy "either $u \neq 0$ or $u = 0$", that

$$\text{either } \tilde{c} \text{ is finite or } \tilde{c} = -\infty,$$

and that one can not have $\tilde{c} = +\infty$.

Consider first the case when $\tilde{c}$ is finite; then use the dicotomy "either $u \neq 0$ or $u = 0$": if $u = 0$, then by (3.38) $\tilde{c} = -\infty$, which is not the case; therefore

$$\tilde{c} \text{ finite} \Rightarrow u \neq 0.$$

Consider then the case where $\tilde{c} = -\infty$; then use the dicotomy "either $u \neq 0$ or $u = 0$": if $u \neq 0$, then by (3.34) $\tilde{c}$ is finite, which is not the case; therefore

$$\tilde{c} = -\infty \Rightarrow u = 0.$$

As far as $\phi_{n''}(u_{n''})$ is concerned, one deduces from (3.32) and (3.33) and from the dicotomy "either $u \neq 0$ or $u = 0$" that, as $n'' \to +\infty$,

$$\text{either } \phi_{n''}(u_{n''}) \rightharpoonup \phi(u) \text{ weakly in } L^2(0,L) \text{ or } \phi_{n''}(u_{n''}) \to +\infty \text{ uniformly in } [0,L].$$

A proof similar to the proof made just above for $\tilde{c}$ leads to

$$\phi_{n''}(u_{n''}) \rightharpoonup \phi(u) \text{ weakly in } L^2(0,L), \text{ as } n'' \to +\infty \Rightarrow u \neq 0,$$

and to

$$\phi_{n''}(u_{n''}(x)) \to +\infty \text{ uniformly in } [0,L], \text{ as } n'' \to +\infty \Rightarrow u = 0.$$

This completes the proof of the two equivalences (3.32) and (3.33) and of Proposition 3.6. □

At the end of this section, one could think that, except maybe in some very special cases, every limit u of weak solutions of approximations of problem (2.1) which satisfy (3.1)-(3.3) and (3.4) is always a weak solution of problem (2.1) in the sense of Definition 2.6.

We will see in Section 4 below that this is not the case, and that for a large class of functions $g \in L^2(0,L)$ (see Theorem 4.1), and for a large class of functions ϕ (see Theorem 4.4), every limit of approximations is $u \equiv 0$. This is unexpected.

We will also see in Section 7 below that for an another large class of functions ϕ and for another large class of functions g, there exists at least one weak solution of problem (2.1) in the sense of Definition 2.6. This will be also unexpected.

4. Non-existence results

In this section we give two results of non-existence of a weak solution of problem (2.1) in the sense of Definition 2.6.

Our first non-existence result states, in particular, that there is no weak solution of problem (2.1) in the sense of Definition 2.6 when $g \in L^\infty(0,L)$. This result is obtained independently of the nonlinearity ϕ, provided $\phi(0) = +\infty$, i.e (2.6) holds true.

Theorem 4.1 (Non-existence when g is bounded from below). *Assume that hypothesis (2.2) holds true, and that the data (a, g, ϕ) satisfy hypotheses (2.3)-(2.5) and (2.6). Assume moreover that exists $M > 0$ such that*

$$g(x) \geq -M \quad \text{for a.e. } x \in (0,L). \tag{4.1}$$

Then it does not exist any weak solution of problem (2.1) in the sense of Definition 2.6.

Remark 4.2. Observe that Theorem 4.1 implies that if, for a given nonlinearity ϕ, $\hat{u}$ is a weak solution of problem (2.1) in the sense of Definition 2.6 corresponding to a source term g (we will see in Section 7 below that there exist many data (a, g, ϕ) for which there exist weak solutions of problem (2.1) in the sense of Definition 2.6), one can not hope to approximate $\hat{u}$ by approximating $\hat{g}$ by any sequence $\hat{g}_n$ which approximate $\hat{g}$ (weakly or strongly) in $L^2(0,L)$: in view of Theorem 4.1, it is indeed sufficient to approximate $\hat{g}$ by a sequence $\hat{g}_n \in L^\infty(0,L)$ which converges to g (weakly or even strongly) in $L^2(0,L)$, since for those $\hat{g}_n$ there is no weak solution u_n of problem (2.1) in the sense of Definition 2.6 with the source term g_n.

□

The proof of Theorem 4.1 will use the following Lemma:

Lemma 4.3 (**The forbidden region**). *Assume that hypothesis* (2.2) *holds true, and consider data* (a, l, ϕ) *which satisfy hypotheses* (2.3)-(2.5) *and* (2.6). *Let* w *which satisfies*

$$\begin{cases} w \in H^1(0,L)\,, \quad \phi(w) \in L^2(0,L), \\ a(x)\dfrac{dw}{dx} = \phi(w) + l \quad in \quad \mathcal{D}'(0,L)\,. \end{cases} \tag{4.2}$$

Let A, B *and* x_0 *be such that*

$$0 \le A < B \le L, \quad x_0 \in [A,B], \quad w(x_0) = 0\,. \tag{4.3}$$

If l *satisfies*

$$\exists\, M > 0, \quad l(x) \ge -M \quad \textit{for a.e. } x \in [A,B]\,, \tag{4.4}$$

then one has

$$\begin{cases} \forall\ k > 0, \quad \exists\ \delta > 0 \quad \textit{such that} \\ \dfrac{dw}{dx} \ge k, \quad \textit{a.e. } x \in [x_0 - \delta, x_0 + \delta] \cap [A,B], \\ w(x) \ge k(x - x_0), \ \forall\ x \in [x_0, x_0 + \delta] \cap [A,B], \\ w(x) \le k(x - x_0), \ \forall\ x \in [x_0 - \delta, x_0] \cap [A,B]. \end{cases} \tag{4.5}$$

Observe that formula (4.5) implies that the graph of the function w can not enter in the *forbidden region* colored red in Figure 1, when x_0, which is a zero of w, can be either an interior point of $[A,B]$ or an extremity of $[A,B]$, namely $x_0 = A$ or $x_0 = B$.

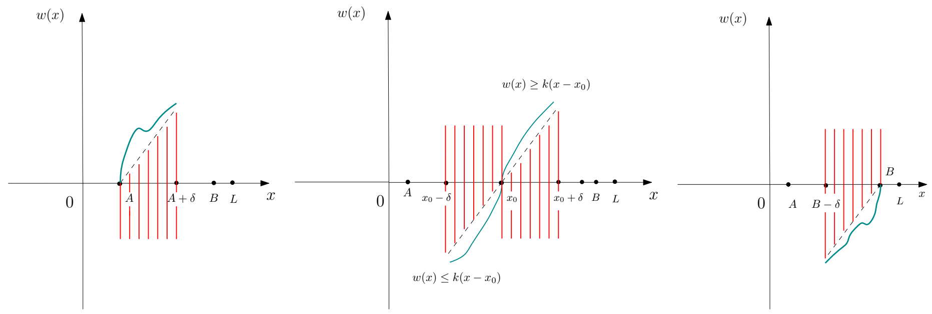

FIGURE 1. Visualizing the statement of Lemma 4.3:
the extreme case $x_0 = A$, the case $x_0 \in (A,B)$ and the extreme case $x_0 = B$.

Proof of Lemma 4.3. The main idea of the proof is that the facts that $\phi(w(x_0)) = +\infty$ and $l \ge -M$ and the equation of the second line of (4.2), formally imply that $\dfrac{dw}{dx}(x_0) = +\infty$, and therefore that $\dfrac{dw}{dx}(x)$ is very large when x is close to x_0. Let us write this idea in a correct mathematical form.

Because $N = 1$ one has $H^1(A,B) \subset C^0([A,B])$ by Morrey's theorem (see (1.23)). Since $w(x_0) = 0$, then for every fixed $\varepsilon > 0$ there exists $\delta > 0$, $\delta = \delta(\varepsilon)$, such that

$$\forall\, x \in \mathcal{V}_\delta = [x_0 - \delta, x_0 + \delta] \cap [A,B], \quad |w(x)| \le \varepsilon.$$

Then,

$$\forall\, x \in \mathcal{V}_\delta \quad |\phi(w(x))| \ge \inf_{t \in [-\varepsilon, +\varepsilon]} \phi(t),$$

so that in view of (4.4) and (4.2), one has

$$a(x)\frac{dw}{dx} \ge \inf_{t \in [-\varepsilon, +\varepsilon]} \phi(t) - M, \text{for a.e. } \ x \in \mathcal{V}_\delta. \tag{4.6}$$

Since $\inf_{t\in[-\varepsilon,+\varepsilon]} \phi(t) \to +\infty$ as $\varepsilon \to 0$, the left hand side of (4.6) is non-negative for ε sufficiently small, so that $\dfrac{dw}{dx} \geq 0$ and, by (2.3), for ε sufficiently small, one has

$$\beta \frac{dw}{dx} \geq a(x)\frac{dw}{dx} \geq \inf_{t\in[-\varepsilon,+\varepsilon]} \phi(t) - M, \quad \text{for a.e. } x \in \mathcal{V}_\delta.$$

Dividing by β and choosing ε sufficiently small, this implies that, for every $k > 0$, one has for some $\delta > 0$

$$\frac{dw}{dx} \geq k \quad \text{a.e. in } \mathcal{V}_\delta\,, \tag{4.7}$$

which immediately implies, using also $w(x_0) = 0$, that also the two latest lines of (4.5) hold true. Lemma 4.3 is proved.

□

Proof of Theorem 4.1. Assume by contradiction that there exists some u which is a weak solution of problem (2.1) in the sense of Definition 2.6. Then, by Proposition 2.13, u satisfies

$$\begin{cases} u \in H^1(0,L), \quad \phi(u) \in L^2(0,L), \\ u(0) = u(L) = 0, \\ a(x)\dfrac{du}{dx} = \phi(u) + g + c \quad \text{in } \mathcal{D}'(0,L), \end{cases}$$

where c is given in the last line of (2.31).

Apply Lemma 4.3 with $w = u$, $l = g + c$, $A = 0$, $B = L$, and $x_0 = 0$. Fixing any $k > 0$, one obtains that for some $\delta_0 > 0$ with $\delta_0 < L$,

$$u(x) \geq kx > 0, \quad \forall\, x \in (0, \delta_0]\,. \tag{4.8}$$

Let us define the set $X \subset [0, L]$ and the number y by

$$X = \{x : x \in [\delta_0, L],\, u(x) = 0\}$$
$$y = \inf_{x\in X} x.$$

The set X is non-empty since $L \in X$; therefore y is correctly defined. In the case where y is not a minimum, let x_n be a minimizing sequence, i.e. a sequence which satisfies

$$x_n \in X, \text{ i.e. } x_n \in [\delta_0, L],\, u(x_n) = 0, \text{ and } x_n \to y.$$

Then $y \in [\delta_0, L]$ and $u(y) = 0$ since u is continuous. Since $u(\delta_0) \geq k\delta_0 > 0$ by (4.8), one has $L \geq y > \delta_0 > 0$. The same holds in the case where y is a minimum.

Then apply again Lemma 4.3, now with $w = u$, $l = g + c$, $A = \delta_0$, $B = y$, and $x_0 = y$. Fixing any $k > 0$, one obtains that for some $\delta_y > 0$ with $y - \delta_y \geq \delta_0$,

$$u(x) \leq k(x - y) < 0, \quad \forall\, x \in [y - \delta_y, y)\,. \tag{4.9}$$

Now observe that $u(\delta_0) > 0$ and $u(y - \delta_y) < 0$. Since u is continuous there exists some y_0 such that $\delta_0 < y_0 < y - \delta_y < y$ with $u(y_0) = 0$, which contradicts the definition of y. Theorem 4.1 is proved.

□

Our second non-existence result is obtained instead independently of the source term g. It asserts that when the singularity of ϕ at $s = 0$ is too strong, and, more precisely, when ϕ is not integrable both in 0^+ and 0^-, then it does not exist any weak solution of problem (2.1) in the sense of Definition 2.6.

Theorem 4.4 (**Non-existence when the singularity is too strong**). *Assume that hypothesis* (2.2) *holds true, and that the data* (a, g, ϕ) *satisfy hypotheses* (2.3)-(2.5) *and* (2.6). *Assume moreover that* ϕ *satisfies*

$$\int_0^{+\delta} \phi(t)\, dt = +\infty\,, \ \forall\delta,\ 0 < \delta < 1, \tag{4.10}$$

and

$$\int_{-\delta}^{0} \phi(t)\, dt = +\infty, \ \forall\delta,\ 0 < \delta < 1. \tag{4.11}$$

Then it does not exist any weak solution of problem (2.1) *in the sense of Definition 2.6.*

Remark 4.5. In the model case (2.10) where the function ϕ is given by $\phi = \phi_\gamma$ defined by

$$\phi_\gamma(s) = \frac{c}{|s|^\gamma} + \varphi(s), \text{ with } c > 0,\ \gamma > 0,\ \varphi \in C^0(\mathbb{R}), \tag{4.12}$$

hypotheses (4.10) and (4.11) are satisfied if and only if $\gamma \geq 1$.

In this case the proof of Theorem 4.4 is very simple. Assume indeed by contradiction that u is a weak solution of problem (2.1) in the sense of Definition 2.6, and let $x_0 \in [0, L]$ be such that $u(x_0) = 0$. Recalling Morrey's embedding $H^1_0(0, L) \subset C^{0,\frac{1}{2}}([0, L])$ (see (1.23)), one has

$$|u(x)| = |u(x) - u(x_0)| \leq \|u\|_{C^{0,\frac{1}{2}}([0,L])} |x - x_0|^{\frac{1}{2}}.$$

Using (4.12) and recalling that $u \in H^1(0, L)$ and therefore $\varphi(u)$ is bounded, one has

$$\phi_\gamma(u(x)) = \frac{c}{|u(x)|^\gamma} + \varphi(u(x)) \geq \frac{c}{\|u\|^\gamma_{C^{0,\frac{1}{2}}([0,L])} |x - x_0|^{\frac{\gamma}{2}}} + \inf_{|s| \leq \|u\|_{L^\infty(0,L)}} |\varphi(s)|,$$

from which one deduces that $\phi_\gamma(u(x)) \notin L^2(0, L)$ if $\gamma \geq 1$, a contradiction.

Observe that this proof continues to hold in the case where (2.10) is only assumed to be in force on a neighborhood of $s = 0$.

□

Remark 4.6. Let us remark that, when ϕ satisfies hypothesis (2.5), it is equivalent to make hypotheses (4.10) and (4.11) for every δ, $0 < \delta < 1$, or to assume that there exists some $\delta_0 > 0$ such that hypothesis (4.10) and (4.11) hold true for this fixed δ_0.

□

Theorem 4.4 immediately follows from the following proposition, which has its own interest.

Proposition 4.7. *Assume that hypothesis* (2.2) *holds true, and that the data* (a, g, ϕ) *satisfy hypotheses* (2.3)-(2.5) *and* (2.6). *Assume moreover that* ϕ *satisfies* (4.10). *Then every possible weak solution of problem* (2.1) *in the sense of Definition 2.6 is non-positive.*

Similarly, assume that ϕ *satisfies* (4.11). *Then every possible weak solution of problem* (2.1) *in the sense of Definition 2.6 is non-negative.*

Proposition 4.7 is itself an immediate consequence of the following result, where the set $\mathcal{U}$ is defined by

$$\mathcal{U} = \{u \in H^1_0(0, L) \text{ such that } \ \phi(u) \in L^2(0, L)\}; \tag{4.13}$$

observe that every solution u of problem (2.1) in the sense of Definition 2.6 belongs to $\mathcal{U}$, while $0 \notin \mathcal{U}$.

Proposition 4.8. *Assume that hypothesis* (2.2) *holds true, let* ϕ *be a nonlinearity satisfying* (2.5) *and* (2.6)*, and let* $u \in \mathcal{U}$. *Then if* ϕ *satisfies* (4.10)*, one has*

$$u(x) \leq 0, \quad \forall x \in [0, L]; \tag{4.14}$$

if ϕ *satisfies* (4.11)*, one has*

$$u(x) \geq 0, \quad \forall x \in [0, L]. \tag{4.15}$$

Proof of Proposition 4.8. We will prove that hypothesis (4.10) implies (4.14). In the case where (4.11) is assumed instead of (4.10), the proof of (4.15) is similar.

Assume that (4.10) holds true and assume by contradiction that $u \in \mathcal{U}$ is such that for some $x_0 \in (0, L)$ one has $u(x_0) > 0$.

Define y_0 by

$$y_0 = \inf\{x \in [0, x_0] \text{ such that } u(z) > 0 \text{ for all } z \in (x, x_0]\} \tag{4.16}$$

(one could also consider $y_1 = \sup\{x \in [x_0, L]$ such that $u(z) > 0$ for all $z \in [x_0, x))$. Observe that y_0 is well defined since the set $\{x \in [0, x_0]$ such that $u(z) > 0$ for all $z \in (x, x_0]\}$ is not empty as it contains x_0, and that $0 \leq y_0 < x_0$ since u is continuous. One has

$$u(y_0) = 0; \tag{4.17}$$

indeed, since $u(x) > 0$ for every $x \in (y_0, x_0]$, one has $u(y_0) \geq 0$; but if $u(y_0) > 0$, there exists $\delta > 0$ such that $u(x) > 0$ for every $x \in [y_0 - \delta, y_0 + \delta]$, a contradiction with the definition (4.16) of y_0.

On the other hand, fix $\delta > 0$ and define the function $\hat{\psi}_\delta :]0, +\infty[\mapsto \mathbb{R}$ by

$$\hat{\psi}_\delta(s) = \int_s^\delta \phi(t)dt, \quad \forall s > 0,$$

or equivalently by

$$\hat{\psi}_\delta(\delta) = 0, \quad \hat{\psi}'_\delta(s) = -\phi(s), \quad \forall s > 0;$$

(please do not confuse this function $\hat{\psi}_\delta$ with the function ψ_n used in the proof of Lemma 2.12 above, whose definition is recalled in (5.17) below). Then (4.10) is equivalent to

$$\hat{\psi}_\delta(s) \to +\infty \quad \text{as } s \to 0,\ s > 0. \tag{4.18}$$

Recall that $u \in H_0^1(\Omega) \subset\subset C^0([0, L])$, and for η such that $0 < \eta < x_0 - y_0$, define the two real numbers $\underline{u}_\eta$ and $\overline{u}$ by

$$\underline{u}_\eta = \min_{x \in [y_0+\eta, x_0]} u(x), \quad \overline{u} = \max_{x \in [y_0, x_0]} u(x),$$

and observe that

$$\forall \eta, \quad 0 < \eta < x_0 - y_0, \text{ one has } \ 0 < \underline{u}_\eta \leq u(x) \leq \overline{u} < +\infty, \quad \forall x \in [y_0 + \eta, x_0].$$

Define also the two real numbers

$$\underline{\phi} = \min_{s \in [0, \overline{u}]} \phi(s), \quad \overline{\phi}_\eta = \max_{s \in [\underline{u}_\eta, \overline{u}]} \phi(s),$$

and observe that $\overline{\phi}_\eta$ is finite for every η, $0 < \eta < x_0 - y_0$, even if it is unbounded as $\eta \to 0$.

Then since $u \in H^1(y_0 + \eta, x_0)$ and since

$$\phi \in C_b^0([\underline{u}_\eta, \overline{u}]), \quad \text{which implies that} \quad \hat{\psi}_\delta \in C^1([\underline{u}_\eta, \overline{u}]),$$

one has the chain rule

$$\phi(u)\frac{du}{dx} = -\hat{\psi}'_\delta(u)\frac{du}{dx} = -\frac{d\hat{\psi}_\delta(u)}{dx} \quad \text{in } \ L^2(y_0 + \eta, x_0),$$

and therefore

$$\int_{y_0+\eta}^{x_0} \phi(u)\frac{du}{dx}dx = \int_{y_0+\eta}^{x_0} -\frac{d\hat{\psi}_\delta(u)}{dx}dx = \hat{\psi}_\delta(u(y_0+\eta)) - \hat{\psi}_\delta(u(x_0)), \quad \forall \eta, \quad 0 < \eta < x_0 - y_0.$$

Now $\hat{\psi}_\delta(u(x_0))$ is finite, while in view of (4.17) and (4.18) one has

$$u(y_0 + \eta) \to 0 \quad \text{and} \quad \hat{\psi}_\delta(u(y_0 + \eta)) \to +\infty, \quad \text{as} \quad \eta \to 0, \quad \eta > 0,$$

which implies that

$$\int_{y_0+\eta}^{x_0} \phi(u)\frac{du}{dx}dx \to +\infty, \quad \text{as } \eta \to 0,\ \eta > 0, \tag{4.19}$$

a contradiction since $u \in \mathcal{U}$. This proves Proposition 4.8.

□

5. Studying an (associated) ODE

In this section we will study an Ordinary Differential Equation (ODE) that for the moment we formally write as

$$\begin{cases} a(x)\dfrac{dv}{dx} = \phi(v) + h(x) & \text{in } (0, L)\,, \\ v(0) = 0. \end{cases} \tag{5.1}$$

Under convenient hypotheses, we will prove an existence result, an a priori estimate, and two stability results (Subsection 5.1), a positivity result, a comparison result, and an uniqueness result (Subsection 5.2). For the sake of exposition these results are summarized in the brief Subsection 5.3.

The ODE (5.1) is clearly strongly related to problem (2.1), see e.g. (2.30) in Proposition 2.13 above. But in the present section we will not try to make any connection between the two problems and we will study the ODE (5.1) for itself. The results of the present section will then be exploited in the following sections, and in particular in Section 6 to obtain multiplicity results for problem (2.1).

In order to emphasize the difference between the present study of ODE (5.1) and the study of problem (2.1), we will denote by v (and not by u as in problem (2.1)) the solution of the ODE (5.1). We will also denote by (a, h, ϕ) (and not by (a, g, ϕ)) the data for ODE (5.1).

In the whole section we will assume that the data (a, h, ϕ) satisfy the following hypotheses (see Remark 5.2 below for a comparison with the hypotheses (2.3)-(2.5) on the data (a, g, ϕ) for problem (2.1)):

$$a \in L^\infty(0, L),\ \exists\ \alpha, \beta,\ 0 < \alpha < \beta,\ \alpha \le a(x) \le \beta \quad \text{a.e. } x \in (0, L), \tag{5.2}$$

$$h \in L^2(0, L), \tag{5.3}$$

$$\begin{cases} \phi : \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\},\ \phi \ \text{ is continuous with values in } \mathbb{R} \cup \{+\infty\}, \\ \phi(s) < +\infty, \quad \forall s \in \mathbb{R}, \quad s \neq 0\,, \end{cases} \tag{5.4}$$

$$\int_0^{+\delta} \phi(t)\, dt < +\infty, \quad \int_{-\delta}^0 \phi(t)\, dt < +\infty,\ \forall \delta,\ 0 < \delta < 1, \tag{5.5}$$

$$\phi \in C_b^0(\mathbb{R} \backslash (-\delta, \delta)), \quad \forall\ \delta > 0. \tag{5.6}$$

Remark 5.1. When $\phi(0) < +\infty$, then (5.5) is automatically satisfied, and, due to (5.6), one has $\phi \in C_b^0(\mathbb{R})$. In this case the results of the present section are classical. But the interest is actually in the case

$$\phi(0) = +\infty,$$

where the results of this section are new.

□

Remark 5.2. Hypothesis (5.2) on a, (5.3) on h, and (5.4) on ϕ are identical to hypotheses (2.3) on a, (2.4) on g, and (2.5) on ϕ made in Section 2 above.

In contrast, hypotheses (5.5) and (5.6) on ϕ are new and restrictive in comparison with the hypotheses made in Section 2 above.

Hypothesis (5.5) it is quite natural in this context; recall in fact that it is proved in Theorem 4.4 above that if ϕ satisfies (4.10) and (4.11), namely if

$$\int_0^{+\delta} \phi(t)\, dt = +\infty, \quad \int_{-\delta}^0 \phi(t)\, dt = +\infty,\ \forall \delta,\ 0 < \delta < 1, \tag{5.7}$$

(compare with hypothesis (5.5)), then problem (2.1) does not have any weak solution in the sense of Definition 2.6.

Concerning hypothesis (5.6), observe that this new hypothesis impose a restriction on the function ϕ in comparison with the hypotheses made in Section 2: indeed hypothesis (5.6) impose that ϕ is bounded at $s = -\infty$ and $s = +\infty$, or in other terms that $\phi \in C_b^0(\mathbb{R} \backslash (-\delta, +\delta))$ for every $\delta > 0$ (compare with (2.9)). However this restriction can be considered as tolerable due to the uniform boundedness of every possible solutions of problem (5.1) (see Remark 2.5 above).

□

Remark 5.3. Observe that, when ϕ satisfies hypothesis (5.4), it is equivalent to make hypothesis (5.5) for every δ, $0 < \delta < 1$, and hypothesis (5.6) for every δ, $\delta > 0$, or to assume that there exists some $\delta_0 > 0$ such that hypotheses (5.5) and (5.6) hold true for this δ_0.

□

Remark 5.4. In the ODE (5.1) we have assumed that the (Cauchy) initial condition is $v(0) = 0$. Results similar to the ones stated and proved in the present section could be obtained for any arbitrary initial condition $v(0) = v_0 \in \mathbb{R}$. We do not consider this possibility here since our interest in the present paper is only in the case where $v(0) = 0$.

□

5.1. **Existence of a solution of the Cauchy problem** (5.8)**.** The mathematical (correct) formulation of initial value problem associated to the ODE in (5.1) that we will use in this paper is the following: we look for a function v which satisfies

$$\begin{cases} v \in H^1(0,L), \ \phi(v) \in L^2(0,L)\,, \\ a(x)\dfrac{dv}{dx} = \phi(v) + h \ \text{ in } \ \mathcal{D}'(0,L), \\ v(0) = 0\,. \end{cases} \tag{5.8}$$

In this subsection we will prove the following existence result:

Theorem 5.5 (**Existence**)**.** *Assume that* (2.2) *holds true, and that the data* (a, h, ϕ) *satisfy hypotheses* (5.2)-(5.6)*. Then there exists at least a solution of problem* (5.8)*.*

The proof of the existence Theorem 5.5 is based on the two propositions 5.9 and 5.11 below. Before stating and proving these two propositions, let us state and prove a lemma which looks natural but is not so easy to obtain due to the possible singularity of the function ϕ.

Let us define the function $\psi : \mathbb{R} \mapsto \mathbb{R}$ by

$$\psi(s) = \int_0^s \phi(t)dt, \quad \forall\, s \in \mathbb{R}; \tag{5.9}$$

note that in view of hypothesis (5.5) the function ϕ is integrable both in 0^+ and 0^-, and that the function ψ therefore satisfies

$$\psi \in W^{1,1}_{\text{loc}}(\mathbb{R}) \subset C^0(\mathbb{R}) \ \text{ with } \ \psi(0) = 0\,. \tag{5.10}$$

Lemma 5.6. *Assume that* (2.2) *holds true, and that* ϕ *satisfies hypotheses* (5.4)-(5.6)*. Let* z *satisfying*

$$z \in H^1(0,L) \textit{ with } \phi(z) \in L^2(0,L). \tag{5.11}$$

Then the function ψ *defined by* (5.9) *satisfies*

$$\psi(z) \in W^{1,1}(0,L) \textit{ with } \frac{d\psi(z)}{dx} = \phi(z)\frac{dz}{dx} \ \textit{ in } \ \mathcal{D}'(0,L), \tag{5.12}$$

a result which in particular implies that

$$\int_0^L \phi(z)\frac{dz}{dx}dx = \psi(z(L)) - \psi(z(0)). \tag{5.13}$$

Remark 5.7. When $\phi(0) < +\infty$, the result (5.12) is classical since then $\phi \in C^0_b(\mathbb{R})$ in view of (5.4) and (5.6). Indeed in this case one has $\psi' = \phi \in C^0_b(\mathbb{R})$, which implies that $\psi \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$; the classical chain rule in $H^1(0,L)$ then implies that

$$\psi(z) \in H^1(0,L) \ \text{ with } \ \frac{d\psi(z)}{dx} = \phi(z)\frac{dz}{dx} \ \text{ in } \ \mathcal{D}'(0,L),$$

which implies (5.12).

□

Remark 5.8. If $z \in H^1_0(0,L)$, the result (5.13) implies that

$$\int_0^L \phi(z)\frac{dz}{dx}dx = \psi(0) - \psi(0) = 0 - 0 = 0.$$

This result is nothing but Lemma 2.12 above, which we recall was proven under the sole hypothesis (2.5) (identical to (5.4)) without making hypotheses (5.5) and (5.6) on ϕ.

□

Proof of Lemma 5.6. Step 1. Fix $R > 0$. Then, one has

$$\psi(s) = \int_0^s \phi(t)dt = \int_0^{+R} \phi(t)dt + \int_{+R}^s \phi(t)dt, \quad \forall s \geq +R,$$

which, thanks to (5.6), implies that

(5.14)
$$\begin{cases} |\psi(s)| \le \displaystyle\int_0^{+R} |\phi(t)|dt + \|\phi\|_{L^\infty(+R,+\infty)}\ (s-R) \le \\ \\ \qquad\qquad \le \|\phi\|_{L^1(0,+R)} + \|\phi\|_{L^\infty(+R,+\infty)}\ |s|\,,\ \forall\, s \ge +R, \end{cases}$$

a result which in fact holds true for every $s \ge 0$.

Similarly, one has, for $s \le -R$

$$\psi(s) = \int_0^s \phi(t)dt = \int_0^{-R} \phi(t)dt + \int_{-R}^s \phi(t)dt, \quad \forall s \le -R,$$

which implies that

(5.15)
$$\begin{cases} |\psi(s)| \le \displaystyle\int_{-R}^{0} |\phi(t)|dt + \|\phi\|_{L^\infty(-\infty,-R)}\ (-R-s) \le \\ \\ \qquad\qquad \le \|\phi\|_{L^1(-R,0)} + \|\phi\|_{L^\infty(-\infty,-R)}\ |s|\,,\ \forall\, s \le -R, \end{cases}$$

a result which in fact holds true for every $s \le 0$.

These two results imply that when ϕ satisfies (5.4)-(5.6), one has

$$|\psi(s)| \le \|\phi\|_{L^1(-R,+R)} + \|\phi\|_{L^\infty(\mathbb{R}\setminus[-R,+R])}\ |s|\,, \quad \forall\, R > 0, \quad \forall\, s \in \mathbb{R}. \tag{5.16}$$

Step 2. Let T_n be the truncation at height n defined by (1.24). Then $T_n(\phi) \in C_b^0(\mathbb{R})$.

Defining, as it was done in the proof of Lemma 2.12, the function[3] ψ_n by

$$\psi_n(s) = \int_0^s T_n(\phi(t))\, dt, \quad \forall s \in \mathbb{R}, \tag{5.17}$$

one has $\psi_n \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$, so that, as observed in the Remark 5.7 where $\phi \in C_b^0(\mathbb{R})$, one has

$$\psi_n(z) \in H^1(0,L) \ \text{ with } \ \frac{d\psi_n(z)}{dx} = T_n(\phi(z))\frac{dz}{dx} \ \text{ in } \ \mathcal{D}'(0,L). \tag{5.18}$$

Let us now use the fact that $\phi(z) \in L^2(0,L)$.

Since

$$|T_n(\phi(z(x)))| \le |\phi(z(x))|, \ \text{ a.e. } \ x \in (0,L) \ \text{ with } \ T_n(r) \to r \text{ as } n \to +\infty, \ \forall r \in \mathbb{R},$$

Lebesgue's dominated convergence theorem implies that

$$T_n(\phi(z))\frac{dz}{dx} \to \phi(z)\frac{dz}{dx} \ \text{ strongly in } \ L^1(0,L). \tag{5.19}$$

On the other hand, in view of (5.16), one has

(5.20)
$$\begin{aligned} |\psi_n(s)| &\le \|T_n(\phi)\|_{L^1(-R,+R)} + \|T_n(\phi)\|_{L^\infty(\mathbb{R}\setminus[-R,+R])}|s| \le \\ &\le \|\phi\|_{L^1(-R,+R)} + \|\phi\|_{L^\infty(\mathbb{R}\setminus[-R,+R])}\ |s|, \quad \forall k > 0, \ \forall s \in \mathbb{R}. \end{aligned}$$

Since $z \in H^1(0,L) \subset L^\infty(0,L)$, this implies that

$$\psi_n(z) \ \text{ is bounded in } \ L^\infty(0,L). \tag{5.21}$$

Also, for every $s \in \mathbb{R}$, one has by (5.5)

$$\phi \in L^1(0,s) \ \text{ if } \ s > 0, \ \text{ and } \ \phi \in L^1(s,0) \ \text{ if } \ s < 0,$$

while

$$\begin{cases} |T_n(\phi(t))| \le |\phi(t)|\,, \forall\, t \in (0,s) \ \text{ if } \ s > 0, \ \text{ and } \ \forall t \in (s,0) \ \text{ if } \ s < 0, \\ \text{with } T_n(r) \to r \text{ as } n \to +\infty, \ \forall r \in \mathbb{R}, \end{cases}$$

[3] Please do not confuse this function ψ_n with the function $\hat{\psi}_\delta$ used in the proof of Proposition 4.8

so that Lebesgue's dominated convergence theorem (in $L^1(0,s)$ and in $L^1(s,0)$) implies that

$$\psi_n(s) = \int_0^s T_n(\phi(t))dt \to \int_0^s \phi(t)dt = \psi(s), \quad \forall s \in \mathbb{R}. \tag{5.22}$$

From (5.21) and (5.22) one deduces (using again Lebesgue's dominated convergence theorem) that

$$\psi_n(z) \to \psi(z) \quad \text{strongly in} \quad L^p(0,L) \quad \forall p,\ 1 \le p < +\infty. \tag{5.23}$$

This fact implies that $\dfrac{d\psi_n(z)}{dx}$ converges to $\dfrac{d\psi(z)}{dx}$ in $\mathcal{D}'(0,L)$ and also, by (5.18) and (5.19), strongly in $L^1(0,L)$; this implies that $\dfrac{d\psi(z)}{dx} = \phi(z)\dfrac{dz}{dx}$, and therefore that $\psi(z) \in W^{1,1}(0,L)$.

Lemma 5.6 is proved.

□

Proposition 5.9 (**A priori estimate**). *Assume that* (2.2) *holds true, and that the data* (a,h,ϕ) *satisfy hypotheses* (5.2)-(5.6)*. If* v *is any solution of the problem* (5.8)*, then, for any given* $R>0$*,* v *satisfies*

$$\|v\|_{H^1(0,L)} \le C_R\,, \tag{5.24}$$

where C_R *is given by*

$$C_R = (L+1)\left(\frac{\sqrt{L}\|\phi\|_{L^\infty(\mathbb{R}\setminus[-R,+R])} + \|h\|_{L^2(0,L)}}{\alpha} + \frac{\sqrt{\|\phi\|_{L^1(-R,+R)}}}{\sqrt{\alpha}}\right), \tag{5.25}$$

which depends only on R*,* L*,* α*,* $\|h\|_{L^2(0,L)}$*,* $\|\phi\|_{L^1(-R,+R)}$*, and* $\|\phi\|_{L^\infty(\mathbb{R}\setminus(-R,+R))}$*.*

Proof of Proposition 5.9. Multiplying pointwise the second line of (5.8) by $\dfrac{dv}{dx}$ and integrating between 0 and L, we get

$$\int_0^L a(x)\left|\frac{dv}{dx}\right|^2 dx = \int_0^L \phi(v)\frac{dv}{dx}dx + \int_0^L h(x)\frac{dv}{dx}dx\,. \tag{5.26}$$

Using in (5.26) the coercivity of a (see (5.2)), the result (5.13), $\psi(v(0)) = \psi(0) = 0$, and the Cauchy-Schwartz inequality implies that

$$\alpha\int_0^L \left|\frac{dv}{dx}\right|^2 dx \le |\psi(v(L))| + \|h\|_{L^2(0,L)}\left\|\frac{dv}{dx}\right\|_{L^2(0,L)}. \tag{5.27}$$

Since $v \in H^1(0,L)$ with $v(0)=0$, (5.16) combined with Morrey's estimate (1.22) (which continues to hold true for $z \in H^1(0,L)$ with $z(0)=0$ without assuming that $z(L)=0$), yields

$$\begin{aligned} &|\psi(v(L))| \le \|\phi\|_{L^1(-R,+R)} + \|\phi\|_{L^\infty(\mathbb{R}\setminus[-R,+R])}\ |v(L)| \le \\ &\le \|\phi\|_{L^1(-R,+R)} + \|\phi\|_{L^\infty(\mathbb{R}\setminus[-R,+R])}\sqrt{L}\left\|\frac{dv}{dx}\right\|_{L^2(0,L)}, \quad \forall R>0. \end{aligned} \tag{5.28}$$

Turning back to (5.27) we have proved that, for every $R>0$

$$\alpha\left\|\frac{dv}{dx}\right\|^2_{L^2(0,L)} \le \left(\sqrt{L}\|\phi\|_{L^\infty(\mathbb{R}\setminus[-R,+R])} + \|h\|_{L^2(0,L)}\right)\left\|\frac{dv}{dx}\right\|_{L^2(0,L)} + \|\phi\|_{L^1(-R,+R)}.$$

From this inequality, using the fact that for $\alpha>0$, $B>0$, $\Gamma>0$, one has

$$\alpha X^2 \le BX + \Gamma, \quad X \ge 0 \Longleftrightarrow 0 \le X \le \frac{B+\sqrt{B^2+4\alpha\Gamma}}{2\alpha}$$

and then the inequality

$$\frac{B+\sqrt{B^2+4\alpha\Gamma}}{2\alpha} \le \frac{B+B+2\sqrt{\alpha\Gamma}}{2\alpha} = \frac{B}{\alpha} + \frac{\sqrt{\Gamma}}{\sqrt{\alpha}},$$

one deduces that v satisfies

$$\left\|\frac{dv}{dx}\right\|_{L^2(0,L)} \leq \frac{\sqrt{L}\|\phi\|_{L^\infty(\mathbb{R}\setminus[-R,+R])} + \|h\|_{L^2(0,L)}}{\alpha} + \frac{\sqrt{\|\phi\|_{L^1(-R,+R)}}}{\sqrt{\alpha}}\,, \forall\, R > 0\,. \tag{5.29}$$

Combined with the Poincaré inequality (1.21) (which continues to hold true for $z \in H^1(0,L)$ with $z(0) = 0$ without assuming that $z(L) = 0$), namely

$$\|z\|_{L^2(0,L)} \leq L \left\|\frac{dz}{dx}\right\|_{L^2(0,L)}, \tag{5.30}$$

formula (5.29) gives the desired a priori estimate (5.24) with C_R given by (5.25).

□

Remark 5.10. Observe that as far as Lemma 5.9 is concerned it is an a priori estimate for any possible solution of the problem (5.8).

Observe that the proof of this result is a proof in the spirit of an a priori estimate for a PDE, rather then for an ODE.

□

Proposition 5.11 (Passage to the limit). *Assume that* (2.2) *holds true, and that the data* (a, h, ϕ) *satisfy hypotheses* (5.2)–(5.6)*. Let* h_k *be a sequence which satisfies*

$$h_k \in L^2(0,L), \quad h_k \rightharpoonup h \;\; \textit{weakly in} \;\; L^2(0,L). \tag{5.31}$$

Let also ϕ_k *be a reasonable sequence of approximations of* ϕ *which satisfies* (5.4)–(5.6) *for every* k*. Assume that there exists some* $R^* > 0$ *and some* $C^* > 0$ *such that, one has*

$$(L+1)\left(\frac{\sqrt{L}\|\phi_k\|_{L^\infty(\mathbb{R}\setminus[-R^*,+R^*])} + \|h_k\|_{L^2(0,L)}}{\alpha} + \frac{\sqrt{\|\phi_k\|_{L^1(-R^*,+R^*)}}}{\sqrt{\alpha}}\right) \leq C^*, \quad \forall k\,. \tag{5.32}$$

Consider a sequence of solutions v_k *of the ODE problem*

$$\begin{cases} v_k \in H^1(0,L),\; \phi_k(v_k) \in L^2(0,L)\,, \\ a(x)\dfrac{dv_k}{dx} = \phi_k(v_k) + h_k \;\; \textit{in} \;\; \mathcal{D}'(0,L), \\ v_k(0) = 0\,. \end{cases} \tag{5.33}$$

Then there exists a function v *and a subsequence, denoted by* $v_{k'}$*, such that*

$$v_{k'} \rightharpoonup v \;\; \textit{weakly in} \;\; H^1(0,L), \tag{5.34}$$

where v *is a solution of the ODE* (5.8)*, that is*

$$\begin{cases} v \in H^1(0,L),\; \phi(v) \in L^2(0,L)\,, \\ a(x)\dfrac{dv}{dx} = \phi(v) + h \;\; \textit{in} \;\; \mathcal{D}'(0,L), \\ v(0) = 0\,. \end{cases} \tag{5.35}$$

Moreover, if further to hypothesis (5.31)*, the sequence* h_k *satisfies*

$$h_k \to h \quad \textit{strongly in} \;\; L^2(0,L), \tag{5.36}$$

and if, further to hypotheses (5.4)–(5.6)*, the sequence* ϕ_k *satisfies*

$$\phi_k(s) \to \phi(s) \;\; \textit{strongly in} \;\; L^1_{\text{loc}}(\mathbb{R}). \tag{5.37}$$

then one has the strong convergence

$$v_{k'} \to v \quad \textit{strongly in} \;\; H^1(0,L). \tag{5.38}$$

Remark 5.12. Note that in view of hypotheses (5.4) and (5.5) the funciton ϕ satisfies $\phi \in L^1_{\text{loc}}(\mathbb{R})$. Therefore hypothesis (5.37) makes sense.

Proof. In view of (5.24), (5.25), and (5.32) we have

$$\|v_k\|_{H^1(0,L)} \le C^* .$$

Therefore, there exists a $v \in H^1(0,L)$ and a subsequence, denoted by $v_{k'}$, such that

$$\begin{cases} v_{k'} \rightharpoonup v \text{ weakly in } H^1(0,L), \\ v_{k'} \to v \text{ strongly in } L^2(0,L) \text{ and a.e. in } (0,L). \end{cases}$$

Moreover,

$$\phi_k(v_k) = a(x)\frac{dv_k}{dx} - h_k \;\text{ is bounded in } L^2(0,L),$$

so that, since ϕ_k is a reasonable sequence of approximations of ϕ, one gets

$$\begin{cases} \phi_{k'}(v_{k'}) \to \phi(v) & \text{a.e. in } (0,L), \\ \phi_{k'}(v_{k'}) \rightharpoonup \phi(v) & \text{weakly in } L^2(0,L), \end{cases}$$

which allows one to pass to the limit in (5.33) and to prove that v satisfies (5.35).

In order to prove the strong convergence (5.38), assume now hypothesis (5.36), namely that the sequence h_k converges strongly in $L^2(0,L)$, as well as thypothesis (5.37), namely that the sequence ϕ_k converges strongly in $L^1_{loc}(\mathbb{R})$. Multiplying pointwise the equation in (5.33) by $\dfrac{dv_k}{dx}$ and integrating on $(0,L)$ we get

$$\int_0^L a(x)\frac{dv_k}{dx}\frac{dv_k}{dx}dx = \int_0^L \phi_k(v_k)\frac{dv_k}{dx}dx + \int_0^L h_k(x)\frac{dv_k}{dx}dx\,.$$

Define now in this proof the function $\overline{\psi}_k$ by

$$\overline{\psi}_k(s) = \int_0^s \phi_k(t)\,dt,\ \forall k \in \mathbb{R},$$

(please do not confuse this function $\overline{\psi}_k$, neither with the function ψ_n defined by (5.17) and used in the proof of Lemma 2.12 and of Lemma 5.6, nor with the function $\hat{\psi}_\delta$ used in the proof of Proposition 4.8).

Using formula (5.13) and $\overline{\psi}_k(v_k(0)) = \overline{\psi}_k(0) = 0$, we get

$$\int_0^L a(x)\frac{dv_k}{dx}\frac{dv_k}{dx}dx = \overline{\psi}_k(v_k(L)) + \int_0^L h_k(x)\frac{dv_k}{dx}dx\,. \tag{5.39}$$

Since the subsequence $v_{k'}$ converges weakly in $H^1(0,L)$ (see (5.34)) and therefore by Morrey's embedding strongly in $C^0([0,L])$, since the sequnce ϕ_k converges strongly in $L^1_{loc}(\mathbb{R})$, which implies that the sequence $\overline{\psi}_k$ converges strongly in $W^{1,1}_{loc}(\mathbb{R})$, and therefore strongly in $C^0_{loc}(\mathbb{R})$, and finally since the sequence h_k is assumed to converge strongly to h in $L^2(0,L)$ (see (5.36)), the right hand side of (5.39) converges to

$$\psi(v(L)) + \int_0^L h(x)\frac{dv}{dx}dx,$$

which is nothing but

$$\int_0^L a(x)\frac{dv}{dx}\frac{dv}{dx}dx,$$

(multiply pointwise the second line of (5.8) by $\dfrac{dv}{dx}$ and integrate between 0 and L, which is allowed), we have proved that

$$\int_0^L a(x)\frac{dv_{k'}}{dx}\frac{dv_{k'}}{dx}\,dx \to \int_0^L a(x)\frac{dv}{dx}\frac{dv}{dx}\,dx. \tag{5.40}$$

Passing to the limit in

$$\int_0^L a(x)\left(\frac{dv_{k'}}{dx} - \frac{dv}{dx}\right)\left(\frac{dv_{k'}}{dx} - \frac{dv}{dx}\right)dx \ge \alpha\int_0^L \left(\frac{dv_{k'}}{dx} - \frac{dv}{dx}\right)^2 dx$$

with the help of (5.34) and (5.40) proves (5.38).

Proposition 5.11 is proved.

□

Proof of Theorem 5.5. In order to prove the existence theorem 5.5 we will apply three times Proposition 5.11 to the following sequences of approximations:

• Firstly, for any $n \in \mathbb{N}$, we define $\phi_n : \mathbb{R} \mapsto \mathbb{R}$ by

$$\phi_n(s) = T_n(\phi(s))\,,$$

where T_n is the truncation at level n defined by (1.24).

• Then, for n fixed, for any $m \in \mathbb{N}$, we define $\phi_{n,m} : \mathbb{R} \mapsto \mathbb{R}$ by

$$\phi_{n,m}(s) = \begin{cases} \phi_n(-m) & \text{if } s < -m, \\ \phi_n(s) & \text{if } |s| \leq m, \\ \phi_n(m) & \text{if } s > m\,. \end{cases}$$

• Finally, for n and m fixed, for any $\varepsilon > 0$, we define $\phi_{n,m,\varepsilon} : \mathbb{R} \mapsto \mathbb{R}$ by

$$\phi_{n,m,\varepsilon} = \phi_{n,m} * \rho_\varepsilon,$$

where ρ_ε is a standard sequence of mollifiers.

We will first pass to the limit in ε for n and m fixed.

Recalling the fifth example in Remark 2.17, one observes that the function $\phi_{n,m} \in C^0(\mathbb{R})$ and that the function $\phi_{n,m,\varepsilon} \in \mathrm{Lip}(\mathbb{R})$. Therefore there exists a unique solution $v_{n,m,\varepsilon}$ of problem (5.8) for the function $\phi_{n,m,\varepsilon}$. For n and m fixed, the sequence $\phi_{n,m,\varepsilon}$ is a good sequence of approximations of $\phi_{n,m}$, which satisfies

$$\|\phi_{n,m,\varepsilon}\|_{L^\infty(\mathbb{R})} \leq \|\phi_{n,m}\|_{L^\infty(\mathbb{R})} \leq n.$$

Therefore, for n and m fixed, (5.32) is satisfied with a constant C^* given by

$$(L+1)\left(\frac{\sqrt{L}n + \|h\|_{L^2(0,L)}}{\alpha} + \frac{\sqrt{2R^* n}}{\sqrt{\alpha}}\right).$$

An application of Proposition 5.11 proves the existence of a solution $v_{n,m}$ of problem (5.8) for the function $\phi_{n,m}$.

We will then pass to the limit in m for n fixed.

For n fixed, the sequence $\phi_{n,m}$ is a good sequence of approximations of ϕ_n, which satisfies, for $m > R^*$,

$$\phi_{n,m} = \phi_n \ \text{ in } \ (-R^*, +R^*),$$

$$\|\phi_{n,m}\|_{L^\infty(\mathbb{R}\setminus[-R^*,+R^*])} \leq \|\phi_n\|_{L^\infty(\mathbb{R}\setminus[-R^*,+R^*])}.$$

Therefore, for n fixed and $m > R^*$, (5.32) is satisfied with a constant C^* given by

$$(L+1)\left(\frac{\sqrt{L}\|\phi_n\|_{L^\infty(\mathbb{R}\setminus[-R^*,+R^*])} + \|h\|_{L^2(0,L)}}{\alpha} + \frac{\sqrt{\|\phi_n\|_{L^1(-R^*,+R^*)}}}{\sqrt{\alpha}}\right).$$

An application of Proposition 5.11 proves the existence of a solution v_n of problem (5.8) for the function ϕ_n.

We will finally pass to the limit in n.

The sequence ϕ_n is a good sequence of approximations of ϕ (recall the first example in Remark 2.17), which satisfies

$$|\phi_n(s)| \leq |\phi(s)|, \quad \forall s \in \mathbb{R}.$$

Therefore (5.32) is satisfied with a constant C^* given by

$$(L+1)\left(\frac{\sqrt{L}\|\phi\|_{L^\infty(\mathbb{R}\setminus[-R^*,+R^*])} + \|h\|_{L^2(0,L)}}{\alpha} + \frac{\sqrt{\|\phi\|_{L^1(-R^*,+R^*)}}}{\sqrt{\alpha}}\right).$$

An application of Proposition 5.11 proves the existence of a solution v of problem (5.8) for the function ϕ, namely the Theorem 5.5.

□

5.2. **Positivity, comparison and uniqueness of the solutions of** (5.8)**.** In this subsection, further to hypotheses (5.2)–(5.6) on the data (a, g, ϕ), we shall assume (2.6), that is

$$\phi(0) = +\infty, \tag{5.41}$$

i.e. that ϕ is singular in $s = 0$, and that

$$\forall\, \eta \in (0, L), \quad \exists\, M_\eta > 0, \text{ such that } \; h(x) \geq -M_\eta \quad \forall\, x \in [0, L - \eta]; \tag{5.42}$$

i.e. that h may blow down to $-\infty$ only for $x = L$.

Remark 5.13. Observe that hypothesis (5.42) looks similar to hypothesis (4.1) of Theorem 4.1, but that these two hypoteses are used in the study of two different problems (namely problem (5.8) and problem (2.1)), for which they imply very different consequences. Indeed in (5.42) one assumes $\eta > 0$, so that h may blow down at $s = L$, while, in Theorem 4.1, one assumes that h is globally bounded from below, i.e. that in some sense $\eta = 0$, an assumption which implies that problem (2.1) does not have any weak solution in the sense of Definition 2.6.

□

Proposition 5.14 (**Positivity**)**.** *Assume that* (2.2) *holds true, and that the data* (a, h, ϕ) *satisfy hypotheses* (5.2)*–*(5.6) *and* (5.41)*–*(5.42)*.*

Then any solution v *of the ODE problem* (5.8) *satisfies*

$$v(x) > 0, \quad \forall x \in (0, L). \tag{5.43}$$

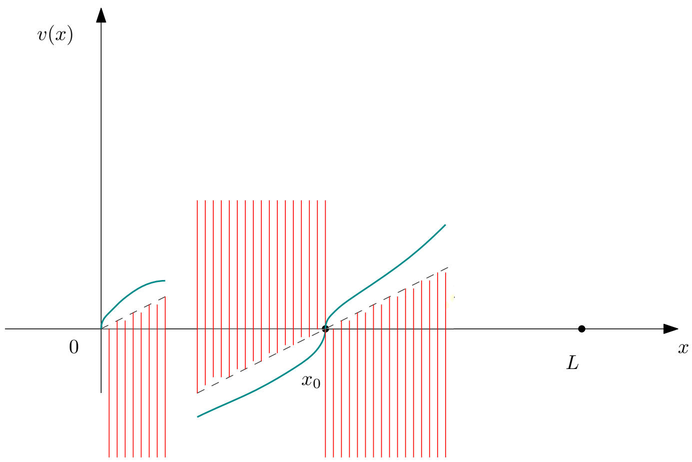

FIGURE 2. The forbidden region in red

Proof. Since all the hypotheses of the existence Theorem 5.5 are assumed in the statement of Proposition 5.14, the ODE problem (5.8) has at least a solution v.

Apply Lemma 4.3 with $w = v$, $l = h$, $A = 0$, $B = L$, and $x_0 = 0$; this is licit in view of hypothesis (5.42). Fixing any $k > 0$, one obtains that for some $\delta_0 > 0$ with $\delta_0 < L$, one has

$$v(x) \geq kx > 0, \quad \forall\, x \in (0, \delta_0]\,. \tag{5.44}$$

The function $v \in C^0([0, \delta_0])$ satisfies (5.44). Then one has the following alternative: either

$$v(x) > 0, \quad \forall\, x \in (0, L), \tag{5.45}$$

or

$$\exists \hat{x}, \quad \hat{x} \in (\delta_0, L), \quad v(\hat{x}) = 0. \tag{5.46}$$

In the second case, we define the set X and the number y by

$$X = \{x : \; x \in [\delta_0, L],\, u(x) = 0\}$$
$$y = \inf_{x \in X} x.$$

The set X is non-empty, since $\hat{x} \in X$; therefore y is correctly defined, $y > \delta_0$, and reasoning as in the proof of Theorem 4.1, y is actually a minimum, i.e. $v(y) = 0$.

Apply now Lemma 4.3 with $w = v$, $l = h$, $A = 0$, $B = L$, and $x_0 = y$. Fixing any $k > 0$, one obtains that for some $\delta_y > 0$, one has

$$v(x) \leq k(x-y) < 0, \quad \forall\, x \in (y-\delta_y, y)\,.$$

a contradiction with the definitions of the set X and of y.

Therefore the second possibility (5.46) is impossible and so (5.45) holds, i.e.

$$v(x) > 0, \quad \forall\, x \in (0, L).$$

This completes the proof of Proposition 5.14.

□

Remark 5.15. Proposition 5.5 can be analogously proved for the "backward" case (starting from L) by performing the change of variable $y = L - x$. In this case, a corresponding result of the one in Lemma 5.14 can be proved provided one assume (5.42) in $[\eta, L]$ instead of $[0, L-\eta]$; as an application of Lemma 4.3 in this case the solution is negative and it cannot be zero up to possibly $x = 0$. □

In the rest of this section, further to hypotheses (5.2)–(5.6) and (5.41)–(5.42), we also assume that $\phi(s)$ is monotone non-increasing for $s > 0$, i.e. that:

$$\phi(s) \geq \phi(t) \quad \text{for} \quad 0 \leq s \leq t\,. \tag{5.47}$$

Proposition 5.16 (Comparison and uniqueness). *Assume that* (2.2) *holds true, and that the data* (a, h_1, ϕ) *and* (a, h_2, ϕ) *satisfy hypotheses* (5.2)–(5.6), (5.41)–(5.42), *and* (5.47). *Let* v_1 *and* v_2 *be solutions of problem* (5.8) *for the data* h_1 *and* h_2. *Assume that*

$$h_1 \leq h_2\,, \tag{5.48}$$

then one has

$$v_1 \leq v_2. \tag{5.49}$$

This comparison result immediatly implies that the solution of ODE problem (5.8) *is unique.*

Proof. By Theorem 5.14 v_1 and v_2 are positive. We take the difference of the second lines of the formulations

$$\begin{cases} v_1 \in H^1(0,L), \ \phi(v_1) \in L^2(0,L)\,, \\ a(x)\dfrac{dv_1}{dx} = \phi(v_1) + h_1 \ \text{ in } \ \mathcal{D}'(0,L), \\ v_1(0) = 0\,, \end{cases}$$

$$\begin{cases} v_2 \in H^1(0,L), \ \phi(v_2) \in L^2(0,L)\,, \\ a(x)\dfrac{dv_2}{dx} = \phi(v_2) + h_2 \ \text{ in } \ \mathcal{D}'(0,L), \\ v_2(0) = 0\,, \end{cases}$$

and we multiply this difference pointwise by $(v_1 - v_2)^+$. Since $(v_1 - v_2)^+ \in H^1(0, L)$, using (5.47) and (5.48), one has

$$\begin{cases} \dfrac{a(x)}{2}\dfrac{d}{dx}((v_1-v_2)^+)^2 = (\phi(v_1) - \phi(v_2))(v_1-v_2)^+ + (h_1-h_2)(v_1-v_2)^+ \leq 0, \\ (v_1-v_2)^+(0) = 0, \end{cases}$$

which easily implies the comparison result.

□

5.3. **Synthesis of the results on the ODE problem** (5.8)**.** To conclude this section, we synthesize the results that we have proved concerning the ODE problem (5.8), namely

$$\begin{cases} v \in H^1(0,L), \ \phi(v) \in L^2(0,L)\,, \\ a(x)\dfrac{dv}{dx} = \phi(v) + h \ \text{ in } \ \mathcal{D}'(0,L), \\ v(0) = 0\,. \end{cases}$$

If we assume that the data satisfy hypotheses (5.2)–(5.6), then the ODE problem (5.8) has at least one solution (Theorem 5.5). Moreover, all the possible solutions of (5.8) satisfy an a priori estimate (Proposition 5.9). These solutions enjoy a stability property (Proposition 5.11), namely the fact that from a sequence of solutions of problems (5.8) relative to a sequence of reasonable approximations ϕ_k of ϕ, one can extract a subsequence which converges to a solution of problem (5.8) relative to ϕ, weakly in $H^1(0,L)$ if the sequence of source terms converges weakly in $L^2(0,L)$, and strongly in $H^1(0,L)$ if the sequence of source terms converges strongly in $L^2(0,L)$ and if the sequence ϕ_k strongly converges in $L^1_{loc}(\mathbb{R})$.

If further to hypotheses (5.4)–(5.6), we assume that the data satisfy hypotheses (5.41) ($\phi(0) = +\infty$) and (5.42) ($h(x) \geq M_\eta$ on $(0, L-\eta)$ for every $\eta > 0$), then every solution of the ODE problem (5.8) is positive in $(0,L)$ (Proposition 5.14).

If in addition to hypotheses (5.4)–(5.6) and (5.41)–(5.42), we assume hypothesis (5.47) (ϕ monotone non-increasing for $s > 0$), then the solutions of the ODE problem (5.8) satisfy a comparison principle and the solution is unique (Proposition 5.16). In this latest case, the stability property described above becomes the continuity (in the weak and in the strong topologies) of the application which from the source term h provides the (unique) solution of the ODE problem (5.8).

6. An unexpected multiplicity result

In this section we show that, under suitable assumptions on ϕ, if there exists a solution of (2.1) in the sense of Definition 2.6 then there are infinitely many solutions of (2.1) in the sense of Definition 2.6. Here we will assume that g satisfies (5.42) with h replaced by g, i.e.

$$\forall\, \eta \in (0,L), \ \ \exists\, M_\eta > 0, \ \text{such that} \ \ g(x) \geq -M_\eta \ \ \forall\, x \in [0, L-\eta]. \tag{6.1}$$

Here is the main result of this section:

Theorem 6.1. *Assume that* (2.2) *holds true, and that the data* (a, g, ϕ) *satisfy hypotheses* (2.3)–(2.6)*,* (5.5)-(5.6)*,* (5.47) *and* (6.1)*.*

Assume that there exists a solution $\overline{u}$ *of problem* (2.1) *in the sense of Definition 2.6. Then there exist infinitely many solutions of* (2.1) *in the sense of Definition 2.6.*

More precisely, there exists a critical value c^* *satisfying*

$$c^* \leq \frac{1}{\sqrt{L}}\left(\frac{\beta}{\alpha} + 1\right) \|g\|_{L^2(0,L)} - \inf_{s\in\mathbb{R}} \phi(s)\,, \tag{6.2}$$

and a function U

$$U : c \in (-\infty, c^*] \longrightarrow U(c) \in H^1_0(0,L) \quad \textit{weakly continuous in } H^1_0(0,L)\,, \tag{6.3}$$

which satisfies

$$U(c) \textit{ is a solution of problem } (2.1) \textit{ in the sense of Definition 2.6}\,, \tag{6.4}$$

$$U(c) \geq 0\,, \tag{6.5}$$

$$\begin{cases} \textit{for any} \ \ c_1, c_2 \in (-\infty, c^*] \ \textit{such that} \ c_1 < c_2, \\ \textit{then} \ U(c_1)(x) < U(c_2)(x) \ \ \textit{for any} \ x \in (0,L)\,, \end{cases} \tag{6.6}$$

$$\begin{cases} U(c) \rightharpoonup 0 \ \ \textit{weakly in } H^1_0(0,L), \\ U(c) \to 0 \ \ \textit{uniformly on } [0,L], \end{cases} \qquad \textit{when } c \to -\infty. \tag{6.7}$$

Moreover

$$\begin{cases} \textit{for any solution } u \textit{ of problem } (2.1) \textit{ in the sense of Definition 2.6 for the data } (a,g,\phi), \\ \textit{there exists (a unique) } c \in (-\infty, c^*] \textit{ such that } u = U(c)\,. \end{cases} \tag{6.8}$$

Proof of Theorem 6.1. We assumed that there exists a solution $\overline{u}$ of problem (2.1) in the sense of Definition 2.6. Then by Proposition 2.13, $\overline{u}$ also solves

$$\begin{cases} \overline{u} \in H^1_0(0,L), \ \ \phi(\overline{u}) \in L^2(0,L), \\ a(x)\dfrac{d\overline{u}}{dx} = \phi(\overline{u}) + g + \overline{c} \ \ \text{in } \mathcal{D}'(0,L), \end{cases} \tag{6.9}$$

for

$$\bar{c} = -\frac{\displaystyle\int_0^L \frac{\phi(\overline{u})}{a(x)}dx + \int_0^L \frac{g}{a(x)}dx}{\displaystyle\int_0^L \frac{1}{a(x)}dx}. \tag{6.10}$$

On the other hand, for every $c \in \mathbb{R}$, by Theorem 5.5 with $h = g + c$ and Proposition 5.16 there exists a unique solution v_c of

$$\begin{cases} v_c \in H^1(0,L), \quad \phi(v_c) \in L^2(0,L), \\ a(x)\dfrac{dv_c}{dx} = \phi(v_c) + g + c \quad \text{in } \mathcal{D}'(0,L), \\ v_c(0) = 0. \end{cases} \tag{6.11}$$

This allows us to define a function V by

$$V : c \in \mathbb{R} \longrightarrow V(c) = v_c \in H^1(0,L).$$

Since, by Lemma 5.14, $V(c) = v_c > 0$ in $(0,L)$, one has

$$V(c)(L) \geq 0 \quad \forall\, c \in \mathbb{R}. \tag{6.12}$$

Observe that $V(c)$ is a solution of problem (2.1) in the sense of Definition 2.6 if and only if $V(c)(L) = 0$.

Moreover, as a consequence of Proposition 5.16, $\overline{u} = V(\bar{c})$, and, in particular, $V(\bar{c})(L) = 0$.

For any $c < \bar{c}$ one has, again by Proposition 5.16, $V(c) \leq V(\bar{c})$. Therefore, by (6.12), we have $V(c)(L) = 0$ and $V(c)$ is a solution of (2.1) in the sense of Definition 2.6.

For any $c \in \mathbb{R}$ we consider the solution $V(c)$ of (6.11) and we define

$$c^* = \sup\{c \in \mathbb{R} : V(c) \text{ satisfies } V(c)(L) = 0\}$$

Observe that $c^* \geq \bar{c}$ where $\bar{c}$ is defined in (6.10).

On the other hand, if $c > c^*$ then $V(c)(L) > 0$ and so, by uniqueness of solutions of (6.11), no solutions of (2.1) do exist.

We show that $c^* < +\infty$. In fact, if u is a solution for (2.1) in the sense of Definition 2.6 then by (2.24) one has

$$\left\|\frac{du}{dx}\right\|_{L^2(0,L)} \leq \frac{1}{\alpha}\|g\|_{L^2(0,L)}. \tag{6.13}$$

Hence, by Proposition 2.13, there exists $c \leq c^*$

$$c = a(x)\frac{du}{dx} - \phi(u) - g \leq \beta\left|\frac{du}{dx}\right| - \inf_{s\in\mathbb{R}}\phi(s) + |g|.$$

Therefore, using (6.13)

$$c \leq \frac{1}{\sqrt{L}}\left(\frac{\beta}{\alpha} + 1\right)\|g\|_{L^2(0,L)} - \inf_{s\in\mathbb{R}}\phi(s),$$

this, recalling Proposition 2.13 and the Definition of c^*, implies that

$$c^* \leq \frac{1}{\sqrt{L}}\left(\frac{\beta}{\alpha} + 1\right)\|g\|_{L^2(0,L)} - \inf_{s\in\mathbb{R}}\phi(s) < +\infty;$$

that is (6.2).

Let us show that c^* is actually a maximum. Let $c_n \nearrow c^*$ so that $V(c_n) \in H_0^1(0,L)$. As c_n are bounded, using (5.24) one gets

$$\|V(c_n)\|_{H_0^1(0,L)} \leq C.$$

Reasoning as before one can pass to the limit in

$$a(x)\frac{dV(c_n)}{dx} = \phi(V(c_n)) + g + c_n \quad \text{in } \mathcal{D}'(0,L)$$

to obtain, using also Proposition 5.16, that

$$a(x)\frac{dV(c^*)}{dx} = \phi(V(c^*)) + g + c^* \quad \text{in } \mathcal{D}'(0,L).$$

The map $U = V_{|(-\infty,c^*]}$ is then well defined by

$$U(c) = V(c) \text{ for any } \ c \in (-\infty, c^*];$$

as there is no ambiguity in view of Proposition 2.13, we will denote by $U(c)$ the function $V(c)$ once referring to the solution of (2.1) in the sense of Definition 2.6. Hence (6.4) and (6.5) are proven and the weak continuity of U in $H_0^1(\Omega)$ is straightforward.

Let us prove (6.6).
By Proposition 5.16 one has that $U(c_1) \leq U(c_2)$. Assuming by contradiction that there exists $x_0 \in (0, L)$ be such that $U(c_1)(x_0) = U(c_2)(x_0)$ then we define

$$\sigma(x) = a(x)\frac{d(U(c_1) - U(c_2))}{dx}(x)$$

so that

$$\sigma(x) = \phi(U(c_1)(x)) - \phi(U(c_2)(x)) + c_1 - c_2;$$

in particular, since σ is continuous on $(0, L)$ and since

$$\sigma(x_0) = c_1 - c_2 < 0, \tag{6.14}$$

One has

$$\sigma(x) < \frac{c_1 - c_2}{2} \quad \forall x \text{ in } \ (x_0 - \delta, x_0 + \delta)$$

for some $\delta > 0$. This implies that

$$\frac{d(U(c_1) - U(c_2))}{dx}(x) \leq \frac{c_1 - c_2}{2\beta} < 0 \ \text{ for a.e. } x \in (x_0 - \delta, x_0 + \delta), \tag{6.15}$$

a contradiction with the fact that $U(c_1) \leq U(c_2)$ as (6.15) gives that $U(c_1)-U(c_2)$ is strictly decreasing around x_0. In fact, let $x \in (x_0 - \delta, x_0)$ and denote by $w = U(c_1) - U(c_2)$; we have $w(x) \leq 0$ and $w(x_0) = 0$. On the other hand

$$w(x) = \int_{x_0}^{x} \frac{dw}{dx}(s)\, ds = -\int_{x}^{x_0} \frac{dw}{dx}(s)\, ds \overset{(6.15)}{>} 0\,.$$

Let us formally prove (6.7). Let $c_n < c^*$ such that $c_n \searrow -\infty$. As before

$$\|U(c_n)\|_{H_0^1(0,L)} \leq \frac{1}{\alpha}\|g\|_{L^2(0,L)}\,.$$

so that, up to a subsequence, $U(c_n)$ converges to u weakly in $H_0^1(0, L)$ and uniformly (by compact embedding). We can apply Theorem 3.4: either $u \equiv 0$ or assume by contradiction that there exists $x_0 \in (0, L)$ such that

$$u(x_0) > 0,$$

and so, in a small neighborhood of x_0 of the form $(x_0 - \delta, x_0 + \delta)$, one has

$$c_n = a(x)\frac{dU(c_n)}{dx} - \phi(U(c_n)) - g\,.$$

i.e., using Proposition 3.6 one has that $|c_n|$ is bounded, that is a contradiction.

In order to conclude the proof of Theorem 6.1 we are left with the proof of (6.8) which easily follows by Proposition 5.16: in fact, if u is a solution of problem (2.1) in the sense of Definition 2.6, then $u = U(c)$ by uniqueness of the solution of problem (5.8).

□

7. Existence of solutions in the sense of Definition 2.6

Up to now we explored the consequences of Theorem 3.4 (Alternative) by assuming the existence of a solution of problem (2.1) in the sense of Definition 2.6, but we never proved the existence of such a solution.

In this section we fix a and ϕ satisfying (2.3) and (2.5) as well as (4.10) and (4.11) (to be compared with Theorem 4.4), and we consider g satisfying (2.4) as a parameter. We aim at constructing two large sets of g's for which a solution in the sense of Definition 2.6 does exist.

7.1. **A first remark about the set of solutions of problem** (2.1) **in the sense of Definition 2.6.** Let us begin with the following remark, which is very simple, but essential.

Remark 7.1. Define the set $\mathcal{G}$ of "good data" by

$$\mathcal{G} = \{g \in L^2(0,L) : \exists \text{ at least one solution } u \text{ of (2.1) in the sense of Definition 2.6}\}, \tag{7.1}$$

and, as in (4.13), the set

$$\mathcal{U} = \{\hat{u} \in H_0^1(0,L) \text{ such that } \phi(\hat{u}) \in L^2(0,L)\}\,. \tag{7.2}$$

It is clear that every u which is a solution of problem (2.1) in the sense of Definition 2.6 is an element of $\mathcal{U}$.

On the other side, for every $u \in \mathcal{U}$, taking any $c \in \mathbb{R}$, and setting

$$g = a(x)\frac{du}{dx} - \phi(u) - c\,, \tag{7.3}$$

it is clear that u is a solution of problem (2.1) in the sense of Definition 2.6 for the source term g. Actually, every $g \in \mathcal{G}$ is of the form (7.3) for some $u \in \mathcal{U}$ and $c \in \mathbb{R}$.

This establishes a very strong relation between the two sets $\mathcal{G}$ and $\mathcal{U}$. In particular $\mathcal{G}$ is non-empty if and only if $\mathcal{U}$ is non-empty.

Therefore the study of the set $\mathcal{U}$ is essential when studying the existence of solutions of problem (2.1) in the sense of Definition 2.6

□

Remark 7.2 (Model example)**.** In the model case where the nonlinearity ϕ is given by

$$\phi(s) = \frac{c}{|s|^\gamma} \quad \text{with} \quad c > 0 \quad \text{and} \quad 0 < \gamma < 1,$$

it is easy to see that u defined by

$$u(x) = Kx^\lambda(L-x)^\lambda \quad \text{with} \quad \frac{1}{2} < \lambda < \frac{1}{2\gamma}, \ \forall K \in \mathbb{R}, \ K \neq 0, \tag{7.4}$$

belongs to $\mathcal{U}$ (since then $\frac{du}{dx} \in L^2(0,L)$ and $\frac{1}{|u|^\gamma} \in L^2(0,L)$ because we have $0 < \gamma < 1$). Therefore, for every $c \in \mathbb{R}$, the function u is a solution of problem (2.1) in the sense of Definition 2.6 for the source term

$$\begin{aligned} g(x) &= a(x)\frac{du}{dx} - \phi(u) - c = \\ &= a(x)\left(K\lambda x^{\lambda-1}(L-x)^\lambda - K\lambda x^\lambda(L-x)^{\lambda-1}\right) - \frac{c}{|K|^\lambda x^{\lambda\gamma}(L-x)^{\lambda\gamma}} - c. \end{aligned}$$

This is a first example of a solution of problem (2.1) in the sense of Definition 2.6, which will be a model for the whole of the present section.

□

7.2. **A large class of good data.** Starting from the idea of the example presented in Remark 7.2 we will show that it is always possible to construct explicit local solutions w^r of problem (2.1) emerging from a point $\overline{x} \in [0,L)$ towards the right side, and w^ℓ coming backward from a point $\overline{y} \in (0,L]$ to the left side provided the datum g is chosen accordingly.

For the sake of exposition we start by showing how these solutions can be constructed in the model case

$$\phi(s) = \frac{c}{|s|^\gamma} \quad \text{with} \quad c > 0 \quad \text{and} \quad 0 < \gamma < 1. \tag{7.5}$$

Define for $y \in [0,L)$ and for some $K^r \in \mathbb{R}$, $K^r \neq 0$, $\lambda^r > 0$ and $\delta > 0$, the function w^r by

$$w^r(x) = K^r(x-y)^{\lambda^r} \quad \text{for } y \le x \le y + \delta\,.$$

Since

$$\frac{dw^r}{dx} = K^r\lambda^r(x-y)^{\lambda^r-1} \quad \text{and} \quad \phi(w^r) = \frac{c}{(|K^r|(x-y)^{\lambda^r})^\gamma} \quad \text{for } y \le x \le y+\delta,$$

we have $\dfrac{dw^r}{dx}$ and $\phi(w^r)$ in $L^2(y, y+\delta)$ if and only if

$$\frac{1}{2} < \lambda^r < \frac{1}{2\gamma}; \tag{7.6}$$

this choice is possible since $0 < \gamma < 1$.

Reasoning in the same way, define for $y \in (0, L]$ and for some $K^\ell \in \mathbb{R}$, $K^\ell \neq 0$, $\lambda^\ell > 0$ and $\delta > 0$, the function w^ℓ by

$$w^\ell(x) = K^\ell (y - x)^{\lambda^\ell} \quad \text{for } y - \delta \le x < y\,,$$

for which we have $\dfrac{dw^\ell}{dx}$ and $\phi(w^\ell)$ in $L^2(y-\delta, y)$ if and only if

$$\frac{1}{2} < \lambda^\ell < \frac{1}{2\gamma}\,.$$

Now, we show how given any x_1 and x_2 with $0 \le x_1 < x_2 \le L$ and $\delta > 0$ with $x_1 + \delta < x_2 - \delta$, one is able to construct a solution of problem (2.1) in the sense of Definition 2.6 in any interval of the form $[x_1, x_2]$ and not only on $[0, L]$. In fact, take any function $w^{int}(x) \in H^1(x_1 + \delta, x_2 - \delta)$ such that $w^{int}(x_1 + \delta) = w_1^r(x_1 + \delta)$ and $w^{int}(x_2 - \delta) = w_2^l(x_2 - \delta)$ where w_1^r and w_2^l are, respectively, the function w^r departing from x_1 and the function w^ℓ arriving in x_2 constructed above. We also request that, for some $\eta > 0$, $w^{int}(x) \ge \eta$ in $x \in (x_1 + \delta, x_2 - \delta)$. Now we define

$$w(x) = \begin{cases} w_1^r & x \in [x_1, x_1 + \delta), \\ w^{int} & x \in [x_1 + \delta, x_2 - \delta], \\ w_2^l & x \in (x_2 - \delta, x_2]. \end{cases} \tag{7.7}$$

Then, if we set

$$g = a(x)\frac{dw}{dx} - \frac{c}{|w|^\gamma} \quad \text{a.e. } x \in (x_1, x_2),$$

it is easy to check that $g \in L^2(x_1, x_2)$, and that w is a solution of

$$\begin{cases} w \in H_0^1(x_1, x_2), \quad \dfrac{c}{|w|^\gamma} \in L^2(x_1, x_2), \\ -\dfrac{d}{dx}\left(a(x)\dfrac{dw}{dx}\right) = -\dfrac{d}{dx}\left(\dfrac{c}{|w|^\gamma}\right) - \dfrac{dg}{dx} \quad \text{in } \mathcal{D}'(x_1, x_2)\,. \end{cases}$$

By modifying the value of λ, with $\frac{1}{2} < \lambda < \frac{1}{2\gamma}$, the value and the sign of K, and reasoning around a finite number of points $x_0 = 0 < x_1 < ... < x_n < x_{n+1} = L$, we can construct a bunch of functions which behave as w between x_i and x_{i+1}, and so a large class of data for which there exists a solution of problem (2.1) in the sense of Definition 2.6.

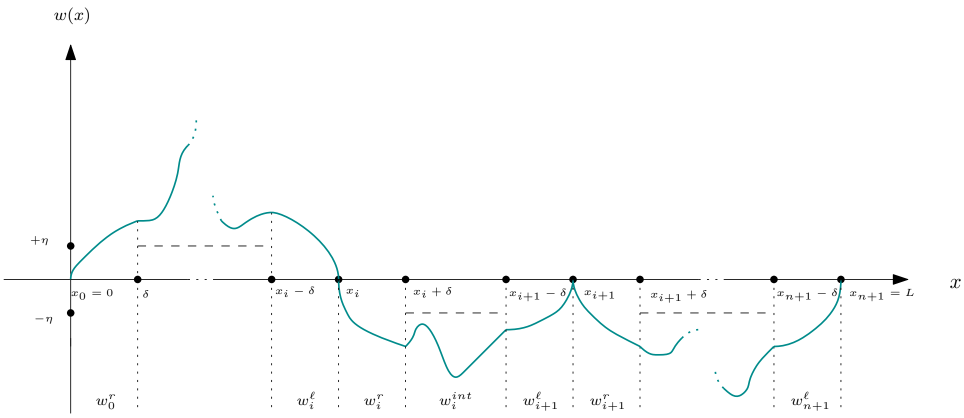

FIGURE 3. Building up the function $w(x)$

More precisely let $x_0 = 0 < x_1 < ... < x_n < x_{n+1} = L$ (i.e. $x_1, ..., x_n$ are the internal points) and let $\delta > 0$ such that

$$x_i + \delta < x_{i+1} - \delta\,, \quad \text{for } 0 \le i \le n\,.$$

Moreover for the internal points x_i, $i=1,...,n$, let us consider λ_i^r, λ_i^ℓ satisfying (7.6), and let K_i^r, K_i^ℓ in $\mathbb{R}\setminus\{0\}$; also consider λ_0^r and λ_{n+1}^ℓ in $\mathbb{R}$ satisfying (7.6), and K_0^r and K_{n+1}^ℓ in $\mathbb{R}\setminus\{0\}$ for the extremal points 0 and L. We assume

$$K_i^r K_{i+1}^\ell > 0, \quad \text{for } i=0,...,n.$$

Now, around each internal point x_i, $i=1,...,n$, we set

$$\begin{aligned} w_i^\ell(x) &= K_i^\ell (x_i - x)^{\lambda_i^\ell} \text{ for } x_i-\delta<x<x_i\,, \\ w_i^r(x) &= K_i^r (x-x_i)^{\lambda_i^r} \text{ for } x_i<x<x_i+\delta, \end{aligned} \tag{7.8}$$

and for the extremities (i.e. $i=0$ and $i=n+1$)

$$\begin{aligned} w_0^r(x) &= K_0^r x^{\lambda_0^r} \text{ for } 0<x<\delta, \\ w_{n+1}^\ell(x) &= K_{n+1}^\ell (L-x)^{\lambda_{n+1}^\ell} \text{ for } L-\delta<x<L\,. \end{aligned} \tag{7.9}$$

In the remaining intervals, which are of the form $(x_i+\delta, x_{i+1}-\delta)$, $i=0,...,n$, we define $w_i^{int}(x)$ as any function in $H^1(x_i+\delta,x_{i+1}-\delta)$ which is continuously joined with the functions w_i^r and w_{i+1}^ℓ defined in (7.8) and (7.9), i.e.

$$w_i^{int}(x_i+\delta) = w_i^r(x_i+\delta) \;\text{ and }\; w_i^{int}(x_{i+1}-\delta) = w_{i+1}^\ell(x_{i+1}-\delta) \;\; \forall\, i=0,...,n\,,$$

and which satisfies, for some $\eta>0$

$$|w_i^{int}(x)| \geq \eta > 0, \quad \text{for any } \; x\in(x_i+\delta, x_{i+1}-\delta).$$

Summarizing we have defined a function $w \in H_0^1(0,L)$ that in any interval (x_i,x_{i+1}), $i=0,...,n$, is given by

$$w(x) = \begin{cases} K_i^r(x-x_i)^{\lambda_i^r}, & x\in(x_i, x_i+\delta], \\ w_i^{int}(x), & x\in(x_i+\delta, x_{i+1}-\delta), \\ K_{i+1}^\ell (x_{i+1}-x)^{\lambda_{i+1}^\ell}, & x\in[x_{i+1}-\delta, x_{i+1}). \end{cases} \tag{7.10}$$

Finally we define the function g by

$$g = \frac{dw}{dx} - \phi(w) \text{ in } \mathcal{D}'(0,L), \tag{7.11}$$

and we observe that $w\in H_0^1(0,L)$ is a solution of problem (2.1) in the sense of Definition 2.6 with g as datum.

We have proved the following result.

Proposition 7.3. *Assume* (2.2)–(2.3) *and* (2.10). *For $n\in\mathbb{N}$, fix n points $x_1,...,x_n$ such that $0=x_0<x_1<...x_n<x_{n+1}=L$. For the internal points x_i, $i=1,...,n$, take λ_i^r, λ_i^ℓ satisfying* (7.6)*, and K_i^r, K_i^ℓ in $\mathbb{R}\setminus\{0\}$. For the external point $x_0=0$ and $x_{n+1}=L$, take λ_0^r and λ_{n+1}^ℓ satisfying* (7.6)*, and K_0^r and K_{n+1}^ℓ in $\mathbb{R}\setminus\{0\}$. Finally assume that*

$$K_i^r K_{i+1}^\ell > 0, \quad \textit{for } i=0,...,n.$$

Then there exists $w\in H_0^1(0,L)$ solution of (2.1) *in the sense of Definition 2.6 with datum $g=\dfrac{dw}{dx}-\phi(w)$ and such that $w(x_i)=0$ for $i=1,...,n$.*

Remark 7.4. Observe that the data g that we have built in the proof of Proposition 7.3 are in general unbounded around each zero of the function w. Indeed, near each zero x_i one has

$$\left|\frac{dw}{dx}\right|(x) \approx (x-x_i)^{\lambda_i^r-1} \;\text{ and }\; \phi(w(x)) \approx (x-x_i)^{-\lambda_i^r\gamma}, \;\text{ for }\; x\in(x_i,x_i+\delta),$$

and

$$\left|\frac{dw}{dx}\right|(x) \approx (x_i-x)^{\lambda_i^\ell-1} \;\text{ and }\; \phi(w(x)) \approx (x_i-x)^{-\lambda_i^\ell\gamma}, \;\text{ for }\; x\in(x_i-\delta,x_i),$$

so that the function $g=a(x)\dfrac{dw}{dx}-\phi(w)$ behaves as the difference or the sum of two (in general different) powers of $(x-x_i)$ or (x_i-x), and therefore is, in general, unbounded around x_i.

Observe neverthless that, in case where

$$a(x) = a \in \mathbb{R} \ \text{ with } \ a > 0 \ \text{ and } \quad \phi(s) = \frac{c}{|s|^\gamma} \ \text{ with } c > 0,$$

the choice

$$w(x) = K|x - x_i|^{\lambda-1}(x - x_i) \ \text{ for } \ x \in (x_i - \delta, x_i + \delta), \text{ with } \lambda = \frac{1}{1+\gamma} \text{ and } K = \left(\frac{a\lambda}{c}\right)^{-\frac{1}{1+\gamma}},$$

produces a function $g = a\dfrac{dw}{dx} - \phi(w)$which satisfies

$$g = 0 \text{ for } \ x \in (x_i - \delta, x_i + \delta).$$

This proves that there exist some cases where the function g can be bounded (or even be identically zero) around a zero x_i of some weak solution of problem (2.1) in the sense of Definition 2.6.

Note in contrast that it can be proven by an argument based on the proof of Lemma 4.3 above and similar to the proof of Theorem 4.1 that it is impossible for the function g to be bounded from below around two consecutive zeros x_i and x_{i+1} of a weak solution of problem (2.1) in the sense of Definition 2.6.

Also observe that in this subsection we choose to present the construction for the model case of ϕ given by (7.5). As a matter of fact this construction can be done for a general ϕ satisfying (2.5)–(2.6), (5.5) and (5.6) using the idea presented in Theorem 7.5 of Subsection 7.3.

□

7.3. **Obtaining solutions for any datum g by modifying it on $[L - \delta, L]$.** In the previous subsection we constructed a bunch of solutions of problem (2.1) in the sense of Definition 2.6. From this construction we obtained by (7.11) a large class of data g for which there exist a solution of problem (2.1) in the sense of Definition 2.6.

In this subsection, we change viewpoint and we construct, for any fixed $g \in L^2(0, L)$ and for any δ with $0 < \delta < L$, a datum $\hat{g}$ which coincides with g on $[0, L - \delta]$ for which problem (2.1) admits a solution in the sense of Definition 2.6. This construction uses an idea similar to the one used in the previous subsection, but now exploits a shooting argument for backward solutions starting from $x = L$ on $[L-\delta, L]$. Moreover, this argument is presented here in the case of a general ϕ satisfying (2.5)–(2.6), (5.5) and (5.6), and not only in the model case of ϕ given by (7.5).

Theorem 7.5. *Assume that* (2.2) *holds true, and that the data* (a, ϕ) *satisfy hypotheses* (2.3), (2.5)–(2.6), (5.5)-(5.6), *and* (5.41)–(5.42). *Then for any* $g \in L^2(0, L)$ *and any* δ *with* $0 < \delta < L$, *there exist* $\hat{g}_\delta$ *in* $\mathcal{G}$ *such that* $\hat{g}_\delta = g$ *in* $[0, L - \delta)$.

Proof. By Theorem 5.5, there exists v such that

$$\begin{cases} v \in H^1(0, L), \quad \phi(v) \in L^2(0, L), \\ a(x)\dfrac{dv}{dx} = \phi(v) + g \ \text{ in } \ \mathcal{D}'(0, L), \\ v(0) = 0\,. \end{cases}$$

By possibly changing δ in a smaller one we can assume that $v(L - \delta) \neq 0$.

Let us assume for a moment that we know a function w_δ which satisfies

$$\begin{cases} w_\delta \in H^1(L - \delta, L), \quad \phi(w_\delta) \in L^2(L - \delta, L), \\ w_\delta(L - \delta) = v(L - \delta)\,, \\ w_\delta(L) = 0\,. \end{cases} \tag{7.12}$$

Then we define $\hat{u}_\delta$ and $\hat{g}_\delta$ by

$$\hat{u}_\delta(x) = \begin{cases} v(x) & \text{for every } x \text{ in } [0, L - \delta], \\ w_\delta(x) & \text{for every } x \text{ in } (L - \delta, L], \end{cases}$$

and

$$\hat{g}_\delta(x) = \begin{cases} g(x) & \text{for a.e. } x \text{ in } (0, L - \delta), \\ a(x)\dfrac{dw_\delta}{dx} - \phi(w_\delta) & \text{for a.e. } x \text{ in } (L - \delta, L). \end{cases}$$

Then $\hat{u}_\delta$ is a solution of problem (2.1) in the sense of Definition 2.6 for the datum $\hat{g}_\delta$. This proves that $\hat{g}_\delta \in \mathcal{G}$.

In order to prove Theorem 7.5, it is then sufficient to construct a function w_δ satisfying (7.12).

To simplify the argument we observe that, by mean of the change of variable $y = L - x$, the construction of such a function w satisfying (7.12) is equivalent to the construction of a function $\overline{w}$ satisfying

$$\begin{cases} \overline{w} \in H^1(0,\delta), \quad \phi(\overline{w}) \in L^2(0,\delta), \\ \overline{w}(\delta) = v(L-\delta)\,, \\ \overline{w}(0) = 0\,, \end{cases} \tag{7.13}$$

where, to ease the notation, we now omit the dependence on the parameter δ.

Assume first that $v(L-\delta) > 0$. We define the following function

$$\phi^{\oplus}(s) = \phi(s) - \inf_{s\in\mathbb{R}} \phi(s) + 1\,; \tag{7.14}$$

it is easy to check that $\phi^{\oplus}$ also satisfies (2.5)–(2.6), (5.5) and (5.6), moreover

$$\phi^{\oplus}(s) \geq 1 \quad \text{for any } s \in \mathbb{R}. \tag{7.15}$$

Now, for a fixed $K > 0$ to be chosen later, Theorem 5.5 implies that there exists at least a solution $w^{\oplus}$ to

$$\begin{cases} w^{\oplus} \in H^1(0,\delta), \quad \phi^{\oplus}(w^{\oplus}) \in L^2(0,\delta), \\ \dfrac{dw^{\oplus}}{dx} = K\phi^{\oplus}(w^{\oplus}) \quad \text{in} \quad \mathcal{D}'(0,L), \\ w^{\oplus}(0) = 0; \end{cases}$$

by Proposition 5.14 one has $w^{\oplus} > 0$ on $(0,\delta)$.

First observe that $\phi(w^{\oplus}) \in L^2(0,\delta)$ if and only if $\phi^{\oplus}(w^{\oplus}) \in L^2(0,\delta)$.

Now, in view of (7.15), we have

$$\frac{1}{\phi^{\oplus}(w^{\oplus})}\frac{dw^{\oplus}}{dx} = K \quad \text{for a.e. } x \in (0,\delta). \tag{7.16}$$

Now observe that

$$0 \leq \frac{1}{\phi^{\oplus}(s)} \leq 1 \quad \text{for any } s \in \mathbb{R}. \tag{7.17}$$

Hence, we define

$$\zeta(s) = \int_0^s \frac{1}{\phi^{\oplus}(r)}dr,$$

and, using (7.15), ζ is strictly increasing on $\mathbb{R}^+$, by (2.6) one has $\zeta(0) = 0$, and, recalling (5.6), for any $\eta > 0$ there exists $c_\eta > 0$ such that

$$\frac{1}{\phi^{\oplus}(r)} \geq \frac{1}{c_\eta} > 0 \quad \text{for any } r \text{ in } [\eta, +\infty),$$

so that we deduce that

$$\lim_{s\to+\infty} \zeta(s) = +\infty,$$

yielding in particular that $\zeta : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a bijection.

Ultimately, by (7.16) one has

$$\begin{cases} \dfrac{d}{dx}\zeta(w^{\oplus}) = K \quad \text{for a.e. } x \in (0,\delta), \\ \zeta(0) = 0\,, \end{cases}$$

i.e.,

$$\zeta(w^{\oplus}(x)) = Kx \quad \text{for a.e. } x \in (0,\delta),$$

that is, choosing

$$K = \frac{\zeta(v(L-\delta))}{\delta},$$

one easily check that $w^{\oplus}$ satisfies (7.13) and we conclude.

In the case where $v(L-\delta)<0$, we define

$$\phi^{\ominus}(s)=\phi^{\oplus}(-s),$$

and we reason as before and we may pick a positive $K>0$ and a solution $\underline{w}$ of

$$\begin{cases} \underline{w}\in H^1(0,\delta), \quad \phi^{\ominus}(\underline{w})\in L^2(0,\delta),\\ \dfrac{d\underline{w}}{dx}=K\phi^{\ominus}(\underline{w}) \ \text{ in } \ \mathcal{D}'(0,L),\\ \underline{w}(0)=0\,, \end{cases}$$

such that $\underline{w}(\delta)=-v(L-\delta)$. To conclude we define $w^{\ominus}=-\underline{w}$ and observing that

$$\phi(w^{\ominus})\in L^2(0,\delta) \iff \phi^{\oplus}(w^{\ominus})\in L^2(0,\delta) \iff \phi^{\ominus}(\underline{w})\in L^2(0,\delta)$$

we get that $w^{\ominus}$ satisfies (7.13).

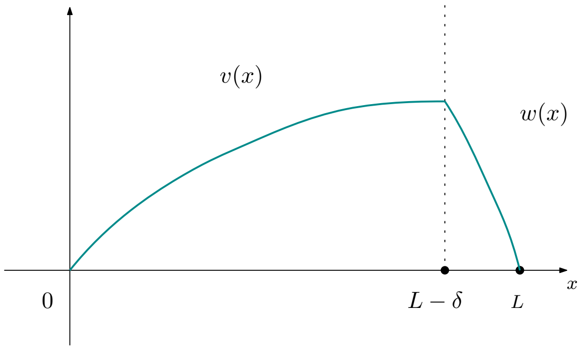

FIGURE 4. Gluing up $v(x)$ and $w(x)$

□

8. Stability and instability of the solutions of approximate equations

The following Proposition 8.1 shows that any solution of (2.1) in the sense of Definition 2.6 is not isolated, or, in other terms, can be obtained as a limit of solutions of convenient approximating problems, each of those problems being different from the limit one.

Proposition 8.1. *Assume that* (2.2) *holds true, and that the data* (a,g,ϕ) *satisfy hypotheses* (2.3)–(2.5)*. Assume also that there exists a weak solution* u *of problem* (2.1) *in the sense of Definition 2.6, i.e. a function* u *which satisfies*

$$\begin{cases} u\in H^1_0(0,L), \quad \phi(u)\in L^2(0,L),\\ \\ -\dfrac{d}{dx}\left(a(x)\dfrac{du}{dx}\right)=-\dfrac{d\phi(u)}{dx}-\dfrac{dg}{dx} \quad \textit{in } \mathcal{D}'(0,L)\,. \end{cases} \tag{8.1}$$

Fix any reasonable approximation ϕ_n *of* ϕ*, and assume that the sequence* ϕ_n *satisfies for some positive constant* C

$$|\phi_n(s)|\le C|\phi(s)| \quad \forall s\in\mathbb{R},\ \forall n. \tag{8.2}$$

Then there exists a sequence g_n *satisfying*

$$g_n\in L^2(0,L) \textit{ with } g_n\to g \textit{ strongly in } L^2(0,L), \tag{8.3}$$

for which there exists a weak solution u_n *in the sense of Definition 2.6 of*

$$\begin{cases} u_n\in H^1_0(0,L), \quad \phi_n(u_n)\in L^2(0,L),\\ \\ -\dfrac{d}{dx}\left(a(x)\dfrac{du_n}{dx}\right)=-\dfrac{d\phi_n(u_n)}{dx}-\dfrac{dg_n}{dx} \quad \textit{in } \mathcal{D}'(0,L)\,, \end{cases} \tag{8.4}$$

such that

$$u_n\to u \quad \textit{strongly in} \quad H^1_0(0,L)\,.$$

Proof. As u is a solution of problem (2.1) in the sense of Definition 2.6, the second line of (8.1) is equivalent to the existence of some $c \in \mathbb{R}$ such that

$$a(x)\frac{du}{dx} = \phi(u) + g + c \quad \text{in} \quad \mathcal{D}'(0, L),$$

in view of Proposition 2.13.

Let ϕ_n be any reasonable approximation of ϕ which satisfies (8.2). Define g_n by

$$g_n = \phi(u) - \phi_n(u) + g;$$

then $u_n = u$ is a weak solution of (8.4) in the sense of Definition 2.6, while g_n converges strongly to g in $L^2(0, L)$ since ϕ_n is a reasonable approximation of ϕ which satisfies the condition of domination (8.2). This proves Proposition 8.1. □

Remark 8.2. Note that the approximation by truncation and the homographic approximation (defined respectively in the first example (2.44) and in the second example (2.46) of Remark 2.17) both satisfy the condition of domination (8.2) for the constant $C = 1$.

Note also that choosing any sequence ϕ_n of reasonable approximations of ϕ which satisfies (8.2) as well as $\phi_n \neq \phi$ for every n proves that any weak solution of (8.1) in the sense of Definition 2.6 is not isolated in the sense defined above.

Finally, the reader could be disturbed by the fact u_n is equal to u for every n. This can be mitigated in the following way: take any interval $[A, B]$ with $0 < A < B < L$ for which $u(x) \geq \delta$ on $[A, B]$ (or for which $u(x) \leq -\delta$ on $[A, B]$) for a certain $\delta > 0$, and replace u by

$$u_n = \begin{cases} u & \text{in } [0, L]\backslash(A, B), \\ v_n & \text{in } [A, B], \end{cases}$$

where $v_n \in H^1(A, B)$ is any sequence such that $(v_n - u)$ tends to 0 strongly in $H_0^1(A, B)$, with $v_n \geq \eta$ (or with $v_n \leq -\eta$) for a certain $\eta > 0$.

□

Proposition 8.1 shows that, given any weak solution u of (8.1) in the sense of Definition 2.6, and any reasonable approximation ϕ_n which satisfies the condition of domination (8.2), one can find a sequence g_n which satisfies (8.3) such that there exists a sequence of solutions of (8.4) in the sense of Definition 2.6 which converges to u in $H_0^1(0, L)$.

In contrast, the following Proposition 8.3 shows that a "bad choice" of g_n produces, for a subsequence n', approximated problems (8.4) in which $g_{n'}$ converges strongly to g while $u_{n'}$ converges weakly to 0 in $H_0^1(0, L)$.

Let us stress that in Proposition 8.1 the sequence ϕ_n can be any reasonable approximation of ϕ which satisfies a condition of domination (in the sense of Lebesgue's theorem), and that in Proposition 8.3 the sequence ϕ_n can be any good approximation of ϕ, but that in contrast, in both cases, the sequence g_n has to be chosen according to the choice of the sequence ϕ_n.

Proposition 8.3. *Assume that* (2.2) *holds true, and that the data* (a, g, ϕ) *satisfy hypotheses* (2.3)–(2.6). *Assume also that there exists a weak solution u of problem* (2.1) *in the sense of Definition 2.6, i.e. a function u which satisfies*

$$\begin{cases} u \in H_0^1(0, L), \quad \phi(u) \in L^2(0, L), \\ \\ -\dfrac{d}{dx}\left(a(x)\dfrac{du}{dx}\right) = -\dfrac{d\phi(u)}{dx} - \dfrac{dg}{dx} \quad \text{in } \mathcal{D}'(0, L)\,. \end{cases} \tag{8.5}$$

Fix any good approximation ϕ_n *of* ϕ. *Then one can extract a subsequence* n' *and find a sequence* $g_{n'}$ *such that*

$$g_{n'} \in L^2(0, L) \quad g_{n'} \to g \quad \text{strongly in} \quad L^2(0, L),$$

for which for any sequence $u_{n'}$ *of classical weak solutions of the approximating problems*

$$\begin{cases} u_{n'} \in H_0^1(0, L), \\ \\ -\dfrac{d}{dx}\left(a(x)\dfrac{du_{n'}}{dx}\right) = -\dfrac{d\phi_{n'}(u_{n'})}{dx} - \dfrac{dg_{n'}}{dx} \quad \text{in } \mathcal{D}'(0, L)\,, \end{cases} \tag{8.6}$$

(such classical weak solutions exist in view of Proposition 2.15), one has

$$u_{n'} \rightharpoonup 0 \ \ in \ \ H^1_0(0,L)\,.$$

Proof. Fix any sequence $\overline{g}_n$ such that

$$\overline{g}_n \in L^\infty(0,L) \quad \text{with} \ \ \overline{g}_n \to g \text{ strongly in } L^2(0,L)\,.$$

For every fixed $n \in \mathbb{N}$, and for every $k \in \mathbb{N}$, Proposition 2.15 implies that there exists at least one classical weak solution v_n^k of

$$\begin{cases} v_n^k \in H^1_0(0,L), \\ \\ -\dfrac{d}{dx}\left(a(x)\dfrac{dv_n^k}{dx}\right) = -\dfrac{d\phi_k(v_n^k)}{dx} - \dfrac{d\overline{g}_n}{dx} \ \ \text{in } \mathcal{D}'(0,L)\,. \end{cases} \tag{8.7}$$

Moreover any classical weak solution v_n^k of (8.7) satisfies (see (2.36) above)

$$\left\|\frac{dv_n^k}{dx}\right\|_{L^2(0,L)} \le \frac{1}{\alpha}\|\overline{g}_n\|_{L^2(0,L)} \le C, \tag{8.8}$$

where the constant C does not depend neither on n nor on k.

As n is fixed and $\overline{g}_n \in L^\infty(0,L)$, Theorems 3.4 and 4.1 imply that

$$v_n^k \rightharpoonup 0 \ \ \text{weakly in } H^1_0(0,L) \ \ \text{as} \ \ k \to +\infty\,, \tag{8.9}$$

and therefore strongly in $L^2(0,L)$. In particular, if ε_n is a sequence of positive constants which tend to 0, for any fixed n one can pick some $k^*(n)$ such that

$$\|v_n^k\|_{L^2(0,L)} \le \varepsilon_n \ \ \text{for any} \ \ k \ge k^*(n)\,. \tag{8.10}$$

Choose a strictly increasing function $\overline{k} : \mathbb{N} \mapsto \mathbb{N}$ such that

$$\overline{k}(n) \ge k^*(n), \quad \text{for any } n \in \mathbb{N}\,,$$

and denote by $\mathbb{N}' \subset \mathbb{N}$ the image $\overline{k}(\mathbb{N})$ of the (strictly increasing) function $\overline{k}$. Then $\overline{k}$ is a bijection from $\mathbb{N}$ to $\mathbb{N}'$, and its inverse $\overline{k}^{-1}$ is a (strictly increasing) bijection from $\mathbb{N}'$ to $\mathbb{N}$. Denote

$$n' = \overline{k}(n) \ \ \text{and } n = \overline{k}^{-1}(n'), \quad \forall n \in \mathbb{N},$$

and set

$$g_{n'} = \overline{g}_n = \overline{g}_{\overline{k}^{-1}(n')}\,, \quad u_{n'} = u_n^{\overline{k}(n)} = u^{n'}_{\overline{k}^{-1}(n')}, \quad \eta_{n'} = \varepsilon_n = \varepsilon_{\overline{k}^{-1}(n')}, \quad \forall n' \in \mathbb{N}';$$

note that $\eta_{n'} \to 0$ as $n' \to +\infty$.

Then (8.7) with $k = \overline{k}(n)$ reads as

$$\begin{cases} u_{n'} \in H^1_0(0,L), \\ \\ -\dfrac{d}{dx}\left(a(x)\dfrac{du_{n'}}{dx}\right) = -\dfrac{d\phi_{n'}(u_{n'})}{dx} - \dfrac{dg_{n'}}{dx} \ \ \text{in } \mathcal{D}'(0,L), \quad \forall n' \in \mathbb{N}', \end{cases} \tag{8.11}$$

and (8.10) with $k = \overline{k}(n)$ reads as

$$\|u_{n'}\|_{L^2(0,L)} \le \varepsilon_n = \eta_{n'} \quad \forall n' \in \mathbb{N}',$$

which implies, since $u_{n'}$ is bounded in $H^1_0(0,L)$ in view of (8.8), that one has

$$u_{n'} \rightharpoonup 0 \ \ \text{weakly in } H^1_0(0,L) \ \ \text{as} \ \ n' \to +\infty\,.$$

Since $g_{n'}(=\overline{g}_n)$ converges strongly to g in $L^2(0,L)$, Proposition 8.3 is proved.

□

Remark 8.4. In the proof of Propositions 8.1 and 8.3, one fixes the sequence ϕ_n, and the choices of the sequences g_n and $g_{n'}$ are made according to the choice of the given sequence ϕ_n. In contrast we do not know how choose a sequence ϕ_n if the sequence g_n is given.

□

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