

# DISTRIBUTION OF RATIONAL POINTS ON RANDOM FANO HYPERSURFACES

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ABSTRACT. We prove an asymptotic formula for the average number of rational points on Fano hypersurfaces that are contained in a small ball centered at a given adelic point. We also prove an asymptotic formula for the number of hypersurfaces admitting adelic points that are contained in a small ball.

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## 1. INTRODUCTION

Let  $V$  be a projective variety defined over  $\mathbb{Q}$ . The set of rational points  $V(\mathbb{Q})$  has been studied from various points of view. When the variety is in the class that is believed to have many rational points, e.g. low degree hypersurfaces, a fundamental question is how the rational points distribute on the variety. In the framework of Manin's conjecture, we introduce a height function  $H$  and count the number  $N(B)$  of rational points  $x$  such that  $H(x) \leq B$ . The conjecture predicts an asymptotic formula for  $N(B)$  as  $B \rightarrow \infty$ . To study the distribution of rational points more

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precisely, we may embed  $V(\mathbb{Q})$  into the set of adelic points of  $V$ :

$$V(\mathbb{Q}) \hookrightarrow V(\mathbf{A}_{\mathbb{Q}}) = \prod_{p \in M_{\mathbb{Q}}} V(\mathbb{Q}_p)$$

where  $M_{\mathbb{Q}}$  is the set of all places of  $\mathbb{Q}$ . A quantitative form of weak approximation problem is to determine the asymptotic formula of the number of rational points of bounded height contained in a measurable subset (or simply a product of balls) of  $V(\mathbf{A}_{\mathbb{Q}})$ . See, for example, [3],[11],[12] for works in this direction.

We also mention that there is a general equidistribution principle that the distribution of rational points is governed by a naturally constructed measure on  $V(\mathbf{A}_{\mathbb{Q}})$  (cf. [10]).

At this moment, it seems very difficult to prove asymptotic formula of the number of rational points, or equidistribution theorem for a given arbitrary variety. However, it might be possible to prove such a statement in the form that the formula holds on average. Indeed, recently Browning, Le Boudec, and Sawin studied the average of counting functions of all rational points of Fano hypersurfaces, and as a consequence they proved that 100% of Fano hypersurfaces (except cubic surfaces) satisfy the Hasse principle, which is believed to be very difficult to prove for a given Fano variety [4].

Following their idea as well as [14], we investigate the number of rational points on random hypersurfaces that are contained in an adelic neighborhood (actually a box) of a given adelic point. More precisely, we study the average of counting functions of those points with bounded height, where the average is taken over all hypersurfaces. To state our main theorems, let us first introduce the set of all hypersurfaces.

**Definition 1.1.** Let  $n, d \in \mathbb{Z}_{\geq 1}$  be positive integers.

- (1)  $\mathbb{V}_{d,n} = \{V \subset \mathbb{P}_{\mathbb{Q}}^n \mid V \text{ is a hypersurface of degree } d\}$ .
- (2) Let  $R$  be a commutative ring. Let  $X_0, \dots, X_n$  be indeterminate. The set of all degree  $d$  monomials in  $X_0, \dots, X_n$  is denoted by  $\mathcal{M}_{d,n}$ . Ordering the members of  $\mathcal{M}_{d,n}$  by the lexicographic order, we identify

$$(1.1) \quad \left\{ \begin{array}{l} \text{homogeneous polynomials of degree } d \\ \text{in } X_0, \dots, X_n \text{ with coefficients in } R \end{array} \right\} = R^{N_{d,n}}$$

where

$$N_{d,n} := \binom{n+d}{d} = \#\mathcal{M}_{d,n}.$$

For  $\mathbf{a} \in R^{N_{d,n}}$ , we denote the corresponding homogeneous polynomial by  $f_{\mathbf{a}}$ .

When  $R$  is a field and  $\mathbf{a} \in R^{N_{d,n}} \setminus \{0\}$ ,  $\mathbf{a}$  can be considered as a homogeneous coordinates of a point  $a \in \mathbb{P}^{N_{d,n}-1}(R)$ . In this case, the closed subscheme  $V_+(\mathbf{a}) \subset \mathbb{P}_R^n$  defined by  $f_{\mathbf{a}}$  is determined by  $a$  and we write  $V_+(f_a)$  instead of  $V_+(\mathbf{a})$ .

- (3) Any  $V \in \mathbb{V}_{d,n}$  has exactly two defining equations with coefficients in  $\mathbb{Z}$  whose coefficient vectors are primitive. One of such coefficient vectors is denoted by  $\mathbf{a}_V$ . (Thus we have  $V = V_+(\mathbf{a}_V)$ .) For  $A \geq 1$ , we define

$$\mathbb{V}_{d,n}(A) := \{V \in \mathbb{V}_{d,n} \mid \|\mathbf{a}_V\| \leq A\}$$

where  $\|\cdot\|$  is the usual Euclidean norm.

We use the following metric on  $\mathbb{P}^n(\mathbb{Q}_p)$ .

**Definition 1.2.** Let  $p \in M_{\mathbb{Q}}$ . For  $x = (x_0 : \cdots : x_n), y = (y_0 : \cdots : y_n) \in \mathbb{P}^n(\mathbb{Q}_p)$ , we write  $\mathbf{x} = (x_0, \dots, x_n), \mathbf{y} = (y_0, \dots, y_n) \in \mathbb{Q}_p^{n+1}$  and set

$$d_p(x, y) = \frac{\|\mathbf{x} \wedge \mathbf{y}\|_p}{\|\mathbf{x}\|_p \|\mathbf{y}\|_p}.$$

Here, when the place  $p$  is finite, let  $\|\cdot\|_p$  be the max norms on  $\mathbb{Q}_p^{n+1}$  and  $\bigwedge^2 \mathbb{Q}_p^{n+1}$  with respect to the standard basis  $\mathbf{e}_0, \dots, \mathbf{e}_n$  and  $\mathbf{e}_i \wedge \mathbf{e}_j$  with  $0 \leq i < j \leq n$ . When  $p = \infty$ , let  $\|\cdot\|_{\infty}$  be the Euclidean norms on  $\mathbb{R}^{n+1}$  and  $\bigwedge^2 \mathbb{R}^{n+1}$  with respect to the standard basis  $\mathbf{e}_0, \dots, \mathbf{e}_n$  and  $\mathbf{e}_i \wedge \mathbf{e}_j$  with  $0 \leq i < j \leq n$ .

This function  $d_p$  is a well-defined non-negative function on  $\mathbb{P}^n(\mathbb{Q}_p) \times \mathbb{P}^n(\mathbb{Q}_p)$ , symmetric and  $d_p(x, y) = 0$  if and only if  $x = y$ . One can also prove that  $d_p$  satisfies the strong triangle inequality for  $p < \infty$  and the triangle inequality for  $p = \infty$  (see e.g. [18, Section 1.1]) and so  $d_p$  is a genuine metric on  $\mathbb{P}^n(\mathbb{Q}_p)$ . Note that  $d_{\infty}(x, y) = |\sin \theta|$  with the angle  $\theta \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$  between  $\mathbf{x}$  and  $\mathbf{y}$  since

$$(1.2) \quad \|\mathbf{x} \wedge \mathbf{y}\|^2 = \|\mathbf{x}\|^2 \cdot \|\mathbf{y}\|^2 - \langle \mathbf{x}, \mathbf{y} \rangle^2 = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 |\sin \theta|^2$$

where  $\langle \cdot, \cdot \rangle$  is the usual Euclidean inner product and  $\|\cdot\|$  is the usual Euclidean norm.

Now let us introduce the counting function of rational points that are contained in an adelic neighborhood of a given point.

**Definition 1.3.** Let  $n, d \in \mathbb{Z}_{\geq 1}$  and suppose  $d \leq n$  (so that general members of  $\mathbb{V}_{d,n}$  are smooth Fano varieties).

(1) For  $x \in \mathbb{P}^n(\mathbb{Q})$  with primitive homogeneous coordinates  $\mathbf{x} \in \mathbb{Z}^{n+1}$ , we define

$$H(x) = \|\mathbf{x}\|^{n+1-d},$$

where  $\|\cdot\|$  is the usual Euclidean norm. Note that when restricted on a hypersurface of degree  $d$ , this is the multiplicative height function associated with the anti-canonical divisor.

(2) We write

$$\prod_{p \in M_{\mathbb{Q}}}^* (0, 1] := \left\{ (\sigma_p)_p \in \prod_{p \in M_{\mathbb{Q}}} (0, 1] \left| \begin{array}{l} \sigma_p = 1 \text{ for all but finitely many } p \\ \sigma_p = p^{-e_p} \text{ with some } e_p \in \mathbb{Z}_{\geq 0} \text{ for all } p < \infty \end{array} \right. \right\}$$

For  $\sigma = (\sigma_p)_p \in \prod_{p \in M_{\mathbb{Q}}}^* (0, 1]$ , we write

$$\sigma_p = p^{-e_p} \quad \text{for } p < \infty$$

and set

$$q = q(\sigma) := \prod_{\substack{p \in M_{\mathbb{Q}} \\ p < \infty}} p^{e_p}, \quad \mathfrak{q} = \mathfrak{q}(\sigma) := \frac{q(\sigma)}{\sigma_{\infty}} = \prod_{p \in M_{\mathbb{Q}}} \frac{1}{\sigma_p}$$

and

$$\text{Supp } \sigma := \{p \in M_{\mathbb{Q}} \mid \sigma_p < 1\}.$$

(3) For  $\xi = (\xi_p)_{p \in M_{\mathbb{Q}}} \in \prod_{p \in M_{\mathbb{Q}}} \mathbb{P}^n(\mathbb{Q}_p)$  and  $\sigma = (\sigma_p)_{p \in M_{\mathbb{Q}}} \in \prod_{p \in M_{\mathbb{Q}}}^*(0, 1]$ , we let

$$\mathbb{B}_{\mathbf{A}}^n(\xi, \sigma) := \left\{ \eta = (\eta_p)_{p \in M_{\mathbb{Q}}} \in \prod_{p \in M_{\mathbb{Q}}} \mathbb{P}^n(\mathbb{Q}_p) \mid d_p(\xi_p, \eta_p) \leq \sigma_p \text{ for all } p \in M_{\mathbb{Q}} \right\},$$

which we call the adelic ball centered at  $\xi$  of radius  $\sigma$ .

(4) For  $V \in \mathbb{V}_{d,n}$ ,  $B \geq 1$  and an adelic ball  $\mathbb{B}_{\mathbf{A}}^n(\xi, \sigma)$ , we define

$$(1.3) \quad N_V(B; \xi, \sigma) := \#\{x \in V(\mathbb{Q}) \mid H(x) \leq B \text{ and } x \in \mathbb{B}_{\mathbf{A}}^n(\xi, \sigma)\},$$

where we identify  $x \in V(\mathbb{Q})$  with its embedding to  $\prod_{p \in M_{\mathbb{Q}}} \mathbb{P}^n(\mathbb{Q}_p)$ .

We are almost ready to state our first main theorem. The Riemann zeta function is denoted by  $\zeta$ . The Euler's totient function is denoted by  $\varphi$ , and the  $k$ -th Jordan totient function is denoted by  $J_k$ :

$$J_k(q) = q^k \prod_{p|q} \left(1 - \frac{1}{p^k}\right)$$

and  $\varphi(q) = J_1(q)$  for positive integers  $q$ . The volume of unit ball in  $\mathbb{R}^N$  for  $N \in \mathbb{Z}_{\geq 1}$  is denoted by  $V_N$ . The following is our first main theorem.

**Theorem 1.4.** *Let  $n \geq d \geq 2$  with  $(n, d) \neq (2, 2)$ ,  $\xi = (\xi_p)_{p \in M_{\mathbb{Q}}} \in \prod_{p \in M_{\mathbb{Q}}} \mathbb{P}^n(\mathbb{Q}_p)$  and  $\sigma = (\sigma_p)_{p \in M_{\mathbb{Q}}} \in \prod_{p \in M_{\mathbb{Q}}}^*(0, 1]$ . For  $A, B \geq 1$  and  $\varepsilon > 0$ , we have*

$$\sum_{V \in \mathbb{V}_{d,n}(A)} N_V(B; \xi, \sigma) = \tilde{C}_{d,n}(\xi, \sigma) A^{N_{d,n}-1} B(1 + R_{d,n}(A, B; \xi, \sigma))$$

with the error term  $R_{d,n}(A, B; \xi, \sigma)$  bounded as

$$R_{d,n}(A, B; \xi, \sigma) \ll (\mathfrak{q}A^{-1} + \mathfrak{q}B^{-\frac{1}{n}} + A^{-1}B^{\frac{d+1}{(2n+1)(n+1-d)}} + \mathfrak{q}^{n-1}A^{-(2n-1)}B^{\frac{d}{n+1-d}})B^\varepsilon$$

provided

$$A \geq \max(\mathfrak{q}, B^{\frac{d+1}{(2n+1)(n+1-d)}}, \mathfrak{q}^{\frac{n-1}{2n-1}}B^{\frac{d}{(2n-1)(n+1-d)}}) \quad \text{and} \quad B \geq \mathfrak{q}^n,$$

where the coefficient  $\tilde{C}_{d,n}(\xi, \sigma)$  is explicitly given by

$$\tilde{C}_{d,n}(\xi, \sigma) := \frac{V_{N_{d,n}-1}}{4\zeta(N_{d,n}-1)} \frac{\mathfrak{W}_{d,n}(\xi_\infty; \sigma_\infty)\varphi(q)}{J_{n+1}(q)\zeta(n+1)},$$

$$\mathfrak{W}_{d,n}(\xi_\infty, \sigma_\infty) := \int_{\mathcal{C}_{\mathfrak{e}_{n+1}}(\xi_\infty, \sigma_\infty) \cap \mathcal{B}_{n+1}(1)} \frac{d\mathbf{x}}{\|\nu_{d,n}(\mathbf{x})\|}$$

with a homogeneous coordinate  $\xi_\infty \in \mathbb{R}^{n+1}$  of  $\xi_\infty$  (for other notation, see Section 2) and the implicit constant depends only on  $d, n, \varepsilon$ . In particular, we have

$$\tilde{C}_{d,n}(\xi, \sigma) \asymp \frac{\sigma_\infty^n \varphi(q)}{J_{n+1}(q)} \asymp \frac{\varphi(q)}{q} \frac{1}{\mathfrak{q}^n}$$

where the implicit constant depends only on  $d, n, \varepsilon$ .

**Remark 1.5.** For the last assertion, use (6.2) to see  $\mathfrak{W}_{d,n}(\xi_\infty, \sigma_\infty) \asymp \sigma_\infty^n$ .

**Remark 1.6.** In Theorem 8.3 below, we indeed have a better error term estimate than Theorem 1.4. We simplified the error term in Theorem 1.4 by making some error terms larger keeping the admissible range of  $A, B, \mathfrak{q}$ .

When a hypersurface  $V$  has no adelic points  $x = (x_p) \in \prod_{p \in M_{\mathbb{Q}}} V(\mathbb{Q}_p)$  that satisfies the same conditions in (1.3) except the height bound, that is,  $d_p(x_p, \xi_p) \leq \sigma_p$  for all  $p \in M_{\mathbb{Q}}$ , we automatically have  $N_V(B; \xi, \sigma) = 0$ . From this viewpoint, it is natural to consider the average of  $N_V(B; \xi, \sigma)$  over the following set of hypersurfaces rather than  $\mathbb{V}_{d,n}$ :

$$\mathbb{V}_{d,n}^{\text{loc}}(\xi, \sigma) := \{V \in \mathbb{V}_{d,n} \mid \forall p \in M_{\mathbb{Q}}, \exists \eta \in V(\mathbb{Q}_p) \text{ such that } d_p(\eta, \xi_p) \leq \sigma_p\}.$$

For  $A \geq 1$ , we also introduce the set of such hypersurfaces with height  $\leq A$ :

$$\mathbb{V}_{d,n}^{\text{loc}}(A; \xi, \sigma) := \mathbb{V}_{d,n}^{\text{loc}}(\xi, \sigma) \cap \mathbb{V}_{d,n}(A).$$

Our second main theorem is to give an asymptotic formula of the size of this set. To state the theorem, let us introduce some notation.

A point  $a \in \mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p)$  corresponds to a homogeneous polynomial of degree  $d$  up to non-zero multiple via (1.1). One of such polynomial is denoted by  $f_a$ . Then  $V_+(f_a)$  depends only on  $a$ . By the assignment  $a \mapsto V_+(f_a)$ , we identify

$$\{\text{Hypersurfaces in } \mathbb{P}_{\mathbb{Q}_p}^n \text{ of degree } d\} = \mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p).$$

Having this identification in mind, we set

$$Z_p(\xi, \sigma) = \left\{ a \in \mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p) \mid \begin{array}{l} \exists \eta \in V_+(f_a)(\mathbb{Q}_p) \text{ such that} \\ d_p(\eta, \xi_p) \leq \sigma_p \end{array} \right\}.$$

This is the locus consisting of hypersurfaces that admit a  $\mathbb{Q}_p$ -point close to the given point  $\xi_p$ . It is easy to see that this set is closed with respect to the strong topology coming from  $\mathbb{Q}_p$ , cf. Section 10.1. The density of this set with respect to the standard probability measure on  $\mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p)$  is denoted by  $\rho_p(\xi, \sigma)$ . See Definition 10.3.

**Theorem 1.7.** *Let  $A \geq 1$ . Let  $\xi \in \prod_{p \in M_{\mathbb{Q}}} \mathbb{P}^n(\mathbb{Q}_p)$  and  $\sigma \in \prod_{p \in M_{\mathbb{Q}}}^*(0, 1]$ . Let us write*

$$\#\mathbb{V}_{d,n}^{\text{loc}}(A; \xi, \sigma) = \frac{V_{N_{d,n}} \prod_{p \in M_{\mathbb{Q}}} \rho_p(\xi, \sigma)}{2\zeta(N_{d,n})} A^{N_{d,n}} (1 + R(A; \xi, \sigma)).$$

If

$$(1.4) \quad A \gg \mathfrak{q} \quad \text{and} \quad A \geq q ((\log \log 3q)(\log 2A))^{\frac{1}{N_{d,n}-2}},$$

then we have

$$(1.5) \quad R(A; \xi, \sigma) \ll \frac{1}{\log \frac{A}{\mathfrak{q}} \log \log \frac{A}{\mathfrak{q}}} + \frac{\log \log 3A}{\frac{A}{\mathfrak{q}}}.$$

Here the implicit constant in (1.4) is an absolute constant and that of (1.5) depends only on  $d, n$ . Moreover, the size of the coefficient of the main term is

$$\frac{V_{N_{d,n}} \prod_{p \in M_{\mathbb{Q}}} \rho_p(\xi, \sigma)}{2\zeta(N_{d,n})} \asymp \frac{1}{\mathfrak{q}}$$

where the implicit constant depends only on  $d$  and  $n$ .

**Remark 1.8.** It has been proven that positive proportion of hypersurfaces admit adelic points [16, Theorem 3.6]. The above Theorem 1.7 can be regarded as a generalization as well as a quantitative version of this result. The key ingredient of the proof of [16, Theorem 3.6] is [15, Lemma 20]. The proof of Theorem 1.7 goes

the same line, but we have to deal with additional parameters  $\xi$  and  $\sigma$  and make implicit constants independent of them.

By combining Theorem 1.4 and Theorem 1.7, we can obtain the following corollary on the least height of the rational point satisfying the same condition as (1.3) on almost all hypersurfaces:

**Corollary 1.9.** *Let  $n \geq d \geq 2$  and  $(n, d) \neq (2, 2)$ ,  $\xi \in \prod_{p \in M_Q} \mathbb{P}^n(\mathbb{Q}_p)$  and  $\sigma \in \prod_{p \in M_Q}^*(0, 1]$ . For  $A \geq 2$ ,  $0 < \delta < 1$  and  $\varepsilon > 0$ , we have*

$$\mathfrak{M}(V; \xi, \sigma) := \min\{H(x) \mid x \in V(\mathbb{Q}) \text{ and } d_p(x, \xi_p) \leq \sigma_p \text{ for all } p \in S\} \geq \delta \mathfrak{q}^{n-1} A$$

for all  $V \in \mathbb{V}_{d,n}^{\text{loc}}(A; \xi, \sigma)$  but at most  $\ll \delta |\mathbb{V}_{d,n}^{\text{loc}}(A; \xi, \sigma)|$  exceptions provided

$$(1.6) \quad A \geq \frac{\mathfrak{q}^{\max(\theta, 1) + \varepsilon}}{\delta}$$

with

$$\theta = \theta(n, d) := \frac{n^2 - 1}{(2n - 1)(n + 1 - d) - d},$$

where the implicit constant depends only on  $d, n, \varepsilon$ .

*Proof.* We may assume  $A \geq C \mathfrak{q}^{1+\varepsilon}$  for some constant  $C$  depending on  $d, n, \varepsilon$  since otherwise the assumption (1.6) implies  $\delta \geq 1/C$ , so the assertion is trivial. This ensures all assumptions of Theorem 1.7, so we can apply Theorem 1.7 to the current setting. Assume  $A \geq \mathfrak{q}^{1+\varepsilon}/\delta$ . Take  $B = \delta \mathfrak{q}^{n-1} A$ . We then automatically have  $B \geq \mathfrak{q}^{n+\varepsilon}$  since  $A \geq \mathfrak{q}^{1+\varepsilon}/\delta$ . We use Theorem 1.4. To this end, we need to check

$$(1.7) \quad A \geq \max(B^{\frac{d+1}{(2n+1)(n+1-d)}}, \mathfrak{q}^{\frac{n-1}{2n-1}} B^{\frac{d}{(2n-1)(n+1-d)}}).$$

Since  $B = \delta \mathfrak{q}^{n-1} A \leq \mathfrak{q}^{n-1} A$ , we have

$$A \geq B^{\frac{d+1}{(2n+1)(n+1-d)}} \iff A \geq (\mathfrak{q}^{n-1} A)^{\frac{d+1}{(2n+1)(n+1-d)}} \iff A \geq \mathfrak{q}^{\theta_1}$$

with

$$\theta_1 := \frac{(d+1)(n-1)}{(2n+1)(n+1-d) - (d+1)}.$$

We also have

$$A \geq \mathfrak{q}^{\frac{n-1}{2n-1}} B^{\frac{d}{(2n-1)(n+1-d)}} \iff A \geq \mathfrak{q}^{\frac{n-1}{2n-1}} (\mathfrak{q}^{n-1} A)^{\frac{d}{(2n-1)(n+1-d)}} \iff A \geq \mathfrak{q}^{\theta_2}$$

with

$$\theta_2 := \frac{n^2 - 1}{(2n - 1)(n + 1 - d) - d}.$$

Note that  $n \geq d \geq 2$  implies

$$\begin{aligned} \theta_1 &= \frac{(d+1)(n-1)}{(2n-1)(n+1-d) - d + 2(n+1-d) - 1} \\ &\leq \frac{(n+1)(n-1)}{(2n-1)(n+1-d) - d} = \theta_2 = \theta \end{aligned}$$

and so (1.7) is assured by (1.6). By Theorem 1.4 and Theorem 1.7, it suffices to show that the error terms in Theorem 1.4 are all  $\ll B^{-\varepsilon}$ , which is assured by (1.6) again.  $\square$

**Remark 1.10.** In Corollary 1.9, we have

$$\theta \leq 1 \iff n^2 - 1 \leq (2n - 1)(n + 1 - d) - d \iff n \geq 2d - 1.$$

Thus, the conclusion of Corollary 1.9 holds only assuming  $A \geq \mathfrak{q}^{1+\varepsilon}$  if  $n \geq 2d - 1$ . On the other hand, since  $\theta(n, n) = n + 1$ , the exponent  $\theta(n, d)$  is not bounded with respect to  $(n, d)$  without restriction.

**Outline of the proof of the main theorems.** The proof of Theorem 1.4 mainly follows the argument of le Boudec [14] as we do in Section 8. Our starting point is a swap of the summation over hypersurfaces and rational points, which reduces the problem to the lattice point counting as in the last line of (8.1). This gives a decomposition (8.2) of the left-hand side of Theorem 1.4 into the main term  $T$  and the error terms  $E_1, E_2$  defined by (8.3). To deal with these terms, in Section 5, we prepare lattice point counting results with local conditions caused by the weak approximation. The archimedean condition can be dealt with the result of Barroero–Widmer [1] (Lemma 3.8) and some simple volume estimates given in Section 4. For the non-archimedean conditions, a straightforward treatment produces a too large error term, so we overcome this difficulty with an averaging trick over units (Lemma 5.6). By using these results, we can prove an asymptotic formula for the main term  $T$  in Section 6, and we can easily bound the error term  $E_2$  as in (8.5). For the error terms  $E_1$ , we need to control the largest successive minima of the relevant lattices. With the weak approximation, the pointwise bound of the largest successive minima due to le Boudec [14, Lemma 5] is not enough anymore. We thus introduce the method of Browning–le Boudec–Sawin [4, Section 3] to prepare a statistical result on the largest successive minima (Lemma 7.9 with Lemma 3.6). We moreover need to keep track of the effect of the weak approximation, which causes a new difficulty, as we do in Section 7.

The strategy of the proof of Theorem 1.7 is as follows. We divide  $\#\mathbb{V}_{d,n}^{\text{loc}}(A; \xi, \sigma)$  as in (10.2), with a parameter  $P$  which we optimize at the last step of the proof according to the form of the estimates for  $M(A, P)$  and  $E(A, P)$  we obtain. The main term  $M(A, P)$  counts points in  $\mathbb{Z}_{\text{prim}}^{N_{d,n}}$  that are contained in certain  $p$ -adic regions for small  $p$  and  $p = \infty$ . Roughly speaking, the  $p$ -adic region corresponds to the set of hypersurfaces with  $p$ -adic points. To count the lattice points contained in them, we approximate the regions by  $p$ -adic balls for  $p \neq \infty$ , which is done in Section 10.1. To bound the error term  $E(A, P)$ , we use the Ekedahl sieve method. For a point  $\mathbf{a} \in \mathbb{Z}_{\text{prim}}^{N_{d,n}}$ , the condition “ $\exists p \in (P, \infty)$ ,  $[\mathbf{a}] \notin Z_p$ ” can be interpreted to containment in some algebraic subset  $\text{NIP}_{d,n}$  of codimension at least two modulo  $p$ . This follows essentially from the Lang-Weil estimates; see Section 9. Thus, we can use a standard Ekedahl sieve type method, but here we also need to make the estimates independent of  $q, \sigma$ .

**Organization of the paper.** In Section 2, we summarize our notation. In Section 3, we review several facts on geometry of numbers including some basic theories and introduce a theorem on lattice point counting due to Barroero and Widmer. In Section 4, we prove several estimates of volumes of subsets of Euclidean space, which we use later to count lattice points contained in them. In Section 5, we prove several asymptotic formulas of counting functions of lattice points with semialgebraic and congruence conditions. In Section 6, we prove lemmas that we use in the proof of Theorem 1.4. The estimate in Section 7 is our core calculation. In

Section 8, we prove our first main theorem Theorem 1.4. Section 9 is a preparation of the proof of our second main theorem Theorem 1.7. In Section 10, we prove Theorem 1.7.

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## 2. NOTATIONS AND CONVENTIONS

Besides notations and conventions introduced in Section 1 and will be introduced in the latter sections, we use the following notations and conventions.

Throughout the paper,  $d$  and  $n$  denote positive integers. We also use  $M, N$  to denote non-negative integers, which are used for the dimension or the rank of certain vector spaces or modules. The letter  $p$  denotes prime numbers or  $\infty$  unless otherwise specified.

Let  $\mu(n)$  be the Möbius function,  $\tau(n)$  be the number of positive divisors of an integer  $n$ ,  $\varphi(q)$  be the Euler totient function and

$$J_k(q) := q^k \prod_{p|q} \left(1 - \frac{1}{p^k}\right)$$

be the Jordan totient function. For integers  $a, b, \dots, c$ , we denote their greatest common divisor by  $(a, b, \dots, c)$ . When confusion with tuples may occur, we write instead  $\gcd(a, b, \dots, c)$ .

For an integer  $n$  and a positive integer  $q$  such that  $(n, q) = 1$ , we write  $\bar{n} \pmod{q}$  be the multiplicative inverse of  $n \pmod{q}$  in  $\mathbb{Z}/q\mathbb{Z}$ . The symbol

$$\sum_{u \pmod{q}}^* \quad \text{and} \quad \bigcup_{u \pmod{q}}^*$$

denote the sum and union over all reduced residues  $u \pmod{q}$ , i.e. units of the ring  $\mathbb{Z}/q\mathbb{Z}$ . The symbol

$$\sum_{n_1, \dots, n_r}^{(d)}$$

denote the summation where the summation variable runs over suitable dyadic sequences of the form  $n_i = 2^{k_i} a_i$  with  $k_i \in \mathbb{Z}$  and a positive real number  $a_i$ .

Let  $V$  be a real vector space. For  $S \subset V$ , we let  $S_{\mathbb{R}}$  be the  $\mathbb{R}$ -subspace of  $V$  spanned by  $S$  and  $\mathbb{R}S := \{ax \mid a \in \mathbb{R} \text{ and } x \in S\}$ .

Let  $V$  be a real metric vector space of dimension  $r$ , i.e. a real vector space of dimension  $r$  with an inner product. We use the following notation.

- For  $R > 0$ , we set  $\mathcal{B}_V(R) := \{\mathbf{x} \in V \mid \|\mathbf{x}\| \leq R\}$
- $\text{vol}_V$  is the measure corresponding to the Lebesgue measure on  $\mathbb{R}^r$  via an isometric isomorphism  $V \simeq \mathbb{R}^r$ . This is independent of the choice of the isometry.
- For  $\mathbf{u} \in V$  and  $0 \leq \sigma \leq 1$ , we let

$$\mathcal{C}_V(\mathbf{u}, \sigma) := \{\mathbf{x} \in V \mid \|\mathbf{x} \wedge \mathbf{u}\| \leq \sigma \|\mathbf{x}\| \|\mathbf{u}\|\},$$

$$\mathcal{C}_V^\perp(\mathbf{u}, \sigma) := \{\mathbf{x} \in V \mid |\langle \mathbf{x}, \mathbf{u} \rangle| \leq \sigma \|\mathbf{x}\| \|\mathbf{u}\|\}.$$

Here note the the metric on  $V$  naturally induces a metric on  $\bigwedge^2 V$ .

We always use the Euclidean inner product  $\langle \cdot, \cdot \rangle$  and the Euclidean norm  $\|\cdot\|$  to define the metric vector space structure on  $\mathbb{R}^N$  and its subspaces. For  $\mathbb{R}^N$ , we simplify the above notation as follows:

$$\begin{aligned}\mathcal{B}_N(R) &= \mathcal{B}_{\mathbb{R}^N}(R) = \{\mathbf{x} \in \mathbb{R}^N \mid \|\mathbf{x}\| \leq R\}, \\ \text{vol}_N &\text{ denotes the Lebesgue measure on } \mathbb{R}^N, \\ \mathcal{C}_N(\mathbf{u}, \sigma) &= \mathcal{C}_{\mathbb{R}^N}(\mathbf{u}, \sigma), \\ \mathcal{C}_N^\perp(\mathbf{u}, \sigma) &= \mathcal{C}_{\mathbb{R}^N}^\perp(\mathbf{u}, \sigma).\end{aligned}$$

We set  $\text{vol}_0(\mathbb{R}^0) = 1$ . Note that if  $\mathbf{u} \neq 0$ , we have

$$\mathcal{C}_N(\mathbf{u}, \sigma) = \{\mathbf{x} \in \mathbb{R}^N \setminus \{0\} \mid d_\infty(\mathbf{x}, \mathbf{u}) \leq \sigma\} \cup \{0\}.$$

This is the cone with axis  $\mathbb{R}\mathbf{u}$  of angle  $\theta \in [0, \frac{\pi}{2}]$  with  $\sin \theta = \sigma$ .

The set of primitive vectors in  $\mathbb{Z}^N$  is denoted by  $\mathbb{Z}_{\text{prim}}^N$ :

$$\mathbb{Z}_{\text{prim}}^N := \{\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{Z}^N \mid \gcd(x_1, \dots, x_N) = 1\}.$$

More generally, for a free  $\mathbb{Z}$ -module  $M$  of finite rank, we write

$$M_{\text{prim}} := M \setminus \bigcup_{d \geq 2} dM,$$

which is the set of all primitive elements of  $M$ .

For a finitely generated  $\mathbb{Z}$ -module  $M$ ,  $M_{\text{tors}}$  denotes the torsion part of  $M$ .

We use the Veronese embedding given by

$$(2.1) \quad \nu_{d,n}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{\mathcal{M}_{d,n}}; (x_0, \dots, x_n) \mapsto (M(x_0, \dots, x_n))_{M \in \mathcal{M}_{d,n}}.$$

Note that we have

$$(2.2) \quad \|\nu_{d,n}(\mathbf{x})\| \leq \|\mathbf{x}\|^d \leq d! \|\nu_{d,n}(\mathbf{x})\|,$$

which can be seen by squaring and expanding  $\|\mathbf{x}\|^d$ .

Let  $K$  be a field. For  $\mathbf{a} \in K^{n+1} \setminus \{0\}$ , we denote the point in  $\mathbb{P}^n(K)$  with homogeneous coordinates  $\mathbf{a}$  as  $[\mathbf{a}]$ .

We use Landau's symbol " $O$ ", Hardy's symbol " $\asymp$ " and Vinogradov's symbol " $\ll$ " in the standard way. The dependence of the implicit constant on  $a, b, \dots, c$  is denoted by subscript, e.g. " $O_{a,b,\dots,c}$ ", " $\asymp_{a,b,\dots,c}$ " or " $\ll_{a,b,\dots,c}$ ". If Theorem or Lemma is stated with the phrase "where the implicit constant depends on  $a, b, \dots, c$ ", then every implicit constant in the corresponding proof may also depend on  $a, b, \dots, c$  unless otherwise specified.

### 3. PRELIMINARIES ON GEOMETRY OF NUMBERS

In this section, we recall some definitions and known results in geometry of numbers. As a standard reference, see e.g. Cassels's book [5] or [6, Chapter 12].

A lattice  $\Lambda \subset \mathbb{R}^N$  of rank  $r \geq 0$  is a  $\mathbb{Z}$ -submodule of  $\mathbb{R}^N$  of the form  $\Lambda = \mathbb{Z}\mathbf{v}_1 + \dots + \mathbb{Z}\mathbf{v}_r$ , where  $\mathbf{v}_1, \dots, \mathbf{v}_r \in \mathbb{R}^N$  are vectors linearly independent over  $\mathbb{R}$ . For a lattice  $\Lambda \subset \mathbb{R}^N$ , let the determinant  $\det(\Lambda)$  be the  $r$ -dimensional volume with respect to the induced metric on  $\text{span}_{\mathbb{R}}(\Lambda)$  of the fundamental parallelepiped. When  $\Lambda$  is 0, we set  $\det(\Lambda) = 1$ . We say a lattice  $\Lambda \subset \mathbb{R}^N$  is integral if  $\Lambda \subset \mathbb{Z}^N$ .

For a lattice  $\Lambda \subset \mathbb{R}^N$ , we say a sublattice  $\Gamma \subset \Lambda$  is primitive with respect to  $\Lambda$  if the quotient module  $\Lambda/\Gamma$  is torsion-free over  $\mathbb{Z}$ . When  $\Lambda = \mathbb{Z}^N$ , we just say  $\Gamma$  is primitive.

Consider a lattice  $\Lambda \subset \mathbb{R}^N$  of rank  $r$ . We let

$$\lambda_i(\Lambda) := \inf\{\lambda \in \mathbb{R}_{>0} \mid \dim \text{span}_{\mathbb{R}}(\Lambda \cap \mathcal{B}_N(\lambda)) \geq i\} \quad (i = 1, \dots, r),$$

which are called the successive minima of  $\Lambda$ .

**Lemma 3.1** (Minkowski's second theorem). *For a lattice  $\Lambda \subset \mathbb{R}^N$  of rank  $r$ ,*

$$\det(\Lambda) \asymp \lambda_1(\Lambda) \cdots \lambda_r(\Lambda),$$

where the implicit constant depends only on  $r$ .

*Proof.* See Theorem I of Chapter VIII of [5, p. 205]. □

For a lattice  $\Lambda \subset \mathbb{R}^N$ , its  $\mathbb{R}$ -span is denoted by  $\Lambda_{\mathbb{R}}$ .

**Lemma 3.2.** *Every lattice  $\Lambda \subset \mathbb{R}^N$  of rank  $r \geq 0$  has a  $\mathbb{Z}$ -basis  $(\mathbf{v}_1, \dots, \mathbf{v}_r)$  with the following properties:*

- (1) We have  $\|\mathbf{v}_i\| \asymp \lambda_i(\Lambda)$  for  $i = 1, \dots, r$ .
- (2) For  $\nu = 1, \dots, r$ , the sublattice  $\Lambda_{\nu} := \mathbb{Z}\mathbf{v}_1 + \cdots + \mathbb{Z}\mathbf{v}_{\nu} \subset \Lambda$  satisfies

$$\det(\Lambda_{\nu}) \asymp \lambda_1(\Lambda) \cdots \lambda_{\nu}(\Lambda).$$

- (3) For the orthogonal projection  $\pi_{\nu}^{\perp}: \mathbb{R}^N = (\Lambda_{\nu})_{\mathbb{R}} \oplus (\Lambda_{\nu})_{\mathbb{R}}^{\perp} \rightarrow (\Lambda_{\nu})_{\mathbb{R}}^{\perp}$ , we have

$$\|\pi_{\nu}^{\perp}(\mathbf{v}_i)\| \asymp \lambda_i(\Lambda) \quad \text{for } 1 \leq \nu < i \leq r.$$

- (4) For any  $x_1, \dots, x_r \in \mathbb{R}$ , we have

$$\left\| \sum_{i=1}^r x_i \mathbf{v}_i \right\| \asymp \sum_{i=1}^r |x_i| \|\mathbf{v}_i\|.$$

where the implicit constants depend at most on  $r$ .

*Proof.* This is essentially Lemma 12.3 of [6, p. 78]. By sending  $\Lambda$  through some isometry  $\Lambda_{\mathbb{R}} \rightarrow \mathbb{R}^r$  and by dilating, we may assume  $N = r$  and  $\det(\Lambda) = 1$ . We may further rotate  $\mathbb{R}^r$  without loss of generality. By Lemma 12.3 of [6, p. 78], after a suitable rotation, we can take a basis  $(\mathbf{v}_1, \dots, \mathbf{v}_r)$  such that

$$(3.1) \quad \begin{pmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_r^T \end{pmatrix} = \begin{pmatrix} v_{11} & & & \\ v_{21} & v_{21} & & \\ \vdots & \vdots & \ddots & \\ v_{r1} & v_{r2} & \cdots & v_{rr} \end{pmatrix}$$

and

$$(3.2) \quad \|\mathbf{v}_i\| \asymp |v_{ii}| \asymp \lambda_i(\Lambda) \quad \text{for } i = 1, \dots, r.$$

We then check such chosen  $(\mathbf{v}_1, \dots, \mathbf{v}_r)$  satisfies the claimed conditions. The condition (1) is nothing but (3.2). For the condition (2), it suffices to see that (3.1)

and (3.2) gives  $\det(\Lambda_\nu)^2 = \det((v_{ij})_{1 \leq i, j \leq \nu})^2 = (|v_{11}| \cdots |v_{\nu\nu}|)^2 \asymp (\lambda_1 \cdots \lambda_\nu)^2$ . For the condition (3), we see that (3.1) gives

$$\begin{pmatrix} \pi_\nu^\perp(\mathbf{v}_{\nu+1}^T) \\ \pi_\nu^\perp(\mathbf{v}_{\nu+2}^T) \\ \vdots \\ \pi_\nu^\perp(\mathbf{v}_r^T) \end{pmatrix} = \begin{pmatrix} v_{\nu+1, \nu+1} & & & \\ v_{\nu+2, \nu+1} & v_{\nu+2, \nu+2} & & \\ \vdots & \vdots & \ddots & \\ v_{r, \nu+1} & v_{r, \nu+2} & \cdots & v_{rr} \end{pmatrix},$$

so (3.2) implies  $\lambda_i(\Lambda) \ll |v_{ii}| \ll \|\pi_\nu^\perp(\mathbf{v}_i)\| \ll \|\mathbf{v}_i\| \ll \lambda_i(\Lambda)$  for  $\nu < i \leq r$ .

Finally, we prove (4). By the triangle inequality, we have  $\|\sum_{i=1}^r x_i \mathbf{v}_i\| \leq \sum_{i=1}^r |x_i| \|\mathbf{v}_i\|$ . Take  $i_0 \in \{1, \dots, r\}$  such that  $|x_{i_0}| \|\mathbf{v}_{i_0}\| = \max_i |x_i| \|\mathbf{v}_i\|$ . Then

$$\sum_{i=1}^r |x_i| \|\mathbf{v}_i\| \leq r |x_{i_0}| \|\mathbf{v}_{i_0}\| \ll |x_{i_0}| \frac{\det(\Lambda)}{\prod_{\substack{1 \leq i \leq r \\ i \neq i_0}} \lambda_i(\Lambda)}$$

where we use (1) and Minkowski's second (Lemma 3.1). Note that

$$\begin{aligned} \det(\Lambda) &= |\det(\mathbf{v}_1 \cdots \mathbf{v}_r)| = |\det(\mathbf{v}_1 \cdots \tilde{\mathbf{v}}_{i_0}^\perp \cdots \mathbf{v}_r)| \\ &\leq \|\mathbf{v}_1\| \cdots \|\tilde{\mathbf{v}}_{i_0}^\perp\| \cdots \|\mathbf{v}_r\| \ll \prod_{\substack{1 \leq i \leq r \\ i \neq i_0}} \lambda_i(\Lambda) \cdot \|\tilde{\mathbf{v}}_{i_0}^\perp\|, \end{aligned}$$

where  $\mathbf{v}_{i_0} = \tilde{\mathbf{v}}_{i_0} + \tilde{\mathbf{v}}_{i_0}^\perp$  with  $\tilde{\mathbf{v}}_{i_0} \in \sum_{\substack{1 \leq i \leq r \\ i \neq i_0}} \mathbb{R} \mathbf{v}_i$  and  $\tilde{\mathbf{v}}_{i_0}^\perp \in (\sum_{\substack{1 \leq i \leq r \\ i \neq i_0}} \mathbb{R} \mathbf{v}_i)^\perp$ . Here we use (1) for the last inequality. Thus

$$\begin{aligned} \sum_{i=1}^r |x_i| \|\mathbf{v}_i\| &\ll |x_{i_0}| \|\tilde{\mathbf{v}}_{i_0}^\perp\| \\ &\leq \|x_1 \mathbf{v}_1 + \cdots + x_{i_0} \tilde{\mathbf{v}}_{i_0} + \cdots + x_r \mathbf{v}_r + x_{i_0} \tilde{\mathbf{v}}_{i_0}^\perp\| = \left\| \sum_{i=1}^r x_i \mathbf{v}_i \right\|. \end{aligned}$$

This completes the proof.  $\square$

For a non-zero integral vector  $\mathbf{c} \in \mathbb{Z}^N \setminus \{0\}$ , we define

$$\Lambda_{\mathbf{c}} := \{\mathbf{x} \in \mathbb{Z}^N \mid \langle \mathbf{c}, \mathbf{x} \rangle = 0\},$$

which is a primitive integral lattice of  $\mathbb{R}^N$  of rank  $N - 1$ . For integral vectors  $\mathbf{c}_1, \dots, \mathbf{c}_k \in \mathbb{Z}^N$ , let  $\mathcal{G}(\mathbf{c}_1, \dots, \mathbf{c}_k)$  be the greatest common divisor of determinants of  $k$ -minors of the  $N$  by  $k$  matrix  $(\mathbf{c}_1 \cdots \mathbf{c}_k)$ . We have the following:

**Lemma 3.3.** *For vectors  $\mathbf{c}_1, \dots, \mathbf{c}_k \in \mathbb{Z}^N$  linearly independent over  $\mathbb{R}$ , we have*

$$\det(\Lambda_{\mathbf{c}_1} \cap \cdots \cap \Lambda_{\mathbf{c}_k}) = \frac{\det(\mathbb{Z}\mathbf{c}_1 + \cdots + \mathbb{Z}\mathbf{c}_k)}{\mathcal{G}(\mathbf{c}_1, \dots, \mathbf{c}_k)}.$$

*Proof.* See [14, Lemma 4].  $\square$

For  $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1}$ , since  $\nu_{d,n}(\mathbf{x}) \in \mathbb{Z}_{\text{prim}}^{\mathcal{M}_{d,n}}$ , Lemma 3.3 gives

$$(3.3) \quad \det(\Lambda_{\nu_{d,n}(\mathbf{x})}) = \|\nu_{d,n}(\mathbf{x})\|.$$

For an integral vector  $\mathbf{c} \in \mathbb{Z}^N$  and  $q \in \mathbb{N}$ , we also use the (mod  $q$ ) analog

$$\Lambda_{\mathbf{c}}^{(q)} := \{\mathbf{x} \in \mathbb{Z}^N \mid \langle \mathbf{c}, \mathbf{x} \rangle \equiv 0 \pmod{q}\}$$

of  $\Lambda_{\mathbf{c}}$ , which is an integral lattice of rank  $N$ .

**Lemma 3.4.** *For an integral vector  $\mathbf{c} \in \mathbb{Z}^N$  and  $q \in \mathbb{N}$ ,*

$$\det(\Lambda_{\mathbf{c}}^{(q)}) = \frac{q}{\gcd(\mathbf{c}, q)} \quad \text{and} \quad \lambda_1(\Lambda_{\mathbf{c}}^{(q)}) \leq \cdots \leq \lambda_N(\Lambda_{\mathbf{c}}^{(q)}) \leq q.$$

*Proof.* We may assume  $\mathbf{c} \neq 0$ . Take  $A \in \text{GL}_N(\mathbb{Z})$  satisfying

$$\mathbf{c}^T A \equiv (d \cdot u \quad 0 \quad \cdots \quad 0) \pmod{q} \quad \text{with} \quad d := \gcd(\mathbf{c}, q) \text{ and } u \in (\mathbb{Z}/q\mathbb{Z})^\times,$$

which is possible by considering the Smith normal form. We then have

$$\begin{aligned} \Lambda_{\mathbf{c}}^{(q)} &= A \cdot \{\mathbf{x} \in \mathbb{Z}^N \mid (d \cdot u \quad 0 \quad \cdots \quad 0) \mathbf{x} \equiv 0 \pmod{q}\} \\ &= A \cdot \left(\frac{q}{d}\mathbb{Z}\mathbf{e}_1 + \mathbb{Z}\mathbf{e}_2 + \cdots + \mathbb{Z}\mathbf{e}_N\right), \end{aligned}$$

where  $\mathbf{e}_1, \dots, \mathbf{e}_N$  are standard vectors. Therefore, we have

$$\det(\Lambda_{\mathbf{c}}^{(q)})^2 = \det(A)^2 \left(\frac{q}{d}\right)^2 = \left(\frac{q}{d}\right)^2 \quad \text{and so} \quad \det(\Lambda_{\mathbf{c}}^{(q)}) = \frac{q}{d}.$$

For successive minima, it suffices to note that  $q\mathbf{e}_1, \dots, q\mathbf{e}_N \in q\mathbb{Z}^N \subset \Lambda_{\mathbf{c}}^{(q)} \subset \mathbb{Z}^N$ .  $\square$

Note that

$$x \in V(\mathbb{Q}) \iff \mathbf{a}_V \in \Lambda_{\nu_{d,n}(\mathbf{x})}$$

when  $\mathbf{x} \in \mathbb{Z}^{n+1}$  is a homogeneous coordinate of  $x$ . Thus, when taking average over  $V$ , the main difficulty lies in counting points of  $\Lambda_{\nu_{d,n}(\mathbf{x})}$  and its dependence on  $\mathbf{x}$ . As a key tool for studying the successive minima of  $\Lambda_{\nu_{d,n}(\mathbf{x})}$ , we use a result of Browning, le Boudec and Sawin [4]. We recall the following quantity defined in [4]:

**Definition 3.5.** For  $\mathbf{x} \in \mathbb{Z}^{n+1}$  and  $1 \leq r \leq n+1$ , let

$$\mathfrak{d}_r(\mathbf{x}) := \min\{\det(\Lambda) \mid \Lambda \subset \mathbb{Z}^{n+1} \text{ is a rank } r \text{ integral lattice with } \mathbf{x} \in \Lambda\}.$$

It is clear that  $1 \leq \mathfrak{d}_r(\mathbf{x}) \leq \|\mathbf{x}\|$  since we can use some  $r-1$  vectors in the standard basis together with  $\mathbf{x}$  to generate a rank  $r$  integral lattice of determinant  $\leq \|\mathbf{x}\|$  and since the determinant of integral lattice is a positive integer. By using  $\mathfrak{d}_2(\mathbf{x})$ , we can bound the largest successive minima  $\lambda_{N_{d,n-1}}(\Lambda_{\nu_{d,n}(\mathbf{x})})$  as follows:

**Lemma 3.6** (Browning–le Boudec–Sawin). *For  $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1}$ , we have*

$$\lambda_{N_{d,n-1}}(\Lambda_{\nu_{d,n}(\mathbf{x})}) \leq \min\left(n \frac{\|\mathbf{x}\|}{\mathfrak{d}_2(\mathbf{x})}, \|\mathbf{x}\|\right).$$

*Proof.* See Lemma 5 of [14] and Lemma 3.15 of [4].  $\square$

Following Schmidt [19, Section 2], we introduce quotient lattices. Let  $\Gamma$  be a primitive sublattice of a lattice  $\Lambda \subset \mathbb{R}^N$ , and let  $\pi: \mathbb{R}^N \rightarrow \Gamma_{\mathbb{R}}^{\perp}$  be the orthogonal projection. By the primitivity of  $\Gamma$ , this projection induces an isomorphism  $\Lambda/\Gamma \cong \pi(\Lambda)$ . By this isomorphism, we identify the quotient module  $\Lambda/\Gamma$  with a lattice  $\pi(\Lambda) \subset \mathbb{R}^N$ . Such a defined lattice  $\Lambda/\Gamma = \pi(\Lambda)$  is called the quotient lattice. We then have  $\text{rank}(\Lambda/\Gamma) = \text{rank}(\Lambda) - \text{rank}(\Gamma)$  and

$$(3.4) \quad \det(\Lambda/\Gamma) = \frac{\det(\Lambda)}{\det(\Gamma)}.$$

As usual, for  $\Lambda/\Gamma = \pi(\Lambda) \subset \Gamma_{\mathbb{R}}^{\perp}$ , we use the metric structure inherited from  $\mathbb{R}^N$  via the inclusion  $\Gamma_{\mathbb{R}}^{\perp} \subset \mathbb{R}^N$ .

Let us conclude this section by recalling a recent result by Barroero and Widmer on lattice point counting [1]. It provides an asymptotic formula of the number of lattice points contained in semialgebraic sets which is applicable uniformly over lattices and semialgebraic conditions varying semialgebraically. Note that Barroero and Widmer work with general  $\mathfrak{o}$ -minimal structures containing semialgebraic sets, but for our purpose, the semialgebraic sets are sufficient.

**Definition 3.7.** For a linear subspace  $W$  of a real metric vector space  $V$ , consider the orthogonal projection  $\pi_W: V \rightarrow W$ . For a semialgebraic set  $A \subset V$ , let

$$V_\nu(A) = V_{\nu,V}(A) := \sup_{\substack{W: \text{subspace of } V \\ \dim W = \nu}} \text{vol}_W(\pi_W(A)).$$

(This is  $V'_\nu(A)$  of Barroero–Widmer [1].)

**Lemma 3.8.** Consider a semialgebraic set  $Z \subset \mathbb{R}^{M+N}$  with  $M, N \in \mathbb{N}$  such that for any  $\mathbf{T} \in \mathbb{R}^M$ , the fiber  $Z_{\mathbf{T}} := \{\mathbf{x} \in \mathbb{R}^N \mid (\mathbf{T}, \mathbf{x}) \in Z\}$  is bounded. For any  $\mathbf{T} \in \mathbb{R}^M$  and any lattice  $\Lambda \subset \mathbb{R}^N$  of rank  $r$ , we have

$$\#(\Lambda \cap Z_{\mathbf{T}}) = \frac{\text{vol}_{\Lambda_{\mathbb{R}}}(\Lambda_{\mathbb{R}} \cap Z_{\mathbf{T}})}{\det(\Lambda)} + O\left(\sum_{0 \leq \nu < r} \frac{V_{\nu, \Lambda_{\mathbb{R}}}(\Lambda_{\mathbb{R}} \cap Z_{\mathbf{T}})}{\lambda_1(\Lambda) \cdots \lambda_\nu(\Lambda)}\right),$$

where the implicit constant depends only on  $Z$ .

*Proof.* For the full-rank case  $r = N$ , this is Theorem 1.3 of Barroero–Widmer [1, p. 4936] with semialgebraic sets as the  $\mathfrak{o}$ -minimal structure of the theorem except that we bound  $V_\nu(Z_{\mathbf{T}})$  of [1] by  $V_\nu(Z_{\mathbf{T}})$  defined by Definition 3.7. Note that semialgebraic sets forms an  $\mathfrak{o}$ -minimal structure by the Tarski–Seidenberg theorem (see e.g. [20, Corollary 2.11, p. 37]). When  $r < N$ , consider the semialgebraic set

$$\tilde{Z} := \{(\Phi, \mathbf{T}, \mathbf{x}) \in \mathbb{R}^{Nr} \times \mathbb{R}^M \times \mathbb{R}^r \mid \Phi \in O_{N,r} \text{ and } (\mathbf{T}, \Phi(\mathbf{x})) \in Z\},$$

where we identify  $\mathbb{R}^{Nr}$  with the set of  $N \times r$  matrices, and  $O_{N,r}$  is the set of  $N \times r$  matrices  $\Phi$  satisfying  $\Phi^T \cdot \Phi = I_r$ . For any  $(\Phi, \mathbf{T}) \in O_{N,r} \times \mathbb{R}^M$ , by identifying  $\Phi \in O_{N,r}$  with the isometry  $\mathbb{R}^r \rightarrow \mathbb{R}^N$ ;  $\mathbf{x} \mapsto \Phi \cdot \mathbf{x}$ , we have  $\tilde{Z}_{(\Phi, \mathbf{T})} = \Phi^{-1}(Z_{\mathbf{T}})$ , which is bounded. Thus, by taking an isometry  $\Phi_\Lambda: \mathbb{R}^r \rightarrow \Lambda_{\mathbb{R}} \subset \mathbb{R}^N$  and noting that  $\Phi_\Lambda^{-1}(\Lambda)$  is a full-rank lattice of  $\mathbb{R}^r$  with the same determinant and successive minima as  $\Lambda$ , we can apply the assertion for the case  $r = N$  to  $\#(\Lambda \cap Z_{\mathbf{T}}) = \#(\Phi_\Lambda^{-1}(\Lambda) \cap \tilde{Z}_{(\Phi_\Lambda, \mathbf{T})})$  to obtain the assertion for the case  $r < N$ .  $\square$

#### 4. PRELIMINARY LEMMAS FOR THE ARCHIMEDEAN PLACE

We use Lemma 3.8 to separate the effect of archimedean restriction like the condition  $d_\infty(x, \xi_\infty) \leq \sigma_\infty$  in (1.3). In this section, we prepare some lemmas to estimate the volumes which appear in the application of Lemma 3.8.

Let us fix  $N \in \mathbb{Z}_{\geq 2}$ ,  $\boldsymbol{\xi} \in \mathbb{R}^N \setminus \{0\}$  and  $\sigma \in (0, 1]$  throughout this section.

**Lemma 4.1.** We have

$$\begin{aligned} \text{vol}_N(\mathcal{C}_N^\perp(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(1)) &= 2V_{N-1} \int_0^\sigma \left( (1-h^2)^{\frac{N-1}{2}} - \left(\frac{1}{\sigma}\right)^2 - 1)^{\frac{N-1}{2}} h^{N-1} \right) dh \\ &= 2 \left( \frac{N-1}{N} \right) V_{N-1} \sigma \exp(O(\sigma^2)) \asymp \sigma, \end{aligned}$$

where the implicit constant depends only on  $N$ .

*Proof.* Since  $\boldsymbol{\xi} \neq 0$ , we can take an orthonormal basis  $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_{N-1}$  with  $\mathbf{e}_0 := \boldsymbol{\xi}/\|\boldsymbol{\xi}\|$ . Let us parametrize  $\mathbf{x} \in \mathbb{R}^N$  as  $\mathbf{x} = h\mathbf{e}_0 + y_1\mathbf{e}_1 + \dots + y_{N-1}\mathbf{e}_{N-1}$  and write  $\mathbf{y} := (y_1, \dots, y_{N-1})$ . We then have

$$(4.1) \quad \begin{aligned} \mathbf{x} \in \mathcal{C}_N^\perp(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(1) &\iff |\langle \boldsymbol{\xi}, \mathbf{x} \rangle| \leq \sigma \|\mathbf{x}\| \|\boldsymbol{\xi}\| \text{ and } \|\mathbf{x}\| \leq 1 \\ &\iff \left( \left( \frac{1}{\sigma} \right)^2 - 1 \right) h^2 \leq \|\mathbf{y}\|^2 \leq 1 - h^2. \end{aligned}$$

By (4.1) and by noticing

$$\left( \left( \frac{1}{\sigma} \right)^2 - 1 \right) h^2 \leq \|\mathbf{y}\|^2 \leq 1 - h^2 \implies |h| \leq \sigma,$$

we have the first equality

$$\text{vol}_N(\mathcal{C}_N^\perp(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(1)) = 2V_{N-1} \int_0^\sigma \left( (1-h^2)^{\frac{N-1}{2}} - \left( \frac{1}{\sigma} \right)^2 h^{N-1} \right) dh$$

The second equality follows from the first equality if  $\sigma$  is sufficiently small and it is trivial if  $\sigma \gg 1$ . The last estimate is clear.  $\square$

**Lemma 4.2.** *We have*

$$\text{vol}_N(\mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(1)) = \frac{2}{N} V_{N-1} \sigma^{N-1} \exp(O(\sigma^2)) \asymp \sigma^{N-1},$$

where the implicit constant depends only on  $N$ .

*Proof.* By (1.2), we have

$$\begin{aligned} \text{vol}_N(\mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(1)) &= \text{vol}_N((\mathbb{R}^N \setminus \mathcal{C}_N^\perp(\boldsymbol{\xi}, (1-\sigma^2)^{\frac{1}{2}})) \cap \mathcal{B}_N(1)) \\ &= V_N - \text{vol}_N(\mathcal{C}_N^\perp(\boldsymbol{\xi}, (1-\sigma^2)^{\frac{1}{2}}) \cap \mathcal{B}_N(1)). \end{aligned}$$

Therefore, for sufficiently small  $\sigma$ , by Lemma 4.1, we have

$$\begin{aligned} &\text{vol}_N(\mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(1)) \\ &= V_N - 2V_{N-1} \int_0^{(1-\sigma^2)^{\frac{1}{2}}} \left( (1-h^2)^{\frac{N-1}{2}} - \left( \frac{1}{1-\sigma^2} - 1 \right)^{\frac{N-1}{2}} h^{N-1} \right) dh \\ &= \frac{2}{N} V_{N-1} \sigma^{N-1} (1 + O(\sigma^2)) \asymp \sigma^{N-1}, \end{aligned}$$

as claimed. When  $\sigma \gg 1$ , the assertion is then trivial.  $\square$

**Lemma 4.3.** *Let  $W \subset \mathbb{R}^N$  be an  $\mathbb{R}$ -subspace of  $\dim W = \nu \geq 1$ ,  $\pi: \mathbb{R}^N \rightarrow W$  be the orthogonal projection, and  $\tau := \|\pi(\boldsymbol{\xi})\|/\|\boldsymbol{\xi}\| \in [0, 1]$ . Then, for  $X \geq 0$ , we have*

$$\text{vol}_W(\pi(\mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(X))) \ll \left( \frac{\tau}{\sigma} + 1 \right) \cdot (\sigma X)^\nu \ll \frac{1}{\sigma} \cdot (\sigma X)^\nu,$$

where the implicit constants depend only on  $\nu$ .

*Proof.* Take  $\mathbf{x} \in \mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(X) \setminus \{\mathbf{0}\}$  arbitrarily. Decompose  $\mathbf{x}$  as

$$(4.2) \quad \mathbf{x} = \tilde{\mathbf{x}} + \tilde{\mathbf{x}}^\perp \quad \text{with} \quad \tilde{\mathbf{x}} = \frac{\langle \mathbf{x}, \boldsymbol{\xi} \rangle}{\|\boldsymbol{\xi}\|^2} \boldsymbol{\xi} \in \mathbb{R}\boldsymbol{\xi} \quad \text{and} \quad \tilde{\mathbf{x}}^\perp \in (\mathbb{R}\boldsymbol{\xi})^\perp.$$

We then have  $\|\tilde{\mathbf{x}}^\perp\|^2 = \|\mathbf{x} \wedge \boldsymbol{\xi}\|^2 / \|\boldsymbol{\xi}\|^2$ . Since  $\mathbf{x} \in \mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(X)$ , this gives

$$(4.3) \quad \|\tilde{\mathbf{x}}^\perp\| \leq \sigma X.$$

We now consider two cases according to whether  $\tau = 0$  or not.

When  $\tau = 0$ , i.e.  $\pi(\boldsymbol{\xi}) = 0$ , by (4.2) and (4.3), we have  $\|\pi(\mathbf{x})\| \leq \|\pi(\tilde{\mathbf{x}}^\perp)\| \leq \|\tilde{\mathbf{x}}^\perp\| \leq \sigma X$ . This shows  $\pi(\mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(X)) \subset \mathcal{B}_W(\sigma X)$ . Thus,

$$\text{vol}_\nu(\pi(\mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(X))) \leq \text{vol}_\nu(\mathcal{B}_\nu(\sigma X)) \ll (\sigma X)^\nu = \left(\frac{\tau}{\sigma} + 1\right) \cdot (\sigma X)^\nu.$$

This proves the assertion when  $\tau = 0$ .

We next consider the case  $\tau > 0$ . Decompose  $\pi(\mathbf{x})$  orthogonally as

$$(4.4) \quad \pi(\mathbf{x}) = \mathbf{y} + \mathbf{y}^\perp \quad \text{with} \quad \mathbf{y} \in \mathbb{R}\pi(\boldsymbol{\xi}) \quad \text{and} \quad \mathbf{y}^\perp \in (\mathbb{R}\pi(\boldsymbol{\xi}))^\perp.$$

By (4.2) and (4.4), we have

$$\|\mathbf{y}\| \leq \|\pi(\mathbf{x})\| \leq \|\pi(\tilde{\mathbf{x}})\| + \|\pi(\tilde{\mathbf{x}}^\perp)\| \leq \|\pi(\tilde{\mathbf{x}})\| + \|\tilde{\mathbf{x}}^\perp\|$$

By using the definition of  $\tau$  and (4.3), we have

$$(4.5) \quad \|\mathbf{y}\| \leq \tau \|\tilde{\mathbf{x}}\| + \|\tilde{\mathbf{x}}^\perp\| \leq \tau \|\mathbf{x}\| + \|\tilde{\mathbf{x}}^\perp\| \leq (\tau + \sigma)X.$$

Since  $\pi(\tilde{\mathbf{x}}) \in \mathbb{R}\pi(\boldsymbol{\xi})$ , by using (4.3) and (4.4), we have

$$\|\mathbf{y}^\perp\|^2 \leq \|\pi(\tilde{\mathbf{x}}) - \mathbf{y}\|^2 + \|\mathbf{y}^\perp\|^2 = \|\pi(\tilde{\mathbf{x}}^\perp)\|^2 \leq \|\tilde{\mathbf{x}}^\perp\|^2 \leq (\sigma X)^2$$

and so

$$(4.6) \quad \|\mathbf{y}^\perp\| \leq \sigma X.$$

Since the assumption  $\tau > 0$  implies  $\pi(\boldsymbol{\xi}) \neq 0$ , by using (4.6) and (4.5), we have

$$\begin{aligned} \text{vol}_W(\pi(\mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(X))) &\leq \int_{\|\mathbf{y}\| \leq (\tau + \sigma)X} \left( \int_{\|\mathbf{y}^\perp\| \leq \sigma X} d\mathbf{y}^\perp \right) d\mathbf{y} \\ &\ll (\sigma X)^{\nu-1} \int_{\|\mathbf{y}\| \leq (\tau + \sigma)X} d\mathbf{y} \ll \left(\frac{\tau}{\sigma} + 1\right) \cdot (\sigma X)^\nu. \end{aligned}$$

This proves the assertion when  $\tau \neq 0$ . □

**Lemma 4.4.** *Let  $W \subset \mathbb{R}^N$  be an  $\mathbb{R}$ -subspace of  $\dim W = \nu \geq 1$ . Write*

$$(4.7) \quad \boldsymbol{\xi} = \tilde{\boldsymbol{\xi}} + \tilde{\boldsymbol{\xi}}^\perp \quad \text{with} \quad \tilde{\boldsymbol{\xi}} \in W \quad \text{and} \quad \tilde{\boldsymbol{\xi}}^\perp \in W^\perp.$$

Let  $\tau := \|\tilde{\boldsymbol{\xi}}\|/\|\boldsymbol{\xi}\| \in [0, 1]$ . We then have

- (i) When  $0 \leq \tau^2 < 1 - \sigma^2$ , we have  $\mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap W = \{0\}$ .
- (ii) When  $1 - \sigma^2 \leq \tau^2 \leq 1$ , we have

$$\mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap W \subset \mathcal{C}_W(\tilde{\boldsymbol{\xi}}, \tilde{\sigma}),$$

where  $\tilde{\sigma} \in [0, 1]$  is defined by

$$\tilde{\sigma}^2 := \begin{cases} \frac{\sigma^2 + \tau^2 - 1}{\tau^2} & \text{if } \tau > 0, \\ 1 & \text{if } \tau = 0 \text{ (and so } \sigma = 1), \end{cases}$$

In particular, we have

$$\tilde{\xi} = 0 \implies \mathcal{C}_N(\xi, \sigma) \cap W = \begin{cases} W & \text{if } \sigma = 1, \\ \{0\} & \text{if } 0 \leq \sigma < 1, \end{cases}$$

and

$$\tilde{\xi} \neq 0 \implies \mathcal{C}_N(\xi, \sigma) \cap W \subset \mathcal{C}_W(\tilde{\xi}, \sigma).$$

*Proof.* Take  $\mathbf{x} \in \mathcal{C}_N(\xi, \sigma) \cap W$  arbitrarily. By (4.7), we then have

$$\langle \mathbf{x}, \tilde{\xi} \rangle = \langle \mathbf{x}, \xi \rangle.$$

Therefore, by recalling  $\mathbf{x} \in \mathcal{C}_N(\xi, \sigma)$ , we have

$$\begin{aligned} \|\mathbf{x} \wedge \tilde{\xi}\|^2 &= \|\mathbf{x} \wedge \xi\|^2 + \|\mathbf{x}\|^2 \|\tilde{\xi}\|^2 - \|\mathbf{x}\|^2 \|\xi\|^2 \\ &\leq \sigma^2 \|\mathbf{x}\|^2 \|\xi\|^2 + \|\mathbf{x}\|^2 \|\tilde{\xi}\|^2 - \|\mathbf{x}\|^2 \|\xi\|^2 = (\sigma^2 + \tau^2 - 1) \|\mathbf{x}\|^2 \|\xi\|^2. \end{aligned}$$

and so

$$(4.8) \quad 0 \leq \|\mathbf{x} \wedge \tilde{\xi}\|^2 \leq (\sigma^2 + \tau^2 - 1) \|\mathbf{x}\|^2 \|\xi\|^2.$$

When  $0 \leq \tau^2 < 1 - \sigma^2$ , by (4.8) and  $\|\xi\| \neq 0$ , we should have  $\|\mathbf{x}\| = 0$  and so (i) holds. When  $\tau^2 = 1 - \sigma^2$  and  $\sigma < 1$ , by (4.8), we have  $\mathbf{x} \in \mathcal{C}_W(\tilde{\xi}, \tilde{\sigma})$  since  $\|\mathbf{x} \wedge \tilde{\xi}\| = 0$  in this case. When  $\tau^2 = 1 - \sigma^2$  and  $\sigma = 1$ , we have  $\mathbf{x} \in W = \mathcal{C}_W(\tilde{\xi}, \tilde{\sigma})$  since  $\tilde{\sigma} = 1$  in this case. When  $1 - \sigma^2 < \tau \leq 1$ , since  $\tau > 0$ , by (4.8), we have

$$\|\mathbf{x} \wedge \tilde{\xi}\|^2 \leq \left( \frac{\sigma^2 + \tau^2 - 1}{\tau^2} \right) \|\mathbf{x}\|^2 \|\tilde{\xi}\|^2$$

so that  $\mathbf{x} \in \mathcal{C}_W(\tilde{\xi}, \tilde{\sigma})$ . Therefore, (ii) holds as well.  $\square$

## 5. LATTICE POINT COUNTING WITH LOCAL CONDITIONS

In this section, we prepare some asymptotic formulas of the number of lattice points satisfying given semialgebraic and congruence conditions. In order to deal with various types of semialgebraic conditions at the same time, we introduce a class of semialgebraic sets defined as follows:

**Definition 5.1.** A semialgebraic set  $Z \subset \mathbb{R}^{M+N}$  is called a semialgebraic family of homogeneous sets if for any  $\mathbf{T} \in \mathbb{R}^M$ , the fiber

$$Z_{\mathbf{T}} = \{\mathbf{x} \in \mathbb{R}^N \mid (\mathbf{T}, \mathbf{x}) \in Z\}$$

satisfies  $cZ_{\mathbf{T}} = Z_{\mathbf{T}}$  for any  $c \in \mathbb{R}^\times$ , i.e. the fiber  $Z_{\mathbf{T}}$  is invariant under dilation. In other words, if  $(\mathbf{T}, \mathbf{x}) \in Z$  and  $c \in \mathbb{R}^\times$ , then we have  $(\mathbf{T}, c\mathbf{x}) \in Z$ .

We use the following terminology.

**Definition 5.2.** Let  $\Lambda$  be a free  $\mathbb{Z}$ -module of finite rank  $r \geq 0$ . For  $\mathbf{c} \in \Lambda$  and  $q \in \mathbb{Z}_{\geq 1}$ , we say  $\mathbf{c}$  is  $q$ -primitive if the following holds:

$$\mathbf{c} \in d\Lambda \text{ and } d \mid q \implies d = 1.$$

**Remark 5.3.** Under the notation of Definition 5.2, the following are equivalent:

- (1) The vector  $\mathbf{c}$  is  $q$ -primitive.
- (2) Either  $\mathbf{c} = 0$  and  $q = 1$ , or  $\mathbf{c} \neq 0$  and if  $\mathbf{c} = l\mathbf{v}$  for some  $l \in \mathbb{Z}$  and primitive  $\mathbf{v} \in \Lambda$ , then  $(l, q) = 1$ .
- (3) Either  $\mathbf{c} = 0$  and  $q = 1$ , or  $\mathbf{c} \neq 0$  and  $(\#(\Lambda/\mathbb{Z}\mathbf{c})_{\text{tors}}, q) = 1$ .

- (4) Either  $\mathbf{c} = 0$  and  $q = 1$ , or  $\mathbf{c} \neq 0$  and the image of  $\mathbf{c}$  in  $\Lambda/q\Lambda$  can be extended to a  $\mathbb{Z}/q\mathbb{Z}$ -basis of  $\Lambda/q\Lambda$ .

We use the following notation for the volumes used below.

**Definition 5.4.** Let  $\Lambda \subset \Gamma \subset \mathbb{R}^N$  be lattices. For  $\mathbf{c} \in \Gamma$ ,  $q \in \mathbb{N}$ , and  $Z \subset \mathbb{R}^N$ , let

$$\begin{aligned} \mathfrak{V}(\mathbf{c}, q; Z) &= \mathfrak{V}(\Lambda, \Gamma; \mathbf{c}, q; Z) := \text{vol}_{\Lambda_{\mathbb{R}}}(\text{span}_{\mathbb{R}}(\Lambda \cap (\mathbf{c} + q\Gamma)) \cap Z \cap \mathcal{B}_N(1)), \\ \mathfrak{V}_{\nu}(\mathbf{c}, q; Z) &= \mathfrak{V}_{\nu}(\Lambda, \Gamma; \mathbf{c}, q; Z) := V_{\nu, \Lambda_{\mathbb{R}}}(\text{span}_{\mathbb{R}}(\Lambda \cap (\mathbf{c} + q\Gamma)) \cap Z \cap \mathcal{B}_N(1)). \end{aligned}$$

Note that if  $\Lambda \cap (\mathbf{c} + q\Gamma) \neq \emptyset$ , then

$$\mathfrak{V}(\mathbf{c}, q; Z) = \text{vol}_{\Lambda_{\mathbb{R}}}(Z \cap \mathcal{B}_{\Lambda_{\mathbb{R}}}(1)) \quad \text{and} \quad \mathfrak{V}_{\nu}(\mathbf{c}, q; Z) = V_{\nu, \Lambda_{\mathbb{R}}}(Z \cap \mathcal{B}_{\Lambda_{\mathbb{R}}}(1)).$$

**Lemma 5.5.** Consider a semialgebraic family of homogeneous sets  $Z \subset \mathbb{R}^{M+N}$  with  $M, N \in \mathbb{N}$ , a lattice  $\Gamma \subset \mathbb{R}^N$ , and its primitive sublattice  $\Lambda$  of rank  $r \geq 1$ . For  $X \geq 0$ ,  $\mathbf{T} \in \mathbb{R}^M$ ,  $\mathbf{c} \in \Gamma$  and  $q \in \mathbb{N}$ , we have

$$\begin{aligned} \mathcal{N} &:= \#(\Lambda \cap (\mathbf{c} + q\Gamma) \cap Z_{\mathbf{T}} \cap \mathcal{B}_N(X)) \\ &= \frac{\mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}})X^r}{q^r \det(\Lambda)} + O\left(\sum_{1 \leq \nu < r} \frac{\mathfrak{V}_{\nu}(\mathbf{c}, q; Z_{\mathbf{T}})X^{\nu}}{q^{\nu} \lambda_1(\Lambda) \cdots \lambda_{\nu}(\Lambda)} + \mathfrak{Q} + \mathfrak{R}\right), \end{aligned}$$

where

$$\begin{aligned} \mathfrak{Q} &:= \mathbb{1}_{X \geq q\lambda_1(\Lambda)}, \\ \mathfrak{R} &:= \mathbb{1}_{\Lambda \cap (\mathbf{c} + q\Gamma) \cap Z_{\mathbf{T}} \cap \mathcal{B}_N(X) \neq \emptyset} + \mathbb{1}_{X < q\lambda_1(\Lambda)} \times \frac{\mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}})X^r}{q^r \det(\Lambda)} \end{aligned}$$

and the implicit constant depends only on  $Z$  and  $r$ .

*Proof.* We may assume  $\Lambda \cap (\mathbf{c} + q\Gamma) \neq \emptyset$  since otherwise the assertion is trivial. Since we can shift  $\mathbf{c}$  by the elements of  $q\Gamma$  without changing  $\mathbf{c} + q\Gamma$ , we may assume  $\mathbf{c} \in \Lambda$  without loss of generality. Then, for any  $\mathbf{x} \in \Lambda \cap (\mathbf{c} + q\Gamma)$ , we have  $\frac{1}{q}(\mathbf{x} - \mathbf{c}) \in (\frac{1}{q}\Lambda) \cap \Gamma = \Lambda$  by the primitivity of  $\Lambda$ , so  $\Lambda \cap (\mathbf{c} + q\Gamma) = \mathbf{c} + q\Lambda$ .

We first consider the case  $X \geq q\lambda_1(\Lambda)$ . We have

$$\mathcal{N} = \#((\mathbf{c} + q\Lambda) \cap Z_{\mathbf{T}} \cap \mathcal{B}_N(X)) = \#(q\Lambda \cap (Z_{\mathbf{T}} \cap \mathcal{B}_N(X) - \mathbf{c})).$$

Therefore, by considering the semi-algebraic set

$$\tilde{Z} := \left\{ (\mathbf{c}, \mathbf{T}, X, \mathbf{x}) \in \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R} \times \mathbb{R}^N \mid \mathbf{x} \in Z_{\mathbf{T}} \cap \mathcal{B}_N(X) - \mathbf{c} \right\},$$

we have

$$(5.1) \quad \mathcal{N} = \#(q\Lambda \cap \tilde{Z}_{(\mathbf{c}, \mathbf{T}, X)}),$$

where  $\tilde{Z}_{(\mathbf{c}, \mathbf{T}, X)}$  is as in Lemma 3.8. Also, for any  $(\mathbf{c}, \mathbf{T}, X) \in \mathbb{R}^{N+M+1}$ , the fiber  $\tilde{Z}_{(\mathbf{c}, \mathbf{T}, X)}$  is bounded. Thus, by applying Lemma 3.8 to (5.1) with noting that

$$\det(q\Lambda) = q^r \det(\Lambda) \quad \text{and} \quad \lambda_i(q\Lambda) = q\lambda_i(\Lambda) \quad \text{for } i = 1, \dots, r,$$

we obtain

$$\mathcal{N} = \frac{\text{vol}_r(\Lambda_{\mathbb{R}} \cap \tilde{Z}_{(\mathbf{c}, \mathbf{T}, X)})}{q^r \det(\Lambda)} + O\left(\sum_{1 \leq \nu < r} \frac{V_{\nu, \Lambda_{\mathbb{R}}}(\Lambda_{\mathbb{R}} \cap \tilde{Z}_{(\mathbf{c}, \mathbf{T}, X)})}{q^{\nu} \lambda_1(\Lambda) \cdots \lambda_{\nu}(\Lambda)} + 1\right),$$

where  $+1$  corresponds to the term for  $\nu = 0$  of the sum in the error term. Since  $Z$  is a family of homogeneous sets, we have

$$\text{vol}_r(\Lambda_{\mathbb{R}} \cap \tilde{Z}_{(\mathbf{c}, \mathbf{T}, X)}) = \mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}})X^r, \quad V_{\nu, \Lambda_{\mathbb{R}}}(\Lambda_{\mathbb{R}} \cap \tilde{Z}_{(\mathbf{c}, \mathbf{T}, X)}) = \mathfrak{V}_{\nu}(\mathbf{c}, q; Z_{\mathbf{T}})X^{\nu}$$

and  $\Omega = 1$  in the current case, we obtain the assertion if  $X \geq q\lambda_1(\Lambda)$ .

We consider the remaining case  $X < q\lambda_1(\Lambda)$ . In this case, we have

$$\mathcal{N} \ll \mathbb{1}_{\Lambda \cap (\mathbf{c} + q\Gamma) \cap Z_{\mathbf{T}} \cap \mathcal{B}_N(X) \neq \emptyset}$$

since two distinct elements of  $\mathbf{c} + q\Lambda$  are  $\geq q\lambda_1(\Lambda)$  apart. Thus, since the error term  $\mathfrak{R}$  dominates all the other terms, the assertion trivially holds if  $X < q\lambda_1(\Lambda)$ .  $\square$

The approximations at finite primes corresponds not to a single residue class  $\mathbf{c} \pmod{q}$  but to a union of residue classes  $u\mathbf{c} \pmod{q}$  over the units  $u \in (\mathbb{Z}/q\mathbb{Z})^{\times}$  since we count projective points. Thus, we need to take a sum of the asymptotic formula of the type given in Lemma 5.5. If we have the error term  $O(1)$  in Lemma 5.5, this error is piled up to  $O(\varphi(q))$ , which is too large for our purpose. We overcome this defect by using the following averaging trick:

**Lemma 5.6.** *Let  $\Lambda \subset \mathbb{R}^N$  be a lattice of rank  $r \geq 1$ .  $\mathbf{c} \in \Lambda$  and  $q \in \mathbb{N}$  such that  $\mathbf{c}$  is  $q$ -primitive and  $\mathbf{a} + \mathcal{B}_N(T)$  be a ball centered at  $\mathbf{a} \in \mathbb{R}^N$  with radius  $T \geq 0$ . If  $T \leq Cq\lambda_1(\Lambda)$  with some  $C \geq 1$ , then we have*

$$\mathcal{N} := \sum_{u \pmod{q}} \sum_{\substack{\mathbf{x} \in \mathbf{a} + \mathcal{B}_N(T) \\ \mathbf{x} \in u\mathbf{c} + q\Lambda}} 1 \ll \frac{T}{\lambda_1(\Lambda)} + 1,$$

where the implicit constant depends only on  $r$  and  $C$ .

*Proof.* We prove the assertion by induction on the rank  $r$ .

We first consider the initial case  $r = 1$ . Since  $\mathbf{c}$  is  $q$ -primitive, for a given  $\mathbf{x} \in \mathbb{R}^N$ , there are at most one  $u \pmod{q}$  with  $\mathbf{x} \in u\mathbf{c} + q\Lambda$ . Thus, we have

$$\mathcal{N} = \sum_{\substack{\mathbf{x} \in \mathbf{a} + \mathcal{B}_N(T) \\ \mathbf{x} \in u\mathbf{c} + q\Lambda}} \sum_{u \pmod{q}} 1 \ll \sum_{\substack{\mathbf{x} \in \mathbf{a} + \mathcal{B}_N(T) \\ \mathbf{x} \in \Lambda}} 1.$$

By using Lemma 3.8 here, we have the claimed bound for the case  $r = 1$ .

Assume that  $r \geq 2$  and that the assertion holds for the rank  $r - 1$  case. Take a  $\mathbb{Z}$ -basis  $(\mathbf{v}_1, \dots, \mathbf{v}_r)$  of  $\Lambda$  as given in Lemma 3.2. Write

$$\mathbf{c} = c_1\mathbf{v}_1 + \dots + c_r\mathbf{v}_r \text{ with } c_1, \dots, c_r \in \mathbb{Z} \quad \text{and} \quad \tilde{\mathbf{c}} := c_2\mathbf{v}_2 + \dots + c_r\mathbf{v}_r.$$

We then have  $\gcd(c_1, \dots, c_r, q) = 1$  because  $\mathbf{c}$  is  $q$ -primitive. Also, write

$$\mathbf{x} = x_1\mathbf{v}_1 + \mathbf{y} \in \Lambda, \quad \mathbf{a} = a_1\mathbf{v}_1 + \mathbf{b} + \mathbf{a}^{\perp}$$

with

$$x_1 \in \mathbb{Z}, \quad a_1 \in \mathbb{R}, \quad \mathbf{y} \in \tilde{\Lambda}, \quad \mathbf{b} \in \tilde{\Lambda}_{\mathbb{R}}, \quad \tilde{\Lambda} := \mathbb{Z}\mathbf{v}_2 + \dots + \mathbb{Z}\mathbf{v}_r \quad \text{and} \quad \mathbf{a}^{\perp} \in \Lambda_{\mathbb{R}}^{\perp}.$$

By (1) and (4) of Lemma 3.2, if  $\mathbf{x} \in \mathbf{a} + \mathcal{B}_N(T)$ , we have

$$T \geq \|\mathbf{x} - \mathbf{a}\| \geq \|(x_1 - a_1)\mathbf{v}_1 + (\mathbf{y} - \mathbf{b})\| \gg |x_1 - a_1| \|\mathbf{v}_1\| \gg |x_1 - a_1| \lambda_1(\Lambda)$$

and so

$$|x_1 - a_1| \leq C_1 U \quad \text{with} \quad U := \frac{T}{\lambda_1(\Lambda)}$$

for some  $C_1 \in \mathbb{R}$  with  $1 \leq C_1 \ll 1$ . Note that  $a_1$  depends only on  $\mathbf{a}$  and the choice of the  $\mathbb{Z}$ -basis  $(\mathbf{v}_1, \dots, \mathbf{v}_r)$ , so it is independent of  $\mathbf{x}$  and  $T$ . We then have

$$\mathcal{N} \leq \sum_{u \pmod{q}} \sum_{\substack{|x_1 - a_1| \leq C_1 U \\ x_1 \equiv uc_1 \pmod{q}}} \sum_{\substack{\mathbf{y} \in -x_1 \mathbf{v}_1 + \mathbf{a} + \mathcal{B}_N(T) \\ \mathbf{y} \in u\tilde{\mathbf{c}} + q\tilde{\Lambda}}} 1.$$

Since  $x_1 \equiv uc_1 \pmod{q}$ , we can write  $x_1 = (c_1, q)\xi_1$  with  $\xi_1 \in \mathbb{Z}$  to get

$$\mathcal{N} \leq \sum_{|\xi_1 - \frac{a_1}{(c_1, q)}| \leq \frac{C_1 U}{(c_1, q)}} \sum_{\xi_1 \equiv u \frac{c_1}{(c_1, q)} \pmod{\frac{q}{(c_1, q)}}} \sum_{\substack{\mathbf{y} \in -(c_1, q)\xi_1 \mathbf{v}_1 + \mathbf{a} + \mathcal{B}_N(T) \\ \mathbf{y} \in u\tilde{\mathbf{c}} + q\tilde{\Lambda}}} 1.$$

We now write  $u = \xi_1 \frac{c_1}{(c_1, q)} + \frac{q}{(c_1, q)}v$  with  $v \in \mathbb{Z}/(c_1, q)\mathbb{Z}$  and  $\mathbf{y} = \xi_1 \frac{c_1}{(c_1, q)}\tilde{\mathbf{c}} + \mathbf{z}$  with  $\mathbf{z} \in \frac{q}{(c_1, q)}v\tilde{\mathbf{c}} + q\tilde{\Lambda}$ . We then have

$$\mathcal{N} \leq \sum_{|\xi_1 - \frac{a_1}{(c_1, q)}| \leq \frac{C_1 U}{(c_1, q)}} \sum_{v \pmod{(c_1, q)}} \sum_{\substack{\mathbf{z} \in -\xi_1 \frac{c_1}{(c_1, q)}\tilde{\mathbf{c}} - (c_1, q)\xi_1 \mathbf{v}_1 + \mathbf{a} + \mathcal{B}_N(T) \\ \mathbf{z} \in \frac{q}{(c_1, q)}v\tilde{\mathbf{c}} + q\tilde{\Lambda}}} 1.$$

Thus, we can further write  $\mathbf{z} = \frac{q}{(c_1, q)}\tilde{\mathbf{x}}$  with  $\tilde{\mathbf{x}} \in v\tilde{\mathbf{c}} + (c_1, q)\tilde{\Lambda}$  and get

$$(5.2) \quad \mathcal{N} \leq \sum_{|\xi_1 - \frac{a_1}{(c_1, q)}| \leq \frac{C_1 U}{(c_1, q)}} \sum_{v \pmod{(c_1, q)}} \sum_{\substack{\tilde{\mathbf{x}} \in \tilde{\mathbf{a}}(\xi_1) + \mathcal{B}_N(\tilde{T}) \\ \tilde{\mathbf{x}} \in v\tilde{\mathbf{c}} + (c_1, q)\tilde{\Lambda}}} 1,$$

where  $\tilde{\mathbf{a}}(\xi_1) := \frac{1}{q/(c_1, q)}(-\xi_1 \frac{c_1}{(c_1, q)}\tilde{\mathbf{c}} - (c_1, q)\xi_1 \mathbf{v}_1 + \mathbf{a})$  and  $\tilde{T} := \frac{T}{q/(c_1, q)}$ . By  $\text{rank } \tilde{\Lambda} = r - 1$ ,  $\gcd(c_2, \dots, c_r, (c_1, q)) = \gcd(c_1, \dots, c_r, q) = 1$  and  $\tilde{T} \leq C \cdot (c_1, q)\lambda_1(\Lambda) \leq C \cdot (c_1, q)\lambda_1(\tilde{\Lambda})$ , we find that the two inner sum of the right-hand side of (5.2) can be bounded by the induction hypothesis. Since  $\lambda_1(\tilde{\Lambda}) \geq \lambda_1(\Lambda)$ , we thus have

$$\begin{aligned} \mathcal{N} &\ll \left( \frac{T}{(c_1, q)\lambda_1(\Lambda)} + 1 \right) \left( \frac{T}{q/(c_1, q) \cdot \lambda_1(\Lambda)} + 1 \right) \\ &\ll \frac{T^2}{q\lambda_1(\Lambda)^2} + \frac{T}{(c_1, q)\lambda_1(\Lambda)} + \frac{T}{q/(c_1, q) \cdot \lambda_1(\Lambda)} + 1 \ll \frac{T}{\lambda_1(\Lambda)} + 1 \end{aligned}$$

since  $T \ll q\lambda_1(\Lambda)$ . This completes the proof.  $\square$

By taking a sum of Lemma 5.5 with using Lemma 5.6, we obtain the following:

**Lemma 5.7.** *Consider a semialgebraic family of homogeneous sets  $Z \subset \mathbb{R}^{M+N}$  with  $M, N \in \mathbb{N}$ , a lattice  $\Gamma \subset \mathbb{R}^N$  and its primitive sublattice  $\Lambda$  of rank  $r \geq 2$ . Let  $\mathbf{c} \in \Gamma$  and  $q \in \mathbb{N}$  such that  $\mathbf{c}$  is  $q$ -primitive in  $\Gamma$ . For  $X \geq 0$  and  $\mathbf{T} \in \mathbb{R}^M$ , we have*

$$\begin{aligned} &\sum_{u \pmod{q}}^* \#((\Lambda \setminus \{0\}) \cap (u\mathbf{c} + q\Gamma) \cap Z_{\mathbf{T}} \cap \mathcal{B}_N(X)) \\ &= \frac{\varphi(q)\mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}})X^r}{q^r \det(\Lambda)} + O\left( \sum_{2 \leq \nu < r} \frac{\varphi(q)\mathfrak{V}_\nu(\mathbf{c}, q; Z_{\mathbf{T}})X^\nu}{q^\nu \lambda_1(\Lambda) \cdots \lambda_\nu(\Lambda)} + \hat{\mathfrak{R}} + \frac{X}{\lambda_1(\Lambda)} \right), \end{aligned}$$

where  $\hat{\mathfrak{R}}$  is defined by

$$\hat{\mathfrak{R}} := \mathbb{1}_{X < q\lambda_1(\Lambda)} \times \frac{\varphi(q)\mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}})X^r}{q^r \det(\Lambda)}$$

and the implicit constant depends only on  $Z$  and  $r$ .

*Proof.* When  $X < \lambda_1(\Lambda)$ , the left-hand side of the assertion is zero and

$$\frac{\varphi(q)\mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}})X^r}{q^r \det(\Lambda)} \ll \left( \frac{X}{\lambda_1(\Lambda)} \right)^r \ll \frac{X}{\lambda_1(\Lambda)}$$

by Lemma 3.1. Thus, we may assume  $X \geq \lambda_1(\Lambda)$  since otherwise the assertion is trivial. This enables us to replace  $\Lambda \setminus \{0\}$  on the left-hand side of the assertion by  $\Lambda$  since this replacement produce an error term of the size  $\ll 1 \ll \frac{X}{\lambda_1(\Lambda)}$ .

We may also assume  $\Lambda \cap (\mathbf{c} + q\Gamma) \neq \emptyset$  since otherwise  $\Lambda \cap (u\mathbf{c} + q\Gamma) = \emptyset$  for all  $u \in (\mathbb{Z}/q\mathbb{Z})^\times$ , so there is nothing to prove. Then, we replace  $\mathbf{c}$  so that  $\mathbf{c} \in \Lambda$  as in Lemma 5.5, which keeps the  $q$ -primitivity of  $\mathbf{c}$ . We use Lemma 5.5 with  $\mathbf{c}$  replaced by  $u\mathbf{c}$  and take a sum over  $u \in (\mathbb{Z}/q\mathbb{Z})^\times$ . Note that  $\mathfrak{V}(u\mathbf{c}, q; Z_{\mathbf{T}}) = \mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}})$  for any  $u \in (\mathbb{Z}/q\mathbb{Z})^\times$ . The main term is just multiplied by  $\varphi(q)$  and all error terms except  $\mathfrak{Q}, \mathfrak{R}$  are multiplied by  $\varphi(q)$ . It now suffices to show

$$(5.3) \quad \varphi(q)\mathfrak{Q} = \varphi(q)\mathbb{1}_{X \geq q\lambda_1(\Lambda)} \ll \frac{X}{\lambda_1(\Lambda)}$$

and

$$(5.4) \quad E := \sum_{u \pmod{q}}^* \mathbb{1}_{\Lambda \cap (u\mathbf{c} + q\Gamma) \cap Z_{\mathbf{T}} \cap \mathfrak{B}_N(X) \neq \emptyset} \ll \frac{X}{\lambda_1(\Lambda)}.$$

The estimate (5.3) is clear by checking two cases  $X \geq q\lambda_1(\Lambda)$  and  $\lambda_1(\Lambda) \leq X < q\lambda_1(\Lambda)$  separately. We thus prove (5.4). If  $X \geq q\lambda_1(\Lambda)$ , we have  $E \leq q \leq \frac{X}{\lambda_1(\Lambda)}$  as desired. If  $\lambda_1(\Lambda) \leq X < q\lambda_1(\Lambda)$ , by Lemma 5.6, we have

$$E \leq \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in \mathfrak{B}_N(X) \\ \mathbf{x} \in u\mathbf{c} + q\Lambda}} 1 \ll \frac{X}{\lambda_1(\Lambda)} + 1 \ll \frac{X}{\lambda_1(\Lambda)}.$$

This completes the proof.  $\square$

We also need an asymptotic formula for the number of the primitive vectors. The next lemma generalize Lemma 3 of le Boudec [14, p. 663] with local conditions and some minor modification.

**Proposition 5.8.** *Under the same setting and conditions as in Lemma 5.7,*

$$\begin{aligned} \mathcal{N} &:= \sum_{u \pmod{q}}^* \#(\Lambda_{\text{prim}} \cap (u\mathbf{c} + q\Gamma) \cap Z_{\mathbf{T}} \cap \mathfrak{B}_N(X)) \\ &= \frac{\varphi(q)\mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}})X^r}{J_r(q)\zeta(r) \det(\Lambda)} \\ &\quad + O\left( \sum_{1 \leq \nu < r} \frac{\varphi(q)\mathfrak{V}_\nu(\mathbf{c}, q; Z_{\mathbf{T}})X^\nu}{q^\nu \lambda_1(\Lambda) \cdots \lambda_\nu(\Lambda)} + \frac{X}{\lambda_1(\Lambda)} \log\left( \frac{X}{\lambda_1(\Lambda)} + 2 \right) \right), \end{aligned}$$

where the implicit constant depends only on  $Z$  and  $r$ .

*Proof.* As in Lemma 5.7, we may assume  $\mathbf{c} \in \Lambda$ , so  $\Lambda \cap (u\mathbf{c} + q\Gamma) = u\mathbf{c} + q\Lambda$ . Then,

$$(5.5) \quad \mathcal{N} = \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in u\mathbf{c} + q\Lambda \\ \mathbf{x} \in \Lambda_{\text{prim}} \\ \mathbf{x} \in Z_{\mathbf{T}} \\ \|\mathbf{x}\| \leq X}} 1 = \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in u\mathbf{c} + q\Lambda \\ \mathbf{x} \in Z_{\mathbf{T}} \\ 0 < \|\mathbf{x}\| \leq X}} \sum_{\substack{\ell \in \mathbb{N} \\ \mathbf{x} \in \ell\Lambda}} \mu(\ell).$$

By using the condition  $0 < \|\mathbf{x}\| \leq X$ , we get  $\ell\lambda_1(\Lambda) = \lambda_1(\ell\Lambda) \leq \|\mathbf{x}\| \leq X$ . Thus, by writing  $\mathbf{x} = \ell\mathbf{y}$  with  $\mathbf{y} \in \Lambda$  in (5.5), we further have

$$(5.6) \quad \mathcal{N} = \sum_{\ell \leq \frac{X}{\lambda_1(\Lambda)}} \mu(\ell) \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{y} \in \Lambda \\ \ell\mathbf{y} \in u\mathbf{c} + q\Lambda \\ \mathbf{y} \in Z_{\mathbf{T}} \\ 0 < \|\mathbf{y}\| \leq \frac{X}{\ell}}} 1.$$

Since  $Z$  is a semialgebraic family of homogeneous sets, we have  $\ell\mathbf{y} \in Z_{\mathbf{T}}$  iff  $\mathbf{y} \in Z_{\mathbf{T}}$ . Since  $\mathbf{c}$  is  $q$ -primitive in  $\Gamma$ , the conditions  $(u, q) = 1$  and  $\ell\mathbf{y} \in u\mathbf{c} + q\Lambda$  implies  $(\ell, q) = 1$ . Thus, after a permutation of  $u \in (\mathbb{Z}/q\mathbb{Z})^\times$ , we can rewrite (5.6) as

$$(5.7) \quad \mathcal{N} = \sum_{\substack{\ell \leq \frac{X}{\lambda_1(\Lambda)} \\ (\ell, q) = 1}} \mu(\ell) \sum_{u \pmod{q}}^* \#((\Lambda \setminus \{0\}) \cap (u\mathbf{c} + q\Gamma) \cap Z_{\mathbf{T}} \cap \mathcal{B}_N(\frac{X}{\ell})).$$

By Lemma 5.7 and (5.7), we have

$$(5.8) \quad \mathcal{N} = \frac{\varphi(q)\mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}})X^r}{q^r \det(\Lambda)} S_{\text{main}} + O\left(\sum_{1 \leq \nu < r} E_\nu + E_{\mathfrak{R}} + \frac{X}{\lambda_1(\Lambda)} \log\left(\frac{X}{\lambda_1(\Lambda)} + 2\right)\right),$$

where

$$S_{\text{main}} := \sum_{\substack{\ell \leq \frac{X}{\lambda_1(\Lambda)} \\ (\ell, q) = 1}} \frac{\mu(\ell)}{\ell^r}, \quad E_\nu := \sum_{\ell \leq \frac{X}{\lambda_1(\Lambda)}} \frac{\mu^2(\ell)}{\ell^\nu} \frac{\varphi(q)\mathfrak{V}_\nu(\mathbf{c}, q; Z_{\mathbf{T}})X^\nu}{q^\nu \lambda_1(\Lambda) \cdots \lambda_\nu(\Lambda)},$$

$$E_{\mathfrak{R}} := \frac{\varphi(q)\mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}})X^r}{q^r \det(\Lambda)} \sum_{\frac{X}{q\lambda_1(\Lambda)} < \ell \leq \frac{X}{\lambda_1(\Lambda)}} \frac{\mu^2(\ell)}{\ell^r}.$$

For  $S_{\text{main}}$ , by  $r \geq 2$  and Minkowski's second theorem (Lemma 3.1), we have

$$(5.9) \quad \frac{\varphi(q)\mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}})X^r}{q^r \det(\Lambda)} S_{\text{main}} = \frac{\varphi(q)\mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}})X^r}{J_r(q)\zeta(r) \det(\Lambda)} + O\left(\frac{X}{\lambda_1(\Lambda)}\right).$$

For  $E_\nu$  with  $\nu = 1$ , we have

$$(5.10) \quad E_1 \ll \sum_{\ell \leq \frac{X}{\lambda_1(\Lambda)}} \frac{\mu^2(\ell)}{\ell} \frac{X}{\lambda_1(\Lambda)} \ll \frac{X}{\lambda_1(\Lambda)} \log\left(\frac{X}{\lambda_1(\Lambda)} + 2\right)$$

For  $E_\nu$  with  $1 < \nu < r$ , we have

$$(5.11) \quad E_\nu \ll \frac{\varphi(q)\mathfrak{V}_\nu(\mathbf{c}, q; Z_{\mathbf{T}})X^\nu}{q^\nu \lambda_1(\Lambda) \cdots \lambda_\nu(\Lambda)}.$$

For  $E_{\mathfrak{R}}$ , since  $r \geq 2$ , by Minkowski's second theorem (Lemma 3.1), we have

$$(5.12) \quad E_{\mathfrak{R}} \ll \frac{\varphi(q)\mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}})X^r}{q^r \det(\Lambda)} \left(\frac{q\lambda_1(\Lambda)}{X}\right)^{r-1} \ll \frac{X}{\lambda_1(\Lambda)}.$$

On inserting (5.9), (5.10), (5.11) and (5.12) into (5.8), we obtain the lemma.  $\square$

For the ease of readers, we state the special case of Proposition 5.8 without congruence conditions and semialgebraic restriction as follows:

**Proposition 5.9.** Consider a lattice  $\Lambda \subset \mathbb{R}^N$  of rank  $r \geq 2$ . For  $X \geq 0$ , we have

$$\begin{aligned} & \#(\Lambda_{\text{prim}} \cap \mathcal{B}_N(X)) \\ &= \frac{V_r X^r}{\zeta(r) \det(\Lambda)} + O\left(\sum_{2 \leq \nu < r} \frac{X^\nu}{\lambda_1(\Lambda) \cdots \lambda_\nu(\Lambda)} + \frac{X}{\lambda_1(\Lambda)} \log\left(\frac{X}{\lambda_1(\Lambda)} + 2\right)\right), \end{aligned}$$

where the implicit constant depends only on  $r$ .

*Proof.* This follows by Proposition 5.8 with  $Z = \mathbb{R}^{N+1}$ ,  $q = 1$ ,  $\sigma = 1$  and  $\mathbf{T} = 0$ .  $\square$

We now specialize the above setting to the case where the semialgebraic family of homogeneous sets is given by the cone  $\mathcal{C}_N(\boldsymbol{\xi}, \sigma)$ . With this setting, for simplicity, we give only upper bound, which suffices for our purpose.

**Proposition 5.10.** Consider a lattice  $\Gamma \subset \mathbb{R}^N$  and its primitive sublattice  $\Lambda$  of rank  $r \geq 2$ . Let  $\mathbf{c} \in \Gamma$  and  $q \in \mathbb{N}$  be such that  $\mathbf{c}$  is  $q$ -primitive in  $\Gamma$ . For  $\boldsymbol{\xi} \in \mathbb{R}^N \setminus \{0\}$ ,  $\sigma \in (0, 1]$  and  $X \geq 0$ , we have

$$\begin{aligned} & \sum_{u \pmod{q}}^* \#((\Lambda \setminus \{0\}) \cap (u\mathbf{c} + q\Gamma) \cap \mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(X)) \\ & \ll \sum_{2 \leq \nu \leq r} \frac{\varphi(q) \left(\frac{\sigma}{q} X\right)^\nu}{\sigma \lambda_1(\Lambda) \cdots \lambda_\nu(\Lambda)} + \frac{X}{\lambda_1(\Lambda)}, \end{aligned}$$

where the implicit constant depends only on  $r$ .

*Proof.* For  $\nu \in \{1, \dots, \nu\}$ , let us write

$$\begin{aligned} \mathfrak{V}(\mathbf{c}, q; \boldsymbol{\xi}, \sigma) &:= \text{vol}_{\Lambda_{\mathbb{R}}}(\text{span}_{\mathbb{R}}(\Lambda \cap (\mathbf{c} + q\Gamma)) \cap \mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(1)), \\ \mathfrak{V}_\nu(\mathbf{c}, q; \boldsymbol{\xi}, \sigma) &:= V_{\nu, \Lambda_{\mathbb{R}}}(\text{span}_{\mathbb{R}}(\Lambda \cap (\mathbf{c} + q\Gamma)) \cap \mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(1)). \end{aligned}$$

For  $\nu \in \{1, \dots, \nu\}$ , we first show the bound

$$(5.13) \quad \mathfrak{V}_\nu(\mathbf{c}, q; \boldsymbol{\xi}, \sigma) \ll \sigma^{\nu-1}, \quad \text{so particularly, } \mathfrak{V}(\mathbf{c}, q; \boldsymbol{\xi}, \sigma) \ll \sigma^{r-1}.$$

Note that  $\mathfrak{V}_\nu(\mathbf{c}, q; \boldsymbol{\xi}, \sigma) \leq V_{\nu, \Lambda_{\mathbb{R}}}(\Lambda_{\mathbb{R}} \cap \mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(1))$ . Let  $V \subset \Lambda_{\mathbb{R}}$  be an arbitrary  $\nu$ -dimensional subspace and consider the orthogonal projection  $\pi_V: \Lambda_{\mathbb{R}} \rightarrow V$ . It suffices to bound  $\text{vol}_V(\pi_V(\Lambda_{\mathbb{R}} \cap \mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(1)))$ . In order to apply Lemma 4.4, consider the orthogonal decomposition  $\boldsymbol{\xi} = \tilde{\boldsymbol{\xi}} + \tilde{\boldsymbol{\xi}}^\perp$  with  $\tilde{\boldsymbol{\xi}} \in \Lambda_{\mathbb{R}}$  and  $\tilde{\boldsymbol{\xi}}^\perp \in \Lambda_{\mathbb{R}}^\perp$ . When  $\tilde{\boldsymbol{\xi}} = 0$  and  $\sigma = 1$ , by Lemma 4.4 with  $W := \Lambda_{\mathbb{R}}$ , we have

$$\text{vol}_V(\pi_V(\Lambda_{\mathbb{R}} \cap \mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(1))) = \text{vol}_V(\pi_V(\mathcal{B}_N(1))) \ll 1 = \sigma^{\nu-1}.$$

When  $\tilde{\boldsymbol{\xi}} = 0$  and  $0 \leq \sigma < 1$ , by Lemma 4.4 with  $W := \Lambda_{\mathbb{R}}$ , we have

$$\text{vol}_V(\pi_V(\Lambda_{\mathbb{R}} \cap \mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(1))) = \text{vol}_V(\{0\}) = 0 \ll \sigma^{\nu-1}.$$

When  $\tilde{\boldsymbol{\xi}} \neq 0$ , by Lemma 4.4 with  $W := \Lambda_{\mathbb{R}}$  and by Lemma 4.3, we have

$$\text{vol}_V(\pi_V(\Lambda_{\mathbb{R}} \cap \mathcal{C}_N(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_N(1))) \leq \text{vol}_V\left(\pi_V(\mathcal{C}_{\Lambda_{\mathbb{R}}}(\tilde{\boldsymbol{\xi}}, \sigma) \cap \mathcal{B}_N(1))\right) \ll \sigma^{\nu-1}.$$

By these bounds, we obtain (5.13) for all cases.

We now use Lemma 5.7 with a semialgebraic family of homogeneous sets

$$Z := \{(\boldsymbol{\xi}, \sigma, \mathbf{x}) \in \mathbb{R}^N \times (0, 1] \times \mathbb{R}^N \mid \mathbf{x} \in \mathcal{C}_N(\boldsymbol{\xi}, \sigma)\}$$

and  $\mathbf{T} = (\boldsymbol{\xi}, \sigma)$ . For the main term, we have  $\mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}}) = \mathfrak{V}(\mathbf{c}, q; \boldsymbol{\xi}, \sigma) \ll \sigma^{r-1}$  by (5.13). For the sum in the error term, (5.13) gives  $\mathfrak{V}_{\nu}(\mathbf{c}, q; Z_{\mathbf{T}}) = \mathfrak{V}_{\nu}(\mathbf{c}, q; \boldsymbol{\xi}, \sigma) \ll \sigma^{\nu-1}$ . For the error term  $\widehat{\mathfrak{R}}$ , since  $r \geq 2$ , by Lemma 3.1 and (5.13), we have

$$\mathbb{1}_{X < q\lambda_1(\Lambda)} \times \frac{\varphi(q)\mathfrak{V}(\mathbf{c}, q; Z_{\mathbf{T}})X^r}{q^r \det(\Lambda)} = \mathbb{1}_{X < q\lambda_1(\Lambda)} \times \varphi(q) \left( \frac{X}{q\lambda_1(\Lambda)} \right)^r \ll \frac{X}{\lambda_1(\Lambda)}.$$

On inserting the above observations into Lemma 5.7, we obtain the lemma.  $\square$

## 6. SUMS OF RECIPROCAL INCLUDING THE VERONESE EMBEDDING

In this section, we prove some lemmas that we use in Section 8 to prove Theorem 1.4. They will be used to calculate the main term of  $\sum_{V \in \mathbb{V}_{d,n}(A)} N_V(B; \boldsymbol{\xi}, \sigma)$ .

**Lemma 6.1.** *Let  $n, d$  be positive integers with  $n \geq d \geq 2$ . For  $\mathbf{c} \in \mathbb{Z}^{n+1}$  and  $q \in \mathbb{N}$  such that  $\mathbf{c}$  is  $q$ -primitive and for  $\boldsymbol{\xi} \in \mathbb{R}^{n+1} \setminus \{0\}$ ,  $\sigma \in (0, 1]$  and  $X \geq 1$ , we have*

$$(6.1) \quad \sum_{\substack{\mathbf{x} \in \mathbf{c} + q\mathbb{Z}^{n+1} \\ 0 < \|\mathbf{x}\| \leq X \\ \mathbf{x} \in \mathcal{E}_{n+1}(\boldsymbol{\xi}, \sigma)}} \frac{1}{\|\nu_{d,n}(\mathbf{x})\|} = \frac{\mathfrak{W}_{d,n}(\boldsymbol{\xi}, \sigma) X^{n+1-d}}{q^{n+1}} + O\left(\frac{1}{\sigma} \left(\frac{\sigma}{q}\right)^n (X^{n-d} + \log X) + \frac{1}{q} + \mathfrak{R}\right),$$

where the coefficient  $\mathfrak{W}_{d,n}(\boldsymbol{\xi}, \sigma)$  is defined by

$$\mathfrak{W}_{d,n}(\boldsymbol{\xi}, \sigma) := \int_{\mathcal{E}_{n+1}(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_{n+1}(1)} \frac{d\mathbf{x}}{\|\nu_{d,n}(\mathbf{x})\|},$$

the error term  $\mathfrak{R}$  is defined by

$$\mathfrak{R} := \int_1^q \mathbb{1}_{(\mathbf{c} + q\mathbb{Z}^{n+1}) \cap (\mathcal{B}_{n+1}(\min(X, dt^{\frac{1}{d}})) \setminus \{0\}) \neq \emptyset} \frac{dt}{t^2}$$

and the implicit constant depends only on  $d, n$ . Moreover, we have

$$(6.2) \quad \mathfrak{W}_{d,n}(\boldsymbol{\xi}, \sigma) \asymp \sigma^n,$$

where the implicit constant depends only on  $d, n$ .

*Proof.* We first prove the asymptotic formula (6.1). By partial summation, we have

$$(6.3) \quad \sum_{\substack{\mathbf{x} \in \mathbf{c} + q\mathbb{Z}^{n+1} \\ 0 < \|\mathbf{x}\| \leq X \\ \mathbf{x} \in \mathcal{E}_{n+1}(\boldsymbol{\xi}, \sigma)}} \frac{1}{\|\nu_{d,n}(\mathbf{x})\|} = \int_1^\infty \left( \sum_{\substack{\mathbf{x} \in \mathbf{c} + q\mathbb{Z}^{n+1} \\ 0 < \|\mathbf{x}\| \leq X \\ \|\nu_{d,n}(\mathbf{x})\| \leq t \\ \mathbf{x} \in \mathcal{E}_{n+1}(\boldsymbol{\xi}, \sigma)}} 1 \right) \frac{dt}{t^2}.$$

By considering the semialgebraic set

$$Z := \left\{ (\mathbf{c}, \boldsymbol{\xi}, \sigma, X, t, \mathbf{x}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \mid (*) \right\},$$

where  $(*)$  is the condition

$$(*) \quad 0 < \|\mathbf{c} + \mathbf{x}\| \leq X, \quad \|\nu_{d,n}(\mathbf{c} + \mathbf{x})\| \leq t, \quad \|(\mathbf{c} + \mathbf{x}) \wedge \boldsymbol{\xi}\| \leq \sigma \|\mathbf{c} + \mathbf{x}\| \|\boldsymbol{\xi}\|,$$

and by changing the variable via  $\mathbf{x} \rightsquigarrow \mathbf{x} + \mathbf{c}$ , we have

$$(6.4) \quad \sum_{\substack{\mathbf{x} \in \mathbf{c} + q\mathbb{Z}^{n+1} \\ 0 < \|\mathbf{x}\| \leq X \\ \|\nu_{d,n}(\mathbf{x})\| \leq t \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} 1 = \#(q\mathbb{Z}^{n+1} \cap Z_{(\mathbf{c}, \boldsymbol{\xi}, \sigma, X, t)}),$$

where  $Z_{\mathbf{T}}$  is as in Lemma 3.8. By applying Lemma 3.8 to (6.4), we obtain

$$(6.5) \quad \sum_{\substack{\mathbf{x} \in \mathbf{c} + q\mathbb{Z}^{n+1} \\ 0 < \|\mathbf{x}\| \leq X \\ \|\nu_{d,n}(\mathbf{x})\| \leq t \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} 1 = \frac{\text{vol}_{n+1}(Z_{(\mathbf{c}, \boldsymbol{\xi}, \sigma, X, t)})}{q^{n+1}} + O\left(\sum_{0 \leq \nu \leq n} \frac{V_\nu(Z_{(\mathbf{c}, \boldsymbol{\xi}, \sigma, X, t)})}{q^\nu}\right).$$

For the fiber  $Z_{(\mathbf{c}, \boldsymbol{\xi}, \sigma, X, t)}$ , we have

$$(6.6) \quad Z_{(\mathbf{c}, \boldsymbol{\xi}, \sigma, X, t)} = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \mid \begin{array}{l} 0 < \|\mathbf{x}\| \leq X, \|\nu_{d,n}(\mathbf{x})\| \leq t \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma) \end{array} \right\} - \mathbf{c}.$$

We next bound  $V_\nu(Z_{(\mathbf{c}, \boldsymbol{\xi}, \sigma, X, t)})$ . Take a  $\nu$ -dimensional subspace  $W$  of  $\mathbb{R}^{n+1}$  arbitrarily and consider the orthogonal projection  $\pi_W: \mathbb{R}^{n+1} \rightarrow W$ . It suffices to bound  $\text{vol}_W(\pi_W(Z_{(\mathbf{c}, \boldsymbol{\xi}, \sigma, X, t)}))$ . By (2.2), (6.6) and Lemma 4.3, we have

$$(6.7) \quad \begin{aligned} \text{vol}_W(\pi_W(Z_{(\mathbf{c}, \boldsymbol{\xi}, \sigma, X, t)})) &\leq \text{vol}_W\left(\pi_W(\mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_{n+1}(\min(X, dt^{\frac{1}{d}}))\right) \\ &\ll \frac{1}{\sigma}(\sigma \min(X, dt^{\frac{1}{d}}))^\nu \end{aligned}$$

for  $\nu \geq 1$ . Also, we trivially have

$$(6.8) \quad V_0(Z_{(\mathbf{c}, \boldsymbol{\xi}, \sigma, X, t)}) \leq 1.$$

On inserting (6.6), (6.7) and (6.8) into (6.5), we obtain

$$\sum_{\substack{\mathbf{x} \in \mathbf{c} + q\mathbb{Z}^{n+1} \\ 0 < \|\mathbf{x}\| \leq X \\ \|\nu_{d,n}(\mathbf{x})\| \leq t \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} 1 = \frac{1}{q^{n+1}} \int_{\substack{\|\mathbf{x}\| \leq X \\ \|\nu_{d,n}(\mathbf{x})\| \leq t \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} d\mathbf{x} + O(E_1 + E_2),$$

where

$$E_1 := \frac{1}{\sigma} \left( \frac{\sigma}{q} \min(X, dt^{\frac{1}{d}}) \right)^n \quad \text{and} \quad E_2 := \frac{1}{q} \min(X, dt^{\frac{1}{d}})$$

provided  $E_2 \geq \frac{1}{3}$  and so  $E_2 \gg 1$ . On the other hand, when  $E_2 \leq \frac{1}{3}$ , by (2.2), the conditions  $\|\mathbf{x}\| > 0$  and  $\|\nu_{d,n}(\mathbf{x})\| \leq t$  implies  $0 < \|\mathbf{x}\| \leq \frac{q}{3}$ . In this case  $E_2 \leq \frac{1}{3}$ , by Lemma 4.3 and  $\frac{\sigma}{q} \min(X, dt^{\frac{1}{d}}) \leq 1$ , we have

$$\frac{1}{q^{n+1}} \int_{\substack{\|\mathbf{x}\| \leq X \\ \|\nu_{d,n}(\mathbf{x})\| \leq t \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} d\mathbf{x} \ll \frac{1}{\sigma} \left( \frac{\sigma}{q} \min(X, dt^{\frac{1}{d}}) \right)^{n+1} \ll E_1$$

and, since there is at most one point in

$$(\mathbf{c} + q\mathbb{Z}^{n+1}) \cap \mathcal{B}_{n+1}(\min(X, dt^{\frac{1}{d}})) \subset (\mathbf{c} + q\mathbb{Z}^{n+1}) \cap \mathcal{B}_{n+1}(\frac{q}{3}),$$

we have

$$\sum_{\substack{\mathbf{x} \in \mathbf{c} + q\mathbb{Z}^{n+1} \\ 0 < \|\mathbf{x}\| \leq X \\ \|\nu_{d,n}(\mathbf{x})\| \leq t \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} 1 \leq \mathbb{1}_{(\mathbf{c} + q\mathbb{Z}^{n+1}) \cap (\mathcal{B}_{n+1}(\min(X, dt^{\frac{1}{d}})) \setminus \{0\}) \neq \emptyset} =: \tilde{\mathfrak{R}}.$$

Thus, in any case, we have

$$\sum_{\substack{\mathbf{x} \in \mathbf{c} + q\mathbb{Z}^{n+1} \\ 0 < \|\mathbf{x}\| \leq X \\ \|\nu_{d,n}(\mathbf{x})\| \leq t \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} 1 = \frac{1}{q^{n+1}} \int_{\substack{\|\mathbf{x}\| \leq X \\ \|\nu_{d,n}(\mathbf{x})\| \leq t \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} d\mathbf{x} + O(E_1 + E_2 + \tilde{\mathfrak{R}}).$$

We substitute this formula into (6.3) and obtain

$$(6.9) \quad \sum_{\substack{\mathbf{x} \in \mathbf{c} + q\mathbb{Z}^{n+1} \\ 0 < \|\mathbf{x}\| \leq X \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} \frac{1}{\|\nu_{d,n}(\mathbf{x})\|} = I + O(I_0 + I_1 + I_2 + I_{\tilde{\mathfrak{R}}}),$$

where

$$\begin{aligned} I &:= \frac{1}{q^{n+1}} \int_0^\infty \left( \int_{\substack{\|\mathbf{x}\| \leq X \\ \|\nu_{d,n}(\mathbf{x})\| \leq t \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} d\mathbf{x} \right) \frac{dt}{t^2}, & I_0 &:= \frac{1}{q^{n+1}} \int_0^1 \left( \int_{\substack{\|\mathbf{x}\| \leq X \\ \|\nu_{d,n}(\mathbf{x})\| \leq t \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} d\mathbf{x} \right) \frac{dt}{t^2}, \\ I_1 &:= \frac{1}{\sigma} \left( \frac{\sigma}{q} \right)^n \int_1^\infty \min(X, dt^{\frac{1}{d}})^n \frac{dt}{t^2}, & I_2 &:= \frac{1}{q} \int_1^\infty \min(X, dt^{\frac{1}{d}}) \frac{dt}{t^2}, \\ I_{\tilde{\mathfrak{R}}} &:= \int_1^\infty \mathbb{1}_{(\mathbf{c} + q\mathbb{Z}^{n+1}) \cap (\mathcal{B}_{n+1}(\min(X, dt^{\frac{1}{d}})) \setminus \{0\}) \neq \emptyset} \frac{dt}{t^2} \end{aligned}$$

For  $I$ , we swap the integral and change the variable via  $\mathbf{x} \rightsquigarrow X \cdot \mathbf{x}$  to obtain

$$(6.10) \quad I = \frac{1}{q^{n+1}} \int_{\substack{\|\mathbf{x}\| \leq X \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} \frac{d\mathbf{x}}{\|\nu_{d,n}(\mathbf{x})\|} = \frac{\mathfrak{W}_{d,n}(\boldsymbol{\xi}, \sigma) X^{n+1-d}}{q^{n+1}}.$$

For the integral  $I_0$ , by using (2.2) and Lemma 4.3, we bound as

$$(6.11) \quad I_0 \ll \frac{1}{\sigma} \left( \frac{\sigma}{q} \right)^{n+1} \int_0^1 t^{\frac{n+1}{d}-2} dt \ll \frac{1}{\sigma} \left( \frac{\sigma}{q} \right)^{n+1} \ll \frac{1}{\sigma} \left( \frac{\sigma}{q} \right)^n X^{n-d}.$$

For  $I_1$ , we dissect integral at  $t = X^d$  and bound as

$$(6.12) \quad I_1 = \frac{1}{\sigma} \left( \frac{\sigma}{q} \right)^n \left( \int_1^{X^d} t^{\frac{n}{d}-2} dt + X^n \int_{X^d}^\infty \frac{dt}{t^2} \right) \ll \frac{1}{\sigma} \left( \frac{\sigma}{q} \right)^n (X^{n-d} + \log X),$$

where  $\log X$  is introduced to deal with the case  $n = d$ . Similarly,  $I_2$  is

$$(6.13) \quad I_2 \ll \frac{1}{q} \left( \int_1^{X^d} t^{\frac{1}{d}-2} du + X \int_{X^d}^\infty \frac{dt}{t^2} \right) \ll \frac{1}{q}$$

Finally, we bound  $I_{\tilde{\mathfrak{R}}}$  as

$$(6.14) \quad I_{\tilde{\mathfrak{R}}} \ll \mathfrak{R} + \int_q^\infty \frac{dt}{t^2} \ll \mathfrak{R} + \frac{1}{q}.$$

On inserting (6.10), (6.11), (6.12), (6.13) and (6.14) into (6.9), we get (6.1)

We next prove (6.2). For the upper bound, we use (2.2) and Lemma 4.3 to obtain

$$\begin{aligned} \mathfrak{W}_{d,n}(\boldsymbol{\xi}, \sigma) &= \int_{\mathcal{E}_{n+1}(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_{n+1}(1)} \int_{\|\nu_{d,n}(\mathbf{x})\|}^{\infty} \frac{dt}{t^2} d\mathbf{x} \\ &\leq \int_0^{\infty} \left( \int_{\mathcal{E}_{n+1}(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_{n+1}(\min(1, dt^{\frac{1}{d}}))} d\mathbf{x} \right) \frac{dt}{t^2} \\ &\ll \sigma^n \int_0^{\infty} \min(1, dt^{\frac{1}{d}})^{n+1} \frac{dt}{t^2} \ll \sigma^n \left( \int_0^1 t^{\frac{n+1}{d}-2} dt + \int_1^{\infty} \frac{dt}{t^2} \right) \ll \sigma^n \end{aligned}$$

For the lower bound, by (2.2) and Lemma 4.2, we have

$$\mathfrak{W}_{d,n}(\boldsymbol{\xi}, \sigma) \gg \int_{\mathcal{E}_{n+1}(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_{n+1}(1)} d\mathbf{x} \gg \sigma^n.$$

This completes the proof of (6.2).  $\square$

**Corollary 6.2.** *Under the same setting and conditions as in Lemma 6.1, we have*

$$\begin{aligned} &\sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in u\mathbf{c} + q\mathbb{Z}^{n+1} \\ 0 < \|\mathbf{x}\| \leq X \\ \mathbf{x} \in \mathcal{E}_{n+1}(\boldsymbol{\xi}, \sigma)}} \frac{1}{\|\nu_{d,n}(\mathbf{x})\|} \\ &= \frac{\varphi(q) \mathfrak{W}_{d,n}(\boldsymbol{\xi}, \sigma) X^{n+1-d}}{q^{n+1}} + O\left( \left( \frac{\sigma}{q} \right)^{n-1} (X^{n-d} + \log X) + 1 \right), \end{aligned}$$

where the implicit constant depends only on  $d, n$ .

*Proof.* We use Lemma 6.1 with  $\mathbf{c}$  replaced by  $u\mathbf{c}$  and then take a sum over  $u \in (\mathbb{Z}/q\mathbb{Z})^\times$ . The main term is just multiplied by  $\varphi(q)$  and all but the last error term is multiplied by  $\varphi(q) \leq q$ . By Lemma 5.6, the remaining error term is bounded as

$$\leq \int_1^q \left( \sum_{u \pmod{q}}^* \mathbb{1}_{(u\mathbf{c} + q\mathbb{Z}^{n+1}) \cap \mathcal{B}_{n+1}(dt^{\frac{1}{d}}) \neq \emptyset} \right) \frac{dt}{t^2} \ll \int_1^q t^{\frac{1}{d}-2} dt \ll 1.$$

This completes the proof.  $\square$

**Lemma 6.3.** *Under the same setting and conditions as in Lemma 6.1, we have*

$$\begin{aligned} S &:= \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1} \\ \|\mathbf{x}\| \leq X \\ \mathbf{x} \equiv u\mathbf{c} \pmod{q} \\ \mathbf{x} \in \mathcal{E}_{n+1}(\boldsymbol{\xi}, \sigma)}} \frac{1}{\|\nu_{d,n}(\mathbf{x})\|} \\ &= \frac{\mathfrak{W}_{d,n}(\boldsymbol{\xi}, \sigma) \varphi(q) X^{n+1-d}}{J_{n+1}(q) \zeta(n+1)} + O\left( \left( \frac{\sigma}{q} \right)^{n-1} (X^{n-d} + \log X) + 1 \right), \end{aligned}$$

where the implicit constant depends only on  $d, n$ .

*Proof.* Since the cone  $\mathcal{C}_N(\boldsymbol{\xi}, \sigma)$  is invariant under dilation, we have

$$\begin{aligned}
(6.15) \quad S &= \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}^{n+1} \\ 0 < \|\mathbf{x}\| \leq X \\ \mathbf{x} \equiv u\mathbf{c} \pmod{q} \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} \frac{1}{\|\nu_{d,n}(\mathbf{x})\|} \sum_{\ell|\mathbf{x}} \mu(\ell) \\
&= \sum_{\ell \leq X} \frac{\mu(\ell)}{\ell^d} \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}^{n+1} \\ 0 < \|\mathbf{x}\| \leq X/\ell \\ \ell\mathbf{x} \equiv u\mathbf{c} \pmod{q} \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} \frac{1}{\|\nu_{d,n}(\mathbf{x})\|}.
\end{aligned}$$

Since  $\mathbf{c}$  is  $q$ -primitive, the conditions  $(u, q) = 1$  and  $\ell\mathbf{x} \equiv u\mathbf{c} \pmod{q}$  imply  $(\ell, q) = 1$ . Thus, by changing the variable via  $u \rightsquigarrow \ell u$ , we can rewrite (6.15) as

$$S = \sum_{\substack{\ell \leq X \\ (\ell, q) = 1}} \frac{\mu(\ell)}{\ell^d} \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}^{n+1} \\ 0 < \|\mathbf{x}\| \leq X/\ell \\ \mathbf{x} \equiv u\mathbf{c} \pmod{q} \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma)}} \frac{1}{\|\nu_{d,n}(\mathbf{x})\|}.$$

We then apply Corollary 6.2 to the inner sum and get

$$S = \mathfrak{W}_{d,n}(\boldsymbol{\xi}, \sigma) \frac{\varphi(q)}{q^{n+1}} X^{n+1-d} \sum_{\substack{\ell \leq X \\ (\ell, q) = 1}} \frac{\mu(\ell)}{\ell^{n+1}} + O\left(\left(\frac{\sigma}{q}\right)^{n-1} (X^{n-d} + \log X) + 1\right)$$

since  $n \geq d \geq 2$ . By using

$$\sum_{\substack{\ell \leq X \\ (\ell, q) = 1}} \frac{\mu(\ell)}{\ell^{n+1}} = \prod_{p|q} \left(1 - \frac{1}{p^{n+1}}\right) + O\left(\frac{1}{X}\right) = \frac{q^{n+1}}{J_{n+1}(q)\zeta(n+1)} + O\left(\frac{1}{X}\right).$$

and (6.2) for the main term, we obtain the assertion.  $\square$

## 7. DISTRIBUTION OF $\mathfrak{d}_2(\mathbf{x})$ WITH LOCAL CONDITIONS

In this section, as a preparation for the proof of Theorem 1.4, we prove a key lemma Lemma 7.9 on the distribution of  $\mathfrak{d}_2(\mathbf{x})$  over integral vectors  $\mathbf{x}$  satisfying local conditions. It is deduced from Proposition 7.7, which gives an upper bound of the number of primitive sublattices of  $\mathbb{Z}^{n+1}$  with prescribed successive minima that contain points  $\mathbf{x}$  satisfying conditions of the form  $d_p([\mathbf{x}], \xi_p) \leq \sigma_p$ . Lemma 7.9 and Proposition 7.7 correspond to [4, Lemma 3.20] and [4, Lemma 3.19] respectively, though we only treat “ $r = 2$ ” case in Lemma 7.9.

**Definition 7.1.** Let  $M$  be a free  $\mathbb{Z}$ -module of finite rank. Let  $v \in M$ . Define

$$g(v) = g_M(v) := \begin{cases} \#(M/\mathbb{Z}v)_{\text{tors}} & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases}$$

When  $v \neq 0$ , by writing  $v = dw$  with  $d \in \mathbb{Z}$  and  $w \in M_{\text{prim}}$ , we have  $g(v) = |d|$ .

**Definition 7.2.** Let  $n, r$  be integers with  $n \geq 2$  and  $1 \leq r \leq n$ , and  $s_1, \dots, s_r \geq 1$  be real numbers. Let  $\sigma \in (0, 1]$  and  $\boldsymbol{\xi} \in \mathbb{R}^{n+1} \setminus \{0\}$  and let  $q \in \mathbb{N}$  and  $\mathbf{c} \in \mathbb{Z}^{n+1}$  with  $(g_{\mathbb{Z}^{n+1}}(\mathbf{c}), q) = 1$ . Note that if  $\mathbf{c} = 0$ , then  $q = 1$ . Let

$$\mathcal{S}_{r,n} = \mathcal{S}_{r,n}(s_1, \dots, s_r; \mathbf{c}, q; \boldsymbol{\xi}, \sigma)$$

$$:= \left\{ L \subset \mathbb{Z}^{n+1} \left| \begin{array}{l} L : \text{a primitive sublattice of } \mathbb{Z}^{n+1} \text{ of rank } r \\ s_i/2 < \lambda_i(L) \leq s_i \text{ for } i \in \{1, \dots, r\} \\ (L \setminus \{0\}) \cap \bigcup_{u \pmod{q}}^* (u\mathbf{c} + q\mathbb{Z}^{n+1}) \cap \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma) \neq \emptyset \end{array} \right. \right\}.$$

and

$$S_{r,n} = S_{r,n}(s_1, \dots, s_r; \mathbf{c}, q; \boldsymbol{\xi}, \sigma) := \#\mathcal{S}_{r,n}.$$

**Definition 7.3.** For a non-zero sublattice  $L \subset \mathbb{Z}^m$  and  $\boldsymbol{\xi} \in \mathbb{R}^m \setminus \{0\}$ , define

$$d_\infty(L, \boldsymbol{\xi}) := \inf \{d_\infty(\mathbf{x}, \boldsymbol{\xi}) \mid \mathbf{x} \in L \setminus \{0\}\}.$$

Also, for the zero lattice  $L = 0$  and  $\boldsymbol{\xi} \in \mathbb{R}^m \setminus \{0\}$ , we define  $d_\infty(0, \boldsymbol{\xi}) := 1$ .

**Remark 7.4.** For  $\boldsymbol{\xi} \in \mathbb{R}^m$  with  $\|\boldsymbol{\xi}\| = 1$ , we have

$$d_\infty(L, \boldsymbol{\xi}) = \|\boldsymbol{\xi}_2\| \quad \text{by writing } \boldsymbol{\xi} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 \text{ with } \boldsymbol{\xi}_1 \in L_{\mathbb{R}} \text{ and } \boldsymbol{\xi}_2 \in L_{\mathbb{R}}^\perp.$$

Indeed, for  $\mathbf{x} \in L_{\mathbb{R}}$  with  $\|\mathbf{x}\| = 1$ , we have

$$d_\infty(\mathbf{x}, \boldsymbol{\xi})^2 = \frac{\|\mathbf{x}\|^2 \|\boldsymbol{\xi}\|^2 - \langle \mathbf{x}, \boldsymbol{\xi} \rangle^2}{\|\mathbf{x}\|^2 \|\boldsymbol{\xi}\|^2} = 1 - \langle \mathbf{x}, \boldsymbol{\xi}_1 \rangle^2,$$

which takes its minimum  $\|\boldsymbol{\xi}_2\|^2$  when  $\mathbf{x} = \boldsymbol{\xi}_1 / \|\boldsymbol{\xi}_1\|$ .

**Lemma 7.5.** Use the same setting and conditions as in Definition 7.2. For any  $L \in \mathcal{S}_{r,n}$ , there is a sequence of primitive sublattices

$$(7.1) \quad 0 = L_0 \subset L_1 \subset L_2 \subset \dots \subset L_r = L$$

satisfying the following conditions. To state the conditions, let

$$\pi_i: \mathbb{R}^{n+1} \rightarrow (L_i)_{\mathbb{R}}^\perp$$

be the orthogonal projection,  $M_i := \pi_i(\mathbb{Z}^{n+1}) = \mathbb{Z}^{n+1}/L_i$  be the quotient lattices,  $d_i := (g_{M_i}(\pi_i(\mathbf{c})), q)$  for  $i = 0, \dots, r$ , and take a  $\mathbb{Z}$ -basis  $\mathbf{w}_i$  of  $L_i/L_{i-1}$  for each of  $i = 1, \dots, r$ .

- (1) We have  $\text{rank } L_i = i$  for  $i = 0, \dots, r$ .
- (2) We have  $\det L_i \asymp s_1 \cdots s_i$  for  $i = 1, \dots, r$ .
- (3) We have  $\lambda_1(L_i/L_{i-1}) \asymp s_i$  for  $i = 1, \dots, r$ .
- (4) We have  $1 = d_0 \mid d_1 \mid \dots \mid d_{r-1} \mid d_r = q$ .
- (5) We have  $d_\infty(L_1, \boldsymbol{\xi}) \geq \dots \geq d_\infty(L_r, \boldsymbol{\xi})$  and  $d_\infty(L_r, \boldsymbol{\xi}) \leq \sigma$ .
- (6) For  $i = 1, \dots, r$ , there is  $u \in (\mathbb{Z}/(d_i/d_{i-1})\mathbb{Z})^\times$  such that

$$\mathbf{w}_i \equiv u \cdot (d_{i-1}^{-1} \pi_{i-1}(\mathbf{c})) \pmod{(d_i/d_{i-1}) \cdot M_{i-1}}.$$

Here, note that  $d_{i-1}^{-1} \pi_{i-1}(\mathbf{c}) \in M_{i-1}$  and the definition of  $d_i$  implies

$$(g_{M_{i-1}}(d_{i-1}^{-1} \pi_{i-1}(\mathbf{c})), d_i/d_{i-1}) = 1,$$

which is vacuously true if  $\pi_{i-1}(\mathbf{c}) = 0$  since then  $d_{i-1} = d_i = q$ .

- (7) For  $i = 1, 2, \dots, r$ , we have

$$(7.2) \quad d_{\infty, (L_{i-1})_{\mathbb{R}}^\perp}(\mathbf{w}_i, \pi_{i-1}(\boldsymbol{\xi})) d_\infty(L_{i-1}, \boldsymbol{\xi}) \leq d_\infty(L_i, \boldsymbol{\xi}).$$

The implicit constants above depend only on  $r$ .

*Proof.* Take a  $\mathbb{Z}$ -basis  $(\mathbf{v}_1, \dots, \mathbf{v}_r)$  of  $\Lambda := L$  given in Lemma 3.2 and let  $L_i := \mathbb{Z}\mathbf{v}_1 + \dots + \mathbb{Z}\mathbf{v}_i$  for  $i = 0, \dots, r$ . Then, (1), (2) and (3) of the current lemma are satisfied. We check the remaining conditions for such defined  $L_i$ 's.

By the definition of  $g$ , we have  $g_{M_{i-1}}(\pi_{i-1}(\mathbf{c})) \mid g_{M_i}(\pi_i(\mathbf{c}))$  so  $d_{i-1} \mid d_i$ . Since  $L_0 = 0$  and  $(g_{\mathbb{Z}^{n+1}}(\mathbf{c}), q) = 1$ , we have  $d_0 = 1$ . Since  $L_r = L \in \mathcal{S}_{r,n}$ , we have  $d_r = q$ . Thus, (4) is satisfied. We have  $d_\infty(L_{i-1}, \boldsymbol{\xi}) \geq d_\infty(L_i, \boldsymbol{\xi})$  since  $L_{i-1} \subset L_i$ . Since  $L_r = L \in \mathcal{S}_{r,n}$ , we have  $d_\infty(L_r, \boldsymbol{\xi}) \leq \sigma$ . Therefore, (5) is also satisfied.

We next check the condition (6). We identify  $M_{i-1}$  with  $\mathbb{Z}^{n+2-i}$  via a  $\mathbb{Z}$ -basis of  $M_{i-1}$  extending  $\mathbf{w}_i$ . Let us write  $\pi_{i-1}(\mathbf{c}) = (k_1, \dots, k_{n+2-i})$ . Then,

$$\gcd(k_1, \dots, k_{n+2-i}, q) = d_{i-1} \quad \text{and} \quad \gcd(k_2, \dots, k_{n+2-i}, q) = d_i.$$

Thus, we obtain

$$\begin{aligned} d_{i-1}^{-1} \pi_{i-1}(\mathbf{c}) &= (k_1/d_{i-1}, \dots, k_{n+2-i}/d_{i-1}) \\ &\equiv (k_1/d_{i-1}, 0, \dots, 0) \pmod{(d_i/d_{i-1}) \cdot \mathbb{Z}^{n+2-i}} \end{aligned}$$

and  $(k_1/d_{i-1}, d_i/d_{i-1}) = 1$  since  $\pi_{i-1}(\mathbf{c}) = 0$  implies  $d_i/d_{i-1} = 1$ . This proves (6).

We finally check (7). We may assume  $i \geq 2$  since otherwise  $d_\infty(L_{i-1}, \boldsymbol{\xi}) = 1$  and  $d_{\infty, (L_{i-1})_{\mathbb{R}}^\perp}(\mathbf{w}_i, \pi_{i-1}(\boldsymbol{\xi})) = d_\infty(L_i, \boldsymbol{\xi})$ . Also, we may assume  $\|\boldsymbol{\xi}\| = 1$ . Let us write

$$\boldsymbol{\xi} = \boldsymbol{\eta} + \boldsymbol{\eta}' \quad \text{with} \quad \boldsymbol{\eta} \in (L_i)_{\mathbb{R}} \quad \text{and} \quad \boldsymbol{\eta}' \in (L_i)_{\mathbb{R}}^\perp.$$

Since  $\|\boldsymbol{\xi}\| = 1$ , we have  $d_\infty(L_i, \boldsymbol{\xi}) = \|\boldsymbol{\eta}'\|$ . Write similarly

$$\boldsymbol{\eta} = \boldsymbol{\eta}_1 + \boldsymbol{\eta}_2 \quad \text{with} \quad \boldsymbol{\eta}_1 \in (L_{i-1})_{\mathbb{R}} \quad \text{and} \quad \boldsymbol{\eta}_2 \in (L_i)_{\mathbb{R}} \cap (L_{i-1})_{\mathbb{R}}^\perp = \mathbb{R}\mathbf{w}_i$$

We may assume  $\boldsymbol{\eta}_2 + \boldsymbol{\eta}' \neq 0$  since otherwise  $\boldsymbol{\xi} \in (L_{i-1})_{\mathbb{R}}$  and  $d_\infty(L_{i-1}, \boldsymbol{\xi}) = 0$ , so (7.2) is trivial. Then, we have  $d_\infty(L_{i-1}, \boldsymbol{\xi}) = \|\boldsymbol{\eta}_2 + \boldsymbol{\eta}'\|$ . We may further assume  $\boldsymbol{\eta}_2 \neq 0$  since otherwise  $d_\infty(L_i, \boldsymbol{\xi}) = d_\infty(L_{i-1}, \boldsymbol{\xi}) = \|\boldsymbol{\eta}'\|$ , so (7.2) is trivial. We have

$$d_\infty(L_i, \boldsymbol{\xi}) = \|\boldsymbol{\eta}'\| = \|(\boldsymbol{\eta}_2 + \boldsymbol{\eta}') - \boldsymbol{\eta}_2\|.$$

Dividing both side by  $d_\infty(L_{i-1}, \boldsymbol{\xi}) > 0$ , we get

$$\frac{d_\infty(L_i, \boldsymbol{\xi})}{d_\infty(L_{i-1}, \boldsymbol{\xi})} = \left\| \frac{\boldsymbol{\eta}_2 + \boldsymbol{\eta}'}{\|\boldsymbol{\eta}_2 + \boldsymbol{\eta}'\|} - \frac{\boldsymbol{\eta}_2}{\|\boldsymbol{\eta}_2 + \boldsymbol{\eta}'\|} \right\| \geq |\sin(\theta)| = d_\infty(\boldsymbol{\eta}_2 + \boldsymbol{\eta}', \boldsymbol{\eta}_2),$$

where  $\theta$  is the angle between  $\boldsymbol{\eta}_2 + \boldsymbol{\eta}'$  and  $\boldsymbol{\eta}_2$ . Since  $\pi_{i-1}(\boldsymbol{\xi}) = \boldsymbol{\eta}_2 + \boldsymbol{\eta}'$  and  $\boldsymbol{\eta}_2 \in \mathbb{R}\mathbf{w}_i$ , we have  $d_\infty(\boldsymbol{\eta}_2 + \boldsymbol{\eta}', \boldsymbol{\eta}_2) = d_{\infty, (L_{i-1})_{\mathbb{R}}^\perp}(\pi_{i-1}(\boldsymbol{\xi}), v)$ , so (7) follows.  $\square$

**Lemma 7.6.** *Let  $L \subset \mathbb{Z}^{n+1}$  be a primitive sublattice of rank  $i \in \{0, \dots, n-1\}$  and  $M := \mathbb{Z}^{n+1}/L$ . Let  $d \in \mathbb{N}$  and  $\mathbf{c} \in M$  be such that  $(g_M(\mathbf{c}), d) = 1$ . Let  $\boldsymbol{\xi} \in L_{\mathbb{R}}^\perp \setminus \{0\}$  and  $\sigma \in (0, 1]$ . Then, we have*

$$\begin{aligned} \mathcal{N} &:= \# \left\{ \mathbf{v} \in M_{\text{prim}} \left| \begin{array}{l} \|\mathbf{v}\| \leq X \\ \mathbf{v} \equiv u\mathbf{c} \pmod{dM} \text{ for some } u \in (\mathbb{Z}/d\mathbb{Z})^\times \\ d_{\infty, L_{\mathbb{R}}^\perp}(\mathbf{v}, \boldsymbol{\xi}) \leq \sigma \end{array} \right. \right\} \\ &\ll \det L \left( \frac{\varphi(d)\sigma^{n-i}}{d^{n+1-i}} X^{n+1-i} + X \right), \end{aligned}$$

where the implicit constant depends only on  $n$ .

*Proof.* By Proposition 5.10 with  $\Gamma = \Lambda := M$ , we have

$$\begin{aligned} \mathcal{N} &\leq \sum_{u \pmod{q}}^* \#((M \setminus \{0\}) \cap (u\mathbf{c} + dM) \cap \mathcal{C}_{L_{\mathbb{R}}^{\perp}}(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_{L_{\mathbb{R}}^{\perp}}(X)) \\ &\ll \sum_{1 \leq \nu \leq n+1-i} \frac{\varphi(d)(\frac{\sigma}{d}X)^{\nu}}{\sigma \lambda_1(M) \cdots \lambda_{\nu}(M)} + \frac{X}{\lambda_1(M)}. \end{aligned}$$

By using  $\lambda_1(M) \leq \cdots \leq \lambda_{n+1-i}(M) \leq 1$  and (3.4), this gives

$$\begin{aligned} \mathcal{N} &\ll \sum_{1 \leq \nu \leq n+1-i} \frac{\varphi(d)(\frac{\sigma}{d}X)^{\nu}}{\sigma \cdot (1/\det L)} + \frac{X}{1/\det L} \\ &\ll \det L \left( \frac{\varphi(d)\sigma^{n-i}}{d^{n+1-i}} X^{n+1-i} + \frac{\varphi(d)}{d} X + X \right). \end{aligned}$$

Since  $\varphi(d)/d \leq 1$ , the claimed estimate follows.  $\square$

**Proposition 7.7.** *Under the same setting and conditions as in Definition 7.2,*

$$\begin{aligned} &S_{r,n}(s_1, \dots, s_r; \mathbf{c}, q; \boldsymbol{\xi}, \sigma) \\ &\ll \left( \prod_{j=1}^r s_j^{r-j} \right) \left( \left( \frac{\sigma}{q} \right)^{n+1-r} \prod_{1 \leq j \leq r} s_j^{n+2-j} + \tau(q)^{r-1} \left( \log \frac{2}{\sigma} \right)^{r-1} \sum_{l=1}^r s_l \prod_{\substack{1 \leq j \leq r \\ j \neq l}} s_j^{n+2-j} \right) \end{aligned}$$

where the implicit constant depends only on  $n$ .

*Proof.* Let  $k = \lfloor \log_2 \sigma^{-1} \rfloor$  so that  $2^{-k-1} < \sigma \leq 2^{-k}$ . Consider the intervals

$$I_{k+1} := [0, 2^{-k}], \quad I_k := (2^{-k}, 2^{-(k-1)}], \quad \dots, \quad I_2 := (2^{-2}, 2^{-1}], \quad I_1 := (2^{-1}, 1].$$

By taking one sequence (7.1) of Lemma 7.5 for each of  $L \in \mathcal{S}_{r,n}$ , we have

$$(7.3) \quad S_{r,n} \leq \sum_{\substack{\mathbf{i} \in \mathcal{I} \\ \mathbf{d} \in \mathcal{D}}} \sum_{\substack{L_1 \subset \mathbb{Z}^{n+1} \text{ prim.} \\ \text{rank } L_1 = 1 \\ \det L_1 \asymp s_1 \\ (g_{M_1}(\pi_1(\mathbf{c}), q) = d_1 \\ d_{\infty}(L_1, \boldsymbol{\xi}) \in I_{i_1}}} \sum_{\substack{L_1 \subset L_2 \subset \mathbb{Z}^{n+1} \text{ prim.} \\ \text{rank } L_2 = 2 \\ \det L_2 \asymp s_1 s_2 \\ \lambda_1(L_2/L_1) \asymp s_2 \\ (g_{M_2}(\pi_2(\mathbf{c}), q) = d_2 \\ d_{\infty}(L_2, \boldsymbol{\xi}) \in I_{i_2}}} \cdots \sum_{\substack{L_{r-1} \subset L_r \subset \mathbb{Z}^{n+1} \text{ prim.} \\ \text{rank } L_r = r \\ \det L_r \asymp s_1 \cdots s_r \\ \lambda_1(L_r/L_{r-1}) \asymp s_r \\ (g_{M_r}(\pi_r(\mathbf{c}), q) = q \\ d_{\infty}(\boldsymbol{\xi}, L_r) \in I_{k+1}}} 1,$$

where  $M_i := \mathbb{Z}^{n+1}/L_i$ ,  $\mathbf{i} := (i_1, \dots, i_{r-1})$ ,  $\mathbf{d} = (d_1, \dots, d_{r-1})$ ,

$$\mathcal{I} := \{(i_1, \dots, i_{r-1}) \in \mathbb{N}^{r-1} \mid 1 \leq i_1 \leq \cdots \leq i_{r-1} \leq k+1\},$$

$$\mathcal{D} := \{(d_1, \dots, d_{r-1}) \in \mathbb{N}^{r-1} \mid d_1 \mid \cdots \mid d_{r-1} \mid q\},$$

and  $\pi_i: \mathbb{R}^{n+1} \rightarrow (L_i)_{\mathbb{R}}^{\perp}$  are the orthogonal projections.

Let us set

$$\begin{aligned} d_0 &= 1, \quad d_r = q, \quad i_0 = 0, \quad i_r = k+1, \\ \sigma_j &= \min(1, 2^{j-1-i_j+1}) \quad \text{for } j = 1, \dots, r. \end{aligned}$$

Note that if  $\pi_j(\boldsymbol{\xi}) = 0$ , then  $d_{\infty}(L_j, \boldsymbol{\xi}) = 0 \in I_{k+1}$  and thus  $\sigma_{j+1} = 1$ . In the following, we use the condition on  $\mathbf{v} \in \mathbb{Z}^{n+1}/L_j$  of the form

$$d_{\infty, (L_{j-1})_{\mathbb{R}}^{\perp}}^{\perp}(\mathbf{v}, \pi_{j-1}(\boldsymbol{\xi})) \leq \sigma_j$$

coming from (7) of Lemma 7.5. When  $\pi_j(\boldsymbol{\xi}) = 0$ , this should be understood as a vacuous condition.

By Lemma 7.5, for  $j = 1, \dots, r$  and for a fixed  $L_{j-1}$ , the sum in (7.3) over  $L_j$  is bounded by the number of  $\mathbf{v} \in (\mathbb{Z}^{n+1}/L_{j-1})_{\text{prim}}$  satisfying  $\|\mathbf{v}\| \ll s_j$ ,

$\mathbf{v} \equiv u(d_{j-1}^{-1}\pi_{j-1}(\mathbf{c})) \pmod{(d_j/d_{j-1})(\mathbb{Z}^{n+1}/L_{j-1})}$  for some  $u \in (\mathbb{Z}/(d_j/d_{j-1})\mathbb{Z})^\times$  and  $d_{\infty, (L_{j-1})_{\mathbb{R}}^\perp}(\mathbf{v}, \pi_{j-1}(\boldsymbol{\xi})) \leq \sigma_j$ . Thus, by Lemma 7.6, this sum is bounded by

$$\begin{aligned} &\ll \det L_{j-1} \left( \frac{\varphi(d_j/d_{j-1})\sigma_r^{n+1-j}}{(d_j/d_{j-1})^{n+2-j}} s_j^{n+2-j} + s_j \right) \\ &\ll s_1 \cdots s_{j-1} \left( \frac{\varphi(d_j/d_{j-1})\sigma_j^{n+1-j}}{(d_j/d_{j-1})^{n+2-j}} s_j^{n+2-j} + s_j \right). \end{aligned}$$

On inserting these bounds into (7.3), we get

$$\begin{aligned} &S_{r,n} \\ &\ll \sum_{\substack{\mathbf{i} \in \mathcal{F} \\ \mathbf{d} \in \mathcal{D}}} \prod_{j=1}^r s_1 \cdots s_{j-1} \left( \frac{\varphi(d_j/d_{j-1})\sigma_j^{n+1-j}}{(d_j/d_{j-1})^{n+2-j}} s_j^{n+2-j} + s_j \right) \\ (7.4) \quad &\ll \sum_{\substack{\mathbf{i} \in \mathcal{F} \\ \mathbf{d} \in \mathcal{D}}} \left( \prod_{j=1}^r s_j^{r-j} \right) \left( \prod_{j=1}^r \frac{\varphi(d_j/d_{j-1})\sigma_j^{n+1-j}}{(d_j/d_{j-1})^{n+2-j}} s_j^{n+2-j} + \sum_{l=1}^r s_l \prod_{\substack{1 \leq j \leq r \\ j \neq l}} s_j^{n+2-j} \right) \end{aligned}$$

Now, since  $\sigma_j \asymp 2^{i_{j-1} - i_j}$  for  $j = 1, \dots, r$ , we find that

$$\prod_{j=1}^r \sigma_j^{n+1-j} = (\sigma_1 \cdots \sigma_r)^{n+1-r} \prod_{j=1}^{r-1} \sigma_j^{r-j} \ll \sigma^{n+1-r} \frac{1}{2^{\sum_{1 \leq j \leq r-1} i_j}},$$

where the implicit constant depends on  $n$ . Thus we get

$$\begin{aligned} &\sum_{\mathbf{i} \in \mathcal{F}} \prod_{j=1}^r \sigma_j^{n+1-j} \ll \sigma^{n+1-r} \sum_{r-1 \leq m \leq (r-1)(k+1)} \frac{1}{2^m} \sum_{\substack{1 \leq i_1 \leq \dots \leq i_{r-1} \leq k+1 \\ i_1 + \dots + i_{r-1} = m}} 1 \\ (7.5) \quad &\leq \sigma^{n+1-r} \sum_{m=0}^{\infty} \frac{m^{r-2}}{2^m} \ll \sigma^{n+1-r}, \end{aligned}$$

where the implicit constant depends only on  $r$ .

Next, by using the inequality  $\varphi(m)\varphi(n) \leq \varphi(mn)$ , we have

$$\begin{aligned} \prod_{j=1}^r \frac{\varphi(d_j/d_{j-1})}{(d_j/d_{j-1})^{n+2-j}} &= \frac{\varphi(d_1)\varphi(d_2/d_1) \cdots \varphi(d_r/d_{r-1})}{d_1 \cdots d_{r-1} d_r^{n+2-r}} \\ &= \frac{\varphi(d_1)\varphi(d_2/d_1) \cdots \varphi(d_{r-1}/d_{r-2})\varphi(q/d_{r-1})}{d_1 \cdots d_{r-1} q^{n+2-r}} \\ &\leq \frac{\varphi(q)}{q^{n+2-r}} \frac{1}{d_1 \cdots d_{r-1}}. \end{aligned}$$

By using the multiplicativity, we have

$$\begin{aligned} \sum_{\mathbf{d} \in \mathcal{D}} \frac{1}{d_1 \cdots d_{r-1}} &= \prod_{p^v \parallel q} \left( \sum_{0 \leq i_1 \leq \cdots \leq i_{r-1} \leq v} \frac{1}{p^{i_1 + \cdots + i_{r-1}}} \right) \\ &= \prod_{p^v \parallel q} \left( \sum_{m=0}^{\infty} \frac{1}{p^m} \sum_{\substack{0 \leq i_1 \leq \cdots \leq i_{r-1} \leq v \\ i_1 + \cdots + i_{r-1} = m}} 1 \right) \\ &\leq \prod_{p^v \parallel q} \left( 1 + \frac{1}{p} + \sum_{m=2}^{\infty} \frac{(m+1)^{r-2}}{p^m} \right) \ll \prod_{p|q} \left( 1 + \frac{1}{p} \right), \end{aligned}$$

where the implicit constant depends only on  $r$ . Therefore, we have

$$(7.6) \quad \sum_{\mathbf{d} \in \mathcal{D}} \prod_{j=1}^r \frac{\varphi(d_j/d_{j-1})}{(d_j/d_{j-1})^{n+2-j}} \ll \frac{\varphi(q)}{q^{n+2-r}} \prod_{p|q} \left( 1 + \frac{1}{p} \right) \leq \frac{1}{q^{n+1-r}}.$$

By (7.4), (7.5), and (7.6) we get

$$\begin{aligned} &S_{r,n} \\ &\ll \prod_{j=1}^r s_j^{r-j} \left( \left( \frac{\sigma}{q} \right)^{n+1-r} \prod_{1 \leq j \leq r} s_j^{n+2-j} + \tau(q)^{r-1} \left( \log \frac{2}{\sigma} \right)^{r-1} \sum_{l=1}^r s_l \prod_{\substack{1 \leq j \leq r \\ j \neq l}} s_j^{n+2-j} \right) \end{aligned}$$

and we are done.  $\square$

**Definition 7.8.** Let  $n, r$  be integers with  $n \geq 2$  and  $1 \leq r \leq n$ ,  $\sigma \in (0, 1]$ ,  $\boldsymbol{\xi} \in \mathbb{R}^{n+1} \setminus \{0\}$ ,  $q \in \mathbb{N}$  and  $\mathbf{c} \in \mathbb{Z}^{n+1}$  with  $(g_{\mathbb{Z}^{n+1}}(\mathbf{c}), q) = 1$ . For real numbers  $X, \Delta \geq 0$ , we let

$$\begin{aligned} &\mathcal{L}_{r,n}(X, \Delta; \mathbf{c}, q; \boldsymbol{\xi}, \sigma) \\ &:= \left\{ \mathbf{x} \in \left( \bigcup_{u \pmod{q}}^* (u\mathbf{c} + q\mathbb{Z}^{n+1}) \cap \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_{n+1}(X) \right) \mid \mathfrak{d}_r(\mathbf{x}) \leq \Delta \right\}, \end{aligned}$$

and

$$L_{r,n}(X, \Delta; \mathbf{c}, q; \boldsymbol{\xi}, \sigma) := \#\mathcal{L}_{r,n}(X, \Delta; \mathbf{c}, q; \boldsymbol{\xi}, \sigma).$$

**Lemma 7.9.** Let  $n \geq 3$  be an integer,  $\sigma \in (0, 1]$ ,  $\boldsymbol{\xi} \in \mathbb{R}^{n+1} \setminus \{0\}$ ,  $q \in \mathbb{N}$  and  $\mathbf{c} \in \mathbb{Z}^{n+1}$  with  $(g_{\mathbb{Z}^{n+1}}(\mathbf{c}), q) = 1$ . For real numbers  $X, \Delta \geq 1$ , we then have

$$L_{2,n}(X, \Delta; \mathbf{c}, q; \boldsymbol{\xi}, \sigma) \ll \tau(q) (\log \frac{2}{\sigma}) (\log 2\Delta) \left( \left( \frac{\sigma}{q} \right)^n + \frac{\sigma}{q} \frac{1}{\Delta} + \frac{1}{X} \right) \Delta^n X^2,$$

where the implicit constant depends only on  $n$ .

*Proof.* We may assume  $\Delta \leq X$  since otherwise we have  $X^{-1} \cdot \Delta^n X^2 \gg X^{n+1}$  and so the assertion follows by the trivial bound  $L_{2,n}(X, \Delta; \mathbf{c}, q; \boldsymbol{\xi}, \sigma) \ll \#(\mathbb{Z}^{n+1} \cap \mathcal{B}_{n+1}(X)) \ll X^{n+1}$ . We have

$$\begin{aligned} &L_{2,n}(X, \Delta; \mathbf{c}, q; \boldsymbol{\xi}, \sigma) \\ &\ll \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in u\mathbf{c} + q\mathbb{Z}^{n+1} \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma) \\ \|\mathbf{x}\| \leq X}} \sum_{\substack{1 \ll s_1 \ll s_2 \\ s_1 s_2 \ll \Delta \\ s_1 \ll X}} \sum_{\substack{\Lambda \in \mathcal{F}_{2,n}(s_1, s_2; \mathbf{c}, q; \boldsymbol{\xi}, \sigma) \\ \mathbf{x} \in \Lambda}} 1 \end{aligned}$$

$$\ll \sum_{\substack{1 \ll s_1 \ll s_2 \\ s_1 s_2 \ll \Delta \\ s_1 \ll X}}^{(d)} \sum_{\Lambda \in \mathcal{F}_{2,n}(s_1, s_2; \mathbf{c}, q; \boldsymbol{\xi}, \sigma)} \times \sum_{u \pmod{q}}^* \# \left( \Lambda \cap (u\mathbf{c} + q\mathbb{Z}^{n+1}) \cap \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_{n+1}(X) \right),$$

where the sum over  $s_1, s_2$  is taken dyadically. By Proposition 5.10,

$$\begin{aligned} & \sum_{u \pmod{q}}^* \# \left( \Lambda \cap (u\mathbf{c} + q\mathbb{Z}^{n+1}) \cap \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_{n+1}(X) \right) \\ & \ll \sum_{u \pmod{q}}^* \# \left( (\Lambda \setminus \{0\}) \cap (u\mathbf{c} + q\mathbb{Z}^{n+1}) \cap \mathcal{C}_{n+1}(\boldsymbol{\xi}, \sigma) \cap \mathcal{B}_{n+1}(X) \right) + 1 \\ & \ll \frac{\sigma}{q} \frac{X^2}{s_1 s_2} + \frac{X}{s_1} + 1 \ll \frac{\sigma}{q} \frac{X^2}{s_1 s_2} + \frac{X}{s_1}, \end{aligned}$$

where we use the fact that  $u\mathbf{c} + q\mathbb{Z}^{n+1}$  are disjoint and  $X \gg s_1$ . Thus, combined with Proposition 7.7, we have

$$\begin{aligned} & L_{2,n}(X, \Delta; \mathbf{c}, q; \boldsymbol{\xi}, \sigma) \\ & \ll \tau(q) \left( \log \frac{2}{\sigma} \right) \sum_{\substack{1 \ll s_1 \ll s_2 \\ s_1 s_2 \ll \Delta \\ s_1 \ll X}}^{(d)} \left( \left( \frac{\sigma}{q} \right)^{n-1} s_1^{n+2} s_2^n + s_1^2 s_2^n + s_1^{n+2} s_2 \right) \left( \frac{\sigma}{q} \frac{X^2}{s_1 s_2} + \frac{X}{s_1} \right). \end{aligned}$$

Note that the above summand is non-decreasing in  $s_2$  and

$$s_1 \ll s_2 \text{ and } s_1 s_2 \ll \Delta \implies s_1 \ll \Delta^{\frac{1}{2}} \text{ and } s_2 \ll \frac{\Delta}{s_1}.$$

Therefore, we have

$$\begin{aligned} & L_{2,n}(X, \Delta; \mathbf{c}, q; \boldsymbol{\xi}, \sigma) \\ & \ll \tau(q) \left( \log \frac{2}{\sigma} \right) (\log 2\Delta) \\ & \quad \times \sum_{1 \ll s_1 \ll \Delta^{\frac{1}{2}}}^{(d)} \left( \left( \frac{\sigma}{q} \right)^{n-1} \Delta^n s_1^2 + s_1^{-(n-2)} \Delta^n + s_1^{n+1} \Delta \right) \left( \frac{\sigma}{q} \frac{X^2}{\Delta} + \frac{X}{s_1} \right) \\ & \ll \tau(q) \left( \log \frac{2}{\sigma} \right) (\log 2\Delta) \\ & \quad \times \left\{ \sum_{1 \ll s_1 \ll \Delta^{\frac{1}{2}}}^{(d)} \left( \left( \frac{\sigma}{q} \right)^{n-1} \Delta^n s_1^2 + s_1^{n+1} \Delta \right) \left( \frac{\sigma}{q} \frac{X^2}{\Delta} + \frac{X}{s_1} \right) \right. \\ & \quad \left. + \sum_{1 \ll s_1 \ll \Delta^{\frac{1}{2}}}^{(d)} s_1^{-(n-2)} \Delta^n \left( \frac{\sigma}{q} \frac{X^2}{\Delta} + \frac{X}{s_1} \right) \right\} \\ & \ll \tau(q) \left( \log \frac{2}{\sigma} \right) (\log 2\Delta) \\ & \quad \times \left\{ \left( \left( \frac{\sigma}{q} \right)^{n-1} \Delta^{n+1} + \Delta^{\frac{n+3}{2}} \right) \left( \frac{\sigma}{q} \frac{X^2}{\Delta} + \frac{X}{\Delta^{\frac{1}{2}}} \right) + \Delta^n \left( \frac{\sigma}{q} \frac{X^2}{\Delta} + X \right) \right\}. \end{aligned}$$

By  $n \geq 3$  and  $\Delta \geq 1$ , we have

$$\Delta^{\frac{n+3}{2}} \left( \frac{\sigma X^2}{q \Delta} + \frac{X}{\Delta^{\frac{1}{2}}} \right) \ll \Delta^n \left( \frac{\sigma X^2}{q \Delta} + X \right).$$

Therefore, we have

$$\begin{aligned} & L_{2,n}(X, \Delta; \mathbf{c}, q; \boldsymbol{\xi}, \sigma) \\ & \ll \tau(q)(\log \frac{2}{\sigma})(\log 2\Delta) \left( \left( \frac{\sigma}{q} \right)^n + \left( \frac{\sigma}{q} \right)^{n-1} \frac{\Delta^{\frac{1}{2}}}{X} + \frac{\sigma}{q} \frac{1}{\Delta} + \frac{1}{X} \right) \Delta^n X^2. \end{aligned}$$

Furthermore, since  $n \geq 3$  and we are assuming  $\Delta \leq X$ , we have

$$\left( \frac{\sigma}{q} \right)^n + \frac{1}{X} \geq \left( \left( \frac{\sigma}{q} \right)^n \right)^{\frac{n-1}{n}} \left( \frac{1}{X} \right)^{\frac{1}{n}} = \left( \frac{\sigma}{q} \right)^{n-1} \frac{1}{X^{\frac{1}{n}}} \geq \left( \frac{\sigma}{q} \right)^{n-1} \frac{\Delta^{\frac{1}{2}}}{X}$$

and so we obtain the result.  $\square$

## 8. PROOF OF THEOREM 1.4

In this section, we prove our first main theorem Theorem 1.4.

**Lemma 8.1.** *Under the setting of Theorem 1.4, there exist  $\mathbf{c} \in \mathbb{Z}^{n+1} \setminus \{0\}$  and  $\boldsymbol{\xi}_\infty \in \mathbb{R}^{n+1} \setminus \{0\}$  such that  $\gcd(\mathbf{c}, q) = 1$  and for  $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1}$ , we have*

$$\begin{aligned} & d_p([\mathbf{x}], \boldsymbol{\xi}_p) \leq \sigma_p \text{ for all } p \in M_{\mathbb{Q}} \\ & \iff \mathbf{x} \equiv u\mathbf{c} \pmod{q} \text{ for some } u \in (\mathbb{Z}/q\mathbb{Z})^\times \quad \text{and} \quad \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}_\infty, \sigma_\infty). \end{aligned}$$

*Proof.* For each  $p < \infty$  with  $\sigma_p = p^{-e_p} < 1$ , take  $\boldsymbol{\xi}_p \in \mathbb{Z}_p^{n+1} \setminus \{0\}$  with  $\xi_p = [\boldsymbol{\xi}_p]$  and  $\|\boldsymbol{\xi}_p\|_p = 1$ . Then, by the Chinese remainder theorem with moduli  $p^{e_p}$ , we get the desired  $\mathbf{c}$ . For  $\boldsymbol{\xi}_\infty$ , it suffices to take  $\boldsymbol{\xi}_\infty \in \mathbb{R}^{n+1} \setminus \{0\}$  with  $\xi_\infty = [\boldsymbol{\xi}_\infty]$ .  $\square$

**Lemma 8.2.** *Consider an integral lattice  $\Lambda \subset \mathbb{R}^N$  of rank  $r \geq 2$ . For  $A, M \geq 1$  and  $s \in \mathbb{Z}_{\geq 0}$  such that  $A, \lambda_r(\Lambda) \leq M$ , we have*

$$\sum_{1 \leq \nu < r} \frac{A^\nu}{\lambda_1(\Lambda) \cdots \lambda_\nu(\Lambda)} \ll A^r \left( \frac{1}{\det(\Lambda)} \left( \frac{M}{A} \right)^s + \frac{1}{A} \left( \frac{1}{A} \right)^s \right),$$

where the implicit constant depends only on  $r$ .

*Proof.* If  $s \geq r$ , then

$$\begin{aligned} \sum_{1 \leq \nu < r} \frac{A^\nu}{\lambda_1(\Lambda) \cdots \lambda_\nu(\Lambda)} & \leq \sum_{1 \leq \nu < r} \frac{M^\nu}{\lambda_1(\Lambda) \cdots \lambda_\nu(\Lambda)} \\ & \ll \frac{M^r}{\lambda_1(\Lambda) \cdots \lambda_r(\Lambda)} \asymp A^r \frac{1}{\det(\Lambda)} \left( \frac{M}{A} \right)^r \ll A^r \frac{1}{\det(\Lambda)} \left( \frac{M}{A} \right)^s \end{aligned}$$

by Minkowski's second theorem (Lemma 3.1). Thus, we may assume  $0 \leq s < r - 1$ .

We have

$$\sum_{1 \leq \nu < r} \frac{A^\nu}{\lambda_1(\Lambda) \cdots \lambda_\nu(\Lambda)} = \sum_{1 \leq i \leq s} \frac{A^{r-i}}{\lambda_1(\Lambda) \cdots \lambda_{r-i}(\Lambda)} + \sum_{s+1 \leq i < r} \frac{A^{r-i}}{\lambda_1(\Lambda) \cdots \lambda_{r-i}(\Lambda)}.$$

For the former sum, by Minkowski's second theorem (Lemma 3.1) and  $\lambda_r(\Lambda) \leq M$ ,

$$\begin{aligned} \sum_{1 \leq i \leq s} \frac{A^{r-i}}{\lambda_1(\Lambda) \cdots \lambda_{r-i}(\Lambda)} &= \frac{A^r}{\lambda_1(\Lambda) \cdots \lambda_r(\Lambda)} \sum_{i=1}^s \frac{\lambda_{r-i+1}(\Lambda) \cdots \lambda_r(\Lambda)}{A^i} \\ &\ll \frac{A^r}{\det(\Lambda)} \left( \frac{M}{A} \right)^s. \end{aligned}$$

For the latter sum, since  $\Lambda$  is integral, we have  $\lambda_1(\Lambda) \geq 1$  and so

$$\sum_{s+1 \leq i < r} \frac{A^{r-i}}{\lambda_1(\Lambda) \cdots \lambda_{r-i}(\Lambda)} \ll \sum_{s+1 \leq i < r} A^{r-i} \ll A^{r-(s+1)}.$$

By combining the above estimates, we obtain the lemma.  $\square$

**Theorem 8.3.** *Under the setting of Theorem 1.4, we have*

$$S := \sum_{V \in \mathbb{V}_{d,n}(A)} N_V(B; \xi, \sigma) = \tilde{C}_{d,n}(\xi, \sigma) A^{N_{d,n}-1} B(1 + O(R_{d,n} \cdot B^\varepsilon))$$

for  $A, B \geq 2$  provided

$$A \geq \max(\mathfrak{q}, B^{\frac{d+1}{(2n+1)(n+1-d)}}, \mathfrak{q}^{\frac{n-1}{2n-1}} B^{\frac{d}{(2n-1)(n+1-d)}}), \quad B \geq \mathfrak{q}^n \quad \text{with} \quad \mathfrak{q} := \frac{q}{\sigma_\infty},$$

where

$$\begin{aligned} \tilde{C}_{d,n}(\xi, \sigma) &:= \frac{V_{N_{d,n}-1}}{4\zeta(N_{d,n}-1)} \frac{\mathfrak{W}_{d,n}(\xi_\infty; \sigma_\infty) \varphi(q)}{J_{n+1}(q) \zeta(n+1)}, \\ R_{d,n} &:= \mathfrak{q} A^{-1} + \mathfrak{q}^n B^{-1} + \mathfrak{q} B^{-\frac{1}{n+1-d}} + A^{-1} B^{\frac{1}{n(n+1-d)}} + \tilde{R}_{d,n} \end{aligned}$$

with

$$\tilde{R}_{d,n} := \begin{cases} 0 & \text{if } A \geq B^{\frac{1}{n+1-d}}, \\ A^{-(2n+1)} B^{\frac{d+1}{n+1-d}} + \mathfrak{q}^{n-1} A^{-(2n-1)} B^{\frac{d}{n+1-d}} & \text{if } A \leq B^{\frac{1}{n+1-d}} \end{cases}$$

and the implicit constant depends only on  $d, n, \varepsilon$ .

*Proof.* Note that the assumption on the size of  $A$  implies

$$A \geq B^{\frac{d+1}{(2n+1)(n+1-d)}} \geq B^{\frac{d+1}{3n(n+1-d)}} \geq B^{\frac{1}{n(n+1-d)}}.$$

Thus, we have  $A \geq \max(\mathfrak{q}, B^{\frac{1}{n(n+1-d)}})$ , which we use later. By Lemma 8.1, we have

$$\begin{aligned} (8.1) \quad S &= \frac{1}{4} \sum_{\substack{\mathbf{a} \in \mathbb{Z}_{\text{prim}}^{\mathcal{M}_{d,n}} \\ \|\mathbf{a}\| \leq A}} \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1} \\ H(\mathbf{x}) \leq B \\ \mathbf{x} \in u\mathbf{c} + q\mathbb{Z}^{n+1} \\ \mathbf{x} \in \mathcal{C}_{n+1}(\xi_\infty, \sigma_\infty) \\ \mathbf{a} \in \Lambda_{\nu_{d,n}}(\mathbf{x})}} 1 \\ &= \frac{1}{4} \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1} \\ H(\mathbf{x}) \leq B \\ \mathbf{x} \in u\mathbf{c} + q\mathbb{Z}^{n+1} \\ \mathbf{x} \in \mathcal{C}_{n+1}(\xi_\infty, \sigma_\infty)}} \#(\Lambda_{\nu_{d,n}}(\mathbf{x}) \cap \mathbb{Z}_{\text{prim}}^{\mathcal{M}_{d,n}} \cap \mathcal{B}_{N_{d,n}}(A)). \end{aligned}$$

By Proposition 5.9 with  $q = 1$  and  $\sigma = 1$ , we have

$$(8.2) \quad S = T + O(E_1 + E_2),$$

where

$$\begin{aligned}
(8.3) \quad T &:= \frac{V_{N_{d,n}-1}}{4\zeta(N_{d,n}-1)} A^{N_{d,n}-1} \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1} \\ H(\mathbf{x}) \leq B \\ \mathbf{x} \in u\mathbf{c} + q\mathbb{Z}^{n+1} \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}_{\infty}, \sigma_{\infty})}} \frac{1}{\det(\Lambda_{\nu_{d,n}}(\mathbf{x}))}, \\
E_1 &:= \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1} \\ H(\mathbf{x}) \leq B \\ \mathbf{x} \in u\mathbf{c} + q\mathbb{Z}^{n+1} \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}_{\infty}, \sigma_{\infty})}} \sum_{1 \leq \nu < N_{d,n}-1} \frac{A^{\nu}}{\lambda_1(\Lambda_{\nu_{d,n}}(\mathbf{x})) \cdots \lambda_{\nu}(\Lambda_{\nu_{d,n}}(\mathbf{x}))}, \\
E_2 &:= A \log A \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1} \\ H(\mathbf{x}) \leq B \\ \mathbf{x} \in u\mathbf{c} + q\mathbb{Z}^{n+1} \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}_{\infty}, \sigma_{\infty})}} 1.
\end{aligned}$$

For the main term  $T$ , by (3.3) and Lemma 6.3, we have

$$T = \tilde{C}_{d,n}(\xi, \sigma) A^{N_{d,n}-1} B + O(\mathfrak{q}^{-(n-1)} A^{N_{d,n}-1} (B^{\frac{n-d}{n+1-d}} + \log B) + A^{N_{d,n}-1}).$$

By (6.2) and the bound  $\varphi(q) \gg q^{1-\varepsilon}$ , we have  $\tilde{C}_{d,n}(\xi, \sigma) \gg \mathfrak{q}^{-n-\varepsilon}$  and so

$$T = \tilde{C}_{d,n}(\xi, \sigma) A^{N_{d,n}-1} B \left( 1 + O((\mathfrak{q} B^{-\frac{1}{n+1-d}} + \mathfrak{q}^n B^{-1}) B^{\varepsilon}) \right),$$

where we use  $B \geq \mathfrak{q}^n$ .

For the error term  $E_1$ , let us introduce

$$\mu(\mathbf{x}) := n \frac{\|\mathbf{x}\|}{\mathfrak{d}_2(\mathbf{x})}$$

and we further dissect  $E_1$  as

$$E_1 = E_{11} + E_{12},$$

where

$$\begin{aligned}
E_{11} &:= \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1} \\ H(\mathbf{x}) \leq B \\ \mathbf{x} \in u\mathbf{c} + q\mathbb{Z}^{n+1} \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}_{\infty}, \sigma_{\infty}) \\ \lambda_{N_{d,n}-1}(\Lambda_{\nu_{d,n}}(\mathbf{x})) \leq A}} \sum_{1 \leq \nu < N_{d,n}-1} \frac{A^{\nu}}{\lambda_1(\Lambda_{\nu_{d,n}}(\mathbf{x})) \cdots \lambda_{\nu}(\Lambda_{\nu_{d,n}}(\mathbf{x}))}, \\
E_{12} &:= \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1} \\ H(\mathbf{x}) \leq B \\ \mathbf{x} \in u\mathbf{c} + q\mathbb{Z}^{n+1} \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}_{\infty}, \sigma_{\infty}) \\ \lambda_{N_{d,n}-1}(\Lambda_{\nu_{d,n}}(\mathbf{x})) > A}} \sum_{1 \leq \nu < N_{d,n}-1} \frac{A^{\nu}}{\lambda_1(\Lambda_{\nu_{d,n}}(\mathbf{x})) \cdots \lambda_{\nu}(\Lambda_{\nu_{d,n}}(\mathbf{x}))}.
\end{aligned}$$

For the sum  $E_{11}$ , by using (3.3), Lemma 3.1 and Lemma 3.6,

$$E_{11} \ll A^{N_{d,n}-2} \sum_{u \pmod{q}}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1} \\ H(\mathbf{x}) \leq B \\ \mathbf{x} \in u\mathbf{c} + q\mathbb{Z}^{n+1} \\ \mathbf{x} \in \mathcal{C}_{n+1}(\boldsymbol{\xi}_\infty, \sigma_\infty)}} \frac{\mu(\mathbf{x})}{\|\mathbf{x}\|^d}.$$

We then dissect the sum dyadically with writing

$$(8.4) \quad \|\mathbf{x}\| \asymp X \quad \text{and} \quad \mu(\mathbf{x}) \asymp M$$

as

$$E_{11} \ll A^{N_{d,n}-2} \sum_{\substack{1 \ll X^{n+1-d} \ll B \\ 1 \ll M \ll X}}^{(d)} \frac{M}{X^d} L_{2,n} \left( X, \frac{CX}{M}; \mathbf{c}, q; \boldsymbol{\xi}_\infty, \sigma_\infty \right),$$

with some  $1 \leq C \ll 1$ . By Proposition 5.10, Lemma 7.9, we have

$$L_{2,n} \left( X, \frac{CX}{M}; \mathbf{c}, q; \boldsymbol{\xi}_\infty, \sigma_\infty \right) \ll (F_1 + F_2 + F_3) B^\varepsilon + X$$

(note that we can ignore the effect of the zero vector in Proposition 5.10 since  $X \gg 1$ ), where

$$\begin{aligned} F_1 &= F_1(X, M) := \mathfrak{q}^{-n} X^{n+1} \min(M^{-n} X, 1), \\ F_2 &= F_2(X, M) := \mathfrak{q}^{-n} X^{n+1} \min(\mathfrak{q}^{n-1} M^{-(n-1)}, 1), \\ F_3 &= F_3(X, M) := \mathfrak{q}^{-n} X^{n+1} \min(\mathfrak{q}^n M^{-n}, 1). \end{aligned}$$

For  $1 \ll M \ll X$ , we then have

$$M \cdot F_1(X, M) \ll \mathfrak{q}^{-n} X^{n+1+\frac{1}{n}}, \quad M \cdot F_i(X, M) \ll \mathfrak{q}^{-n+1} X^{n+1} \quad \text{for } i = 2, 3$$

and  $M \cdot X \ll X^2$ . We thus have

$$\begin{aligned} E_{11} &\ll \mathfrak{q}^{-n} A^{N_{d,n}-2} \sum_{1 \ll X^{n+1-d} \ll B}^{(d)} (X^{n+1-d} (X^{\frac{1}{n}} + \mathfrak{q}) + \mathfrak{q}^n X^{-(d-2)}) B^\varepsilon \\ &\ll \mathfrak{q}^{-n} A^{N_{d,n}-1} B (A^{-1} B^{\frac{1}{n(n+1-d)}} + \mathfrak{q} A^{-1} + \mathfrak{q}^n (AB)^{-1}) B^\varepsilon. \end{aligned}$$

This completes the estimate for  $E_{11}$ .

We next bound  $E_{12}$ . For  $\mathbf{x}$  counted in  $E_{12}$ , by Lemma 3.6, we have

$$A < \lambda_{N_{d,n}-1}(\Lambda_{\nu_{d,n}(\mathbf{x})}) \leq \|\mathbf{x}\| \leq B^{\frac{1}{n+1-d}}.$$

Thus, we may assume  $A \leq B^{\frac{1}{n+1-d}}$  since otherwise the sum  $E_{12}$  is empty. Also, for  $\mathbf{x}$  counted in  $E_{12}$ , by Lemma 3.6, we have  $\mu(\mathbf{x}) > A$ . We dissect dyadically by using variables (8.4) and use Lemma 8.2 to get

$$E_{12} \ll A^{N_{d,n}-1} \sum_{\substack{1 \ll X^{n+1-d} \ll B \\ A \leq M \ll X}}^{(d)} G \cdot L_{2,n} \left( X, \frac{CX}{M}; \mathbf{c}, q; \boldsymbol{\xi}_\infty, \sigma_\infty \right)$$

with some  $1 \leq C \ll 1$ , where

$$G := \min_{s \in \mathbb{Z}_{\geq 0}} G_s \quad \text{with} \quad G_s := \left( \frac{1}{X^d} \left( \frac{M}{A} \right)^s + \left( \frac{1}{A} \right)^{s+1} \right).$$

By using Lemma 7.9, we have

$$E_{12} \ll A^{N_{d,n-1}} \sum_{\substack{1 \ll X^{n+1-d} \ll B \\ A \leq M \ll X}}^{(d)} G(\tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3 + \tilde{F}_4) B^\varepsilon,$$

where

$$\begin{aligned} \tilde{F}_1 &:= \mathfrak{q}^{-n} X^{n+1} \cdot M^{-n} X, \\ \tilde{F}_2 &:= \mathfrak{q}^{-n} X^{n+1} \cdot \mathfrak{q}^{n-1} M^{-(n-1)}, \\ \tilde{F}_3 &:= \mathfrak{q}^{-n} X^{n+1} \cdot \mathfrak{q}^n M^{-n}. \end{aligned}$$

Note that since  $M \geq A \geq \max(\mathfrak{q}, B^{\frac{1}{n(n+1-d)}}) \geq \max(\mathfrak{q}, X^{\frac{1}{n}})$  in the sum  $E_2$ , there is no benefit to use the trivial bound given by Proposition 5.10 as for  $E_{11}$ . In the sum  $E_{12}$ , by the assumption  $A \geq \mathfrak{q}$ , we have  $\mathfrak{q} \leq A \leq M \ll X$  and so

$$\tilde{F}_3 = \mathfrak{q} M^{-1} \tilde{F}_2 \ll \tilde{F}_2.$$

We thus simply have

$$E_{12} \ll A^{N_{d,n-1}} \mathfrak{q}^\varepsilon \sum_{\substack{1 \ll X^{n+1-d} \ll B \\ A \leq M \ll X}}^{(d)} G(\tilde{F}_1 + \tilde{F}_2) B^\varepsilon,$$

We use the bound  $G \leq G_n$  for  $G\tilde{F}_1$  and  $G \leq G_{n-1}$  for  $G\tilde{F}_2$ . In the expressions  $G_n\tilde{F}_1$  and  $G_{n-1}\tilde{F}_2$  the exponents of  $X$  is  $\geq n+1-d$  and the exponents of  $M$  is  $\leq 0$ . Thus, we have

$$\begin{aligned} E_{12} &\ll A^{N_{d,n-1}} B^\varepsilon (G_n \tilde{F}_1 + G_{n-1} \tilde{F}_2) \Big|_{X=B^{\frac{1}{n+1-d}}, M=A}, \\ &= \mathfrak{q}^{-n} A^{N_{d,n-1}} B \\ &\quad \times \left( (1 + A^{-(n+1)} B^{\frac{d}{n+1-d}}) A^{-n} B^{\frac{1}{n+1-d}} \right. \\ &\quad \left. + (1 + A^{-n} B^{\frac{d}{n+1-d}}) \mathfrak{q}^{n-1} A^{-(n-1)} \right) B^\varepsilon. \end{aligned}$$

Since  $A \geq \max(\mathfrak{q}, B^{\frac{1}{n(n+1-d)}})$ , we have

$$A^{-n} B^{\frac{1}{n+1-d}} \leq A^{-1} B^{\frac{1}{n(n+1-d)}} \quad \text{and} \quad \mathfrak{q}^{n-1} A^{-(n-1)} \leq \mathfrak{q} A^{-1}$$

and so the contribution of terms

$$A^{-n} B^{\frac{1}{n+1-d}}, \quad \mathfrak{q}^{n-1} A^{-(n-1)}$$

are already covered by  $R_{d,n}$ . The remaining terms contribute

$$\mathfrak{q}^{-n} A^{N_{d,n-1}} B (A^{-(2n+1)} B^{\frac{d+1}{n+1-d}} + \mathfrak{q}^{n-1} A^{-(2n-1)} B^{\frac{d}{n+1-d}}) B^\varepsilon$$

which is  $\tilde{R}_{d,n}$  when  $A \leq B^{\frac{1}{n+1-d}}$ .

Finally, for  $E_2$ , we just use Proposition 5.10 with  $B \geq \mathfrak{q}^n$  to obtain

$$\begin{aligned} (8.5) \quad E_2 &\ll A \log A (\mathfrak{q}^{-n} B^{1+\frac{d}{n+1-d}} + B^{\frac{1}{n+1-d}}) \\ &\ll \mathfrak{q}^{-n} A^{N_{d,n-1}} B \cdot A^{-(N_{d,n-3})} B^{\frac{d}{n+1-d}}. \end{aligned}$$

Note that we have  $N_{d,n} \geq nd + 3$ . Indeed,

$$N_{d,n} = \left(\frac{n}{d} + 1\right) \cdots \left(\frac{n}{2} + 1\right) \left(\frac{n}{1} + 1\right) \geq 2^{d-2} \cdot \frac{5}{2}(n+1) \geq \frac{5}{4}d(n+1) \geq dn + 4.$$

Therefore, by using the assumption  $A \geq B^{\frac{1}{n(n+1-d)}}$ , we have

$$A^{-(N_{d,n}-3)} B^{\frac{d}{n+1-d}} \leq A^{-nd} B^{\frac{d}{n+1-d}} \leq A^{-1} B^{\frac{1}{n(n+1-d)}}$$

and so the contribution of  $E_2$  is covered by  $R_{d,n}$ . By combining the above estimates, we obtain the theorem.  $\square$

**Remark 8.4.** Under the assumptions  $n \geq d \geq 2$  and  $B \geq \mathfrak{q}^n$ , we have

$$B^{\frac{d+1}{(2n+1)(n+1-d)}} \leq B^{\frac{1}{n+1-d}},$$

$$\mathfrak{q}^{\frac{n-1}{2n-1}} B^{\frac{d}{(2n-1)(n+1-d)}} \leq B^{\frac{n-1}{n(2n-1)} + \frac{d}{(2n-1)(n+1-d)}} \leq B^{\frac{n-1+d}{(2n-1)(n+1-d)}} \leq B^{\frac{1}{n+1-d}}.$$

Therefore, in Theorem 8.3, we can replace the assumptions on the size of  $A, B, \mathfrak{q}$  to  $A \geq B^{\frac{1}{n+1-d}}$  and  $B \geq \mathfrak{q}^n$  and then remove the error term  $\tilde{R}_{n,d}$ .

**Remark 8.5.** By choosing the parameter  $s$  in Lemma 8.2 more efficiently, we can improve the error term in Theorem 8.3 to get a wider admissible range for  $A, B, \mathfrak{q}$ . However, we took  $s = n$  and  $s = n - 1$  for simplicity.

We now simplify Theorem 8.3 to obtain Theorem 1.4:

*Proof of Theorem 1.4.* It suffices to bound the error terms of Theorem 8.3 by those of Theorem 1.4 under the assumptions on the size of  $A, B, \mathfrak{q}$  in Theorem 8.3. By  $B \geq \mathfrak{q}^n$ , we have

$$\mathfrak{q}^n B^{-1} = (\mathfrak{q} B^{-\frac{1}{n}})^n \leq \mathfrak{q} B^{-\frac{1}{n}}.$$

Also, since  $d \geq 2$ , we have

$$\mathfrak{q} B^{-\frac{1}{n+1-d}} \leq \mathfrak{q} B^{-\frac{1}{n}} \quad \text{and} \quad A^{-1} B^{\frac{1}{n(n+1-d)}} = A^{-1} B^{\frac{3}{3n(n+1-d)}} \leq A^{-1} B^{\frac{d+1}{(2n+1)(n+1-d)}}.$$

Finally, since  $A \geq B^{\frac{d+1}{(2n+1)(n+1-d)}}$ , we have

$$A^{-(2n+1)} B^{\frac{d+1}{n+1-d}} = (A^{-1} B^{\frac{d+1}{(2n+1)(n+1-d)}})^{2n+1} \leq A^{-1} B^{\frac{d+1}{(2n+1)(n+1-d)}}.$$

Thus, any error term of Theorem 8.3 is covered by those of Theorem 1.4.  $\square$

## 9. LOCAL SOLUBILITY AND REDUCIBLE LOCUS

So far we focus on the asymptotic formula of the sum of counting function of rational points on Fano hypersurfaces. Now we turn our attention to the set of hypersurfaces (not necessarily Fano) that have local solutions inside adelic neighbourhoods of given points.

In this section we recall a result on the existence of  $\mathbb{Q}_p$ -points of hypersurfaces, which essentially follows from the Lang-Weil estimate. The contents of this section are well-known to experts (cf. proof of [16, Theorem 3.6]), but we need the scheme “NIP $_{d,n}$ ” defined below, which appears only implicitly in loc. cit.

Let  $n \in \mathbb{Z}_{\geq 0}$  and  $d \in \mathbb{Z}_{\geq 1}$ . Recall we identify (the ring valued points of)  $\mathbb{A}_{\mathbb{Z}}^{N_{d,n}}$  with the set of homogeneous polynomials of degree  $d$  in  $n+1$  variables. Then we have the following morphism of schemes corresponding to the product of polynomials:

$$p_{d',d''} : \mathbb{A}_{\mathbb{Z}}^{N_{d',n}} \times_{\mathbb{Z}} \mathbb{A}_{\mathbb{Z}}^{N_{d'',n}} \longrightarrow \mathbb{A}_{\mathbb{Z}}^{N_{d,n}}$$

where  $d' + d'' = d$ .

**Definition 9.1.** We set

$$\text{NIP}_{d,n} := \bigcup_{\substack{d'+d''=d \\ d',d'' \geq 1}} (\text{scheme theoretic image of } p_{d',d''}) \subset \mathbb{A}_{\mathbb{Z}}^{N_{d,n}}.$$

We equip  $\text{NIP}_{d,n}$  with the reduced scheme structure. Here NIP stands for Non-Irreducible Polynomials.

**Lemma 9.2.** *Let  $n, d \in \mathbb{Z}_{\geq 2}$  and suppose  $(n, d) \neq (2, 2), (2, 3)$ . Then  $\text{NIP}_{d,n}$  is flat over  $\mathbb{Z}$  and*

$$\begin{aligned} \dim \text{NIP}_{d,n} &\leq N_{d,n} - 1 \\ \dim(\text{NIP}_{d,n})_{\mathbb{Q}} &\leq N_{d,n} - 2 \end{aligned}$$

where  $(\text{NIP}_{d,n})_{\mathbb{Q}}$  is the generic fiber of  $\text{NIP}_{d,n}$  over  $\text{Spec } \mathbb{Z}$ .

*Proof.* Let  $X_{d',d''} = (\text{scheme theoretic image of } p_{d',d''})$ . Since

$$\begin{aligned} \dim \text{NIP}_{d,n} &= \max\{\dim X_{d',d''} \mid d' + d'' = d, d', d'' \geq 1\} \\ \dim(\text{NIP}_{d,n})_{\mathbb{Q}} &= \max\{\dim(X_{d',d''})_{\mathbb{Q}} \mid d' + d'' = d, d', d'' \geq 1\}, \end{aligned}$$

to bound the dimension it is enough to show

$$\dim X_{d',d''} \leq N_{d,n} - 1 \quad \text{and} \quad \dim(X_{d',d''})_{\mathbb{Q}} \leq N_{d,n} - 2$$

for all  $d', d'' \geq 1$  with  $d' + d'' = d$ . Note that  $X_{d',d''}$  is an integral scheme because so is  $\mathbb{A}_{\mathbb{Z}}^{N_{d',n}} \times_{\mathbb{Z}} \mathbb{A}_{\mathbb{Z}}^{N_{d'',n}}$ . Therefore the generic point of  $X_{d',d''}$  is mapped to the generic point of  $\text{Spec } \mathbb{Z}$ . By the definition of  $\text{NIP}_{d,n}$ , its irreducible components can be found among  $X_{d',d''}$  and hence  $\text{NIP}_{d,n}$  is flat over  $\text{Spec } \mathbb{Z}$ .

Now, since the structure morphism  $\pi: X_{d',d''} \rightarrow \text{Spec } \mathbb{Z}$  is flat, for any scheme point  $x \in X_{d',d''}$ , we have

$$\dim \mathcal{O}_{X_{d',d''},x} = \dim \mathcal{O}_{\text{Spec } \mathbb{Z},\pi(x)} + \dim \mathcal{O}_{\pi^{-1}(\pi(x)),x}.$$

Thus

$$\dim X_{d',d''} \leq 1 + \max\{\dim \pi^{-1}(\pi(x)) \mid x \in X_{d',d''}\}.$$

Since  $\pi$  is flat, all the non-empty fibers of  $\pi$  have the same dimension (cf. [9, Theorem 14.116]). Thus it is enough to show that the generic fiber of  $\pi$  has dimension at most  $N_{d,n} - 2$ . Since the generic fiber  $(X_{d',d''})_{\mathbb{Q}}$  is the scheme theoretic image of the base change  $(p_{d',d''})_{\mathbb{Q}}$  of  $p_{d',d''}$ , it is enough to show that

$$\dim \mathbb{A}_{\mathbb{Q}}^{N_{d',n}} \times_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}^{N_{d'',n}} = N_{d',n} + N_{d'',n} \leq N_{d,n} - 2,$$

or equivalently,

$$\binom{n+d'}{n} + \binom{n+d''}{n} \leq \binom{n+d}{n} - 2.$$

Noticing

$$\begin{aligned} \binom{n+d'}{n} &= \binom{n+d'-1}{n-1} + \binom{n+d'-2}{n-1} + \cdots + \binom{n}{n-1} + 1 \\ \binom{n+d}{n} &= \binom{n+d-1}{n-1} + \cdots + \binom{n+d''}{n-1} + \binom{n+d''}{n}, \end{aligned}$$

we get

$$\binom{n+d}{n} - \binom{n+d'}{n} - \binom{n+d''}{n} \geq d' \binom{n}{n-2} d'' - 1$$

and the right hand side is at least 2 under our assumption.  $\square$

**Proposition 9.3.** *Let  $n \in \mathbb{Z}_{\geq 2}$  and  $d \in \mathbb{Z}_{\geq 1}$ . There is  $P_0 \geq 1$  with the following property. For all  $p > P_0$  and  $\mathbf{a} \in \mathbb{Z}_p^{N_{d,n}}$ , if  $\mathbf{a} \pmod{p} \in \mathbb{F}_p^{N_{d,n}} \setminus \text{NIP}_{d,n}(\mathbb{F}_p)$ , then we have*

$$V_+(\mathbf{f}_{\mathbf{a}})(\mathbb{Q}_p) \neq \emptyset.$$

(see Definition 1.1 for the definition of the notation  $\mathbf{f}_{\mathbf{a}}$ .)

*Proof.* By the Lang-Weil estimates [13, Theorem 1 and Lemma 1], there is  $P_0 \geq 1$  such that for every prime number  $p > P_0$  and every geometrically integral hypersurface  $X \subset \mathbb{P}_{\mathbb{F}_p}^n$  of degree  $d$ ,  $X$  has a smooth  $\mathbb{F}_p$ -point. Let  $p > P_0$  be an arbitrary prime. If  $\mathbf{a} \pmod{p} \in \mathbb{F}_p^{N_{d,n}} \setminus \text{NIP}_{d,n}(\mathbb{F}_p)$ , then  $V_+(\mathbf{f}_{\mathbf{a} \pmod{p}})$  is geometrically integral. Then  $V_+(\mathbf{f}_{\mathbf{a} \pmod{p}})$  has a smooth  $\mathbb{F}_p$ -point and thus  $V_+(\mathbf{f}_{\mathbf{a}})$  has a  $\mathbb{Q}_p$ -point by the lifting property.  $\square$

## 10. ASYMPTOTIC FORMULA FOR $\#\mathbb{V}_{d,n}^{\text{loc}}(A; \xi, \sigma)$

We give an asymptotic formula of the number of hypersurfaces  $V \subset \mathbb{P}_{\mathbb{Q}}^n$  of bounded height such that  $\prod_{p \leq \infty} V(\mathbb{Q}_p)$  is non-empty and contains a point that is close to a given point of  $\prod_{p \leq \infty} \mathbb{P}^n(\mathbb{Q}_p)$  with respect to the product topology.

As we pointed out in Remark 1.8, our argument follows Poonen–Voloch [16] which uses Lemma 20 of Poonen–Stoll [15, p. 1134], but we need to carry out everything in a quantitative way. The exponent  $g_p$  in Lemma 10.20, which decides the level of  $p$ -adic approximation of the set of hypersurfaces admitting  $p$ -adic points, will be chosen in a standard manner (see (10.23) and (10.28)) as in the previous studies, e.g. the exponent in (2.9) of [4].

### Setup

- Let  $n \geq 3$  and  $d \geq 2$ .
- Let  $\sigma = (\sigma_p)_{p \in M_{\mathbb{Q}}} \in \mathbb{R}^{M_{\mathbb{Q}}}$  such that  $0 < \sigma_p \leq 1$  and  $\sigma_p = 1$  for all but finitely many  $p$ . For  $p \neq \infty$ , let

$$e_p := \min\{e \in \mathbb{Z}_{\geq 0} \mid p^{-e} \leq \sigma_p\}.$$

We have  $p^{-e_p} \leq \sigma_p < p^{-(e_p-1)}$ . Also  $\sigma_p < 1$  if and only if  $e_p \geq 1$ . We set

$$q := \prod_{p < \infty} p^{e_p}.$$

- Let  $S := \{p \in M_{\mathbb{Q}} \mid \sigma_p < 1\} \cup \{\infty\}$ . We write  $S_{\text{fin}} := S \setminus \{\infty\}$ .
- Let  $\xi = (\xi_p)_{p \in M_{\mathbb{Q}}}$ , where  $\xi_p \in \mathbb{P}^n(\mathbb{Q}_p)$ . For  $p \neq \infty$ , we pick  $\xi_p \in \mathbb{Z}_p^{n+1}$  such that  $\|\xi_p\|_p = 1$  and  $[\xi_p] = \xi_p$ . We also pick  $\xi_{\infty} \in \mathbb{R}^{n+1}$  such that  $\|\xi_{\infty}\| = 1$  and  $[\xi_{\infty}] = \xi_{\infty}$ .

**Definition 10.1.** We use the following notation:

$$\begin{aligned} (\mathbb{Z}_p^N)_{\text{prim}} &:= \{\mathbf{a} \in \mathbb{Z}_p^N \mid \|\mathbf{a}\|_p = 1\} \\ (\mathbb{Z}/p^e\mathbb{Z})_{\text{prim}}^N &:= \{\mathbf{a} \in (\mathbb{Z}/p^e\mathbb{Z})^N \mid \mathbf{a} \not\equiv 0 \pmod{p}\} \end{aligned}$$

$$\begin{aligned}
S^{N-1} &:= \{\mathbf{a} \in \mathbb{R}^N \mid \|\mathbf{a}\| = 1\} \\
\mu_p &:= \mu_p^N \quad \text{the Haar measure on } \mathbb{Z}_p^N \text{ such that } \mu_p(\mathbb{Z}_p^N) = 1 \\
\mu_{S^{N-1}} &\quad \text{the spherical measure on } S^{N-1}.
\end{aligned}$$

We also use the following normalized measure on  $\mathbb{P}^{N-1}(\mathbb{Q}_p)$ :

$$\begin{aligned}
(10.1) \quad \mu_{\mathbb{P}^{N-1}} &:= \frac{1}{\mu_p^N((\mathbb{Z}_p^N)_{\text{prim}})} \left( (\mathbb{Z}_p^N)_{\text{prim}} \rightarrow \mathbb{P}^{N-1}(\mathbb{Q}_p) \right)_* \left( \mu_p^N|_{(\mathbb{Z}_p^N)_{\text{prim}}} \right) \\
&= \frac{1}{1-p^{-N}} \left( (\mathbb{Z}_p^N)_{\text{prim}} \rightarrow \mathbb{P}^{N-1}(\mathbb{Q}_p) \right)_* \left( \mu_p^N|_{(\mathbb{Z}_p^N)_{\text{prim}}} \right)
\end{aligned}$$

**Definition 10.2.**

- (1) For  $p \in M_{\mathbb{Q}}$ , we set  $Z_p := \{a \in \mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p) \mid V_+(f_a)(\mathbb{Q}_p) \neq \emptyset\}$ .
- (2) For  $p \in M_{\mathbb{Q}}$  and  $\xi, \sigma$  as before, we set

$$Z_p(\xi, \sigma) := \left\{ a \in \mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p) \mid \begin{array}{l} \exists \eta \in V_+(f_a)(\mathbb{Q}_p) \text{ such that} \\ d_p(\eta, \xi_p) \leq \sigma_p \end{array} \right\}$$

Note that we obviously have  $Z_p(\xi, \sigma) \subset Z_p$ , and  $Z_p(\xi, \sigma) = Z_p$  for  $p \notin S$ .

- (3) For  $p \in M_{\mathbb{Q}} \setminus \{\infty\}$ , we set

$$\begin{aligned}
W_p &:= \left( (\mathbb{Z}_p^{N_{d,n}})_{\text{prim}} \rightarrow \mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p) \right)^{-1} (Z_p) \\
W_p(\xi, \sigma) &:= \left( (\mathbb{Z}_p^{N_{d,n}})_{\text{prim}} \rightarrow \mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p) \right)^{-1} (Z_p(\xi, \sigma)).
\end{aligned}$$

Also, we set

$$\begin{aligned}
W_{\infty} &:= \left( S^{N_{d,n}-1} \rightarrow \mathbb{P}^{N_{d,n}-1}(\mathbb{R}) \right)^{-1} (Z_{\infty}) \\
W_{\infty}(\xi, \sigma) &:= \left( S^{N_{d,n}-1} \rightarrow \mathbb{P}^{N_{d,n}-1}(\mathbb{R}) \right)^{-1} (Z_{\infty}(\xi, \sigma)).
\end{aligned}$$

- (4) For  $P \in \mathbb{R}_{\geq 1}$ , we set

$$M_{\mathbb{Q}, \leq P} := \{p \in M_{\mathbb{Q}} \mid p \text{ is a prime number and } p \leq P\}.$$

**Definition 10.3.** We write

$$\begin{aligned}
\rho_p(\xi, \sigma) &:= \mu_{\mathbb{P}^{N_{d,n}-1}}(Z_p(\xi, \sigma)) \quad \text{for } p \in M_{\mathbb{Q}} \setminus \{\infty\}; \\
\rho_{\infty}(\xi, \sigma) &:= \mu_{S^{N_{d,n}-1}}(W_{\infty}(\xi, \sigma)).
\end{aligned}$$

Using our notation, we can rewrite

$$\#\mathbb{V}_{d,n}^{\text{loc}}(A; \xi, \sigma) = \frac{1}{2} \# \left\{ \mathbf{a} \in \mathbb{Z}_{\text{prim}}^{N_{d,n}} \mid \|\mathbf{a}\| \leq A, \forall p \in M_{\mathbb{Q}}, [\mathbf{a}] \in Z_p(\xi, \sigma) \right\}.$$

For  $A, P \in \mathbb{R}_{\geq 1}$ , then we have

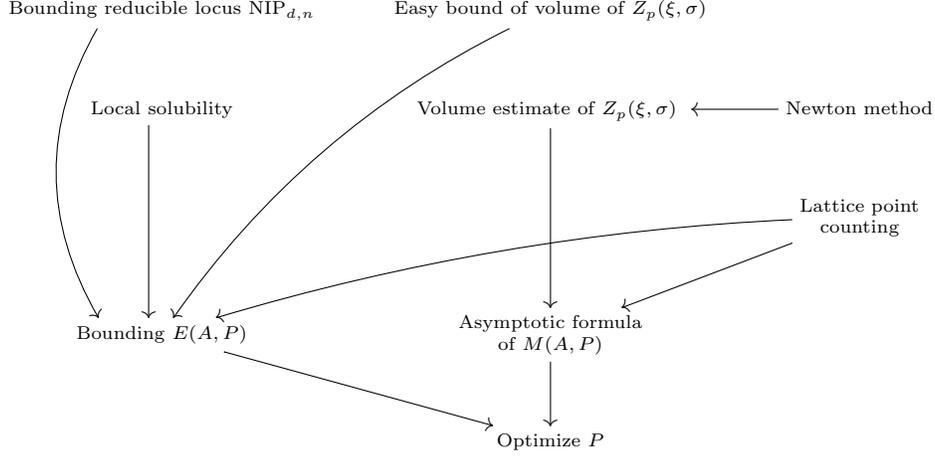
$$(10.2) \quad \#\mathbb{V}_{d,n}^{\text{loc}}(A; \xi, \sigma) = \frac{1}{2}M(A, P) - \frac{1}{2}E(A, P),$$

where

$$M(A, P) := \# \left\{ \mathbf{a} \in \mathbb{Z}_{\text{prim}}^{N_{d,n}} \mid \begin{array}{l} \|\mathbf{a}\| \leq A \\ \forall p \in S \cup M_{\mathbb{Q}, \leq P}, [\mathbf{a}] \in Z_p(\xi, \sigma) \end{array} \right\},$$

$$E(A, P) := \# \left\{ \mathbf{a} \in \mathbb{Z}_{\text{prim}}^{N_{d,n}} \mid \begin{array}{l} \|\mathbf{a}\| \leq A \\ \forall p \in S \cup M_{\mathbb{Q}, \leq P}, [\mathbf{a}] \in Z_p(\xi, \sigma) \\ \exists p \in (P, \infty), [\mathbf{a}] \notin Z_p \end{array} \right\}.$$

We summarize our strategy to get an asymptotic formula of  $\# \mathbb{V}_{d,n}^{\text{loc}}(A; \xi, \sigma)$ , i.e. the strategy of the proof of Theorem 1.7.



10.1. **On the sets  $Z_p, Z_p(\xi, \sigma)$ .** For  $p \in M_{\mathbb{Q}}$ , consider the following diagram

$$\begin{array}{c} \mathcal{X}_p \\ \cap \\ \mathbb{P}_{\mathbb{Q}_p}^n \times_{\mathbb{Q}_p} \mathbb{P}_{\mathbb{Q}_p}^{N_{d,n}-1} \\ \swarrow \text{pr}_1 \quad \searrow \text{pr}_2 \\ \mathbb{P}_{\mathbb{Q}_p}^n \quad \mathbb{P}_{\mathbb{Q}_p}^{N_{d,n}-1} \end{array}$$

where  $\mathcal{X}_p$  is the closed subscheme defined by  $f_{\mathbf{a}}(\mathbf{x}) = 0$ , where  $\mathbf{x}, \mathbf{a}$  are homogeneous coordinates of  $\mathbb{P}^n$  and  $\mathbb{P}^{N_{d,n}-1}$ , respectively. Then we have

$$Z_p = \text{pr}_2(\mathcal{X}_p(\mathbb{Q}_p)) \quad \text{and} \quad Z_p(\xi, \sigma) = \text{pr}_2\left(\mathcal{X}_p(\mathbb{Q}_p) \cap \text{pr}_1^{-1}(B_{d_p}(\xi_p, \sigma_p))\right),$$

where  $B_{d_p}(\xi_p, \sigma_p) = \{\eta \in \mathbb{P}^n(\mathbb{Q}_p) \mid d_p(\eta, \xi_p) \leq \sigma_p\}$ . In particular,  $Z_p, Z_p(\xi, \sigma)$  are closed subsets of  $\mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p)$  with respect to the strong topology.

**Remark 10.4.** By this description, we can say that the boundary of  $Z_p$  and  $Z_p(\xi, \sigma)$  are contained in a proper Zariski closed subsets of  $\mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p)$ . Indeed, the morphism  $\text{pr}_2|_{\mathcal{X}_p}: \mathcal{X}_p \rightarrow \mathbb{P}_{\mathbb{Q}_p}^{N_{d,n}-1}$  is generically smooth because it is generically flat and general geometric fibers are smooth. Pick a dense Zariski open subset  $U \subset \mathbb{P}_{\mathbb{Q}_p}^{N_{d,n}-1}$  such that  $\text{pr}_2|_{\mathcal{X}_p}$  is smooth over  $U$ . Then the boundaries are contained in  $(\mathbb{P}_{\mathbb{Q}_p}^{N_{d,n}-1} \setminus U)(\mathbb{Q}_p)$ . From this fact, we see that the boundaries have measure zero. Later, we will need an estimate of the measure of  $Z_p(\xi, \sigma)$  (Proposition 10.12). To

obtain this, we need to estimate “proportion” of the boundary in the reduction modulo prime powers (see Proposition 10.10).

For the archimedean solubility, we again appeal Lemma 3.8 to. We thus check the archimedean solubility is merely a semialgebraic condition:

**Lemma 10.5.** *The family of subsets*

$$(\mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma)) - \mathbf{a}_0$$

of  $\mathbb{R}^{N_{d,n}}$ , which is independent of  $\xi_p, \sigma_p$  for  $p < \infty$ , parametrized by

$$(A, \sigma_\infty, \boldsymbol{\xi}_\infty, \mathbf{a}_0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{N_{d,n}}$$

is semialgebraic. In particular,  $\mathbb{R}W_\infty, \mathbb{R}W_\infty(\xi, \sigma) \subset \mathbb{R}^{N_{d,n}}$  are semialgebraic.

*Proof.* Let  $\tilde{K} \subset \mathbb{R}^{N_{d,n}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{N_{d,n}} \times \mathbb{R}^{n+1}$  be the set of tuples

$$(\mathbf{a}, A, \sigma_\infty, \boldsymbol{\xi}_\infty, \mathbf{a}_0, \mathbf{x}) \in \mathbb{R}^{N_{d,n}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{N_{d,n}} \times \mathbb{R}^{n+1}$$

satisfying

$$\mathbf{a} + \mathbf{a}_0 \in \mathcal{B}_{N_{d,n}}(A), \quad \mathbf{x} \neq 0, \quad d_\infty([\mathbf{x}], [\boldsymbol{\xi}_\infty]) \leq \sigma_\infty, \quad f_{\mathbf{a}+\mathbf{a}_0}(\mathbf{x}) = 0.$$

These are semialgebraic conditions and thus  $\tilde{K}$  is a semialgebraic set. Then the projection  $K$  of  $\tilde{K}$  to the first five factors is also semialgebraic and the fiber of  $K$  over the point  $(A, \sigma_\infty, \boldsymbol{\xi}_\infty, \mathbf{a}_0)$  is  $(\mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma)) - \mathbf{a}_0$ .

If we remove  $A$  and the condition involving  $A$ , and set  $\mathbf{a}_0 = 0$ , we get the semialgebraicity of  $\mathbb{R}W$  and  $\mathbb{R}W_\infty(\xi, \sigma)$ .  $\square$

In order to determine the magnitude of the main term of Theorem 1.7, we need to get bounds for the local densities  $\rho_p(\xi, \sigma)$ . For the archimedean case, we use the classical Newton method. Recall the set of monomials  $\mathcal{M}_{d,n}$  (Definition 1.1) and Veronese embedding  $\nu_{d,n}$  defined in (2.1). Let us define

$$\nu_{d,n}^{(i)}(\mathbf{x}) := \left( \frac{\partial M}{\partial x_i}(\mathbf{x}) \right)_{M \in \mathcal{M}_{d,n}} \quad \text{for } i = 0, \dots, n$$

so that the partial derivatives  $\nu_{d,n}^{(i)}$  of the Veronese embedding can be expressed as

$$\frac{\partial f_{\mathbf{a}}}{\partial x_i}(\mathbf{x}) = \langle \mathbf{a}, \nu_{d,n}^{(i)}(\mathbf{x}) \rangle.$$

Then, the local density  $\rho_\infty(\xi, \sigma)$  for the archimedean place is bounded as follows:

**Proposition 10.6.** *We have*

$$\rho_\infty(\xi, \sigma) = \mu_{S^{N_{d,n}-1}}(W_\infty(\xi, \sigma)) \asymp \sigma_\infty,$$

where the implicit constant depends only on  $d, n$ .

*Proof.* First we prove  $\rho_\infty(\xi, \sigma) \ll_{d,n} \sigma_\infty$ . If  $\mathbf{a} \in W_\infty(\xi, \sigma)$ , then there is  $\boldsymbol{\eta} \in S^n$  such that  $\langle \mathbf{a}, \nu_{d,n}(\boldsymbol{\eta}) \rangle = 0$  and  $d_\infty([\boldsymbol{\eta}], [\boldsymbol{\xi}_\infty]) \leq \sigma_\infty$ . By replacing  $\boldsymbol{\eta}$  with  $-\boldsymbol{\eta}$ , we may assume  $\langle \boldsymbol{\eta}, \boldsymbol{\xi}_\infty \rangle \geq 0$ . Then we have

$$\begin{aligned} |\langle \mathbf{a}, \nu_{d,n}(\boldsymbol{\xi}_\infty) \rangle| &= |\langle \mathbf{a}, \nu_{d,n}(\boldsymbol{\xi}_\infty) - \nu_{d,n}(\boldsymbol{\eta}) \rangle| \\ &\leq \|\nu_{d,n}(\boldsymbol{\xi}_\infty) - \nu_{d,n}(\boldsymbol{\eta})\| \ll_{d,n} \|\boldsymbol{\xi}_\infty - \boldsymbol{\eta}\| \ll d_\infty([\boldsymbol{\xi}_\infty], [\boldsymbol{\eta}]) \leq \sigma_\infty. \end{aligned}$$

From this, we can easily deduce  $\rho_\infty(\xi, \sigma) \ll_{d,n} \sigma_\infty$ .

Next, we prove  $\rho_\infty(\xi, \sigma) \gg_{d,n} \sigma_\infty$ . For an orthogonal transformation  $R$  on  $\mathbb{R}^{n+1}$ , we can associate a linear automorphism  $\tilde{R}$  on  $\mathbb{R}^{N_{d,n}}$  so that  $\langle \mathbf{a}, \nu_{d,n}(R \cdot \mathbf{x}) \rangle = \langle \tilde{R} \cdot \mathbf{a}, \nu_{d,n}(\mathbf{x}) \rangle$  for all  $\mathbf{a} \in \mathbb{R}^{N_{d,n}}$  and  $\mathbf{x} \in \mathbb{R}^{n+1}$ . Note that  $\tilde{R}$  is continuously depending on  $R$  and the orthogonal transformation on  $\mathbb{R}^{n+1}$  forms a compact group. This gives that the bounds for the operator norms  $\|\tilde{R}\|, \|\tilde{R}^{-1}\| \ll_{d,n} 1$  and the determinants  $|\det \tilde{R}|, |\det \tilde{R}^{-1}| \ll_{d,n} 1$  uniformly over  $R$ . Thus, under the automorphism  $\tilde{R}$ , the density  $\rho_\infty(\xi, \sigma)$  is invariant up to multiple by a positive constant depending only on  $d, n$ . Therefore, by considering an orthogonal transformation of  $\mathbb{R}^{n+1}$  sending  $\xi_\infty$  to  $(1, 0, \dots, 0)$ , we may assume  $\xi_\infty = (1, 0, \dots, 0)$ .

If  $\xi_\infty = (1, 0, \dots, 0)$ , then  $\nu_{d,n}(\xi_\infty)$  and  $\nu_{d,n}^{(1)}(\xi_\infty)$  have the coordinate 1 at the position corresponding to the monomials  $x_0^d$  and  $x_0^{d-d}x_1$ , respectively, and the other coordinates are all zero. Let us consider the set

$$\Sigma(\tau) := \{\mathbf{a} \in S^{N_{d,n}-1} \mid |\langle \mathbf{a}, \nu_{d,n}(\xi_\infty) \rangle| \leq \tau \text{ and } |\langle \mathbf{a}, \nu_{d,n}^{(1)}(\xi_\infty) \rangle| \geq \frac{1}{2}\}$$

for  $\tau > 0$ . Note that then  $\Sigma(\tau)$  is the set of vectors  $\mathbf{a} \in S^{N_{d,n}-1}$  having coordinate of the absolute value  $\leq \tau$  and  $\geq \frac{1}{2}$  at the positions corresponding to the monomials  $x_0^d$  and  $x_0^{d-1}x_1$ . Thus, we have

$$(10.3) \quad \mu_{S^{N_{d,n}-1}}(\Sigma(\tau)) \gg_{d,n} \tau.$$

We are applying Newton's method to show that for  $\mathbf{a} \in \Sigma(\tau)$ , the polynomial  $f_{\mathbf{a}}$  has a solution close to  $\xi_\infty$  if  $\tau$  is small enough. We use Newton's method presented in [7, II.4, p. 49–51]. Let  $f(t) = f_{\mathbf{a}}(1, t, 0, \dots, 0)$ . Then,  $f$  is a polynomial in  $t$  with degree at most  $d$  and

$$|f(0)| = |\langle \mathbf{a}, \nu_{d,n}(\xi_\infty) \rangle| \leq \tau, \quad |f'(0)| = |\langle \mathbf{a}, \nu_{d,n}^{(1)}(\xi_\infty) \rangle| \geq \frac{1}{2},$$

$$|f| = \max\{\text{absolute value of coefficient of } f\} \leq 1.$$

Let  $c := (12d(d-1)^2)^{-1}$ . Then, for  $t, u \in [-c, +c]$ , we have

$$|f'(t) - f'(u)| \leq |t - u| \max_{|t| \leq c} |f''(t)| \leq 2d(d-1)^2 c = \frac{1}{6}.$$

Combined with  $|f'(0)| \geq 1/2$ , we get  $|f'(t)| \geq 1/2 - 2d(d-1)^2 c = \frac{1}{3}$  for  $t \in [-c, +c]$ . Thus, we have  $|f'(t)| \geq 1/3$  and  $|f'(t) - f'(u)| \leq 1/6$  for  $t, u \in [-c, +c]$ . By [7, II.4, p. 49–51], if  $|f(0)| \leq c/6$ , there is  $y \in \mathbb{R}$  such that

$$f(y) = 0 \quad \text{and} \quad |y| \leq c.$$

This means  $f_{\mathbf{a}}$  has a solution  $(1, y, 0, \dots, 0)$ , which satisfies

$$d_\infty([(1, y, 0, \dots, 0)], \xi_\infty) = \frac{|y|}{\sqrt{1+y^2}} \leq c.$$

In particular, we have  $\Sigma(\sigma_\infty/6) \subset W_\infty(\xi, \sigma)$ , so the bound (10.3) gives

$$\mu_{S^{N_{d,n}-1}}(W_\infty(\xi, \sigma)) \geq \mu_{S^{N_{d,n}-1}}(\Sigma(\sigma_\infty/6)) \gg_{n,d} \sigma_\infty$$

as desired.  $\square$

We next estimate  $\rho_p(\xi, \sigma)$  again by using the Newton method, i.e. Hensel's lemma, but the proof is more complicated and broke down into the subsequent several lemmas. Also, the subsequent argument has another purpose to estimate

for the “boundary” of the solubility locus  $Z_p(\xi, \sigma)$ . We use Hensel’s lemma of the form, e.g. given on p. 48 of [17] as follows:

**Lemma 10.7.** *Let  $n, d \in \mathbb{N}$ ,  $p$  be a prime number and  $\mathbf{a} \in \mathbb{Z}_{\text{prim}}^{N_{d,n}}$  and  $f := f_{\mathbf{a}}$ . For any  $\xi \in \mathbb{Z}_p^{n+1}$  with  $\|\xi\|_p = 1$  and  $e, l \in \mathbb{Z}_{\geq 0}$ , if*

$$e > l, \quad |f(\xi)|_p \leq p^{-e} < \max_{0 \leq i \leq n} \left| \frac{\partial f}{\partial x_i}(\xi) \right|_p^2, \quad p^{-l} \leq \max_{0 \leq i \leq n} \left| \frac{\partial f}{\partial x_i}(\xi) \right|_p,$$

then there is  $\tilde{\xi} \in \mathbb{Z}_p^{n+1}$  with  $\|\tilde{\xi}\|_p = 1$  such that

$$f(\tilde{\xi}) = 0 \quad \text{and} \quad d_p(\xi, \tilde{\xi}) \leq p^{-(e-l)}.$$

*Proof.* Without loss of generality, we may assume

$$\max_{0 \leq i \leq n} \left| \frac{\partial f}{\partial x_i}(\xi) \right|_p = \left| \frac{\partial f}{\partial x_0}(\xi) \right|_p.$$

Let us write  $\xi = (\xi_0, \dots, \xi_n)$  and set  $g(t) := f(t, \xi_1, \dots, \xi_n) \in \mathbb{Z}_p[t]$ . We have

$$|g(\xi_0)|_p = |f(\xi)|_p \leq p^{-e} < \left| \frac{\partial f}{\partial x_0}(\xi) \right|_p^2 = |g'(\xi_0)|_p^2.$$

By Hensel’s lemma of the form given on p. 48 of [17], there is  $\tilde{\xi}_0 \in \mathbb{Z}_p$  such that

$$g(\tilde{\xi}_0) = 0 \quad \text{and} \quad |\tilde{\xi}_0 - \xi_0|_p \leq \frac{|g(\xi_0)|_p}{|g'(\xi_0)|_p} = \frac{|f(\xi)|_p}{|(\partial f / \partial x_0)(\xi)|_p} \leq p^{-(e-l)}.$$

Set  $\tilde{\xi} = (\tilde{\xi}_0, \xi_1, \dots, \xi_n)$ . Since  $\|\xi\|_p = 1$  and  $|\tilde{\xi}_0 - \xi_0|_p \leq p^{-(e-l)} < 1$ , we have  $\|\tilde{\xi}\|_p = 1$ . We have  $f(\tilde{\xi}) = g(\tilde{\xi}_0) = 0$ . Moreover, we have

$$d_p(\tilde{\xi}, \xi) = \max_{1 \leq i \leq n} |\tilde{\xi}_0 \xi_i - \xi_i \xi_0|_p \leq |\tilde{\xi}_0 - \xi_0|_p \leq p^{-(e-l)}.$$

as desired.  $\square$

**Proposition 10.8.** *For  $p \in S_{\text{fin}}$  and  $\mathbf{a} \in \mathbb{Z}_p^{N_{d,n}} \setminus \{0\}$ , if  $[\mathbf{a}] \in Z_p(\xi, \sigma)$ , then*

$$|\langle \mathbf{a}, \nu_{d,n}(\xi_p) \rangle|_p \leq \sigma_p \|\mathbf{a}\|_p.$$

*Proof.* Since  $[\mathbf{a}] \in Z_p(\xi, \sigma)$ , there is  $\eta \in \mathbb{Z}_p^{n+1}$  with  $\|\eta\|_p = 1$ ,  $\langle \mathbf{a}, \nu_{d,n}(\eta) \rangle = 0$  and  $d_p([\eta], \xi_p) \leq \sigma_p < 1$ . Note that since  $d_p([\eta], \xi_p) < 1$ , we can take  $\eta$  so that at least one coordinate is unit and equal to the corresponding entry of  $\xi_p$ . Then we have

$$\begin{aligned} |\langle \mathbf{a}, \nu_{d,n}(\xi_p) \rangle|_p &= |\langle \mathbf{a}, \nu_{d,n}(\xi_p) - \nu_{d,n}(\eta) \rangle|_p \\ &\leq \|\mathbf{a}\|_p \|\nu_{d,n}(\xi_p) - \nu_{d,n}(\eta)\|_p \\ &\leq \|\mathbf{a}\|_p \|\xi_p - \eta\|_p = \|\mathbf{a}\|_p d_p([\xi_p], [\eta]) \leq \sigma_p \|\mathbf{a}\|_p \end{aligned}$$

as desired.  $\square$

We then introduce the “boundary balls” for the solubility locus  $Z_p(\xi, \sigma)$ .

**Definition 10.9.** Let  $p$  be a prime number.

(1) Let  $N \in \mathbb{Z}_{\geq 1}$ . For  $\mathbf{a} \in \mathbb{Z}_p^N$  and  $r > 0$ , we write

$$B(\mathbf{a}, r) := \left\{ \mathbf{x} \in \mathbb{Z}_p^N \mid \|\mathbf{x} - \mathbf{a}\|_p \leq r \right\}.$$

Note that if  $r = p^{-v}$  for some  $v \in \mathbb{Z}_{\geq 1}$ , then  $B(\mathbf{a}, p^{-v})$  is determined by the class  $\mathbf{a} \pmod{p^v}$ . For  $\mathbf{a}' \in (\mathbb{Z}/p^v\mathbb{Z})^N$ , we write  $B(\mathbf{a}', p^{-v})$  for the ball  $B(\mathbf{a}, p^{-v})$  where  $\mathbf{a} \in \mathbb{Z}_p^N$  is an arbitrary lift of  $\mathbf{a}'$ .

(2) For  $v \in \mathbb{Z}_{\geq 1}$ , we set

$$\begin{aligned} \partial W_p(p^v) &:= \left\{ \mathbf{a} \in (\mathbb{Z}/p^v\mathbb{Z})^{N_{d,n}} \mid \begin{array}{l} B(\mathbf{a}, p^{-v}) \cap W_p \neq \emptyset \\ B(\mathbf{a}, p^{-v}) \not\subset W_p \end{array} \right\}, \\ \partial W_p(\xi, \sigma)(p^v) &:= \left\{ \mathbf{a} \in (\mathbb{Z}/p^v\mathbb{Z})^{N_{d,n}} \mid \begin{array}{l} B(\mathbf{a}, p^{-v}) \cap W_p(\xi, \sigma) \neq \emptyset \\ B(\mathbf{a}, p^{-v}) \not\subset W_p(\xi, \sigma) \end{array} \right\}. \end{aligned}$$

We now estimate the boundary of  $Z_p(\xi, \sigma)$ . This may be compared to, e.g. Lemma 5.6 of [4, p. 1188] but the exponent  $e$  in [4] corresponds to a larger one  $v_0$  necessary for the Hensel lifting. Also, we use a different reductions to the lower prime powers and use the shifts of  $\mathbf{x}$  (see (10.12)) not only the multiplications by unit (see (10.10)) as in [4], which gives better bounds in our setting.

**Proposition 10.10.**

(1) Let  $p$  be a prime number. For  $v \in \mathbb{Z}_{\geq 1}$  with  $v \geq e_p$ , set

$$(10.4) \quad \tilde{v} := \min(e_p, v - e_p + 1).$$

Then, we have

$$(10.5) \quad \#\partial W_p(\xi, \sigma)(p^v) \leq p^{vN_{d,n} - v - n\tilde{v}} \frac{p\varphi(p^{e_p})}{\varphi(p)p^{e_p}}.$$

In particular,

$$(10.6) \quad \#\partial W_p(p^v) \leq p^{vN_{d,n} - v} \left( \frac{p}{p-1} \right).$$

(2) For  $p \in S_{\text{fin}}$ , we have

$$(10.7) \quad \#\partial W_p(\xi, \sigma)(p^{e_p}) \leq p^{e_p N_{d,n} - e_p - n}.$$

*Proof.* The bound (10.6) follows from (10.5) by setting  $e_p = 0$ . The bound (10.7) follows from (10.5) by setting  $v = e_p$  since  $p \in S_{\text{fin}}$  implies  $e_p \geq 1$ . Thus, it is enough to prove the bound (10.5). We fix  $v \in \mathbb{Z}_{\geq 1}$  with  $v \geq e_p$ . Let

$$(10.8) \quad v_0 := \min\left(\left\lceil \frac{v}{2} \right\rceil, v - e_p + 1\right).$$

**Claim 10.11.** For  $\mathbf{a} \in W_p(\xi, \sigma)$  satisfying

$$B(\mathbf{a}, p^{-v}) \not\subset W_p(\xi, \sigma),$$

there is  $\mathbf{x} \in (\mathbb{Z}_p^{n+1})_{\text{prim}}$  such that

$$f_{\mathbf{a}}(\mathbf{x}) = 0, \quad d_p([\mathbf{x}], [\xi_p]) \leq p^{-e_p} \quad \text{and} \quad \left| \frac{\partial f_{\mathbf{a}}}{\partial x_i}(\mathbf{x}) \right|_p \leq p^{-v_0} \quad \text{for all } 0 \leq i \leq n.$$

*Proof of Claim 10.11.* Since  $\mathbf{a} \in W_p(\xi, \sigma)$ , there is  $\mathbf{x} \in (\mathbb{Z}_p^{n+1})_{\text{prim}}$  such that

$$f_{\mathbf{a}}(\mathbf{x}) = 0 \quad \text{and} \quad d_p([\mathbf{x}], [\xi_p]) \leq p^{-e_p}.$$

Take a point  $\mathbf{b} \in B(\mathbf{a}, p^{-v}) \setminus W_p(\xi, \sigma)$ . Then we have

$$|f_{\mathbf{b}}(\mathbf{x})|_p = |f_{\mathbf{b}}(\mathbf{x}) - f_{\mathbf{a}}(\mathbf{x})|_p = |f_{\mathbf{b}-\mathbf{a}}(\mathbf{x})|_p \leq p^{-v}.$$

We first show

$$(10.9) \quad \left| \frac{\partial f_{\mathbf{b}}}{\partial x_i}(\mathbf{x}) \right|_p \leq p^{-v_0} \quad \text{for all } 0 \leq i \leq n.$$

Suppose there is  $i$  such that

$$\left| \frac{\partial f_{\mathbf{b}}}{\partial x_i}(\mathbf{x}) \right|_p > p^{-v_0} \quad \text{so} \quad \left| \frac{\partial f_{\mathbf{b}}}{\partial x_i}(\mathbf{x}) \right|_p \geq p^{-(v_0-1)}.$$

Since  $2(v_0 - 1) \leq 2(\lceil \frac{v}{2} \rceil - 1) < v$ , we have

$$\left| \frac{\partial f_{\mathbf{b}}}{\partial x_i}(\mathbf{x}) \right|_p^2 > p^{-v} \geq |f_{\mathbf{b}}(\mathbf{x})|.$$

By Lemma 10.7 with  $e = v$ ,  $l = \tilde{v} - 1$ , we get  $\mathbf{y} \in (\mathbb{Z}_p^{n+1})_{\text{prim}}$  such that

$$f_{\mathbf{b}}(\mathbf{y}) = 0 \quad \text{and} \quad d_p([\mathbf{y}], [\mathbf{x}]) \leq p^{-(v-(v_0-1))}.$$

By (10.8), we have

$$d_p([\mathbf{y}], [\xi_p]) \leq \max(d_p([\mathbf{y}], [\mathbf{x}]), d_p([\mathbf{x}], [\xi_p])) \leq \max(p^{-(v-v_0+1)}, p^{-e_p}) = p^{-e_p}.$$

The existence of such  $\mathbf{y}$  contradicts  $\mathbf{b} \notin W_p(\xi, \sigma)$ . Thus, (10.9) follows.

By (10.9), for all  $i$ , we get

$$\begin{aligned} \left| \frac{\partial f_{\mathbf{a}}}{\partial x_i}(\mathbf{x}) \right|_p &= \left| \frac{\partial f_{\mathbf{a}}}{\partial x_i}(\mathbf{x}) - \frac{\partial f_{\mathbf{b}}}{\partial x_i}(\mathbf{x}) + \frac{\partial f_{\mathbf{b}}}{\partial x_i}(\mathbf{x}) \right|_p \\ &\leq \max\left( \left| \frac{\partial f_{\mathbf{a}-\mathbf{b}}}{\partial x_i}(\mathbf{x}) \right|_p, \left| \frac{\partial f_{\mathbf{b}}}{\partial x_i}(\mathbf{x}) \right|_p \right) \leq \max(p^{-v}, p^{-v_0}) \leq p^{-v_0}, \end{aligned}$$

where we use  $\mathbf{b} \in B(\mathbf{a}, p^{-v})$  to bound  $|(\partial f_{\mathbf{a}-\mathbf{b}}/\partial x_i)(\mathbf{x})|_p$ .  $\square$

In light of this claim, we consider the following set

$$U := \left\{ \mathbf{a} \in (\mathbb{Z}/p^v\mathbb{Z})_{\text{prim}}^{N_{d,n}} \left| \begin{array}{l} \exists \mathbf{x} \in (\mathbb{Z}/p^v\mathbb{Z})_{\text{prim}}^{n+1} \text{ s.t.} \\ \mathbf{x} \equiv \xi_p \pmod{p^{e_p}} \\ \langle \mathbf{a}, \nu_{d,n}(\mathbf{x}) \rangle \equiv 0 \pmod{p^v} \\ \langle \mathbf{a}, \nu_{d,n}^{(i)}(\mathbf{x}) \rangle \equiv 0 \pmod{p^{v_0}} \text{ for all } i \end{array} \right. \right\}.$$

By Claim 10.11, we have  $\partial W_p(\xi, \sigma)(p^v) \subset U$ . Thus it is enough to bound  $\#U$ .

For  $\mathbf{a} \in (\mathbb{Z}/p^v\mathbb{Z})_{\text{prim}}^{N_{d,n}}$ , let

$$\mathfrak{D}_{\mathbf{a}}(\xi, \sigma) := \left\{ \mathbf{x} \in (\mathbb{Z}/p^v\mathbb{Z})_{\text{prim}}^{n+1} \left| \begin{array}{l} \mathbf{x} \equiv \xi_p \pmod{p^{e_p}} \\ \langle \mathbf{a}, \nu_{d,n}(\mathbf{x}) \rangle \equiv 0 \pmod{p^v} \\ \langle \mathbf{a}, \nu_{d,n}^{(i)}(\mathbf{x}) \rangle \equiv 0 \pmod{p^{v_0}} \text{ for all } i \end{array} \right. \right\}.$$

For  $\mathbf{a} \in U$ , we first estimate  $\#\mathfrak{D}_{\mathbf{a}}(\xi, \sigma)$  from below by finding symmetries on  $\mathfrak{D}_{\mathbf{a}}(\xi, \sigma)$ . Since  $\langle \mathbf{a}, \nu_{d,n}(\mathbf{x}) \rangle$  is homogeneous in  $\mathbf{x}$ , we have

$$(10.10) \quad \mathbf{x} \in \mathfrak{D}_{\mathbf{a}}(\xi, \sigma) \implies u\mathbf{x} \in \mathfrak{D}_{\mathbf{a}}(\xi, \sigma) \text{ for } u \in (\mathbb{Z}/p^v\mathbb{Z})^\times \text{ and } u \equiv 1 \pmod{p^{e_p}},$$

where  $u \equiv 1 \pmod{p^{e_p}}$  is necessary to keep the condition  $\mathbf{x} \equiv \boldsymbol{\xi}_p \pmod{p^{e_p}}$ . We next consider the shift  $\mathbf{x} + p^g\mathbf{y}$  of  $\mathbf{x} \in \mathfrak{D}_{\mathbf{a}}(\xi, \sigma)$  for some  $g \in \{1, \dots, v\}$  and  $\mathbf{y} = (y_0, \dots, y_n) \in (\mathbb{Z}/p^{v-g}\mathbb{Z})^{n+1}$ . We then have

$$\langle \mathbf{a}, \nu_{d,n}(\mathbf{x} + p^g\mathbf{y}) \rangle \equiv \langle \mathbf{a}, \nu_{d,n}(\mathbf{x}) \rangle + \sum_{i=0}^n \langle \mathbf{a}, \nu_{d,n}^{(i)}(\mathbf{x}) \rangle p^g y_i \pmod{p^{2g}}$$

Thus, if  $2g \geq v$  and  $v_0 + g \geq v$ , then  $\langle \mathbf{a}, \nu_{d,n}(\mathbf{x} + p^g\mathbf{y}) \rangle \equiv 0 \pmod{p^v}$ . Also, if  $2g \geq v$ , then  $g \geq \lceil v/2 \rceil \geq v_0$ . Thus, we have

$$\langle \mathbf{a}, \nu_{d,n}^{(i)}(\mathbf{x} + p^g\mathbf{y}) \rangle \equiv \langle \mathbf{a}, \nu_{d,n}^{(i)}(\mathbf{x}) \rangle \equiv 0 \pmod{p^{v_0}}$$

for all  $i$ . Hence, by taking the condition  $\mathbf{x} \equiv \boldsymbol{\xi}_p \pmod{p^{e_p}}$  into account, if we set

$$(10.11) \quad g_0 := \max\left(\left\lceil \frac{v}{2} \right\rceil, v - v_0, e_p\right) = \max\left(\left\lceil \frac{v}{2} \right\rceil, e_p\right),$$

where the last equality holds by (10.8), then we have

$$(10.12) \quad \mathbf{x} \in \mathfrak{D}_{\mathbf{a}}(\xi, \sigma) \implies \mathbf{x} + p^{g_0}\mathbf{y} \in \mathfrak{D}_{\mathbf{a}}(\xi, \sigma) \text{ for } \mathbf{y} \in (\mathbb{Z}/p^{v-g_0}\mathbb{Z})^{n+1}.$$

We thus obtained symmetries (10.10) and (10.12). Now, if  $\mathbf{a} \in U$ , we can take  $\mathbf{x} \in \mathfrak{D}_{\mathbf{a}}(\xi, \sigma)$ . Then, the symmetries (10.10) and (10.12) imply

$$\mathfrak{D}_{\mathbf{a}}(\xi, \sigma) \supset \left\{ u\mathbf{x} + p^{g_0}\mathbf{y} \in (\mathbb{Z}/p^v\mathbb{Z})_{\text{prim}}^{n+1} \left| \begin{array}{l} u \in (\mathbb{Z}/p^v\mathbb{Z})^\times \text{ with } u \equiv 1 \pmod{p^{e_p}} \\ \mathbf{y} \in (\mathbb{Z}/p^{v-g_0}\mathbb{Z})^{n+1} \end{array} \right. \right\}.$$

Note that the action of

$$(\mathbf{y}, u) \in G := (\mathbb{Z}/p^{v-g_0}\mathbb{Z})^{n+1} \rtimes \{u \in (\mathbb{Z}/p^v\mathbb{Z})^\times \mid u \equiv 1 \pmod{p^{e_p}}\}$$

to a point  $\mathbf{x} \in (\mathbb{Z}/p^v\mathbb{Z})_{\text{prim}}^{n+1}$  given by  $(\mathbf{y}, u) \cdot \mathbf{x} := u\mathbf{x} + p^{g_0}\mathbf{y}$  has the stabilizer

$$\{(u, p^{-g_0}(1-u)\mathbf{x}) \in G \mid u \in (\mathbb{Z}/p^v\mathbb{Z})^\times \text{ and } u \equiv 1 \pmod{p^{g_0}}\}$$

of the size  $\varphi(p^v)/\varphi(p^{g_0})$ . Thus, we have

$$\#\mathfrak{D}_{\mathbf{a}}(\xi, \sigma) \geq p^{(v-g_0)(n+1)} \frac{\varphi(p^{g_0})}{\varphi(p^{e_p})} \quad \text{if } \mathbf{a} \in U.$$

We therefore have

$$(10.13) \quad \begin{aligned} \#U &\leq p^{-(v-g_0)(n+1)} \frac{\varphi(p^{e_p})}{\varphi(p^{g_0})} \sum_{\mathbf{a} \in (\mathbb{Z}/p^v\mathbb{Z})_{\text{prim}}^{N_{d,n}}} \#\mathfrak{D}_{\mathbf{a}}(\xi, \sigma) \\ &= p^{-(v-g_0)(n+1)} \frac{\varphi(p^{e_p})}{\varphi(p^{g_0})} \sum_{\substack{\mathbf{x} \in (\mathbb{Z}/p^v\mathbb{Z})_{\text{prim}}^{n+1} \\ \mathbf{x} \equiv \boldsymbol{\xi}_p \pmod{p^{e_p}}} } \#K(\mathbf{x}), \end{aligned}$$

where

$$(10.14) \quad K(\mathbf{x}) := \left\{ \mathbf{a} \in (\mathbb{Z}/p^v\mathbb{Z})_{\text{prim}}^{N_{d,n}} \left| \begin{array}{l} \langle \mathbf{a}, \nu_{d,n}(\mathbf{x}) \rangle \equiv 0 \pmod{p^v} \\ \langle \mathbf{a}, \nu_{d,n}^{(i)}(\mathbf{x}) \rangle \equiv 0 \pmod{p^{v_0}} \text{ for all } i \end{array} \right. \right\}.$$

We next bound  $\#K(\mathbf{x})$  for a given  $\mathbf{x} \in (\mathbb{Z}/p^v\mathbb{Z})_{\text{prim}}^{n+1}$ . We may assume  $(x_0, p) = 1$  without loss of generality. Let  $a_0, a_1, \dots, a_n$  be the coordinates of  $\mathbf{a} \in (\mathbb{Z}/p^v)^{N_{d,n}}$  corresponding to the monomials  $X_0^d, X_0^{d-1}X_1, \dots, X_0^{d-1}X_n$  and write the remaining part of  $f_{\mathbf{a}}(\mathbf{x}) = \langle \mathbf{a}, \nu_{d,n}(\mathbf{x}) \rangle$  as

$$g_{\mathbf{a}}(\mathbf{x}) := \sum_{\substack{M \in \mathcal{M}_{d,n} \\ M \notin \{X_0^d, X_0^{d-1}X_1, \dots, X_0^{d-1}X_n\}}} a_M M(\mathbf{x}).$$

Then, the linear equations in (10.14) imply

$$\begin{aligned} \langle \mathbf{a}, \nu_{d,n}(\mathbf{x}) \rangle &\equiv 0 \pmod{p^v} \\ \iff a_0 x_0^d &\equiv -(a_1 x_0^{d-1} x_1 + \dots + a_n x_0^{d-1} x_n) - g_{\mathbf{a}}(\mathbf{x}) \pmod{p^v} \end{aligned}$$

and

$$\langle \mathbf{a}, \nu_{d,n}^{(i)}(\mathbf{x}) \rangle \equiv 0 \pmod{p^{v_0}} \iff a_i x_0^{d-1} \equiv -\frac{\partial g_{\mathbf{a}}}{\partial x_i} \pmod{p^{v_0}}$$

where  $i \in \{1, \dots, n\}$  for the latter equation. Thus, we have

$$(10.15) \quad \#K(\mathbf{x}) \leq p^{v(N_{d,n-1}) - v_0 n}.$$

By combining (10.13) and (10.15), we now have

$$\begin{aligned} \#U &\leq p^{v(N_{d,n-1}) - v_0 n - (v - g_0)(n+1)} \frac{\varphi(p^{e_p})}{\varphi(p^{g_0})} \sum_{\substack{\mathbf{x} \in (\mathbb{Z}/p^v\mathbb{Z})^{n+1} \\ \mathbf{x} \equiv \xi_p \pmod{p^{e_p}}} } 1 \\ &= p^{v(N_{d,n-1}) - v_0 n + (g_0 - e_p)n} \frac{p^{g_0} \varphi(p^{e_p})}{\varphi(p^{g_0}) p^{e_p}} \\ &= p^{v(N_{d,n-1}) - v_0 n + (g_0 - e_p)n} \frac{p \varphi(p^{e_p})}{\varphi(p) p^{e_p}} \end{aligned}$$

since  $g_0 \geq 1$ . For the exponent, by recalling (10.11) of  $g_0$ , we have

$$\begin{aligned} v(N_{d,n-1}) - v_0 n + (g_0 - e_p)n &= v(N_{d,n-1}) - v_0 n + \max\left(\left\lceil \frac{v}{2} \right\rceil - e_p, 0\right)n \\ &= v(N_{d,n-1}) - n \min\left(v_0 - \left\lceil \frac{v}{2} \right\rceil + e_p, v_0\right). \end{aligned}$$

Since  $\min(e_p, v - e_p + 1) \leq \frac{e_p + (v - e_p + 1)}{2} = \frac{v+1}{2}$  so

$$\min(e_p, v - e_p + 1) \leq \left\lceil \frac{v}{2} \right\rceil,$$

the definition (10.4) of  $v_0$  give

$$\min\left(v_0 - \left\lceil \frac{v}{2} \right\rceil + e_p, v_0\right) = \min\left(e_p, \left\lceil \frac{v}{2} \right\rceil + 1, v_0\right) = \min(e_p, v_0) = \tilde{v}.$$

Therefore, we arrive at

$$\#\partial W_p(\xi, \sigma)(p^v) \leq \#U \leq p^{vN_{d,n} - v - n\tilde{v}} \frac{p \varphi(p^{e_p})}{\varphi(p) p^{e_p}}$$

as desired.  $\square$

**Proposition 10.12.** For  $p \in S_{\text{fin}}$ , the local density  $\rho_p(\xi, \sigma) = \mu_{\mathbb{P}_{\mathbb{Q}_p}^{N_{d,n}-1}}(Z_p(\xi, \sigma))$  is bounded as

$$(10.16) \quad \left( \frac{1 - p^{-(N_{d,n}-1)} - p^{-n}}{1 - p^{-N_{d,n}}} \right) \frac{1}{p^{e_p}} \leq \rho_p(\xi, \sigma) \leq \left( \frac{1 - p^{-(N_{d,n}-1)}}{1 - p^{-N_{d,n}}} \right) \frac{1}{p^{e_p}}.$$

*Proof.* Consider the following set:

$$Y := \left\{ \mathbf{a} \in (\mathbb{Z}_p^{N_{d,n}})_{\text{prim}} \mid \langle \mathbf{a}, \nu_{d,n}(\xi_p) \rangle \equiv 0 \pmod{p^{e_p}} \right\}.$$

Then we have

**Claim 10.13.**

- (1)  $Y = \left\{ \mathbf{a} \in (\mathbb{Z}_p^{N_{d,n}})_{\text{prim}} \mid B(\mathbf{a}, p^{-e_p}) \cap W_p(\xi, \sigma) \neq \emptyset \right\}$ .
- (2)  $\mu_p(Y) - p^{-(e_p+n)} \leq \mu_p(W_p(\xi, \sigma)) \leq \mu_p(Y)$ .
- (3)  $\mu_p(Y) = (1 - p^{-(N_{d,n}-1)})p^{-e_p}$ .

*Proof of Claim 10.13.* (1) Let  $Y'$  denote the right hand side. Suppose  $\mathbf{a} \in Y'$ . Take a point  $\mathbf{b} \in B(\mathbf{a}, p^{-e_p}) \cap W_p(\xi, \sigma)$ . Then we have

$$\langle \mathbf{a}, \nu_{d,n}(\xi_p) \rangle \equiv \langle \mathbf{b}, \nu_{d,n}(\xi_p) \rangle \equiv 0 \pmod{p^{e_p}}$$

and thus  $\mathbf{a} \in Y$ . Conversely suppose  $\mathbf{a} \in Y$ . Then we can write  $\langle \mathbf{a}, \nu_{d,n}(\xi_p) \rangle = p^{e_p} \alpha$  for some  $\alpha \in \mathbb{Z}_p$ . We shall find  $\mathbf{c} \in \mathbb{Z}_p^{N_{d,n}}$  such that  $\langle \mathbf{a} + p^{e_p} \mathbf{c}, \nu_{d,n}(\xi_p) \rangle = 0$ . We have  $\langle \mathbf{a} + p^{e_p} \mathbf{c}, \nu_{d,n}(\xi_p) \rangle = p^{e_p} \alpha + p^{e_p} \langle \mathbf{c}, \nu_{d,n}(\xi_p) \rangle$ . Since  $\|\xi_p\|_p = 1$ , we have  $\|\nu_{d,n}(\xi_p)\|_p = 1$ , so  $\langle \cdot, \nu_{d,n}(\xi_p) \rangle: \mathbb{Z}_p^{N_{d,n}} \rightarrow \mathbb{Z}_p$  is surjective. Therefore, we can find  $\mathbf{c} \in \mathbb{Z}_p^{N_{d,n}}$  such that  $\langle \mathbf{c}, \nu_{d,n}(\xi_p) \rangle = -\alpha$  and we are done.

(2) Obviously, we have

$$\bigcup_{\substack{\mathbf{a} \in (\mathbb{Z}_p^{N_{d,n}})_{\text{prim}} \\ B(\mathbf{a}, p^{-e_p}) \subset W_p(\xi, \sigma)}} B(\mathbf{a}, p^{-e_p}) \subset W_p(\xi, \sigma) \subset \bigcup_{\substack{\mathbf{a} \in (\mathbb{Z}_p^{N_{d,n}})_{\text{prim}} \\ B(\mathbf{a}, p^{-e_p}) \cap W_p(\xi, \sigma) \neq \emptyset}} B(\mathbf{a}, p^{-e_p}) = Y.$$

Thus, we have

$$\mu_p(Y) - \mu_p \left( \bigcup_{\mathbf{a} \in \partial W_p(\xi, \sigma)(p^{e_p})} B(\mathbf{a}, p^{-e_p}) \right) \leq \mu_p(W_p(\xi, \sigma)) \leq \mu_p(Y).$$

By (2) of Proposition 10.10, we have

$$\mu_p \left( \bigcup_{\mathbf{a} \in \partial W_p(\xi, \sigma)(p^{e_p})} B(\mathbf{a}, p^{-e_p}) \right) \leq p^{e_p N_{d,n} - e_p - n} \cdot p^{-e_p N_{d,n}} = p^{-(e_p+n)}$$

and we are done.

(3) We have

$$Y = \left\{ \mathbf{a} \in \mathbb{Z}_p^{N_{d,n}} \mid \langle \mathbf{a}, \nu_{d,n}(\xi_p) \rangle \equiv 0 \pmod{p^{e_p}} \right\} \\ \setminus p \left\{ \mathbf{a} \in \mathbb{Z}_p^{N_{d,n}} \mid \langle \mathbf{a}, \nu_{d,n}(\xi_p) \rangle \equiv 0 \pmod{p^{e_p-1}} \right\}.$$

Thus,  $\mu_p(Y) = p^{-e_p} - p^{-(e_p-1)} \cdot p^{-N_{d,n}}$  as desired.  $\square$

From this claim, we obtain

$$\left(1 - \frac{1}{p^{N_{d,n}-1}} - \frac{1}{p^n}\right) \frac{1}{p^{e_p}} \leq \mu_p(W_p(\xi, \sigma)) \leq \left(1 - \frac{1}{p^{N_{d,n}-1}}\right) \frac{1}{p^{e_p}}.$$

Then, from this bound, we can deduce the bound (10.16) by recalling (10.1).  $\square$

**Lemma 10.14.** *We have*

$$\mu_{\mathbb{P}_{\mathbb{Q}_p}^{N_{d,n}-1}}(\mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p) \setminus Z_p) \ll \frac{1}{p^2}$$

where the implicit constant depends only on  $d, n$ .

*Proof.* By (10.1), it is enough to show that

$$\mu_p((\mathbb{Z}_p^{N_{d,n}})_{\text{prim}} \setminus W_p) \ll \frac{1}{p^2}.$$

By Proposition 9.3, we have

$$(\mathbb{Z}_p^{N_{d,n}})_{\text{prim}} \setminus W_p \subset \{\mathbf{a} \in \mathbb{Z}_p^{N_{d,n}} \mid \mathbf{a} \pmod{p} \in \text{NIP}_{d,n}(\mathbb{F}_p)\}$$

provided  $p$  is sufficiently large in terms of  $d$  and  $n$ . Therefore,

$$\begin{aligned} \mu_p((\mathbb{Z}_p^{N_{d,n}})_{\text{prim}} \setminus W_p) &\leq \mu_p(\{\mathbf{a} \in \mathbb{Z}_p^{N_{d,n}} \mid \mathbf{a} \pmod{p} \in \text{NIP}_{d,n}(\mathbb{F}_p)\}) \\ &= \frac{\#\text{NIP}_{d,n}(\mathbb{F}_p)}{p^{N_{d,n}}} \ll \frac{p^{N_{d,n}-2}}{p^{N_{d,n}}} = \frac{1}{p^2}, \end{aligned}$$

where we use Lemma 9.2 to bound  $\#\text{NIP}_{d,n}(\mathbb{F}_p)$ .  $\square$

**10.2. Bounding  $E(A, P)$ .** In this subsection, we prove the following upper bound of  $E(A, P)$ . The idea of the proof is based on the method called Ekedahl sieve, originally carried out in [8]. A quantitative version is developed in [2, Theorem 3.3], and our  $E(A, P)$  has similar form with the set counted there. We interpret our conditions “[ $\mathbf{a}$ ]  $\in Z_p(\xi, \sigma)$ ” and “[ $\mathbf{a}$ ]  $\notin Z_p$ ” into congruence conditions and containment in some algebraic sets. This leads us to more similar setting as in [2, Theorem 3.3], but we have to count lattice points in semialgebraic sets with some congruence conditions.

**Proposition 10.15.** *There exists  $P_0 \in \mathbb{R}_{\geq 1}$  that depends only on  $d, n$  such that for all  $A \geq 1$  and  $P > P_0$ , we have*

$$E(A, P) \ll \frac{1}{P \log 2P} \frac{\sigma_\infty A^{N_{d,n}}}{q} + A^{N_{d,n}-1} \log \log 3A$$

where the implicit constant depends only on  $d, n$ .

We are proving this proposition in several steps.

Recall that we defined the subscheme  $\text{NIP}_{d,n} \subset \mathbb{A}_{\mathbb{Z}}^{N_{d,n}}$ , which contains all non-irreducible polynomials. By Lemma 9.2, we have  $\dim(\text{NIP}_{d,n})_{\mathbb{Q}} \leq N_{d,n} - 2$ , where  $(\text{NIP}_{d,n})_{\mathbb{Q}}$  is the generic fiber over  $\mathbb{Z}$ . In particular, we can pick a linear projection

$$\pi_{\mathbb{Q}}: \mathbb{A}_{\mathbb{Q}}^{N_{d,n}} \longrightarrow \mathbb{A}_{\mathbb{Q}}^{N_{d,n}-1}$$

such that the induced morphism

$$(\text{NIP}_{d,n})_{\mathbb{Q}} \longrightarrow \mathbb{A}_{\mathbb{Q}}^{N_{d,n}-1}$$

is finite. By composition with a linear automorphism if necessary, we may assume that there is a linear morphism  $\pi: \mathbb{A}_{\mathbb{Z}}^{N_{d,n}} \rightarrow \mathbb{A}_{\mathbb{Z}}^{N_{d,n}-1}$  such that the base change of  $\pi$  to  $\mathbb{Q}$  is  $\pi_{\mathbb{Q}}$ . By a standard argument of “passage to the limit”, there exists  $R$ , a localization of  $\mathbb{Z}$  by a single element, with the following properties. Consider the following diagram:

$$\begin{array}{ccccc}
(\mathrm{NIP}_{d,n})_{\mathbb{Q}} & \longrightarrow & (\mathrm{NIP}_{d,n})_R & \longrightarrow & \mathrm{NIP}_{d,n} \\
\cap & & \cap & & \cap \\
\mathbb{A}_{\mathbb{Q}}^{N_{d,n}} & \longrightarrow & \mathbb{A}_R^{N_{d,n}} & \longrightarrow & \mathbb{A}_{\mathbb{Z}}^{N_{d,n}} \\
\pi_{\mathbb{Q}} \downarrow & & \pi_R \downarrow & & \downarrow \pi \\
\mathbb{A}_{\mathbb{Q}}^{N_{d,n}-1} & \longrightarrow & \mathbb{A}_R^{N_{d,n}-1} & \longrightarrow & \mathbb{A}_{\mathbb{Z}}^{N_{d,n}-1}
\end{array}$$

where  $(\mathrm{NIP}_{d,n})_R, \pi_R$  are the base change to  $R$ . Then  $\pi_R|_{(\mathrm{NIP}_{d,n})_R}$  is a finite morphism and the coordinate ring  $\mathcal{O}((\mathrm{NIP}_{d,n})_R)$  of  $(\mathrm{NIP}_{d,n})_R$  is generated by, say,  $r$  elements over  $\mathcal{O}(\mathbb{A}_R^{N_{d,n}-1})$ . Since  $\dim(\mathrm{NIP}_{d,n})_{\mathbb{Q}} \leq N_{d,n} - 2$ ,  $\mathrm{NIP}_{d,n}$  does not dominate  $\mathbb{A}_{\mathbb{Z}}^{N_{d,n}-1}$ . We pick a non-zero polynomial  $f \in \mathcal{O}(\mathbb{A}_{\mathbb{Z}}^{N_{d,n}-1})$  such that

$$\pi(\mathrm{NIP}_{d,n}) \subset V(f) \subset \mathbb{A}_{\mathbb{Z}}^{N_{d,n}-1}.$$

Here  $V(f)$  is the closed subscheme of  $\mathbb{A}_{\mathbb{Z}}^{N_{d,n}-1}$  defined by  $f$ .

Now we take  $P_0 \in \mathbb{R}_{\geq 1}$  so that Proposition 9.3 works for this  $P_0$  and also any prime  $p > P_0$  generates a prime ideal of  $R$ .

Choosing  $P_0$  as above, for  $A \geq 1$  and  $P > P_0$ , we can use Proposition 9.3 to get

$$E(A, P) \leq \# \left\{ \mathbf{a} \in \mathbb{Z}^{N_{d,n}} \left| \begin{array}{l} 0 < \|\mathbf{a}\| \leq A \\ \forall p \in S, [\mathbf{a}] \in Z_p(\xi, \sigma) \\ \exists p \in (P, \infty) \setminus S \text{ s.t. } \mathbf{a} \pmod{p} \in \mathrm{NIP}_{d,n}(\mathbb{F}_p) \end{array} \right. \right\}.$$

We then decompose the sum into two parts according to the size of “bad”  $p$  as

$$E(A, P) \leq E_1(A, P) + E_2(A, P),$$

where

$$\begin{aligned}
E_1(A, P) &:= \# \left\{ \mathbf{a} \in \mathbb{Z}^{N_{d,n}} \left| \begin{array}{l} 0 < \|\mathbf{a}\| \leq A \\ \forall p \in S, [\mathbf{a}] \in Z_p(\xi, \sigma) \\ \exists p \in (P, A] \setminus S \text{ s.t. } \mathbf{a} \pmod{p} \in \mathrm{NIP}_{d,n}(\mathbb{F}_p) \end{array} \right. \right\}, \\
E_2(A, P) &:= \# \left\{ \mathbf{a} \in \mathbb{Z}^{N_{d,n}} \left| \begin{array}{l} \|\mathbf{a}\| \leq A \\ \exists p \in (\max\{P, A\}, \infty) \text{ s.t. } \mathbf{a} \pmod{p} \in \mathrm{NIP}_{d,n}(\mathbb{F}_p) \end{array} \right. \right\}.
\end{aligned}$$

Note that we weakened the condition on  $\mathbf{a}$  in the definition of  $E_2(A, P)$ .

We first bound  $E_2(A, P)$ .

**Lemma 10.16.** *Let  $P_0$  as above. For  $A \geq 1$  and  $P > P_0$ , we have*

$$E_2(A, P) \ll A^{N_{d,n}-1},$$

where the implicit constant depends only on  $d, n$ .

*Proof.* We may assume  $A \geq 2$ . We further divide  $E_2(A, P)$  into two parts as follows:

$$E_2(A, P) \leq E_{21} + E_{22},$$

where

$$E_{21} := \# \left\{ \mathbf{a} \in \mathbb{Z}^{N_{d,n}} \left| \begin{array}{l} \|\mathbf{a}\| \leq A \text{ and } f(\pi(\mathbf{a})) \neq 0 \\ \exists p \in (\max\{P, A\}, \infty) \text{ s.t. } \mathbf{a} \pmod{p} \in \text{NIP}_{d,n}(\mathbb{F}_p) \end{array} \right. \right\},$$

$$E_{22} := \#\{\mathbf{a} \in \mathbb{Z}^{N_{d,n}} \mid \|\mathbf{a}\| \leq A \text{ and } f(\pi(\mathbf{a})) = 0\}.$$

Since  $f \circ \pi$  is a non-zero polynomial function on  $\mathbb{A}_{\mathbb{Z}}^{N_{d,n}}$ , we have

$$(10.17) \quad E_{22} \ll A^{N_{d,n}-1},$$

where the implicit constant depends only on  $d, n, \pi, f$ .

Since  $\pi: \mathbb{Z}^{N_{d,n}} \rightarrow \mathbb{Z}^{N_{d,n}-1}$  is a linear map, there is a non-negative real number which we denote by  $\|\pi\|$  such that  $\|\pi(\mathbf{x})\| \leq \|\pi\| \|\mathbf{x}\|$  for all  $\mathbf{x} \in \mathbb{Z}^{N_{d,n}}$ . We have

$$(10.18) \quad \begin{aligned} E_{21} &\leq \sum_{p > \max\{P, A\}} \sum_{\substack{\mathbf{a} \in \mathbb{Z}^{N_{d,n}} \\ \|\mathbf{a}\| \leq A \\ f(\pi(\mathbf{a})) \neq 0 \\ \mathbf{a} \pmod{p} \in \text{NIP}_{d,n}(\mathbb{F}_p)}} 1 \\ &\leq \sum_{\substack{\mathbf{b} \in \mathbb{Z}^{N_{d,n}-1} \\ f(\mathbf{b}) \neq 0 \\ \|\mathbf{b}\| \leq \|\pi\| A}} \sum_{\substack{p > \max\{P, A\} \\ p \mid f(\mathbf{b})}} \sum_{\substack{\mathbf{a} \in \mathbb{Z}^{N_{d,n}} \\ \|\mathbf{a}\| \leq A \\ \mathbf{a} \pmod{p} \in \text{NIP}_{d,n}(\mathbb{F}_p) \\ \pi(\mathbf{a}) = \mathbf{b}}} 1. \end{aligned}$$

Since  $p > \max\{P, A\} \geq P > P_0$ , every fiber of

$$\pi: \text{NIP}_{d,n}(\mathbb{F}_p) \rightarrow \mathbb{A}^{N_{d,n}-1}(\mathbb{F}_p)$$

consists at most  $r$  elements. Noting  $p > A$ , we get

$$\sum_{\substack{\mathbf{a} \in \mathbb{Z}^{N_{d,n}} \\ \|\mathbf{a}\| \leq A \\ \mathbf{a} \pmod{p} \in \text{NIP}_{d,n}(\mathbb{F}_p) \\ \pi(\mathbf{a}) = \mathbf{b}}} 1 \ll 1$$

where the implicit constant depends only on  $d, n, r$ . Thus (10.18) is bounded as

$$\ll \sum_{\substack{\mathbf{b} \in \mathbb{Z}^{N_{d,n}-1} \\ f(\mathbf{b}) \neq 0 \\ \|\mathbf{b}\| \leq \|\pi\| A}} \sum_{\substack{p > \max\{P, A\} \\ p \mid f(\mathbf{b})}} 1.$$

Let

$$\alpha(\mathbf{b}) = \sum_{\substack{p > \max\{P, A\} \\ p \mid f(\mathbf{b})}} 1.$$

Then, for  $\mathbf{b} \in \mathbb{Z}^{N_{d,n}-1}$  with  $f(\mathbf{b}) \neq 0$ , we get

$$A^{\alpha(\mathbf{b})} \leq \prod_{\substack{p > \max\{P, A\} \\ p \mid f(\mathbf{b})}} p \leq |f(\mathbf{b})| \ll A^{\deg f} \quad \text{so} \quad \alpha(\mathbf{b}) \ll 1,$$

where the implicit constant depends only on  $\pi, f$ . Therefore, we get

$$(10.19) \quad E_{21} \ll \sum_{\substack{\mathbf{b} \in \mathbb{Z}^{N_{d,n}-1} \\ f(\mathbf{b}) \neq 0 \\ \|\mathbf{b}\| \leq \|\pi\|A}} 1 \ll A^{N_{d,n}-1}$$

where the implicit constants depend only on  $d, n, r, \pi, f$ .

Thus, by combining (10.17) and (10.19), we get

$$E_2(A, P) \ll A^{N_{d,n}-1}$$

where the implicit constant depends only on  $d, n, r, \pi, f$ . Since we chose  $\pi, f, r$  depending only on  $d, n$ , we are done.  $\square$

Next we are going to bound  $E_1(A, P)$ .

**Lemma 10.17.** *For  $A \geq 1$  and  $P \geq 1$ , we have*

$$E_1(A, P) \ll \frac{1}{P \log 2P} \frac{\sigma_\infty A^{N_{d,n}}}{q} + A^{N_{d,n}-1} \log \log 3A,$$

where the implicit constant depends only on  $d, n$ .

*Proof.* Note that  $[\mathbf{a}] \in Z_\infty(\xi, \sigma) \iff \mathbf{a} \in \mathbb{R}W_\infty(\xi, \sigma)$ . For the condition  $[\mathbf{a}] \in Z_\infty(\xi, \sigma)$  with  $p \in S_{\text{fin}}$ , we use Proposition 10.8. By Chinese remainder theorem, we can pick a  $\xi_0 \in \mathbb{Z}^{N_{d,n}}$  such that

$$\xi_0 \equiv \xi_p \pmod{p^{e_p}} \quad \text{for } p \in S_{\text{fin}}.$$

Note that  $\gcd(q, \xi_0) = 1$  since  $\|\xi_p\|_p = 1$ . We then get

$$E_1(A, P) \leq \# \left\{ \mathbf{a} \in \mathbb{Z}^{N_{d,n}} \left| \begin{array}{l} \mathbf{a} \in \mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma) \\ \langle \mathbf{a}, \nu_{d,n}(\xi_0) \rangle \equiv 0 \pmod{q} \\ \exists p \in (P, A] \setminus S \text{ s.t. } \mathbf{a} \pmod{p} \in \text{NIP}_{d,n}(\mathbb{F}_p) \end{array} \right. \right\}.$$

Let

$$\Lambda := \{ \mathbf{a} \in \mathbb{Z}^{N_{d,n}} \mid \langle \mathbf{a}, \nu_{d,n}(\xi_0) \rangle \equiv 0 \pmod{q} \}.$$

This is a lattice of rank  $N_{d,n}$ . Using this lattice, we see

$$(10.20) \quad E_1(A, P) \leq \sum_{\substack{P < p \leq A \\ p \notin S}} \sum_{x \in \text{NIP}_{d,n}(\mathbb{F}_p)} \#\mathcal{N}(p, x),$$

where

$$\mathcal{N}(p, x) := \{ \mathbf{a} \in \Lambda \cap \mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma) \mid \mathbf{a} \pmod{p} = x \}.$$

The set  $\mathcal{N}(p, x)$  is non-empty only if

$$\#\{ \mathbf{a} \in \Lambda \mid \mathbf{a} \pmod{p} = x \} \neq \emptyset.$$

When this is the case, pick  $\mathbf{a}_0 \in \Lambda$  such that  $\mathbf{a}_0 \pmod{p} = x$ . Then we have

$$\begin{aligned} \mathcal{N}(p, x) &= \Lambda \cap \mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma) \cap (\mathbf{a}_0 + p\mathbb{Z}^{N_{d,n}}) \\ &= (\mathbf{a}_0 + p\Lambda) \cap \mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma), \end{aligned}$$

where we use  $\Lambda \cap p\mathbb{Z}^{N_{d,n}} = p\Lambda$ , which follows from  $p \nmid q$ . Thus,

$$\#\mathcal{N}(p, x) = \# \left( p\Lambda \cap ((\mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma)) - \mathbf{a}_0) \right).$$

We use Lemma 3.8 to estimate this number. By Lemma 10.5, the set

$$(\mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma)) - \mathbf{a}_0$$

comes from a semi-algebraic family of subsets of  $\mathbb{R}^{N_{d,n}}$  parametrized by  $A, \sigma_\infty, \xi_\infty$ , and  $\mathbf{a}_0$ . By Lemma 3.8, we get

$$\begin{aligned} & \#\mathcal{N}(p, x) \\ &= \frac{\text{vol}_{N_{d,n}}(\mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma))}{\det(p\Lambda)} + O\left(\sum_{\nu=0}^{N_{d,n}-1} \frac{V_\nu(\mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma))}{\lambda_1(p\Lambda) \cdots \lambda_\nu(p\Lambda)}\right) \\ &\ll \frac{\sigma_\infty A^{N_{d,n}}}{p^{N_{d,n}} \det \Lambda} + O\left(\sum_{\nu=0}^{N_{d,n}-1} \frac{A^\nu}{p^\nu}\right) \ll \frac{\sigma_\infty A^{N_{d,n}}}{qp^{N_{d,n}}} + \frac{A^{N_{d,n}-1}}{p^{N_{d,n}-1}}, \end{aligned}$$

where we use Proposition 10.6 for the first “ $\ll$ ” and the fact  $\det \Lambda = q$  and  $p \leq A$  for the second “ $\ll$ ”.

Plugging the above estimate into (10.20), we get

$$E_1(A, P) \ll \sum_{P < p \leq A} \#\text{NIP}_{d,n}(\mathbb{F}_p) \left( \frac{\sigma_\infty A^{N_{d,n}}}{qp^{N_{d,n}}} + \frac{A^{N_{d,n}-1}}{p^{N_{d,n}-1}} \right).$$

Since  $\dim(\text{NIP}_{d,n})_{\mathbb{Q}} \leq N_{d,n} - 2$  (Lemma 9.2), we have

$$\#\text{NIP}_{d,n}(\mathbb{F}_p) \ll p^{N_{d,n}-2}.$$

Therefore, by using the bounds

$$\sum_{P < p \leq A} \frac{1}{p^2} \leq \sum_{P < p} \frac{1}{p^2} \ll \frac{1}{P \log 2P} \quad \text{and} \quad \sum_{P < p \leq A} \frac{1}{p} \leq \sum_{p \leq A} \frac{1}{p} \ll \log \log 3A,$$

which can be proven by the prime number theorem, we get

$$\begin{aligned} E_1(A, P) &\ll \sum_{P < p \leq A} \left( \frac{\sigma_\infty A^{N_{d,n}}}{q} \frac{1}{p^2} + A^{N_{d,n}-1} \frac{1}{p} \right) \\ &\ll \frac{1}{P \log 2P} \frac{\sigma_\infty A^{N_{d,n}}}{q} + A^{N_{d,n}-1} \log \log 3A. \end{aligned}$$

This completes the proof.  $\square$

*Proof of Proposition 10.15.* This follows from Lemma 10.17 and Lemma 10.16.  $\square$

**10.3. Asymptotic formula for  $M(A, P)$ .** Recall that

$$\begin{aligned} M(A, P) &:= \#\{\mathbf{a} \in \mathbb{Z}_{\text{prim}}^{N_{d,n}} \mid \|\mathbf{a}\| \leq A \quad \text{and} \quad \forall p \in S \cup M_{\mathbb{Q}, \leq P}, [\mathbf{a}] \in Z_p(\xi, \sigma)\} \\ &= \#\left\{ \mathbf{a} \in \mathbb{Z}_{\text{prim}}^{N_{d,n}} \left| \begin{array}{l} \|\mathbf{a}\| \leq A, \quad \mathbf{a} \in \mathbb{R}W_\infty(\xi, \sigma) \\ \forall p \in S_{\text{fin}} \cup M_{\mathbb{Q}, \leq P}, \quad \mathbf{a} \in W_p(\xi, \sigma) \end{array} \right. \right\}. \end{aligned}$$

We are going to get an asymptotic formula for  $M(A, P)$ . The difficulty is that we do not know the precise shape of the subsets  $W_p(\xi, \sigma) \subset \mathbb{Z}_p^{N_{d,n}}$ . We approximate the set  $W_p(\xi, \sigma)$  by finitely many disjoint balls with the same radius from inside and outside. Being contained in a  $p$ -adic ball is equivalent to a certain congruence condition, so we can apply Proposition 5.8 to count lattice points.

Let us start with preparing the balls. Consider the following two sets of balls in  $\mathbb{Z}_p^{N_{d,n}}$ , which approximate the sets  $W_p(\xi, \sigma)$  from inside and outside: for prime  $p$  and  $v_p \in \mathbb{Z}_{\geq 1}$ , we define

$$\begin{aligned}\Omega_0(p^{v_p}) &:= \{B(\mathbf{c}, p^{-v_p}) \mid \mathbf{c} \in \mathbb{Z}_p^{N_{d,n}}, B(\mathbf{c}, p^{-v_p}) \subset W_p(\xi, \sigma)\}, \\ \Omega_1(p^{v_p}) &:= \{B(\mathbf{c}, p^{-v_p}) \mid \mathbf{c} \in \mathbb{Z}_p^{N_{d,n}}, B(\mathbf{c}, p^{-v_p}) \cap W_p(\xi, \sigma) \neq \emptyset\}.\end{aligned}$$

**Lemma 10.18.** *Let  $p$  be a prime. For  $v_p \in \mathbb{Z}_{\geq 1}$ , we set*

$$\tilde{v}_p := \min(e_p, v_p - e_p + 1) \quad \text{and} \quad \nu(v_p, e_p) := v_p + n\tilde{v}_p.$$

Then, the following hold:

(1) We have

$$\#(\Omega_1(p^{v_p}) \setminus \Omega_0(p^{v_p})) \leq \frac{2p^{v_p N_{d,n}}}{p^{\nu(v_p, e_p)}}.$$

(2) For the local density  $\rho_p(\xi, \sigma) := \mu_{\mathbb{P}_{\mathbb{Q}_p}^{N_{d,n}-1}}(Z_p(\xi, \sigma))$ , we have

$$\begin{aligned}\frac{\#\Omega_1(p^{v_p})}{p^{v_p N_{d,n}}} &\leq (1 - p^{-N_{d,n}})\rho_p(\xi, \sigma) + \frac{2}{p^{\nu(v_p, e_p)}}, \\ \frac{\#\Omega_0(p^{v_p})}{p^{v_p N_{d,n}}} &\geq (1 - p^{-N_{d,n}})\rho_p(\xi, \sigma) - \frac{2}{p^{\nu(v_p, e_p)}}.\end{aligned}$$

*Proof.* (1) follows from Proposition 10.10 since

$$\frac{p\varphi(p^{e_p})}{\varphi(p)p^{e_p}} \leq \frac{p}{p-1} \leq 2.$$

For (2), recall that

$$\mu_p(B(\mathbf{c}, p^{-v_p})) = p^{-v_p N_{d,n}}$$

for all  $\mathbf{c} \in \mathbb{Z}_p^{N_{d,n}}$ . Thus, combined with (1), we get

$$\begin{aligned}\mu_p(W_p(\xi, \sigma)) &\geq \frac{\#\Omega_0(p^{v_p})}{p^{v_p N_{d,n}}} \geq \frac{\#\Omega_1(p^{v_p})}{p^{v_p N_{d,n}}} - \frac{2}{p^{\nu(v_p, e_p)}}, \\ \mu_p(W_p(\xi, \sigma)) &\leq \frac{\#\Omega_1(p^{v_p})}{p^{v_p N_{d,n}}} \leq \frac{\#\Omega_0(p^{v_p})}{p^{v_p N_{d,n}}} + \frac{2}{p^{\nu(v_p, e_p)}}.\end{aligned}$$

We then obtain (2) by recalling (10.1).  $\square$

**Proposition 10.19.** *Let  $v_p \in \mathbb{Z}_{\geq 1}$  for each  $p \in S_{\text{fin}} \cup M_{\mathbb{Q}, \leq P}$ . Set*

$$q' := \prod_{p \in S_{\text{fin}} \cup M_{\mathbb{Q}, \leq P}} p^{v_p}.$$

Recall the definitions of the local densities

$$\rho_\infty(\xi, \sigma) = \mu_{S^{N_{d,n}-1}}(W_\infty(\xi, \sigma)) \quad \text{and} \quad \rho_p(\xi, \sigma) = \mu_{\mathbb{P}_{\mathbb{Q}_p}^{N_{d,n}-1}}(Z_p(\xi, \sigma)) \quad \text{for } p \neq \infty.$$

For  $A, P \geq 1$ , we have

$$M(A, P) \leq \frac{V_{N_{d,n}} \prod_{p \in M_{\mathbb{Q}}} \rho_p(\xi, \sigma)}{\zeta(N_{d,n})} A^{N_{d,n}} \Theta_{d,n}^+(\xi, \sigma) \left(1 + O\left(\frac{1}{P \log 2P}\right)\right)$$

$$\times \left( 1 + O \left( \frac{J_{N_{d,n}}(q')}{\rho_\infty(\xi, \sigma) q'^{N_{d,n}-1} A} + \frac{J_{N_{d,n}}(q') \log 2A}{\rho_\infty(\xi, \sigma) \varphi(q') A^{N_{d,n}-1}} \right) \right)$$

and

$$M(A, P) \geq \frac{V_{N_{d,n}} \prod_{p \in M_{\mathbb{Q}}} \rho_p(\xi, \sigma)}{\zeta(N_{d,n})} A^{N_{d,n}} \Theta_{d,n}^-(\xi, \sigma) \left( 1 + O \left( \frac{1}{P \log 2P} \right) \right) \\ \times \left( 1 + O \left( \frac{J_{N_{d,n}}(q')}{\rho_\infty(\xi, \sigma) q'^{N_{d,n}-1} A} + \frac{J_{N_{d,n}}(q') \log 2A}{\rho_\infty(\xi, \sigma) \varphi(q') A^{N_{d,n}-1}} \right) \right),$$

where

$$\Theta_{d,n}^\pm(\xi, \sigma) := \prod_{p \in S_{\text{fin}} \cup M_{\mathbb{Q}, \leq P}} \left( 1 \pm \frac{2}{(1 - p^{-N_{d,n}}) \rho_p(\xi, \sigma) p^{\nu(v_p, e_p)}} \right)$$

and the implicit constants depend only on  $d, n$ .

*Proof.* Let us write  $\mathcal{S} = S_{\text{fin}} \cup M_{\mathbb{Q}, \leq P}$ . We have

$$M(A, P) \leq \sum_{(B_p) \in \prod_{p \in \mathcal{S}} \Omega_1(p^{v_p})} \# \left\{ \mathbf{a} \in \mathbb{Z}_{\text{prim}}^{N_{d,n}} \mid \begin{array}{l} \mathbf{a} \in \mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma) \\ \forall p \in \mathcal{S}, \mathbf{a} \in B_p \end{array} \right\}, \\ M(A, P) \geq \sum_{(B_p) \in \prod_{p \in \mathcal{S}} \Omega_0(p^{v_p})} \# \left\{ \mathbf{a} \in \mathbb{Z}_{\text{prim}}^{N_{d,n}} \mid \begin{array}{l} \mathbf{a} \in \mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma) \\ \forall p \in \mathcal{S}, \mathbf{a} \in B_p \end{array} \right\}.$$

We identify a ball in  $\mathbb{Z}_p^{N_{d,n}}$  of radius  $p^{-v_p}$  with an element of  $(\mathbb{Z}/p^{v_p}\mathbb{Z})^{N_{d,n}}$ . By Chinese remainder theorem, we have  $\prod_{p \in \mathcal{S}} (\mathbb{Z}/p^{v_p}\mathbb{Z})^{N_{d,n}} \simeq (\mathbb{Z}/q'\mathbb{Z})^{N_{d,n}}$ . Let  $\mathcal{C}_i$  ( $i = 0, 1$ ) be the image of  $\prod_{p \in \mathcal{S}} \Omega_i(p^{v_p})$  ( $i = 0, 1$ ) by this bijection:

$$\begin{array}{ccc} \prod_{p \in \mathcal{S}} (\mathbb{Z}/p^{v_p}\mathbb{Z})^{N_{d,n}} & \xrightarrow{\sim} & (\mathbb{Z}/q'\mathbb{Z})^{N_{d,n}} \\ \cup & & \cup \\ \prod_{p \in \mathcal{S}} (\mathbb{Z}/p^{v_p}\mathbb{Z})_{\text{prim}}^{N_{d,n}} & \xrightarrow{\sim} & (\mathbb{Z}/q'\mathbb{Z})_{\text{prim}}^{N_{d,n}} \\ \cup & & \cup \\ \prod_{p \in \mathcal{S}} \Omega_i(p^{v_p}) & \xrightarrow{\sim} & \mathcal{C}_i \end{array}$$

Here note that all the elements of  $\Omega_i(p^{v_p})$  are contained in  $(\mathbb{Z}/p^{v_p}\mathbb{Z})_{\text{prim}}^{N_{d,n}}$  because  $W_p(\xi, \sigma) \subset (\mathbb{Z}_p^{N_{d,n}})_{\text{prim}}$ . Also, by the definition of  $W_p(\xi, \sigma)$ , the sets  $\mathcal{C}_i$  are stable under multiplication by any elements of  $(\mathbb{Z}/q'\mathbb{Z})^\times$ . With this notation, we get

$$M(A, P) \\ \leq \sum_{\mathbf{c} \in \mathcal{C}_1} \# \left\{ \mathbf{a} \in \mathbb{Z}_{\text{prim}}^{N_{d,n}} \mid \begin{array}{l} \mathbf{a} \in \mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma) \\ \mathbf{a} \equiv \mathbf{c} \pmod{q'} \end{array} \right\} \\ = \sum_{\mathbf{c} \in \mathcal{C}_1} \# \left( \mathbb{Z}_{\text{prim}}^{N_{d,n}} \cap \mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma) \cap (\mathbf{c} + q'\mathbb{Z}^{N_{d,n}}) \right)$$

$$\begin{aligned}
&= \frac{1}{\varphi(q')} \sum_{u \in (\mathbb{Z}/q'\mathbb{Z})^\times} \sum_{\mathbf{c} \in \mathcal{C}_1} \# \left( \mathbb{Z}_{\text{prim}}^{N_{d,n}} \cap \mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma) \cap (u\mathbf{c} + q'\mathbb{Z}^{N_{d,n}}) \right) \\
&= \frac{1}{\varphi(q')} \sum_{\mathbf{c} \in \mathcal{C}_1} \sum_{u \in (\mathbb{Z}/q'\mathbb{Z})^\times} \# \left( \mathbb{Z}_{\text{prim}}^{N_{d,n}} \cap \mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma) \cap (u\mathbf{c} + q'\mathbb{Z}^{N_{d,n}}) \right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&M(A, P) \\
&= \frac{1}{\varphi(q')} \sum_{\mathbf{c} \in \mathcal{C}_0} \sum_{u \in (\mathbb{Z}/q'\mathbb{Z})^\times} \# \left( \mathbb{Z}_{\text{prim}}^{N_{d,n}} \cap \mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma) \cap (u\mathbf{c} + q'\mathbb{Z}^{N_{d,n}}) \right).
\end{aligned}$$

Note that  $\mathbb{R}W_\infty(\xi, \sigma)$  form a semialgebraic family of homogeneous sets when  $\xi$  and  $\sigma$  varies. Applying Proposition 5.8 with  $\Gamma = \Lambda = \mathbb{Z}^{N_{d,n}}$ , for all  $A \geq 1$  we have

$$\begin{aligned}
&\sum_{u \in (\mathbb{Z}/q'\mathbb{Z})^\times} \# \left( \mathbb{Z}_{\text{prim}}^{N_{d,n}} \cap \mathcal{B}_{N_{d,n}}(A) \cap \mathbb{R}W_\infty(\xi, \sigma) \cap (u\mathbf{c} + q'\mathbb{Z}^{N_{d,n}}) \right) \\
&= \frac{\varphi(q') \text{vol}_{N_{d,n}}(\mathbb{R}W_\infty(\xi, \sigma) \cap \mathcal{B}_{N_{d,n}}(1)) A^{N_{d,n}}}{J_{N_{d,n}}(q') \zeta(N_{d,n})} + O \left( \frac{\varphi(q') A^{N_{d,n}-1}}{q'^{N_{d,n}-1}} + A \log 2A \right) \\
&= \frac{\varphi(q') \rho_\infty(\xi, \sigma) V_{N_{d,n}} A^{N_{d,n}}}{J_{N_{d,n}}(q') \zeta(N_{d,n})} + O \left( \frac{\varphi(q') A^{N_{d,n}-1}}{q'^{N_{d,n}-1}} + A \log 2A \right),
\end{aligned}$$

where the implicit constants depend only on  $d, n$ . Therefore we get

$$\begin{aligned}
(10.21) \quad M(A, P) &\leq \#\mathcal{C}_1 \left( \frac{\rho_\infty(\xi, \sigma) V_{N_{d,n}} A^{N_{d,n}}}{J_{N_{d,n}}(q') \zeta(N_{d,n})} + O \left( \frac{A^{N_{d,n}-1}}{q'^{N_{d,n}-1}} + \frac{A \log 2A}{\varphi(q')} \right) \right), \\
M(A, P) &\geq \#\mathcal{C}_0 \left( \frac{\rho_\infty(\xi, \sigma) V_{N_{d,n}} A^{N_{d,n}}}{J_{N_{d,n}}(q') \zeta(N_{d,n})} + O \left( \frac{A^{N_{d,n}-1}}{q'^{N_{d,n}-1}} + \frac{A \log 2A}{\varphi(q')} \right) \right).
\end{aligned}$$

Now by Lemma 10.18 we have

$$\begin{aligned}
\#\mathcal{C}_1 &= \prod_{p \in \mathcal{S}} \#\Omega_1(p^{v_p}) \leq \prod_{p \in \mathcal{S}} p^{v_p N_{d,n}} \left( (1 - p^{-N_{d,n}}) \rho_p(\xi, \sigma) + \frac{2}{p^{\nu(v_p, e_p)}} \right) \\
&= J_{N_{d,n}}(q') \prod_{p \in \mathcal{S}} \rho_p(\xi, \sigma) \cdot \Theta_{d,n}^+(\xi, \sigma)
\end{aligned}$$

and

$$\begin{aligned}
\#\mathcal{C}_0 &= \prod_{p \in \mathcal{S}} \#\Omega_0(p^{v_p}) \geq \prod_{p \in \mathcal{S}} p^{v_p N_{d,n}} \left( (1 - p^{-N_{d,n}}) \rho_p(\xi, \sigma) - \frac{2}{p^{\nu(v_p, e_p)}} \right) \\
&= J_{N_{d,n}}(q') \prod_{p \in \mathcal{S}} \rho_p(\xi, \sigma) \cdot \Theta_{d,n}^-(\xi, \sigma).
\end{aligned}$$

Plugging these bounds into (10.21), we get

$$\begin{aligned}
(10.22) \quad M(A, P) &\leq \prod_{p \in \mathcal{S}} \rho_p(\xi, \sigma) \Theta_{d,n}^+(\xi, \sigma) (M_0 + O(M_E)), \\
M(A, P) &\geq \prod_{p \in \mathcal{S}} \rho_p(\xi, \sigma) \Theta_{d,n}^-(\xi, \sigma) (M_0 + O(M_E)).
\end{aligned}$$

where

$$M_0 := \frac{\rho_\infty(\xi, \sigma) V_{N_{d,n}} A^{N_{d,n}}}{\zeta(N_{d,n})},$$

$$M_E := \frac{J_{N_{d,n}}(q') A^{N_{d,n}-1}}{q'^{N_{d,n}-1}} + \frac{J_{N_{d,n}}(q') A \log 2A}{\varphi(q')}.$$

We then replace  $\prod_{p \in \mathcal{S}} \rho_p(\xi, \sigma)$  with  $\prod_{p \in M_{\mathbb{Q}} \setminus \{\infty\}} \rho_p(\xi, \sigma)$ . By using

$$\prod_{p > P} \left(1 + O\left(\frac{1}{p^2}\right)\right) = 1 + O\left(\frac{1}{P \log 2P}\right) \quad \text{for } P \geq 1,$$

which can be proven by the prime number theorem, and by Lemma 10.14, we have

$$\begin{aligned} \prod_{\substack{p > P \\ p \notin S}} \rho_p(\xi, \sigma) &= \prod_{\substack{p > P \\ p \notin S}} \mu_{\mathbb{P}_{\mathbb{Q}_p}^{N_{d,n}-1}}(Z_p) = \prod_{\substack{p > P \\ p \notin S}} \left(1 - \mu_{\mathbb{P}_{\mathbb{Q}_p}^{N_{d,n}-1}}(\mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p) \setminus Z_p)\right) \\ &= 1 + O\left(\frac{1}{P \log 2P}\right) \end{aligned}$$

for  $P \geq 1$ . Replacing  $\prod_{p \in \mathcal{S}} \rho_p(\xi, \sigma)$  with

$$\left(\prod_{\substack{p \in M_{\mathbb{Q}} \\ p < \infty}} \rho_p(\xi, \sigma)\right) \left(\prod_{\substack{p > P \\ p \notin S}} \rho_p(\xi, \sigma)\right)^{-1} = \left(\prod_{\substack{p \in M_{\mathbb{Q}} \\ p < \infty}} \rho_p(\xi, \sigma)\right) \left(1 + O\left(\frac{1}{P \log 2P}\right)\right)$$

in (10.22), we get the desired inequalities.  $\square$

Now we are going to choose  $v_p$ 's, which means we are adjusting the radii of the small balls utilized to approximate the sets  $W_p(\xi, \sigma)$ . Note that as  $v_p$ 's increase,  $\nu(v_p, e_p)$  and  $q'$  are getting large. This means the product factor

$$\Theta_{d,n}^{\pm}(\xi, \sigma) = \prod_{p \in S_{\text{fin}} \cup M_{\mathbb{Q}, \leq P}} \left(1 \pm \frac{2}{(1 - p^{-N_{d,n}}) \rho_p(\xi, \sigma) p^{\nu(v_p, e_p)}}\right)$$

in Proposition 10.19 approaches 1 as we make  $v_p$ 's larger. However, this increase in  $v_p$ 's also make the big-O terms in Proposition 10.19 large. Therefore, our task is to choose  $v_p$ 's in a manner that balances these effects. We are choosing  $v_p$ 's as in the form specified in the following lemma.

**Lemma 10.20.** *Take  $g_p \geq 1$  for each  $p \leq P$  and set*

$$v_p := \begin{cases} g_p & \text{when } p \leq P \text{ and } p \notin S, \\ e_p + g_p & \text{when } p \leq P \text{ and } p \in S, \\ e_p & \text{when } p > P \text{ and } p \in S. \end{cases}$$

Note that then we have  $v_p \in \mathbb{Z}_{\geq 1}$  and  $v_p \geq e_p$  for all  $p < \infty$ . Then, we have

$$\begin{aligned} \nu(v_p, e_p) &= g_p && \text{when } p \leq P \text{ and } p \notin S, \\ \nu(v_p, e_p) &\geq e_p + g_p + n && \text{when } p \leq P \text{ and } p \in S, \\ \nu(v_p, e_p) &= e_p + n && \text{when } p > P \text{ and } p \in S. \end{aligned}$$

*Proof.* Recall  $\tilde{v}_p$  of Lemma 10.18. When  $p \leq P$  and  $p \notin S$ , we have  $\tilde{v}_p = 0$ , so

$$\nu(v_p, e_p) = \nu(g_p, 0) = g_p.$$

When  $p \leq P$  and  $p \in S$ , since  $v_p \geq e_p$ , we have  $\tilde{v}_p \geq 1$ , so

$$\nu(v_p, e_p) \geq v_p + n = e_p + g_p + n.$$

Finally, when  $p > P$  and  $p \in S$ , we have  $\tilde{v}_p = 1$ , so

$$\nu(v_p, e_p) = v_p + n = e_p + n.$$

This completes the proof.  $\square$

**Lemma 10.21.** *We have*

$$\begin{aligned} (1 - p^{-N_{d,n}})\rho_p(\xi, \sigma) &\gg \frac{1}{p^{e_p}} && \text{if } p \in S_{\text{fin}}, \\ (1 - p^{-N_{d,n}})\rho_p(\xi, \sigma) &\gg 1 && \text{if } p \notin S, \end{aligned}$$

where the implicit constants depend at most on  $d, n$ .

*Proof.* First inequality follows from Proposition 10.12. Second inequality follows from Lemma 10.14 with noting that  $\rho_p(\xi, \sigma) \neq 0$  for all  $p < \infty$  by Remark 10.4 or Proposition 10.12.  $\square$

**Lemma 10.22.** *There is a constant  $C \geq 1$  depending only on  $d, n$  with the following property. For any  $V \geq 2$  and  $P \geq 1$ , there are  $v_p \in \mathbb{Z}_{\geq 1}$  for each  $p \in S_{\text{fin}} \cup M_{\mathbb{Q}, \leq P}$  such that*

$$\sum_{p \in S_{\text{fin}} \cup M_{\mathbb{Q}, \leq P}} \frac{2}{(1 - p^{-N_{d,n}})\rho_p(\xi, \sigma)p^{\nu(v_p, e_p)}} \leq C \left( \frac{\pi(P)}{V} + \frac{2}{P^{n-1}} \right)$$

and

$$q' \leq q(PV)^{\pi(P)},$$

where  $\pi(P)$  is the number of prime numbers less than or equal to  $P$ .

*Proof.* Let us write

$$a_p = \frac{2}{(1 - p^{-N_{d,n}})\rho_p(\xi, \sigma)p^{\nu(v_p, e_p)}}.$$

We choose  $v_p$ 's as in Lemma 10.20 and choose  $g_p$ 's later. By Lemma 10.21, we have

$$\sum_{p \in S_{\text{fin}} \cup M_{\mathbb{Q}, \leq P}} a_p \ll \sum_{\substack{p \leq P \\ p \notin S}} \frac{1}{p^{\nu(v_p, e_p)}} + \sum_{\substack{p \in S_{\text{fin}} \\ p \leq P}} \frac{p^{e_p}}{p^{\nu(v_p, e_p)}} + \sum_{\substack{p \in S_{\text{fin}} \\ p > P}} \frac{p^{e_p}}{p^{\nu(v_p, e_p)}}.$$

Then, by Lemma 10.20, we can further bound

$$\sum_{p \in S_{\text{fin}} \cup M_{\mathbb{Q}, \leq P}} a_p \ll \sum_{\substack{p \leq P \\ p \notin S}} \frac{1}{p^{g_p}} + \sum_{\substack{p \in S_{\text{fin}} \\ p \leq P}} \frac{1}{p^{g_p+n}} + \sum_{\substack{p \in S_{\text{fin}} \\ p > P}} \frac{1}{p^n} \leq \sum_{p \leq P} \frac{1}{p^{g_p}} + \sum_{p > P} \frac{1}{p^n}.$$

Now for a given  $V \geq 2$ , let  $g_p \in \mathbb{Z}_{\geq 1}$  be such that

$$(10.23) \quad p^{g_p-1} < V \leq p^{g_p}.$$

Then, we get

$$\sum_{p \in S_{\text{fin}} \cup M_{\mathbb{Q}, \leq P}} a_p \ll \frac{\pi(P)}{V} + \frac{1}{P^{n-1}}.$$

To get the bound of  $q'$ , note that (10.23) implies  $p^{g_p} < pV$ . Thus, we have

$$\begin{aligned} q' &= \prod_{p \in S_{\text{fin}} \cup M_{\mathbb{Q}, \leq P}} p^{v_p} = \prod_{\substack{p \leq P \\ p \notin S}} p^{g_p} \prod_{\substack{p \in S_{\text{fin}} \\ p \leq P}} p^{e_p + g_p} \prod_{\substack{p \in S_{\text{fin}} \\ p > P}} p^{e_p} \\ &= q \prod_{p \leq P} p^{g_p} \leq q \prod_{p \leq P} (pV) \leq q(PV)^{\pi(P)} \end{aligned}$$

as desired. This completes the proof.  $\square$

**Proposition 10.23.** *There is a constant  $C > 2$  depending only on  $d, n$  with the following property. For any  $A, P \geq 1$  and  $V \geq 2$  such that*

$$\frac{\pi(P)}{V} + \frac{1}{P^{n-1}} \leq \frac{1}{C},$$

we have

$$\begin{aligned} M(A, P) &= \frac{V_{N_{d,n}} \prod_{p \in M_{\mathbb{Q}}} \rho_p(\xi, \sigma)}{\zeta(N_{d,n})} A^{N_{d,n}} \left( 1 + O \left( \frac{1}{P \log 2P} + \frac{\pi(P)}{V} \right. \right. \\ &\quad \left. \left. + \frac{q(PV)^{\pi(P)}}{\sigma_{\infty} A} + \frac{q^{N_{d,n}-1} (\log \log 3q) (PV)^{\pi(P)N_{d,n}} \log 2A}{\sigma_{\infty} A^{N_{d,n}-1}} \right) \right) \end{aligned}$$

*Proof.* The task is to simplify the last three factors in the upper and lower bounds of  $M(A, P)$  in the statement of Proposition 10.19. In view of Lemma 10.22, given  $A, P \geq 1$  and  $V \geq 2$ , if

$$\frac{\pi(P)}{V} + \frac{1}{P^{n-1}}$$

is sufficiently small (depending only on  $d, n$ ), we can choose  $v_p \in \mathbb{Z}_{\geq 1}$  so that

$$\prod_{p \in S_{\text{fin}} \cup M_{\mathbb{Q}, \leq P}} \left( 1 \pm \frac{2}{(1 - p^{-N_{d,n}}) \rho_p(\xi, \sigma) p^{\nu(v_p, e_p)}} \right) = 1 + O \left( \frac{\pi(P)}{V} + \frac{1}{P^{n-1}} \right)$$

and

$$(10.24) \quad q' \leq q(PV)^{\pi(P)}.$$

Here we use the fact  $\prod_{1 \leq i \leq l} (1 + a_i) = 1 + O(\sum_{1 \leq i \leq l} |a_i|)$  for real numbers  $a_i$  with  $\sum_{1 \leq i \leq l} |a_i| < 1/2$  for example. Note that the implicit constant is absolute constant, so in particular independent of  $l$ .

Since

$$\frac{\pi(P)}{V} + \frac{1}{P^{n-1}} \ll 1 \quad \text{and} \quad \frac{1}{P \log 2P} \ll 1,$$

we get

$$\left( 1 + O \left( \frac{\pi(P)}{V} + \frac{1}{P^{n-1}} \right) \right) \left( 1 + O \left( \frac{1}{P \log 2P} \right) \right)$$

$$\begin{aligned}
& \times \left( 1 + O \left( \frac{J_{N_{d,n}}(q')}{\rho_\infty(\xi, \sigma) q'^{N_{d,n}-1} A} + \frac{J_{N_{d,n}}(q') \log 2A}{\rho_\infty(\xi, \sigma) \varphi(q') A^{N_{d,n}-1}} \right) \right) \\
& = 1 + O \left( \frac{\pi(P)}{V} + \frac{1}{P^{n-1}} + \frac{1}{P \log 2P} + \frac{q'}{\sigma_\infty A} + \frac{q'^{N_{d,n}-1} \log \log 3q' \log 2A}{\sigma_\infty A^{N_{d,n}-1}} \right) \\
& = 1 + O \left( \frac{1}{P \log 2P} + \frac{\pi(P)}{V} + \frac{q(PV)^{\pi(P)}}{\sigma_\infty A} \right. \\
& \quad \left. + \frac{q^{N_{d,n}-1} (\log \log 3q) (PV)^{\pi(P) N_{d,n}} \log 2A}{\sigma_\infty A^{N_{d,n}-1}} \right),
\end{aligned}$$

where for the first equality, we use Proposition 10.6 and the fact  $J_{N_{d,n}}(q')/\varphi(q') \ll q'^{N_{d,n}-1} \log \log 3q'$ , and for the second equality, we use (10.24), the bound

$$\log \log 3q' \ll (\log \log 3q) (PV)^{\pi(P)}$$

and our assumption  $n \geq 3$ . Plugging this into the two inequalities in Proposition 10.19, we are done.  $\square$

**10.4. Asymptotic formula of  $\#\mathbb{V}_{d,n}^{\text{loc}}(A; \xi, \sigma)$ .** We can finally prove Theorem 1.7.

*Proof of Theorem 1.7.* Recall that we have

$$\#\mathbb{V}_{d,n}^{\text{loc}}(A; \xi, \sigma) = \frac{1}{2} M(A, P) - \frac{1}{2} E(A, P)$$

for  $A, P \geq 1$  as shown in (10.2). By Proposition 10.15, Proposition 10.23, there are constants  $P_0, C_0 \geq 1$  depending only on  $d, n$  such that for

$$A \geq 1, \quad P \geq P_0, \quad V \geq C_0 \pi(P),$$

we have

$$\begin{aligned}
\#\mathbb{V}_{d,n}^{\text{loc}}(A; \xi, \sigma) &= \frac{V_{N_{d,n}} \prod_{p \in M_{\mathbb{Q}}} \rho_p(\xi, \sigma)}{2\zeta(N_{d,n})} A^{N_{d,n}} \left( 1 + O \left( \frac{1}{P \log 2P} + \frac{\pi(P)}{V} \right. \right. \\
& \quad \left. \left. + \frac{q(PV)^{\pi(P)}}{\sigma_\infty A} + \frac{q^{N_{d,n}-1} (\log \log 3q) (PV)^{\pi(P) N_{d,n}} \log 2A}{\sigma_\infty A^{N_{d,n}-1}} \right) \right) \\
& \quad + O \left( \frac{1}{P \log 2P} \frac{\sigma_\infty A^{N_{d,n}}}{q} + A^{N_{d,n}-1} \log \log 3A \right).
\end{aligned}$$

By Proposition 10.12, Proposition 10.6, and Lemma 10.14, we have

$$\frac{V_{N_{d,n}} \prod_{p \in M_{\mathbb{Q}}} \rho_p(\xi, \sigma)}{2\zeta(N_{d,n})} \asymp_{d,n} \frac{\sigma_\infty}{q}.$$

Thus if we write

$$\#\mathbb{V}_{d,n}^{\text{loc}}(A; \xi, \sigma) = \frac{V_{N_{d,n}} \prod_{p \in M_{\mathbb{Q}}} \rho_p(\xi, \sigma)}{2\zeta(N_{d,n})} A^{N_{d,n}} (1 + R(A; \xi, \sigma)),$$

then

$$(10.25) \quad R(A; \xi, \sigma) \ll \frac{1}{P \log 2P} + \frac{\pi(P)}{V} + \frac{q(PV)^{\pi(P)}}{\sigma_\infty A} + \frac{q \log \log 3A}{\sigma_\infty A}$$

$$+ \frac{q^{N_{d,n}-1}(\log \log 3q)(PV)^{\pi(P)N_{d,n}} \log 2A}{\sigma_{\infty} A^{N_{d,n}-1}}$$

provided  $A \geq 1$ ,  $P \geq P_0$ , and  $V \geq C_0\pi(P)$ .

We want to find an upper bound of the infimum of the right hand side of (10.25) when  $V$  and  $P$  vary. We set  $V = P^2$ . (Larger  $V$  does not improve the upper bound because the second term is the only term decreasing with respect to  $V$  and the second term is smaller than the first term if  $V \geq P^2$ . Also, it turns out that smaller  $V$  does not improve the bound as well.) Note that by enlarging  $P_0$  in advance, we have  $P^2 \geq C_0\pi(P)$  for all  $P \geq P_0$ , which we need for setting  $V = P^2$ . Then for  $A \geq 1$  and  $P \geq P_0$ , we have

$$\begin{aligned} R(A; \xi, \sigma) &\ll \frac{1}{P \log 2P} + \frac{qP^{3\pi(P)}}{\sigma_{\infty} A} \\ &\quad + \frac{q^{N_{d,n}-1}(\log \log 3q)P^{3\pi(P)N_{d,n}} \log 2A}{\sigma_{\infty} A^{N_{d,n}-1}} + \frac{q \log \log 3A}{\sigma_{\infty} A}. \end{aligned}$$

By the prime number theorem, further enlarging  $P_0$ , we have

$$P^{3\pi(P)} \leq e^{4P} \quad \text{and} \quad P^{3\pi(P)N_{d,n}} \leq e^{4N_{d,n}P}$$

for  $P \geq P_0$ . Thus

$$(10.26) \quad R(A; \xi, \sigma) \ll \frac{1}{P \log 2P} + \frac{q}{\sigma_{\infty} A} e^{4P} \\ + \frac{q^{N_{d,n}-1}(\log \log 3q) \log 2A}{\sigma_{\infty} A^{N_{d,n}-1}} e^{4N_{d,n}P} + \frac{q \log \log 3A}{\sigma_{\infty} A}$$

for  $A \geq 1$  and  $P \geq P_0$ . Let us set

$$\begin{aligned} f(P) &:= \frac{1}{P \log 2P}, & g(P) &:= \frac{q}{\sigma_{\infty} A} e^{4P}, \\ h(P) &:= \frac{q^{N_{d,n}-1}(\log \log 3q) \log 2A}{\sigma_{\infty} A^{N_{d,n}-1}} e^{4N_{d,n}P}. \end{aligned}$$

Since  $f$  is decreasing and  $g, h$  are increasing with respect to  $P$ , we have

$$(10.27) \quad \inf_{P \geq P_0} (f(P) + g(P) + h(P)) \\ \ll \inf_{P \geq P_0} f(P) + \inf_{P \geq P_0} g(P) + \inf_{P \geq P_0} h(P) + \alpha + \beta$$

where

$$\alpha := \begin{cases} f(P_*) & \text{if } \exists P_* \geq P_0 \text{ such that } f(P_*) = g(P_*), \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta := \begin{cases} f(P_*) & \text{if } \exists P_* \geq P_0 \text{ such that } f(P_*) = h(P_*), \\ 0 & \text{otherwise.} \end{cases}$$

If  $\sigma_{\infty} A/q \gg 1$  as assumed in in (1.4), we can take the logarithm of the equality  $f(P) = g(P)$  for several times and see that  $f(P) = g(P)$  implies

$$(10.28) \quad 4P + \log P + \log \log 2P = \log \frac{\sigma_{\infty} A}{q} \quad \text{and} \quad \log P + O(1) = \log \log \frac{\sigma_{\infty} A}{q}.$$

Thus we get

$$\alpha \ll \frac{1}{\log \frac{\sigma_\infty A}{q} \log \log \frac{\sigma_\infty A}{q}}.$$

Note that under the condition (1.4), we have

$$K := \frac{\sigma_\infty A^{N_{d,n}-1}}{q^{N_{d,n}-1} (\log \log 3q) \log 2A} \geq \frac{\sigma_\infty A}{q} \gg 1.$$

Therefore, we can solve  $f(P) = h(P)$  in the same way and get

$$\beta \ll \frac{1}{\log K \log \log K} \leq \frac{1}{\log \frac{\sigma_\infty A}{q} \log \log \frac{\sigma_\infty A}{q}}.$$

Going back to (10.26) and (10.27), we get

$$\begin{aligned} R(A; \xi, \sigma) &\ll \frac{q}{\sigma_\infty A} + \frac{q^{N_{d,n}-1} (\log \log 3q) \log 2A}{\sigma_\infty A^{N_{d,n}-1}} \\ &\quad + \frac{1}{\log \frac{\sigma_\infty A}{q} \log \log \frac{\sigma_\infty A}{q}} + \frac{q \log \log 3A}{\sigma_\infty A}. \end{aligned}$$

By condition (1.4), the second term is bounded by the first term. Since the first term is obviously bounded by the last term, we get

$$R(A; \xi, \sigma) \ll \frac{1}{\log \frac{\sigma_\infty A}{q} \log \log \frac{\sigma_\infty A}{q}} + \frac{q \log \log 3A}{\sigma_\infty A}.$$

This completes the proof.  $\square$

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