ISOPARAMETRIC HYPERSURFACES OF $\mathbb{H}^n \times \mathbb{R}$ AND $\mathbb{S}^n \times \mathbb{R}$

RONALDO F. DE LIMA AND GIUSEPPE PIPOLI

Abstract. We classify the isoparametric hypersurfaces and the homogeneous hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$ and $\mathbb{S}^n \times \mathbb{R}$, $n \geq 2$, by establishing that any such hypersurface has constant angle function and constant principal curvatures.

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1. Introduction

The construction and classification of isoparametric hypersurfaces in general Riemannian manifolds has become a matter of great interest in submanifold theory since the early works by Cartan on this subject. In a series of four remarkable papers [3]–[6], published in the late 1930's, he classified the isoparametric hypersurfaces of the n-hyperbolic spaces $\mathbb{H}^n$, and brought to light that the classification of the isoparametric hypersurfaces of the n-spheres $\mathbb{S}^n$ is a rather intricate problem. As a matter of fact, this classification has been built through many works over the last decades, and only recently it was announced to be complete (see [7, 9]).

The theory of isoparametric hypersurfaces connects with many branches of physics and mathematics. Recently, it has been applied to establish existence and classification results to various extrinsic geometric flows in Riemannian manifolds, such as the mean curvature flow, the inverse mean curvature flow, elliptic Weingarten flows, and higher order mean curvature flows (see [1, 10, 11, 14, 24, 30]).

By definition, an isoparametric hypersurface has constant mean curvature, as do any nearby (locally defined) hypersurface which is parallel to it. Homogeneous hypersurfaces, that is, those which are codimension one orbits of isometric actions on the ambient space, are well known examples of isoparametric hypersurfaces. It turns out that all isoparametric hypersurfaces of Euclidean space $\mathbb{R}^n$ (classified by Segre [33]), as well as those of hyperbolic space $\mathbb{H}^n$ (classified by Cartan [3]), are homogeneous. In $\mathbb{S}^n$, one has the two types, homogeneous (classified by Hsiang and Lawson [25]) and nonhomogeneous (cf. [22, 29]).

Besides being isoparametric, homogeneous hypersurfaces have constant principal curvatures. However, the constancy of the principal curvatures implies neither being isoparametric nor homogeneous. For instance, in simply connected space forms, as proved by Cartan, a hypersurface is isoparametric if and only if it has constant principal curvatures. On the other hand, as we pointed out, there are nonhomogeneous isoparametric hypersurfaces in $\mathbb{S}^n$. Also, Rodríguez-Vázquez [31] showed that, for each $n \geq 3$, there exists an n-dimensional torus which contains a non-isoparametric hypersurface whose principal curvatures are constant. In addition, Guimarães, Santos and Santos [24] applied the theory of mean curvature flow to

obtain a class of Riemannian manifolds that admit nonisoparametric hypersurfaces with constant principal curvatures. It should also be mentioned that, in certain Riemannian manifolds, such as Damek-Ricci spaces, there exist isoparametric hypersurfaces whose principal curvatures are not constant functions (see [16, 23]).

In [20], Domínguez-Vázquez and Manzano classified the isoparametric surfaces of all simply connected homogeneous 3-manifolds with 4-dimensional isometry group. These are the so called $\mathbb{E}(\kappa,\tau)$ spaces with $\kappa-4\tau^2\neq 0$, which include the products $\mathbb{H}^2\times\mathbb{R}$ and $\mathbb{S}^2\times\mathbb{R}$. Their result also provides the classification of the homogeneous surfaces of these spaces, as well as of the surfaces having constant principal curvatures. Other results on classification of isoparametric or constant principal curvature hypersurfaces of Riemannian manifolds of nonconstant sectional curvature were obtained in [8, 17, 18, 19, 28, 32].

Inspired by Domínguez-Vázquez and Manzano's work, we aim to establish here the following classification result for hypersurfaces of the products $\mathbb{Q}_\epsilon^n\times\mathbb{R}$, where $\mathbb{Q}_\epsilon^n$ stands for the $n(\geq 2)$-dimensional simply connected space form of constant sectional curvature $\epsilon=\pm 1$, that is, the hyperbolic space $\mathbb{H}^n$ for $\epsilon=-1$, and the sphere $\mathbb{S}^n$ for $\epsilon=1$:

Theorem 1. *Let Σ be a connected hypersurface of $\mathbb{Q}_\epsilon^n\times\mathbb{R}$. Then the following are equivalent:*

(i) *Σ is isoparametric.*

(ii) *Σ has constant angle and constant principal curvatures.*

(iii) *Σ is an open subset of one of the following complete hypersurfaces:*

  (a) *a horizontal slice $\mathbb{Q}_\epsilon^n\times\{t_0\}$,*

  (b) *a vertical cylinder over a complete isoparametric hypersurface of $\mathbb{Q}_\epsilon^n$,*

  (c) *a parabolic bowl of $\mathbb{H}^n\times\mathbb{R}$ (see Fig. 1).*

Moreover, in the hyperbolic case $\epsilon=-1$, the condition

(iv) *Σ is an open subset of a homogeneous hypersurface*

is also equivalent to (i)–(iii)*. In the spherical case $\epsilon=1$,* (iv) *is equivalent to*

(v) *Σ is an open subset of either a horizontal slice or a vertical cylinder over a complete homogeneous hypersurface of $\mathbb{S}^n$.*

The most delicate part of the proof of Theorem 1, which we do in Proposition 7, is showing that the connected isoparametric hypersurfaces of $\mathbb{Q}_\epsilon^n\times\mathbb{R}$ have constant angle function. To accomplish that, we proceed as Domínguez-Vázquez and Manzano in [20]. More precisely, we apply Jacobi field theory for reducing the proof to the resolution of an algebraic problem. In the n-dimensional setting, such problem is considerably more involved than its 2-dimensional analogue, which compelled us to approach it differently. Our trick then was to consider an alternate formulation on which the corresponding algebraic equations are all linear. In this way, the solution became attainable, although this linear problem were still arduous (see Section 6).

For the remaining of the proof of Theorem 1, we apply some results obtained in [15, 36], including the one that characterizes constant angle hypersurfaces of the products $\mathbb{Q}_\epsilon^n\times\mathbb{R}$ as horizontal slices, vertical cylinders or vertical graphs built on parallel hypersurfaces of $\mathbb{Q}_\epsilon^n$ (cf. Sec. 2.3).

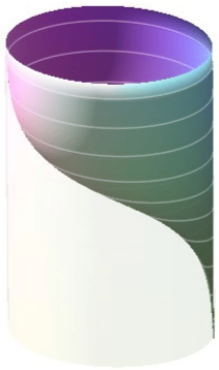


FIGURE 1. The depicted hypersurface, called a *parabolic bowl*, is a homogeneous entire vertical graph in $\mathbb{H}^n \times \mathbb{R}$ whose level hypersurfaces are parallel horospheres, and whose vertical translations define a nonsingular isoparametric foliation of $\mathbb{H}^n \times \mathbb{R}$ (we are grateful to João P. dos Santos for this plot).

Theorem 1 provides an explicit classification of the isoparametric hypersurfaces and of the homogeneous hypersurfaces of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, since such classes of hypersurfaces are completely classified in $\mathbb{Q}^n_\epsilon$ [(i)]. The case $n = 2$, of course, is contained in the main result by Domínguez-Vázquez and Manzano [20]. We included it here due to the fact that our proof differs from theirs in some substantial parts. For $n = 3$, it was proved in [28] that hypersurfaces with constant principal curvatures have constant angle. Considering this result, we can drop the assumption on the constancy of the angle in Theorem 1 when $n = 3$. Finally, we remark that Theorem 1 also gives that all isoparametric hypersurfaces of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ have constant scalar curvature, since the scalar curvature of a hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ satisfies a relation (see, e.g., equality (6.2) in [13]) which implies that any such hypersurface having constant angle and constant principal curvatures is necessarily of constant scalar curvature.

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2. PRELIMINARIES

2.1. **Tensor Curvature of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$.** Given a field $X \in \mathfrak{X}(\mathbb{Q}^n_\epsilon \times \mathbb{R})$, we shall denote by X^h its component which is tangent to $\mathbb{Q}^n_\epsilon$ (called *horizontal*), that is,

$$X^h := X - \langle X, \partial_t \rangle \partial_t,$$

[(i)] Due to some controversial results by Siffert [34, 35], there is no general agreement that the isoparametric hypersurfaces of the sphere $\mathbb{S}^{13}$ are indeed classified.

where ∂_t denotes the gradient of the projection $\pi_{\mathbb{R}}$ of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ on its second factor $\mathbb{R}$. Note that ∂_t is a unit parallel field on $\mathbb{Q}_\epsilon^n \times \mathbb{R}$. When $X = X^h$, we say that the field X is *horizontal*.

Since the first factor of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ has constant sectional curvature ϵ and its second factor is one-dimensional, the Riemannian curvature tensor R_ϵ of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ is

$$R_\epsilon(X,Y)Z = \epsilon(\langle X^h, Z^h\rangle Y^h - \langle Y^h, Z^h\rangle X^h) \quad \forall X, Y, Z \in \mathfrak{X}(\mathbb{Q}_\epsilon^n \times \mathbb{R}).$$

2.2. **Hypersurfaces of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$.** Given an oriented hypersurface Σ of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ (endowed with the standard product metric), set N for its unit normal field and A for its shape operator with respect to N, that is

$$AX = -\overline{\nabla}_X N, \;\; X \in \mathfrak{X}(\Sigma),$$

where $\overline{\nabla}$ is the Levi-Civita connection of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$, and $\mathfrak{X}(\Sigma)$ is the tangent bundle of Σ. The principal curvatures of Σ, i.e., the eigenvalues of the shape operator A, will be denoted by $k_1, \ldots, k_n$. In this setting, the non-normalized *mean curvature* H of Σ is defined as the sum of its principal curvatures, that is,

$$H := k_1 + \cdots + k_n.$$

The *height function* ϕ and the *angle function* Θ of Σ are defined by

$$\phi := \pi_{\mathbb{R}}|_\Sigma, \quad \Theta := \langle N, \partial_t\rangle.$$

Denoting by ∇ the gradient on $C^\infty(\Sigma)$ and writing $T := \nabla\phi$, the identity

$$T = \partial_t - \Theta N$$

holds everywhere on Σ. In particular, $\|T\|^2 = 1 - \Theta^2$.

2.3. **Graphs on parallel hypersurfaces.** Let $\mathscr{F} := \{M_s \subset \mathbb{Q}_\epsilon^n \,;\, s \in I\}$ be a family of parallel hypersurfaces of $\mathbb{Q}_\epsilon^n$, where $I \subset \mathbb{R}$ is an open interval. Given a smooth function ϕ on I, let

$$f\colon M_{s_0} \times I \to \mathbb{Q}_\epsilon^n \times \mathbb{R}, \;\; s_0 \in I,$$

be the immersion given by

$$f(p,s) := (\exp_p(s\eta_{s_0}(p)), \phi(s)), \;\; (p,s) \in M_{s_0} \times I,$$

where exp denotes the exponential map of $\mathbb{Q}_\epsilon^n$, and η_{s_0} is the unit normal of M_{s_0}. The hypersurface $\Sigma = f(M_{s_0} \times I)$ is a vertical graph over an open set of $\mathbb{Q}_\epsilon^n$ whose level hypersurfaces are the parallels M_s to M_{s_0}.

Definition 1. With the above notation, we call Σ an (M_s, ϕ)-*graph* and say that

$$\varrho(s) := \frac{\phi'(s)}{\sqrt{1 + (\phi')^2(s)}}, \;\; s \in I, \tag{1}$$

is the ϱ-*function* of Σ.

Given an (M_s, ϕ)-graph Σ in $\mathbb{Q}_\epsilon^n \times \mathbb{R}$, it is easily seen that $N = -\varrho(s)\eta_s(p) + \Theta\partial_t$ is a unit normal to it. In particular, one has that the equality

$$\varrho^2 + \Theta^2 = 1 \tag{2}$$

holds everywhere on Σ.

It was proved in [12] that, with this orientation, the principal curvature functions $k_i = k_i(p,s)$ of Σ at a point $(p,s) \in M_{s_0} \times I$ are:

$$k_i = -\varrho(s)k_i^s(p), \;\; i \in \{1, \ldots, n-1\}, \quad \text{and} \quad k_n = \varrho'(s),$$

where $k_i^s(p)$ is the i-th principal curvature of the parallel M_s at $\exp_p(s\eta_{s_0}(p))$.

We point out that, from (1), one has

$$\phi(s) = \int_{s_0}^{s} \frac{\varrho(u)}{\sqrt{1-\varrho^2(u)}} du + \phi(s_0), \quad s_0, s \in I.$$

Hence, an (M_s, ϕ)-graph is determined by its ϱ-function.

3. Isoparametric hypersurfaces of constant angle

In this section, we prove Propositions 4 and 5, which establish the existence of the parabolic bowl as described in Figure 1, and its uniqueness — together with horizontal slices and vertical cylinders over isoparametric hypersurfaces of $\mathbb{Q}_\epsilon^n$ — as a hypersurface of constant angle and constant principal curvatures in $\mathbb{Q}_\epsilon^n \times \mathbb{R}$. In the proofs, the following two lemmas will play a crucial role. They first appeared in [36], and later in [15] in a more general form. For notational purposes, we refer to the corresponding results of [15].

Lemma 2 (Theorem 7 of [15])**.** *Let $\{M_s\,;\, s \in I\}$ be a family of isoparametric hypersurfaces of $\mathbb{Q}_\epsilon^n$. Consider the first order differential equation*

$$y' = H^s y + H, \quad s \in I, \tag{3}$$

where H^s denotes the mean curvature of M_s and H is a constant. In this setting, if $\varrho\colon I \to (0,1) \subset \mathbb{R}$ is a solution to (3)*, then the (M_s, ϕ)-graph determined by ϱ has constant mean curvature H. Conversely, if an (M_s, ϕ)-graph Σ has constant mean curvature H, then $\{M_s\,;\, s \in I\}$ is isoparametric and the ϱ-function of Σ is a solution to* (3)*.*

Lemma 3 (Corollary 4 of [15])**.** *Let Σ be a connected hypersurface of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ whose angle function is constant. Then, one of the following occurs:*

- *Σ is an open set of a horizontal slice $\mathbb{Q}_\epsilon^n \times \{t_0\}$;*
- *Σ is a vertical cylinder over a hypersurface of $\mathbb{Q}_\epsilon^n$;*
- *Σ is locally an (M_s, ϕ)-graph with ϕ' a nonzero constant.*

Now, we are in a position to state and prove the announced Propositions 4 and 5.

Proposition 4. *Given $H \in (0, n-1)$, there exists an entire (M_s, ϕ)-graph Σ_H in $\mathbb{H}^n \times \mathbb{R}$ (to be called a* parabolic bowl*) of constant mean curvature H, which is homogeneous and has constant angle. In particular, Σ_H is isoparametric and has constant principal curvatures.*

Proof. Let $\mathscr{F} := \{M_s\,;\, s \in \mathbb{R}\}$ be a family of parallel horospheres of $\mathbb{H}^n$. Then, considering the "outward orientation" in each M_s, for all $s \in \mathbb{R}$ and all $p \in M_s$, we have that $k_i^s(p) = -1$ for all $i \in 1, \ldots, n-1$. Therefore, given $H \in (0, n-1)$, the ODE (3) associated to $\mathscr{F}$ and H is

$$y' = -(n-1)y + H. \tag{4}$$

Clearly, the constant function $\varrho = H/(n-1) < 1$ is a solution to (4). Hence, by Lemma 2, the entire (M_s, ϕ)-graph Σ_H of $\mathbb{H}^n \times \mathbb{R}$ which is determined by ϱ has constant mean curvature H. Moreover, by (2), Σ_H has constant angle function.

Finally, the homogeneity of Σ_H follows from its invariance by the one-parameter group of parabolic translations of $\mathbb{H}^n$ which fix each horosphere M_s (extended slicewise to $\mathbb{H}^n \times \mathbb{R}$), as well as by the isometries $\Psi_u \times \tau_u$, $u \in \mathbb{R}$, where

$$\{\Psi_u\,;\, u \in \mathbb{R}\} \subset \mathrm{Iso}(\mathbb{H}^n)$$

is the flow defined by the unit normals of the horospheres M_s, and τ_u is the vertical translation $(p,t) \mapsto (p, t + \varrho u)$ of $\mathbb{H}^n \times \mathbb{R}$. □

When $n = 2$, our parabolic bowl Σ_H corresponds to the surface $P_{H,-1,0}$ of [20], and to the surface P_H of [27]. They also appear in [28], for $n = 3$, and in [8], for $n \geq 2$. In all of these occurrences, the hypersurface is under no specific designation, except in [20], where they are called *parabolic helicoids*. Our chosen nomenclature comes from the extrinsic geometric flows theory, for it was shown in [14] that, for each $r \in \{1, \ldots, n-1\}$, there exists a parabolic bowl Σ_{H_r} of constant r-th mean curvature H_r such that $H_r = \Theta$ on Σ_{H_r}. As a consequence, Σ_{H_r} is a translating soliton to mean curvature flow of r-th order. In this context, entire graphs with this property are called *bowl solitons*. We should also mention that any parallel to a given parabolic bowl is nothing but a vertical translation of it. Therefore, analogously to the slices $\mathbb{H}^n \times \{t\}$, $t \in \mathbb{R}$, the family of all parallels to a parabolic bowl defines a nonsingular isoparametric foliation of $\mathbb{H}^n \times \mathbb{R}$.

Proposition 5. *Let Σ be a connected hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ with constant angle function. Then, Σ is isoparametric if and only if it has constant principal curvatures. If so, Σ is an open set of one of the following hypersurfaces:*

(i) *a horizontal slice $\mathbb{Q}^n_\epsilon \times \{t_0\}$,*

(ii) *a vertical cylinder over a complete isoparametric hypersurface of $\mathbb{Q}^n_\epsilon$,*

(iii) *a parabolic bowl if $\epsilon = -1$.*

Proof. By Lemma 3, Σ is an open set of a horizontal slice, a vertical cylinder over a hypersurface of $\mathbb{Q}^n_\epsilon$, or is locally an (M_s, ϕ)-graph such that ϕ' is constant. If the first occurs, we are done. Assume then that Σ is a cylinder $\Sigma_0 \times \mathbb{R}$, where Σ_0 is a hypersurface of $\mathbb{Q}^n_\epsilon$. It is easily seen that Σ is isoparametric (resp. has constant principal curvatures) if and only if Σ_0 is isoparametric (resp. has constant principal curvatures). However, in $\mathbb{Q}^n_\epsilon$, to be isoparametric and to have constant principal curvatures are equivalent conditions. Besides, we have from the classification of isoparametric hypersurfaces of space forms that any isoparametric hypersurface of $\mathbb{Q}^n_\epsilon$ is necessarily an open set of a complete isoparametric hypersurface.

Let us suppose now that Σ is locally an (M_s, ϕ)-graph of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, $s \in I$, such that ϕ' is a nonzero constant. Then, from (1), the ϱ-function of Σ is a nonzero constant as well. If either Σ is isoparametric or has constant principal curvatures, then its mean curvature H is constant. In this case, it follows from Lemma 2 that the hypersurfaces M_s are isoparametric and that ϱ satisfies $0 = \varrho' = \varrho H^s + H$, which implies that the mean curvature H^s of M_s is a constant independent of s. Again, by considering the classification of isoparametric hypersurfaces of $\mathbb{Q}^n_\epsilon$, one concludes that $\{M_s\,;\, s \in I\}$ is necessarily a family of parallel horospheres of $\mathbb{H}^n$, so that Σ is an open set of a parabolic bowl of $\mathbb{H}^n \times \mathbb{R}$. □

Remark 6. The first part of Proposition 5 is essentially the content of Corollary 5.2 of [8]. Nonetheless, the proof which is given there is rather distinct from ours.

4. The constancy of the angle function

This section will be entirely dedicated to the proof of the following proposition, which constitutes our main result for establishing Theorem 1.

Proposition 7. *The angle function of any connected isoparametric hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ is constant.*

Proof of Proposition 7. Let Σ be a connected oriented isoparametric hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. Since its angle function Θ is continuous, it suffices to prove that Θ is locally constant on Σ. We can assume that $\Theta^2 \neq 1$, so that the gradient T of the height function of Σ never vanishes. In this setting, define $\Phi^r \colon \Sigma \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ by

$$\Phi^r(p) = \exp_p(rN_p),$$

where N_p is the unit normal field of Σ at $p \in \Sigma$ and exp stands for the exponential map of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. Passing to an open subset of Σ, we can assume that, for a small $\delta > 0$, and for all $r \in (-\delta, \delta)$, the map Φ^r is well defined and $\Sigma_r := \Phi^r(\Sigma)$ is an embedded hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ which lies at distance $|r|$ from Σ.

Given $p \in \Sigma$, let $\gamma_p(r)$ be the geodesic of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ such that $\gamma_p(0) = p$ and $\gamma_p'(0) = N_p$, that is, $\gamma_p(r) = \Phi^r(p)$. It is easily seen that the unit normal to Σ^r at $\gamma_p(r)$ is $N(r) := \gamma_p'(r)$. In particular, $N(r)$ is parallel along γ_p. Since ∂_t is parallel on $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, this gives that the angle function of Σ_r is constant along γ_p. Consequently, the gradient of the height function of Σ_r at $\gamma_p(r)$ is the parallel transport $T(r)$ of T along γ_p.

Set $U_1(r) = T(r)/|T(r)|$ and let $N(r), U_1(r), U_2(r), \ldots, U_n(r)$ be an orthonormal parallel frame along γ_p. Notice that, for all $i \in \{2, \ldots, n\}$, $U_i(r)$ is horizontal. Indeed, for such an i, $0 = \langle U_i(r), T(r)\rangle = \langle U_i, \partial_t\rangle$.

For any $j \in \{1, \ldots, n\}$, let $\zeta_j = \zeta_j(r)$ be the Jacobi field along γ_p such that

$$\zeta_j(0) = U_j(0) \quad \text{and} \quad \zeta_j'(0) = -AU_j(0), \tag{5}$$

where A is the shape operator of Σ. Then, for any such j, ζ_j satisfies

$$\zeta_j'' + R_\epsilon(\gamma_p', \zeta_j)\gamma_p' = 0. \tag{6}$$

In addition, for all $r \in (-\delta, \delta)$, one has $\langle \zeta_j(r), N(r)\rangle = 0$. Therefore, there exist smooth functions $b_{ij} = b_{ij}(r)$ such that

$$\zeta_j = \sum_{i=1}^{n} b_{ij} U_i, \;\; j \in \{1, \ldots, n\}. \tag{7}$$

Furthermore, since all U_i are parallel along γ_p, we have

$$\zeta_j'' = \sum_{i=1}^{n} b_{ij}'' U_i, \;\; j \in \{1, \ldots, n\}. \tag{8}$$

Now, aiming at the Jacobi equation (6), we compute

$$R_\epsilon(\gamma_p', \zeta_j)\gamma_p' = R_\epsilon(N, \zeta_j)N = \epsilon(\langle N^h, N^h\rangle \zeta_j^h - \langle \zeta_j^h, N^h\rangle N). \tag{9}$$

However, $N^h = N - \Theta\partial_t$, so that $\langle N^h, N^h\rangle = 1 - \Theta^2 = \|T\|^2$. Besides, considering (7) and the fact that U_i is horizontal for all $i \geq 2$, we have:

$$\zeta_j^h = \zeta_j - \langle \zeta_j, \partial_t\rangle \partial_t = \zeta_j - b_{1j}\langle U_1, \partial_t\rangle \partial_t = \zeta_j - b_{1j}\|T\|\partial_t.$$

In particular,

$$\langle \zeta_j^h, N^h\rangle = \langle \zeta_j - b_{1j}\|T\|\partial_t, N - \Theta\partial_t\rangle = -\Theta\langle \zeta_j, \partial_t\rangle = -\Theta b_{1j}\|T\|.$$

It follows from (9) and the above equalities that,

$$\begin{aligned}\epsilon R_\epsilon(\gamma_p', \zeta_j)\gamma_p' &= \|T\|^2\zeta_j - b_{1j}\|T\|\partial_t + \Theta b_{1j}\|T\|N = \|T\|^2\zeta_j - b_{1j}\|T\|T \\ &= \|T\|^2(\zeta_j - b_{1j}U_1) = \sum_{i=2}^{n}\|T\|^2 b_{ij}U_i.\end{aligned}$$

This last equality and (6) then give

$$\zeta_j'' = \sum_{i=2}^{n} -\epsilon\|T\|^2 b_{ij}U_i. \tag{10}$$

Now, set (a_{ij}) for the (symmetric) matrix of A with respect to the orthonormal basis $\{U_1(0), \ldots, U_n(0)\}$, that is,

$$AU_j(0) = \sum_{i=1}^{n} a_{ij}U_i(0).$$

Considering this last equality and comparing (8) with (10), we conclude that ζ_j is a Jacobi field satisfying the initial conditions (5) if and only if the coefficients b_{ij} are solutions of an initial value problem. Namely,

$$\begin{cases} b_{1j}''(r) &= 0; \\ b_{ij}''(r) &= -\epsilon\|T\|^2 b_{ij}(r), \quad \text{if } i \geq 2; \\ b_{ij}(0) &= \delta_{ij}; \\ b_{ij}'(0) &= -a_{ij}. \end{cases} \tag{11}$$

Defining the function

$$\tau := -\epsilon\|T\|^2 = -\epsilon(1 - \Theta^2) \neq 0,$$

the solutions of (11) are

$$\begin{cases} b_{1j}(r) &= \delta_{1j} - a_{1j}r; \\ b_{ij}(r) &= \delta_{ij}c_\tau(r) - a_{ij}s_\tau(r) \quad \text{if } i \geq 2; \end{cases} \tag{12}$$

where s_τ and c_τ are the functions:

(13)

$$s_\tau(r) := \begin{cases} \frac{1}{\sqrt{\tau}}\sinh(\sqrt{\tau}\, r) & \text{if} \quad \tau > 0, \\ \frac{1}{\sqrt{-\tau}}\sin(\sqrt{-\tau}\, r) & \text{if} \quad \tau < 0, \end{cases} \qquad c_\tau(r) := \begin{cases} \cosh(\sqrt{\tau}\, r) & \text{if} \quad \tau > 0, \\ \cos(\sqrt{-\tau}\, r) & \text{if} \quad \tau < 0. \end{cases}$$

Notice that the derivatives of s_τ and c_τ satisfy:

$$s_\tau'(r) = c_\tau(r) \quad \text{and} \quad c_\tau'(r) = \tau s_\tau(r) \ \ \forall r \in \mathbb{R}. \tag{14}$$

Given $r \in (-\delta, \delta)$, let $B(r)$ and $C(r)$ be the linear operators of $T_{\gamma_p(r)}\Sigma_r$ which take the basis $\{U_1(r), \ldots, U_n(r)\}$ to $\{\zeta_1(r), \ldots, \zeta_n(r)\}$ and $\{\zeta_1'(r), \ldots, \zeta_n'(r)\}$, respectively. Considering (7) and the fact that each U_i is parallel along γ_p, we conclude that their matrices with respect to this basis are

$$B(r) = (b_{ij}(r)) \quad \text{and} \quad C(r) = (b_{ij}'(r)), \ \ i, j \in \{1, \ldots, n\},$$

where b_{ij} are the functions defined in (12).

In the above setting, Jacobi field theory applies, giving that $B(r)$ is nonsingular for each $r \in (-\delta, \delta)$, and that the shape operator of Σ^r is $A^r = -C(r)B(r)^{-1}$ (see [2, Theorem 10.2.1]). In particular,

$$H(r) = \operatorname{trace} A^r = -\operatorname{trace}\left(C(r)B(r)^{-1}\right) = -\frac{\frac{d}{dr}(\det B(r))}{\det B(r)},$$

where, in the last equality, we used the fact that $\nabla \det(B) = (\det B)\operatorname{adj}(B^{-1})$.

Defining $D(r) = \det B(r)$, it follows from the above that the function

$$f(r) = D'(r) + H(r)D(r)$$

vanishes identically. Since $D' = f - HD = -HD$, one has

$$f' = D'' + H'D + HD' = D'' + (H' - H^2)D.$$

Therefore, for all $k \in \mathbb{N}$,

$$0 = f^{(k)} = D^{(k+1)} + \phi_k D, \quad \phi_k = \phi_k(H, H', \ldots, H^{(k)}), \tag{15}$$

where $f^{(k)}$ denotes the k-th derivative of f.

Now, considering the equalities (12), we write the matrix $B(r) = (b_{ij}(r))$ as

$$B(r) = \left[\begin{array}{c|ccc} 1 - a_{11}r & -a_{12}r & \ldots & -a_{1n}r \\ \hline -a_{12}s_\tau(r) & & & \\ \vdots & & \delta_{ij}c_\tau(r) - a_{ij}s_\tau(r) & \\ -a_{1n}s_\tau(r) & & & \end{array}\right]. \tag{16}$$

Set

$$d_k = -\phi_k(H(0), H'(0), \ldots, H^k(0)), \quad k \in \mathbb{N}.$$

The following lemma will play a fundamental role in our final argument.

Lemma 8. *Suppose that n is odd. Then, for every positive integer s_0, there exists $s_* \geq s_0$ such that $d_{s_*} \neq 0$.*

Proof. Assume, by contradiction, that

$$d_s = 0 \quad \forall s \geq s_0.$$

Then, by (15),

$$D^{(s+1)}(0) = 0 \quad \forall s \geq s_0. \tag{17}$$

Fix an arbitrary point $p \in \Sigma$. Since the functions c_τ and s_τ are real analytic, so is D. Then, it follows from (17) that $D = D(r)$ agrees with a polynomial in a neighborhood of $r = 0$.

First, we shall classify the symmetric matrices A for which D is a polynomial in r, then we will show that none of them can be the second fundamental form of an isoparametric hypersurface.

Write the matrix in (16) in block form as

$$B(r) = \begin{bmatrix} 1 - ar & -rv^T \\ -s_\tau(r)v & c_\tau(r)I_m - s_\tau(r)A_0 \end{bmatrix},$$

where $m = n - 1$ and

$$a = a_{11}, \qquad v = (a_{12}, \ldots, a_{1n})^T, \qquad A_0 = (a_{ij})_{2 \leq i,j \leq n}.$$

Set

$$M(r) = c_\tau(r)I_m - s_\tau(r)A_0, \qquad \varphi(r) = \det M(r).$$

The standard determinant formula for block matrices, together with the matrix determinant lemma, gives

$$D(r) = \varphi(r) - r\psi(r) \quad \text{and} \quad \psi(r) = a\varphi(r) + s_\tau(r) v^T \operatorname{adj}(M(r)) v. \tag{18}$$

We claim that

$$\varphi \equiv 1 \quad \text{and} \quad \psi \equiv a. \tag{19}$$

Indeed, let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of the symmetric matrix A_0, and let $\widetilde{v} = (\widetilde{v}_1, \dots, \widetilde{v}_m)$ be the coordinates of v with respect to an orthonormal eigenbasis of A_0. Then

$$\varphi(r) = \prod_{\ell=1}^{m} \big(c_\tau(r) - \lambda_\ell s_\tau(r)\big), \tag{20}$$

whereas

$$s_\tau(r) v^T \operatorname{adj}(M(r)) v = \sum_{\ell=1}^{m} \widetilde{v}_\ell^{\,2} s_\tau(r) \prod_{\substack{k=1 \\ k\neq\ell}}^{m} \big(c_\tau(r) - \lambda_k s_\tau(r)\big). \tag{21}$$

Set

$$\omega = \begin{cases} \sqrt{\tau}, & \tau > 0, \\ \sqrt{-\tau}, & \tau < 0. \end{cases}$$

For $\tau < 0$, we have

$$c_\tau(r) = \frac{1}{2}\left(e^{i\omega r} + e^{-i\omega r}\right), \qquad s_\tau(r) = \frac{1}{2i\omega}\left(e^{i\omega r} - e^{-i\omega r}\right), \tag{22}$$

whereas, for $\tau > 0$,

$$c_\tau(r) = \frac{1}{2}\left(e^{\omega r} + e^{-\omega r}\right), \qquad s_\tau(r) = \frac{1}{2\omega}\left(e^{\omega r} - e^{-\omega r}\right). \tag{23}$$

It follows from (20)–(23) that there exist constants α_j, β_j such that

$$\varphi(r) = \sum_{j=0}^{m} \alpha_j e^{\nu_j r}, \qquad \psi(r) = \sum_{j=0}^{m} \beta_j e^{\nu_j r},$$

where

$$\nu_j = \begin{cases} (m-2j)\sqrt{\tau}, & \tau > 0, \\ i(m-2j)\sqrt{-\tau}, & \tau < 0. \end{cases}$$

Note that $\nu_0, \dots, \nu_m$ are pairwise distinct. Moreover, since $m = n - 1$ is even,

$$\nu_j = 0 \quad \Longleftrightarrow \quad j = \frac{m}{2}.$$

Using $D = \varphi - r\psi$, we obtain

$$0 = \sum_{\substack{0\le j\le m \\ j\neq m/2}} (\alpha_j - r\beta_j) e^{\nu_j r} + \left(\alpha_{m/2} - r\beta_{m/2} - D(r)\right).$$

Recall the standard uniqueness property of exponential polynomials: if $\mu_1, \dots, \mu_\ell$ are pairwise distinct and

$$\sum_{j=1}^{\ell} P_j(r) e^{\mu_j r} = 0$$

on an open interval, where the P_j are polynomials, then $P_j = 0$ for every j. Since D is a polynomial near $r = 0$, this property applied to the preceding equality yields

$$\alpha_j = \beta_j = 0 \quad \forall j \neq \frac{m}{2}, \qquad D(r) = \alpha_{m/2} - r\beta_{m/2}.$$

Thus D is affine and

$$\varphi(r) = \alpha_{m/2}, \qquad \psi(r) = \beta_{m/2}.$$

In particular, both φ and ψ are constant. Since

$$\varphi(0) = 1 \qquad \text{and} \qquad \psi(0) = a,$$

we conclude that (19) holds.

Suppose first that $\tau < 0$, and set $z = e^{i\omega r}$. It follows from (20) and (22) that

$$1 = \varphi(r) = 2^{-m} \prod_{j=1}^{m} \left[\left(1 + i\frac{\lambda_j}{\omega}\right) z + \left(1 - i\frac{\lambda_j}{\omega}\right) z^{-1} \right]. \tag{24}$$

However, the coefficient of z^m in the right-hand side is

$$2^{-m} \prod_{j=1}^{m} \left(1 + i\frac{\lambda_j}{\omega}\right),$$

which is nonzero because every λ_j is real. Thus the right-hand side of (24) cannot be constant. This contradiction rules out the case $\tau < 0$.

We must therefore have $\tau > 0$. Set $z = e^{\omega r}$. From (20) and (23), we obtain

$$1 = \varphi(r) = 2^{-m} \prod_{j=1}^{m} \left[\left(1 - \frac{\lambda_j}{\omega}\right) z + \left(1 + \frac{\lambda_j}{\omega}\right) z^{-1} \right]. \tag{25}$$

The coefficients of $z^{\pm m}$ on the right-hand side of (25) vanish, so

$$2^{-m} \prod_{j=1}^{m} \left(1 \mp \frac{\lambda_j}{\omega}\right) = 0.$$

Hence, after reordering the indices, we may assume that $\lambda_1 = \omega$ and $\lambda_m = -\omega$. The corresponding factors are $2z^{-1}$ and $2z$, whose product is 4. Thus,

$$1 = 2^{-(m-2)} \prod_{j=2}^{m-1} \left[\left(1 - \frac{\lambda_j}{\omega}\right) z + \left(1 + \frac{\lambda_j}{\omega}\right) z^{-1} \right].$$

If $m > 2$, comparing the coefficients of $z^{\pm(m-2)}$ gives

$$2^{-(m-2)} \prod_{j=2}^{m-1} \left(1 \mp \frac{\lambda_j}{\omega}\right) = 0.$$

Therefore, again, after reordering the remaining indices, we may assume that $\lambda_2 = \omega$ and $\lambda_{m-1} = -\omega$.

Since m is even, repeating this argument removes all m factors in pairs, each corresponding to one eigenvalue ω and one eigenvalue $-\omega$. Hence the eigenvalues of A_0 are ω and $-\omega$, each with multiplicity $m/2$.

Now, by (18) and (19),

$$s_\tau(r) v^T \operatorname{adj}(M(r)) v = 0.$$

Differentiating with respect to r, we obtain

$$c_\tau(r)v^T \operatorname{adj}(M(r))v + s_\tau(r)v^T \frac{d}{dr}\big(\operatorname{adj}(M(r))\big)v = 0.$$

Setting $r = 0$ then yields

$$0 = v^T \operatorname{adj}(I_m)v = v^T v = \|v\|^2.$$

Hence $v = 0$, and therefore $U_1 = T/\|T\|$ is a principal direction. Moreover,

$$AU_1 = aU_1 \qquad \text{and} \qquad \operatorname{spec}\Big(A|_{U_1^\perp}\Big) = \{\sqrt{\tau}, -\sqrt{\tau}\}, \tag{26}$$

with

$$\operatorname{mult}_{A|_{U_1^\perp}}(\sqrt{\tau}) = \operatorname{mult}_{A|_{U_1^\perp}}(-\sqrt{\tau}) = \frac{m}{2}\cdot \tag{27}$$

Since the point $p \in \Sigma$ was arbitrary, conclusions (26)–(27) hold at every point of Σ. The case $\tau < 0$ having been ruled out, we necessarily have $\epsilon = -1$, so that

$$\Sigma \subset \mathbb{H}^n \times \mathbb{R} \qquad \text{and} \qquad \tau = \|T\|^2.$$

Let M_c be a connected component of a regular slice of Σ, namely,

$$M_c = \Sigma \cap \big(\mathbb{H}^n \times \{c\}\big).$$

For every $X \in TM_c = U_1^\perp$, we have

$$X(\Theta) = -\langle AX, T\rangle = 0,$$

for $U_1 = T/\|T\|$ is a principal direction. Hence Θ, and therefore

$$\|T\| = \sqrt{1 - \Theta^2},$$

is constant along M_c.

A horizontal unit normal to $M_c \subset \mathbb{H}^n$ is

$$\eta = \frac{N - \Theta\partial_t}{\|T\|}\cdot$$

Let S denote the shape operator of $M_c \subset \mathbb{H}^n$ with respect to η. Since Θ and $\|T\|$ are constant along M_c, and ∂_t is parallel, we have, for every $X \in TM_c$,

$$SX = -\overline{\nabla}_X \eta = \frac{1}{\|T\|}AX = \frac{1}{\sqrt{\tau}}AX.$$

This equality, together with (26)–(27), implies that M_c has precisely two constant principal curvatures, 1 and -1, each with multiplicity $m/2$. Thus M_c is an isoparametric hypersurface of $\mathbb{H}^n \times \{c\} \simeq \mathbb{H}^n$. However, the classification of isoparametric hypersurfaces of $\mathbb{H}^n$ shows that no such hypersurface exists (cf. [3], [7, Thm. 7]). This contradiction completes the proof of the lemma. □

Expanding $D(r) = \det B(r)$ with respect to the first row of $B(r)$ and considering the equalities (14), one can easily prove by induction that, for any integers $n \geq 2$ and $k \geq 1$, and for any $\ell \in \{0, \ldots, n-1\}$, there are coefficients $\alpha_{\ell,k}, \beta_{\ell,k}$, which do not depend on r, such that

$$D^{(k)}(r) = \sum_{\ell=0}^{n-1} (\alpha_{\ell,k} + \beta_{\ell,k} r) s_\tau^\ell(r) c_\tau^{n-1-\ell}(r). \tag{28}$$

Taking the first $2n-1$ derivatives of the (constant) function f at $r=0$, and recalling that

$$d_k = -\phi_k(H(0), H'(0), \dots, H^{(k)}(0)),$$

it follows from (15) and (28) that

$$0 = f^{(k)}(0) = \alpha_{0,k+1} - d_k \quad \text{for every } k \in \{1, \dots, 2n-1\}. \tag{29}$$

In addition, as shown in Lemma 9 of the appendix, the coefficients $\alpha_{\ell,k}$ and $\beta_{\ell,k}$ satisfy recursive equations which allow us to express each of them in terms of $\alpha_{\ell,0}$ and $\beta_{\ell,0}$. In this way, for any $k \in \mathbb{N}$, after $k+1$ steps, we can write $\alpha_{0,k+1}$ as the linear combination

$$\alpha_{0,k+1} = \sum_{\ell=0}^{n-1} (p_{k+1,\ell}\alpha_{\ell,0} + q_{k+1,\ell}\beta_{\ell,0}), \tag{30}$$

where the coefficients $p_{k+1,\ell}$ and $q_{k+1,\ell}$ depend only on k, n, ℓ, and τ. Moreover, we have that $\alpha_{0,0} = D(0) = 1$ and, by (29), that $\alpha_{0,k+1}$ coincides with the constant d_k. Therefore, the vector

$$x_0 = (\alpha_{1,0}, \dots, \alpha_{n-1,0}, \beta_{0,0}, \dots, \beta_{n-1,0})$$

is a solution to the linear system

$$Mx = P, \quad x \in \mathbb{R}^{2n-1}, \tag{31}$$

whose augmented matrix $[M|P]$ has vector rows $L_1, \dots, L_{2n-1} \in \mathbb{R}^{2n}$, where

$$L_k := (p_{k+1,1}, p_{k+1,2}, \dots, p_{k+1,n-1}, q_{k+1,0}, q_{k+1,1}, \dots, q_{k+1,n-1}, d_k - p_{k+1,0}).$$

In what follows, by means of a thorough analysis of the system (31), we shall show that $\tau = \tau(p)$ is necessarily a root of an algebraic equation, and so τ must be constant on Σ. To that end, it will be convenient to consider first the cases $n=2$ and $n=3$.

Case 1: $n = 2$**.**

As we mentioned before, this case was considered in [20]. We include it here to better illustrate our strategy, which is distinct from the one employed there.

For $n=2$, the equalities of Lemma 9 in the appendix yield

- $\alpha_{0,2} = 2\beta_{1,0} + \tau\alpha_{0,0}$;
- $\alpha_{0,3} = \tau\alpha_{1,0} + 3\tau\beta_{0,0}$;
- $\alpha_{0,4} = 4\tau\beta_{1,0} + \tau^2\alpha_{0,0}$.

These equalities imply that the augmented matrix $[M|P]$ of the system (31) is

$$[M|P] = \left[\begin{array}{ccc|c} 0 & 0 & 2 & d_1 - \tau \\ \tau & 3\tau & 0 & d_2 \\ 0 & 0 & 4\tau & d_3 - \tau^2 \end{array}\right],$$

where $d_i = -\phi_i(0)$, $i \in \{1,2,3\}$. However, it is easily seen that $\det M = 0$. Hence, denoting by M_j the matrix obtained from M by replacing its j-th column with P, we have that $\det M_j = 0$ for all $j \in \{1,2,3\}$. Otherwise, by Cramer's rule, the system $Mx = P$ would have no solution, thereby contradicting the existence of the solution $x_0 = (\alpha_{1,0}, \beta_{0,0}, \beta_{1,0})$. In particular, we have

$$0 = \det M_2 = 2\tau^3 - 4d_1\tau^2 + 2d_3\tau, \tag{32}$$

so that τ is a root of a polynomial equation. This proves Proposition 7 for $n=2$.

Case 2: $n = 3$**.**

Firstly, we point out that, as shown in Proposition 10 in the appendix, the following equalities hold:

- $L_{2s-1} = (0, 2^{2s-1}\tau^{s-1}, 0, s2^{2s-1}\tau^{s-1}, 0, d_{2s-1} - 2^{2s-1}\tau^s)$;
- $L_{2s} = (2^{2s}\tau^s, 0, (2s+1)2^{2s-1}\tau^s, 0, (2s+1)2^{2s-1}\tau^{s-1}, d_{2s})$.

Proceeding as in the previous case, we conclude that, for $n = 3$, the augmented matrix $[M|P]$ is

$$[M|P] = \left[\begin{array}{ccccc|c} 0 & 2 & 0 & 2 & 0 & d_1 - 2\tau \\ 4\tau & 0 & 6\tau & 0 & 6 & d_2 \\ 0 & 8\tau & 0 & 16\tau & 0 & d_3 - 8\tau^2 \\ 16\tau^2 & 0 & 40\tau^2 & 0 & 40\tau & d_4 \\ 0 & 32\tau^2 & 0 & 96\tau^2 & 0 & d_5 - 32\tau^3 \end{array}\right].$$

Again, we have $\det M = 0$, so that $\det M_j = 0$ for any $j \in \{1, \ldots, 5\}$, for

$$x_0 = (\alpha_{1,0}, \alpha_{2,0}, \beta_{0,0}, \beta_{1,0}, \beta_{2,0})$$

is a solution to (31). Since the 3rd and 5th column vectors of M are linearly dependent, for $j \notin \{3, 5\}$, the equality $\det M_j = 0$ holds for any τ. On the other hand, a direct computation gives

$$\det M_5 = -\det M_3 = d_1 2^{14}\tau^6 - d_3 2^{13}\tau^5 + d_5 2^{10}\tau^4. \tag{33}$$

Thus, if d_1, d_3 and d_5 are not all zero, any of the equalities $\det M_j = 0$, $j \in \{3, 5\}$, gives that τ is a nonzero root of a polynomial equation. However, if $d_1 = d_3 = d_5 = 0$, the determinants of all M_j will vanish identically. To overcome this problem, we replace the system $Mx = P$ with a suitable system $\overline{M}x = \overline{P}$ as follows.

Given $s > 3$, consider the system $\overline{M}x = \overline{P}$, where $[\overline{M}|\overline{P}]$ is obtained from $[M|P]$ by replacing its fifth row vector L_5 with L_{2s-1}. The above reasoning applied to $\overline{M}x = \overline{P}$ leads to the same conclusion: if d_1, d_3 and d_{2s-1} are not all zero, then τ is a nonzero root of a polynomial equation. So, assume that

$$d_k = 0 \quad \text{for every odd } \ k \geq 1.$$

Then, it follows from Lemma 8 that there exists $s_* = 2s > 4$ such that

$$d_{2s} \neq 0.$$

Consider the system $\overline{M}x = \overline{P}$, where $[\overline{M}|\overline{P}]$ is obtained from $[M|P]$ by replacing its fifth row vector L_5 with L_{2s}. Setting

$$\lambda = 2^{2s}\tau^s, \quad \mu = (2s+1)2^{2s-1}\tau^s \quad \text{and} \quad \nu = (2s+1)2^{2s-1}\tau^{s-1},$$

$[\overline{M}|\overline{P}]$ takes the form

$$[\overline{M}|\overline{P}] = \left[\begin{array}{ccccc|c} 0 & 2 & 0 & 2 & 0 & d_1 - 2\tau \\ 4\tau & 0 & 6\tau & 0 & 6 & d_2 \\ 0 & 8\tau & 0 & 16\tau & 0 & d_3 - 8\tau^2 \\ 16\tau^2 & 0 & 40\tau^2 & 0 & 40\tau & d_4 \\ \lambda & 0 & \mu & 0 & \nu & d_{2s} \end{array}\right].$$

Noticing that $\nu\tau = \mu$, one has $\det \overline{M} = 2^{14}\tau^3(\nu\tau - \mu) = 0$. Hence, since x_0 is also a solution to $\overline{M}x = \overline{P}$, we must have

$$0 = \det \overline{M}_3 = d_2(2-s)2^{2s+8}\tau^{s+2} + d_4(s-1)2^{2s+6}\tau^{s+1} - d_{2s}2^{10}\tau^3. \tag{34}$$

Since $d_{2s} \neq 0$, the right-hand side of this equality is a polynomial, which forces τ to be constant. This proves Proposition 7 in the case $n = 3$.

Next, we treat the general case $n > 3$. Our goal is to show that the augmented matrix $[M \mid P]$ of size $(2n-1) \times 2n$ has the same key properties as in the cases $n = 2$ and $n = 3$, which led to the polynomial identities (32) (for $n = 2$) and (33)–(34) (for $n = 3$). This is more involved; we establish the needed facts in the appendix (Section 6) via a sequence of lemmas and propositions. For the application here, we split the proof for $n > 3$ into two parts.

Case 3: *$n > 3$, n even.*

By Proposition 20 (i)(a), $\det M = 0$. Since x_0 solves $Mx = P$, it follows that $\det M_j = 0$ for all $j \in \{1, \dots, 2n-1\}$. Let $j = j_*$ be the index provided by Proposition 20 (i)(b). Then the equality $\det M_{j_*} = 0$ is a nontrivial algebraic equation in the variable τ.

Case 4: *$n > 3$, n odd.*

By Lemma 8, there exists $s_* \geq 2n$ such that $d_{s_*} \neq 0$. Let $M(s_*)$ be the matrix obtained from M by replacing its last row with L_{s_*} (omitting the last entry). By Proposition 20 (ii)(a), we have $\det M(s_*) = 0$. So, as before,

$$\det M_j(s_*) = 0 \quad \forall j \in \{1, \dots, 2n-1\}.$$

Finally, setting $j = n$, it follows from Proposition 20 (ii)(b) that the equality

$$\det M_n(s_*) = 0$$

yields a nontrivial algebraic equation in τ.

This concludes the proof of Proposition 7. □

5. Proof of Theorem 1

In this brief section we prove Theorem 1, which we restate here for the reader's convenience:

Theorem 1. *Let Σ be a connected hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. Then the following are equivalent:*

(i) *Σ is isoparametric.*

(ii) *Σ has constant angle and constant principal curvatures.*

(iii) *Σ is an open subset of one of the following complete hypersurfaces:*

  (a) *a horizontal slice $\mathbb{Q}^n_\epsilon \times \{t_0\}$,*

  (b) *a vertical cylinder over a complete isoparametric hypersurface of $\mathbb{Q}^n_\epsilon$,*

  (c) *a parabolic bowl of $\mathbb{H}^n \times \mathbb{R}$ (see Fig. 1).*

Moreover, in the hyperbolic case $\epsilon = -1$, the condition

(iv) *Σ is an open subset of a homogeneous hypersurface*

is also equivalent to (i)–(iii)*. In the spherical case $\epsilon = 1$,* (iv) *is equivalent to*

(v) *Σ is an open subset of either a horizontal slice or a vertical cylinder over a complete homogeneous hypersurface of $\mathbb{S}^n$.*

Proof. (i) ⇒ (ii). If Σ is isoparametric, then by Proposition 7 it has constant angle; hence, by the first part of Proposition 5, it has constant principal curvatures.

(ii) ⇒ (iii). This follows directly from the second part of Proposition 5.

(iii) $\Rightarrow$ (iv) (for $\epsilon = -1$). In $\mathbb{H}^n$, a complete hypersurface is isoparametric if and only if it is homogeneous. Therefore any vertical cylinder in $\mathbb{H}^n \times \mathbb{R}$ over a complete isoparametric hypersurface of $\mathbb{H}^n$ is homogeneous; slices are trivially homogeneous; and parabolic bowls are homogeneous (as already verified). Hence all hypersurfaces listed in (iii) are homogeneous in $\mathbb{H}^n \times \mathbb{R}$.

(iv) $\Rightarrow$ (i). It is well known that this holds in general, i.e., any homogeneous hypersurface of a Riemannian manifold is isoparametric.

(iv) $\Leftrightarrow$ (v) (for $\epsilon = 1$). The implication (v) $\Rightarrow$ (iv) is immediate. Conversely, assume $\Sigma \subset \mathbb{S}^n \times \mathbb{R}$ is homogeneous. From (iv) $\Rightarrow$ (i) $\Rightarrow$ (iii), Σ must be one of the hypersurfaces in (a) or (b). But a vertical cylinder in $\mathbb{S}^n \times \mathbb{R}$ over a complete isoparametric hypersurface Σ_0 of $\mathbb{S}^n$ is homogeneous if and only if Σ_0 is homogeneous. This proves the equivalence. □

6. Appendix

In this appendix, we prove some results which we have used in the proof of Proposition 7. We keep the notation of the preceding section.

To start with, define the row vector

$$\mathfrak{L}_{k-1} = (p_{k,0}, \ldots, p_{k,n-1}, q_{k,0}, \ldots, q_{k,n-1}), \;\; k \geq 2, \tag{35}$$

and denote by $Z = [p_{k,\ell}, q_{k,\ell}]$ the $(2n-1) \times 2n$ matrix with rows $\mathfrak{L}_{k-1}$, $2 \leq k \leq 2n$:

$$Z = [p_{k,\ell}, q_{k,\ell}] = \begin{bmatrix} \mathfrak{L}_1 \\ \vdots \\ \mathfrak{L}_{2n-1} \end{bmatrix}. \tag{36}$$

Our interest in the matrix Z relies on the equality $Z = [-P_\tau, M]$, where

$$P_\tau := P - P_d, \quad P_d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{2n-1} \end{bmatrix}. \tag{37}$$

In what follows, we establish a series of lemmas that will lead to a complete description of Z (see Propositions 10 and 13).

Lemma 9. *Let $\alpha_{\ell,k}$ and $\beta_{\ell,k}$ be the coefficient functions defined in* (28)*, that is,*

$$D^{(k)}(r) = \sum_{\ell=0}^{n-1} (\alpha_{\ell,k} + \beta_{\ell,k} r) s_\tau^\ell(r) c_\tau^{n-1-\ell}(r). \tag{38}$$

Then, for any $n, k \in \mathbb{N}$, the following equalities hold:

$$\begin{aligned}
\alpha_{0,k+1} &= \beta_{0,k} + \alpha_{1,k};\\
\alpha_{\ell,k+1} &= \beta_{\ell,k} + (\ell+1)\alpha_{\ell+1,k} + \tau(n-\ell)\alpha_{\ell-1,k} \quad \text{if } \ell \in \{1, \ldots, n-2\};\\
\alpha_{n-1,k+1} &= \beta_{n-1,k} + \tau\alpha_{n-2,k};
\end{aligned}$$

$$\begin{aligned}
\beta_{0,k+1} &= \beta_{1,k};\\
\beta_{\ell,k+1} &= (\ell+1)\beta_{\ell+1,k} + \tau(n-\ell)\beta_{\ell-1,k} \qquad \text{if } \ell \in \{1, \ldots, n-2\};\\
\beta_{n-1,k+1} &= \tau\beta_{n-2,k}.
\end{aligned}$$

Proof. Considering (38), we have that

$$\begin{aligned}
D^{(k+1)}(r) &= \sum_{\ell=0}^{n-1} \big[\beta_{\ell,k} s_\tau^\ell(r) c_\tau^{n-1-\ell}(r) + \ell(\alpha_{\ell,k} + \beta_{\ell,k} r) s_\tau^{\ell-1}(r) c_\tau^{n-\ell}(r) \\
&\qquad + (n-1-\ell)(\alpha_{\ell,k} + \beta_{\ell,k} r) s_\tau^{\ell+1}(r) c_\tau^{n-2-\ell}(r)\big] \\
&= \sum_{\ell=0}^{n-1} \beta_{\ell,k} s_\tau^\ell(r) c_\tau^{n-1-\ell}(r) + \sum_{p=0}^{n-1} p(\alpha_{p,k} + \beta_{p,k} r) s_\tau^{p-1}(r) c_\tau^{n-p}(r) \\
&\quad + \sum_{q=0}^{n-1} (n-1-q)(\alpha_{q,k} + \beta_{q,k} r) s_\tau^{q+1}(r) c_\tau^{n-2-q}(r) \\
&= \sum_{\ell=0}^{n-1} \beta_{\ell,k} s_\tau^\ell(r) c_\tau^{n-1-\ell}(r) + \sum_{\ell=0}^{n-2} \ell(\alpha_{\ell+1,k} + \beta_{\ell+1,k} r) s_\tau^\ell(r) c_\tau^{n-1-\ell}(r) \\
&\quad + \sum_{\ell=1}^{n-1} (n-\ell)(\alpha_{\ell-1,k} + \beta_{\ell-1,k} r) s_\tau^\ell(r) c_\tau^{n-1-\ell}(r) \\
&= (\beta_{0,k} + \alpha_{1,k} + \beta_{1,k} r) c_\tau^{n-1}(r) + (\beta_{n-1,k} + \tau\alpha_{n-2,k} + \tau\beta_{n-2,k} r) s_\tau^{n-1}(r) \\
&\quad + \sum_{\ell=1}^{n-2} [\beta_{\ell,k} + (\ell+1)(\alpha_{\ell+1,k} + \beta_{\ell+1,k} r) \\
&\qquad\qquad + \tau(n-\ell)(\alpha_{\ell-1,k} + \beta_{\ell-1,k} r)] s_\tau^\ell(r) c_\tau^{n-1-\ell}(r).
\end{aligned}$$

The result follows by comparison of the coefficients of the last equality with those of $D^{(k+1)}$ as given in (38). □

Now, with our purpose of describing the matrix Z, we establish some fundamental properties of the coefficients $p_{k,\ell}$ and $q_{k,\ell}$ defined by the equality

$$\alpha_{0,k+1} = \sum_{\ell=0}^{n-1} (p_{k+1,\ell} \alpha_{\ell,0} + q_{k+1,\ell} \beta_{\ell,0}). \tag{39}$$

We start with the case $n = 3$, in which one has

$$p_{2,0} = 2\tau,\ p_{2,1} = 0,\ p_{2,2} = 2,\ q_{2,0} = 0,\ q_{2,1} = 2, \text{ and } q_{2,2} = 0. \tag{40}$$

Besides, for all $k \geq 2$, we have from Lemma 9 that

$$\begin{aligned}
\alpha_{0,k+1} &= \sum_{\ell=0}^{2} (p_{k,\ell} \alpha_{\ell,1} + q_{k,\ell} \beta_{\ell,1}) \\
&= 2\tau p_{k,1} \alpha_{0,0} + (p_{k,0} + \tau p_{k,2}) \alpha_{1,0} + 2 p_{k,1} \alpha_{2,0} \\
&\quad + (p_{k,0} + 2\tau q_{k,1}) \beta_{0,0} + (p_{k,1} + q_{k,0} + \tau q_{k,2}) \beta_{1,0} + (p_{k,2} + 2 q_{k,1}) \beta_{2,0}.
\end{aligned}$$

Comparing this last equality with (39) for $n = 3$, we conclude that, for all $k \geq 2$, the following identities hold:

$$
\begin{aligned}
p_{k+1,0} &= 2\tau p_{k,1} \\
p_{k+1,1} &= p_{k,0} + \tau p_{k,2} \\
p_{k+1,2} &= 2p_{k,1} \\
\\
q_{k+1,0} &= p_{k,0} + 2\tau q_{k,1} \\
q_{k+1,1} &= p_{k,1} + q_{k,0} + \tau q_{k,2} \\
q_{k+1,2} &= p_{k,2} + 2q_{k,1}.
\end{aligned} \tag{41}
$$

By means of the equalities in (41), we can compute explicitly the coefficients $p_{k,\ell}$ and $q_{k,\ell}$ as shown in the proposition below.

Proposition 10. *For any $k \geq 2$, $s \geq 1$ and $\ell \in \{0, 1, 2\}$, the following hold:*

(i) *If $k + \ell$ is odd, then $p_{k,\ell} = 0$, if $k + \ell$ is even, then $q_{k,\ell} = 0$;*

(ii) $p_{2s,0} = \tau p_{2s,2} = 2^{2s-1}\tau^s$;

(iii) $p_{2s+1,1} = 2^{2s}\tau^s$;

(iv) $q_{2s+1,0} = \tau q_{2s+1,2} = (2s + 1)2^{2s-1}\tau^s$;

(v) $q_{2s,1} = s2^{2s-1}\tau^{s-1}$.

Proof. (i) We proceed by induction on k. The result is true if $k = 2$ by the initial conditions (40). Suppose now that for any ℓ such that $k+\ell$ is odd we have $p_{k,\ell} = 0$ and for any ℓ such that $k + \ell$ is even we have $q_{k,\ell} = 0$. The first result follows by applying the inductive hypothesis to the equalities for $p_{k+1,\ell}$ in (41). Indeed, in these equalities, just the p-functions appear, and the parity of the sum of the indices does not change. Similarly, in the equation for $q_{k+1,\ell}$, the parity of the sum of the indices of the q-functions doesn't change, whereas it changes for the p-functions.

(ii) We prove the two equalities separately. By (41), we have

$$p_{2s+2,0} = 2\tau p_{2s+1,1} = \tau p_{2s+2,2}.$$

For the second equality, we proceed by induction on s. The case $s = 1$ is true because of the initial conditions (40). Now, let $s \geq 1$, and suppose that $p_{2s,0} = 2^{2s-1}\tau^s$. Applying (41) twice and considering the previous equality, we have from the inductive hypothesis that

$$
\begin{aligned}
p_{2s+2,0} &= 2\tau p_{2s+1,1} \\
&= 2\tau(p_{2s,0} + \tau p_{2s,2}) \\
&= 2\tau(2^{2s-1}\tau^s + 2^{2s-1}\tau^s) \;=\; 2^{2s-1}\tau^{s-1}.
\end{aligned}
$$

(iii) Fix $s \geq 1$. By (41) and the proved equality (ii), we have

$$p_{2s+1,1} = p_{2s,0} + \tau p_{2s,2} = 2(2^{2s-1}\tau^s) = 2^{2s}\tau^s.$$

(iv) The first equality can be proved with the help of (41) and equality (ii):

$$q_{2s+1,0} = p_{2s,0} + 2\tau q_{2s,1} = \tau p_{2s,2} + \tau q_{2s,1} = \tau q_{2s+1,2}.$$

For the second equality, we proceed by induction on s. From (40) and (41), we have:

$$q_{3,0} = p_{2,0} + 2\tau q_{2,1} = 4\tau.$$

Suppose the result is true for $s \geq 1$. By (41), the previous equality, and (ii), we have:

$$\begin{aligned} q_{2s+3,0} &= p_{2s+2,0} + 2\tau q_{2s+2,1} \\ &= 2^{2s+1}\tau^{s+1} + 2\tau(p_{2s+1,1} + q_{2s+1,0} + \tau q_{2s+1,2}) \\ &= 2^{2s+2}\tau^{s+1} + 4\tau q_{2s+1,0} \\ &= 2^{2s+2}\tau^{s+1} + 4\tau(2^{2s-1}\tau^{s-1}(2s+1)) \\ &= 2^{2s+1}\tau^{s}(2s+3). \end{aligned}$$

(v) From (41), (iii), and (iv), we get

$$\begin{aligned} q_{2s,1} &= p_{2s-1,1} + q_{2s-1,0} + \tau q_{2s-1,2} \\ &= 2^{2s-2}\tau^{s-1} + 2q_{2s-1,0} \\ &= 2^{2s-2}\tau^{s-1} + 2((2s-1)2^{2s-3}\tau^{s-1}) \\ &= s2^{2s-1}\tau^{s-1}, \end{aligned}$$

and this finishes the proof. □

Next, we apply Lemma 9 to establish relations between the coefficient functions of the expression of $\alpha_{0,k+1}$ as in (39) for any $n \geq 4$.

Lemma 11. *Let $p_{k+1,\ell}$ and $q_{k+1,\ell}$ be the coefficient functions defined in* (30)*, i.e.,*

$$\alpha_{0,k+1} = \sum_{\ell=0}^{n-1}(p_{k+1,\ell}\alpha_{\ell,0} + q_{k+1,\ell}\beta_{\ell,0}).$$

Then, for all $n \geq 4$ and $k \geq 2$, the following equalities hold:

$$\begin{aligned} p_{2,0} &= (n-1)\tau, \quad p_{2,2} = 2 \quad \text{and} \quad p_{2,\ell} = 0 \text{ if } \ell \neq 0, 2 \\ p_{k+1,0} &= (n-1)\tau p_{k,1} \\ p_{k+1,\ell} &= \ell \cdot p_{k,\ell-1} + (n-1-\ell)\tau p_{k,\ell+1}, \quad \text{if } \ell \in \{1, \ldots, n-2\} \\ p_{k+1,n-1} &= (n-1)p_{k,n-2} \end{aligned}$$

$$\begin{aligned} q_{2,1} &= 2 \quad \text{and} \quad q_{2,\ell} = 0 \text{ if } \ell \neq 1 \\ q_{k+1,0} &= p_{k,0} + (n-1)\tau q_{k,1} \\ q_{k+1,\ell} &= p_{k,\ell} + \ell \cdot q_{k,\ell-1} + (n-1-\ell)\tau q_{k,\ell+1}, \quad \text{if } \ell \in \{1, \ldots, n-2\} \\ q_{k+1,n-1} &= p_{k,n-1} + (n-1)q_{k,n-2} \end{aligned}$$

Proof. We have that

$$\alpha_{0,k+1} = \sum_{\ell=0}^{n-1}(p_{k+1,\ell}\alpha_{\ell,0} + q_{k+1,\ell}\beta_{\ell,0}),$$

but, by Lemma 9 we can also prove that

$$\begin{aligned}
\alpha_{0,k+1} &= \sum_{\ell=0}^{n-1}(p_{k,\ell}\alpha_{\ell,1} + q_{k,\ell}\beta_{\ell,1}) \\
&= p_{k,0}(\beta_{0,0} + \alpha_{1,0}) + p_{k,n-1}(\beta_{n-1,0} + \tau\alpha_{n-2,0}) \\
&\quad + \sum_{\ell=1}^{n-2} p_{k,\ell}(\beta_{\ell,0} + (\ell+1)\alpha_{\ell+1,0} + (n-\ell)\tau\alpha_{\ell-1,0}) \\
&\quad + q_{k,0}\beta_{1,0} + q_{k,n-1}\tau\beta_{n-2,0} \\
&\quad + \sum_{\ell=1}^{n-2} q_{k,\ell}((\ell+1)\beta_{\ell+1,0} + (n-\ell)\tau\beta_{\ell-1,0}) \\
&= (n-1)\tau p_{k,1}\alpha_{0,0} + (p_{k,0} + (n-2)\tau p_{k,2})\alpha_{1,0} \\
&\quad + \sum_{\ell=2}^{n-3}(\ell\cdot p_{k,\ell-1} + (n-1-\ell)\tau p_{k,\ell+1})\alpha_{\ell,0} \\
&\quad + ((n-2)p_{k,n-3} + \tau p_{k,n-1})\alpha_{n-2,0} + ((n-1)p_{k,n-2})\alpha_{n-1,0} \\
&\quad + (p_{k,0} + (n-1)\tau q_{k,1})\beta_{0,0} + (p_{k,1} + q_{k,0} + (n-2)\tau q_{k,2})\beta_{1,0} \\
&\quad + \sum_{\ell=2}^{n-3}(p_{k,\ell} + \ell\cdot q_{k,\ell-1} + (n-1-\ell)\tau q_{k,\ell+1})\beta_{\ell,0} \\
&\quad + (p_{k,n-2} + (n-2)q_{k,n-3} + \tau q_{k,n-1})\beta_{n-2,0} \\
&\quad + (p_{k,n-1} + (n-1)q_{k,n-2})\beta_{n-1,0}
\end{aligned}$$

In the last equality, if $n = 4$, then the summand from $\ell = 2$ to $\ell = n-3$ should be ignored. The result follows by comparison of the corresponding coefficients. □

Example 12. By means of the relations established in Lemma 11, we can obtain the matrix $Z = [p_{k,\ell}, q_{k,\ell}]$ for any value of $n \geq 2$. For n from 2 to 5, for instance, Z is given by the matrices:

- $\begin{bmatrix} \tau & 0 & 0 & 2 \\ 0 & \tau & 3\tau & 0 \\ \tau^2 & 0 & 0 & 4\tau \end{bmatrix}$;
- $\begin{bmatrix} 2\tau & 0 & 2 & 0 & 2 & 0 \\ 0 & 4\tau & 0 & 6\tau & 0 & 6 \\ 8\tau^2 & 0 & 8\tau & 0 & 16\tau & 0 \\ 0 & 16\tau^2 & 0 & 40\tau^2 & 0 & 40\tau \\ 32\tau^3 & 0 & 32\tau^2 & 0 & 96\tau^2 & 0 \end{bmatrix}$;
- $\begin{bmatrix} 3\tau & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 7\tau & 0 & 6 & 9\tau & 0 & 6 & 0 \\ 21\tau^2 & 0 & 20\tau & 0 & 0 & 28\tau & 0 & 24 \\ 0 & 61\tau^2 & 0 & 60\tau & 105\tau^2 & 0 & 100\tau & 0 \\ 183\tau^3 & 0 & 182\tau^2 & 0 & 0 & 366\tau^2 & 0 & 360\tau \\ 0 & 547\tau^3 & 0 & 546\tau^2 & 1281\tau^3 & 0 & 1274\tau^2 & 0 \\ 1641\tau^4 & 0 & 1640\tau^3 & 0 & 0 & 4376\tau^3 & 0 & 4368\tau^2 \end{bmatrix}$;

- $$\begin{bmatrix} 4\tau & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 10\tau & 0 & 6 & 0 & 12\tau & 0 & 6 & 0 & 0 \\ 40\tau^2 & 0 & 32\tau & 0 & 24 & 0 & 40\tau & 0 & 24 & 0 \\ 0 & 136\tau^2 & 0 & 120\tau & 0 & 200\tau^2 & 0 & 160\tau & 0 & 120 \\ 544\tau^3 & 0 & 512\tau^2 & 0 & 480\tau & 0 & 816\tau^2 & 0 & 720\tau & 0 \\ 0 & 2080\tau^3 & 0 & 2016\tau^2 & 0 & 3808\tau^3 & 0 & 3584\tau^2 & 0 & 3360\tau \\ 8320\tau^4 & 0 & 8192\tau^3 & 0 & 8064\tau^2 & 0 & 16640\tau^3 & 0 & 16128\tau^2 & 0 \\ 0 & 32896\tau^4 & 0 & 32640\tau^3 & 0 & 74880\tau^2 & 0 & 73728\tau^3 & 0 & 72576\tau^2 \\ 131584\tau^5 & 0 & 131072\tau^4 & 0 & 130560\tau^3 & 0 & 328960\tau^2 & 0 & 326400\tau^3 & 0 \end{bmatrix}.$$

In the next proposition, we establish some fundamental properties of the coefficients $p_{k,\ell}$, $q_{k,\ell}$ that will lead to a general description of the matrix Z. These properties can be checked in the matrices of the above example.

Proposition 13. *Given $n \geq 4$ and $k \geq 2$, the following assertions hold:*

(i) *$p_{k,\ell} = 0$ if $k + \ell$ is odd, and $q_{k,\ell} = 0$ if $k + \ell$ is even;*

(ii) *$p_{k,\ell} = q_{k,\ell} = 0$ for all $\ell > k$;*

(iii) *$p_{k,k} = k!$ and $q_{k+1,k} = (k + 1)!$ for all $k \geq 2$;*

(iv) *if $k - \ell = 2s$, then $p_{k,\ell} = \sigma^m_{k,\ell}(n)\tau^s$, where $\sigma^m_{k,\ell}$ is a polynomial of degree $m \geq s$ with positive leading coefficient;*

(v) *if $k - \ell = 2s + 1$, then $q_{k,\ell} = \omega^m_{k,\ell}(n)\tau^s$, where $\omega^m_{k,\ell}$ is a polynomial of degree $m \geq s$ with positive leading coefficient.*

Proof. (i) Analogous to the one given for Proposition 10-(i).

(ii) We proceed by induction on k. The result is true if $k = 2$, by the initial conditions in Lemma 11. Suppose now that, for any $\ell > k$, we have $p_{k,\ell} = 0$. In this setting, for any $\ell > k+1$, we have from Lemma 11 that $p_{k+1,\ell} = ap_{k,\ell-1} + b\tau p_{k,\ell+1}$, where a and b are positive integers. Since $\ell + 1 > \ell - 1 > k$, it follows from the inductive hypothesis that $p_{k+1,\ell} = 0$. The assertion on $q_{k+1,\ell}$ is proved analogously by using induction and the result just proved for $p_{k+1,\ell}$.

(iii) We proceed by induction on k. For $k = 2$ the result is immediate. Suppose that it is true for $k \geq 2$. Under these conditions, one concludes that the result is true for $k + 1$ by applying the recursive formulas of Lemma 11 and the proved item (ii).

(iv) We proceed by induction on k: suppose that for any ℓ such that $k - \ell = 2s$ the result is true. Then,

$$\begin{aligned} p_{k+1,\ell+1} &= (\ell + 1)p_{k,\ell} + (n - 2 - \ell)\tau p_{k,\ell+2} \\ &= (\ell + 1)\sigma^{m_1}_{k,\ell}(n)\tau^s + (n - 2 - \ell)\tau\sigma^{m_2}_{k,\ell+2}(n)\tau^{s-1} \\ &= \sigma^m_{k+1,\ell+1}(n)\tau^s, \end{aligned}$$

where $\sigma^m_{k+1,\ell+1}$ is the polynomial of degree $m = \max\{m_1, m_2\} > s$ defined by

$$\sigma^m_{k+1,\ell+1} = (\ell + 1)\sigma^{m_1}_{k,\ell} + (n - 2 - \ell)\sigma^{m_2}_{k,\ell+2}.$$

Clearly, the leading coefficient of $\sigma^m_{k+1,\ell+1}$ is positive. Therefore, the result is true for $k + 1$.

(v) Analogous to (iv). □

Now, we proceed to determine the rank of the matrix $Z = [-P_\tau|M]$ defined in (36). It will be convenient to reinterpret the recursive formulas of Lemma 11 in vectorial form. With that in mind, we consider the decomposition $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$

and set $e_1 = (1, \dots, 0) \in \mathbb{R}^n$. From the definition (35) of the row vectors $\mathfrak{L}_k$, after a straightforward computation we have that the row vectors $\mathfrak{L}_k$ relate as

$$\mathfrak{L}_k = \mathfrak{L}_{k-1} Q = (e_1, 0) Q^{k+1}, \;\; k \geq 2,$$

where Q is the $2n \times 2n$ matrix defined by

$$Q = \left[\begin{array}{c|c} \mathcal{K} & I_n \\ \hline O_n & \mathcal{K} \end{array}\right], \tag{42}$$

being the blocks 0_n and I_n the null and identity $n \times n$ matrices, respectively, and $\mathcal{K} = (k_{ij})$ the $n \times n$ matrix defined by the equalities

$$k_{ij} = \begin{cases} (n-j)\tau & \text{if } j = i-1, \\ i & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases}$$

that is,

$$\mathcal{K} = \begin{bmatrix} 0 & 1 & 0 & & 0 & 0 & 0 \\ (n-1)\tau & 0 & 2 & \cdots & 0 & 0 & 0 \\ 0 & (n-2)\tau & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & n-2 & 0 \\ 0 & 0 & 0 & \cdots & 2\tau & 0 & n-1 \\ 0 & 0 & 0 & & 0 & \tau & 0 \end{bmatrix}.$$

Remark 14. In the case $\tau = 1$, the transpose of $\mathcal{K}$ is known as the *Kac* or *Sylvester-Kac matrix*. Due to that nomenclature, for $n \geq 2$ and $\tau \neq 0$, we shall call $\mathcal{K}$ the τ-*Kac matrix* of order n. We add that the Kac matrix appears in many different contexts as, for example, in the description of random walks on a hypercube (see [21] and the references therein).

Lemma 15. *The τ-Kac matrix $\mathcal{K}$ of order n has the following properties:*

(i) *It has n simple eigenvalues $\lambda_0, \dots, \lambda_{n-1}$, which are*

$$\lambda_\ell = (n - 1 - 2\ell)\sqrt{\tau}, \quad \ell \in \{0, 1, \dots, n-1\}.$$

In particular λ_ℓ is real if $\tau > 0$, and purely imaginary if $\tau < 0$.

(ii) *Its rank is n, if n is even, and $n-1$ if n is odd. In particular, $\mathcal{K}$ is nonsingular if and only if n is even.*

(iii) *The coordinates of $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$ with respect to the basis of its eigenvectors are all different from zero.*

Proof. We shall show (i) through a straight adaptation of the beautiful proof given for [21, Theorem 2.1]. To that end, we first define the functions (seen as vectors)

$$\omega_\ell(x) = s_\tau^\ell(x) c_\tau^{n-1-\ell}(x), \quad \ell \in \{0, \dots, n-1\},$$

where s_τ and c_τ are the functions defined in (13). Clearly, the set $\mathcal{B} = \{\omega_0, \dots, \omega_{n-1}\}$ is linearly independent, and so it generates a vector space V of dimension n. In addition, we have that

$$\frac{d}{dx}\omega_\ell(x) = \ell\omega_{\ell-1}(x) + (n-1-\ell)\tau\omega_{\ell+1},$$

from which we conclude that d/dx is an operator on V whose matrix with respect to the basis $\mathcal{B}$ is the τ-Kac matrix $\mathcal{K}$.

Now, considering the equality

$$e^{(n-1-2\ell)\sqrt{\tau}x} = (c_\tau(x) + \sqrt{\tau}s_\tau(x))^{n-1-\ell}(c_\tau(x) - \sqrt{\tau}s_\tau(x))^\ell,$$

we have that, for each $\ell \in \{0, \ldots, n-1\}$, the function $x \mapsto e^{(n-1-2\ell)\sqrt{\tau}x}$ belongs to V, and it is an eigenvector of d/dx with eigenvalue $(n-1-2\ell)\sqrt{\tau}$. This proves (i).

The statement (ii) follows directly from (i).

Finally, to prove (iii), we identify $\mathbb{R}^n$ with V, so that e_1 becomes the first vector $\omega_0(x) = c_\tau^{n-1}(x)$ of $\mathcal{B}$. Since $c_\tau(x) = \frac{1}{2}\left(e^{\sqrt{\tau}x} + e^{-\sqrt{\tau}x}\right)$, we have from the binomial formula that

$$c_\tau^{n-1}(x) = \frac{1}{2^{n-1}} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} e^{(n-1-2\ell)\sqrt{\tau}x},$$

which clearly proves (iii). □

Let $\{v_0, \ldots, v_{n-1}\} \subset \mathbb{R}^n$ be the basis of eigenvectors of the τ-Kac matrix $\mathcal{K}$. Consider the decomposition $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ and define the following vectors:

$$x_\ell = (v_\ell, 0) \quad \text{and} \quad y_\ell = (0, v_\ell), \ \ \ell \in \{0, \ldots, n-1\}. \tag{43}$$

As a direct consequence of Lemma 15 and the block structure of the $2n \times 2n$ matrix Q defined in (42), we have:

Corollary 16. *The following assertions hold true.*

(i) *The matrix Q is nonsingular if and only if n is even.*

(ii) *For any $\ell \in \{0, \ldots, n-1\}$, the vectors x_ℓ and y_ℓ defined in* (43) *satisfy:*

$$x_\ell Q = \lambda_\ell x_\ell + y_\ell \quad \text{and} \quad y_\ell Q = \lambda_\ell y_\ell,$$

i.e. x_ℓ is a generalized eigenvector of Q, whereas y_ℓ is an eigenvector of Q.

(iii) *Regarding the coordinates of $e_1 = (e_1, 0) \in \mathbb{R}^{2n}$ with respect to the basis*

$$\mathcal{B} = \{x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}\},$$

those with respect to the generalized eigenvectors x_ℓ never vanish, whereas the ones with respect to the eigenvectors y_ℓ are all zero.

Next, we establish a result that plays a crucial role in the proof of Proposition 20. In its proof, we shall consider a certain type of generalized Vandermonde matrix as defined in [26]. First, let us recall that, given n pairwise distinct numbers $x_0, \ldots, x_{n-1}$, the $n \times n$ Vandermonde matrix $V(x_0, \ldots, x_{n-1})$ is defined by

$$V(x_0, \ldots, x_{n-1}) = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & \ldots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & x_1^3 & \ldots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & x_{n-1}^3 & \ldots & x_{n-1}^{n-1} \end{bmatrix},$$

and its determinant is given by

$$\det V(x_0, \ldots, x_{n-1}) = \prod_{i<j} (x_j - x_i),$$

which implies that $V(x_0, \ldots, x_{n-1})$ is nonsingular.

The generalized Vandermonde matrix $V_2(x_0,\dots,x_{n-1})$ of type 2 is the following $2n\times 2n$ matrix:

$$V_2(x_0,\dots,x_{n-1})=\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & \dots & x_0^{2n-1}\\ 0 & 1 & 2x_0 & 3x_0^2 & \dots & (2n-1)x_0^{2n-2}\\ 1 & x_1 & x_1^2 & x_1^3 & \dots & x_1^{2n-1}\\ 0 & 1 & 2x_1 & 3x_1^2 & \dots & (2n-1)x_1^{2n-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & x_{n-1} & x_{n-1}^2 & x_{n-1}^3 & \dots & x_{n-1}^{2n-1}\\ 0 & 1 & 2x_{n-1} & 3x_{n-1}^2 & \dots & (2n-1)x_{n-1}^{2n-2} \end{bmatrix}.$$

Notice that each x_i defines a pair of row vectors L_i, L_{i+1} which satisfy

$$L_{i+1}=\frac{\partial L_i}{\partial x_i}\cdot$$

It can be proved that (see [26])

$$\det V_2(x_0,\dots,x_{n-1})=\prod_{i<j}(x_j-x_i)^4>0,$$

which implies that $V_2(x_0,\dots,x_{n-1})$ is nonsingular as well.

Proposition 17. *Setting $e_1=(e_1,0)\in\mathbb{R}^{2n}$, the following assertions hold.*

(i) *If n is even, for any positive integer s, the set*

$$\{e_1Q^s, e_1Q^{s+1},\dots,e_1Q^{s+2n-1}\}$$

is linearly independent.

(ii) *If n is odd, let $s\geq 2n$ and define*

$$\Lambda=\{e_1Q^2,e_1Q^3,\dots,e_1Q^{2n-1}\},\quad \Lambda_s=\Lambda\cup\{e_1Q^s\}.$$

Then, denoting by $C_1,\dots,C_{2n}$ the column vectors of the matrix $Z(s)$ whose rows are the vectors of Λ_s, the following hold:

(a) *Λ is linearly independent, whereas Λ_s is linearly dependent;*

(b) *C_1 is in the span of the odd columns $C_3,C_5,\dots,C_n$;*

(c) *C_{n+1} is in the span of the even columns $C_{n+3},C_{n+5},\dots,C_{2n}$.*

Proof. (i) When n is even, we know from Corollary 16-(i) that Q is invertible. So, it suffices to prove (i) for $s=0$. Notice that, for any $\ell\in\{0,\dots,n-1\}$, one has $x_\ell Q^2=\lambda_\ell^2 x_\ell+2\lambda_\ell y_\ell$. Therefore, by induction, one has that the equality

$$x_\ell Q^k=\lambda_\ell^k x_\ell+k\lambda_\ell^{k-1}y_\ell \tag{44}$$

holds for any positive integer k.

Now, consider the following vector equation of variables $c_0,\dots,c_{2n-1}$:

$$\sum_{k=0}^{2n-1}c_k e_1Q^k=0. \tag{45}$$

We have from Corollary 16-(iii) that

$$e_1 = \sum_{\ell=0}^{n-1} a_\ell x_\ell, \tag{46}$$

with $a_\ell \neq 0$ for any $\ell \in \{0, \ldots, n-1\}$.

Setting $\bar{x}_\ell = a_\ell x_\ell$ and $\bar{y}_\ell = a_\ell y_\ell$, we get from equalities (44)–(46) that

$$\sum_{\ell=0}^{n-1} \sum_{k=0}^{2n-1} \left(\lambda_\ell^k c_k \bar{x}_\ell + k\lambda_\ell^{k-1} c_k \bar{y}_\ell\right) = 0,$$

which implies that (45) is equivalent to the homogeneous linear system of equations:

$$\sum_{k=0}^{2n-1} \lambda_\ell^k c_k = 0, \quad \sum_{k=0}^{2n-1} k\lambda_\ell^{k-1} c_k = 0, \quad \ell \in \{0, \ldots, n-1\}. \tag{47}$$

The matrix of coefficients of the system (47) is the generalized Vandermonde matrix $V_2(\lambda_0, \ldots, \lambda_{n-1})$, and so it is invertible, since the eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$ are pairwise distinct. Thus, $c_k = 0$ for all $k \in \{0, \ldots, 2n-1\}$, which proves (i).

(ii)-(a) Assume now that n is odd and consider the following vector equation of $2n-1$ variables $c_2, \ldots, c_{2n-1}, c_s$:

$$\sum_{k=2}^{2n-1} c_k e_1 Q^k + c_s e_1 Q^s = 0. \tag{48}$$

Proceeding as in the case n even, we conclude that the equation (48) is equivalent to the linear system:

$$\sum_{k=2}^{2n-1} \lambda_\ell^k c_k + \lambda_\ell^s c_s = 0, \quad \sum_{k=2}^{2n-1} k\lambda_\ell^{k-1} c_k + s\lambda_\ell^{s-1} c_s = 0, \quad \ell \in \{0, \ldots, n-1\}. \tag{49}$$

Since n is odd, we have from Corollary 16 that $\lambda_{(n-1)/2} = 0$, so that (49) is a homogeneous linear system of $2n-2$ equations with $2n-1$ unknowns. The matrix R of coefficients of the system (49) can be decomposed into $n-1$ blocks as

$$R = \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_\ell \\ \vdots \\ B_{n-1} \end{bmatrix}, \quad \ell \neq \frac{n-1}{2},$$

where the generic block B_ℓ is the $2 \times (2n-1)$ matrix given by

$$B_\ell = \begin{bmatrix} \lambda_\ell^2 & \lambda_\ell^3 & \cdots & \lambda_\ell^{2n-1} & \lambda_\ell^s \\ 2\lambda_\ell & 3\lambda_\ell^2 & \cdots & (2n-1)\lambda_\ell^{2n-2} & s\lambda_\ell^{s-1} \end{bmatrix}.$$

It is immediate that $\operatorname{rank} R \leq 2n-2$, proving that Λ_s is linearly dependent. To prove that Λ is linearly independent, consider the matrix R_{2n-1} obtained from R by removing the last of its $2n-1$ columns. Each block of R_{2n-1} can be modified

through elementary operations on its rows, so that the row-equivalent resulting matrix $\overline{R}_{2n-1}$ is composed by $n-1$ blocks of $2\times(2n-2)$ matrices of the type

$$\begin{bmatrix} 1 & \lambda_\ell & \lambda_\ell^2 & \cdots & \lambda_\ell^{2n-3} \\ 0 & 1 & 2\lambda_\ell & \cdots & (2n-3)\lambda_\ell^{2n-4} \end{bmatrix},$$

from which we conclude that $\overline{R}_{2n-1}$ is the generalized Vandermonde matrix

$$V_2(\lambda_0,\ldots,\lambda_{(n-3)/2},\lambda_{(n+1)/2},\ldots,\lambda_{n-1}).$$

Therefore, $\overline{R}_{2n-1}$ is nonsingular, which implies that $\operatorname{rank} R_{2n-1} = 2n-2$, as we wished to prove.

(ii)-(b) Set $n=2m+1$. The matrix composed by the rows $e_1Q^2,\ldots,e_1Q^{2n-1},e_1Q^s$ has $2n$ columns. Since we are only interested in the first n, it will be convenient to work in $\mathbb{R}^n$ by identifying x_ℓ with v_ℓ. We also point out that $C_1,\ \ldots\ ,C_n$ are the columns of the matrix whose rows are $e_1\mathcal{K}^2,\ldots,e_1\mathcal{K}^{2n-1},e_1\mathcal{K}^s$, $s\geq 2n$. As in the above argument, the last row will be immaterial. So, without loss of generality, we can assume $s=2n$, in which case $C_1,\ldots,C_n$ are nothing but the first n columns of the matrix Z defined in (36).

We claim that the span of the set $\{C_{2i+1}\}_{i=0,\ldots,m}$ has dimension m. From the considerations of the preceding paragraph, it suffices to show that the span of the rows $\{e_1\mathcal{K}^{2i}\}_{i\in\{1,\ldots m+1\}}$ has dimension m. Indeed, observing the zero entries of these row vectors as determined in Proposition 13, we have to consider just the odd rows, since we are only interested in the odd columns (cf. the matrices Z in Example 12 for the cases $n=3,5$).

As before, we consider the vector equation of $m+1$ variables $c_1,\ldots,c_{m+1}$:

$$\sum_{j=1}^{m+1} c_j e_1\mathcal{K}^{2j} = 0,$$

which is equivalent to the linear system of $2m$ equations with $m+1$ unknowns:

$$\sum_{j=1}^{m+1} \lambda_\ell^{2j} c_j = 0, \quad \ell\in\{0,\ldots,n\}. \tag{50}$$

The ℓ-th row of the matrix R of coefficients of the system (50) is

$$\left[\ \lambda_\ell^2 \quad \lambda_\ell^4 \quad \cdots \quad \lambda_\ell^{2m+2}\ \right].$$

In addition, by Lemma 15, $\lambda_m = 0$ and, for any eigenvalue $\lambda\neq 0$, $-\lambda$ is an eigenvalue as well. Therefore, each one of these rows appears twice, showing that the rank of R is at most m.

Now, consider the m rows which are distinct from each other. Extract the factor λ_ℓ^2 from each one of them and ignore the last column. The resultant matrix is easily seen to be an $m\times m$ standard Vandermonde matrix, and so it is nonsingular. Therefore, the rank of R is m and the claim is proved.

To conclude the proof of (ii)-(b), we have just to observe that, by Proposition 13, all entries of $Z=[p_{k,\ell},q_{k,\ell}]$ in the $(2n-1)\times n$ block of the coefficients $p_{k,\ell}$ which lie above the "factorial diagonal" (i.e., the one of entries $2!,3!,\ldots,(n-1)!$) are zero. Indeed, this property of Z clearly implies that the set $\{C_{2i+1}\}_{i=1,\ldots,m}$ is linearly independent. Consequently, C_1 is in the span of $\{C_{2i+1}\}_{i=1,\ldots,m}$, since the span of $\{C_{2i+1}\}_{i=0,\ldots,m}$ has dimension m.

(ii)-(c) Similarly, we claim that the span of the set $\{C_{n+2i+1}\}_{i=0,\dots,m}$ has dimension m. We start the proof of this claim by noting that, for any j, we have

$$Q^j = \left[\begin{array}{c|c} \mathcal{K}^j & j\mathcal{K}^{j-1} \\ \hline 0_n & \mathcal{K}^j \end{array}\right].$$

As before, we can work in $\mathbb{R}^n$ and assume $s = 2n$. In this setting, $C_{n+1}, \dots, C_{2n}$ are the columns of the matrix whose rows are $2e_1\mathcal{K}, 3e_1\mathcal{K}^2, \dots, 2ne_1\mathcal{K}^{2n-1}$. Since now we are interested just in even columns, it is enough to consider just even rows. Therefore, this time the starting vectorial equation is

$$\sum_{j=1}^{m+1} (2j+1) e_1 \mathcal{K}^{2j} = 0,$$

which leads to the homogeneous linear system whose matrix of coefficients is composed by rows of the type

$$\left[\begin{array}{cccc} 3\lambda_\ell^2 & 5\lambda_\ell^4 & \cdots & (2m+3)\lambda_\ell^{2m+2} \end{array}\right].$$

Once again, since just even powers of λ_ℓ appear, by Lemma 15, we have exactly one zero row (that one for $\ell = m$) and that each non-zero row appears twice. Considering then the m rows which are distinct from each other, we get the matrix

$$\left[\begin{array}{cccc} 3\lambda_0^2 & 5\lambda_0^4 & \cdots & (2m+3)\lambda_0^{2m+2} \\ \vdots & \vdots & \ddots & \vdots \\ 3\lambda_{m-1}^2 & 5\lambda_{m-1}^4 & \cdots & (2m+3)\lambda_{m-1}^{2m+2} \end{array}\right].$$

For any ℓ and j, extract the factors λ_ℓ^2 from the ℓ-th row and $2j+1$ from the j-th column. In this way, we reduce this matrix to the standard Vandermonde matrix, showing that it has maximal rank m. □

Lemma 18. *Given $m \in \mathbb{N}$, $i, j \in \{1, \dots, m\}$, and $h_{ij}, b_i, c_j \in \mathbb{R}$, consider the $m \times m$ matrices $X = (h_{ij}\tau^{b_i + c_j})_{i,j}$ and $Y = (h_{ij})_{i,j}$. Then,*

$$\det X = \tau^{(\sum_{i=1}^n b_i + \sum_{j=1}^n c_j)} \det Y.$$

Proof. For any $i, j \in \{1, \dots, m\}$, extract τ^{b_i} from the i-th row and τ^{c_j} from the j-th column, then apply the standard properties of the determinant. □

Remark 19. Any square submatrix of Z can be written as the matrix X in the statement of Lemma 18. Moreover, for all indices i and j, both b_i and c_j can be written as $\gamma/2$, where γ is a nonnegative integer. In particular, any minor of Z is either zero or a monomial in τ of the form $\mu\tau^{\gamma/2}$, $\mu \neq 0$. An easy way to see that is by writing the zeros of Z as a product $0\tau^{p/q}$, where p and q are suitable integers. Then, we have that

$$b_i = \frac{i-1}{2}, \quad c_j = \gamma_1(j)/2,$$

where $\gamma_1(j)/2$ is the power of τ in the first entry of the j-th column. For instance, when $n = 4$, we can verify this property by rewriting the matrix Z as (see

Example 12):

$$\begin{bmatrix} 3\tau & 0\tau^{\frac{1}{2}} & 2 & 0\tau^{-\frac{1}{2}} & 0\tau^{\frac{1}{2}} & 2 & 0\tau^{-\frac{1}{2}} & 0\tau^{-1} \\ 0\tau^{\frac{3}{2}} & 7\tau & 0\tau^{\frac{1}{2}} & 6 & 9\tau & 0\tau^{\frac{1}{2}} & 6 & 0\tau^{-\frac{1}{2}} \\ 21\tau^2 & 0\tau^{\frac{3}{2}} & 20\tau & 0\tau^{\frac{1}{2}} & 0\tau^{\frac{3}{2}} & 28\tau & 0\tau^{\frac{1}{2}} & 24 \\ 0\tau^{\frac{5}{2}} & 61\tau^2 & 0\tau^{\frac{3}{2}} & 60\tau & 105\tau^2 & 0\tau^{\frac{3}{2}} & 100\tau & 0\tau^{\frac{1}{2}} \\ 183\tau^3 & 0\tau^{\frac{5}{2}} & 182\tau^2 & 0\tau^{\frac{3}{2}} & 0\tau^{\frac{5}{2}} & 366\tau^2 & 0\tau^{\frac{3}{2}} & 360\tau \\ 0\tau^{\frac{7}{2}} & 547\tau^3 & 0\tau^{\frac{5}{2}} & 546\tau^2 & 1281\tau^3 & 0\tau^{\frac{3}{2}} & 1274\tau^2 & 0\tau^{\frac{3}{2}} \\ 1641\tau^4 & 0\tau^{\frac{7}{2}} & 1640\tau^3 & 0\tau^{\frac{5}{2}} & 0\tau^{\frac{7}{2}} & 4376\tau^3 & 0\tau^{\frac{5}{2}} & 4368\tau^2 \end{bmatrix}.$$

Recall that $M(s)$ denotes the $(2n-1)\times(2n-1)$ matrix obtained from the matrix $Z(s)$ defined in the statement of Proposition 17-(ii) by exclusion of its first column.

Proposition 20. *The matrices M and $M(s)$ have the following properties:*

(i) *If n is even, one has:*

(a) *M has rank $2n-2$;*

(b) *there exists $j_* \in \{1,\dots,2n-1\}$ such that*

$$\det M_{j_*} = \mu_0\tau^{\gamma_0} + \sum_{i=1}^{2n-1} \mu_i d_i \tau^{\gamma_i},$$

where $\mu_0 \neq 0, \mu_1, \dots, \mu_{2n-1}$ and $\gamma_0 > \gamma_1 > \cdots > \gamma_{2n-1} > 0$ are all integers.

(ii) *If n is odd, for any $s \geq 2n$, one has:*

(a) *$M(s)$ has rank $2n-2$;*

(b) *the determinant of $M_n(s)$ is given by*

$$\det M_n(s) = \mu_s d_s \tau^{\gamma_s} + \sum_{i=1}^{2n-2} \mu_i d_i \tau^{\gamma_i},$$

where $\mu_1, \dots, \mu_{2n-2}, \mu_s \neq 0$ and $\gamma_1 > \cdots > \gamma_{2n-2} > \gamma_s > 0$ are all integers.

Proof. (i)-(a) Looking at the position of the zero entries of M as given in Proposition 13-(i), one easily sees that its odd rows define a set of n vectors spanning an $(n-1)$-dimensional vector space. Hence, they are linearly dependent, that is, the rank of M is at most $2n-2$. However, Proposition 17-(i) for $s=2$ gives that Z has rank $2n-1$, so that the rank of M is exactly $2n-2$.

(i)-(b) Denote by M_j^τ (resp. M_j^d) the matrix obtained from M by replacing its j-th column with the column matrix P_τ (resp. P_d) defined in (37). Since, as seen above, $Z = [-P_\tau, M]$ has maximal rank $2n-1$, there exists $j_* \in \{1,\dots,2n-1\}$ such that $\det M_{j_*}^\tau \neq 0$. By applying Lemma 18 to the matrix $M_{j_*}^\tau$ (see also Remark 19), we conclude that there exist integers $\mu_0 \neq 0$ and $\gamma_0 > 0$ such that $\det M_{j_*}^\tau = \mu_0 \tau^{\frac{\gamma_0}{2}}$.

With the notation of Lemma 18, we have for the matrix $X = M$ that

$$2b_i = i-1, \quad 2c_j = \begin{cases} 2-j & \text{if } j \in \{1,\dots,n-1\}; \\ n+1-j & \text{if } j \in \{n,\dots,2n-1\}. \end{cases} \tag{51}$$

To obtain the constants $\mu_1, \ldots, \mu_{2n-1}$ and $\gamma_1, \ldots, \gamma_{2n-1}$ as in the statement, it suffices to expand $\det M^d_{j_*}$ with respect to its j_*-th column, and then apply Lemma 18 to any minor occurring in such Laplace expansion. Due to the relations (51), it is clear that the exponents $\gamma_i = \gamma_i(j_*)$ are all integers. The same is true for the coefficients μ_i, since any entry of the matrix M is either an integer or an integer multiple of some power of τ.

It follows from the first equality in (51) that the function $i \mapsto b_i$ is strictly increasing, so that $i \mapsto \gamma_i$ is strictly decreasing (notice that the power $\gamma_i/2$ comes from the minor that suppresses the i-th row of M). It is also clear that $\gamma_0 = \gamma_1 + 1$. Finally, considering all values of j and writing $\gamma_i = \gamma_i(j)$, we have from the second equality in (51) that the smallest value of $\gamma_{2n-1}(j)$ is achieved for $j = 1$ or $j = n$.

Taking all that into account, we have

$$\begin{aligned}\gamma_{2n-1} &= \gamma_{2n-1}(j_*) \geq 2\sum_{i=1}^{2n-2} b_i + 2\sum_{j=2}^{2n-1} c_j \\ &= \frac{1}{2}\left[(2n-2)(2n-3) - (n-2)(n-3) - n(n-3)\right] \\ &= n(n-1) > 0.\end{aligned}$$

(ii)-(a) It follows directly from items (a) and (b) of Proposition 17-(ii).

(ii)-(b) Defining $M_j(s)$, $M^\tau_j(s)$ and $M^d_j(s)$ analogously to M_j, M^τ_j and M^d_j, we have from Proposition 17-(ii) that $\det M^\tau_j(s) = 0$ for all j, in particular, for $j = n$. So,

$$\det M_n(s) = \det M^d_n(s).$$

Henceforth, proceeding just as in the proof of (i)-(a), we get the equality for $\det M_n$ as in the statement. It only remains to prove that $\mu_s \neq 0$. However, μ_s is the determinant of the submatrix of $M(s)$ obtained by removing its last row and its n-th column (be aware of the ordering of the columns: the n-th column of $M(s)$ is the $(n+1)$-th column of $Z(s)$) and, by Proposition 17-(ii), this submatrix is non-singular. □

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(A1) Departamento de Matemática - Universidade Federal do Rio Grande do Norte, Lagoa Nova - 59072-970.

Email address: ronaldo.freire@ufrn.br

(A2) Department of Information Engineering, Computer Science and Mathematics, Università degli Studi dell'Aquila, via Vetoio, Coppito - 67100.

Email address: giuseppe.pipoli@univaq.it