A three-dimensional chemotaxis-Navier–Stokes model with sub-logarithmic sensitivity: Global solvability, eventual smoothness and asymptotic stabilization

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Abstract: An initial-boundary problem associated with the chemotaxis-Navier–Stokes system $n_t+u\cdot\nabla n=\Delta n-\nabla\cdot(n\chi(c)\nabla c), c_t+u\cdot\nabla c=\Delta c-nc$ and $u_t+(u\cdot\nabla)u=\Delta u+\nabla P+n\nabla\phi$ is considered in a bounded domain $\Omega\subset\mathbb{R}^3$ with smooth boundary, subject to no-flux/no-flux/Dirichlet boundary conditions, where $\chi(c)=\frac{\chi_0}{c^\theta}$ with $\chi_0>0$ and $\theta\in[0,1)$. Being different from precedent findings in which the system is investigated in the spatially two-dimensional setting, and in which the signal-consumption mechanism $-nc$ in the second equation is replaced by $-f(n)c$ under specified hypothesis on f in conjunction with the chemotactic sensitivity function χ in the first equation in the spatially three-dimensional context, energy-based analytical approaches used therein seem to be inaccessible to the problem considered herein. Accordingly, with an appropriate choice of a weighted function, a weighted functional is introduced for establishing essential a priori estimates under suitable smallness assumptions on the initial data. Relying on these estimates, the initial-boundary problem with $\theta\in[0,\frac{1}{2})$ is proved to be globally solved by a generalized solution which will become a classical solution after some relaxation time, and which will stabilize to a constant equilibria exponentially with a certain convergence rate as $t\to\infty$.

Key words: chemotaxis; Navier–Stokes; global existence; generalized solution; eventual regularity; stabilization

2020 Mathematics Subject Classification: Primary, 35B65, 35B40; Second, 35K55, 92C17, 35Q30, 35Q92

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1 Introduction

Chemotaxis is a kind of oriented or partially oriented movement in populations of microorganism toward the gradients of the signal which is produced or consumed by the microorganism, as described in the following cross-diffusion system

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\ c_t = \Delta c + h(n,c), \end{cases} \tag{1.1}$$

with $h(n,c) = -c + n$ representing the case of signal-production ([14]), or with $h(n,c) = -cn$ standing for the situation of signal-consumption ([15]). If the response of the microorganism to the chemotactic stimulus approximately complies with Weber–Fechner law, the signal-dependent chemotactic sensitivity χ in the first equation of (1.1) fulfills

$$\chi(c) = \frac{\chi_0}{c^\theta} \tag{1.2}$$

with $\theta = 1$ and some constant $\chi_0 > 0$. In biology, the importance of this type of chemotactic sensitivity appears to be due to intrinsic mathematical correlation to the formation of wave-like behavior, as emphasized independently through modeling methods in [13], [23] and [24]. From a viewpoint of theoretical analysis, the significance of (1.2) with $\theta = 1$ and $\chi_0 > 0$ consists in the effects that it exerts in the cross-diffusion on global existence and qualitative behavior of the solutions to (1.1) and corresponding variants thereof, such as [43], [32], [9],[19], [21], [50] and [49] in the case when $h(n,c) = -c + n$, or as [48] for $h(n,c) = -cn$.

In the surrounding of fluid flows, with (1.1) coupling to a fluid equation in mathematics, i.e.

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\ c_t + u \cdot \nabla c = \Delta c + h(n,c), \\ u_t + \kappa(u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\phi, \end{cases} \tag{1.3}$$

where the function χ fulfills (1.2) with $\theta = 1$ and $\chi_0 > 0$, corresponding solutions thereof are shown to enjoy the features partially resembling that of the solutions constructed in the fluid-free situation, for example, global existence in the classical sense when $h(n,c) = -c + n$ in the spatially two-dimensional setting ([4],[25]) or in a generalized sense for a three-dimensional simplified Stokes-version thereof ([3]), and globally generalized existence in conjunction with corresponding large time behavior for $h(n,c) = -cn$ in the spatially two-dimensional context ([35],[2],[27]).

If $\theta \in [0,1)$ in (1.2), corresponding parameter function χ could be called sub-logarithmic sensitivity as compared to the logarithmic signal-gradient appearing in the cross-diffusion term when $\theta = 1$, i.e. $-\chi_0 \nabla \cdot (n\nabla \ln c)$. In view of the fact that $\theta < 1$ shows weaker singularity than that in the case when $\theta = 1$, it is a natural conjecture that the solutions of (1.3) with $0 \le \theta < 1$ should exhibit better properties than that presented by the solutions to (1.3) when $\theta = 1$, which has been verified in some recent studies. In particular, in the case when $h(n,c) = -cn$, results on global boundedness in higher-dimensional setting under appropriate porous medium type diffusion, which, nevertheless, seem unavailable for logarithmic sensitivity ([20]), can be achieved for χ satisfying (1.2) with $\theta \in [0,1)$ ([39]), even in the context of three-dimensional Stokes fluid ([40],[34]). Apart from that, for $h(n,c)=-c+n$ and χ satisfying (1.2) with $\chi_0>0$ and $\theta=1$, as far as we concerned, without any aid of dampening mechanisms, outcomes on large time behavior of the globally generalized solutions to (1.1) appear to be barren in the spatially two-dimensional setting, however, for $\theta\in(0,\frac{1}{2})$, eventual smoothness and stabilization could be detected for appropriately small initial data ([22]).

In fact, besides the findings [40] and [34] which have been mentioned above on global boundedness within weak frameworks for three-dimensional simplfied Stokes-variants of (1.3) with nonlinear diffusion and sub-logarithmic sensitivity, system (1.3) and its closely related versions are proved to be globally solvable in the classical sense when the spatial dimesion $N=2$ ([1],[26]), with the solutions to corresponding Stokes version being bounded and approaching to some homogeneous equilibria as time goes to infinity ([26]). In the case when $N=3$ and $\kappa=1$, to the best of our knowledge, merely a claim on global solvability in some weak sense appears to be available for a related variant of (1.3) in which $-nc$ on the righ-hand side of the second equation is replaced by $-f(n)c$ with the function f fulfilling a certain structural relationship with the chemotactic sensitivity function χ ([16]). So in this work we are concerned with the issues that whether system (1.3) remains globally solvable under quite mild assumptions on χ, and that how will corresponding solutions behavior in large time if (1.3) could be solved globally. Specially, with $\Omega\subset\mathbb{R}^3$ being a bounded domain with smooth boundary, the problem considered herein is associated with the following system

$$\begin{cases} n_t+u\cdot\nabla n=\Delta n-\chi_0\nabla\cdot(\frac{n}{c^\theta}\nabla c), & x\in\Omega,\ t>0,\\ c_t+u\cdot\nabla c=\Delta c-cn, & x\in\Omega,\ t>0,\\ u_t+(u\cdot\nabla)u=\Delta u+\nabla P+n\nabla\phi,\quad \nabla\cdot u=0 & x\in\Omega,\ t>0,\end{cases}\tag{1.4}$$

in conjunction with the initial-boundary conditions as follows:

$$n(x,0)=n_0(x),\ c(x,0)=c_0(x)\ \text{ and }\ u(x,0)=u_0(x)\ \text{ for }\ x\in\Omega,\tag{1.5}$$

as well as

$$\frac{\partial n}{\partial\nu}=\frac{\partial c}{\partial\nu}=0\ \text{ and }\ u=0\ \text{ on }\ \partial\Omega\times(0,\infty),\tag{1.6}$$

where ϕ is a given function in the third equation of (1.4) denotes the gravitational potential and satisfies

$$\phi\in W^{2,\infty}(\Omega),\tag{1.7}$$

and where the initial data (n_0,c_0,u_0) fulfills

$$\begin{cases} n_0\in C^0(\overline{\Omega})\ \text{ with }\ n_0>0\ \text{ in }\ \overline{\Omega},\\ c_0\in W^{1,\infty}(\Omega)\ \text{ with }\ c_0>0\ \text{ in }\ \overline{\Omega}\ \text{ such that }\ \sqrt{c_0}\in W^{1,2}(\Omega),\ \text{as well as}\\ u_0\in W^{2,2}(\Omega;\mathbb{R}^3)\cap W^{1,2}_{0,\sigma}(\Omega),\end{cases}\tag{1.8}$$

with $W^{1,2}_{0,\sigma}(\Omega):=W^{1,2}_0(\Omega;\mathbb{R}^3)\cap L^2_\sigma(\Omega)$ in which $L^2_\sigma(\Omega):=\{\psi\in L^2(\Omega;\mathbb{R}^3)|\nabla\cdot\psi=0\}$.

Unlike in the spatially two-dimensional setting where (1.4) admits a quasi-energy structure which exactly underlies the derivation of the outcomes on globally classical solvability of (1.4) with

$\kappa \in \{0,1\}$ and on stabilization toward a homogeneous equilibria when $\kappa = 0$ ([26]), similar energy-based analytical approaches cannot be facilitated in the present three-dimensional context, because in combination with the mass-preservation features of the first equation the dissipative effects of Δn appear to be insufficient to achieve essential a priori estimates for deriving consequences on global solvability even in a fairly generalized framework, and hence we resort to an idea that devotes to tracking the evolution of the following weighted functional

$$\int_\Omega n^l \rho(c), \quad l > 1, \tag{1.9}$$

with a suitable choice of the function ρ, as used previously for corresponding variants of (1.4) with non-singular sensitivity ([37],[6],[51],[17]). Despite the first utilization to the model with singular sensitivity, as far as we concerned, being similar as the case of non-singular sensitivity, this approach could provide temporally uniform $L^l(\Omega)$-estimates of n as well as time-independent spatio-temporal estimates of $\nabla n^{\frac{l}{2}}$ for some $l > 1$ provided that $\theta \in [0, \frac{1}{2})$ and that c_0 satisfies appropriate smallness assumptions, which plays the role of a cornerstone in the analysis of global solvability and qualitative behavior in large time, so as to achieve our goals in this paper. In particular, results on global solvability of (1.4), (1.5) and (1.6) can be stated as follows.

Theorem 1.1 *Let $\theta \in [0, \frac{1}{2})$. Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, and that (1.7) holds. Then there exists $\delta_0 = \delta_0(\omega) > 0$ such that if n_0, c_0 and u_0 satisfy (1.8) and*

$$\|c_0\|^{1-\theta}_{L^\infty(\Omega)} < \delta_0(\omega), \tag{1.10}$$

the problem (1.4), (1.5) and (1.6) admits at least one global generalized solution in the context specified by Definition 5.1 below.

Remark. In fact, Theorem 1.1 is valid for any $l > 1$. Since $l \to \infty$ implies $\delta_0 \to 0$ in (3.2), it can be inferred that δ_0 is nonincreasing with respect to $l > 1$, and hence in order to warrant global solvability for the initial data as large as possible, the parameter $l > 1$ is chosen to be sufficiently close to 1 in Theorem 1.1.

Recalling a recent finding ([16]) in which global weak solutions have been established under the restriction of $\theta < \frac{1}{2}\left(\sqrt{\frac{11}{3}} - 1\right)$ along with specified structural assumptions on χ, one can find that despite requiring suitably small initial data in the sense of (1.10) the derivation of global solvability herein seems to admit larger θ without any structural hypothesis on χ, due to the observation that $\frac{1}{2}\left(\sqrt{\frac{11}{3}} - 1\right) < \frac{1}{2}\left(\sqrt{\frac{12}{3}} - 1\right) = \frac{1}{2}$. Beyond this, for smaller initial data, corresponding generalized solutions would possess the following ultimate regularity and asymptotic stabilization.

Theorem 1.2 *Let $\theta \in [0, \frac{1}{2})$ and $l = 2$ in (3.2) below, and let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that (1.7) is valid. Then for each triple (n_0, c_0, u_0) of functions fulfilling (1.8) and (1.10), one can find $t_\star := t_\star(n_0, c_0, u_0) > 0$ such that the global generalized solutions constructed by Theorem 1.1 comply with*

$$n \in C^{2,1}(\overline{\Omega} \times [t_\star, \infty)), \ c \in C^{2,1}(\overline{\Omega} \times [t_\star, \infty)) \quad \text{and} \quad u \in C^{2,1}(\overline{\Omega} \times [t_\star, \infty); \mathbb{R}^3), \tag{1.11}$$

and that with suitable choice of $P \in C^{1,0}(\overline{\Omega} \times [t_\star, \infty))$ the quadruple (n, c, u, P) solves (1.4), (1.5) and (1.6) classically in $\overline{\Omega} \times [t_\star, \infty)$. Beyond this, if $\overline{n_0} := \frac{1}{|\Omega|}\int_\Omega n_0$ and $\lambda_1 > 0$ denotes the first *nonzero eigenvalue of* $-\Delta$ *in* Ω *under homogeneous Neumann boundary conditions, then there exists* $T_* > t_\star$ *such that for* $\zeta_n \in \left(0, \min\{\lambda_1, (\frac{3}{4}-\theta)\overline{n_0}\}\right)$, $\zeta_c \in \left(0, \min\{\lambda_1, \frac{3\overline{n_0}}{4}\}\right)$ *and* $\zeta_u \in (0, \min\{\mu_0, \zeta_n\})$ *with* μ_0 *taken by Lemma 7.6 below, one can find* $C = C(\zeta_n, \zeta_c, \zeta_u) > 0$ *having the properties that*

$$\|n(\cdot,t) - \overline{n_0}\|_{L^\infty(\Omega)} \le Ce^{-\zeta_n(t-T_*)} \quad \textit{for all} \quad t > T_* + 1, \tag{1.12}$$

that

$$\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le Ce^{-\zeta_c(t-T_*)} \quad \textit{for all} \quad t > T_* + 1, \tag{1.13}$$

and that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le Ce^{-\zeta_u(t-T_*)} \quad \textit{for all} \quad t > T_* + 1. \tag{1.14}$$

In Section 2, for all $\varepsilon \in (0,1)$, we introduce an approximated problem which is proved to be globally solvable in the classical sense through a combination of contraction fixed point arguments with regularity properties of homogeneous Neumann heat semigroup and Stokes semigroup. In Section 3, for the components n_ε and c_ε of the approximated solutions established in Section 2, we construct a weighted functional as (1.9) by choosing an appropriate weighted function ρ, and derive basic but important estimates for n_ε based on the analysis of (1.9). Upon a transformation of the regularized problem into another one with nonsingular repulsive sensitivity, a functional in forms of

$$\int_\Omega \ln(n_\varepsilon + 1) + \int_\Omega |\nabla v_\varepsilon|^2 \tag{1.15}$$

could be established for achieving some time-dependent regularity properties of the approximated solutions in Section 4, which together with the estimates derived in Section 2 guarantee the global solvability of (1.4), (1.5) and (1.6) in a generalized sense by means of a suitable extraction procedure in Section 5. In Section 6, taking the decay properties implied by the spatio-temporal estimates attained in Section 3 as a starting point, we proceed along a bootstrap type argument to achieve $C^{2+\gamma,1+\frac{\gamma}{2}}$-regularity properties for the approximated solutions, so as to derive the conclusions on ultimate smoothness of the generalized solutions according to the Arzela-Ascoli theorem. In Section 7, relying on the decay features provided by the spatio-temporal estimates established in Section 3, the generalized solutions can be shown to approach to a constant equilibria as time goes to infinity, and corresponding convergence rate could be found via applications of smoothness features of homogeneous Neumann heat semigroup and Stokes semigroup to the variation-of-constants representation of each equation in (7.16) below.

2 Global solvability of approximated problems

To begin with, let us introduce a regularized problem of (1.4), (1.5) and (1.6) through an approximation procedure resembling that used in [44] and [27] (or in [48] for fluid-free situations). With $\varepsilon \in (0,1)$ representing the approximation index, the analysis of the regularized system could provide ε-dependent a priori estimates which are sufficient to verify global solvability in the classical sense by means of a bootstrap-type strategy ([36],[48]). In the present context, corresponding regularized problems of (1.4), (1.5) and (1.6) appear as

$$
\begin{cases}
\partial_t n_\varepsilon + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \chi_0 \nabla \cdot (\frac{n_\varepsilon F'_\varepsilon(n_\varepsilon)}{c_\varepsilon^\theta} \nabla c_\varepsilon), & x \in \Omega,\ t > 0, \\
\partial_t c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - F_\varepsilon(n_\varepsilon) c_\varepsilon, & x \in \Omega,\ t > 0, \\
\partial_t u_\varepsilon + (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla \phi, & x \in \Omega,\ t > 0, \\
\nabla \cdot u_\varepsilon = 0, & x \in \Omega,\ t > 0, \\
\dfrac{\partial n_\varepsilon}{\partial \nu} = \dfrac{\partial c_\varepsilon}{\partial \nu} = 0,\ u_\varepsilon = 0, & x \in \partial\Omega,\ t > 0, \\
n_\varepsilon(x,0) = n_0(x), \quad c_\varepsilon(x,0) = c_0(x), \quad u_\varepsilon(x,0) = u_0(x), & x \in \Omega,
\end{cases}
\tag{2.1}
$$

where we utilize $F_\varepsilon \in C^\infty([0,\infty))$ determined by

$$
F_\varepsilon(s) := \int_0^s \varrho(\varepsilon\sigma) d\sigma, \qquad s \geq 0 \tag{2.2}
$$

with $\varrho \in C_0^\infty([0,\infty))$ fulfilling $\varrho \equiv 1$ in $[0,1]$ and $\varrho \equiv 0$ in $[2,\infty)$, which guarantees that

$$
F_\varepsilon(s) = s \quad \text{for all} \quad s \in [0, \frac{1}{\varepsilon}], \tag{2.3}
$$

that

$$
0 \leq F'_\varepsilon \leq 1 \quad \text{for all} \quad s \in [0,\infty), \tag{2.4}
$$

and that

$$
F'_\varepsilon(s) = 0 \quad \text{for all} \quad s \geq \frac{2}{\varepsilon}, \tag{2.5}
$$

and which also implies that for any $s \geq 0$

$$
F_\varepsilon(s) \nearrow s \quad \text{and} \quad F'_\varepsilon(s) \nearrow 1 \quad \text{as} \quad \varepsilon \searrow 0. \tag{2.6}
$$

In addition, in the third equation of (2.1), we make use of Y_ε to abbreviate the Yosida approximation, i.e.

$$
Y_\varepsilon \varphi := (1 + \varepsilon A)^{-1} \varphi, \quad \varphi \in L^2_\sigma(\Omega),\ \varepsilon \in (0,1), \tag{2.7}
$$

where here and throughout the sequel $A := -\mathcal{P}\Delta$ stands for the realization of the Stokes operator with $\mathcal{P}$ denoting the Helmholtz projection in $L^2(\Omega;\mathbb{R}^3)$, and with respective domain in $L^p(\Omega;\mathbb{R}^3)$ for $p > 1$ defined by $D(A_p) = W^{2,p}(\Omega;\mathbb{R}^3) \bigcap W_0^{1,p}(\Omega) \bigcap L^p_\sigma(\Omega)$ in which $L^p_\sigma(\Omega) := \{\psi \in L^p(\Omega;\mathbb{R}^3) | \nabla \cdot \psi = 0\}$, as well as with $A^\alpha = A_p^\alpha, \alpha \in \mathbb{R}$ standing for corresponding fractional powers thereof.

In fact, the choices of F_ε and ϱ as in (2.2), which determine the value of F'_ε, in special warrant the globally classical solvability of (2.1) for each $\varepsilon \in (0,1)$.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and let (n_0, c_0, u_0) fulfill (1.8). Then for each $\varepsilon \in (0,1)$, (2.1) admits a unique globally classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ complying with*

$$
\begin{cases}
n_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty)) \bigcap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\
c_\varepsilon \in \bigcap_{q>3} C^0([0,\infty); W^{1,q}(\Omega)) \bigcap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\
u_\varepsilon \in \bigcap_{\alpha \in (\frac{3}{4},1)} C^0([0,\infty); D(A^\alpha)) \bigcap C^{2,1}(\overline{\Omega} \times (0,\infty); \mathbb{R}^3) \quad \textit{and} \\
P_\varepsilon \in C^{1,0}(\overline{\Omega} \times (0,\infty)),
\end{cases}
\tag{2.8}
$$

and with

$$n_\varepsilon > 0 \quad in \ \ \overline{\Omega} \times (0, \infty), \quad \int_\Omega n_\varepsilon(\cdot, t) = \int_\Omega n_0 \quad for\ all \ \ t > 0 \tag{2.9}$$

as well as

$$0 < c_\varepsilon \le \|c_0\|_{L^\infty(\Omega)} \quad in \ \ \overline{\Omega} \times (0, \infty). \tag{2.10}$$

Proof. In line with a well-established argument on local solvability and extensibility of a related chemotaxis-fluid model, such as [42, Lemma 2.1], there exist $T_{\max,\varepsilon} \in (0, \infty]$ and a unique quadruple $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ of functions

$$\begin{cases} n_\varepsilon \in C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \bigcap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \\ c_\varepsilon \in \bigcap_{q>3} C^0([0, T_{\max,\varepsilon}); W^{1,q}(\Omega)) \bigcap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \\ u_\varepsilon \in \bigcap_{\alpha \in (\frac{3}{4}, 1)} C^0([0, T_{\max,\varepsilon}); D(A^\alpha)) \bigcap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon}); \mathbb{R}^3), \\ P_\varepsilon \in C^{1,0}(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \end{cases} \tag{2.11}$$

such that $n_\varepsilon > 0$ and $c_\varepsilon > 0$ in $\overline{\Omega} \times (0, T_{\max,\varepsilon})$, that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (2.1) classically in $\overline{\Omega} \times [0, T_{\max,\varepsilon})$, and that n_ε and c_ε satisfy

$$\int_\Omega n_\varepsilon(\cdot, t) = \int_\Omega n_0 \quad \text{for all} \ \ t \in (0, T_{\max,\varepsilon}) \tag{2.12}$$

and

$$0 < c_\varepsilon \le \|c_0\|_{L^\infty(\Omega)} \quad \text{in} \ \ \overline{\Omega} \times (0, T_{\max,\varepsilon}), \tag{2.13}$$

respectively. Apart from that, with $q > 3$ and $\alpha \in (\frac{3}{4}, 1)$, either

$$\begin{aligned} T_{\max,\varepsilon} = \infty \ \ \text{or} \quad & \lim_{t \nearrow T_{\max,\varepsilon}} (\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot, t)\|_{W^{1,q}(\Omega)} + \|A^\alpha u_\varepsilon(\cdot, t)\|_{L^2(\Omega)}) \to \infty \\ \text{or} \quad & \liminf_{t \nearrow T_{\max,\varepsilon}} \inf_{x \in \Omega} c_\varepsilon(x, t) = 0 \end{aligned} \tag{2.14}$$

is valid.

Thanks to (2.5), an application of the maximum principle to the first equation in (2.1) yields $C_n = C_n(\varepsilon) := \max\left\{\|n_0\|_{L^\infty(\Omega)}, \frac{2}{\varepsilon}\right\} > 0$ satisfying

$$n_\varepsilon \le C_n \quad \text{for all} \ \ t \in (0, T_{\max,\varepsilon}). \tag{2.15}$$

Since $u_\varepsilon = 0$ on $\partial\Omega$, the Poincaré inequality provides $C_1 > 0$ such that

$$\int_\Omega |u_\varepsilon(\cdot, t)|^2 \le C_1 \int_\Omega |\nabla u_\varepsilon(\cdot, t)|^2 \quad \text{for all} \ \ t \in (0, T_{\max,\varepsilon}). \tag{2.16}$$

Testing the third equation by u_ε, we invoke Young's inequality along with (2.15) and (2.16) to derive that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon(\cdot, t)|^2 + \int_\Omega |\nabla u_\varepsilon(\cdot, t)|^2 &= \int_\Omega n_\varepsilon(\cdot, t) u_\varepsilon(\cdot, t) \cdot \nabla\phi \\ &\le \frac{1}{4C_1}\int_\Omega |u_\varepsilon(\cdot, t)|^2 + C_1 C_n^2 \|\nabla\phi\|_{L^\infty(\Omega)}^2 |\Omega| \\ &\le \frac{1}{4}\int_\Omega |\nabla u_\varepsilon(\cdot, t)|^2 + C_1 C_n^2 \|\nabla\phi\|_{L^\infty(\Omega)}^2 |\Omega| \quad \text{for all} \ \ t \in (0, T_{\max,\varepsilon}), \end{aligned}$$

which in combination with (2.16) implies that

$$\frac{d}{dt}\int_\Omega |u_\varepsilon(\cdot,t)|^2 + \frac{1}{C_1}\int_\Omega |u_\varepsilon(\cdot,t)|^2 + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \le 2C_1C_n^2\|\nabla\phi\|^2_{L^\infty(\Omega)}|\Omega| \tag{2.17}$$

for all $t\in(0,T_{\max,\varepsilon})$. Thereby, an ODE comparison argument in conjunction with (1.8) immediately achieves

$$\int_\Omega |u_\varepsilon(\cdot,t)|^2 \le \max\left\{\int_\Omega u_0^2, 2C_1^2C_n^2\|\nabla\phi\|^2_{L^\infty(\Omega)}|\Omega|\right\} =: C_2(\varepsilon) \quad \text{for all} \ \ t\in(0,T_{\max,\varepsilon}). \tag{2.18}$$

Let $\tau := \min\{1, \frac{T_{\max,\varepsilon}}{2}\}$. Then upon a direct integration on $(t,t+\tau)$ for each $t\in(0,T_{\max,\varepsilon}-\tau)$, it follows from (2.16) and (2.17) that

$$\int_t^{t+\tau}\int_\Omega |\nabla u_\varepsilon|^2 \le 2C_2 + 4C_1C_n^2\|\nabla\phi\|^2_{L^\infty(\Omega)}|\Omega| =: C_3(\varepsilon) \quad \text{for all} \ \ t\in(0,T_{\max,\varepsilon}-\tau). \tag{2.19}$$

In view of the boundedness properties of Y_ε from $L^2_\sigma(\Omega)$ to $W^{2,2}(\Omega;\mathbb{R}^3)\bigcap W^{1,2}_{0,\sigma}(\Omega)\hookrightarrow L^\infty(\Omega;\mathbb{R}^3)$, there exists $C_4 := C_4(\varepsilon)>0$ such that we combine with (2.18) to have

$$\|Y_\varepsilon u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_4\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_4C_2^{\frac{1}{2}} \quad \text{for all} \ \ t\in(0,T_{\max,\varepsilon}). \tag{2.20}$$

With $g_\varepsilon(x,t) := \mathcal{P}[-(Y_\varepsilon u_\varepsilon(x,t)\cdot\nabla)u_\varepsilon(x,t) + n_\varepsilon(x,t)\nabla\phi]$, $(x,t)\in\Omega\times(0,T_{\max,\varepsilon})$, a collection of (2.20), (2.19), (2.15) and (1.7) thus entails that

$$\begin{aligned}\int_t^{t+\tau}\int_\Omega |g_\varepsilon|^2 \le& 2C_4^2C_2\int_t^{t+\tau}\int_\Omega |\nabla u_\varepsilon|^2 + 2C_n^2\|\nabla\phi\|^2_{L^\infty(\Omega)}\\ \le& 2(C_4^2C_2C_3 + C_n^2\|\nabla\phi\|^2_{L^\infty(\Omega)}) =: C_5(\varepsilon) \quad \text{for all} \ \ t\in(0,T_{\max,\varepsilon}-\tau).\end{aligned} \tag{2.21}$$

Since the Poincaré inequality warrants the existence of $C_6>0$ satisfying

$$\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \le C_6\int_\Omega |Au_\varepsilon(\cdot,t)|^2 \quad \text{for all} \ \ t\in(0,T_{\max,\varepsilon}),$$

we take $-Au_\varepsilon$ as a testing function for the third equation in (2.1) and derive from integration by parts and Young's inequality that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 =& -\int_\Omega |Au_\varepsilon(\cdot,t)|^2 + \int_\Omega Au_\varepsilon(\cdot,t)\cdot g_\varepsilon(\cdot,t)\\ \le& -\frac{1}{2}\int_\Omega |Au_\varepsilon(\cdot,t)|^2 + \frac{1}{2}\int_\Omega |g_\varepsilon(\cdot,t)|^2\\ \le& -\frac{1}{2C_6}\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 + \frac{1}{2}\int_\Omega |g_\varepsilon(\cdot,t)|^2 \quad \text{for all} \ \ t\in(0,T_{\max,\varepsilon}).\end{aligned} \tag{2.22}$$

Combining (2.22) with (2.21), we invoke [36, Lemma 3.4] to obtain that

$$\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \le \int_\Omega |\nabla u_0|^2 + \frac{C_5}{1-e^{-\frac{\tau}{C_6}}} =: C_7(\varepsilon) \quad \text{for all} \ \ t\in(0,T_{\max,\varepsilon}),$$

whence this along with (2.20), (2.15) and (1.7) implies that

$$\int_\Omega |g_\varepsilon(\cdot,t)|^2 \le 2C_4^2C_2C_7 + 2C_n^2\|\nabla\phi\|^2_{L^\infty(\Omega)} =: C_8(\varepsilon) \quad \text{for all} \ \ t\in(0,T_{\max,\varepsilon}). \tag{2.23}$$

For any fixed $\alpha \in (\frac{3}{4}, 1)$, a utilization of standard L^p-L^q estimates of the Dirichlet Stokes semigroup $(e^{-tA})_{t\geq 0}$ ([30],[12, p.201]) together with (2.23) and (1.8) provides $\mu > 0, C_9 > 0$ and $C_{10} > 0$ such that

$$\begin{aligned}
\|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} =& \left\|A^\alpha e^{-tA}u_0 + \int_0^t A^\alpha e^{-(t-s)A} g_\varepsilon(\cdot,s)ds\right\|_{L^2(\Omega)} \\
\leq& \|A^\alpha e^{-tA}u_0\|_{L^2(\Omega)} + \int_0^t \|A^\alpha e^{-(t-s)A} g_\varepsilon(\cdot,s)ds\|_{L^2(\Omega)} ds \\
\leq& C_9\|A^\alpha u_0\|_{L^2(\Omega)} + C_9\int_0^t (t-s)^{-\alpha} e^{-\mu(t-s)}\|g_\varepsilon(\cdot,s)\|_{L^2(\Omega)}ds \\
\leq& C_9 C_{10} + C_9 C_8^{\frac{1}{2}}\int_0^t (t-s)^{-\alpha}e^{-\mu(t-s)}ds \\
\leq& C_{11} \quad \text{for all} \ \ t \in (0, T_{\max,\varepsilon}),
\end{aligned} \tag{2.24}$$

with $C_{11} = C_{11}(\varepsilon) := C_9C_{10} + C_9C_8^{\frac{1}{2}} \int_0^\infty \sigma^{-\alpha}e^{-\mu\sigma}d\sigma < \infty$. In light of the embedding $D(A^\alpha) \hookrightarrow L^\infty(\Omega)$ due to $\alpha \in (\frac{3}{4}, 1)$, (2.24) ensures that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq C_{12} \quad \text{for all} \ \ t \in (0, T_{\max,\varepsilon}), \tag{2.25}$$

with some $C_{12} = C_{12}(\varepsilon) > 0$. Now, with B standing for the realization of $-\Delta + 1$ subject to homogeneous Neumann boundary conditions in $L^q(\Omega)$, for any fixed $\beta \in (\frac{1}{2}, 1)$, $q > 3$ and $T \in (\frac{3\tau}{4}, T_{\max,\varepsilon})$, we let

$$M(T) := \sup_{t\in(\frac{3\tau}{4},T)} \|B^\beta c_\varepsilon(\cdot,t)\|_{L^q(\Omega)}. \tag{2.26}$$

Based on corresponding variation-of-constants representation of the second equation in (2.1), it can be deduced from smoothing features of the Neumann heat semigroup $(e^{t\Delta})_{t\geq 0}$ in Ω ([8]) that there exists $C_{13} > 0$ such that

$$\begin{aligned}
\|B^\beta c_\varepsilon(\cdot,t)\|_{L^q(\Omega)} =& \Bigg\|B^\beta e^{-B(t-\tau)}c(\cdot,\tau) - \int_\tau^t B^\beta e^{-B(t-s)}[F_\varepsilon(n_\varepsilon(\cdot,s)) - 1]c_\varepsilon(\cdot,s)ds \\
&- \int_\tau^t B^\beta e^{-B(t-s)}[u_\varepsilon(\cdot,s)\cdot\nabla c_\varepsilon(\cdot,s)]ds\Bigg\|_{L^q(\Omega)} \\
\leq& \|B^\beta e^{-B(t-\tau)}c(\cdot,\tau)\|_{L^q(\Omega)} + \int_\tau^t \left\|B^\beta e^{-B(t-s)}[F_\varepsilon(n_\varepsilon(\cdot,s)) - 1]c_\varepsilon(\cdot,s)\right\|_{L^q(\Omega)} ds \\
&+ \left(\int_\tau^{\frac{3\tau}{4}} + \int_{\frac{3\tau}{4}}^t\right) \left\|B^\beta e^{-B(t-s)}[u_\varepsilon(\cdot,s)\cdot\nabla c_\varepsilon(\cdot,s)]\right\|_{L^q(\Omega)} ds \\
\leq& C_{13}(t-\tau)^{-\beta}\|c(\cdot,\tau)\|_{L^q(\Omega)} + C_{13}\int_\tau^t (t-s)^{-\beta}\|c_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}\left[\|F_\varepsilon(n_\varepsilon(\cdot,s))\|_{L^q(\Omega)}\right. \\
&\left.+|\Omega|^{\frac{1}{q}}\right]ds + C_{13}\left(\int_\tau^{\frac{3\tau}{4}} + \int_{\frac{3\tau}{4}}^t\right)(t-s)^{-\beta}\|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla c_\varepsilon(\cdot,s)\|_{L^q(\Omega)}ds
\end{aligned} \tag{2.27}$$

for all $t \in (\frac{3\tau}{4}, T)$. Since (2.2) implies $F_\varepsilon(s) \leq s$ for all $s \geq 0$, (2.27) along with (2.13), (2.25), (2.15) and (2.11) provides $C_{14} = C_{14}(\varepsilon) > 0$ satisfying

$$\|B^\beta c_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \leq C_{14} + C_{14}\sup_{s\in(\frac{3\tau}{4},T)}\|\nabla c_\varepsilon(\cdot,s)\|_{L^q(\Omega)} \tag{2.28}$$

for all $t \in (\frac{3\tau}{4}, T)$, where in accordance with [8, Theorem 2.14.1], there exists $C_{15} > 0$ such that we combine with (2.13) to obtain

$$\begin{aligned}\|\nabla c_\varepsilon(\cdot,s)\|_{L^q(\Omega)} \leq & C_{15}\|B^\beta c_\varepsilon(\cdot,s)\|_{L^q(\Omega)}^{\frac{1}{2\beta}}\|c_\varepsilon(\cdot,s)\|_{L^q(\Omega)}^{\frac{2\beta-1}{2\beta}} \\ \leq & C_{15}\|c_0\|_{L^\infty(\Omega)}^{\frac{2\beta-1}{2\beta}}|\Omega|^{\frac{2\beta-1}{2q\beta}}\|B^\beta c_\varepsilon(\cdot,s)\|_{L^q(\Omega)}^{\frac{1}{2\beta}} \quad \text{for all} \quad s \in (\frac{3\tau}{4}, T).\end{aligned}$$

Thereby, in view of (2.26), for all $T \in (\frac{3\tau}{4}, T_{\max,\varepsilon})$, (2.28) further implies that

$$M(T) \leq C_{16} + C_{16}M^{\frac{1}{2\beta}}(T)$$

with some $C_{16} = C_{16}(\varepsilon) > 0$, which immediately entails

$$M(T) \leq \max\{1, (2C_{16})^{\frac{2\beta}{2\beta-1}}\} =: C_{17}(\varepsilon) \quad \text{for all} \quad T \in (\frac{3\tau}{4}, T_{\max,\varepsilon}).$$

This in conjunction with the embedding $D(B_q^\beta) \hookrightarrow W^{1,q}(\Omega)$ guarantees the existence of $C_{18} = C_{18}(\varepsilon) > 0$ fulfilling

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \leq C_{18} \quad \text{for all} \quad t \in (\frac{3\tau}{4}, T_{\max,\varepsilon}), \tag{2.29}$$

and whereafter collecting (2.15), (2.13), (2.29) and (2.24) shows that

$$\lim_{t \nearrow T_{\max,\varepsilon}} (\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}) < \infty. \tag{2.30}$$

Thanks to (2.15), the second equation of (2.1) admits a sub-solution in $\Omega \times (0, T_{\max,\varepsilon})$, that is

$$\underline{c}(x,t) := \left\{\min_{z\in\overline{\Omega}} c_0(z)\right\} \cdot e^{-C_n t}, \quad x \in \Omega, \ t \in (0, T_{\max,\varepsilon}),$$

as shown in [48], and thus it follows immediately that

$$c_\varepsilon(x,t) \geq \left\{\min_{z\in\overline{\Omega}} c_0(z)\right\} \cdot e^{-C_n t} \quad \text{for all} \quad (x,t) \in \Omega \times (0, T_{\max,\varepsilon}) \quad \text{and} \quad \varepsilon \in (0,1). \tag{2.31}$$

Now, we let $T := T_{\max,\varepsilon} < \infty$. Then from (2.31) and (1.8), it is clear that

$$0 < c_\varepsilon(x,t) \not\to 0 \quad \text{as} \quad t \nearrow T \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad \varepsilon \in (0,1). \tag{2.32}$$

Thereby, in view of (2.30) and (2.32), we conclude from (2.14) that $T_{\max,\varepsilon} = \infty$ which together with (2.11), (2.12) and (2.13) indicates that Lemma 2.1 is indeed valid and hence completes the proof. $\square$

3 Temporally uniform estimates for n_ε

This section devotes to detecting the evolution of the weighted functional (1.9), with a suitably chosen weighted function ρ satisfying the properties in Lemma 3.3 below. Firstly, as a preparation, let us provide a conclusion on monotonicity of the following function.

Lemma 3.1 *With constants $a > 0$ and $b > 0$, the function*

$$z(s) := \frac{\tan(as)}{bs}, \quad s \in (0, \frac{\pi}{2a})$$

is increasing with respect to s.

Proof. Let $y(s) := as - \frac{1}{2}\sin(2as),\ s \in [0, \frac{\pi}{2a})$. Then

$$y'(s) = a \cdot \{1 - \cos(2as)\} > 0 \quad \text{for all} \quad s \in (0, \frac{\pi}{2a}),$$

which in conjunction with $y(0) = 0$ implies that

$$y(s) = as - \frac{1}{2}\sin(2as) > 0 \quad \text{for all} \quad s \in (0, \frac{\pi}{2a}),$$

and whereby

$$z'(s) = \frac{as - \frac{1}{2}\sin(2as)}{bs^2\cos^2(as)} > 0 \quad \text{for all} \quad s \in (0, \frac{\pi}{2a}),$$

from which Lemma 3.1 follows. □

With Lemma 3.1 at hand, it is possible to verify the existence of a fixed point for the function as follows

$$\begin{aligned} f(M) :=& \frac{(1-\theta)^{\frac{1}{2}}(l-1)^{\frac{1}{2}}[4\theta M(1-\eta) + \chi_0^2 l(l-1)(1-\theta)]^{\frac{1}{2}}}{2[1+\eta(l-1)]^{\frac{1}{2}}\|c_0\|_{L^\infty(\Omega)}^{1-\theta}} \\ & \cdot \tan\left\{\frac{[1+\eta(l-1)]^{\frac{1}{2}}[4\theta M(1-\eta) + \chi_0^2 l(l-1)(1-\theta)]^{\frac{1}{2}}\|c_0\|_{L^\infty(\Omega)}^{1-\theta}}{2(1-\eta)(1-\theta)^{\frac{3}{2}}(l-1)^{\frac{1}{2}}}\right\}, \quad M \geq 0, \end{aligned} \tag{3.1}$$

with appropriate choices of $l > 1, 0 < \theta < 1, 0 < \eta < 1$ and c_0.

Lemma 3.2 *Let $\theta \in [0, \frac{1}{2})$ and $l > 1$. Then with some $M_* > \frac{\chi_0^2 l(l-1)(1-\theta)}{4(1-2\theta)}$, if*

$$\|c_0\|_{L^\infty(\Omega)}^{1-\theta} < \frac{2(1-\theta)^{\frac{3}{2}}(l-1)^{\frac{1}{2}}}{[4\theta M_* + \chi_0^2 l(l-1)(1-\theta)]^{\frac{1}{2}}} \cdot \omega =: \delta_0(\omega) \tag{3.2}$$

holds for some $\omega > 0$, one can find $M_0 \in (0, M_*)$ and $\eta_0 \in (0,1)$ such that for all $\eta \in (0, \eta_0)$

$$f(M_0) = M_0, \tag{3.3}$$

where f is defined by (3.1).

Proof. Thanks to $M_* > \frac{\chi_0^2 l(l-1)(1-\theta)}{4(1-2\theta)}$, there exists $\eta_1 \in (0,1)$ small enough such that for any $\eta \in (0, \eta_1)$

$$M_* > \frac{\chi_0^2 l(l-1)(1-\theta)}{4(1-\eta)(1-2\theta)} \tag{3.4}$$

is valid, which is equivalent to

$$\frac{4\theta M_*(1-\eta) + \chi_0^2 l(l-1)(1-\theta)}{4(1-\eta)(1-\theta)} < M_*,$$

so that we can pick sufficiently small $\kappa > 0$ fulfilling

$$\frac{4\theta M_*(1-\eta) + \chi_0^2 l(l-1)(1-\theta)}{4(1-\eta)(1-\theta)} \cdot (1+\kappa) < M_*. \tag{3.5}$$

In view of the well-known fact that $\lim_{s\to 0}\frac{\tan s}{s}=1$, it is possible to take appropriately small $\omega>0$ and $\kappa>0$ such that
$$\frac{\tan\omega}{\omega}<1+\kappa. \tag{3.6}$$
Apart from that, (3.2) allows for a choice of $\eta_2\in(0,1)$ adequately small so as to satisfy
$$\|c_0\|_{L^\infty(\Omega)}^{1-\theta}<\frac{2(1-\eta)(1-\theta)^{\frac{3}{2}}(l-1)^{\frac{1}{2}}}{[1+\eta(l-1)]^{\frac{1}{2}}[4\theta M_*(1-\eta)+\chi_0^2 l(l-1)(1-\theta)]^{\frac{1}{2}}}\cdot\omega \quad \text{for all} \quad \eta\in(0,\eta_2),$$
which implies
$$\frac{[1+\eta(l-1)]^{\frac{1}{2}}[4\theta M_*(1-\eta)+\chi_0^2 l(l-1)(1-\theta)]^{\frac{1}{2}}\|c_0\|_{L^\infty(\Omega)}^{1-\theta}}{2(1-\eta)(1-\theta)^{\frac{3}{2}}(l-1)^{\frac{1}{2}}}<\omega \quad \text{for all} \quad \eta\in(0,\eta_2). \tag{3.7}$$
Now, let $\eta_0:=\min\{\eta_1,\eta_2\}$. Then for each $\eta\in(0,\eta_0)$, since the function f in (3.1) can be rewritten as
$$\begin{aligned} f(M)=&\frac{4\theta M(1-\eta)+\chi_0^2 l(l-1)(1-\theta)}{4(1-\eta)(1-\theta)}\\ &\cdot\frac{\tan\left\{\frac{[1+\eta(1-l)]^{\frac{1}{2}}[4\theta M(1-\eta)+\chi_0^2 l(l-1)(1-\theta)]^{\frac{1}{2}}\|c_0\|_{L^\infty(\Omega)}^{1-\theta}}{2(1-\eta)(1-\theta)^{\frac{3}{2}}(l-1)^{\frac{1}{2}}}\right\}}{\left\{\frac{[1+\eta(1-l)]^{\frac{1}{2}}[4\theta M(1-\eta)+\chi_0^2 l(l-1)(1-\theta)]^{\frac{1}{2}}\|c_0\|_{L^\infty(\Omega)}^{1-\theta}}{2(1-\eta)(1-\theta)^{\frac{3}{2}}(l-1)^{\frac{1}{2}}}\right\}},\end{aligned}$$
and since the function $\frac{\tan s}{s}$ is increasing with repect to $s\in(0,\frac{\pi}{2})$ according to Lemma 3.1 with $a=b=1$, whereupon combining with (3.7), (3.6) and (3.5) we obtain that
$$\begin{aligned} f(M_*)<&\frac{4\theta M_*(1-\eta)+\chi_0^2 l(l-1)(1-\theta)}{4(1-\eta)(1-\theta)}\cdot\frac{\tan\omega}{\omega}\\ <&\frac{4\theta M_*(1-\eta)+\chi_0^2 l(l-1)(1-\theta)}{4(1-\eta)(1-\theta)}\cdot(1+\kappa)\\ <&M_*.\end{aligned} \tag{3.8}$$
Again from (3.1), it follows that
$$f(0)=\frac{\chi_0 l^{\frac{1}{2}}(1-\theta)(l-1)}{2[1+\eta(l-1)]^{\frac{1}{2}}\|c_0\|_{L^\infty(\Omega)}^{1-\theta}}\tan\left\{\frac{\chi_0 l^{\frac{1}{2}}[1+\eta(l-1)]^{\frac{1}{2}}\|c_0\|_{L^\infty(\Omega)}^{1-\theta}}{2(1-\eta)(1-\theta)}\right\}>0, \tag{3.9}$$
due to the positivity of c_0 in (1.8), $\chi_0>0, l>1, \theta<\frac{1}{2}$ and $0<\eta<1$. Thereby, abbreviating $F(M):=f(M)-M,\ M\geq 0$, we see from (3.8) and (3.9) that $F(M_*)<0$ and $F(0)>0$, which along with the continuity of F guarantees the existence of some $M_0\in(0,M_*)$ fulfilling $F(M_0)=0$, and thus (3.3) is proved. $\square$

Now, relying on Lemma 3.1 and Lemma 3.2, the weighted function ρ in (1.9) can be chosen as follows, so as to achieve the main objectives of this section by tracking the evolution of (1.9).

Lemma 3.3 *Let $\theta\in[0,\frac{1}{2})$ and $l>1$, and let $\eta_0\in(0,1)$ be chosen as in Lemma 3.2. With $M_*>0$ and $M_0\in(0,M_*)$ provided by Lemma 3.2, assume that (3.2) holds for some*

$\omega \in \left(0, \frac{\pi}{2}\left\{\frac{4\theta M_*+\chi_0^2 l(l-1)(1-\theta)}{4\theta M_0+\chi_0^2 l(l-1)(1-\theta)}\right\}^{\frac{1}{2}}\right)$. *Then for arbitrarily small* $\eta \in (0,\eta_0)$ *we can find a function* $\rho(\cdot) : [0, \|c_0\|_{L^\infty(\Omega)}] \to [1,+\infty)$ *which fulfills*

$$\rho(s) > 1, \quad \rho'(s) > 0 \quad \textit{for all} \;\; s \in (0, \|c_0\|_{L^\infty(\Omega)}) \tag{3.10}$$

and

$$\frac{\rho''(s) - \frac{\chi_0 l \rho'(s) F_\varepsilon'(n_\varepsilon)}{s^\theta}}{l(1-\eta)(l-1)\rho(s)} \geq \left\{\frac{2\rho'(s) - \frac{\chi_0 (l-1)\rho(s) F_\varepsilon'(n_\varepsilon)}{s^\theta}}{2(1-\eta)(l-1)\rho(s)}\right\}^2 \quad \textit{for all} \;\; s \in (0, \|c_0\|_{L^\infty(\Omega)}). \tag{3.11}$$

Proof. Let

$$\rho(s) := e^{\vartheta\left(\frac{s^{1-\theta}}{1-\theta}\right)}, \quad s \geq 0 \tag{3.12}$$

with

$$\begin{aligned}\vartheta(\iota) :=& \frac{-(1-\eta)(l-1)}{1+\eta(l-1)} \\ & \cdot \ln\left\{\cos\left(\frac{[1+\eta(l-1)]^{\frac{1}{2}}[4\theta M_0(1-\eta)+\chi_0^2 l(l-1)(1-\theta)]^{\frac{1}{2}}}{2(1-\eta)(1-\theta)^{\frac{1}{2}}(l-1)^{\frac{1}{2}}}\cdot \iota\right)\right\}, \quad \iota \in \left[0, \frac{\|c_0\|^{1-\theta}_{L^\infty(\Omega)}}{1-\theta}\right].\end{aligned} \tag{3.13}$$

Since $\omega < \frac{\pi}{2}\left\{\frac{4\theta M_*+\chi_0^2 l(l-1)(1-\theta)}{4\theta M_0+\chi_0^2 l(l-1)(1-\theta)}\right\}^{\frac{1}{2}}$ and (3.2) enable us to pick sufficiently small $\eta \in (0,\eta_0)$ fulfilling

$$0 < \omega < \frac{\pi}{2}\cdot\left\{\frac{4\theta M_*(1-\eta)+\chi_0^2 l(l-1)(1-\theta)}{4\theta M_0(1-\eta)+\chi_0^2 l(l-1)(1-\theta)}\right\}^{\frac{1}{2}}$$

and

$$\|c_0\|^{1-\theta}_{L^\infty(\Omega)} < \frac{2(1-\eta)(1-\theta)^{\frac{3}{2}}(l-1)^{\frac{1}{2}}}{[1+\eta(l-1)]^{\frac{1}{2}}[4\theta M_*(1-\eta)+\chi_0^2 l(l-1)(1-\theta)]^{\frac{1}{2}}}\cdot\omega,$$

for all $\iota \in \left[0, \frac{\|c_0\|^{1-\theta}_{L^\infty(\Omega)}}{1-\theta}\right]$ we have

$$\begin{aligned}0 <& \frac{[1+\eta(l-1)]^{\frac{1}{2}}[4\theta M_0(1-\eta)+\chi_0^2 l(l-1)(1-\theta)]^{\frac{1}{2}}}{2(1-\eta)(1-\theta)^{\frac{1}{2}}(l-1)^{\frac{1}{2}}}\cdot\iota \\ \leq& \frac{[1+\eta(l-1)]^{\frac{1}{2}}[4\theta M_0(1-\eta)+\chi_0^2 l(l-1)(1-\theta)]^{\frac{1}{2}}\|c_0\|^{1-\theta}_{L^\infty(\Omega)}}{2(1-\eta)(1-\theta)^{\frac{3}{2}}(l-1)^{\frac{1}{2}}} \\ <& \left\{\frac{4\theta M_0(1-\eta)+\chi_0^2 l(l-1)(1-\theta)}{4\theta M_*(1-\eta)+\chi_0^2 l(l-1)(1-\theta)}\right\}^{\frac{1}{2}}\cdot\omega \\ <& \frac{\pi}{2},\end{aligned}$$

whence with $\iota := \frac{s^{1-\theta}}{1-\theta}$ the function ϑ defined by (3.13) satisfies

$$\vartheta\left(\frac{s^{1-\theta}}{1-\theta}\right) > 0 \quad \text{for all} \;\; s \in (0, \|c_0\|_{L^\infty(\Omega)}),$$

which together with (3.12) implies that

$$\rho(s) > 1 \quad \text{for all} \;\; s \in (0, \|c_0\|_{L^\infty(\Omega)}). \tag{3.14}$$

In addition, due to for $\iota = \frac{s^{1-\theta}}{1-\theta} \in \left(0, \frac{\|c_0\|^{1-\theta}_{L^\infty(\Omega)}}{1-\theta}\right)$,

$$\begin{aligned}\vartheta'(\iota) =& \frac{(l-1)^{\frac{1}{2}}[4\theta M_0(1-\eta)+\chi_0^2 l(l-1)(1-\theta)]^{\frac{1}{2}}}{2[1+\eta(l-1)]^{\frac{1}{2}}(1-\theta)^{\frac{1}{2}}} \\ &\cdot \tan\left\{\frac{[1+\eta(l-1)]^{\frac{1}{2}}[4\theta M_0(1-\eta)+\chi_0^2 l(l-1)(1-\theta)]^{\frac{1}{2}}}{2(1-\eta)(1-\theta)^{\frac{1}{2}}(l-1)^{\frac{1}{2}}}\cdot\iota\right\} > 0,\end{aligned} \tag{3.15}$$

it follows that

$$\rho'(s) = e^{\vartheta\left(\frac{s^{1-\theta}}{1-\theta}\right)} \cdot \vartheta'(\frac{s^{1-\theta}}{1-\theta}) \cdot s^{-\theta} > 0 \quad \text{for all} \quad s \in (0, \|c_0\|_{L^\infty(\Omega)}). \tag{3.16}$$

Thereby, (3.10) results from (3.14) and (3.16). Now, upon elementary computations, one can find that (3.4) is equivalent to

$$\begin{aligned}4(1-\eta)(l-1)\vartheta''(\iota) \geq& 4[1+\eta(l-1)]\vartheta'^2(\iota) + \frac{4\theta(1-\eta)(l-1)\vartheta'(\iota)}{(1-\theta)\iota} \\ &- 4\eta\chi_0 l(l-1)F'_\varepsilon(n_\varepsilon)\vartheta'(\iota) + \chi_0^2 l(l-1)^2 F'_\varepsilon(n_\varepsilon),\end{aligned}$$

with $\iota = \frac{s^{1-\theta}}{1-\theta}$ for all $s \in (0, \|c_0\|_{L^\infty(\Omega)})$, whereupon for proving (3.4) it is sufficient to achieve

$$4(1-\eta)(l-1)\vartheta''(\iota) \geq 4[1+\eta(l-1)]\vartheta'^2(\iota) + \frac{4\theta(1-\eta)(l-1)}{1-\theta} \cdot \frac{\vartheta'(\iota)}{\iota} + \chi_0^2 l(l-1)^2, \tag{3.17}$$

due to (2.4) and the nonnegativity of $4\eta\chi_0 l(l-1)F'_\varepsilon(n_\varepsilon)\vartheta'(\iota)$. On the one hand, according to Lemma 3.1, one can observe from the formula of ϑ' in (3.15) that the function $\frac{\vartheta'(\iota)}{\iota}$ is increasing with respect to ι, and hence we combine with (3.1) and Lemma 3.2 to infer that

$$\frac{\vartheta'(\iota)}{\iota} < \frac{\vartheta'(\frac{\|c_0\|^{1-\theta}_{L^\infty(\Omega)}}{1-\theta})}{\frac{\|c_0\|^{1-\theta}_{L^\infty(\Omega)}}{1-\theta}} = f(M_0) = M_0 \quad \text{for all} \quad \iota \in \left(0, \frac{\|c_0\|^{1-\theta}_{L^\infty(\Omega)}}{1-\theta}\right).$$

On the other hand, it is not difficult to verify that ϑ actually solves the problem

$$\begin{cases} 4(1-\eta)(l-1)\vartheta''(\iota) = 4[1+\eta(l-1)]\vartheta'^2(\iota) + \frac{4\theta(1-\eta)(l-1)}{1-\theta}\cdot M_0 + \chi_0^2 l(l-1)^2, \\ \vartheta(0) = 0, \quad \vartheta'(0) = 0. \end{cases}$$

As a consequence, (3.17) holds, which implies (3.11) and thus completes the proof. $\square$

Lemma 3.4 *Let $\theta \in [0, \frac{1}{2})$ and $l > 1$. Suppose that (3.2) is valid for $\omega > 0$ chosen by Lemma 3.3. Then there exists $C = C(l) > 0$ such that*

$$\int_\Omega n_\varepsilon^l(\cdot, t) \leq C \quad \textit{for all} \quad t > 0 \quad \textit{and} \quad \varepsilon \in (0,1), \tag{3.18}$$

and that

$$\int_0^\infty \int_\Omega n_\varepsilon^{l-2}|\nabla n_\varepsilon|^2 \leq C \quad \textit{for all} \quad \varepsilon \in (0,1). \tag{3.19}$$

Proof. Based on the definition of ρ in (3.12) and on the first two equations in (2.1), we integrate by parts to have

$$
\begin{aligned}
\frac{1}{l}\frac{d}{dt}\int_\Omega n_\varepsilon^l\rho(c_\varepsilon) =& \int_\Omega n_\varepsilon^{l-1}\rho(c_\varepsilon)\cdot n_{\varepsilon t} + \frac{1}{l}\int_\Omega n_\varepsilon^l\rho'(c_\varepsilon)\cdot c_{\varepsilon t}\\
=& \int_\Omega n_\varepsilon^{l-1}\rho(c_\varepsilon)\cdot\left\{-u_\varepsilon\cdot\nabla n_\varepsilon + \nabla\cdot\left(\nabla n_\varepsilon - \frac{\chi_0 n_\varepsilon F_\varepsilon'(n_\varepsilon)}{c_\varepsilon^\theta}\nabla c_\varepsilon\right)\right\}\\
&+\frac{1}{l}\int_\Omega n_\varepsilon^l\rho'(c_\varepsilon)\cdot\{-u_\varepsilon\cdot\nabla c_\varepsilon + \Delta c_\varepsilon - F_\varepsilon(n_\varepsilon)c_\varepsilon\}\\
=& -\frac{1}{l}\left\{\int_\Omega \rho(c_\varepsilon)u_\varepsilon\cdot\nabla n_\varepsilon^l + \int_\Omega n_\varepsilon^l u_\varepsilon\cdot\nabla\rho(c_\varepsilon)\right\} - (l-1)\int_\Omega n_\varepsilon^{l-2}\rho(c_\varepsilon)|\nabla n_\varepsilon|^2\\
&-2\int_\Omega n_\varepsilon^{l-1}\rho'(c_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon + \chi_0(l-1)\int_\Omega \frac{n_\varepsilon^{l-1}F_\varepsilon'(n_\varepsilon)\rho(c_\varepsilon)}{c_\varepsilon^\theta}\nabla n_\varepsilon\cdot\nabla c_\varepsilon\\
&+\chi_0\int_\Omega\frac{n_\varepsilon^l F_\varepsilon'(n_\varepsilon)\rho'(c_\varepsilon)}{c_\varepsilon^\theta}|\nabla c_\varepsilon|^2 - \frac{1}{l}\int_\Omega n_\varepsilon^l\rho''(c_\varepsilon)|\nabla c_\varepsilon|^2 - \frac{1}{l}\int_\Omega n_\varepsilon^l F_\varepsilon(n_\varepsilon)c_\varepsilon\rho'(c_\varepsilon)
\end{aligned}
$$

for all $t>0$ and $\varepsilon\in(0,1)$, where from $\nabla\cdot u_\varepsilon = 0$ and upon integration by parts it follows that

$$
-\frac{1}{l}\left\{\int_\Omega \rho(c_\varepsilon)u_\varepsilon\cdot\nabla n_\varepsilon^l + \int_\Omega n_\varepsilon^l u_\varepsilon\cdot\nabla\rho(c_\varepsilon)\right\} = -\frac{1}{l}\int_\Omega u_\varepsilon\cdot\nabla\cdot\{n_\varepsilon^l\rho(c_\varepsilon)\} = 0
$$

for all $t>0$ and $\varepsilon\in(0,1)$, whence combining with the nonnegativity of $\frac{1}{l}\int_\Omega n_\varepsilon^l F_\varepsilon(n_\varepsilon)c_\varepsilon\rho'(c_\varepsilon)$ we further obtain

$$
\begin{aligned}
\frac{1}{l}\frac{d}{dt}\int_\Omega n_\varepsilon^l\rho(c_\varepsilon) \le& -\eta(l-1)\int_\Omega n_\varepsilon^{l-2}\rho(c_\varepsilon)|\nabla n_\varepsilon|^2 - (1-\eta)(l-1)\int_\Omega n_\varepsilon^{l-2}\rho(c_\varepsilon)|\nabla n_\varepsilon|^2\\
&-\int_\Omega\left\{\frac{\rho''(c_\varepsilon)}{l} - \frac{\chi_0 F_\varepsilon'(n_\varepsilon)\rho'(c_\varepsilon)}{c_\varepsilon^\theta}\right\}n_\varepsilon^l|\nabla c_\varepsilon|^2\\
&-\int_\Omega\left\{2\rho'(c_\varepsilon) - \frac{\chi_0(l-1)F_\varepsilon'(n_\varepsilon)\rho(c_\varepsilon)}{c_\varepsilon^\theta}\right\}n_\varepsilon^{l-1}\nabla n_\varepsilon\cdot\nabla c_\varepsilon\\
=& -\eta(l-1)\int_\Omega n_\varepsilon^{l-2}\rho(c_\varepsilon)|\nabla n_\varepsilon|^2 - (1-\eta)(l-1)\int_\Omega n_\varepsilon^{l-2}\rho(c_\varepsilon)\Bigg\{|\nabla n_\varepsilon|^2\\
&+\frac{2\rho'(c_\varepsilon) - \frac{\chi_0(l-1)F_\varepsilon'(n_\varepsilon)\rho(c_\varepsilon)}{c_\varepsilon^\theta}}{(1-\eta)(l-1)\rho(c_\varepsilon)}n_\varepsilon\nabla n_\varepsilon\cdot\nabla c_\varepsilon + \frac{\rho''(c_\varepsilon) - \frac{l\chi_0 F_\varepsilon'(n_\varepsilon)\rho'(c_\varepsilon)}{c_\varepsilon^\theta}}{l(1-\eta)(l-1)\rho(c_\varepsilon)}n_\varepsilon^2|\nabla c_\varepsilon|^2\Bigg\}
\end{aligned}
\tag{3.20}
$$

for all $t>0$ and $\varepsilon\in(0,1)$ as well as arbitrary $\eta\in(0,\eta_0)$ with $\eta_0>0$ as taken in the proof of Lemma 3.3. Thanks to Lemma 3.3, (3.20) actually implies

$$
\begin{aligned}
\frac{1}{l}\frac{d}{dt}\int_\Omega n_\varepsilon^l\rho(c_\varepsilon) \le& -\eta(l-1)\int_\Omega n_\varepsilon^{l-2}\rho(c_\varepsilon)|\nabla n_\varepsilon|^2\\
&-(1-\eta)(l-1)\int_\Omega n_\varepsilon^{l-2}\rho(c_\varepsilon)\left|\nabla n_\varepsilon + \frac{2\rho'(c_\varepsilon) - \frac{\chi_0(l-1)F_\varepsilon'(n_\varepsilon)\rho(c_\varepsilon)}{c_\varepsilon^\theta}}{2(1-\eta)(l-1)\rho(c_\varepsilon)}n_\varepsilon\nabla c_\varepsilon\right|^2\\
\le& -\eta(l-1)\int_\Omega n_\varepsilon^{l-2}\rho(c_\varepsilon)|\nabla n_\varepsilon|^2
\end{aligned}
\tag{3.21}
$$

for all $t>0$ and $\varepsilon\in(0,1)$. In view of (3.10) and (2.10), we integrate (3.21) on $(0,t)$ for each $t>0$ to attain

$$
\int_\Omega n_\varepsilon^l(\cdot,t) + \eta l(l-1)\int_0^t\int_\Omega n_\varepsilon^{l-2}|\nabla n_\varepsilon|^2 \le \rho(\|c_0\|_{L^\infty(\Omega)})\int_\Omega n_0^l \quad \text{for all} \quad \varepsilon\in(0,1),
$$

which thus implies (3.18) and (3.19) because of (1.8) and the arbitrariness of t. The proof is completed. $\square$

4 Transformed problems of (2.1)

In order to prove global solvability of (1.4), (1.5) and (1.6), besides the estimates established before, it is also essential to provide some other ε-independent bounds on the basis of an equivalent form of (2.1), that is

$$\begin{cases} \partial_t n_\varepsilon + u_\varepsilon\cdot\nabla n_\varepsilon = \Delta n_\varepsilon + \chi_0\|c_0\|^{1-\theta}_{L^\infty(\Omega)}\nabla\cdot(n_\varepsilon F'_\varepsilon(n_\varepsilon)e^{-(1-\theta)v_\varepsilon}\nabla v_\varepsilon), & x\in\Omega,\ t>0,\\ \partial_t v_\varepsilon + u_\varepsilon\cdot\nabla v_\varepsilon = \Delta v_\varepsilon - |\nabla v_\varepsilon|^2 + F_\varepsilon(n_\varepsilon), & x\in\Omega,\ t>0,\\ \partial_t u_\varepsilon + (Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u_\varepsilon = 0, & x\in\Omega,\ t>0,\\ \dfrac{\partial n_\varepsilon}{\partial\nu} = \dfrac{\partial v_\varepsilon}{\partial\nu} = 0,\ u_\varepsilon = 0, & x\in\partial\Omega,\ t>0,\\ n_\varepsilon(x,0)=n_0(x),\quad v_\varepsilon(x,0)=v_0(x),\quad u_\varepsilon(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{4.1}$$

which is achieved through a standard transformation of variables, i.e.

$$v_\varepsilon(x,t) := -\ln\left(\frac{c_\varepsilon(x,t)}{\|c_0\|_{L^\infty(\Omega)}}\right),\quad (x,t)\in\overline{\Omega}\times[0,\infty),\quad \varepsilon\in(0,1). \tag{4.2}$$

Now, let us establish the estimates that are sufficient to provide compactness which allows for applications of standard extraction procedures to derive the global generalized solutions complying with Definition 5.1 below. Firstly, by tracking the evolution of (1.15), we can attain the following ε-independent bounds.

Lemma 4.1 *Assume that $(n_\varepsilon, c_\varepsilon, u_\varepsilon)_{\varepsilon\in(0,1)}$ are the family of global classical solutions constructed in Lemma 2.1. Then with $(v_\varepsilon)_{\varepsilon\in(0,1)}$ defined by (4.2) and for each $T>0$, there exists $C=C(n_0,v_0,T)>0$ such that*

$$\int_0^T\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} \le C(n_0,v_0,T)\quad \textit{for all}\ \ \varepsilon\in(0,1), \tag{4.3}$$

and that

$$\int_\Omega v_\varepsilon(\cdot,T) + \int_0^T\int_\Omega |\nabla v_\varepsilon|^2 \le C(n_0,v_0,T)\quad \textit{for all}\ \ \varepsilon\in(0,1). \tag{4.4}$$

Moreover,

$$\int_0^T\int_\Omega |\nabla c_\varepsilon|^2 \le \frac12\int_\Omega c_0^2\quad \textit{for all}\ \ T>0\ \ \textit{and}\ \ \varepsilon\in(0,1). \tag{4.5}$$

Proof. In conjunction with (2.6) and (2.9), a straightforward integration of the second equation in (4.1) yields that

$$\begin{aligned} \int_\Omega v_\varepsilon(\cdot,T) + \int_0^T\int_\Omega |\nabla v_\varepsilon|^2 &= \int_\Omega v_0 + \int_0^T\int_\Omega F_\varepsilon(n_\varepsilon)\\ &\le \int_\Omega v_0 + \int_0^T\int_\Omega n_\varepsilon\\ &= \int_\Omega v_0 + T\cdot\left\{\int_\Omega n_0\right\} \end{aligned} \tag{4.6}$$

for any $T > 0$ and all $\varepsilon \in (0,1)$. Based on the first equation in (4.1), we integrate by parts and invoke Young's inequality to have

$$\begin{aligned}\frac{d}{dt}\int_\Omega \ln(n_\varepsilon+1) =& \int_\Omega \frac{1}{n_\varepsilon+1}\cdot\left\{\Delta n_\varepsilon + \chi_0\|c_0\|^{1-\theta}_{L^\infty(\Omega)}\nabla\cdot(n_\varepsilon F'_\varepsilon(n_\varepsilon)e^{-(1-\theta)v_\varepsilon}\nabla v_\varepsilon) - u_\varepsilon\cdot\nabla n_\varepsilon\right\}\\ =& \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} + \chi_0\|c_0\|^{1-\theta}_{L^\infty(\Omega)}\int_\Omega \frac{n_\varepsilon F'_\varepsilon(n_\varepsilon)e^{-(1-\theta)v_\varepsilon}}{(n_\varepsilon+1)^2}\nabla n_\varepsilon\cdot\nabla v_\varepsilon\\ \geq& \frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} - \frac{\chi_0^2\|c_0\|^{2(1-\theta)}_{L^\infty(\Omega)}}{2}\int_\Omega \frac{n_\varepsilon^2|\nabla v_\varepsilon|^2}{(n_\varepsilon+1)^2}\\ \geq& \frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} - \frac{\chi_0^2\|c_0\|^{2(1-\theta)}_{L^\infty(\Omega)}}{2}\int_\Omega |\nabla v_\varepsilon|^2\end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0,1)$, and whereafter an integration on $(0,T)$ for each $T > 0$ entails that

$$\int_\Omega \ln(n_0+1) + \frac{1}{2}\int_0^T\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} \leq \int_\Omega \ln(n_\varepsilon(\cdot,T)+1) + \frac{\chi_0^2\|c_0\|^{2(1-\theta)}_{L^\infty(\Omega)}}{2}\int_0^T\int_\Omega |\nabla v_\varepsilon|^2$$

for all $\varepsilon \in (0,1)$, which along with the fact that $\ln(s+1) \leq s$ for $s \geq 0$, (2.9), (4.6) and the nonnegativity of $\int_\Omega \ln(n_0+1)$ implies that

$$\int_0^T\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} \leq 2\int_\Omega n_0 + \chi_0^2\|c_0\|^{2(1-\theta)}_{L^\infty(\Omega)}\left\{\int_\Omega v_0 + T\cdot\left(\int_\Omega n_0\right)\right\} \tag{4.7}$$

for any $T > 0$ and all $\varepsilon \in (0,1)$. Thereby, (4.3) and (4.4) follow from (4.7) and (4.6), respectively, with $C(n_0,v_0,T) := \max\left\{\int_\Omega v_0 + T\cdot\left\{\int_\Omega n_0\right\}, 2\int_\Omega n_0 + \chi_0^2\|c_0\|^{2(1-\theta)}_{L^\infty(\Omega)}\left\{\int_\Omega v_0 + T\cdot\left(\int_\Omega n_0\right)\right\}\right\}$. For the second equation in (2.1), we test it by c_ε and integrate on $(0,T)$ for any $T > 0$ to achieve

$$\begin{aligned}\frac{1}{2}\int_\Omega c_\varepsilon^2(\cdot,T) + \int_0^T\int_\Omega |\nabla c_\varepsilon|^2 =& \frac{1}{2}\int_\Omega c_0^2 - \int_0^T\int_\Omega F_\varepsilon(n_\varepsilon)c_\varepsilon\\ \leq& \frac{1}{2}\int_\Omega c_0^2 \quad \text{for all } \varepsilon \in (0,1),\end{aligned}$$

due to the nonnegativity of $\int_0^T\int_\Omega F_\varepsilon(n_\varepsilon)c_\varepsilon$, and hence (4.5) holds because of $\frac{1}{2}\int_\Omega c_\varepsilon^2(\cdot,T) \geq 0$. □

With the aid of Lemma 3.4, further spatio-temporal estimates of n_ε and some basic regularity properties of u_ε follow from an interpolation-type argument and standard testing procedures, respectively.

Lemma 4.2 *Let $l > 1$, and let (3.2) hold for $\omega > 0$ provided by Lemma 3.3. Then for each $T > 0$ and all $\varepsilon \in (0,1)$, one can find $C = C(l,T) > 0$ such that*

$$\int_0^T\int_\Omega n_\varepsilon^{\frac{5l}{3}} \leq C. \tag{4.8}$$

Proof. By the Gagliardo–Nirenberg inequality and Lemma 3.4, we deduce that

$$\begin{aligned}
\int_0^T\int_\Omega n_\varepsilon^{\frac{5l}{3}} =&\int_0^T \|n_\varepsilon^{\frac{l}{2}}(\cdot,t)\|^{\frac{10}{3}}_{L^{\frac{10}{3}}(\Omega)}dt\\
\le& C_1\int_0^T\left\{\|\nabla n_\varepsilon^{\frac{l}{2}}(\cdot,t)\|^{\frac{3}{5}}_{L^2(\Omega)}\|n_\varepsilon^{\frac{l}{2}}(\cdot,t)\|^{\frac{2}{5}}_{L^2(\Omega)}+\|n_\varepsilon^{\frac{l}{2}}(\cdot,t)\|_{L^2(\Omega)}\right\}^{\frac{10}{3}}dt\\
\le& C_2\int_0^T\int_\Omega|\nabla n_\varepsilon^{\frac{l}{2}}|^2+C_2T\\
=&\frac{C_2l^2}{4}\int_0^T\int_\Omega n_\varepsilon^{l-2}|\nabla n_\varepsilon|^2+C_2T\\
\le& C_3+C_2T \quad \text{for any } \ T>0 \ \text{ and } \ \varepsilon\in(0,1)
\end{aligned}$$

with $C_1=C_1(l)>0, C_2=C_2(l)>0$ and $C_3=C_3(l)>0$, which implies (4.8) and thus completes the proof. $\square$

Lemma 4.3 *Let $l\in(1,\frac{6}{5})$. If (3.2) is valid for $\omega>0$ satisfying the hypothesis of Lemma 3.3, then there exists $C>0$ with the properties that*

$$\int_\Omega|u_\varepsilon(\cdot,t)|^2\le C \quad \textit{for all} \ \ t>0 \ \ \textit{and} \ \ \varepsilon\in(0,1), \tag{4.9}$$

and that

$$\int_t^{t+1}\int_\Omega|\nabla u_\varepsilon|^2\le C \quad \textit{for all} \ \ t>0 \ \ \textit{and} \ \ \varepsilon\in(0,1). \tag{4.10}$$

Moreover, for any $T>0$, one can find $C=C(T)>0$ satisfying

$$\int_0^T\int_\Omega|u_\varepsilon|^{\frac{10}{3}}\le C \quad \textit{for all} \ \ \varepsilon\in(0,1). \tag{4.11}$$

Proof. Thanks to $\nabla\cdot u_\varepsilon=0$, we test the third equation by u_ε and apply the Hölder inequality, the Poincaré–Sobolev inequality along with Young's inequality to obtain that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega|u_\varepsilon(\cdot,t)|^2+\int_\Omega|\nabla u_\varepsilon(\cdot,t)|^2=&\int_\Omega n_\varepsilon(\cdot,t)u_\varepsilon(\cdot,t)\cdot\nabla\phi\\
\le&\|u_\varepsilon(\cdot,t)\|_{L^6(\Omega)}\|n_\varepsilon(\cdot,t)\|_{L^{\frac{6}{5}}(\Omega)}\|\nabla\phi\|_{L^\infty(\Omega)}\\
\le& C_1\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\|n_\varepsilon(\cdot,t)\|_{L^{\frac{6}{5}}(\Omega)}\|\nabla\phi\|_{L^\infty(\Omega)}\\
\le&\frac{1}{2}\int_\Omega|\nabla u_\varepsilon(\cdot,t)|^2+\frac{C_1^2\|\nabla\phi\|^2_{L^\infty(\Omega)}}{2}\|n_\varepsilon(\cdot,t)\|^2_{L^{\frac{6}{5}}(\Omega)}
\end{aligned} \tag{4.12}$$

for all $t>0$ and $\varepsilon\in(0,1)$, with some $C_1>0$. Since $1<l<\frac{6}{5}$ implies $2<\frac{12}{5l}<6$ and $\frac{6-5l}{2l}<1$, applications of the Gagliardo–Nirenberg inequality, Lemma 3.4 and Young's inequality provide

$C_2 > 0$ and $C_3 > 0$ such that

$$\begin{aligned}
&\frac{C_1^2\|\nabla\phi\|^2_{L^\infty(\Omega)}}{2}\|n_\varepsilon(\cdot,t)\|^2_{L^{\frac{6}{5}}(\Omega)}\\
=&\frac{C_1^2\|\nabla\phi\|^2_{L^\infty(\Omega)}}{2}\|n_\varepsilon^{\frac{l}{2}}(\cdot,t)\|^{\frac{4}{l}}_{L^{\frac{12}{5l}}(\Omega)}\\
\le&\frac{C_1^2\|\nabla\phi\|^2_{L^\infty(\Omega)}C_2}{2}\left\{\|\nabla n_\varepsilon^{\frac{l}{2}}(\cdot,t)\|^{\frac{6-5l}{4}}_{L^2(\Omega)}\|n_\varepsilon^{\frac{l}{2}}(\cdot,t)\|^{\frac{5l-2}{4}}_{L^2(\Omega)}+\|n_\varepsilon^{\frac{l}{2}}(\cdot,t)\|_{L^2(\Omega)}\right\}^{\frac{4}{l}}\\
\le&\frac{C_1^2C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}}{2}\left\{\int_\Omega|\nabla n_\varepsilon^{\frac{l}{2}}(\cdot,t)|^2+1\right\}^{\frac{6-5l}{2l}}\\
\le&\frac{C_1^2C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}}{2}\int_\Omega|\nabla n_\varepsilon^{\frac{l}{2}}(\cdot,t)|^2+C_1^2C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}
\end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0, 1)$. Thereby, (4.12) further implies

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega|u_\varepsilon(\cdot,t)|^2+\int_\Omega|\nabla u_\varepsilon(\cdot,t)|^2\\
\le&\frac{l^2C_1^2C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}}{4}\int_\Omega n_\varepsilon^{l-2}(\cdot,t)|\nabla n_\varepsilon(\cdot,t)|^2+2C_1^2C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}
\end{aligned}\tag{4.13}$$

for all $t > 0$ and $\varepsilon \in (0, 1)$. By the Poincaré inequality and Lemma 3.4, we can find $C_4 > 0$ and $C_5 > 0$ such that

$$C_4\int_\Omega|u_\varepsilon(\cdot,t)|^2\le\int_\Omega|\nabla u_\varepsilon(\cdot,t)|^2\quad\text{and}\quad\int_t^{t+1}\int_\Omega n_\varepsilon^{l-2}|\nabla n_\varepsilon|^2\le C_5\tag{4.14}$$

for all $t > 0$ and $\varepsilon \in (0, 1)$, and whereafter letting $y_\varepsilon(t) := \int_\Omega |u_\varepsilon(\cdot,t)|^2$ and $g_\varepsilon(t) := \frac{l^2C_1^2C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}}{4}\int_\Omega n_\varepsilon^{l-2}(\cdot,t)|\nabla n_\varepsilon(\cdot,t)|^2 + 2C_1^2C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}$ for all $t > 0$ and $\varepsilon \in (0, 1)$ we combine (4.13) with (4.14) to have

$$y_\varepsilon'(t)+C_4y_\varepsilon(t)\le g_\varepsilon(t)\quad\text{for all}\quad t>0,\tag{4.15}$$

where

$$\int_t^{t+1}g_\varepsilon(s)ds\le C_5\quad\text{for all}\quad t>0,$$

with some $C_5 > 0$ because of (4.14). According to [36, Lemma 3.4], it follows from (4.15) and (1.8) that

$$y_\varepsilon(t)=\int_\Omega|u_\varepsilon(\cdot,t)|^2\le\int_\Omega u_0^2+\frac{C_5}{1-e^{-C_4}}=:C_6\quad\text{for all}\ \ t>0\ \ \text{and}\ \ \varepsilon\in(0,1),$$

which shows (4.9) and thus yields (4.10) upon an integration of (4.13) on $(t, t + 1)$ for each $t > 0$. As a consequence of (4.9) and (4.10), (4.11) can be established in line with the arguments of [44, Lemma 3.10]. The proof is completed. □

For achieving the compactness that could ensure the performance of a standard extraction procedure in the derivation of limit functions, it is essential to provide regularity features for time derivatives of $\ln(n_\varepsilon + 1), v_\varepsilon, c_\varepsilon$ and u_ε.

Lemma 4.4 *Let $l \in (1, \frac{6}{5})$. Assume that (3.2) is fulfilled for $\omega > 0$ picked by Lemma 3.3. Then for any $T > 0$ and all $\varepsilon \in (0,1)$ there exists $C = C(T) > 0$ such that*

$$\begin{aligned}&\|\partial_t \ln(n_\varepsilon + 1)\|_{L^1((0,T);(W^{3,2}(\Omega))^*)} + \|\partial_t v_\varepsilon\|_{L^1((0,T);(W^{2,2}(\Omega))^*)}\\&+ \|\partial_t c_\varepsilon\|_{L^2((0,T);(W^{2,2}(\Omega))^*)} + \|\partial_t u_\varepsilon\|_{L^{\frac{5}{4}}\left((0,T);(W^{1,5}_{0,\sigma}(\Omega))^*\right)} \le C.\end{aligned} \tag{4.16}$$

Proof. Based on the first equation in (4.1), for any fixed $\psi \in C^\infty(\overline{\Omega})$ and all $t > 0$, it can be deduced from integration by parts, the Cauchy–Schwarz inequality as well as $\nabla \cdot u_\varepsilon = 0$ that

$$\begin{aligned}\left|\int_\Omega \partial_t \ln(n_\varepsilon + 1) \cdot \psi\right| =& \Bigg|\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon + 1)^2} \cdot \psi - \int_\Omega \frac{\nabla n_\varepsilon}{n_\varepsilon + 1} \cdot \nabla\psi\\&+\chi_0 \|c_0\|^{1-\theta}_{L^\infty(\Omega)} \int_\Omega \frac{n_\varepsilon F'_\varepsilon(n_\varepsilon) e^{-(1-\theta)v_\varepsilon}}{(n_\varepsilon + 1)^2} (\nabla n_\varepsilon \cdot \nabla v_\varepsilon)\psi\\&-\chi_0 \|c_0\|^{1-\theta}_{L^\infty(\Omega)} \int_\Omega \frac{n_\varepsilon F'_\varepsilon(n_\varepsilon) e^{-(1-\theta)v_\varepsilon}}{n_\varepsilon + 1} \nabla v_\varepsilon \cdot \nabla\psi\\&+ \int_\Omega \ln(n_\varepsilon + 1)(u_\varepsilon \cdot \nabla\psi)\Bigg|\\\le& \left\{\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon + 1)^2}\right\} \cdot \|\psi\|_{L^\infty(\Omega)} + \left\{\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon + 1)^2}\right\}^{\frac{1}{2}} \cdot \|\nabla\psi\|_{L^2(\Omega)}\\&+ \chi_0 \|c_0\|^{1-\theta}_{L^\infty(\Omega)} \left\{\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon + 1)^2}\right\}^{\frac{1}{2}} \cdot \left\{\int_\Omega |\nabla v_\varepsilon|^2\right\}^{\frac{1}{2}} \cdot \|\psi\|_{L^\infty(\Omega)}\\&+ \chi_0 \|c_0\|^{1-\theta}_{L^\infty(\Omega)} \left\{\int_\Omega |\nabla v_\varepsilon|^2\right\}^{\frac{1}{2}} \cdot \|\nabla\psi\|_{L^2(\Omega)}\\&+ \left\{\int_\Omega \ln^{\frac{15}{8}}(n_\varepsilon + 1)\right\}^{\frac{8}{15}} \cdot \left\{\int_\Omega |u_\varepsilon|^{\frac{10}{3}}\right\}^{\frac{3}{10}} \cdot \|\nabla\psi\|_{L^6(\Omega)}\end{aligned}$$

for all $\varepsilon \in (0,1)$. In view of the embedding $W^{3,2}(\Omega) \hookrightarrow W^{1,6}(\Omega) \hookrightarrow L^\infty(\Omega)$ in the three-dimensional setting and of the fact that $\ln^{\frac{15}{8}}(s+1) \le \left(\frac{15}{8e}\right)^{\frac{15}{8}}(s+1)$ for all $s \ge 0$, there exists $C_1 > 0$ such that by means of Young's inequality we have

$$\begin{aligned}\|\partial_t \ln(n_\varepsilon + 1)\|_{(W^{3,2}(\Omega))^*} \le& C_1 \cdot \Bigg\{\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon + 1)^2} + \left(\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon + 1)^2}\right)^{\frac{1}{2}}\\&+\chi_0 \|c_0\|^{1-\theta}_{L^\infty(\Omega)} \left(\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon + 1)^2}\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v_\varepsilon|^2\right)^{\frac{1}{2}}\\&+\chi_0 \|c_0\|^{1-\theta}_{L^\infty(\Omega)} \left(\int_\Omega |\nabla v_\varepsilon|^2\right)^{\frac{1}{2}} + \frac{15}{8e}\left(\int_\Omega (n_\varepsilon + 1)\right)^{\frac{8}{15}} \left(\int_\Omega |u_\varepsilon|^{\frac{10}{3}}\right)^{\frac{3}{10}}\Bigg\}\\\le& C_1 \cdot \Bigg\{3\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon + 1)^2} + 2\chi_0^2 \|c_0\|^{2(1-\theta)}_{L^\infty(\Omega)} \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega |u_\varepsilon|^{\frac{10}{3}}\\&+ \left(\frac{15}{8e}\right)^{\frac{10}{7}} \left(\int_\Omega (n_\varepsilon + 1)\right)^{\frac{16}{21}} + 2\Bigg\}\end{aligned} \tag{4.17}$$

for all $t > 0$ and $\varepsilon \in (0,1)$, whereupon in combination with Lemma 4.1, Lemma 4.3 and (2.9) an integration of (4.17) on $(0,T)$ for each $T > 0$ shows the boundedness properties of $\partial_t \ln(n_\varepsilon + 1)$

in $L^1\left((0,T);(W^{3,2}(\Omega))^*\right)$. Apart from that, since both $W^{2,2}(\Omega)\hookrightarrow W^{1,2}(\Omega)$ and $W^{2,2}(\Omega)\hookrightarrow L^\infty(\Omega)$ remain valid in the spatially three-dimensional context, the bounds for $\partial_t v_\varepsilon$ and $\partial_t c_\varepsilon$ within respective topology as exhibited in (4.16) can be achieved on the basis of Lemma 4.1, Lemma 4.3, (2.10) and (2.9) according to [35, Lemma 2.4] and [27, Lemma 3.7]. For $\partial_t u_\varepsilon$, thanks to Lemma 4.3 and Lemma 4.2, corresponding boundedness features in $L^{\frac{5}{4}}\left((0,T);(W^{1,5}_{0,\sigma}(\Omega))^*\right)$ result from [44, Lemma 3.11] directly. $\square$

5 A generalized solution notion. Proof of Theorem 1.1

Now, we are in the position to present the notion of global generalized solutions to (1.4), (1.5) and (1.6), which is designed in an analogous manner as that used in [48] (or [35, 27]) for corresponding solutions to closely related problems in the spatially two-dimensional setting. Beyond this, we intend to verify that the limit functions of $n_\varepsilon, c_\varepsilon$ and u_ε indeed form a global generalized solutions of (1.4), (1.5) and (1.6) in line with the notion.

Here and throughout the sequel, for vectors ξ and υ, we use $\xi\otimes\upsilon$ to denote the matrix with corresponding elements $(\xi\otimes\upsilon)_{kl}:=\xi_k\upsilon_l$ for $k,l\in\{1,2,3\}$.

Definition 5.1 *Let $\theta\in[0,\frac{1}{2})$, and let n_0, c_0 and u_0 satisfy (1.8). Then a triple (n,c,u) of functions*

$$\begin{cases} n\in L^1_{loc}(\overline{\Omega}\times[0,\infty)),\\ c\in L^\infty_{loc}(\overline{\Omega}\times[0,\infty))\cap L^2_{loc}([0,\infty);W^{1,2}(\Omega)),\\ u\in L^1_{loc}([0,\infty);W^{1,1}_0(\Omega;\mathbb{R}^3)),\end{cases}\tag{5.1}$$

fulfilling

$$n\geq 0,\ \ c>0\ \ \text{and}\ \ \nabla\cdot u=0\quad \text{a.e. in}\ \ \Omega\times(0,\infty)\tag{5.2}$$

as well as

$$\nabla\ln(n+1)\in L^2_{loc}(\overline{\Omega}\times[0,\infty))\ \ \text{and}\ \ \nabla c^{1-\theta}\in L^2_{loc}(\overline{\Omega}\times[0,\infty)),\tag{5.3}$$

is called a global generalized solution of (1.4), (1.5) and (1.6), if n complies with

$$\int_\Omega n(\cdot,t)=\int_\Omega n_0\quad \text{for a.e.}\ \ t>0,\tag{5.4}$$

if for each nonnegative $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$ the inequality

$$\begin{aligned}&-\int_0^\infty\int_\Omega\ln(n+1)\varphi_t-\int_\Omega\ln(n_0+1)\varphi(\cdot,0)\\ \geq&\int_0^\infty\int_\Omega|\nabla\ln(n+1)|^2\varphi-\int_0^\infty\int_\Omega\nabla\ln(n+1)\cdot\nabla\varphi-\frac{\chi_0}{1-\theta}\int_0^\infty\int_\Omega\frac{n}{n+1}\left(\nabla\ln(n+1)\cdot\nabla c^{1-\theta}\right)\varphi\\ &+\frac{\chi_0}{1-\theta}\int_0^\infty\int_\Omega\frac{n}{n+1}\nabla c^{1-\theta}\cdot\nabla\varphi+\int_0^\infty\int_\Omega\ln(n+1)(u\cdot\nabla\varphi)\end{aligned}\tag{5.5}$$

is fulfilled, if for all $\varphi\in L^\infty(\Omega\times[0,\infty))\cap L^2\left((0,\infty);W^{1,2}(\Omega)\right)$ the identity

$$\int_0^\infty\int_\Omega c\varphi_t+\int_\Omega c_0\varphi(\cdot,0)=\int_0^\infty\int_\Omega\nabla c\cdot\nabla\varphi+\int_0^\infty\int_\Omega nc\varphi-\int_0^\infty\int_\Omega cu\cdot\nabla\varphi\tag{5.6}$$

is valid, and if for any $\varphi \in C_0^\infty\left(\Omega \times [0,\infty);\mathbb{R}^3\right)$ *satisfying* $\nabla\cdot\varphi \equiv 0$ *the equality*

$$-\int_0^\infty\int_\Omega u\cdot\varphi_t - \int_\Omega u_0\cdot\varphi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla u\cdot\nabla\varphi + \int_0^\infty\int_\Omega (u\otimes u)\cdot\nabla\varphi + \int_0^\infty\int_\Omega n(\nabla\phi\cdot\varphi) \tag{5.7}$$

holds.

Lemma 5.2 *Let* $\theta \in [0,\frac{1}{2})$ *and* $l \in (1,\frac{6}{5})$. *Suppose that (3.2) holds with* $\omega > 0$ *chosen by Lemma 3.3. Then there exist a quadruple* (n,c,v,u) *of functions and* $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ *fulfilling* $\varepsilon_j \searrow 0$ *as* $j\to\infty$, *such that* $n \geq 0$ *and* $v \geq 0$ *a.e. in* $\Omega\times(0,\infty)$, *and such that as* $\varepsilon_j \searrow 0$

$$n_\varepsilon \to n \quad \textit{in} \quad L^{\frac{5}{3}}_{loc}(\overline{\Omega}\times[0,\infty)) \quad \textit{and a.e. in} \quad \Omega\times(0,\infty), \tag{5.8}$$

$$\ln(n_\varepsilon+1) \rightharpoonup \ln(n+1) \quad \textit{in} \quad L^2_{loc}([0,\infty);W^{1,2}(\Omega)), \tag{5.9}$$

$$\ln(n_\varepsilon+1) \to \ln(n+1) \quad \textit{in} \quad L^2_{loc}(\overline{\Omega}\times[0,\infty)), \tag{5.10}$$

$$v_\varepsilon \to v \quad \textit{in} \quad L^1_{loc}(\overline{\Omega}\times[0,\infty)) \quad \textit{and a.e. in}\ \Omega\times(0,\infty), \tag{5.11}$$

$$v_\varepsilon(\cdot,t) \to v(\cdot,t) \quad \textit{in} \quad L^1(\Omega) \quad \textit{for a.e.} \quad t\in(0,\infty), \tag{5.12}$$

$$\nabla v_\varepsilon \rightharpoonup \nabla v \quad \textit{in} \quad L^2_{loc}(\overline{\Omega}\times[0,\infty)), \tag{5.13}$$

$$\nabla v_\varepsilon \to \nabla v \quad \textit{in} \quad L^2_{loc}(\overline{\Omega}\times[0,\infty)), \tag{5.14}$$

and

$$u_\varepsilon \to u \quad \textit{in} \quad L^2_{loc}(\overline{\Omega}\times[0,\infty)) \quad \textit{and a.e. in}\ \Omega\times(0,\infty), \tag{5.15}$$

$$u_\varepsilon \rightharpoonup u \quad \textit{in} \quad L^{\frac{10}{3}}_{loc}(\overline{\Omega}\times[0,\infty)) \quad \textit{and a.e. in}\ \Omega\times(0,\infty), \tag{5.16}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \quad \textit{in} \quad L^2_{loc}(\overline{\Omega}\times[0,\infty)), \tag{5.17}$$

$$u_\varepsilon(\cdot,t) \to u(\cdot,t) \quad \textit{in} \quad L^2(\Omega) \quad \textit{for a.e.} \quad t\in(0,\infty), \tag{5.18}$$

as well as

$$c_\varepsilon \to c \quad \textit{in} \quad L^1_{loc}(\overline{\Omega}\times[0,\infty)) \quad \textit{and a.e. in}\ \Omega\times(0,\infty), \tag{5.19}$$

$$c_\varepsilon \overset{*}{\rightharpoonup} c \quad \textit{in} \quad L^\infty\left(\Omega\times(0,\infty)\right), \tag{5.20}$$

$$c_\varepsilon \rightharpoonup c \quad \textit{in} \quad L^2_{loc}([0,\infty);W^{1,2}(\Omega)), \tag{5.21}$$

where

$$c = \|c_0\|_{L^\infty(\Omega)}\cdot e^{-v}. \tag{5.22}$$

Proof. From the boundedness properties of $(n_\varepsilon^{\frac{5}{3}})_{\varepsilon\in(0,1)}$ in $L^l_{loc}(\overline{\Omega}\times[0,\infty))$ as implied by Lemma 4.2, we infer that upon an appropriate extraction if necessary it follows that

$$n_\varepsilon \rightharpoonup n \quad \text{in} \quad L^{\frac{5}{3}}_{loc}(\overline{\Omega}\times[0,\infty)) \quad \text{as} \quad \varepsilon=\varepsilon_j\searrow 0, \tag{5.23}$$

and that

$$n_\varepsilon^{\frac{5}{3}} \rightharpoonup n^{\frac{5}{3}} \quad \text{in} \quad L^l_{loc}(\overline{\Omega}\times[0,\infty)) \quad \text{as} \quad \varepsilon=\varepsilon_j\searrow 0, \tag{5.24}$$

where (5.24) implies that

$$\int_0^T\int_\Omega n_\varepsilon^{\frac{5}{3}} \to \int_0^T\int_\Omega n^{\frac{5}{3}} \quad \text{as} \quad \varepsilon=\varepsilon_j\searrow 0 \quad \text{for any} \quad T>0. \tag{5.25}$$

Thereby, in view of the uniform convexity of $L^{\frac{5}{3}}(\Omega\times(0,T))$ with arbitrary fixed $T>0$, (5.8) is valid from (5.23) and (5.25) according to [5, Proposition 3.32]. Since (2.9) implies the existence of $C_1>0$ satisfying

$$\int_\Omega \ln^2(n_\varepsilon(\cdot,t)+1)dt\le C_1 \quad \text{for all} \quad t>0 \quad \text{and} \quad \varepsilon\in(0,1),$$

we combine with (2.10), Lemma 4.1, Lemma 4.3 and Lemma 4.4 to verify the validaty of (5.9)–(5.13) and (5.15)–(5.21) by picking suitable subsequence from $(\varepsilon_j)_{j\in\mathbb{N}}$ as well as by three straightforward applications of the Aubin–Lions lemma ([33]), and hence $v\ge 0$ results from (5.11) along with the nonnegativity of $(v_\varepsilon)_{\varepsilon\in(0,1)}$. Thanks to (4.2), (5.22) follows from (5.11) and (5.19). Now, as proceeding in the reasoning of [48, Lemmas 2.9] (or [35, Lemmas 2.9]), we let $(\delta_l)_{l\in\mathbb{N}}\subset(0,1)$ denote an arbitrary fixed sequence fulfilling $\delta_l\searrow 0$ as $l\to\infty$. Then it is not difficult to infer that for any $l\in\mathbb{N}$ there exists a null set $\aleph_l\subset(0,\infty)$ such that $(0,\infty)\backslash\aleph_l$ is the set of Lebegue point for $0<t\mapsto\int_\Omega \ln\{c(x,t)+\delta_l\}$. In addition, (5.11) allows for a choice of $\tilde{\aleph}\subset(0,\infty)$ having the property that $v(\cdot,\tilde{t})\in L^1(\Omega)$ for any $\tilde{t}\in(0,\infty)\backslash\tilde{\aleph}$. From (4.9) and (5.18), for each $\phi(t)\in W^{1,\infty}(\mathbb{R})$, we can find $C_1>0$ such that for all $\delta\in(\delta_l)_{l\in\mathbb{N}}$

$$\begin{aligned}\left\|\frac{\phi(t)c_\varepsilon(\cdot,t)u_\varepsilon(\cdot,t)}{(c_\varepsilon(\cdot,t)+\delta)^2}-\frac{\phi(t)cu}{(c+\delta)^2}\right\|_{L^2(\Omega)}\le&\frac{\|\phi\|_{L^\infty(\mathbb{R})}}{\delta}\left(\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}+\|u(\cdot,t)\|_{L^2(\Omega)}\right)\\ \le&\frac{2\|\phi\|_{L^\infty(\mathbb{R})}}{\delta}\sup_{\varepsilon'\in(0,1)}\|u_{\varepsilon'}(\cdot,t)\|_{L^2(\Omega)}\le\frac{C_1}{\delta}\end{aligned}$$

for a.e. $t\in(0,\infty)$ and all $\varepsilon\in(0,1)$, due to $0<\frac{\varpi}{(\varpi+\delta)^2}<\frac{1}{\varpi+\delta}<\frac{1}{\delta}$ with ϖ representing c_ε or c for all $(x,t)\in\Omega\times(0,\infty)$ and $\varepsilon\in(0,1)$, whence along with (5.15) and (5.19) an application of the dominated convergence theorem yields that for any $T>0$

$$\frac{\phi(t)c_\varepsilon u_\varepsilon}{(c_\varepsilon+\delta)^2}\to\frac{\phi(t)cu}{(c+\delta)^2} \quad \text{in} \quad L^2(\Omega\times(0,T)) \quad \text{as} \quad \varepsilon=\varepsilon_j\searrow 0,$$

which together with (5.21) implies

$$\int_0^T\int_\Omega\frac{\phi(t)c_\varepsilon u_\varepsilon\cdot\nabla c_\varepsilon}{(c_\varepsilon+\delta)^2}\to\int_0^T\int_\Omega\frac{\phi(t)cu\cdot\nabla c}{(c+\delta)^2} \quad \text{as} \quad \varepsilon=\varepsilon_j\searrow 0. \tag{5.26}$$

Apart from that, in light of $\nabla\cdot u_\varepsilon=0$, integrating by parts entails

$$\begin{aligned}\int_0^T\int_\Omega\frac{\phi(t)c_\varepsilon u_\varepsilon\cdot\nabla c_\varepsilon}{(c_\varepsilon+\delta)^2}=&-\int_0^T\int_\Omega\phi(t)c_\varepsilon u_\varepsilon\cdot\nabla\left(\frac{1}{c_\varepsilon+\delta}\right)\\ =&\int_0^T\int_\Omega\frac{\phi(t)u_\varepsilon\cdot\nabla c_\varepsilon}{c_\varepsilon+\delta}\\ =&\int_0^T\int_\Omega\phi(t)u_\varepsilon\cdot\nabla\ln(c_\varepsilon+\delta)=0\end{aligned}$$

for any $T>0$ and each $\phi(t)\in W^{1,\infty}(\mathbb{R})$ as well as all $\varepsilon\in(0,1)$, and thus we infer from (5.26) that

$$\int_0^T\int_\Omega \frac{\phi(t)cu\cdot\nabla c}{(c+\delta)^2}=0 \quad \text{for any} \quad T>0 \quad \text{and each} \quad \phi(t)\in W^{1,\infty}(\mathbb{R}). \tag{5.27}$$

For each fixed $t_0\in(0,\infty)\backslash\left(\tilde{\aleph}\cup\bigcup_{l\in\mathbb{N}}\aleph_l\right)$, and for $\iota\in(0,1)$, we define $\upsilon\in W^{1,\infty}(\mathbb{R})$ as in the argument of [48, Lemma 2.9], that is

$$\upsilon_\iota(t):=\begin{cases}1, & \text{if } \ t\le t_0,\\ \frac{t_0+\iota-t}{\iota}, & \text{if } \ t\in(t_0,t_0+\iota),\\ 0, & \text{if } \ t\ge t_0+\iota.\end{cases} \tag{5.28}$$

Since $0<w\mapsto \ln w$ is concave, it follows that for all $\delta\in(\delta_l)_{l\in\mathbb{N}}$ and any $\tau>0$ the limit function c fulfills

$$\ln\{c(x,t)+\delta\}-\ln\{c(x,t-\tau)+\delta\}\le\frac{1}{c(x,t-\tau)+\delta}\cdot\{c(x,t)-c(x,t-\tau)\} \tag{5.29}$$

for all $(x,t)\in\Omega\times(0,\infty)$, where in order to make c be defined on all of $\Omega\times\mathbb{R}$ we let

$$c(x,t):=c_0(x) \quad \text{for} \quad (x,t)\in\Omega\times(-\infty,0]. \tag{5.30}$$

For any $\tau\in(0,t_0)$, we multiply (5.29) by $\frac{\upsilon_\iota(t)}{\tau}$ to obtain that for each $\iota\in(0,1)$

$$\begin{aligned}M(\iota,\tau):=&\frac{1}{\tau}\int_0^\infty\int_\Omega\upsilon_\iota(t)\ln\{c(\cdot,t)+\delta\}-\frac{1}{\tau}\int_0^\infty\int_\Omega\upsilon_\iota(t)\ln\{c(\cdot,t-\tau)+\delta\}\\ \le&\frac{1}{\tau}\int_0^\infty\int_\Omega\upsilon_\iota(t)\cdot\{c(\cdot,t)-c(\cdot,t-\tau)\}\cdot\frac{1}{c(\cdot,t-\tau)+\delta}=:N(\iota,\tau).\end{aligned} \tag{5.31}$$

In view of (5.30) and the fact that $\upsilon_\iota\equiv1$ on $(0,\tau)\subset(0,t_0)$, we replace $t-\tau$ with τ to deduce that

$$\begin{aligned}M(\iota,\tau)=&\frac{1}{\tau}\int_0^\infty\int_\Omega\upsilon_\iota(t)\ln\{c(\cdot,t)+\delta\}-\frac{1}{\tau}\int_0^\infty\int_\Omega\upsilon_\iota(t+\tau)\ln\{c(\cdot,t)+\delta\}\\ &-\frac{1}{\tau}\int_{-\tau}^0\int_\Omega\upsilon_\iota(t+\tau)\ln\{c(\cdot,t)+\delta\}\\ =&-\int_0^\infty\int_\Omega\frac{\upsilon_\iota(t+\tau)-\upsilon_\iota(t)}{\tau}\cdot\ln\{c(\cdot,t)+\delta\}-\int_\Omega\ln(c_0+\delta),\end{aligned}$$

whereupon by $\ln(c+\delta)\le\ln(\|c_0\|_{L^\infty(\Omega)}+1)$ due to (2.10), (5.20) and $\delta\in(\delta_l)_{l\in\mathbb{N}}\subset(0,1)$, and by

$$\frac{\upsilon_\iota(\cdot+\tau)-\upsilon_\iota}{\tau}\overset{*}{\rightharpoonup}\upsilon_\iota' \quad \text{in} \quad L^\infty(\mathbb{R}) \quad \text{as} \quad \tau\searrow0, \tag{5.32}$$

it can be derived that

$$M(\iota,\tau)\to\frac{1}{\iota}\int_{t_0}^{t_0+\iota}\int_\Omega\ln\{c(\cdot,t)+\delta\}-\int_\Omega\ln\{c_0+\delta\} \quad \text{as} \quad \tau\searrow0. \tag{5.33}$$

For the second integral on the right-hand side of (5.31), we once more substitute $t-\tau$ by τ and make use of (5.30) to have

$$\begin{aligned}-\frac{1}{\tau}\int_0^\infty\int_\Omega\upsilon_\iota(t)\cdot\frac{c(\cdot,t-\tau)}{c(\cdot,t-\tau)+\delta}=&-\frac{1}{\tau}\int_0^\infty\int_\Omega\upsilon_\iota(t+\tau)\cdot\frac{c(\cdot,t)}{c(\cdot,t)+\delta}-\frac{1}{\tau}\int_{-\tau}^0\int_\Omega\upsilon_\iota(t+\tau)\cdot\frac{c(\cdot,t)}{c(\cdot,t)+\delta}\\ =&-\frac{1}{\tau}\int_0^\infty\int_\Omega\upsilon_\iota(t+\tau)\cdot\frac{c(\cdot,t)}{c(\cdot,t)+\delta}-\int_\Omega\frac{c_0}{c_0+\delta}\end{aligned}$$

for all $\iota \in (0,1)$ and $\tau \in (0,t_0)$, and hence

$$
\begin{aligned}
N(\iota,\tau) = &- \int_0^\infty \int_\Omega \frac{\upsilon_\iota(t+\tau)-\upsilon_\iota(t)}{\tau} \cdot \frac{c(\cdot,t)}{c(\cdot,t)+\delta} \\
&- \int_0^\infty \int_\Omega \upsilon_\iota(t) c(\cdot,t) \cdot \frac{1}{\tau} \left\{ \frac{1}{c(\cdot,t)+\delta} - \frac{1}{c(\cdot,t-\tau)+\delta} \right\} \\
&- \int_\Omega \frac{c_0}{c_0+\delta} =: N_1(\iota,\tau) + N_2(\iota,\tau) + N_3(\iota,\tau)
\end{aligned}
\tag{5.34}
$$

for all $\iota \in (0,1)$ and $\tau \in (0,t_0)$. Thanks to (5.32) and $0 < \frac{c}{c+\delta} < \frac{1}{\delta}$ for all $\tau \in (0,t_0)$, it follows that

$$
N_1(\iota,\tau) \to \frac{1}{\iota} \int_{t_0}^{t_0+\iota} \int_\Omega \frac{c}{c+\delta} \quad \text{as } \ \tau \searrow 0. \tag{5.35}
$$

Letting

$$
T_\tau[\zeta](x,t) := \frac{1}{\tau} \int_{t-\tau}^t \zeta(x,s)ds, \quad (x,t) \in \Omega \times (0,\infty), \quad \zeta \in L^1_{loc}(\overline{\Omega} \times \mathbb{R})
$$

and

$$
\varphi(x,t) := \upsilon_\iota(t) \cdot T_\tau \left[\frac{1}{c+\delta} \right](x,t), \quad (x,t) \in \Omega \times (0,\infty),
$$

we can infer from the regularity properties attained for c that φ actually satisfies the regularity requirements as a test function in (5.6). Since

$$
-\upsilon_\iota(t) \cdot \frac{1}{\tau} \left\{ \frac{1}{c(\cdot,t)+\delta} - \frac{1}{c(\cdot,t-\tau)+\delta} \right\} = -\varphi_t + \upsilon_\iota'(t) \cdot T_\tau \left[\frac{1}{c+\delta} \right] \quad \text{in } \ \Omega \times (0,\infty),
$$

we again use (5.30) to obtain

$$
\begin{aligned}
N_2(\iota,\tau) = &- \int_0^\infty \int_\Omega c\varphi_t + \int_0^\infty \int_\Omega \upsilon_\iota'(t) c(\cdot,t) T_\tau \left[\frac{1}{c+\delta} \right](\cdot,t) \\
= &\int_\Omega c_0 \varphi(\cdot,0) - \int_0^\infty \int_\Omega \nabla c \cdot \nabla \varphi - \int_0^\infty \int_\Omega nc\varphi + \int_0^\infty \int_\Omega cu \cdot \nabla \varphi \\
&+ \int_0^\infty \int_\Omega \upsilon_\iota'(t) c(\cdot,t) T_\tau \left[\frac{1}{c+\delta} \right](\cdot,t) \\
= &\int_\Omega \frac{c_0}{c_0+\delta} + \int_0^\infty \int_\Omega \upsilon_\iota(t) \nabla c(\cdot,t) \cdot T_\tau \left[\frac{\nabla c}{(c+\delta)^2} \right](\cdot,t) \\
&- \int_0^\infty \int_\Omega \upsilon_\iota(t) n(\cdot,t) c(\cdot,t) T_\tau \left[\frac{1}{c+\delta} \right](\cdot,t) - \int_0^\infty \int_\Omega \upsilon_\iota(t) cu \cdot T_\tau \left[\frac{\nabla c}{(c+\delta)^2} \right](\cdot,t) \\
&+ \int_0^\infty \int_\Omega \upsilon_\iota'(t) c(\cdot,t) T_\tau \left[\frac{1}{c+\delta} \right](\cdot,t) \quad \text{for all } \ \iota \in (0,1) \ \text{ and } \ \tau \in (0,t_0),
\end{aligned}
\tag{5.36}
$$

where due to $\frac{\nabla c}{(c+\delta)^2} \in L^2_{loc}\left(\overline{\Omega} \times \mathbb{R}\right)$ and $\frac{1}{c+\delta} \in L^\infty\left(\Omega \times \mathbb{R}\right)$, we deduce in line with the argument of [46, Lemma 10.2] to have

$$
\int_0^\infty \int_\Omega \upsilon_\iota(t) \nabla c(\cdot,t) \cdot T_\tau \left[\frac{\nabla c}{(c+\delta)^2} \right](\cdot,t) \to \int_0^\infty \int_\Omega \upsilon_\iota(t) \frac{|\nabla c|^2}{(c+\delta)^2} \quad \text{as } \ \tau \searrow 0,
$$

$$
- \int_0^\infty \int_\Omega \upsilon_\iota(t) n(\cdot,t) c(\cdot,t) T_\tau \left[\frac{1}{c+\delta} \right](\cdot,t) \to - \int_0^\infty \int_\Omega \upsilon_\iota(t) \frac{nc}{c+\delta} \quad \text{as } \ \tau \searrow 0,
$$

and

$$\int_0^\infty \int_\Omega \upsilon_\iota(t) cu \cdot T_\tau \left[\frac{\nabla c}{(c+\delta)^2}\right](\cdot,t) \to \int_0^\infty \int_\Omega \upsilon_\iota(t) \frac{cu\cdot \nabla c}{(c+\delta)^2} \quad \text{as} \ \ \tau \searrow 0,$$

as well as

$$\int_0^\infty \int_\Omega \upsilon_\iota'(t) c(\cdot,t) T_\tau \left[\frac{1}{c+\delta}\right](\cdot,t) \to \int_0^\infty \int_\Omega \frac{\upsilon_\iota'(t) c}{c+\delta} = -\frac{1}{\iota}\int_{t_0}^{t_0+\delta}\int_\Omega \frac{c}{c+\delta} \quad \text{as} \ \ \tau \searrow 0.$$

Thereby, collecting (5.31), (5.33)–(5.36) and (5.27) with $\phi(t)$ replaced by $\upsilon_\iota(t)$, we take $\tau \searrow 0$ to derive

$$\begin{aligned}
&\frac{1}{\iota}\int_{t_0}^{t_0+\iota}\int_\Omega \ln\{c(\cdot,t)+\delta\} - \int_\Omega \ln\{c_0+\delta\} \\
\le & \frac{1}{\iota}\int_{t_0}^{t_0+\iota}\int_\Omega \frac{c}{c+\delta} + \left\{\int_\Omega \frac{c_0}{c_0+\delta} + \int_0^\infty\int_\Omega \upsilon_\iota(t)\frac{|\nabla c|^2}{(c+\delta)^2}\right. \\
& \left. - \int_0^\infty\int_\Omega \upsilon_\iota(t)\frac{nc}{c+\delta} - \frac{1}{\iota}\int_{t_0}^{t_0+\delta}\int_\Omega \frac{c}{c+\delta}\right\} - \int_\Omega \frac{c_0}{c_0+\delta} \\
= & \int_0^\infty \int_\Omega \upsilon_\iota(t)\frac{|\nabla c|^2}{(c+\delta)^2} - \int_0^\infty\int_\Omega \upsilon_\iota(t)\frac{nc}{c+\delta} \quad \text{for any} \ \ \iota \in (0,1).
\end{aligned}$$

Since the choice of $t_0 \in (0,\infty)\backslash \left(\tilde{\aleph} \cup \bigcup_{l\in\mathbb{N}} \aleph_l\right)$ implies the Lebesgue point property of t_0, we invoke Beppo Levi's theorem and let $\iota \searrow 0$ to infer that

$$\int_\Omega \ln\{c(\cdot,t_0)+\delta\} - \int_\Omega \ln\{c_0+\delta\} \le \int_0^{t_0}\int_\Omega \frac{|\nabla c|^2}{(c+\delta)^2} - \int_0^{t_0}\int_\Omega \frac{nc}{c+\delta} \quad \text{for any} \ \ \delta \in (\delta_l)_{l\in\mathbb{N}}. \tag{5.37}$$

Taking $\delta = \delta_l \searrow 0$ on both sides of (5.37) and once more drawing on Beppo Levi's theorem as well as combining with (2.12), (5.8) and (5.22) yield that

$$\begin{aligned}
\int_0^{t_0}\int_\Omega |\nabla v|^2 + \int_\Omega v(\cdot,t_0) - |\Omega| \ln\|c_0\|_{L^\infty(\Omega)} = & \int_0^{t_0}\int_\Omega \frac{|\nabla c|^2}{c^2} - \int_\Omega \ln c(\cdot,t_0) \\
\ge & \int_0^{t_0}\int_\Omega n - \int_\Omega \ln c_0 \\
= & t_0 \cdot \int_\Omega n_0 - |\Omega|\ln\|c_0\|_{L^\infty(\Omega)} + \int_\Omega v_0,
\end{aligned}$$

where $\int_\Omega v(\cdot,t_0) < \infty$ because of $t_0 \notin \tilde{\aleph}$, whence with $\aleph := \tilde{\aleph} \cup \bigcup_{l\in\mathbb{N}} \aleph_l$ it is valid that

$$\int_\Omega v(\cdot,t_0) + \int_0^{t_0}\int_\Omega |\nabla v|^2 \ge \int_\Omega v_0 + t_0 \cdot \int_\Omega n_0 \quad \text{for any} \ \ t_0 \in (0,\infty)\backslash\aleph. \tag{5.38}$$

Thus, in accordance with the arguments of [35, Lemma 2.10], a combination of (5.38) with Lemma 4.2, (5.12) and (5.13) entails (5.14). The proof is completed. $\square$

Lemma 5.3 *Let $\theta \in [0,\frac{1}{2})$ and $l \in \left(1,\frac{6}{5}\right)$. If (3.2) is satisfied for $\omega > 0$ taken by Lemma 3.3, then the triple (n,c,u) of functions provided by Lemma 5.2 solves (1.4), (1.5) and (1.6) in the sense of Definition 5.1.*

Proof. The inclusions claimed by (5.1) can be inferred directly from (5.8), (5.20), (5.21), (5.15) and (5.17). Thanks to (5.8) and the positivity of $(n_\varepsilon)_{\varepsilon\in(0,1)}$, $n \ge 0$ clearly holds. As for $c > 0$, it results from (5.22) and the positivity requirement of c_0 in (1.8). Recalling $\nabla \cdot u_\varepsilon = 0$ in (2.1), one can thus conclude $\nabla \cdot u = 0$ a.e. in $\Omega \times (0,\infty)$ from (5.17). Since

$$\nabla c_\varepsilon^{1-\theta} = -(1-\theta)\|c_0\|_{L^\infty(\Omega)}^{1-\theta} e^{-(1-\theta)v_\varepsilon} \nabla v_\varepsilon$$

for all $t > 0$ and $\varepsilon \in (0,1)$, due to $v_\varepsilon(x,t) = -\ln(\frac{c(x,t)}{\|c_0\|_{L^\infty(\Omega)}})$, and since $e^{-(1-\theta)v_\varepsilon} \le 1$ and $e^{-(1-\theta)v_\varepsilon} \to e^{-(1-\theta)v}$ a.e. in $\Omega \times (0,\infty)$ because of $v_\varepsilon \ge 0$ and (5.11) respectively, we combine with (5.14) and deduce from [46, Lemma A.4] that

$$\nabla c_\varepsilon^{1-\theta} = -(1-\theta)\|c_0\|_{L^\infty(\Omega)}^{1-\theta} e^{-(1-\theta)v_\varepsilon} \nabla v_\varepsilon \to -(1-\theta)\|c_0\|_{L^\infty(\Omega)}^{1-\theta} e^{-(1-\theta)v} \nabla v = \nabla c^{1-\theta} \tag{5.39}$$

in $L^2_{loc}(\overline{\Omega} \times [0,\infty))$, which along with (5.9) implies the inclusions in (5.3). In light of (5.8), n straightforwardly inherits the mass conservation property from n_ε, so (5.4) is valid.

For each fixed $0 \le \varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, we take $\frac{\varphi}{n_\varepsilon + 1}$ as a testing function for the first equation in (2.1) and obtain from integration by parts that

$$\begin{aligned}
\int_0^\infty \int_\Omega |\nabla \ln(n_\varepsilon+1)|^2 \varphi = & -\int_0^\infty \int_\Omega \ln(n_\varepsilon+1) \cdot \varphi_t - \int_\Omega \ln(n_0+1)\varphi(x,0) \\
& + \int_0^\infty \int_\Omega \nabla \ln(n_\varepsilon+1) \cdot \nabla\varphi + \frac{\chi_0}{1-\theta} \int_0^\infty \int_\Omega \frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1} (\nabla \ln(n_\varepsilon+1) \cdot \nabla c_\varepsilon^{1-\theta})\varphi \\
& - \frac{\chi_0}{1-\theta} \int_0^\infty \int_\Omega \frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1} \nabla c_\varepsilon^{1-\theta} \cdot \nabla\varphi - \int_0^\infty \int_\Omega \ln(n_\varepsilon+1)(u_\varepsilon \cdot \nabla\varphi)
\end{aligned} \tag{5.40}$$

for all $\varepsilon \in (0,1)$. Since a collection of (2.4), (2.6) and (5.8) implies $0 \le \frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1} \le 1$ and $\frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1} \to \frac{n}{n+1}$ a.e. in $\Omega \times (0,\infty)$ as $\varepsilon = \varepsilon_j \searrow 0$, we combine with (5.9) and once more invoke [46, Lemma A.4] to have

$$\frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1} \nabla c_\varepsilon^{1-\theta} \to \frac{n}{n+1} \nabla c^{1-\theta} \quad \text{in } L^2_{loc}(\overline{\Omega} \times [0,\infty))$$

as $\varepsilon = \varepsilon_j \searrow 0$, which thus in conjunction with (5.9) yields that as $\varepsilon = \varepsilon_j \searrow 0$ both

$$\int_0^\infty \int_\Omega \frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1} (\nabla \ln(n_\varepsilon+1) \cdot \nabla c_\varepsilon^{1-\theta})\varphi \to \int_0^\infty \int_\Omega \frac{n}{n+1} (\nabla \ln(n+1) \cdot \nabla c^{1-\theta})\varphi$$

and

$$\int_0^\infty \int_\Omega \frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1} \nabla c_\varepsilon^{1-\theta} \cdot \nabla\varphi \to \int_0^\infty \int_\Omega \frac{n}{n+1} \nabla c^{1-\theta} \cdot \nabla\varphi$$

hold. Moreover, from (5.10) and (5.15), it follows that

$$\int_0^\infty \int_\Omega \ln(n_\varepsilon+1)(u_\varepsilon \cdot \nabla\varphi) \to \int_0^\infty \int_\Omega \ln(n+1)(u \cdot \nabla\varphi)$$

as $\varepsilon = \varepsilon_j \searrow 0$. As for the rest of the integrals in (5.40), corresponding convergence can be achieved in accordance with the reasoning of [48, Lemma 2.10] (or [35, Theorem 1.1]), with the aid of (5.9) and (5.10). Consequently, (5.5) is valid. For each $T > 0$ and all $\varepsilon \in (0,1)$, it is not difficult to deduce that

$$\begin{aligned}
\|F_\varepsilon(n_\varepsilon) - n\|_{L^{\frac{5}{3}}(\Omega\times(0,T))} \le & \|F_\varepsilon(n_\varepsilon) - F_\varepsilon(n)\|_{L^{\frac{5}{3}}(\Omega\times(0,T))} + \|F_\varepsilon(n) - n\|_{L^{\frac{5}{3}}(\Omega\times(0,T))} \\
\le & \|F_\varepsilon'\|_{L^\infty((0,\infty))} \|n_\varepsilon - n\|_{L^{\frac{5}{3}}(\Omega\times(0,T))} + \|F_\varepsilon(n) - n\|_{L^{\frac{5}{3}}(\Omega\times(0,T))},
\end{aligned}$$

where (2.4) along with (5.8) implies that

$$\|F_\varepsilon'\|_{L^\infty((0,\infty))}\|n_\varepsilon - n\|_{L^{\frac{5}{3}}(\Omega\times(0,T))} \to 0 \quad \text{as} \ \ \varepsilon = \varepsilon_j \searrow 0,$$

and where by means of the dominated convergence theorem it can be inferred from the fact that $F_\varepsilon(s) \le s$ for $s \ge 0$, (5.8) and (2.6) that

$$\|F_\varepsilon(n) - n\|_{L^{\frac{5}{3}}(\Omega\times(0,T))} \to 0 \quad \text{as} \ \ \varepsilon = \varepsilon_j \searrow 0,$$

and hence we have

$$F_\varepsilon(n_\varepsilon) \to n \quad \text{in} \ \ L^{\frac{5}{3}}_{loc}(\overline{\Omega}\times[0,\infty)) \ \ \text{as} \ \ \varepsilon = \varepsilon_j \searrow 0. \tag{5.41}$$

In view of the observation that the second and the third equations in (2.1) resemble that respectively in [44, (2.9)], we recall the arguments of [44, Lemma 4.1]to see that both (5.6) and (5.7) can be derived from (5.8), (5.15), (5.16), (5.17), (5.19), (5.20), (5.21) and (5.41), and thus complete the proof. □

Proof of Theorem 1.1. Theorem 1.1 is a straightforward consequence of Lemma 5.3. □

6 Ultimate regularity of (n, c, u)

6.1 Preliminaries

In this section, we try to detect the eventual regularity properties of (n, c, u) by means of a bootstrap-type argument. This heavily relies on uniform spatial-temporal estimates for $|\nabla n_\varepsilon|^2$, which mainly reflects in the derivation of eventual smallness properties for two temporally average integrals (Lemma 6.2) as well as in the subsequent establishment of eventual bounds for $\nabla c_\varepsilon^{1-\theta}$ (Lemmas 6.5–6.6). It can be observed from Lemma 3.4 that the hypothesis $l > 1$ is not sufficient to provide the time-independent bounds of $\int_0^t\int_\Omega |\nabla n_\varepsilon|^2$ for all $\varepsilon \in (0,1)$, so here we provide a corollary of Lemma 3.4 at first, corresponding to the special case of $l = 2$.

Corollary 6.1 *Let $\theta \in [0, \frac{1}{2})$ and $l = 2$. Assume that (3.2) holds for $\omega > 0$ picked by Lemma 3.3. Then one can find $C > 0$ fulfilling*

$$\int_\Omega n_\varepsilon^2(\cdot,t) \le C \quad \textit{for all} \quad t > 0 \quad \textit{and} \quad \varepsilon \in (0,1), \tag{6.1}$$

and

$$\int_0^\infty\int_\Omega |\nabla n_\varepsilon|^2 \le C \quad \textit{for all} \quad \varepsilon \in (0,1). \tag{6.2}$$

Proof. Corollary 6.1 follows directly by taking $l = 2$ in the argument of Lemma 3.4. □

Lemma 6.2 *Let $\theta \in [0, \frac{1}{2})$ and $l = 2$. Assume that (3.2) is valid with $\omega > 0$ fulfilling the assumptions of Lemma 3.3. Then for any $\varsigma > 0$, one can choose $t_\star := t_\star(\varsigma, n_0, u_0) > 0$ such that*

$$\int_t^{t+1}\int_\Omega (n_\varepsilon - \overline{n_0})^2 < \varsigma \quad \textit{for all} \ \ t > t_\star \ \ \textit{and} \ \ \varepsilon \in (0,1), \tag{6.3}$$

and that

$$\int_t^{t+1}\int_\Omega |\nabla u_\varepsilon|^2 < \varsigma \quad \textit{for all} \ \ t > t_\star \ \ \textit{and} \ \ \varepsilon \in (0,1). \tag{6.4}$$

Proof. From Corollary 6.1, it follows that

$$\int_0^\infty \int_\Omega |\nabla n_\varepsilon|^2 \le C_1 \quad \text{for all} \;\; \varepsilon \in (0,1)$$

with some $C_1 = C_1(n_0) > 0$, which not only implies that

$$\int_t^{t+1} \int_\Omega |\nabla n_\varepsilon|^2 \le C_1 \quad \text{for all} \;\; t > 0 \;\; \text{and} \;\; \varepsilon \in (0,1), \tag{6.5}$$

but also warrants that for each $\varsigma_1 > 0$ there exists $t_1 = t_1(\varsigma_1) > 0$ such that

$$\int_t^{t+1} \int_\Omega |\nabla n_\varepsilon|^2 < \varsigma_1 \quad \text{for all} \;\; t > t_1 \;\; \text{and} \;\; \varepsilon \in (0,1). \tag{6.6}$$

In combination with (2.9), the Poincaré inequality provides $C_2 > 0$ satisfying

$$\int_\Omega |n_\varepsilon(\cdot,t) - \overline{n_0}|^2 \le C_2 \int_\Omega |\nabla n_\varepsilon(\cdot,t)|^2 \quad \text{for all} \;\; t > 0 \;\; \text{and} \;\; \varepsilon \in (0,1),$$

where $\overline{n_0} = \frac{1}{|\Omega|}\int_\Omega n_0$, whence we obtain from (6.5) and (6.6) that

$$\int_t^{t+1} \int_\Omega |n_\varepsilon - \overline{n_0}|^2 \le C_1 C_2 \quad \text{for all} \;\; t > 0 \;\; \text{and} \;\; \varepsilon \in (0,1), \tag{6.7}$$

and that

$$\int_t^{t+1} \int_\Omega |n_\varepsilon - \overline{n_0}|^2 \le C_2 \varsigma_1 \quad \text{for all} \;\; t > t_1 \;\; \text{and} \;\; \varepsilon \in (0,1), \tag{6.8}$$

respectively. Since $\nabla \cdot u_\varepsilon = 0$, and since $\overline{n_0} > 0$ is a constant, we test the third equation in (2.1) by u_ε to have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon(\cdot,t)|^2 + \int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 = \int_\Omega (n_\varepsilon(\cdot,t) - \overline{n_0}) u_\varepsilon \cdot \nabla\phi \tag{6.9}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Another application of the Poincaré inequality yields $C_3 > 0$ such that

$$\int_\Omega |u_\varepsilon(\cdot,t)|^2 \le C_3 \int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \quad \text{for all} \;\; t > 0 \;\; \text{and} \;\; \varepsilon \in (0,1), \tag{6.10}$$

whereupon we invoke (1.7) along with Young's inequality to attain

$$\begin{aligned} \int_\Omega (n_\varepsilon(\cdot,t) - \overline{n_0}) u_\varepsilon \cdot \nabla\phi \le & \|\nabla\phi\|_{L^\infty(\Omega)} \int_\Omega |n_\varepsilon(\cdot,t) - \overline{n_0}| \cdot |u_\varepsilon(\cdot,t)| \\ \le & \frac{1}{2C_3}\int_\Omega |u_\varepsilon(\cdot,t)|^2 + \frac{C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}}{2}\int_\Omega (n_\varepsilon(\cdot,t) - \overline{n_0})^2 \\ \le & \frac{1}{2}\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 + \frac{C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}}{2}\int_\Omega (n_\varepsilon(\cdot,t) - \overline{n_0})^2 \end{aligned} \tag{6.11}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Substituting (6.11) into (6.9), we derive

$$\frac{d}{dt}\int_\Omega |u_\varepsilon(\cdot,t)|^2 + \int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \le C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega (n_\varepsilon(\cdot,t) - \overline{n_0})^2$$

for all $t > 0$ and $\varepsilon \in (0,1)$, which in conjunction with (6.10) implies

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |u_\varepsilon(\cdot,t)|^2 + \frac{1}{2C_3}\int_\Omega |u_\varepsilon(\cdot,t)|^2 + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \\ \le& C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega (n_\varepsilon(\cdot,t)-\overline{n_0})^2\end{aligned} \tag{6.12}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Recalling (6.7), we apply [47, Lemma 3.1] to (6.12) to obtain that

$$\int_\Omega |u_\varepsilon(\cdot,t)|^2 \le e^{-\frac{t}{2C_3}}\int_\Omega |u_0|^2 + \frac{C_1C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}}{1-e^{-\frac{1}{2C_3}}} =: C_4 \tag{6.13}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Now, for each $\varsigma_2 > 0$, let us choose $t_2 > t_1$ sufficiently large fulfilling

$$C_4 \cdot e^{-\frac{t_2-t_1}{2C_3}} < \frac{\varsigma_2}{4}.$$

Then for all $t \ge t_2$ once more drawing on [47, Lemma 3.1], we combine with (6.8) and (6.13) to have

$$\begin{aligned}\int_\Omega |u_\varepsilon(\cdot,t)|^2 \le& e^{-\frac{t-t_1}{2C_3}}\int_\Omega |u(\cdot,t_1)|^2 + \frac{\varsigma_1 C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}}{1-e^{-\frac{1}{2C_3}}} \\ \le& e^{-\frac{t_2-t_1}{2C_3}}\cdot C_4 + \frac{\varsigma_1 C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}}{1-e^{-\frac{1}{2C_3}}} \\ <& \frac{\varsigma_2}{4} + \frac{\varsigma_1 C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}}{1-e^{-\frac{1}{2C_3}}}\end{aligned}$$

for all $\varepsilon \in (0,1)$, and hence taking $\varsigma_1 := \frac{\varsigma_2(1-e^{-\frac{1}{2C_3}})}{4C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}(2-e^{-\frac{1}{2C_3}})}$ we integrate (6.12) on $(t,t+1)$ for each $t \ge t_2$ and employ (6.8) to achieve that

$$\begin{aligned}\frac{1}{2}\int_t^{t+1}\int_\Omega |\nabla u_\varepsilon|^2 \le& \int_\Omega |u_\varepsilon(\cdot,t)|^2 + \varsigma_1 C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)} \\ <& \frac{\varsigma_2}{4} + \frac{\varsigma_1 C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)}}{1-e^{-\frac{1}{2C_3}}} + \varsigma_1 C_2C_3\|\nabla\phi\|^2_{L^\infty(\Omega)} \\ <& \frac{\varsigma_2}{4} + \frac{\varsigma_2}{4} = \frac{\varsigma_2}{2}\end{aligned} \tag{6.14}$$

for all $\varepsilon \in (0,1)$. Thereby, with $\varsigma := \min\{C_2\varsigma_1, \varsigma_2\}$, (6.3) and (6.4) result from (6.8) and (6.14), respectively. The proof is completed. $\Box$

Based on Lemma 6.2, we are able to achieve the following temporally ultimate estimates for u_ε through a well-established argument (cf. the proof of [47, Lemma 3.3]).

Lemma 6.3 *Let $\theta \in [0, \frac{1}{2})$ and $l = 2$, and let (3.2) hold for $\omega > 0$ provided by Lemma 3.3. Then there exists $\eta_1 \in (0, \frac{1}{4})$ such that for any $\varsigma > 0$ one can take $t_\star := t_\star(\varsigma, n_0, u_0) > 0$ with the property that for each $\alpha \in [\frac{1}{4}, \frac{1}{4}+\eta_1]$, any $r \in [3, 3+\eta_1]$ and all $\varepsilon \in (0,1)$*

$$\|u_\varepsilon(\cdot,t)\|_{L^r(\Omega)} + \|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \varsigma \quad \textit{for all} \;\; t > t_\star. \tag{6.15}$$

Proof. The detailed reasoning can be found in [47]. $\Box$

6.2 Further ultimate regularity of $(n_\varepsilon, c_\varepsilon, u_\varepsilon)_{\varepsilon\in(0,1)}$

Now, again relying on the decay features provided by Corollary 6.1, we can turn the bounds as asserted by Lemma 6.3 into eventual Hölder regularity.

Lemma 6.4 *Let $\theta \in [0, \frac{1}{2})$ and $l = 2$. Suppose that (3.2) is satisfied for $\omega > 0$ chosen by Lemma 3.3. Then there exists $\gamma \in (0,1), t_\star := t_\star(n_0, u_0) > 0$ and $C > 0$ such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \textit{for all} \;\; t \ge t_\star \;\; \textit{and} \;\; \varepsilon \in (0,1), \tag{6.16}$$

and that

$$\|u_\varepsilon\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}\times[t,t+1])} \le C \quad \textit{for all} \;\; t \ge t_\star \;\; \textit{and} \;\; \varepsilon \in (0,1). \tag{6.17}$$

Proof. With $\eta_1 \in (0, \frac{1}{4})$ as picked in Lemma 6.3, we let $\alpha_0 := \frac{1}{4} + \frac{3\eta_1}{4}, \alpha := \frac{3}{4} - \frac{1}{2+\eta_1}, r_1 := 3 + 3\eta_1, r_2 := 3 + 2\eta_1$ and $r_3 := 3 + \eta_1$. Then upon elementary calculations, we have

$$2\alpha_0 - \frac{3}{2} = \frac{1+3\eta_1}{2} - \frac{3}{2} > \frac{1+3\eta_1}{2(1+\eta_1)} - \frac{3}{2} = -\frac{3}{r_1}, \tag{6.18}$$

$$r_3 := 3 + \eta_1 < \frac{9}{\eta_1} + 15 + 6\eta_1 = \frac{r_1 r_2}{r_1 - r_2} \tag{6.19}$$

and

$$\frac{1}{2} > \alpha = \frac{3}{4} - \frac{1}{2+\eta_1} > \frac{1}{4} \tag{6.20}$$

due to $\eta_1 \in (0, \frac{1}{4})$, as well as

$$\begin{aligned} -\alpha - \frac{1}{2} - \frac{3}{2}\left(\frac{1}{r_2} - \frac{1}{6}\right) &= -\frac{3}{4} + \frac{3}{6+3\eta_1} - \frac{1}{2} - \frac{3}{2}\left(\frac{1}{3+2\eta_1} - \frac{1}{6}\right) \\ &> -\frac{3}{4} + \frac{3}{6+4\eta_1} - \frac{1}{2} - \frac{3}{6+4\eta_1} + \frac{1}{4} \\ &= -1. \end{aligned} \tag{6.21}$$

From Lemma 6.3 and Corollary 6.1, it can be inferred that there exist $t_1 > 0$ and $C_1 > 0$ such that

$$\|A^{\alpha_0} u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_1 \quad \text{and} \quad \|u_\varepsilon(\cdot,t)\|_{L^{r_3}(\Omega)} \le C_1 \quad \text{for all} \;\; t \ge t_1 \;\; \text{and} \;\; \varepsilon \in (0,1), \tag{6.22}$$

as well as that

$$\int_t^{t+1}\int_\Omega |\nabla n_\varepsilon|^2 \le C_1 \quad \text{for all} \;\; t \ge t_1 \;\; \text{and} \;\; \varepsilon \in (0,1), \tag{6.23}$$

where along with (2.9) and the Poincaré–Sobolev inequality, (6.23) implies that

$$\int_t^{t+1} \|n_\varepsilon(\cdot,s) - \overline{n_0}\|^2_{L^6(\Omega)} ds \le C_2 \int_t^{t+1}\int_\Omega |\nabla n_\varepsilon|^2 \le C_1 C_2 \quad \text{for all} \;\; t \ge t_1 \;\; \text{and} \;\; \varepsilon \in (0,1), \tag{6.24}$$

with some $C_2 > 0$. Now, for all $\varepsilon \in (0,1)$ and each $T > t_2 := t_1 + 1$, we let

$$U_\varepsilon(T) := \sup_{t\in(t_2,T)} \|A^{\alpha_0} u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}, \quad T > t_2, \;\; \varepsilon \in (0,1). \tag{6.25}$$

Thanks to $(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon\equiv\nabla\cdot(Y_\varepsilon u_\varepsilon\otimes u_\varepsilon)$ (cf. [30, p.165]), the variation-of-constants representation of the third equation appears as

$$\begin{aligned}u_\varepsilon(\cdot,t)=&e^{-A}u_\varepsilon(\cdot,t-1)-\int_{t-1}^t e^{-(t-s)A}\cdot\mathcal{P}[\nabla\cdot(Y_\varepsilon u_\varepsilon(\cdot,s)\otimes u_\varepsilon(\cdot,s))]ds\\&+\int_{t-1}^t e^{-(t-s)A}\cdot\mathcal{P}[(n_\varepsilon(\cdot,s)-\overline{n_0})\nabla\phi]ds\\=:&u_{1\varepsilon}(\cdot,t)+u_{2\varepsilon}(\cdot,t)+u_{3\varepsilon}(\cdot,t)\quad\text{for all }\ t\geq t_2\ \text{ and }\ \varepsilon\in(0,1),\end{aligned}\tag{6.26}$$

due to the fact that $\overline{n_0}=\frac{1}{|\Omega|}\int_\Omega n_0>0$ is a constant. Applying A^α on both sides of (6.26) and drawing on standard L^p-L^q estimates for the Stokes semigroup, we can find $C_3>0$ satisfying

$$\|A^\alpha u_{1\varepsilon}(\cdot,t)\|_{L^6(\Omega)}\leq C_3\|u_\varepsilon(\cdot,t-1)\|_{L^{r_3}(\Omega)}\leq C_1C_3\quad\text{for all }\ t\geq t_2\ \text{ and }\ \varepsilon\in(0,1).\tag{6.27}$$

Similarly,

$$\|A^\alpha u_{2\varepsilon}(\cdot,t)\|_{L^6(\Omega)}\leq C_4\int_{t-1}^t(t-s)^{-\alpha-\frac12-\frac32\left(\frac{1}{r_2}-\frac16\right)}\|Y_\varepsilon u_\varepsilon(\cdot,s)\otimes u_\varepsilon(\cdot,s)\|_{L^{r_2}(\Omega)}ds\tag{6.28}$$

for all $t\geq t_2$ and $\varepsilon\in(0,1)$, with some $C_4>0$. Since A^{α_0} and Y_ε are commutative on $D(A_2^{\alpha_0})$, and since Y_ε is nonexpansive on $L^2_\sigma(\Omega)$, we invoke the Hölder inequality together with the embedding $D(A_2^{\alpha_0})\hookrightarrow L^{r_1}(\Omega;\mathbb{R}^3)$ which is guaranteed by (6.18) to obtain $C_5>0$ and $C_6>0$ such that by (6.22) we derive

$$\begin{aligned}\|Y_\varepsilon u_\varepsilon(\cdot,s)\otimes u_\varepsilon(\cdot,s)\|_{L^{r_2}(\Omega)}\leq&C_5\|Y_\varepsilon u_\varepsilon(\cdot,s)\|_{L^{r_1}(\Omega)}\|u_\varepsilon(\cdot,s)\|_{L^{\frac{r_1r_2}{r_1-r_2}}(\Omega)}\\\leq&C_6\|A^{\alpha_0}Y_\varepsilon u_\varepsilon(\cdot,s)\|_{L^2(\Omega)}\|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}^{1-\frac{(r_1-r_2)r_3}{r_1r_2}}\|u_\varepsilon(\cdot,s)\|_{L^{r_3}(\Omega)}^{\frac{(r_1-r_2)r_3}{r_1r_2}}\\\leq&C_6\|A^{\alpha_0}u_\varepsilon(\cdot,s)\|_{L^2(\Omega)}\|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}^{1-\frac{(r_1-r_2)r_3}{r_1r_2}}\|u_\varepsilon(\cdot,s)\|_{L^{r_3}(\Omega)}^{\frac{(r_1-r_2)r_3}{r_1r_2}}\\\leq&C_1^{1+\frac{(r_1-r_2)r_3}{r_1r_2}}C_6\|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}^{1-\frac{(r_1-r_2)r_3}{r_1r_2}}\quad\text{for all }\ s\geq t_1\ \text{ and }\ \varepsilon\in(0,1).\end{aligned}\tag{6.29}$$

In view of (6.25), (6.21) and the embedding $D(A_6^\alpha)\hookrightarrow L^\infty(\Omega)$ that is warranted by (6.20), we insert (6.29) into (6.28) to achieve that

$$\|A^\alpha u_{2\varepsilon}(\cdot,t)\|_{L^6(\Omega)}\leq C_7U_\varepsilon^{1-\frac{(r_1-r_2)r_3}{r_1r_2}}(T)\quad\text{for all }\ t\in(t_2,T)\ \text{ and }\ \varepsilon\in(0,1),\tag{6.30}$$

with some $C_7>0$. Thanks to $\alpha<\frac12$ which is implied by $\eta_1<\frac14$, it can be derived from standard smoothing features of the Stokes semigroup ([30],[12, p. 201]), the boundedness properties of $\mathcal{P}$ in $L^6(\Omega;\mathbb{R}^3)$ ([11]), (1.7), the Hölder inequality and (6.24) that

$$\begin{aligned}\|A^\alpha u_{3\varepsilon}(\cdot,t)\|_{L^6(\Omega)}\leq&C_8\int_{t-1}^t(t-s)^{-\alpha}\|\mathcal{P}[(n_\varepsilon(\cdot,s)-\overline{n_0})\nabla\phi]\|_{L^6(\Omega)}ds\\\leq&C_9\int_{t-1}^t(t-s)^{-\alpha}\|n_\varepsilon(\cdot,s)-\overline{n_0}\|_{L^6(\Omega)}ds\\\leq&C_9\left(\int_{t-1}^t(t-s)^{-2\alpha}ds\right)^{\frac12}\left(\int_{t-1}^t\|n_\varepsilon(\cdot,s)-\overline{n_0}\|_{L^6(\Omega)}^2\right)^{\frac12}\\\leq&\frac{C_9C_1^{\frac12}C_2^{\frac12}}{(1-2\alpha)^{\frac12}}\quad\text{for all }\ t\geq t_2\ \text{ and }\ \varepsilon\in(0,1),\end{aligned}\tag{6.31}$$

with $C_8 > 0$ and $C_9 > 0$. Collecting (6.26), (6.27), (6.30) and (6.31), we can find $C_{10} > 0$ fulfilling

$$\|A^\alpha u_\varepsilon(\cdot,t)\|_{L^6(\Omega)} \le C_{10} + C_{10} U_\varepsilon^{1-\frac{(r_1-r_2)r_3}{r_1 r_2}}(T) \quad \text{for all } \ t \in (t_2, T) \ \text{ and } \ \varepsilon \in (0,1), \tag{6.32}$$

from which it follows that

$$U_\varepsilon(T) \le \max\{1, (2C_{10})^{\frac{r_1 r_2}{(r_1-r_2)r_3}}\} \quad \text{for each } \ T > t_2 \ \text{ and all } \ \varepsilon \in (0,1). \tag{6.33}$$

Thereby, again by the embedding $D(A_6^\alpha) \hookrightarrow L^\infty(\Omega)$, (6.33) implies (6.16). Apart from that, for $\gamma_1 \in (0, 2\alpha - \frac{1}{2})$, (6.33) together with the embedding $D(A_6^\alpha) \hookrightarrow C^{\gamma_1}(\overline{\Omega}; \mathbb{R}^3)$ also provides $C_{11} > 0$ having the properties that

$$\|u_\varepsilon(\cdot,t)\|_{C^{\gamma_1}(\overline{\Omega})} \le C_{11} \quad \text{for all } \ t \ge t_2 \ \text{ and } \ \varepsilon \in (0,1). \tag{6.34}$$

Based on the previous choices of the parameters fulfilling (6.18)–(6.21) and on the estimates achieved in (6.22) and (6.24), a standard modification of the above arguments ([8]) enables us to find $C_{12} > 0$ and $\gamma_2 \in (0,1)$ such that for all $\varepsilon \in (0,1)$

$$\|A^\alpha u_\varepsilon(\cdot,t) - A^\alpha u_\varepsilon(\cdot,t')\|_{L^6(\Omega)} \le C_{12}(t-t')^{\gamma_2} \quad \text{for any } \ t' \ge t_2 \ \text{ and } \ t \in [t', t'+1]. \tag{6.35}$$

With appropriate smallness choice of $\gamma \in (0,1)$, (6.17) thus results from (6.34) and (6.35). $\square$

With Lemma 6.4 at hand, we intend to detect eventual regularity properties of n_ε, which also rests on appropriate ultimate estimates of $\nabla c_\varepsilon^{1-\theta}$ besides Lemma 6.4, so in the following let us provide temporally eventual L^4-bounds for $\nabla c_\varepsilon^{1-\theta}$ at first.

Lemma 6.5 *Let $\theta \in [0, \frac{1}{2})$ and $l = 2$. If (3.2) is fulfilled with $\omega > 0$ picked by Lemma 3.3, then one can find $C > 0$ with the properties that*

$$\int_\Omega \frac{|\nabla c_\varepsilon(\cdot,t)|^2}{c_\varepsilon(\cdot,t)} \le C \quad \textit{for all } \ t > 0 \ \textit{ and } \ \varepsilon \in (0,1), \tag{6.36}$$

and that

$$\int_t^{t+1} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \le C \quad \textit{for all } \ t > 0 \ \textit{ and } \ \varepsilon \in (0,1). \tag{6.37}$$

Proof. Upon integration by parts, we derive from the second equation in (2.1) that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} =& 2\int_\Omega \frac{\nabla c_\varepsilon \cdot \nabla c_{\varepsilon t}}{c_\varepsilon} - \int_\Omega \frac{|\nabla c_\varepsilon|^2 c_{\varepsilon t}}{c_\varepsilon^2} \\
=& 2\int_\Omega \frac{\nabla c_\varepsilon}{c_\varepsilon} \cdot \nabla(\Delta c_\varepsilon - F_\varepsilon(n_\varepsilon)c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon) - \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(\Delta c_\varepsilon - F_\varepsilon(n_\varepsilon)c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon) \\
=& -2\int_\Omega \frac{|\Delta c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}\Delta c_\varepsilon - 2\int_\Omega \nabla F_\varepsilon(n_\varepsilon) \cdot \nabla c_\varepsilon - \int_\Omega \frac{F_\varepsilon(n_\varepsilon)|\nabla c_\varepsilon|^2}{c_\varepsilon} \\
&+ 2\int_\Omega \frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon \cdot \nabla c_\varepsilon) - \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon \cdot \nabla c_\varepsilon)
\end{aligned} \tag{6.38}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Recalling [26, Lemma 3.2], we combine with (2.10) to obtain $C_1 > 0$ and $C_2 > 0$ such that

$$-2\int_\Omega \frac{|\Delta c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}\Delta c_\varepsilon \le -C_1 \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + C_2 \quad \text{for all } \ t > 0 \ \text{ and } \ \varepsilon \in (0,1). \tag{6.39}$$

Apart from that, along with (2.4) and (2.10), two applications of Young's inequality entail that

$$
\begin{aligned}
-2\int_\Omega \nabla F_\varepsilon(n_\varepsilon)\cdot\nabla c_\varepsilon =&-2\int_\Omega F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon\\
\le&\varsigma\int_\Omega|\nabla c_\varepsilon|^2+C_3\int_\Omega|\nabla n_\varepsilon|^2\\
\le&\varsigma\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\varsigma\int_\Omega c_\varepsilon^3+C_3\int_\Omega|\nabla n_\varepsilon|^2\\
\le&\varsigma\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\varsigma\|c_0\|^3_{L^\infty(\Omega)}|\Omega|+C_3\int_\Omega|\nabla n_\varepsilon|^2,
\end{aligned}
\tag{6.40}
$$

with any $\varsigma>0$ and $C_3=C_3(\varsigma)>0$. Thanks to $\nabla\cdot u_\varepsilon=0$, it follows from integration by parts that

$$
\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon)=-\int_\Omega|\nabla c_\varepsilon|^2u_\varepsilon\cdot\nabla\left(\frac{1}{c_\varepsilon}\right)=\int_\Omega\frac{u_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2}{c_\varepsilon}=2\int_\Omega\frac{u_\varepsilon\cdot D^2c_\varepsilon\cdot\nabla c_\varepsilon}{c_\varepsilon}
$$

for all $t>0$ and $\varepsilon\in(0,1)$, and hence

$$
\begin{aligned}
&2\int_\Omega\frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon\cdot\nabla c_\varepsilon)-\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon)\\
=&2\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon)-2\int_\Omega\frac{\nabla c_\varepsilon\cdot\nabla(u_\varepsilon\cdot\nabla c_\varepsilon)}{c_\varepsilon}-\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon)\\
=&\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon)-2\int_\Omega\frac{\nabla c_\varepsilon\cdot\nabla u_\varepsilon\cdot\nabla c_\varepsilon}{c_\varepsilon}-2\int_\Omega\frac{u_\varepsilon\cdot D^2c_\varepsilon\cdot\nabla c_\varepsilon}{c_\varepsilon}\\
=&-2\int_\Omega\frac{\nabla c_\varepsilon\cdot\nabla u_\varepsilon\cdot\nabla c_\varepsilon}{c_\varepsilon}\quad\text{for all }\ t>0\ \text{ and }\ \varepsilon\in(0,1).
\end{aligned}
\tag{6.41}
$$

Again by Young's inequality and (2.10), we obtain from (6.41) that

$$
\begin{aligned}
2\int_\Omega\frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon\cdot\nabla c_\varepsilon)-\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon)=&-2\int_\Omega\frac{\nabla c_\varepsilon\cdot\nabla u_\varepsilon\cdot\nabla c_\varepsilon}{c_\varepsilon}\\
\le&\varsigma\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+C_4\int_\Omega c_\varepsilon|\nabla u_\varepsilon|^2\\
\le&\varsigma\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+C_4\|c_0\|_{L^\infty(\Omega)}\int_\Omega|\nabla u_\varepsilon|^2
\end{aligned}
\tag{6.42}
$$

for all $t>0$ and $\varepsilon\in(0,1)$, where $\varsigma>0$ is arbitrary and $C_4=C_4(\varsigma)>0$. Moreover, for each $\varsigma>0$, another utilization of Young's inequality together with (2.10) yields $C_5=C_5(\varsigma)>0$ satisfying

$$
\begin{aligned}
\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}\le&\varsigma\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+C_5\int_\Omega c_\varepsilon\\
\le&\varsigma\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+C_5\|c_0\|_{L^\infty(\Omega)}|\Omega|\quad\text{for all }\ t>0\ \text{ and }\ \varepsilon\in(0,1).
\end{aligned}
\tag{6.43}
$$

Picking $\varsigma=\frac{C_1}{4}$, we substitute (6.39), (6.40), (6.42) and (6.43) into (6.38) to achieve that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\frac{C_1}{4}\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}\\
\le&C_2+\frac{C_1\|c_0\|^3_{L^\infty(\Omega)}|\Omega|}{4}+C_3\int_\Omega|\nabla n_\varepsilon|^2+C_4\|c_0\|_{L^\infty(\Omega)}\int_\Omega|\nabla u_\varepsilon|^2\\
&+C_5\|c_0\|_{L^\infty(\Omega)}|\Omega|\quad\text{for all }\ t>0\ \text{ and }\ \varepsilon\in(0,1).
\end{aligned}
\tag{6.44}
$$

Since Corollary 6.1 ensures the existence of $C_6 > 0$ fulfilling

$$\int_t^{t+1} \int_\Omega |\nabla n_\varepsilon|^2 \le C_6 \quad \text{for all} \ \ t > 0 \ \ \text{and} \ \ \varepsilon \in (0,1),$$

and since Lemma 4.3 provides $C_7 > 0$ such that

$$\int_t^{t+1} \int_\Omega |\nabla u_\varepsilon|^2 \le C_7 \quad \text{for all} \ \ t > 0 \ \ \text{and} \ \ \varepsilon \in (0,1),$$

we let $C_8 := C_2 + \frac{C_1 \|c_0\|^3_{L^\infty(\Omega)} |\Omega|}{4} + C_3 C_6 + C_4 C_7 \|c_0\|_{L^\infty(\Omega)} + C_5 \|c_0\|_{L^\infty(\Omega)} |\Omega|$ and apply [36, Lemma 3.4] to (6.44) to derive that

$$\int_\Omega \frac{|\nabla c_\varepsilon(\cdot,t)|^2}{c_\varepsilon(\cdot,t)} \le \int_\Omega \frac{|\nabla c_0|^2}{c_0} + \frac{C_8}{1-e^{-1}} \quad \text{for all} \ \ t > 0 \ \ \text{and} \ \ \varepsilon \in (0,1), \tag{6.45}$$

which along with (1.8) shows (6.36). Combined with (1.8) and (6.45), an integration of (6.44) on $(t, t+1)$ implies that

$$\begin{aligned}
\int_t^{t+1} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \le& \frac{4}{C_1} \left\{ \int_\Omega \frac{|\nabla c_\varepsilon(\cdot,t)|^2}{c_\varepsilon(\cdot,t)} + C_8 \right\} \\
\le& \frac{4}{C_1} \left\{ \int_\Omega \frac{|\nabla c_0|^2}{c_0} + \frac{C_8}{1-e^{-1}} + C_8 \right\} \quad \text{for all} \ \ t > 0 \ \ \text{and} \ \ \varepsilon \in (0,1),
\end{aligned}$$

as claimed. □

Lemma 6.6 *Let $\theta \in [0, \frac{1}{2})$ and $l = 2$. Assume that (3.2) holds for $\omega > 0$ taken by Lemma 3.3. Then there exist $t_\star = t_\star(n_0, c_0, u_0) > 0$ and $C > 0$ such that*

$$\|\nabla c_\varepsilon^{1-\theta}(\cdot,t)\|_{L^4(\Omega)} \le C \quad \text{for all} \ \ t > t_\star \ \ \text{and} \ \ \varepsilon \in (0,1). \tag{6.46}$$

Proof. Upon integration by parts, it can be derived from the second equation of (2.1) and from the equality $\nabla c_\varepsilon \cdot \nabla \Delta c_\varepsilon = \frac{1}{2} \Delta |\nabla c_\varepsilon|^2 - |D^2 c_\varepsilon|^2$ that

$$\begin{aligned}
\frac{d}{dt} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} =& 4 \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3} \nabla c_\varepsilon \cdot \nabla \left\{ \Delta c_\varepsilon - F_\varepsilon(n_\varepsilon) c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon \right\} \\
&- 3 \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^4} \cdot \left\{ \Delta c_\varepsilon - F_\varepsilon(n_\varepsilon) c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon \right\} \\
=& 2 \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3} \Delta |\nabla c_\varepsilon|^2 - 4 \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3} |D^2 c_\varepsilon|^2 - 4 \int_\Omega \frac{F'_\varepsilon(n_\varepsilon) |\nabla c_\varepsilon|^2}{c_\varepsilon^2} \nabla c_\varepsilon \cdot \nabla n_\varepsilon \\
&- 4 \int_\Omega \frac{F_\varepsilon(n_\varepsilon) |\nabla c_\varepsilon|^4}{c_\varepsilon^3} - 4 \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3} \nabla c_\varepsilon \cdot \nabla (u_\varepsilon \cdot \nabla c_\varepsilon) \\
&- 3 \int_\Omega \frac{|\nabla c_\varepsilon|^4 \Delta c_\varepsilon}{c_\varepsilon^4} + 3 \int_\Omega \frac{F_\varepsilon(n_\varepsilon) |\nabla c_\varepsilon|^4}{c_\varepsilon^3} + 3 \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^4} (u_\varepsilon \cdot \nabla c_\varepsilon) \\
=& - 2 \int_\Omega \frac{\left|\nabla |\nabla c_\varepsilon|^2\right|^2}{c_\varepsilon^3} - 4 \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3} |D^2 c_\varepsilon|^2 + 6 \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^4} \nabla c_\varepsilon \cdot \nabla |\nabla c_\varepsilon|^2 \\
&- 3 \int_\Omega \frac{|\nabla c_\varepsilon|^4 \Delta c_\varepsilon}{c_\varepsilon^4} - \int_\Omega \frac{F_\varepsilon(n_\varepsilon) |\nabla c_\varepsilon|^4}{c_\varepsilon^3} + 2 \int_{\partial\Omega} \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3} \cdot \frac{\partial |\nabla c_\varepsilon|^2}{\partial \nu} \\
&- 4 \int_\Omega \frac{F'_\varepsilon(n_\varepsilon) |\nabla c_\varepsilon|^2}{c_\varepsilon^2} \nabla c_\varepsilon \cdot \nabla n_\varepsilon - 4 \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3} \nabla c_\varepsilon \cdot \nabla (u_\varepsilon \cdot \nabla c_\varepsilon) \\
&+ 3 \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^4} (u_\varepsilon \cdot \nabla c_\varepsilon) \quad \text{for all} \ \ t > 0 \ \ \text{and} \ \ \varepsilon \in (0,1).
\end{aligned} \tag{6.47}$$

For the first four integrals on the right-hand side of (6.47), we dispose in line with the reasoning of [40, Lemma 3.4] to obtain $C_1 > 0$ such that

$$
\begin{aligned}
&-2\int_\Omega \frac{\big|\nabla|\nabla c_\varepsilon|^2\big|^2}{c_\varepsilon^3} - 4\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2 c_\varepsilon|^2 + 6\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^4}\nabla c_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 - 3\int_\Omega \frac{|\nabla c_\varepsilon|^4\Delta c_\varepsilon}{c_\varepsilon^4} \\
\le& -C_1\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2 c_\varepsilon|^2 - C_1\int_\Omega \frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} \quad \text{for all } \ t>0 \ \text{ and } \ \varepsilon\in(0,1).
\end{aligned}
\tag{6.48}
$$

Since [28, Lemma 4.2] provides $C_2 > 0$ satisfying $\frac{\partial|\nabla\psi|^2}{\partial\nu} \le C_2|\nabla\psi|^2$ for each $\psi \in C^2(\overline{\Omega})$ fulfilling $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega$, and since a combination of [7, Theorem 4.24 (i)] with [7, Proposition 4.22 (ii)] implies the compactness of the embedding $W^{m+\frac12,2}(\Omega) \hookrightarrow L^2(\partial\Omega)$ with $m > 0$, we can find $C_3 > 0$ such that

$$
2\int_{\partial\Omega} \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}\cdot\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} \le 2C_2\int_{\partial\Omega}\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} = 512C_2\left\||\nabla c_\varepsilon^{\frac14}|^2\right\|_{L^2(\partial\Omega)}^2 \le C_3\left\||\nabla c_\varepsilon^{\frac14}|^2\right\|_{W^{m+\frac12,2}(\Omega)}^2
\tag{6.49}
$$

for all $t > 0$ and $\varepsilon \in (0, 1)$, and whereafter for $m \in (0, \frac12)$ an application of the Gagliardo–Nirenberg inequality yields that

$$
\begin{aligned}
C_3\left\||\nabla c_\varepsilon^{\frac14}|^2\right\|_{W^{m+\frac12,2}(\Omega)}^2 \le& C_5\left\{\left\|\nabla|\nabla c_\varepsilon^{\frac14}|^2\right\|_{L^2(\Omega)}^{m+\frac12}\left\||\nabla c_\varepsilon^{\frac14}|^2\right\|_{L^2(\Omega)}^{\frac12-m} + \left\||\nabla c_\varepsilon^{\frac14}|^2\right\|_{L^2(\Omega)}\right\}^2 \\
\le& 2C_5\left\{\left\|\nabla|\nabla c_\varepsilon^{\frac14}|^2\right\|_{L^2(\Omega)}^{2m+1}\left\||\nabla c_\varepsilon^{\frac14}|^2\right\|_{L^2(\Omega)}^{1-2m} + \left\||\nabla c_\varepsilon^{\frac14}|^2\right\|_{L^2(\Omega)}^2\right\}
\end{aligned}
\tag{6.50}
$$

for all $t > 0$ and $\varepsilon \in (0, 1)$, with some $C_5 > 0$. Applying Young's inequality to the right-hand side of (6.50), we further have

$$
\begin{aligned}
&2C_5\left\{\left\|\nabla|\nabla c_\varepsilon^{\frac14}|^2\right\|_{L^2(\Omega)}^{2m+1}\left\||\nabla c_\varepsilon^{\frac14}|^2\right\|_{L^2(\Omega)}^{1-2m} + \left\||\nabla c_\varepsilon^{\frac14}|^2\right\|_{L^2(\Omega)}^2\right\} \\
\le& \varsigma_1\int_\Omega\left|\nabla|\nabla c_\varepsilon^{\frac14}|^2\right|^2 + C_6(\varsigma_1)\int_\Omega|\nabla c_\varepsilon^{\frac14}|^4 + 2C_5\int_\Omega|\nabla c_\varepsilon^{\frac14}|^4 \\
=& \frac{\varsigma_1}{256}\int_\Omega\left|\frac{2\nabla c_\varepsilon\cdot D^2 c_\varepsilon}{c_\varepsilon^{\frac32}} - \frac32\cdot\frac{\nabla c_\varepsilon|\nabla c_\varepsilon|^2}{c_\varepsilon^{\frac52}}\right|^2 + \frac{C_6(\varsigma_1)+2C_5}{256}\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \\
\le& \frac{\varsigma_1}{32}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2 c_\varepsilon|^2 + \frac{9\varsigma_1}{512}\int_\Omega\frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + \frac{C_6(\varsigma_1)+2C_5}{256}\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \\
\le& \frac{\varsigma_1}{32}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2 c_\varepsilon|^2 + \frac{\varsigma_1}{32}\int_\Omega\frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + C_7(\varsigma_1)\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}
\end{aligned}
\tag{6.51}
$$

for all $t > 0$ and $\varepsilon \in (0, 1)$, where $\varsigma_1 > 0$ is arbitrary and $C_6 = C_6(\varsigma_1) > 0$ as well as $C_7 = C_7(\varsigma_1) > 0$. With the choice of $\varsigma_1 := 4C_1$, (6.51) together with (6.36) provides $C_8 > 0$ such that

$$
\begin{aligned}
&2C_5\left\{\left\|\nabla|\nabla c_\varepsilon^{\frac14}|^2\right\|_{L^2(\Omega)}^{2m+1}\left\||\nabla c_\varepsilon^{\frac14}|^2\right\|_{L^2(\Omega)}^{1-2m} + \left\||\nabla c_\varepsilon^{\frac14}|^2\right\|_{L^2(\Omega)}^2\right\} \\
\le& \frac{C_1}{8}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2 c_\varepsilon|^2 + \frac{C_1}{8}\int_\Omega\frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + C_8
\end{aligned}
$$

for all $t > 0$ and $\varepsilon \in (0,1)$, which thus along with (6.49) implies

$$2\int_{\partial\Omega} \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3} \cdot \frac{\partial |\nabla c_\varepsilon|^2}{\partial \nu} \le \frac{C_1}{8}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2 c_\varepsilon|^2 + \frac{C_1}{8}\int_\Omega \frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + C_8 \tag{6.52}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. For each $\varsigma_2 > 0$, we invoke Young's inequality and combine with (2.4) and (2.10) to obtain $C_9 = C_9(\varsigma_2) > 0$ fulfilling

$$\begin{aligned} -4\int_\Omega \frac{F_\varepsilon'(n_\varepsilon)|\nabla c_\varepsilon|^2}{c_\varepsilon^2}\nabla c_\varepsilon \cdot \nabla n_\varepsilon \le& \varsigma_2 \int_\Omega \frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + C_9(\varsigma_2)\int_\Omega c_\varepsilon |\nabla n_\varepsilon|^2 \\ \le& \varsigma_2 \int_\Omega \frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + C_9(\varsigma_2)\|c_0\|_{L^\infty(\Omega)}\int_\Omega |\nabla n_\varepsilon|^2 \end{aligned} \tag{6.53}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Since Lemma 6.4 provides $t_1 = t_1(n_0, u_0) > 0$ and $C_{10} > 0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_{10} \quad \text{for all} \;\; t > t_1 \;\; \text{and} \;\; \varepsilon \in (0,1), \tag{6.54}$$

we again make use of (2.10) and Young's inequality to attain

$$\begin{aligned} 3\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^4}(u_\varepsilon \cdot \nabla c_\varepsilon) \le& 3C_{10}\int_\Omega \frac{|\nabla c_\varepsilon|^5}{c_\varepsilon^4} \\ \le& 3C_{10}\|c_0\|_{L^\infty(\Omega)}^{\frac{1}{6}} \int_\Omega \frac{|\nabla c_\varepsilon|^5}{c_\varepsilon^{\frac{25}{6}}} \\ \le& \varsigma_2 \int_\Omega \frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + C_{11}(\varsigma_2) \end{aligned} \tag{6.55}$$

for all $t > t_1$ and $\varepsilon \in (0,1)$, with $C_{11} = C_{11}(\varsigma_2) > 0$. Through integration by parts, we have

$$\begin{aligned} -4\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}\nabla c_\varepsilon \cdot \nabla(u_\varepsilon \cdot \nabla c_\varepsilon) =& 4\int_\Omega \frac{|\nabla c_\varepsilon|^2 \Delta c_\varepsilon}{c_\varepsilon^3}(u_\varepsilon \cdot \nabla c_\varepsilon) + 8\int_\Omega \frac{\nabla c_\varepsilon \cdot D^2 c_\varepsilon \cdot \nabla c_\varepsilon}{c_\varepsilon^3}(u_\varepsilon \cdot \nabla c_\varepsilon) \\ &- 12\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^4}(u_\varepsilon \cdot \nabla c_\varepsilon) \quad \text{for all} \;\; t > 0 \;\; \text{and} \;\; \varepsilon \in (0,1), \end{aligned} \tag{6.56}$$

where from $|\Delta c_\varepsilon| \le \sqrt{3}|D^2 c_\varepsilon|$, (6.54), (2.10), Young's inequality and (6.36) it follows that

$$\begin{aligned} 4\int_\Omega \frac{|\nabla c_\varepsilon|^2 \Delta c_\varepsilon}{c_\varepsilon^3}(u_\varepsilon \cdot \nabla c_\varepsilon) \le& 4\sqrt{3}C_{10}\int_\Omega \frac{|\nabla c_\varepsilon|^3}{c_\varepsilon^3}|D^2 c_\varepsilon| \\ \le& \varsigma_2 \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2 c_\varepsilon|^2 + C_{12}(\varsigma_2)\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \\ \le& \varsigma_2 \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2 c_\varepsilon|^2 + \varsigma_2 \int_\Omega \frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + C_{13}(\varsigma_2)\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} \\ \le& \varsigma_2 \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2 c_\varepsilon|^2 + \varsigma_2 \int_\Omega \frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + C_{14}(\varsigma_2) \end{aligned}$$

with $C_{12} = C_{12}(\varsigma_2) > 0, C_{13} = C_{13}(\varsigma_2) > 0$ and $C_{14} = C_{14}(\varsigma_2) > 0$ for all $t > t_1$ and $\varepsilon \in (0,1)$, that

$$\begin{aligned} 8\int_\Omega \frac{\nabla c_\varepsilon \cdot D^2 c_\varepsilon \cdot \nabla c_\varepsilon}{c_\varepsilon^3}(u_\varepsilon \cdot \nabla c_\varepsilon) \le& 8C_{10}\int_\Omega \frac{|\nabla c_\varepsilon|^3 |D^2 c_\varepsilon|}{c_\varepsilon^3} \\ \le& \varsigma_2 \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2 c_\varepsilon|^2 + C_{15}(\varsigma_2)\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \\ \le& \varsigma_2 \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2 c_\varepsilon|^2 + \varsigma_2 \int_\Omega \frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + C_{16}(\varsigma_2)\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} \\ \le& \varsigma_2 \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2 c_\varepsilon|^2 + \varsigma_2 \int_\Omega \frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + C_{17}(\varsigma_2) \end{aligned}$$

with $C_{15} = C_{15}(\varsigma_2) > 0, C_{16} = C_{16}(\varsigma_2) > 0$ and $C_{17} = C_{17}(\varsigma_2) > 0$ for all $t > t_1$ and $\varepsilon \in (0,1)$, and that

$$\begin{aligned}
-12\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^4}(u_\varepsilon\cdot\nabla c_\varepsilon) \le & 12C_{10}\int_\Omega \frac{|\nabla c_\varepsilon|^5}{c_\varepsilon^4} \\
\le & 12C_{10}\|c_0\|_{L^\infty(\Omega)}^{\frac{1}{6}}\int_\Omega \frac{|\nabla c_\varepsilon|^5}{c_\varepsilon^{\frac{25}{6}}} \\
\le & \varsigma_2\int_\Omega \frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + C_{18}(\varsigma_2)
\end{aligned}$$

with $C_{18} = C_{18}(\varsigma_2) > 0$ for all $t > t_1$ and $\varepsilon \in (0,1)$, whence (6.56) implies

$$-4\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}\nabla c_\varepsilon\cdot\nabla(u_\varepsilon\cdot\nabla c_\varepsilon) \le 2\varsigma_2\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2c_\varepsilon|^2 + 3\varsigma_2\int_\Omega \frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + C_{19}(\varsigma_2) \tag{6.57}$$

for all $t > t_1$ and $\varepsilon \in (0,1)$, where $C_{19}(\varsigma_2) := C_{14}(\varsigma_2)+C_{17}(\varsigma_2)+C_{18}(\varsigma_2)$. With $\varsigma_2 := \frac{C_1}{8}$, a collection of (6.47), (6.48), (6.52), (6.53), (6.57) and (6.55) entails that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{C_1}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + \frac{5C_1}{8}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2c_\varepsilon|^2 \\
\le & C_9(\varsigma_2)\|c_0\|_{L^\infty(\Omega)}\int_\Omega |\nabla n_\varepsilon|^2 + C_8 + C_{11}(\varsigma_2) + C_{19}(\varsigma_2)
\end{aligned}\tag{6.58}$$

for all $t > t_1$ and $\varepsilon \in (0,1)$, where in view of

$$\begin{aligned}
\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \le & \frac{C_1}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + \frac{1}{C_1}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} \\
\le & \frac{C_1}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^6}{c_\varepsilon^5} + C_{20} \quad \text{for all } \ t > 0 \ \text{ and } \ \varepsilon \in (0,1),
\end{aligned}$$

with $C_{20} > 0$ provided by Young's inequality and (6.36), and of the nonnegativity of $\frac{5C_1}{8}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^3}|D^2c_\varepsilon|^2$, we further derive from (6.58) that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \\
\le & C_9(\varsigma_2)\|c_0\|_{L^\infty(\Omega)}\int_\Omega |\nabla n_\varepsilon|^2 + C_8 + C_{11}(\varsigma_2) + C_{19}(\varsigma_2) + C_{20}
\end{aligned}\tag{6.59}$$

for all $t > t_1$ and $\varepsilon \in (0,1)$. Since (6.37) allows for a choice of $t_2 > t_1$ fulfilling

$$\int_{t_2}^{t_2+1}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \le C \quad \text{for all } \ \varepsilon \in (0,1),$$

so that for each $\varepsilon \in (0,1)$ there exists $t_\varepsilon \in (t_2, t_2+1)$ such that

$$\int_\Omega \frac{|\nabla c_\varepsilon(\cdot,t_\varepsilon)|^4}{c_\varepsilon^3(\cdot,t_\varepsilon)} \le C_{21}$$

with some $C_{21} > 0$, and since Corollary 6.1 allows for a choice of $C_{22} > 0$ satisfying

$$\int_t^{t+1}\int_\Omega |\nabla n_\varepsilon|^2 \le C_{22} \quad \text{for all } \ t > 0 \ \text{ and } \ \varepsilon \in (0,1),$$

we let $C_{23} := C_8 + C_{11}(\varsigma_2) + C_{19}(\varsigma_2) + C_{20}$ and apply [36, Lemma 3.4] to (6.59) to achieve that

$$\begin{aligned}\int_\Omega \frac{|\nabla c_\varepsilon(\cdot,t)|^4}{c_\varepsilon^3(\cdot,t)} \leq & e^{-(t-t_\varepsilon)} \cdot C_{21} + \frac{C_{22}C_9(\varsigma_2)\|c_0\|_{L^\infty(\Omega)} + C_{23}}{1-e^{-1}} \\ \leq & C_{21} + \frac{C_{22}C_9(\varsigma_2)\|c_0\|_{L^\infty(\Omega)} + C_{23}}{1-e^{-1}} =: C_{24}\end{aligned} \tag{6.60}$$

for all $t \geq t_\varepsilon$. Thanks to $\theta \in [0, \frac{1}{2})$, it is clear that

$$\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^{4\theta}} = \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \cdot c_\varepsilon^{3-4\theta} \leq \|c_0\|_{L^\infty(\Omega)}^{3-4\theta} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}$$

for all $t > 0$ and $\varepsilon \in (0,1)$, which thus in conjunction with (6.60) implies (6.46), with $t_\star = t_2+1 > t_\varepsilon$. The proof is completed. □

Now, we are in the position to establish temporally eventual L^∞-bounds for n_ε.

Lemma 6.7 *Let $\theta \in [0, \frac{1}{2})$ and $l = 2$, and let (3.2) be valid with $\omega > 0$ provided by Lemma 3.3. Then one can find $t_\star = t_\star(n_0, c_0, u_0) > 0$ and $C > 0$ such that*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq C \quad \textit{for all} \;\; t > t_\star \;\; \textit{and} \;\; \varepsilon \in (0,1). \tag{6.61}$$

Proof. Due to $\theta \in [0, \frac{1}{2})$ and $l = 2$, by Corollary 6.1, Lemma 6.4 and Lemma 6.6, we can find $t_1 = t_1(n_0, c_0, u_0) > 0$ and $C_1 > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq C_1 \quad \text{for all} \;\; t > 0 \;\; \text{and} \;\; \varepsilon \in (0,1), \tag{6.62}$$

and that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq C_1 \quad \text{and} \quad \|\nabla c_\varepsilon^{1-\theta}(\cdot,t)\|_{L^4(\Omega)} \leq C_1 \quad \text{for all} \;\; t \geq t_1 \;\; \text{and} \;\; \varepsilon \in (0,1). \tag{6.63}$$

Now, for $t_2 := t_1 + 1$, we let

$$\Gamma_\varepsilon(T) := \sup_{t\in(t_2,T)} \|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}, \quad T > t_2, \; \varepsilon \in (0,1), \tag{6.64}$$

and

$$g_\varepsilon(t) := \frac{\chi_0}{1-\theta}\nabla c_\varepsilon^{1-\theta}(\cdot,t) + u_\varepsilon(\cdot,t) \quad \text{for all} \;\; t > 0 \;\; \text{and} \;\; \varepsilon \in (0,1). \tag{6.65}$$

Then the variation-of-constants representation of the first equation in (2.1) can be rewritten as

$$n_\varepsilon(\cdot,t) = e^{\Delta} n_\varepsilon(\cdot,t-1) - \int_{t-1}^t e^{(t-s)\Delta}\nabla \cdot \{g_\varepsilon(\cdot,s)n_\varepsilon(\cdot,s)\}ds \tag{6.66}$$

for all $t \geq t_2$ and $\varepsilon \in (0,1)$, where $(e^{\tau\Delta})_{\tau\geq 0}$ stands for the Neumann heat semigroup in Ω. By means of a standard L^2-L^∞ smoothing feature of $(e^{\tau\Delta})_{\tau\geq 0}$, we can find $C_2 > 0$ such that

$$\|e^{\Delta} n_\varepsilon(\cdot,t-1)\|_{L^\infty(\Omega)} \leq C_2\|n_\varepsilon(\cdot,t-1)\|_{L^2(\Omega)} \leq C_2C_1 \quad \text{for all} \;\; t \geq t_2 \;\; \text{and} \;\; \varepsilon \in (0,1), \tag{6.67}$$

because of (6.62). Observing from (6.63) and (6.65) that

$$\|g_\varepsilon(t)\|_{L^4(\Omega)} \leq C_3 \quad \text{for all} \;\; t \geq t_2 \;\; \text{and} \;\; \varepsilon \in (0,1),$$

with some $C_3 > 0$, for each fixed $l_* \in (3,4)$ and any given $T > t_2$ we invoke [10, Lemma 3.3] together with the Hölder inequality, (6.64) and (6.65) that

$$\begin{aligned}
\left\|\int_{t-1}^{t} e^{(t-s)\Delta}\nabla\cdot\{g_\varepsilon(\cdot,s)n_\varepsilon(\cdot,s)\}ds\right\|_{L^\infty(\Omega)} \le & C_4\int_{t-1}^{t}(t-s)^{-\frac{1}{2}-\frac{3}{2l_*}}\|g_\varepsilon(\cdot,s)n_\varepsilon(\cdot,s)\|_{L^{l_*}(\Omega)}ds \\
\le & C_4\int_{t-1}^{t}(t-s)^{-\frac{1}{2}-\frac{3}{2l_*}}\|g_\varepsilon(\cdot,s)\|_{L^4(\Omega)}\|n_\varepsilon(\cdot,s)\|_{L^{\frac{4l_*}{4-l_*}}(\Omega)}ds \\
\le & C_4C_3\int_{t-1}^{t}(t-s)^{-\frac{1}{2}-\frac{3}{2l_*}}\|n_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}^{1-\frac{4-l_*}{2l_*}}\|n_\varepsilon(\cdot,s)\|_{L^2(\Omega)}^{\frac{4-l_*}{2l_*}}ds \\
\le & C_1^{\frac{4-l_*}{2l_*}}C_3C_4\Gamma_\varepsilon^{1-\frac{4-l_*}{2l_*}}(T)\cdot\int_{t-1}^{t}(t-s)^{-\frac{1}{2}-\frac{3}{2l_*}}ds
\end{aligned} \tag{6.68}$$

for all $t \in [t_2, T)$ and $\varepsilon \in (0,1)$, with $C_4 > 0$ provided by [10, Lemma 3.3]. Since $l_* > 3$ warrants the finiteness of $\int_{t-1}^{t}(t-s)^{-\frac{1}{2}-\frac{3}{2l_*}}ds$, it can be inferred from (6.66), (6.67) and (6.68) that there exists $C_5 > 0$ with the property that

$$\sup_{x\in\Omega} n_\varepsilon(x,t) \le C_5 + C_5\Gamma_\varepsilon^{1-\frac{4-l_*}{2l_*}}(T) \quad \text{for all } t \ge [t_2,T) \text{ and } \varepsilon \in (0,1),$$

which actually implies

$$\Gamma_\varepsilon(T) \le \max\{1, (2C_5)^{\frac{2l_*}{4-l_*}}\} \quad \text{for each } T > t_2 \text{ and all } \varepsilon \in (0,1).$$

As a consequence, we achieve (6.61) by letting $T \to \infty$. $\square$

6.3 Higher order ultimate regularity of $(n_\varepsilon, c_\varepsilon, u_\varepsilon)_{\varepsilon\in(0,1)}$

The temporally ultimate estimates attained previously are sufficient to allow direct utilizations of consequences on parabolic Hölder regularity, and of Schauder theories for the linear inhomogeneous heat and the Stokes evolution equations, so as to achieve higher order regularity properties of $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ in large time.

Lemma 6.8 *Let $\theta \in [0, \frac{1}{2})$ and $l = 2$. Suppose that (3.2) is fulfilled with $\omega > 0$ picked by Lemma 3.3. Then there exist $\gamma \in (0,1)$ and $t_\star = t_\star(n_0, c_0, u_0) > 0$ such that for any $T > 0$ one can find $C = C(T) > 0$ satisfying*

$$\|n_\varepsilon\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[t_\star,T])} + \|c_\varepsilon\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[t_\star,T])} + \|u_\varepsilon\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega}\times[t_\star,T])} \le C \tag{6.69}$$

for all $\varepsilon \in (0,1)$.

Proof. As in the reasoning of Lemma 6.7, we write the first equation of (2.1) in the version of

$$\partial_t n_\varepsilon = \Delta n_\varepsilon - \nabla\cdot(n_\varepsilon g_\varepsilon(x,t)), \quad x \in \Omega, \quad t > 0, \quad \varepsilon \in (0,1), \tag{6.70}$$

with

$$g_\varepsilon(x,t) := \frac{\chi_0}{1-\theta}\nabla c_\varepsilon^{1-\theta}(x,t) + u_\varepsilon(x,t), \quad x \in \Omega, \quad t > 0, \quad \varepsilon \in (0,1). \tag{6.71}$$

From Lemma 6.4, Lemma 6.6 and Lemma 6.7, it can be inferred straightforwardly that there exist $t_1 = t_1(n_0, c_0, u_0) > 0$ and $C_1 > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_1 \quad \text{and} \quad \|g_\varepsilon(\cdot,t)\|_{L^4(\Omega)} \le C_1 \quad \text{for all } t \ge t_1 \text{ and } \varepsilon \in (0,1). \tag{6.72}$$

Thereby, in accordance with [29, Theorem 1.3], we can find $\gamma_1 \in (0,1)$ such that for any $T > t_2 = t_2(n_0, c_0, u_0) := t_1 + 1$ there exists $C_2 = C_2(T) > 0$ with the property that

$$\|n_\varepsilon\|_{C^{\gamma_1, \frac{\gamma_1}{2}}(\overline{\Omega}\times[t_2,T])} \le C_2 \quad \text{for all } \ \varepsilon \in (0,1). \tag{6.73}$$

This in conjunction with the regularity properties of F_ε and with Lemma 6.4 implies the existence of $\gamma_2 \in (0,1)$ and $t_3 > t_2$ such that for any $T > t_3$ both $F_\varepsilon(n_\varepsilon)$ and u_ε comply with ε-independent regularity of $C^{\gamma_2, \frac{\gamma_2}{2}}$, and hence we invoke parabolic Schauder theory ([18]) to choose $\gamma_3 \in (0,1)$ such that for each $T > t_4 = t_4(n_0, c_0, u_0) := t_3 + 1$ and all $\varepsilon \in (0,1)$

$$\|c_\varepsilon\|_{C^{2+\gamma_3, 1+\frac{\gamma_3}{2}}(\overline{\Omega}\times[t_4,T])} \le C_3 \tag{6.74}$$

holds with some $C_3 = C_3(T) > 0$. Thanks to (6.72) and the fact that $F_\varepsilon(s) \le s$ for $s \ge 0$, it follows from the second equation in (2.1) that

$$\partial_t c_\varepsilon \ge \Delta c_\varepsilon - C_1 c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon, \quad \text{in } \ \Omega \times [t_1, \infty) \ \text{ for all } \ \varepsilon \in (0,1),$$

which combined with the comparison principle provides a subsolution, that is $\underline{c}(x,t) := C_4 e^{-C_1 t}$, $(x,t) \in \overline{\Omega} \times [t_1, \infty)$ with $C_4 := \inf_{x\in\Omega} c_0(x) > 0$ because of (1.8), and whereby for each $T > t_1$ we have

$$c_\varepsilon \ge C_5(T) := C_4 e^{-C_1 T} \quad \text{in } \ \Omega \times [t_1, T] \ \text{ for all } \ \varepsilon \in (0,1). \tag{6.75}$$

From (6.74) and (6.75), it is not difficult to infer that for each $T > t_4$ and all $\varepsilon \in (0,1)$ there exists $C_6 = C_6(T) > 0$ fulfilling

$$\|\nabla \ln c_\varepsilon\|_{C^{1+\gamma_3,\gamma_3}(\overline{\Omega}\times[t_4,T])} = \left\| \frac{\nabla c_\varepsilon}{c_\varepsilon} \right\|_{C^{1+\gamma_3,\gamma_3}(\overline{\Omega}\times[t_4,T])} \le C_6. \tag{6.76}$$

Letting $G_\varepsilon := c_\varepsilon^{1-\theta}$, we derive from an elementary computation that

$$\partial_t G_\varepsilon = \Delta G_\varepsilon + (\theta \nabla \ln c_\varepsilon - u_\varepsilon) \cdot \nabla G_\varepsilon - (1-\theta) F_\varepsilon(n_\varepsilon) G_\varepsilon \quad \text{in } \ \Omega \times (0,\infty) \ \text{ for all } \ \varepsilon \in (0,1), \tag{6.77}$$

where Lemma 6.4 along with (6.76), (6.73) and the fact that $F_\varepsilon \in C^\infty([0,\infty))$ shows that both of the coefficients functions $\theta \nabla \ln c_\varepsilon - u_\varepsilon$ and $(1-\theta) F_\varepsilon(n_\varepsilon)$ enjoy ε-independent $C^{\gamma_4, \frac{\gamma_4}{2}}$-regularity properties in $\overline{\Omega} \times [t_5, T]$, with some $\gamma_4 \in (0,1)$ and $t_5 > t_4$ as well as any fixed $T > t_5$. Another application of parabolic Schauder theory to (6.77) provides $\gamma_5 \in (0,1)$ with the property that for each $T > t_6 = t_6(n_0, c_0, u_0) := t_5 + 1$ we can choose $C_7 = C_7(T) > 0$ such that

$$\|G_\varepsilon\|_{C^{2+\gamma_5, 1+\frac{\gamma_5}{2}}(\overline{\Omega}\times[t_6,T])} \le C_7 \ \text{ for all } \ \varepsilon \in (0,1). \tag{6.78}$$

Since (6.70) can be also rewritten as

$$\partial_t n_\varepsilon = \Delta n_\varepsilon - g_\varepsilon(x,t) \nabla n_\varepsilon - h_\varepsilon(x,t), \quad x \in \Omega, \quad t > 0, \quad \varepsilon \in (0,1),$$

with g_ε as in (6.71) and with

$$\begin{aligned} h_\varepsilon(x,t) :=& \frac{\chi_0 n_\varepsilon(x,t)}{1-\theta} \Delta c_\varepsilon^{1-\theta}(x,t) \\ =& \frac{\chi_0 n_\varepsilon(x,t)}{1-\theta} \Delta G_\varepsilon(x,t), \quad x \in \Omega, \quad t > 0, \quad \varepsilon \in (0,1), \end{aligned}$$

and since Lemma 6.4 together with (6.78) and (6.73) guarantees the existence of $\gamma_6 \in (0,1)$ such that for any $T > t_6$ and all $\varepsilon \in (0,1)$

$$\|g_\varepsilon\|_{C^{\gamma_6,\frac{\gamma_6}{2}}(\overline{\Omega}\times[t_6,T])} + \|h_\varepsilon\|_{C^{\gamma_6,\frac{\gamma_6}{2}}(\overline{\Omega}\times[t_6,T])} \le C_8$$

with $C_8 = C_8(T) > 0$, we once more employ parabolic Schauder theory to find $\gamma_7 \in (0,1)$ complying with the property that for all $T > t_7 = t_7(n_0,c_0,u_0) := t_6 + 1$ there exists $C_9 = C_9(T) > 0$ fulfilling

$$\|n_\varepsilon\|_{C^{2+\gamma_7,1+\frac{\gamma_7}{2}}(\overline{\Omega}\times[t_7,T])} \le C_9 \quad \text{for all} \quad \varepsilon \in (0,1). \tag{6.79}$$

In light of Lemma 6.4 and (6.73), a standard argument concerning with Schauder estimates for the linear inhomogeneous Stokes evolution equation ([31]) admits a choice of $\gamma_8 \in (0,1)$ such that for each $T > t_8 = t_8(n_0,c_0,u_0) := t_7 + 1$ and all $\varepsilon \in (0,1)$

$$\|u_\varepsilon\|_{C^{2+\gamma_8,1+\frac{\gamma_8}{2}}(\overline{\Omega}\times[t_8,T])} \le C_{10} \quad \text{for all} \quad \varepsilon \in (0,1)$$

holds with some $C_{10} = C_{10}(T) > 0$, which in conjunction with (6.79) and (6.74) implies (6.69) and thus completes the proof. $\square$

6.4 Ultimate regularity of (n,c,u)

By means of the Arzelà–Ascoli theorem, conclusions on eventual regularity of (n,c,u) result from the compactness provided by Lemma 6.8 directly.

Lemma 6.9 *Let $\theta \in [0,\frac{1}{2})$ and $l = 2$. If (3.2) holds for $\omega > 0$ chosen by Lemma 3.3, then one can find $t_\star = t_\star(n_0,c_0,u_0) > 0$ having the property that for each $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ complying with $\varepsilon_j \to 0$ as $j \to \infty$ there exists a quadruple $(n,c,u,P) \in \left(C^{2,1}(\overline{\Omega}\times[t_\star,\infty))\right)^2 \times C^{2,1}(\overline{\Omega}\times[t_\star,\infty);\mathbb{R}^3) \times C^{1,0}(\overline{\Omega}\times[t_\star,\infty))$ of functions and a subsequence $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ such that as $\varepsilon = \varepsilon_{j_k} \to 0$*

$$\begin{cases} n_\varepsilon \to n \quad \textit{in} \quad C^{2,1}(\overline{\Omega}\times[t_\star,\infty)), \\ c_\varepsilon \to c \quad \textit{in} \quad C^{2,1}(\overline{\Omega}\times[t_\star,\infty)) \quad \textit{and} \\ u_\varepsilon \to u \quad \textit{in} \quad C^{2,1}(\overline{\Omega}\times[t_\star,\infty);\mathbb{R}^3), \end{cases} \tag{6.80}$$

and such that (n,c,u,P) solves (1.4), (1.5) and (1.6) classically in $\overline{\Omega}\times[t_\star,\infty)$.

Proof. In accordance with the Arzelà–Ascoli theorem, Lemma 6.8 along with Lemma 5.2 allows for an extraction of $(\varepsilon_{j_k})_{k\in\mathbb{N}} \subset (\varepsilon_j)_{j\in\mathbb{N}}$ such that as $\varepsilon = \varepsilon_{j_k} \to 0$ the convergence asserted by (6.80) is fulfilled, we thus achieve the classical solvability of (1.4), (1.5) and (1.6) in $\overline{\Omega}\times[t_\star,\infty)$ by letting $\varepsilon = \varepsilon_{j_k} \to 0$ separately in each representation of (2.1), with appropriate construction of P through the approach used in [30, 33]. $\square$

7 Asymptotic stability. Proof of Theorem 1.2

The aim of this section is to provide convergence rate of (n,c,u) toward a homogeneous constant equilibria as $t \to \infty$. Firstly, let us detect the constant equilibria based on the fundamental relaxation properties implied by Corollary 6.1 and by the absorption mechanism in the second equation of (1.4).

Lemma 7.1 *Let $\theta \in [0, \frac{1}{2})$ and $l = 2$, and let n, c and u be the limit functions provided by Lemma 6.9. Assume that (3.2) is valid for $\omega > 0$ complying with the hypothesis of Lemma 3.3. We have*

$$n(\cdot,t) \to \overline{n_0}, \quad c(\cdot,t) \to 0 \quad \textit{and} \quad u(\cdot,t) \to 0 \quad \textit{in} \quad L^\infty(\Omega) \quad \textit{as} \quad t \to \infty. \tag{7.1}$$

Moreover, with $t_\star = t_\star(n_0, c_0, u_0) > 0$ as chosen by Lemma 6.9, there exists $C > 0$ satisfying

$$\int_\Omega \frac{|\nabla c(\cdot,t)|^4}{c^3(\cdot,t)} \leq C \quad \textit{for all} \quad t \geq t_\star. \tag{7.2}$$

Proof. Corollary 6.1 ensures the existence of $C_1 > 0$ such that

$$\int_0^\infty \int_\Omega |\nabla n_\varepsilon|^2 \leq C_1 \quad \text{for all} \quad \varepsilon \in (0,1), \tag{7.3}$$

which implies

$$\int_t^{t+1} \int_\Omega |\nabla n_\varepsilon|^2 \to 0 \quad \text{as} \quad t \to \infty. \tag{7.4}$$

By the Poincaré inequality and (2.9), we can find $C_2 > 0$ fulfilling

$$\int_t^{t+1} \int_\Omega |n_\varepsilon - \overline{n_0}|^2 \leq C_2 \int_t^{t+1} \int_\Omega |\nabla n_\varepsilon|^2 \quad \text{for all} \ \ t > 0 \ \ \text{and} \ \ \varepsilon \in (0,1),$$

whence combining with (7.4) we obtain that

$$\int_t^{t+1} \|n_\varepsilon(\cdot,s) - \overline{n_0}\|^2_{L^2(\Omega)} ds \to 0 \quad \text{as} \quad t \to \infty. \tag{7.5}$$

From Lemma 6.8, it can be inferred that there exists $t_1 = t_1(n_0, c_0, u_0) > 0$ such that $g(t) := \|n_\varepsilon(\cdot,t) - \overline{n_0}\|^2_{L^2(\Omega)}$ is uniformly continuous on (t_1, ∞), and hence thanks to (7.5) we recall [45, Lemma 8.1] to see that actually $g(t) \to 0$ as $t \to \infty$, i.e.

$$n_\varepsilon(\cdot,t) \to \overline{n_0} \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad t \to \infty. \tag{7.6}$$

Since Lemma 6.8 implies the fact that with some $t_2 = t_2(n_0, c_0, u_0) > t_1$, $(n_\varepsilon(\cdot,t))_{t>t_2}$ is relatively compact in $C^0(\overline{\Omega})$, we thus deduce from (7.6) and an Ehrling-type lemma (cf. [26, Proof of Theorem 1.2]) that

$$\|n_\varepsilon(\cdot,t) - \overline{n_0}\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad t \to \infty.$$

This together with Lemma 6.9 entails that

$$\|n(\cdot,t) - \overline{n_0}\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad t \to \infty. \tag{7.7}$$

Testing the third equation in (2.1) by u_ε, we attain

$$\frac{1}{2}\int_\Omega |u_\varepsilon(\cdot,t)|^2 + \int_0^t \int_\Omega |\nabla u_\varepsilon|^2 \leq \frac{1}{2}\int_\Omega |u_0|^2 + \int_0^t \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\phi \tag{7.8}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Again invoking the Poincaré inequality, we obtain $C_3 > 0$ with the property that

$$\int_\Omega |u_\varepsilon(\cdot,t)|^2 \leq C_3 \int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \quad \text{for all} \ \ t > 0 \ \ \text{and} \ \ \varepsilon \in (0,1). \tag{7.9}$$

For the second integral on the right-hand side of (7.8), due to $\nabla \cdot u_\varepsilon = 0$, we integrate by parts and combine with Young's inequality, (1.7), (7.9) and (7.3) to derive that

$$\begin{aligned}\int_0^t \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \phi =& - \int_0^t \int_\Omega \phi u_\varepsilon \cdot \nabla n_\varepsilon \\ \leq& \frac{1}{2C_3} \int_0^t \int_\Omega |u_\varepsilon|^2 + \frac{C_3 \|\phi\|^2_{L^\infty(\Omega)}}{2} \int_0^t \int_\Omega |\nabla n_\varepsilon|^2 \\ \leq& \frac{1}{2} \int_0^t \int_\Omega |\nabla u_\varepsilon|^2 + \frac{C_1 C_3 \|\phi\|^2_{L^\infty(\Omega)}}{2} \quad \text{for all} \quad t > 0 \quad \text{and} \quad \varepsilon \in (0,1).\end{aligned} \tag{7.10}$$

Substituting (7.10) into (7.8) yields that

$$\int_0^t \int_\Omega |\nabla u_\varepsilon|^2 \leq \int_\Omega |u_0|^2 + C_1 C_3 \|\phi\|^2_{L^\infty(\Omega)} =: C_4 \quad \text{for all} \quad t > 0 \quad \text{and} \quad \varepsilon \in (0,1),$$

because of the nonnegativity of $\int_\Omega |u_\varepsilon|^2$, and whereby relying on (7.9), Lemma 6.8 and Lemma 6.9 we proceed along the arguments of (7.7) to achieve that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad t \to \infty. \tag{7.11}$$

Multiplying the second equation in (2.1) by c_ε, we integrate by parts to have

$$\int_0^\infty \int_\Omega |\nabla c_\varepsilon|^2 + \int_0^\infty \int_\Omega F_\varepsilon(n_\varepsilon) c_\varepsilon \leq \frac{1}{2} \int_\Omega c_0^2 \quad \text{for all} \quad \varepsilon \in (0,1). \tag{7.12}$$

In line with the reasoning of [41], (7.12) combined with Lemma 6.7 can yield that

$$\|c_\varepsilon(\cdot, t)\|_{L^1(\Omega)} \to 0 \quad \text{as} \quad t \to \infty. \tag{7.13}$$

Since Lemma 6.8 together with the Arzelà–Ascoli theorem implies the relative compactness of $(c_\varepsilon(\cdot, t))_{t > t_3}$ in $C^0(\overline{\Omega})$, with some $t_3 = t_3(n_0, c_0, u_0) > t_2$, we infer from (7.13) along with an Ehrling-type lemma that

$$\|c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad t \to \infty,$$

and thus by Lemma 6.9, it follows that

$$\|c(\cdot, t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad t \to \infty. \tag{7.14}$$

Consequently, (7.1) is valid from (7.7), (7.14) and (7.11).

Recalling (6.60) in the proof of Lemma 6.6, we can find $t_\varepsilon > 0$ with the property that there exists $C_5 > 0$ fulfilling

$$\int_\Omega \frac{|\nabla c_\varepsilon(\cdot, t)|^4}{c_\varepsilon^3(\cdot, t)} \leq C_5 \quad \text{for all} \quad t \geq t_\varepsilon \quad \text{and} \quad \varepsilon \in (0,1). \tag{7.15}$$

Since Lemma 6.9 provides $t_\star = t_\star(n_0, c_0, u_0) > t_\varepsilon$ such that $c_\varepsilon \to c$ in $C^2_{loc}\big(\overline{\Omega} \times [t_\star, \infty)\big)$ as $\varepsilon \to 0$, with $c > 0$ being valid from (5.22) and (1.8), (7.2) thus results from Fatou's lemma in conjunction with (7.15). The proof is completed. $\square$

Now, with $t_\star = t_\star(n_0, c_0, u_0) > 0$ chosen by Lemma 6.9, we take $t_\star$ as an initial instant and consider the following initial-boundary value problem

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \chi_0 \nabla\cdot(\frac{n}{c^\theta}\nabla c), & x\in\Omega,\ t>t_\star,\\ c_t + u\cdot\nabla c = \Delta c - cn, & x\in\Omega,\ t>t_\star,\\ u_t + (u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\phi, \quad \nabla\cdot u = 0, & x\in\Omega,\ t>t_\star,\\ \frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = 0, \quad u = 0, & x\in\partial\Omega,\ t>t_\star,\\ n_{t_\star}(x) = n(\cdot,t_\star), c_{t_\star}(x) = c(\cdot,t_\star), u_{t_\star}(x) = u(\cdot,t_\star), & x\in\Omega. \end{cases} \tag{7.16}$$

Based on the second equation in (7.16), a comparison-type argument resembling that of [52, Lemma 2.1] yields the convergence rate of c with respect to t as follows.

Lemma 7.2 *Let $\theta\in[0,\frac12)$ and $l=2$, and let (3.2) be satisfied for $\omega>0$ provided by Lemma 3.3. Then for any $\varsigma\in(0,\overline{n_0})$ there exists $T_\varsigma > t_\star$ such that for all $t\ge T_\varsigma$*

$$\|c(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} e^{-(\overline{n_0}-\varsigma)(t-T_\varsigma)} \tag{7.17}$$

holds.

Proof. For the subsystem of (7.16), i.e.

$$\begin{cases} c_t + u\cdot\nabla c = \Delta c - cn, & x\in\Omega,\ t>t_\star,\\ \frac{\partial c}{\partial\nu} = 0, & x\in\partial\Omega,\ t>t_\star,\\ c_{t_\star}(x) = c(\cdot,t_\star), & x\in\Omega, \end{cases}$$

one can derive (7.17) from a similar application of the convergence concerning with n in (7.1), as in the proof of [52, Lemma 2.1]. $\square$

Thanks to $\theta\in[0,\frac12)$, the ultimate boundedness property of $\int_\Omega \frac{|\nabla c|^4}{c^3}$ in Lemma 7.1 enables us to turn the convergence rate of c claimed by Lemma 7.2 into that of ∇c and of $\nabla c^{1-\theta}$ within appropriate topology.

Lemma 7.3 *Let $\theta\in[0,\frac12)$ and $l=2$. Suppose that (3.2) is fulfilled for $\omega>0$ picked by Lemma 3.3. Then for each $\varsigma>0$ one can find $T_\varsigma>t_\star$ with the property that there exists $C>0$ satisfying*

$$\|\nabla c(\cdot,t)\|_{L^4(\Omega)} \le C\|c_0\|_{L^\infty(\Omega)}^{\frac34} e^{-\frac34(\overline{n_0}-\varsigma)(t-T_\varsigma)} \quad \text{for all } \ t\ge T_\varsigma \tag{7.18}$$

and

$$\|\nabla c^{1-\theta}(\cdot,t)\|_{L^4(\Omega)} \le C\|c_0\|_{L^\infty(\Omega)}^{\frac34-\theta} e^{-(\frac34-\theta)(\overline{n_0}-\varsigma)(t-T_\varsigma)} \quad \text{for all } \ t\ge T_\varsigma. \tag{7.19}$$

Proof. Since Lemma 7.1 provides $C_1>0$ such that

$$\int_\Omega \frac{|\nabla c(\cdot,t)|^4}{c^3(\cdot,t)} \le C_1 \quad \text{for all} \quad t\ge t_\star, \tag{7.20}$$

we combine with Lemma 7.2 to deduce that

$$\begin{aligned}\int_\Omega |\nabla c(\cdot,t)|^4 &= \int_\Omega \frac{|\nabla c(\cdot,t)|^4}{c^3(\cdot,t)}\cdot c^3(\cdot,t)\\ &\le \|c(\cdot,t)\|_{L^\infty(\Omega)}^3 \int_\Omega \frac{|\nabla c(\cdot,t)|^4}{c^3(\cdot,t)}\\ &\le C_1\|c_0\|_{L^\infty(\Omega)}^3 e^{-3(\overline{n_0}-\varsigma)(t-T_\varsigma)} \quad \text{for all } \ t\ge T_\varsigma,\end{aligned}$$

which implies (7.18). Observing from $\theta \in [0, \frac{1}{2})$ that $3 - 4\theta > 0$, we once more make use of (7.20) and Lemma 7.2 to obtain that

$$\begin{aligned}\int_\Omega \frac{|\nabla c(\cdot,t)|^4}{c^{4\theta}(\cdot,t)} =& \int_\Omega \frac{|\nabla c(\cdot,t)|^4}{c^3(\cdot,t)} \cdot c^{3-4\theta}(\cdot,t) \\ \leq & \|c(\cdot,t)\|^{3-4\theta}_{L^\infty(\Omega)} \int_\Omega \frac{|\nabla c(\cdot,t)|^4}{c^3(\cdot,t)} \\ \leq & C_1 \|c_0\|^{3-4\theta}_{L^\infty(\Omega)} e^{-(3-4\theta)(\overline{n_0}-\varsigma)(t-T_\varsigma)} \quad \text{for all } \ t \geq T_\varsigma,\end{aligned}$$

as asserted. The proof is completed. □

With Lemmas 7.2–7.3 at hand, it is possible to derive the convergence rate of c and n with respect to t in respective topology as stated in Theorem 1.2 through appropriate utilization of smoothing features of the homogeneous Neumann heat semigroup.

Lemma 7.4 *Let $\theta \in [0, \frac{1}{2})$ and $l = 2$, and let $\zeta_c \in \big(0, \min\{\lambda_1, \frac{3\overline{n_0}}{4}\}\big)$ with $\lambda_1 > 0$ denoting the first nonzero eigenvalue of $-\Delta$ in Ω subject to homogeneous Neumann boundary conditions. If (3.2) holds for $\omega > 0$ chosen by Lemma 3.3, then for any $\varsigma > 0$ there exists $T_\varsigma > t_\star$ such that one can find $C > 0$ with the property that*

$$\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq Ce^{-\zeta_c(t-T_\varsigma)} \quad \textit{for all } \ t \geq T_\varsigma + 1. \tag{7.21}$$

Proof. By $\zeta_c < \frac{3\overline{n_0}}{4}$, we can pick $\varsigma > 0$ small enough such that

$$\zeta_c \leq \frac{3}{4}(\overline{n_0} - \varsigma). \tag{7.22}$$

Apart from that, Lemma 6.9 allows for the choice of $C_1 > 0$ fulfilling

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \leq C_1 \quad \text{for all } \ t \geq t_\star \tag{7.23}$$

and

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C_1 \quad \text{for all } \ t \geq t_\star. \tag{7.24}$$

For $\varsigma > 0$ chosen by (7.22), we let $T_\varsigma > 0$ be corresponding instant as taken in Lemmas 7.2–7.3. Then from the variation-of-constants representation of the second equation in (7.16), i.e.

$$c(\cdot,t) = e^{-(t-T_\varsigma)}c(\cdot,T_\varsigma) - \int_{T_\varsigma}^t e^{(t-s)\Delta} \{n(\cdot,s)c(\cdot,s) + u(\cdot,s)\nabla c(\cdot,s)\}\, ds \quad \text{for all } \ t \geq T_\varsigma,$$

it can be inferred that

$$\begin{aligned}\|\nabla c(\cdot,t)\|_{L^\infty(\Omega)} =& \|\nabla e^{-(t-T_\varsigma)}c(\cdot,T_\varsigma)\|_{L^\infty(\Omega)} + \int_{T_\varsigma}^t \|\nabla e^{(t-s)\Delta} n(\cdot,s)c(\cdot,s)\|_{L^\infty(\Omega)} ds \\ &+ \int_{T_\varsigma}^t \|\nabla e^{(t-s)\Delta} u(\cdot,s)\nabla c(\cdot,s)\|_{L^\infty(\Omega)} ds \quad \text{for all } \ t \geq T_\varsigma.\end{aligned} \tag{7.25}$$

Since (2.10) along with (5.20) implies

$$\|c(\cdot,t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} \quad \text{for all } \ t > 0,$$

we invoke [38, Lemma 1.3(iii)] to obtain $C_2 > 0$ such that for all $t \geq T_\varsigma + 1$

$$\begin{aligned}\|\nabla e^{-(t-T_\varsigma)} c(\cdot, T_\varsigma)\|_{L^\infty(\Omega)} \leq & C_2\{1 + (t - T_\varsigma)^{-\frac{7}{8}}\} e^{-\lambda_1(t-T_\varsigma)} \|c(\cdot, t)\|_{L^4(\Omega)} \\ \leq & 2C_2 \|c_0\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{4}} e^{-\lambda_1(t-T_\varsigma)} \\ \leq & 2C_2 \|c_0\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{4}} e^{-\zeta_c(t-T_\varsigma)},\end{aligned} \tag{7.26}$$

due to $\zeta_c < \lambda_1$. Again employing [38, Lemma 1.3(ii)], we deduce from (7.22), (7.23) and Lemma 7.2 that

$$\begin{aligned}&\int_{T_\varsigma}^t \|\nabla e^{(t-s)\Delta} n(\cdot, s) c(\cdot, s)\|_{L^\infty(\Omega)} ds \\ \leq & C_2 \int_{T_\varsigma}^t \{1 + (t-s)^{-\frac{7}{8}}\} e^{-\lambda_1(t-s)} \|n(\cdot, s) c(\cdot, s)\|_{L^4(\Omega)} ds \\ \leq & C_2 |\Omega|^{\frac{1}{4}} \int_{T_\varsigma}^t \{1 + (t-s)^{-\frac{7}{8}}\} e^{-\lambda_1(t-s)} \|n(\cdot, s)\|_{L^\infty(\Omega)} \|c(\cdot, s)\|_{L^\infty(\Omega)} ds \\ \leq & C_1 C_2 |\Omega|^{\frac{1}{4}} \|c_0\|_{L^\infty(\Omega)} \int_{T_\varsigma}^t \{1 + (t-s)^{-\frac{7}{8}}\} e^{-\lambda_1(t-s)} \cdot e^{-(\overline{n_0} - \varsigma)(s - T_\varsigma)} ds \\ \leq & C_1 C_2 |\Omega|^{\frac{1}{4}} \|c_0\|_{L^\infty(\Omega)} \int_{T_\varsigma}^t \{1 + (t-s)^{-\frac{7}{8}}\} e^{-(\lambda_1 - \zeta_c)(t-s)} ds \cdot e^{-\zeta_c(t-T_\varsigma)} \\ \leq & C_3 \cdot e^{-\zeta_c(t-T_\varsigma)} \quad \text{for all} \quad t > T_\varsigma,\end{aligned} \tag{7.27}$$

with $C_3 := C_1 C_2 |\Omega|^{\frac{1}{4}} \|c_0\|_{L^\infty(\Omega)} \int_0^\infty \{1 + \sigma^{-\frac{7}{8}}\} e^{-(\lambda_1 - \zeta_c)\sigma} d\sigma < \infty$. Similarly, by means of [38, Lemma 1.3(ii)] in conjunction with (7.24), (7.18) and (7.22), it follows that

$$\begin{aligned}&\int_{T_\varsigma}^t \|\nabla e^{(t-s)\Delta} u(\cdot, s) \nabla c(\cdot, s)\|_{L^\infty(\Omega)} ds \\ \leq & C_2 \int_{T_\varsigma}^t \{1 + (t-s)^{-\frac{7}{8}}\} e^{-\lambda_1(t-s)} \|u(\cdot, s) \nabla c(\cdot, s)\|_{L^4(\Omega)} ds \\ \leq & C_2 \int_{T_\varsigma}^t \{1 + (t-s)^{-\frac{7}{8}}\} e^{-\lambda_1(t-s)} \|u(\cdot, s)\|_{L^\infty(\Omega)} \|\nabla c(\cdot, s)\|_{L^4(\Omega)} ds \\ \leq & C_1 C_2 C_4 \|c_0\|_{L^\infty(\Omega)}^{\frac{3}{4}} \int_{T_\varsigma}^t \{1 + (t-s)^{-\frac{7}{8}}\} e^{-\lambda_1(t-s)} \cdot e^{-\frac{3}{4}(\overline{n_0} - \varsigma)(s - T_\varsigma)} ds \\ \leq & C_1 C_2 C_4 \|c_0\|_{L^\infty(\Omega)}^{\frac{3}{4}} \int_{T_\varsigma}^t \{1 + (t-s)^{-\frac{7}{8}}\} e^{-(\lambda_1 - \zeta_c)(t-s)} ds \cdot e^{-\zeta_c(t-T_\varsigma)} \\ \leq & C_5 \cdot e^{-\zeta_c(t-T_\varsigma)} \quad \text{for all} \quad t > T_\varsigma,\end{aligned} \tag{7.28}$$

where $C_4 > 0$ is provided by Lemma 7.3 and $C_5 := C_1 C_2 C_4 \|c_0\|_{L^\infty(\Omega)}^{\frac{3}{4}} \int_0^\infty \{1 + \sigma^{-\frac{7}{8}}\} e^{-(\lambda_1 - \zeta_c)\sigma} d\sigma < \infty$. Collecting (7.25)–(7.28) and letting $C_6 := 2C_2 |\Omega|^{\frac{1}{4}} \|c_0\|_{L^\infty(\Omega)} + C_3 + C_5$, we have

$$\|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq C_6 e^{-\zeta_c(t-T_\varsigma)} \quad \text{for all} \quad t \geq T_\varsigma + 1. \tag{7.29}$$

From (7.22), it is clear that $\zeta_c < \overline{n_0} - \varsigma$, whence Lemma 7.2 implies that

$$\|c(\cdot, t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} e^{-\zeta_c(t-T_\varsigma)} \quad \text{for all} \quad t \geq T_\varsigma,$$

which along with (7.29) entails (7.21) and thus completes the proof. $\square$

Lemma 7.5 *Let $\theta \in [0, \frac{1}{2})$ and $l = 2$, and let $\zeta_n \in \big(0, \min\{\lambda_1, (\frac{3}{4} - \theta)\overline{n_0}\}\big)$ with $\lambda_1 > 0$ representing the first nonzero eigenvalue of $-\Delta$ in Ω under homogeneous Neumann boundary conditions. Assume that (3.2) is satisfied for $\omega > 0$ taken by Lemma 3.3. Then for arbitrary $\varsigma > 0$ one can find $T_\varsigma > t_\star$ having the property that there exists $C > 0$ such that*

$$\|n(\cdot,t) - \overline{n_0}\|_{L^\infty(\Omega)} \le Ce^{-\zeta_n(t-T_\varsigma)} \quad \text{for all} \ \ t \ge T_\varsigma. \tag{7.30}$$

Proof. For any $\varsigma_1 > 0$, the convergence of u in (7.1) warrants the existence of $T_{\varsigma_1} > t_\star$ fulfilling

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} < \varsigma_1 \quad \text{for all} \ \ t \ge T_{\varsigma_1}. \tag{7.31}$$

According to Lemma 7.3, for each $\varsigma_2 > 0$, one can also pick $T_{\varsigma_2} > t_\star$ such that

$$\|\nabla c^{1-\theta}(\cdot,t)\|_{L^4(\Omega)} \le C_1\|c_0\|_{L^\infty(\Omega)}^{\frac{3}{4}-\theta} e^{-(\frac{3}{4}-\theta)(\overline{n_0}-\varsigma_2)(t-T_{\varsigma_2})} \tag{7.32}$$

with some $C_1 > 0$ for all $t \ge T_{\varsigma_2}$. Since $\zeta_n < (\frac{3}{4} - \theta)\overline{n_0}$ allows for a choice of $\varsigma \in (0, \min\{\varsigma_1, \varsigma_2\})$ sufficiently small so as to satisfy

$$\zeta_n \le (\frac{3}{4} - \theta)(\overline{n_0} - \varsigma), \tag{7.33}$$

it is not difficult to infer that for this $\varsigma > 0$ we can find $T_\varsigma > \max\{T_{\varsigma_1}, T_{\varsigma_2}\}$ such that both (7.31) and (7.32) are fulfilled, that is

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} < \varsigma \quad \text{and} \quad \|\nabla c^{1-\theta}(\cdot,t)\|_{L^4(\Omega)} \le C_1\|c_0\|_{L^\infty(\Omega)}^{\frac{3}{4}-\theta} e^{-(\frac{3}{4}-\theta)(\overline{n_0}-\varsigma)(t-T_\varsigma)} \tag{7.34}$$

for all $t \ge T_\varsigma$. Now, we let $N(x,t) := n(x,t) - \overline{n_0}$, and obtain from $\nabla \cdot u = 0$ and the first equation in (7.16) that

$$N_t + u \cdot \nabla N = \Delta N - \chi_0 \nabla \cdot \left(\frac{n}{1-\theta}\nabla c^{1-\theta}\right), \quad x \in \Omega, \ \ t \ge T_\varsigma, \tag{7.35}$$

and hence from corresponding variation-of-constants representation we derive that

$$\begin{aligned}\|N(\cdot,t)\|_{L^\infty(\Omega)} \le& \|e^{\Delta(t-T_\varsigma)}N(\cdot,T_\varsigma)\|_{L^\infty(\Omega)} + \int_{T_\varsigma}^t \left\|e^{\Delta(t-s)}\nabla \cdot \left(\frac{\chi_0 n(\cdot,s)}{1-\theta}\nabla c^{1-\theta}(\cdot,s)\right)\right\|_{L^\infty(\Omega)} ds \\ &+ \int_{T_\varsigma}^t \|e^{\Delta(t-s)}\nabla \cdot (u(\cdot,s)N(\cdot,s))\|_{L^\infty(\Omega)} ds \quad \text{for all} \ \ t \ge T_\varsigma.\end{aligned} \tag{7.36}$$

Recalling Lemma 6.9, we can find $C_2 > 0$ with the properties that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \le C_2 \quad \text{and} \quad \|N(\cdot,t)\|_{L^\infty(\Omega)} \le C_2 \quad \text{for all} \ \ t \ge t_\star, \tag{7.37}$$

and whereby by [38, Lemma 1.3(i)] and $\zeta_n < \lambda_1$ we have

$$\begin{aligned}\|e^{\Delta(t-T_\varsigma)}N(\cdot,T_\varsigma)\|_{L^\infty(\Omega)} \le& C_3 e^{-\lambda_1(t-T_\varsigma)}\|N(\cdot,T_\varsigma)\|_{L^\infty(\Omega)} \\ \le& C_2C_3 e^{-\lambda_1(t-T_\varsigma)} \\ \le& C_2C_3 e^{-\zeta_n(t-T_\varsigma)} \quad \text{for all} \ \ t \ge T_\varsigma,\end{aligned} \tag{7.38}$$

with some $C_3 > 0$. Once more employing [38, Lemma 1.3(iv)], we combine with (7.37), (7.34) and (7.33) to deduce that

$$
\begin{aligned}
&\int_{T_\varsigma}^{t} \left\| e^{\Delta(t-s)}\nabla\cdot\left(\frac{\chi_0 n(\cdot,s)}{1-\theta}\nabla c^{1-\theta}(\cdot,s)\right)\right\|_{L^\infty(\Omega)} ds \\
\le & \frac{C_4\chi_0}{1-\theta}\int_{T_\varsigma}^{t}\{1+(t-s)^{-\frac{7}{8}}\}e^{-\lambda_1(t-s)}\|n(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla c^{1-\theta}(\cdot,s)\|_{L^4(\Omega)}ds \\
\le & \frac{C_1C_2C_4\chi_0\|c_0\|_{L^\infty(\Omega)}^{\frac{3}{4}-\theta}}{1-\theta}\int_{T_\varsigma}^{t}\{1+(t-s)^{-\frac{7}{8}}\}e^{-\lambda_1(t-s)}e^{-(\frac{3}{4}-\theta)(\overline{n_0}-\varsigma)(s-T_\varsigma)}ds \\
\le & \frac{C_1C_2C_4\chi_0\|c_0\|_{L^\infty(\Omega)}^{\frac{3}{4}-\theta}}{1-\theta}\int_{T_\varsigma}^{t}\{1+(t-s)^{-\frac{7}{8}}\}e^{-(\lambda_1-\zeta_n)(t-s)}ds\cdot e^{-\zeta_n(t-T_\varsigma)} \\
\le & C_5\cdot e^{-\zeta_n(t-T_\varsigma)} \quad \text{for all} \quad t\ge T_\varsigma,
\end{aligned}
\tag{7.39}
$$

where $C_4>0$ is provided by [38, Lemma 1.3(iv)] and $C_5 := \frac{C_1C_2C_4\chi_0\|c_0\|_{L^\infty(\Omega)}^{\frac{3}{4}-\theta}}{1-\theta}\int_0^\infty\{1+\sigma^{-\frac{7}{8}}\}e^{-(\lambda_1-\zeta_n)\sigma} d\sigma<\infty$ due to $\zeta_n<\lambda_1$. Now, for $\Gamma_0>0$ fulfilling

$$
\Gamma_0 \ge 3(C_2C_3+C_5), \tag{7.40}
$$

we abbreviate

$$
\hat{T} := \sup\left\{T>T_\varsigma \,\middle|\, \|N(\cdot,t)\|_{L^\infty(\Omega)}\le \Gamma_0 e^{-\zeta_n(t-T_\varsigma)} \quad \text{for all} \quad t\in(T_\varsigma,T]\right\}. \tag{7.41}
$$

Along with (7.34), we again invoke [38, Lemma 1.3(iv)] to attain $C_6>0$ such that for each $t\in(T_\varsigma,\hat{T})$ with $\hat{T}$ defined by (7.41)

$$
\begin{aligned}
&\int_{T_\varsigma}^{t} \|e^{\Delta(t-s)}\nabla\cdot(u(\cdot,s)N(\cdot,s))\|_{L^\infty(\Omega)}ds \\
\le & C_6\int_{T_\varsigma}^{t}\{1+(t-s)^{-\frac{1}{2}}\}e^{-\lambda_1(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\|N(\cdot,s)\|_{L^\infty(\Omega)}ds \\
\le & \varsigma C_6\Gamma_0\int_{T_\varsigma}^{t}\{1+(t-s)^{-\frac{1}{2}}\}e^{-\lambda_1(t-s)}e^{-\zeta_n(s-T_\varsigma)}ds \\
\le & \varsigma C_6\Gamma_0\int_{T_\varsigma}^{t}\{1+(t-s)^{-\frac{1}{2}}\}e^{-(\lambda_1-\zeta_n)(t-s)}ds\cdot e^{-\zeta_n(t-T_\varsigma)} \\
\le & \varsigma C_6C_7\Gamma_0 e^{-\zeta_n(t-T_\varsigma)},
\end{aligned}
\tag{7.42}
$$

with $C_7 := \int_0^\infty\{1+\sigma^{-\frac{1}{2}}\}e^{-(\lambda_1-\zeta_n)\sigma}d\sigma<\infty$ because of $\zeta_n<\lambda_1$. Thereby, for sufficiently small $\varsigma<\frac{C_2C_3+C_5}{2C_6C_7\Gamma_0}$ and all $t\in(T_\varsigma,\hat{T})$, it follows from (7.38), (7.39) and (7.42) that

$$
\|N(\cdot,t)\|_{L^\infty(\Omega)} < \frac{3(C_2C_3+C_5)}{2}e^{-\zeta_n(t-T_\varsigma)} \le \frac{\Gamma_0}{2}e^{-\zeta_n(t-T_\varsigma)}.
$$

In view of (7.41) and the fact that N is continuous with respect to t, this is absurd if $\hat{T}<\infty$, and thus $\hat{T}=\infty$, which proves (7.30). □

By means of standard regularity properties of the Stokes semigroup, appropriate applications of Lemma 7.1 and Lemma 7.4 achieve the following convergence rate of u.

Lemma 7.6 *Let $\theta \in [0, \frac{1}{2})$ and $l = 2$, and let $\zeta_u \in (0, \min\{\zeta_n, \mu_0\})$ with $\zeta_n > 0$ and $\mu_0 > 0$ provided by Lemma 7.5 and by standard L^p-L^q estimates for the Stokes semigroup, respectively. Suppose that (3.2) is valid for $\omega > 0$ picked by Lemma 3.3. Then for each $\varsigma > 0$ there exists $T_\varsigma > t_\star$ such that one can find $C > 0$ satisfying*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le Ce^{-\zeta_u(t-T_\varsigma)} \quad \textit{for all} \;\; t \ge T_\varsigma + 1. \tag{7.43}$$

Proof. For some fixed $r_0 > 3$, it is possible to choose $\alpha \in (\frac{3}{2r_0}, \frac{1}{2})$, which immediately implies

$$2\alpha > \frac{3}{r_0} \tag{7.44}$$

and $\frac{3}{3+r_0(1-2\alpha)} < 1$ as well as $\frac{3}{3+2r_0(1-\alpha)} < 1$. Picking $a_1 \in (\frac{3}{3+r_0(1-2\alpha)}, 1)$ and $a_2 \in (\frac{3}{3+2r_0(1-\alpha)}, 1)$, we let

$$r_1 := r_0 a_1 \quad \text{and} \quad r_2 := r_0 a_2.$$

Then it is not difficult to verify that

$$-\alpha - \frac{1}{2} - \frac{3}{2}(\frac{1}{r_1} - \frac{1}{r_0}) > -\alpha - \frac{1}{2} - \frac{3}{2r_0}\left(\frac{3+r_0(1-2\alpha)}{3} - 1\right) = -1 \tag{7.45}$$

and that

$$-\alpha - \frac{3}{2}(\frac{1}{r_2} - \frac{1}{r_0}) > -\alpha - \frac{3}{2r_0}\left(\frac{3+2r_0(1-\alpha)}{3} - 1\right) = -1. \tag{7.46}$$

Since (7.44) allows for a choice of $\gamma \in (0, 2\alpha - \frac{3}{r_0})$ such that the embedding $D(A^\alpha_{r_0}) \hookrightarrow C^\gamma(\overline{\Omega};\mathbb{R}^3)$ holds, we are able to find $C_1 > 0$ fulfilling

$$\|\psi\|_{L^\infty(\Omega)} \le \|\psi\|_{C^\gamma(\overline{\Omega};\mathbb{R}^3)} \le C_1\|A^\alpha\psi\|_{L^{r_0}(\Omega)} \quad \text{for all} \;\; \psi \in D(A^\alpha_{r_0}). \tag{7.47}$$

According to Lemma 6.9, there exist $t_\star = t_\star(n_0, c_0, u_0) > 0$ and $C_2 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_2 \quad \text{and} \quad \|n(\cdot,t)\|_{L^\infty(\Omega)} \le C_2 \quad \text{for all} \;\; t \ge t_\star. \tag{7.48}$$

Moreover, for any $\varsigma > 0$, Lemma 7.1 also provides some $T_\varsigma > t_\star$ with the property that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} < \varsigma \quad \text{for all} \;\; t \ge T_\varsigma. \tag{7.49}$$

Thanks to $\overline{n_0} \in \mathbb{R}$, the third equation in (7.16) can be rewritten as

$$u_t - Au = -\mathcal{P}[(u\cdot\nabla)u + (n - \overline{n_0})\nabla\phi] \quad \text{for all} \;\; t > t_\star. \tag{7.50}$$

Applying A^α on both sides of the variation-of-constants representation of (7.50), we derive from the resulting equality that

$$\begin{aligned}\|A^\alpha u(\cdot,t)\|_{L^{r_0}(\Omega)} \le& \|A^\alpha e^{-(t-T_\varsigma)}u(\cdot,T_\varsigma)\|_{L^{r_0}(\Omega)} + \int_{T_\varsigma}^t \left\|A^\alpha e^{-(t-s)A}\mathcal{P}[(u(\cdot,s)\cdot\nabla)u(\cdot,s)]\right\|_{L^{r_0}(\Omega)} ds \\ &+ \int_{T_\varsigma}^t \left\|A^\alpha e^{-(t-s)A}\mathcal{P}[(n(\cdot,s) - \overline{n_0})\nabla\phi]\right\|_{L^{r_0}(\Omega)} ds \quad \text{for all} \;\; t > T_\varsigma.\end{aligned} \tag{7.51}$$

By (7.48) and standard smoothing features of the Stokes semigroup ([30],[29, p.201]), it follows that

$$\begin{aligned}\|A^{\alpha}e^{-(t-T_\varsigma)}u(\cdot,T_\varsigma)\|_{L^{r_0}(\Omega)} \le& C_3(t-T_\varsigma)^{-\alpha}e^{-\mu_0(t-T_\varsigma)}\|u(\cdot,T_\varsigma)\|_{L^{r_0}(\Omega)}\\ \le& C_2C_3|\Omega|^{\frac{1}{r_0}}e^{-\mu_0(t-T_\varsigma)}\\ \le& C_2C_3|\Omega|^{\frac{1}{r_0}}e^{-\zeta_u(t-T_\varsigma)} \quad \text{for all} \ \ t>T_\varsigma+1,\end{aligned} \tag{7.52}$$

with $\mu_0>0$ and some $C_3>0$, because of $\zeta_u<\mu_0$. Combining with (1.7), the boundedness feature of $\mathcal{P}$ in $L^{r_0}(\Omega)$ ([11]) and Lemma 7.5, we draw on standard L^p-L^q estimates for the Stokes semigroup to obtain $C_4>0$ and $C_5>0$ such that

$$\begin{aligned}&\int_{T_\varsigma}^{t}\left\|A^{\alpha}e^{-(t-s)A}\mathcal{P}[(n(\cdot,s)-\overline{n_0})\nabla\phi]\right\|_{L^{r_0}(\Omega)}ds\\ \le& C_4\int_{T_\varsigma}^{t}(t-s)^{-\alpha-\frac{3}{2}(\frac{1}{r_2}-\frac{1}{r_0})}e^{-\mu_0(t-s)}\|n(\cdot,s)-\overline{n_0}\|_{L^{r_2}(\Omega)}ds\\ \le& C_4C_5|\Omega|^{\frac{1}{r_2}}\int_{T_\varsigma}^{t}(t-s)^{-\alpha-\frac{3}{2}(\frac{1}{r_2}-\frac{1}{r_0})}e^{-\mu_0(t-s)}e^{-\zeta_n(s-T_\varsigma)}ds\\ \le& C_4C_5|\Omega|^{\frac{1}{r_2}}\int_{T_\varsigma}^{t}(t-s)^{-\alpha-\frac{3}{2}(\frac{1}{r_2}-\frac{1}{r_0})}e^{-(\mu_0-\zeta_u)(t-s)}ds\cdot e^{-\zeta_u(t-T_\varsigma)}\\ \le& C_6\cdot e^{-\zeta_u(t-T_\varsigma)} \quad \text{for all} \ \ t\ge T_\varsigma+1\end{aligned} \tag{7.53}$$

with $C_6:=C_4C_5|\Omega|^{\frac{1}{r_2}}\int_0^\infty \sigma^{-\alpha-\frac{3}{2}(\frac{1}{r_2}-\frac{1}{r_0})}e^{-(\mu_0-\zeta_u)\sigma}d\sigma<\infty$, because of $0<\zeta_u<\min\{\zeta_n,\mu_0\}$ and (7.46). Being similar as the arguments of Lemma 7.5, we choose $\Gamma_0>0$ large enough so as to satisfy

$$\Gamma_0\ge 3C_1(C_2C_3+C_6), \tag{7.54}$$

and let

$$\hat{T}:=\sup\left\{T>T_\varsigma+1\middle|\|u(\cdot,t)\|_{L^\infty(\Omega)}\le\Gamma_0e^{-\zeta_u(t-T_\varsigma)} \quad \text{for all} \ \ t\in(T_\varsigma+1,T]\right\}. \tag{7.55}$$

From $\nabla\cdot u=0$, it can be deduced that $(u\cdot\nabla)u\equiv\nabla\cdot(u\otimes u)$ ([30, cf. p.265]), whence combining with the boundedness property of $\mathcal{P}$, the Hölder inequlity, (7.49) and (7.55) we utilize standard

L^p-L^q estimates for the Stokes semigroup to have

$$
\begin{aligned}
&\int_{T_\varsigma}^{t}\left\|A^\alpha e^{-(t-s)A}\mathcal{P}[(u(\cdot,s)\cdot\nabla)u(\cdot,s)]\right\|_{L^{r_0}(\Omega)}ds\\
\le&\int_{T_\varsigma}^{t}\left\|A^\alpha e^{-(t-s)A}[(u(\cdot,s)\cdot\nabla)u(\cdot,s)]\right\|_{L^{r_0}(\Omega)}ds\\
\le&\int_{T_\varsigma}^{t}\left\|A^\alpha e^{-(t-s)A}\nabla\cdot(u(\cdot,s)\otimes u(\cdot,s))\right\|_{L^{r_0}(\Omega)}ds\\
\le&C_7\int_{T_\varsigma}^{t}(t-s)^{-\alpha-\frac{1}{2}-\frac{3}{2}(\frac{1}{r_1}-\frac{1}{r_0})}e^{-\mu_0(t-s)}\|u(\cdot,s)\otimes u(\cdot,s)\|_{L^{r_1}(\Omega)}ds\\
\le&C_8\int_{T_\varsigma}^{t}(t-s)^{-\alpha-\frac{1}{2}-\frac{3}{2}(\frac{1}{r_1}-\frac{1}{r_0})}e^{-\mu_0(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\|u(\cdot,s)\|_{L^{r_1}(\Omega)}ds\\
\le&\varsigma C_8\Gamma_0|\Omega|^{\frac{1}{r_1}}\int_{T_\varsigma}^{t}(t-s)^{-\alpha-\frac{1}{2}-\frac{3}{2}(\frac{1}{r_1}-\frac{1}{r_0})}e^{-\mu_0(t-s)}\cdot e^{-\zeta_u(s-T_\varsigma)}ds\\
\le&\varsigma C_8\Gamma_0|\Omega|^{\frac{1}{r_1}}\int_{T_\varsigma}^{t}(t-s)^{-\alpha-\frac{1}{2}-\frac{3}{2}(\frac{1}{r_1}-\frac{1}{r_0})}e^{-(\mu_0-\zeta_u)(t-s)}ds\cdot e^{-\zeta_u(t-T_\varsigma)}\\
\le&\varsigma C_8\Gamma_0 C_9\cdot e^{-\zeta_u(t-T_\varsigma)}\quad\text{for all}\quad t\in(T_\varsigma+1,\hat{T}),
\end{aligned}
\tag{7.56}
$$

with $C_9:=|\Omega|^{\frac{1}{r_1}}\int_0^\infty\sigma^{-\alpha-\frac{1}{2}-\frac{3}{2}(\frac{1}{r_1}-\frac{1}{r_0})}e^{-(\mu_0-\zeta_u)\sigma}d\sigma<\infty$, due to $0<\zeta_u<\mu_0$ and (7.45). Now, if $\varsigma>0$ is small enough such that $\varsigma<\frac{C_1(C_2C_3+C_6)}{2C_8C_9\Gamma_0}$, we collect (7.52), (7.53), (7.54) and (7.56), and invoke (7.47) to achieve

$$
\begin{aligned}
\|u(\cdot,s)\|_{L^\infty(\Omega)}\le&C_1\|A^\alpha u(\cdot,t)\|_{L^{r_0}(\Omega)}\\
\le&\frac{3C_1(C_2C_3+C_6)}{2}e^{-\zeta_u(t-T_\varsigma)}\\
\le&\frac{\Gamma_0}{2}e^{-\zeta_u(t-T_\varsigma)}\quad\text{for all}\quad t\in(T_\varsigma+1,\hat{T}).
\end{aligned}
\tag{7.57}
$$

Since u is continuous with respect to t, (7.57) contradicts to the definition of $\hat{T}$ when $\hat{T}<\infty$, which implies that actually $\hat{T}=\infty$, and hence (7.43) is valid. The proof is completed. $\Box$

Proof of Theorem 1.2. Theorem 1.2 straightforwardly results from Lemma 6.9, Lemma 7.4, Lemma 7.5 and Lemma 7.6. $\Box$

8 Acknowledgements

The author would like to express his sincere thanks to the anonymous reviewer for his/her helpful suggestions.

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