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Defect of compactness for Sobolev spaces on manifolds with bounded geometry

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Abstract. *Defect of compactness*, relative to an embedding of two Banach spaces $E \hookrightarrow F$, is the difference between a weakly convergent sequence in E and its weak limit, taken up to a remainder that vanishes in the norm of F. For a number of known embeddings, Sobolev embeddings in particular, defect of compactness takes form of a *profile decomposition* - a sum of clearly structured terms with asymptotically disjoint supports, called *elementary concentrations*. In this paper we construct a profile decomposition for the Sobolev space $H^{1,2}(M)$ of a Riemannian manifold with bounded geometry, in the form of a sum of elementary concentrations associated with concentration profiles defined on manifolds induced by a limiting procedure at infinity, and thus different from M. The profiles satisfy an inequality of Plancherel type: the sum of the quadratic forms of Laplace-Beltrami operators for the profiles on their respective manifolds is bounded by the quadratic form of the Laplace-Beltrami operator of the sequence. A similar relation, related to the Brezis–Lieb Lemma, holds for the L^p-norms of profiles on the respective manifolds.

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1. Introduction

Defect of compactness, relative to an embedding of two Banach spaces $E \hookrightarrow F$, is a difference $u_k - u$ between a weakly convergent sequence $u_k \rightharpoonup u$ in E and its weak limit, taken up to a suitable remainder that vanishes in the norm of F. In particular, if the embedding is compact and E is reflexive, the defect of compactness is null. For many embeddings there exist well-structured representations of the defect of compactness, known as *profile decompositions*. Best studied are profile decompositions relative to Sobolev embeddings, which are sums of terms with asymptotically disjoint supports, called *elementary concentrations* or *bubbles*. Profile decompositions were originally motivated by studies of concentration phe-

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nomena in PDE in the early 1980's by Uhlenbeck, Brezis, Coron, Nirenberg, Aubin and Lions, and they play significant role in verification of convergence of functional sequences in applied analysis, particularly when the information available via the classical concentration compactness method is not enough detailed.

Profile decompositions are known to exist when the embedding $E \hookrightarrow F$ is *cocompact* relative to some group $\mathcal{G}$ of bijective isometries on E. An embedding $E \hookrightarrow F$ is called $\mathcal{G}$-cocompact if any sequence (u_k) in E satisfying $g_k u_k \rightharpoonup 0$ for any sequence of operators (g_k) in $\mathcal{G}$ vanishes in the norm of F. (It is easy to verify, for example, that $\ell^\infty(\mathbb{Z})$ is cocompactly embedded into itself relative to the group of shifts $\mathcal{G} = \{(a_n) \mapsto (a_{n+m})\}_{m\in\mathbb{Z}}$.) The earliest cocompactness result for functional spaces known to the authors is the proof of cocompactness of embedding of the inhomogeneous Sobolev space $H^{1,p}(\mathbb{R}^N)$, $N > p$, into L^q, $q \in (p, p^*)$, where $p^* = \frac{pN}{N-p}$, relative to the group of shifts $u \mapsto u(\cdot - y)$, $y \in \mathbb{R}^N$, by E. Lieb [15] (the term *cocompactness* itself appeared in literature only in the last decade). A profile decomposition relative to a group $\mathcal{G}$ of bijective isometries represents defect of compactness as a sum of *elementary concentrations*, or *bubbles*, $\sum_{n\in\mathbb{N}} g_k^{(n)} w^{(n)}$ with some $g_k^{(n)} \in \mathcal{G}$ and $w^{(n)} \in E$, $k \in \mathbb{N}$, $n \in \mathbb{N}$. The elements $w^{(n)}$, called *concentration profiles*, are then obtained as weak limits of $(g_k^{(n)})^{-1} u_k$ as $k \to \infty$. Typical examples of groups $\mathcal{G}$, involved in profile decompositions, are the above mentioned group of shifts and the rescaling group, which is a product group of shifts and dilations $u \mapsto t^r u(t\cdot)$, $t > 0$, where $r = \frac{N-p}{p}$ for $\dot{H}^{1,p}(\mathbb{R}^N)$, $N > p$.

Existence of profile decompositions for general bounded sequences in $\dot{H}^{1,p}(\mathbb{R}^N)$ equipped with the rescaling group was proved by Solimini [20], and later, independently, but with a weaker form of asymptotics, in [11] and [14] ([14] also extended the result to fractional Sobolev spaces). It was first observed in [16] that profile decomposition (and thus concentration phenomena in general) can be understood in functional-analytic terms, rather than in specific function spaces. The result of [16] was extended in [21] to uniformly convex Banach spaces with the Opial condition (without the Opial condition a profile decomposition still exists but in terms of the less-known Delta convergence instead of weak convergence). However, despite the general character of the statement in [21], it does not apply to several known profile decompositions, in particular, when the space E is not reflexive (*e.g.*, [2]), when one has only a semigroup of isometries (*e.g.*, [1]), or when the profile decomposition can be expressed without a group (*e.g.*, Struwe [22]).

The present paper follows the direction started by the work of Struwe, to study profile decompositions in the Sobolev space of a non-compact Riemannian manifold that possibly *lacks* a nontrivial isometry group. When the isometry group $\mathrm{Iso}(M)$ of manifold M is sufficiently rich, namely, if

$$M = \bigcup_{\eta\in\mathrm{Iso}(M)} \eta K \text{ for some compact set } K \subset M, \tag{1.1}$$

it is shown in [8] that Sobolev embedding $H^{1,2}(M) \hookrightarrow L^p(M)$, $2 < p < \frac{2N}{N-2}$, $N \geq 2$, becomes cocompact relative to the action of $\mathrm{Iso}(M)$. In this case a profile decomposition is immediate from the functional-analytic statement of [16].

In what follows we use the standard invariant norm of $H^{1,2}(M)$, $\|u\|_{1,2} = \left(\int_M (|du|^2 + |u|^2)\mathrm{d}v_g\right)^{1/2}$, where dv_g is the Riemannian measure on M, and we always assume that $N \geq 2$. We quote the result of [8], with the property of unconditional convergence added from the general profile decomposition in [21].

Theorem 1.1. *Let M be a complete Riemannian manifold with a countable group G of isometries satisfying* (1.1)*, and let (u_k) be a bounded sequence in $H^{1,2}(M)$. Then there exists $w^{(n)} \in H^{1,2}(M)$, $g_k^{(n)} \in G$, $k, n \in \mathbb{N}$, such that for a renumbered subsequence*

$$g_k^{(1)} = \mathrm{id}, \ \left(g_k^{(n)^{-1}} g_k^{(m)}\right)_k \text{ is discrete for } n \neq m, \tag{1.2}$$

$$w^{(n)} = \text{w-lim}\, u_k \circ g_k^{(n)} \tag{1.3}$$

$$\sum_{n\in\mathbb{N}} \|w^{(n)}\|_{1,2}^2 \leq \limsup \|u_k\|_{1,2}^2 \tag{1.4}$$

$$u_k - \sum_{n\in\mathbb{N}} w^{(n)} \circ g_k^{(n)^{-1}} \to 0 \text{ in } L^p(M),\ 2 < p < 2^*, \tag{1.5}$$

and the series $\sum_{n\in\mathbb{N}} w \circ g_k^{(n)}$ converges unconditionally and uniformly with respect to k.

In particular, (1.1) holds, implying the assertion of the theorem, when $\mathrm{Iso}(M)$ is transitive, *i.e.*, M is homogeneous space, *e.g.*, if M is $\mathbb{R}^N$ or the hyperbolic space $\mathbb{H}^N$. When a non-compact manifold M has no nontrivial isometries, it does not of course mean that the Sobolev embedding $H^{1,2}(M) \hookrightarrow L^p(M)$, $2 < p < 2^*$, is compact, as we demonstrate in the Example 2.3 below. Thus the question remains if one can express the corresponding defect of compactness in a form similar to profile decomposition of (1.5). In this paper we answer this question positively for manifolds of bounded geometry, as defined below. Absence of a group of isometries comes, however at some cost, which is transparent already from Struwe's profile decomposition in [22], where profiles are functions on the tangent space of M at the points of concentration: in general, absence of a non-compact group $\mathcal{G}$ of isometries that may produce blowup sequences of the form $g_k w \rightharpoonup 0$, $g_k \in \mathcal{G}$, corresponds to emergence of concentration profiles $w^{(n)}$ supported on metric structures different from M. This is indeed the case in the present paper that deals with profile decomposition relative to the embedding $H^{1,2}(M) \hookrightarrow L^p(M)$ when M is a Riemannian manifold of bounded geometry.

The subject of the paper was proposed to one of the authors (C.T.) a number of years ago by Richard Schoen [17].

The paper is organized as follows. In Section 2 we give an analog of the co-compactness property expressed without invoking the isometry group, in terms of the "spotlight vanishing" Lemma 2.4, which naturally requires the manifold to have bounded geometry. This lemma motivates our construction of profile decomposition in the main result of the paper, Theorem 4.5, based on patching of local profiles moving along the manifold. In Section 3 we define the manifolds at infinity needed to formulate Theorem 4.5. Manifolds at infinity play the same role in description of elementary concentrations based on quasi-translations as the tangent space plays in the descriptions of elementary concentrations based on dilations in [22]. In Section 4 we state the main result, as well as provide construction of global profiles as functions on the manifolds at infinity, rather than on the manifold M itself. Section 5 contains technical statements concerning reconstruction of the original sequence from its local profiles. Proof of Theorem 4.5 is given in the Section 6. In Section 7 we show that if M satisfies (1.1), then Theorem 1.1 is a particular case of Theorem 4.5. Appendix contains some elementary properties of manifolds of bounded geometry, existence of a suitable uniform covering, and a gluing theorem used in the construction of manifolds at infinity.

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2. A "spotlight" lemma and preliminary discussion

Let M be a smooth, complete N-dimensional Riemannian manifold with metric g and a positive injectivity radius $r(M)$. In what follows $B(x,r)$ will denote a geodesic ball in M and Ω_r will denote the ball in $\mathbb{R}^N$ of radius r centered at the origin. Let $r \in (0, r(M))$ be fixed. Then the Riemannian exponential map $\exp_x$ is a diffeomorphism of $\{v \in T_xM : g_x(v,v) < r\}$ onto $B(x,r)$. For each $x \in M$ we choose an orthonormal basis for T_xM which yields an identification $i_x : \mathbb{R}^N \to T_xM$. Then $e_x : \Omega_r \to B(x,r)$ will denote geodesic normal coordinates at x given by $e_x = \exp_x \circ i_x$. We do not require smoothness of the map i_x with respect to x, since in the arguments x will be taken from a discrete subset of M.

From now on we assume that M is a connected non-compact manifold of bounded geometry. The latter is defined as follows, *e.g.*, *cf.* [18].

Definition 2.1. A smooth Riemannian manifold M is of bounded geometry if the following two conditions are satisfied:

(i) The injectivity radius $r(M)$ of M is positive;
(ii) Every covariant derivative of the Riemann curvature tensor R^M of M is bounded, *i.e.*, $\nabla^k R^M \in L^\infty(M)$ for every $k = 0, 1, \ldots$

Note that a Riemannian manifold of bounded geometry is always complete. On every paracompact manifold M one can define a Riemannian metric tensor g such that (M, g) is a manifold of bounded geometry, *cf.* [12]. We refer the reader to the appendix for elementary properties of manifolds of bounded geometry used in this paper. Here we recall only the notion of the discretization of the manifold that is crucial for our constructions.

Definition 2.2. A subset Y of a Riemannian manifold M is called an ε-discretization of M, $\varepsilon > 0$, if the distance between any two distinct points of Y is greater than or equal to $\varepsilon/2$ and

$$M = \bigcup_{y \in Y} B(y, \varepsilon).$$

Any connected Riemannian manifold M has an ε-discretization for any $\varepsilon > 0$, and if M is of bounded geometry then for any $t \geq 1$ the covering $\{B(y, t\varepsilon)|\}_{y \in Y}$ is uniformly locally finite, *cf.* Lemma 8.3.

Example 2.3. Let M be a non-compact manifold of bounded geometry, let $w \in C_0^1(\Omega_r) \setminus \{0\}$, let (x_k) be a discrete sequence on M, and let $u_k = w \circ e_{x_k}^{-1}$. Then it is easy to see that $u_k \rightharpoonup 0$ while $\|u_k\|_p$ is bounded away from zero by (8.4). In other words, for non-compact manifolds of bounded geometry presence of a *local* concentration profile w results in a nontrivial defect of compactness.

The main result of the paper, Theorem 4.5, is an analog of Theorem 1.1 based on local concentration profiles in the spirit of Example 2.3. Once we subtract from the sequence all suitably patched local "runaway bumps" of the form $w \circ e_{y_k}^{-1}$, the remainder sequence (v_k) is expected to have no nonzero local profiles left, in other word, to satisfy $v_k \circ e_{y_k} \rightharpoonup 0$ in $H^{1,2}(\Omega_\rho)$ with some $\rho > 0$. This is a condition related to the one in the cocompactness [8, Lemma 2.6], and it implies that (v_k) vanishes in $L^p(M)$. In strict terms we have the following "spotlight vanishing" lemma. In what follows 2^* denotes the Sobolev conjugate of 2, *i.e.*, $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$.

Lemma 2.4 ("Spotlight lemma"). *Let M be an N-dimensional Riemannian manifold of bounded geometry and let $Y \subset M$ be a r-discretization of M, $r < r(M)$. Let (u_k) be a bounded sequence in $H^{1,2}(M)$. Then, $u_k \to 0$ in $L^p(M)$ for any $p \in (2, 2^*)$ if and only if $u_k \circ e_{y_k} \rightharpoonup 0$ in $H^{1,2}(\Omega_r)$ for any sequence (y_k), $y_k \in Y$.*

Proof. Let us fix $p \in (2, 2^*)$ and assume that $u_k \circ e_{y_k} \rightharpoonup 0$ in $H^{1,2}(\Omega_r)$ for any sequence (y_k), $y_k \in Y$. The local Sobolev embedding theorem and the boundedness of the geometry of M implies that there exists $C > 0$ independent of $y \in M$ such that

$$\int_{B(y,r)} |u_k|^p \mathrm{d}v_g \leq C \int_{B(y,r)} \left(|du_k|^2 + |u_k|^2\right) \mathrm{d}v_g \left(\int_{B(y,r)} |u_k|^p \mathrm{d}v_g\right)^{1-2/p}.$$

Adding the terms in left and right hand side over $y \in Y$ we have

$$\int_M |u_k|^p \mathrm{d}v_g \leq C \int_M \left(|du_k|^2 + |u_k|^2\right) \mathrm{d}v_g \ \sup_{y \in Y} \left(\int_{B(y,r)} |u_k|^p \mathrm{d}v_g\right)^{1-2/p}. \quad (2.1)$$

Boundedness of the sequence (u_k) in $H^{1,2}(M)$ implies that the supremum of the right hand side is finite. So for any u_k we can find a sequence $y_k \in Y$, $k \in \mathbb{N}$, such that

$$\sup_{y \in Y} \int_{B(y,r)} |u_k|^p \mathrm{d}v_g \leq 2^{\frac{p}{p-2}} \int_{B(y_k,r)} |u_k|^p \mathrm{d}v_g. \quad (2.2)$$

By compactness of the Sobolev embedding $H^{1,2}(\Omega_r) \hookrightarrow L^p(\Omega_r)$ and weak convergence of the sequence in $H^{1,2}(\Omega_r)$ we have $u_k \circ e_{y_k} \to 0$ in $L^p(\Omega_r)$, and thus, $\int_{B(y_k,r)} |u_k|^p \mathrm{d}v_g \to 0$. Combining this with (2.1) and (2.2) we have $u_k \to 0$ in $L^p(M)$.

Assume now that $u_k \to 0$ in $L^p(M)$. Boundedness of the geometry of M implies for any sequence (y_k) that $u_k \circ e_{y_k} \to 0$ in $L^p(\Omega_r)$. On the other hand boundedness of the sequence u_k in $H^{1,2}(M)$ and boundedness of geometry give us boundedness of any sequence $(u_k \circ e_{y_k})$ in $H^{1,2}(\Omega_r)$. By continuity of the embedding $H^{1,2}(\Omega_r) \hookrightarrow L^p(\Omega_r)$ we get $u_k \circ e_{y_k} \rightharpoonup 0$ in $H^{1,2}(\Omega_r)$. □

The main result of this paper, Theorem 4.5, requires a definition of a manifold at infinity of M associated with a given discrete sequence (y_k) in M, as well as a proof that such manifold exists. These are given in Section 3. Thus we dedicate the rest of this section to discussing the place of our settings (subcritical Sobolev embedding, manifold of bounded geometry) in the context of existing or possible results concerning profile decompositions in Sobolev spaces of Riemannian manifolds.

Struwe [22] (see also the exposition in the book [6]) provided a profile decomposition for the limiting case $p = 2^*$ of the Sobolev embedding on a compact manifold for a particular class of sequences, generalized in a recent paper [5] where profile decomposition is given for any bounded sequence in $H^{1,2}$ of a compact manifold. By means of a finite partition of unity and the exponential map this profile decomposition follows from the profile decomposition for the limiting Sobolev embedding for the case of a bounded domain in $\mathbb{R}^N$. This, in turn, is a consequence of the profile decomposition for the embedding $\dot{H}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ based on the rescaling group which is a product group of shifts $u \mapsto u(\cdot - y)$, $y \in \mathbb{R}^N$, and dilations $u \mapsto t^{\frac{N-2}{2}} u(t\cdot)$, $t > 0$. However, for sequences supported in a bounded domain of $\mathbb{R}^N$ profile decomposition cannot contain shifts to infinity or deflations $u \mapsto t_k^{\frac{N-2}{2}} u(t_k\cdot)$, $t_k \to 0$, or superpositions thereof, so it consists only of blowup terms $u \mapsto t_k^{\frac{N-2}{2}} u(t_k\cdot)$, $t_k \to \infty$, with bounded (or, equivalently, modulo vanishing remainder, constant) shifts.

By analogy with the case $M = \mathbb{R}^N$, one could expect that generalizing Struwe's profile decomposition to a non-compact manifold would mean finding a way to express loss of compactness with respect to shifts along the manifold in combination with changes of scale responsible for loss of compactness in the limiting case $p = 2^*$. While one can easily define a blowup of a local profile traveling along points $y_k \in M$ as $x \mapsto t_k^{\frac{N-2}{2}} w(t_k e_{y_k}^{-1}(x))$ by $t_k^{-\frac{N-2}{2}} u_k(e_{y_k}(t_k^{-1}\cdot)) \rightharpoonup w$ in $H^{1,2}(\Omega_r)$ with $t_k \to \infty$, this construction does not extend to the opposite end of scale, *i.e.*, $t_k \to 0$ and has no simple counterpart in the non-Euclidean case: a putative deflating transformation must be substantially dependent on the geometry of the manifold at every point.

In this paper we provide a profile decomposition only for subcritical Sobolev embeddings, which in the Euclidean case involve only the group of shifts. We use the exponential map to define a *local* counterpart of translations "along" a sequence of points $y_k \in M$, namely, a "spotlight" sequence $u_k \circ e_{y_k} : \Omega_r \to B(y_k, r)$. Like in [5, 22], reconstruction of the original sequence from its concentration profiles involves patching the (local) profiles, composed with the inversed exponential map, by a partition of unity on M.

Without the assumption of bounded geometry, bounded sequences in $H^{1,2}(M)$ do not admit, in general, a profile decomposition for the mere reason that there might be no embedding $H^{1,2}(M) \hookrightarrow L^p(M)$ except for the trivial case $p = 2$. Even if the embedding exists, but the geometry is not bounded, local translations along the manifold may induce complicated - nonlinear and anisotropic - changes of scale, which are likely to affect the expression for the defect of compactness. The critical case $p = 2^*$ of the problem has to cope not only with this difficulty, as well as with the already mentioned issue of additional loss of compactness due a putative non-Euclidean analog of deflations (the opposite end of scale to blowups) in the Euclidean space.

3. Manifolds at infinity

In what follows we will consider the radius $\rho < \frac{r(M)}{8}$ and $\hat{\rho}$-discretization Y of M, $\frac{\rho}{2} < \hat{\rho} < \rho$, and we will use the notation $\mathbb{N}_0 \stackrel{\text{def}}{=} \mathbb{N} \cup \{0\}$.

Definition 3.1. Let $(y_k)_{k\in\mathbb{N}}$ be a sequence in Y that is an enumeration of an infinite subset of Y. A countable family $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$ of sequences on Y is called *a trailing system* for $(y_k)_{k\in\mathbb{N}}$ if for every $k \in \mathbb{N}$ $(y_{k;i})_{i\in\mathbb{N}_0}$ is an ordering of Y by the distance from y_k, that is, an enumeration of Y such that $d(y_{k;i}, y_k) \le d(y_{k;i+1}, y_k)$ for all $i \in \mathbb{N}_0$. In particular, $y_{k;0} = y_k$.

Note that any enumeration of the infinite subset of Y admits a trailing system: it can be constructed inductively, by starting with $y_{k;0} = y_k$ and, given $i \in \mathbb{N}_0$, choosing $y_{k;i+1}$ as any point $y \in Y \setminus \{y_{k;0}, \ldots, y_{k;i}\}$ with the least value of $d(y, y_k)$, $i \in \mathbb{N}_0$. The trailing system is generally not uniquely defined when for some $k \in \mathbb{N}$ there are several points of Y with the same distance from y_k.

Lemma 3.2. *Let $(y_k)_{k\in\mathbb{N}}$ be a sequence in a discretization Y that is an enumeration of an infinite subset of Y and let $(y_{k;i})_{k\in\mathbb{N}}$, $i \in \mathbb{N}_0$, be its trailing system. There exists a renamed subsequence of $(y_k)_{k\in\mathbb{N}}$ with the following property: for any $i \in \mathbb{N}_0$ there exists a finite subset J_i of $\mathbb{N}_0$ such that*

$$B(y_{k;i}, \rho) \cap B(y_{k;j}, \rho) \neq \emptyset \iff j \in J_i\,. \tag{3.1}$$

Proof. Let us fix i. If the ball $B(y_{k;j}, \rho)$ intersects $B(y_{k;i}, \rho)$ then $B(y_{k;\ell}, \rho/2) \subset B(y_k, d(y_k, y_{k;i}) + 3\rho)$ for any $\ell \in \{0, 1, \ldots, j\}$. The geometry of M is bounded so the respective volumes of the balls $B(y_{k;\ell}, \rho/4)$ are bounded from bellow by a constant depending on ρ but independent of the balls. Note that these balls are pairwise disjoint. Moreover the Ricci curvature of M is bounded from below, so by the Bishop-Gromov volume comparison theorem the volume of any ball $B(y_{k;\ell}, r)$ can be estimated from above by the constant depending only on the radius. In consequence

$$C\, j \ \le \sum_{\ell=0}^{j} \mathrm{vol}(B(y_{k;\ell}, \rho/4) \le \mathrm{vol}\Big(B\big(y_k, d(y_k, y_{k;i}) + 3\rho\big)\Big) \le C_i, \qquad (3.2)$$

and the constant C_i is independent of k. Let $J_{k;i} = \{j : B(y_{k;i}, \rho) \cap B(y_{k;j}, \rho) \neq \emptyset\}$. Then for any k we have $J_{k;i} \subset [0, C_i/C]$. Therefore there exists a subsequence $k_1, k_2, \cdots$ such that $J_{k_\ell,i} = J_{k_\nu,i}$ for any ℓ and ν. We put $J_i = J_{k_1,i}$.

The assertion of the lemma follows now from the standard diagonalization argument. □

We will always assume throughout the paper that the sequence we work with satisfies the above property. This can be done since passing to subsequence never spoils our construction.

With a given trailing system $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$ we associate a manifold $M_\infty^{(y_{k;i})}$ defined by gluing data that will be constructed below. In the construction we will use definitions from the second part of Appendix.

When we define the manifold $M_\infty^{(y_{k;i})}$ we assume that we work with a sequence satisfying (3.1). The following subset of $\mathbb{N}_0^2$ is essential for the construction:

$$\mathcal{K} = \bigcup_{i=0}^{\infty} \big\{(i, j) : \ j \in J_i\big\}.$$

If $(i, j) \in \mathcal{K}$, then passing to a subsequence for any $\xi, \eta \in \Omega_{2\rho}$ we have

$$d(e_{y_{k;j}}\xi, e_{y_{k;i}}\eta) \le d(e_{y_{k;j}}\xi, y_{k;j}) + d(y_{k;j}, y_{k;i}) + d(y_{k;i}, e_{y_{k;i}}\eta) < 6\rho < \frac{3r(M)}{4}.$$

Therefore, on a subsequence, we may consider a diffeomorphism

$$\psi_{ij,k} \stackrel{\text{def}}{=} e_{y_{k;i}}^{-1} \circ e_{y_{k;j}} : \bar{\Omega}_{2\rho} \to \Omega_a, \qquad a = \frac{3}{4} r(M).$$

To each pair $(i, j) \in \mathcal{K}$ we associate a subset Ω_{ji} of $\Omega_{2\rho}$ and a diffeomorphism ψ_{ij} defined on Ω_{ji} whenever the latter is nonempty.

By boundedness of the geometry, *cf.* Lemma 8.2, and the Ascoli-Arzela theorem, there is a renamed subsequence of $(\psi_{ij,k})_{k\in\mathbb{N}}$ that converges in $C^\infty(\bar{\Omega}_{2\rho})$ to some smooth function $\psi_{ij} : \bar{\Omega}_{2\rho} \to \Omega_a$, and, moreover, we may assume that the same extraction of $(\psi_{ji,k})_{k\in\mathbb{N}}$ converges in $C^\infty(\bar{\Omega}_{2\rho})$ as well. Note that Lemma 8.2 gives that for any $\alpha \in \mathbb{N}_0^N$ there exists a constant $C_\alpha > 0$, such that

$$|D^\alpha \psi_{ij}(\xi)| \le C_\alpha \text{ whenever } i, j \in \mathbb{N}_0, \ \xi \in \Omega_\rho.$$

We define $\Omega_{ij} \overset{\text{def}}{=} \psi_{ij}(\Omega_\rho) \cap \Omega_\rho$. This set may generally be empty. Let us define a set that we will invoke in our application of Corollary 8.10 that will follow:

$$\mathbb{K} \overset{\text{def}}{=} \{(i, j) \in \mathcal{K} : \ \Omega_{ij} \neq \emptyset\}. \tag{3.3}$$

To prove the cocycle condition for the gluing data we should extract subsequences in a more restrictive way. First we consider a subsequence $\psi^1_{01,k}$ of $\psi_{01,k}$ that converges to ψ_{01} and note that on the same subsequence we have convergence of $\psi^1_{10,k}$ to ψ_{10}. Fix an enumeration $n \mapsto (i_n, j_n)$ of the set of all indices $(i, j) \in \mathbb{K}$, $i < j$, and extract the convergent subsequence $\psi^{n+1}_{i_\ell j_\ell,k}$ of the subsequence $\psi^n_{i_\ell j_\ell,k}$ from the previous extraction step, for $\ell = 0, \ldots, n+1$. Then the diagonal sequence $\psi^k_{i_\ell j_\ell,k}$ will converge to $\psi_{i_\ell j_\ell}$ for any $\ell \in \mathbb{N}$.

By the definition of Ω_{ij} and ψ_{ij} we have $\psi_{ij} \circ \psi_{ji} = \text{id}$ on Ω_{ij} and $\psi_{ji} \circ \psi_{ij} = \text{id}$ on Ω_{ji}. Therefore $\psi_{ji} = \psi^{-1}_{ij}$ in restriction to Ω_{ij}, and ψ_{ji} is a diffeomorphism between Ω_{ij} and Ω_{ji}. Note that this construction gives that $\psi_{ii} = \text{id}$, $\Omega_{ii} = \Omega_\rho$ for all $i \in \mathbb{N}_0$. Thus conditions (i-iii) of Corollary 8.10 are satisfied.

Note also that the second step of the constructions implies

$$\begin{aligned}\psi_{\ell i} &= \lim_{k\to\infty} e^{-1}_{y_{k;\ell}} \circ e_{y_{k;i}} = \lim_{k\to\infty} e^{-1}_{y_{k;\ell}} \circ e_{y_{k;j}} \circ e^{-1}_{y_{k;j}} \circ e_{y_{k;i}} \\ &= \lim_{k\to\infty} e^{-1}_{y_{k;\ell}} \circ e_{y_{k;j}} \circ \lim_{k\to\infty} e^{-1}_{y_{k;j}} \circ e_{y_{k;i}} = \psi_{\ell j} \circ \psi_{ji},\end{aligned}$$

and

$$\psi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \psi_{ij}(\psi_{ji}(\Omega_\rho) \cap \Omega_\rho \cap \psi_{jk}(\Omega_\rho) \cap \Omega_\rho) = \Omega_{ij} \cap \Omega_{ik},$$

which proves condition (iv) of Corollary 8.10.

Let $x \in \partial\Omega_{ij} \cap \Omega_\rho$. Since $\partial\Omega_{ij} \subset \partial\psi_{ij}(\Omega_\rho) \cup \partial\Omega_\rho$ and Ω_ρ is open we conclude that $x \in \partial\psi_{ij}(\Omega_\rho) = \psi_{ij}(\partial\Omega_\rho)$. Thus $\psi_{ji}(x) \in \partial\Omega_\rho$. This proves the condition (v) of Corollary 8.10.

We have thus proved the following proposition, *cf.* Corollary 8.10.

Proposition 3.3. *Let M be a Riemannian manifold with bounded geometry and let Y be its discretization.*

For any trailing system $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$ related to the sequence (y_k) in Y there exists a smooth manifold $M^{(y_{k;i})}_\infty$ with an atlas $\{(U_i, \tau_i)\}_{i\in\mathbb{N}_0}$ such that:

1) *$\tau_i(U_i) = \Omega_\rho$;*
2) *there exists a renamed subsequence of k such that for any two charts (U_i, τ_i) and (U_j, τ_j) with $U_i \cap U_j \neq \emptyset$ the corresponding transition map $\psi_{ij} : \tau_j(U_j \cap U_i) \to \tau_i(U_j \cap U_i)$ is given by the C^∞-limit*

$$\psi_{ij} = \lim_{k\to\infty} e^{-1}_{y_{k;i}} \circ e_{y_{k;j}}.$$

For convenience we will also widely use the "inverse" charts $\varphi_i = \tau_i^{-1}$ so that $\varphi_j^{-1} \circ \varphi_i = \psi_{ji} : \Omega_{ij} \to \Omega_{ji}$.

Now define the Riemannian metric on $M_\infty^{(y_{k;i})}$ in two steps as follows. First for any $i \in \mathbb{N}_0$ we define a metric tensor $g^{(i)}$ on Ω_ρ and afterwards we pull it back onto $U_i = \varphi_i(\Omega_\rho) \subset M_\infty^{(y_{k;i})}$ via φ_i^{-1} and prove the compatibility conditions.

Tensor $g^{(i)}$ is defined as a C^∞-limit on a suitable renamed subsequence:

$$\widetilde{g}^{(i)}_\xi(v, w) \stackrel{\text{def}}{=} \lim_{k\to\infty} g_{e_{y_{k;i}(\xi)}}\big(de_{y_{k;i}}(v), de_{y_{k;i}}(w)\big), \ \xi \in \Omega_\rho \text{ and } v, w \in \mathbb{R}^N. \tag{3.4}$$

Existence of the limit follows from the boundedness of the geometry of the manifold M since the coefficients of the tensors $g_{e_{y_{k;i}}}$ form a bounded family of functions in the spaces $C^\infty(\Omega_\rho)$. Using the standard diagonalization procedure we can choose the same subsequence for any i. Furthermore, $\widetilde{g}^{(i)}$ is a bilinear symmetric positive-definite form. Since we used in the definition (3.4) normal coordinates, we have $\widetilde{g}^{(i)}_0(v, v) = |v|^2$. In consequence, by the boundedness of geometry, $\widetilde{g}^{(i)}_\xi[v, v] \geq \frac{1}{2}|v|^2$ in Ω_ρ for all $i \in \mathbb{N}_0$, provided that ρ is fixed sufficiently small.

Now we can define a metric $\widetilde{g}$ on $M_\infty^{(y_{k;i})}$ by the following relation

$$\begin{gathered}\widetilde{g}_x(v, w) \stackrel{\text{def}}{=} \widetilde{g}^{(i)}_{\varphi_i^{-1}(x)}\big(d\varphi_i^{-1}(v), d\varphi_i^{-1}(w)\big),\\ x \in \varphi_i(\Omega_\rho) \subset M_\infty^{(y_{k;i})} \text{ and } v, w \in T_x M_\infty^{(y_{k;i})}.\end{gathered} \tag{3.5}$$

To prove that the Riemannian metric is well defined we should verify the compatibility relation on overlapping charts, *i.e.*, that

$$\begin{gathered}\widetilde{g}^{(i)}_{\varphi_i^{-1}(x)}(d\varphi_i^{-1}v, d\varphi_i^{-1}w) = \widetilde{g}^{(j)}_{\varphi_j^{-1}(x)}(d\varphi_j^{-1}v, d\varphi_i^{-1}w),\\ \text{if} \quad x \in \varphi_i(\Omega_\rho) \cap \varphi_j(\Omega_\rho) \text{ and } v, w \in T_x M_\infty^{(y_{k;i})}.\end{gathered} \tag{3.6}$$

But $\varphi_j^{-1} \circ \varphi_i = \psi_{ji}$, so it suffices to prove that

$$\widetilde{g}^{(i)}_\xi(v, w) = \widetilde{g}^{(j)}_{\psi_{ji}(\xi)}(d\psi_{ji} v, d\psi_{ji} w), \qquad \text{with} \qquad v, w \in T_\xi \Omega_\rho. \tag{3.7}$$

Let $e^{-1}_{y_{k;j}} \circ e_{y_{k;i}}(\xi) = \eta_k$ then $\psi_{j,i}(\xi) = \lim_{k\to\infty} \eta_k$ and $e_{y_{k;i}}(\xi) = e_{y_{k;j}}(\eta_k)$. In consequence

$$\begin{aligned}\widetilde{g}^{(i)}_\xi(v, w) &= \lim_{k\to\infty} g_{e_{y_{k;i}}(\xi)}(de_{y_{k;i}} v, de_{y_{k;i}} w) = \\ &= \lim_{k\to\infty} g_{e_{y_{k;j}}(\eta_k)}(de_{y^{-1}_{k;j}} e_{y_{k;i}} v, de_{y^{-1}_{k;j}} \circ e_{y_{k;i}} w) = \\ &= g_{\psi_{j,i}(\xi)}(d\psi_{ji} v, d\psi_{ji} w).\end{aligned} \tag{3.8}$$

Definition 3.4. A manifold at infinity $M_\infty^{(y_{k;i})}$ of a manifold M with bounded geometry, generated by a trailing system $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$ of a sequence (y_k) in Y, is the differentiable manifold given by Theorem 8.10, supplied with a Riemannian metric tensor $\widetilde{g}$ defined by (3.5).

For the given chart (Ω_ρ, τ_i) components of the metric tensor $\widetilde{g}$ are defined by formula (3.4), *cf.* (3.5). Let $\xi = 0$. The maps $e_{y_{k;i}}$ are normal coordinates systems, so for any k components $g_{\ell,m}$ of the metric tensor g satisfy $g_{\ell,m}(0) = \delta_{\ell,m}$ and $\partial_n g_{\ell,m}(0) = 0$. So by identity (3.4) we get

$$\widetilde{g}_{\ell,m}(0) = \delta_{\ell,m} \qquad \text{and} \qquad \partial_n \widetilde{g}_{\ell,m}(0) = 0\,.$$

Moreover the components $g_{\ell,m}$ are a bounded set in $C^\infty(\Omega_\rho)$ so all the set of $\widetilde{g}_{\ell,m}$ is also bounded in $C^\infty(\Omega_\rho)$.

For any k and i, $(\Omega_\rho, e_{y_{k;i}})$ is a normal coordinate system, so for any unit vector v we have on that ball $\Gamma^n_{m,\ell}(tv)v_\ell v_m = 0, 0 \le t \le \rho$, where $\Gamma^n_{m,\ell}$ denotes Christoffel symbols of a given Riemannian metric on M. But Christoffel symbols can be expressed in terms of components of Riemannian metric tensor and their derivatives, so the Christoffel symbols $\widetilde{\Gamma}^n_{m,\ell}$ of the manifold $M_\infty^{(y_{k;i})}$ are limit values in C^∞ of the Christoffel symbols $\Gamma^n_{m,\ell}$ of the manifold M. Therefore $t \mapsto tv$, $0 \le t \le \rho$, are geodesic curves also for $M_\infty^{(y_{k;i})}$ in the coordinates (Ω_ρ, φ_i). Thus the injectivity radius of $M_\infty^{(y_{k:i})}$ is not smaller then ρ and (Ω_ρ, φ_i) is a normal system of coordinates.

In terms of the definition above the argument of this subsection proves the following statement.

Proposition 3.5. *Let M be a Riemannian manifold with bounded geometry and let Y be its $\hat{\rho}$-discretization, $\frac{\rho}{2} < \hat{\rho} < \rho < \frac{r(M)}{8}$. Then for every discrete sequence (y_k) in Y and its trailing system $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$ there exists a renamed subsequence (y_k) that generates a Riemannian manifold at infinity $M_\infty^{(y_{k;i})}$ of the manifold M. The manifold $M_\infty^{(y_{k;i})}$ has bounded geometry and its injectivity radius is greater or equal than ρ.*

Remark 3.6. Let M' be another manifold such that M and M' have respective compact subsets M_0 and M'_0 such that $M \setminus M_0$ is isometric to $M' \setminus M'_0$, i. e. let M' and M coincide up to a compact perturbation. Then their respective manifolds at infinity for the same trailing systems coincide. From this follows that manifold at infinity of the manifold M is not necessarily diffeomorphic to M.

4. Local and global profiles. Formulation of the main result

In this section we state our main result. We will use the notation introduced in the last section. In particular we will work with discrete sequences of points and related trailing systems described in Definition 3.1.

Definition 4.1. Let M be a manifold of bounded geometry and Y be its discretization. Let (u_k) be a bounded sequence in $H^{1,2}(M)$. Let (y_k) be a sequence of points in Y and let $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$ be its trailing system. One says that $w_i \in H^{1,2}(\Omega_\rho)$ is *a local profile* of (u_k) relative to a trailing sequence $(y_{k;i})_{k\in\mathbb{N}}$, if, on a renamed subsequence, $u_k \circ e_{y_{k;i}} \rightharpoonup w_i$ in $H^{1,2}(\Omega_\rho)$ as $k \to \infty$. If (y_k) is a renamed (diagonal) subsequence such that $u_k \circ e_{y_{k;i}} \rightharpoonup w_i$ in $H^{1,2}(\Omega_\rho)$ as $k \to \infty$ for all $i \in \mathbb{N}_0$, then the family $\{w_i\}_{i\in\mathbb{N}_0}$ is called an *array of local profiles of* (u_k) *relative to the trailing system* $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$ *of the sequence* (y_k).

Proposition 4.2. *Let M be a manifold of bounded geometry and let Y its discretization. Let (u_k) be a bounded sequence in $H^{1,2}(M)$. Let $\{w_i\}_{i\in\mathbb{N}_0}$ be an array of local profiles of (u_k) associated with a trailing system $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$ related to the sequence (y_k) in Y. Then there exists a function $w : M_\infty^{(y_{k;i})} \to \mathbb{R}$ such that $w\circ\varphi_i = w_i$, $i \in \mathbb{N}_0$, where $\varphi_i : \Omega_\rho \to M_\infty^{(y_{k;i})}$ are local coordinate maps of $M_\infty^{(y_{k;i})}$.*

Proof. Functions w_i are defined on Ω_ρ that is a domain of definition of φ_i. Set $w \overset{\text{def}}{=} w_i \circ \varphi_i^{-1}$ on $\varphi_i^{-1}(\Omega_\rho)$ and note that if $x \in \varphi_i^{-1}(\Omega_\rho) \cap \varphi_j^{-1}(\Omega_\rho)$ for some $j \in \mathbb{N}_0$, then $\varphi_i(x) \in \Omega_{ij}$, $\varphi_j(x) \in \Omega_{ji}$, and, using the a.e. convergence of $u_k \circ e_{y_{k;i}}$ and $u_k \circ e_{y_{k;j}}$ to w_i and w_j respectively, and the uniform convergence of $e_{y_{k;i}}^{-1} e_{y_{k;j}}$ to ψ_{ij}, we have

$$
\begin{aligned}
w_j \circ \varphi_j^{-1} &= \lim_{k\to\infty} u_k \circ e_{y_{k;j}} \circ \varphi_j^{-1} = \lim_{k\to\infty} u_k \circ e_{y_{k;i}} \circ e_{y_{k;i}}^{-1} \circ e_{y_{k;j}} \circ \varphi_j^{-1} \\
&= w_i \circ \psi_{ij} \circ \varphi_j^{-1} = w_i \circ \varphi_i^{-1} \circ \varphi_j \circ \varphi_j^{-1} = w_i \circ \varphi_i^{-1}
\end{aligned}
$$

almost everywhere in $\varphi_i^{-1}(\Omega_\rho) \cap \varphi_j^{-1}(\Omega_\rho)$. □

Definition 4.3. Let $\{w_i\}_{i\in\mathbb{N}_0}$ be a local profile array of a bounded sequence (u_k) in $H^{1,2}(M)$ relative to a trailing system $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$. The function $w : M_\infty^{(y_{k;i})} \to \mathbb{R}$ given by Proposition 4.2 is called the *global profile* of the sequence (u_k) relative to $(y_{k;i})$.

Let us fix a smooth partition of unity $\{\chi_y\}_{y\in Y}$ subordinated to the uniformly finite covering of M by geodesic balls $\{B(y,\rho)\}_{y\in Y}$, given by Lemma 8.4.

Definition 4.4. Let M be a manifold of bounded geometry and let Y be its discretization. Let $M_\infty^{(y_{k;i})}$ be a manifold at infinity of M generated by a trailing system $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$. An *elementary concentration* associated with a function $w : M_\infty^{(y_{k;i})} \to \mathbb{R}$ is a sequence $(W_k)_{k\in\mathbb{N}}$ of functions $M \to \mathbb{R}$ given by

$$
W_k = \sum_{i\in\mathbb{N}_0} \chi_{y_{k;i}} w \circ \varphi_i \circ e_{y_{k;i}}^{-1}, \qquad k \in \mathbb{N}. \tag{4.1}
$$

where φ_i are the local coordinate maps of manifold $M_\infty^{(y_{k;i})}$.

In heuristic terms, after we find limits w_i, $i \in \mathbb{N}_0$, of the sequence (u_k) under the "trailing spotlights" $(e_{y_{k;i}})_{k\in\mathbb{N}_0}$ that follow different trailing sequences $(y_{k;i})_{k\in\mathbb{N}}$ of (y_k), we give an approximate reconstruction W_k of u_k "centered" on the moving center y_k of the "core spotlight". We do that by first splitting w into local profiles $w \circ \varphi_i$, $i \in \mathbb{N}_0$, on the set Ω_ρ, casting them onto the manifold M in the vicinity of $y_{k;i}$ by composition with $e^{-1}_{y_{k;i}}$, and patching all such compositions together by the partition of unity on M. Such reconstruction approximates u_k on geodesic balls $B(y_k, R)$ with any $R > 0$, but it ignores the values of u_k for k large on the balls $B(y'_k, R)$, with $d(y_k, y'_k) \to \infty$, where u_k is approximated by a different local concentration. It has been shown in [8] for the case of manifold M with cocompact action of a group of isometries (in particular, for homogeneous spaces) that a global reconstruction of u_k, up to a remainder vanishing in $L^p(M)$, is a sum of elementary concentrations associated with all such mutually decoupled sequences.

Similarly, the profile decomposition theorem below, which is the main result of this paper, says that any bounded sequence (u_k) in $H^{1,2}(M)$ has a subsequence that, up to a remainder vanishing in $L^p(M)$, $p \in (2, 2^*)$, equals a sum of decoupled elementary concentrations.

In the theorem and next sections we will work with countable families of discrete sequences of the set Y. To each sequence we assign a trailing system so in consequence also a the manifold at infinity. To simplify the notation we will index the sequences in Y, the related trailing systems the corresponding manifolds, concentration profiles on these manifolds, etc. by n, *i.e.*, we will write $y_k^{(n)}$, $y_{k;i}^{(n)}$, $M_\infty^{(n)}$, $w^{(n)}$, etc.

Theorem 4.5. *Let M be a manifold of bounded geometry and let Y be its discretization. Let (u_k) be a sequence in $H^{1,2}(M)$ weakly convergent to some function $w^{(0)}$ in $H^{1,2}(M)$. Then there exists a renamed subsequence of (u_k), sequences $(y_k^{(n)})_{k\in\mathbb{N}}$ in Y , and associated with them global profiles $w^{(n)}$ on the respective manifolds at infinity $M_\infty^{(n)}$, $n \in \mathbb{N}$, such that $d(y_k^{(n)}, y_k^{(m)}) \to \infty$ when $n \neq m$, and*

$$u_k - w^{(0)} - \sum_{n\in\mathbb{N}} W_k^{(n)} \to 0 \text{ in } L^p(M),\ \ p \in (2, 2^*), \tag{4.2}$$

where $W_k^{(n)} = \sum_{i\in\mathbb{N}_0} \chi_i^{(n)} w^{(n)} \circ \varphi_i^{(n)} \circ e^{-1}_{y_{k;i}^{(n)}}$ are elementary concentrations, $\varphi_i^{(n)}$ are the local coordinates of the manifolds $M_\infty^{(n)}$ and $\{\chi_i^{(n)}\}_{i\in\mathbb{N}_0}$ are the corresponding partitions of unity satisfying (8.3)*. The series $\sum_{n\in\mathbb{N}} W_k^{(n)}$ converges in $H^{1,2}(M)$ unconditionally and uniformly in $k \in \mathbb{N}$. Moreover,*

$$\|w^{(0)}\|^2_{H^{1,2}(M)} + \sum_{n=1}^{\infty} \|w^{(n)}\|^2_{H^{1,2}(M_\infty^{(n)})} \leq \limsup \|u_k\|^2_{H^{1,2}(M)}\ , \tag{4.3}$$

and

$$\int_M |u_k|^p d\mathrm{d}v_g \to \int_M |w^{(0)}|^p \mathrm{d}v_g + \sum_{n=1}^{\infty} \int_{M_\infty^{(n)}} |w^{(n)}|^p \mathrm{d}v_{g^{(n)}}. \tag{4.4}$$

5. Auxiliary statements concerning profile decomposition

In Sections 5, 6 and 7 we assume that conditions of Theorem 4.5 hold true. First we prove the inequality for the norms introduced in Lemma 8.6.

Lemma 5.1. *Let (u_k) be a bounded sequence in $H^{1,2}(M)$, let $M_\infty^{(y_{k;i})}$ be a manifold at infinity of M generated by a trailing system $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$, and let $w \in H^{1,2}(M_\infty^{(y_{k;i})})$ be the associated global profile of (u_k). Then*

$$\liminf |||u_k|||^2_{H^{1,2}(M)} \geq |||w|||^2_{H^{1,2}\left(M_\infty^{(y_{k;i})}\right)}.$$

Proof. Let $\{\chi_y\}_{y\in Y}$ be the partition of unity given by Lemma 8.4, and let us enumerate it for each $k \in \mathbb{N}$ according to the enumeration $\{y_{k;i}\}_{i\in\mathbb{N}_0}$ of Y, namely $i \mapsto \chi_{y_{k;i}}$, $i \in \mathbb{N}_0$. In other words, for every k the set $\{\chi_{y_{k;i}}\}_{i\in\mathbb{N}_0}$ equals the set $\{\chi_y\}_{y\in Y}$, and only its enumeration depends on the given trailing system $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$. By Ascoli–Arzela theorem, we can define for any i a function η_i on Ω_ρ by the formula

$$\eta_i = \lim_{k\to\infty} \chi_{y_{k;i}} \circ e_{y_{k;i}}. \tag{5.1}$$

The functions η_i are smooth functions supported in Ω_ρ. Moreover, using the diagonalization argument if needed, we get

$$\eta_i = \lim_{k\to\infty} \chi_{y_{k;i}} \circ e_{y_{k;j}} \circ e^{-1}_{y_{k;j}} \circ e_{y_{k;i}} = \eta_j \circ \psi_{ji}.$$

Since $\sum_{i\in\mathbb{N}_0} \chi_{y_{k;i}} \circ e_{y_{k;j}} = 1$ on Ω_ρ for any $j \in \mathbb{N}_0$, we have in the limit $\sum_{i\in\mathbb{N}_0:\,(i,j)\in\mathbb{K}} \eta_i \circ \psi_{ij} = 1$ on Ω_ρ, *cf.* Lemma 3.2. So the family of the functions

$$\chi_i^{(y_{k;i})} \stackrel{\text{def}}{=} \eta_i \circ \varphi_i^{-1}, \qquad i \in \mathbb{N}_0 \tag{5.2}$$

is a partition of unity on $M_\infty^{(y_{k;i})}$, subordinated to the covering $\{\varphi_i(\Omega_\rho)\}_{i\in\mathbb{N}_0}$ of $M_\infty^{(y_{k;i})}$, and it is easy to see that it satisfies (8.3).

Both the manifolds M and $M_\infty^{(y_{k;i})}$ have bounded geometry, and therefore

$$\begin{aligned}
\liminf_{k\to\infty} |||\, u_k \,|||^2_{H^{1,2}(M)} &= \liminf_{k\to\infty} \sum_{i\in\mathbb{N}_0} \|\left(\chi_{y_{k;i}} u_k\right) \circ e_{y_{k;i}}\|^2_{H^{1,2}(\mathbb{R}^N)} \\
&\geq \sum_{i\in\mathbb{N}_0} \liminf_{k\to\infty} \|\left(\chi_{y_{k;i}} u_k\right) \circ e_{y_{k;i}}\|^2_{H^{1,2}(\mathbb{R}^N)} \\
&\geq \sum_{i\in\mathbb{N}_0} \|\eta_i w_i\|^2_{H^{1,2}(\mathbb{R}^N)} \\
&= \sum_{i\in\mathbb{N}_0} \|\chi_i^{(y_{k;i})} w \circ \varphi_i\|^2_{H^{1,2}(\mathbb{R}^N)} \\
&\geq |||\, w \,|||^2_{H^{1,2}\left(M_\infty^{(y_{k;i})}\right)}
\end{aligned} \tag{5.3}$$

□

Lemma 5.2. *Let $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$ be a trailing system for a discrete sequence (y_k) and let $w \in H^{1,2}(M_\infty^{(y_{k;i})})$. Then the elementary concentration $W_k^{(y_{k;i})}$ associated with this system belongs to $H^{1,2}(M)$). Moreover there is a positive constant C independent of k and i such that*

$$\|W_k^{(y_{k;i})}\|_{H^{1,2}(M)} \leq C\,\|w\|_{H^{1,2}\left(M_\infty^{(y_{k;i})}\right)}. \tag{5.4}$$

If $(y'_k)_{k\in\mathbb{N}}$ is a discrete sequence such that $d(y_k, y'_k) \to \infty$, then the elementary concentration $W_k^{(y_{k;i})}$ satisfies

$$W_k^{(y_{k;i})} \circ e_{y'_k} \to 0$$

in $H^{1,2}(\Omega_\rho)$.

Proof. We recall that

$$W_k^{(y_{k;i})} = \sum_{i\in\mathbb{N}_0} \chi_{y_{k;i}}\, w \circ \varphi_i \circ e_{y_{k;i}}^{-1}, \tag{5.5}$$

cf. (4.1). The functions $\chi_{y_{k;i}} \circ e_{y_{k;i}}$ are smooth compactly supported functions on Ω_ρ and the family $\left\{\chi_{y_{k;i}} \circ e_{y_{k;i}}\right\}$ is a bounded set in $C^\infty(\Omega_\rho)$. By the boundedness of the geometry, *cf.* Lemma 3.2 and Lemma 8.6, and using (5.2), we have

$$\begin{aligned} \|\chi_{y_{k;i}} \circ e_{y_{k;i}}\, w \circ \varphi_i\|^2_{H^{1,2}(\mathbb{R}^N)} &\leq C\|\chi_{y_{k;i}} \circ e_{y_{k;i}} \circ \tau_i w\|^2_{H^{1,2}\left(M_\infty^{(y_{k;i})}\right)} \\ &\leq C \sum_{j:\,(i,j)\in\mathbb{K}} \|\chi_i^{(y_{k;i})} w\|^2_{H^{1,2}\left(M_\infty^{(y_{k;i})}\right)}. \end{aligned} \tag{5.6}$$

So using once more Lemma 8.6 we get

$$\begin{aligned} \|W_k^{(y_{k;i})}\|^2_{H^{1,2}(M)} &\leq C \sum_i \|\chi_{y_{k;i}} \circ e_{y_{k;i}}\, w \circ \varphi_i\|^2_{H^{1,2}\left(\mathbb{R}^N\right)} \\ &\leq C \sum_i \sum_{j:\,(i,j)\in\mathbb{K}} \|\chi_j^{(y_{k;i})} w\|^2_{H^{1,2}\left(M_\infty^{(y_{k;i})}\right)} \\ &\leq C\|w\|^2_{H^{1,2}\left(M_\infty^{(y_{k;i})}\right)}. \end{aligned} \tag{5.7}$$

This proves (5.4).

Let $\epsilon > 0$. If follows from (5.7) that there exist $N_\epsilon \in \mathbb{N}$ independent of k such that

$$\sum_{i\geq N_\epsilon} \|\chi_{y_{k;i}} \circ e_{y_{k;i}}\, w \circ \varphi_i\|^2_{H^{1,2}\left(\mathbb{R}^N\right)} \leq \epsilon. \tag{5.8}$$

By (5.5) we have

$$W_k^{(y_{k;i})} \circ e_{y'_k} = \sum_{i \in I_k} \big(\chi_{y_{k;i}} w \circ \varphi_i \circ e^{-1}_{y_{k;i}}\big) \circ e_{y'_k}, \tag{5.9}$$

where $I_k = \{i : B(y'_k, \rho) \cap B(y_{k;i}, \rho) \neq \emptyset\}$. Since $d(y_k, y'_k) \to \infty$, we have

$$\sup_{i \le N_\epsilon} d(y_{k;i}, y'_k) \ge d(y_k, y'_k) - 2N_\epsilon \rho \to \infty$$

as $k \to \infty$, and thus $B(y'_k, \rho) \cap B(y_{k;i}, \rho) = \emptyset$ for all $i \le N_\epsilon$ if k is sufficiently large. Then $\sum_{i=1}^{N_\epsilon} (\chi_{y_{k;i}} w \circ \varphi_i) \circ e^{-1}_{y_{k;i}} \circ e_{y'_k} = 0$ for all k large, which together with (5.8) proves the lemma. □

Lemma 5.3. *Let w be a profile of the sequence (u_k), given by Proposition* 4.2 *relative to a trailing system $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$, and let W_k be the associated concentration sequence. The following holds true:*

$$\lim_{k\to\infty} \langle u_k, W_k\rangle_{H^{1,2}(M)} = \|w\|^2_{H^{1,2}\left(M_\infty^{(y_{k;i})}\right)}. \tag{5.10}$$

Proof. We use for each $k \in \mathbb{N}$ an enumeration of the covering $\{B(y, \rho)\}_{y\in Y}$ by the points $y_{k;i}$ from the trailing system $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$. Taking into account that, as $k \to \infty$, $u_k \circ y_{k;j} \rightharpoonup w_j$, $e^{-1}_{y_{k;i}} \circ e_{y_{k;j}} \to \psi_{ij}$, and $w_i \circ \psi_{ij} = w_j$, we have

$$\begin{aligned}\langle u_k, W_k\rangle_{H^{1,2}(M)} &= \sum_{j\in\mathbb{N}_0} \int_{B(y_{k;j},\rho)} \chi_{y_{k;j}}(x) u_k(x) W_k(x) \mathrm{d}v_g(x) \\ &\quad + \sum_{j\in\mathbb{N}_0} \int_{B(y_{k;j},\rho)} \chi_{y_{k;j}}(x) g\big(du_k(x), dW_k(x)\big) \mathrm{d}v_g(x),\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}\|w\|^2_{H^{1,2}(M_\infty^{(y_{k;i})})} &= \sum_{j\in\mathbb{N}_0} \int_{B(y_{k;j},\rho)} \chi_j^{(y_{k;j})}(x) |w(x)|^2 \mathrm{d}v_{\tilde g^{(n)}}(x) \\ &\quad + \sum_{j\in\mathbb{N}_0} \int_{B(y_{k;j},\rho)} \chi_j^{(y_{k;i})}(x) g\big(dw(x), dw(x)\big) \mathrm{d}v_{\tilde g^{(n)}}(x),\end{aligned} \tag{5.12}$$

where the functions $\chi_j^{(y_{k;i})}$ are defined by the formulae (5.1)-(5.2) relative to the trailing system $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$.

Both coverings are uniformly locally finite, so it is sufficient to prove local identities

$$\lim_{k\to\infty} \int_{B(y_{k;j},\rho)} \chi_{y_{k;j}}(x) u_k(x) W_k(x) \mathrm{d}v_g(x) = \int_{B(y_{k;j},\rho)} \chi_j^{(y_{k;i})}(x) |w(x)|^2 \mathrm{d}v_{\tilde g^{(n)}}(x) \tag{5.13}$$

and

$$\lim_{k\to\infty}\int_{B(y_{k;j},\rho)}\chi_{y_{k;j}}(x)g\big(du_k(x),dW_k(x)\big)\mathrm{d}v_g(x) \\ =\int_{B(y_{k;j},\rho)}\chi_j^{(y_{k;i})}(x)g\big(dw(x),dw(x)\big)\mathrm{d}v_{\tilde{g}^{(n)}}(x). \tag{5.14}$$

In the first case we have and using the expression $o^w(1)$ for any sequence of functions that converges weakly to zero in $H^{1,2}(\Omega_\rho)$,

$$\begin{aligned}
&\int_{\Omega_\rho}\chi_{y_{k;j}}\circ e_{y_{k;j}}(\xi)u_k\circ e_{y_{k;j}}(\xi)\\
&\quad\times\sum_{i\in\mathbb{N}_0}[\chi_{y_{k;i}}\ w\circ\varphi_i\circ e^{-1}_{y_{k;i}})]\circ e_{y_{k;j}}(\xi)\sqrt{g(\xi)}\ \mathrm{d}\xi\\
&=\int_{\Omega_\rho}\chi_{y_{k;j}}\circ e_{y_{k;j}}(\xi)(w_j+o^w(1))(\xi)\\
&\quad\times\sum_{i\in\mathbb{N}_0}\chi_{y_{k;i}}\circ e_{y_{k;j}}\ w_i\circ(\psi_{ij}+o(1))(\xi)\sqrt{g(\xi)}\ \mathrm{d}\xi\\
&=\int_{\Omega_\rho}\chi_{y_{k;j}}\circ e_{y_{k;j}}(\xi)(w_j+o^w(1))(\xi)\,(w_j+o(1))(\xi)\\
&\quad\times\sqrt{(\widetilde{g}+o(1))(\xi)}\ \mathrm{d}\xi\longrightarrow\int_{\Omega_\rho}\chi_j^{(y_{k;j})}\circ\varphi_j(\xi)|w_j|^2\sqrt{\widetilde{g}(\xi)}\ \mathrm{d}\xi\,,
\end{aligned}$$

where the last equality follows from the identity $\sum_{i\in\mathbb{N}_0}\chi_{y_{k;i}}\circ e_{y_{k;j}}=1$ on Ω_ρ, *cf.* Lemma 3.2. This proves (5.13).

To prove (5.14) we first note that

$$\begin{aligned}
&\sum_{\nu,\mu=1}^{N}g^{\nu,\mu}(\xi)\partial_\nu(u_k\circ e_{y_{k;j}})(\xi)\partial_\mu(W_k\circ e_{y_{k;j}})(\xi)\\
&=\sum_{\nu,\mu=1}^{N}g^{\nu,\mu}(\xi)\partial_\nu(u_k\circ e_{y_{k;j}})(\xi)\\
&\quad\times\partial_\mu\Big(\sum_{i\in\mathbb{N}_0}[\chi_{y_{k;i}}\ w\circ\varphi_i\circ e^{-1}_{y_{k;i}})]\circ e_{y_{k;j}}\Big)(\xi)\\
&=\sum_{\nu,\mu=1}^{N}g^{\nu,\mu}(\xi)\partial_\nu\Big((w_j+o^w(1))\circ e_{y_{k;j}}\Big)(\xi)\\
&\quad\times\partial_\mu\Big(\chi_{y_{k;i}}\circ e_{y_{k;j}}(\xi)\ w_i\circ(\psi_{ij}+o(1))\Big)(\xi)\\
&=\sum_{\nu,\mu=1}^{N}g^{\nu,\mu}(\xi)\partial_\nu\Big((w_j+o^w(1))\circ e_{y_{k;j}}\Big)(\xi)\partial_\mu\Big(w_j+o(1))\Big)(\xi).
\end{aligned}$$

In consequence

$$\int_{\Omega_\rho} \chi_{y_{k;j}} \circ e_{y_{k;j}}(\xi) \sum_{\nu,\mu=1}^{N} g^{\nu,\mu}(\xi)\partial_\nu(u_k \circ e_{y_{k;j}})(\xi)\partial_\mu(W_k \circ e_{y_{k;j}})(\xi)\sqrt{g(\xi)}\, \mathrm{d}\xi$$
$$= \int_{\Omega_\rho} \chi_{y_{k;j}} \circ e_{y_{k;j}}(\xi) \sum_{\nu,\mu=1}^{N} g^{\nu,\mu}(\xi)\partial_\nu\Big((w_j + o^w(1)) \circ e_{y_{k;j}}\Big)(\xi)$$
$$\times \partial_\mu\Big((w_j + o(1)) \circ e_{y_{k;j}}\Big)(\xi)\sqrt{\widetilde{g}(\xi) + o(1)}\, \mathrm{d}\xi$$
$$\longrightarrow \int_{\Omega_\rho} \chi_j^{(y_{k;i})} \circ \varphi_j(\xi) \sum_{\nu,\mu=1}^{N} \widetilde{g}^{\nu,\mu}(\xi)\partial_\nu w \circ \varphi_j(\xi)\partial_\mu w \circ \varphi_j(\xi)\sqrt{\widetilde{g}(\xi)}\, \mathrm{d}\xi.$$

Combining the last calculations with (5.11)-(5.14) we arrive at (5.10). □

Lemma 5.4. *Let w be a profile of the sequence u_k, given by Proposition 4.2 relative to a trailing system $\{(y_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$, and let W_k be the associated concentration sequence. The following holds true:*

$$\lim_{k\to\infty} \|W_k\|^2_{H^{1,2}(M)} = \|w\|^2_{H^{1,2}\left(M_\infty^{(y_{i;k})}\right)}. \tag{5.15}$$

Proof. We can proceed in the similar way as in the proof of Lemma 5.3. Once more we can reduce the argumentation to the local identities using (5.12) and

$$\begin{aligned}\|W_k\|^2_{H^{1,2}(M)} = &\sum_{j\in\mathbb{N}_0} \int_{B(y_{k;j},\rho)} \chi_{y_{k;j}}(x)|W_k(x)|^2 \mathrm{d}v_g(x)\\ &+ \sum_{j\in\mathbb{N}_0} \int_{B(y_{k;j},\rho)} \chi_{y_{k;j}}(x) g\big(dW_k(x) dW_k(x)\big)\mathrm{d}v_g(x).\end{aligned} \tag{5.16}$$

We have

$$\int_{\Omega_\rho} \chi_{y_{k;j}} \circ e_{y_{k;j}}(\xi) \sum_{\nu,\mu=1}^{N} g^{\nu,\mu}(\xi)\partial_\nu\big(W_k \circ e_{y_{k;j}}\big)(\xi)\partial_\mu\big(W_k \circ e_{y_{k;j}}\big)(\xi)\sqrt{g(\xi)}\, \mathrm{d}\xi$$
$$= \int_{\Omega_\rho} \chi_{y_{k;j}} \circ e_{y_{k;j}}(\xi) \sum_{\nu,\mu=1}^{N} g^{\nu,\mu}(\xi)\partial_\nu\Big((w_j + o(1))\Big)(\xi)$$
$$\times \partial_\mu\Big((w_j + o(1))\Big)\sqrt{\widetilde{g}(\xi) + o(1)}\, \mathrm{d}\xi$$
$$\longrightarrow \int_{\Omega_\rho} \chi_j^{(y_{k;i})} \circ \varphi_j(\xi) \sum_{\nu,\mu=1}^{N} \widetilde{g}^{\nu,\mu}(\xi)\partial_\nu\big(w \circ \varphi_j\big)(\xi)\partial_\mu\big(w \circ \varphi_j\big)(\xi)\sqrt{\widetilde{g}(\xi)}\, \mathrm{d}\xi.$$

Also as above,

$$\int_{\Omega_\rho} \chi_{y_{k;j}} \circ e_{y_{k;j}}(\xi) \,\Big| \sum_{i\in\mathbb{N}_0} [\chi_{y_{k;i}} \; w \circ \varphi_i \circ e^{-1}_{y_{k;i}})] \circ e_{y_{k;j}}(\xi) \Big|^2 \sqrt{g}(\xi)\, \mathrm{d}\xi$$

$$= \int_{\Omega_\rho} \chi_{y_{k;j}} \circ e_{y_{k;j}}(\xi) \Big| \sum_{i\in\mathbb{N}_0} \chi_{y_{k;i}} \circ e_{y_{k;j}}(\xi)\; w_i \circ (\psi_{ij} + o(1))(\xi) \Big| \sqrt{g}(\xi)\, \mathrm{d}\xi$$

$$= \int_{\Omega_\rho} \chi_{y_{k;j}} \circ e_{y_{k;j}}(\xi) |(w_j + o(1))(\xi)|^2 \sqrt{(\tilde{g} + o(1))(\xi)}\, \mathrm{d}\xi$$

$$\longrightarrow \int_{\Omega_\rho} \chi_j^{(y_{k;i})} \circ \varphi_j(\xi) |(w_j(\xi)|^2 \sqrt{\tilde{g}(\xi)}\, \mathrm{d}\xi. \qquad \square$$

Below we consider a countable family of trailing systems $\{(y^{(n)}_{k;j})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$, $n \in \mathbb{N}$, and will abbreviate the notation of the associated manifolds at infinity, $M_\infty^{(y^{(n)}_{k;j})}$, as $M^{(n)}_\infty$. This convention will also extend to all other objects generated by trailing systems $\{(y^{(n)}_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$, but not to objects indexed by points in Y, such as $\chi_{y^{(n)}_{k;i}}$.

Lemma 5.5. *Assume that $u_k \rightharpoonup 0$. Assume that trailing systems $\{(y^{(n)}_{k;i})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$ of discrete sequences $(y^{(n)}_k)_{k\in\mathbb{N}}$, $n \in \mathbb{N}$, generate local profiles $\{w^{(n)}_i\}_{i\in\mathbb{N}_0}$, such that $d(y^{(n)}_k, y^{(\ell)}_k) \to \infty$ when $n \neq \ell$. Then*

$$\sum_{n=1}^m \|w^{(n)}\|^2_{H^{1,2}(M^{(n)}_\infty)} \le \limsup \|u_k\|^2_{H^{1,2}(M)}. \tag{5.17}$$

Proof. Consider for each $n = 1, \dots, m$ the elementary concentrations $W^{(n)}_k = \sum_{i\in\mathbb{N}_0} \chi_{y^{(n)}_{k;i}} w^{(n)}_i \circ e^{-1}_{y^{(n)}_{k;i}}$, $w^{(n)}_i = w^{(n)} \circ \varphi^{(n)}_i$, where $\{\varphi_i, \Omega_\rho\}_{i\in\mathbb{N}_0}$ is the atlas of the manifold at infinity $M^{(n)}_\infty \overset{\text{def}}{=} M_\infty^{(y^{(n)}_{k;i})}$, and let us expand by bilinearity the trivial inequality

$$\Big\| u_k - \sum_{n=1}^m W^{(n)}_k \Big\|^2_{H^{1,2}(M)} \ge 0.$$

For convenience, the subscript in the Sobolev norm will be omitted for the rest of this proof. We have then

$$2\sum_{n=1}^m \langle u_k, W^{(n)}_k\rangle - \sum_{n=1}^m \|W^{(n)}_k\|^2 \le \|u_k\|^2 + \sum_{n\neq\ell} \langle W^{(n)}_k, W^{(\ell)}_k\rangle. \tag{5.18}$$

Applying Lemmas 5.3 and 5.4 we have

$$\sum_{n=1}^{m} \|w^{(n)}\|^2_{H^{1,2}\left(M_\infty^{(n)}\right)} \le \|u_k\|^2 + \sum_{n\neq\ell} \left\langle W_k^{(n)}, W_k^{(\ell)} \right\rangle + o(1). \tag{5.19}$$

In order to prove the lemma it suffices therefore to show that $\left\langle W_k^{(n)}, W_k^{(\ell)} \right\rangle \to 0$ whenever $n \neq \ell$.

Since $d(y_k^{(n)}, y_k^{(\ell)}) \to \infty$, we also have $d(y_{k;i}^{(n)}, y_{k;j}^{(\ell)}) \to \infty$ for any $i, j \in \mathbb{N}_0$. Let $\epsilon > 0$ and let $N_\epsilon \in \mathbb{N}$ be such that, in view of Lemma 5.1,

$$\begin{aligned} &\sum_{i\ge N_\epsilon} \int_{\Omega\rho} \chi_i^{(n)}(\xi) \sum_{\nu,\mu=1}^{N} g^{\nu\mu}(\xi)\partial_n(w_i^{(n)})(\xi)\partial_\mu(w_i^{(n)})(\xi)) \\ &+ |w_i^{(n)}(\xi)|^2]\sqrt{g(\xi)}\mathrm{d}\xi \le \epsilon,\ n = 1, \dots, m. \end{aligned} \tag{5.20}$$

Let $W_k^{(n)} = W_k^{(n)'} + W_k^{(n)''}$ where

$$W_k^{(n)'} = \sum_{i<N_\epsilon} \left(\chi_{y_{k;i}^{(n)}} w_i^{(n)} \circ e^{-1}_{y_{k;i}^{(n)}} \right) \text{ and } W_k^{(n)''} = \sum_{i\ge N_\epsilon} \left(\chi_{y_{k;i}^{(n)}} w_i^{(n)} \circ e^{-1}_{y_{k;i}^{(n)}} \right)$$

and note that for all k sufficiently large, $W_k^{(n)'}$ and $W_k^{(\ell)'}$ have disjoint supports. Thus

$$\left|\left\langle W_k^{(n)}, W_k^{(\ell)} \right\rangle\right| \le 2S_k T_k + T_k^2, \tag{5.21}$$

where $S_k = \max_{n=1,\dots m} \|W_k^{(n)'}\|$ and $T_k = \max_{n=1,\dots m} \|W_k^{(n)''}\|$. The estimate for S_k is readily provided by repeating verbally the argument of Lemma 5.4, which gives

$$S_k^2 \le \max_{n=1,\dots,m} \|w^{(n)}\|^2_{H^{1,2}\left(M_\infty^{(n)}\right)} + o(1),$$

so S_k is bounded by $C\|u_k\| + o(1)$ due to Lemma 5.1, while a similar adaptation of Lemma 5.4 to summation for $i \ge N_\epsilon$ yields that T_k^2 is bounded, up to vanishing terms, by the left hand side of (5.20), and thus $T_k \le \sqrt{\epsilon} + o(1)$. Thus from (5.21) we have

$$\left|\left\langle W_k^{(n)}, W_k^{(\ell)} \right\rangle\right| \le C\sqrt{\epsilon}\left(\|u_k\| + \sqrt{\epsilon} + o(1)\right),$$

which implies, in turn, that $\limsup_{k\to\infty} |\langle W_k^{(n)}, W_k^{(\ell)} \rangle| \le C\sqrt{\epsilon}$, and since ϵ is arbitrary, we have $\langle W_k^{(n)}, W_k^{(\ell)} \rangle \to 0$ for $n \neq \ell$, which completes the proof. □

Before we begin the proof of Theorem 4.5, we introduce the following technical definition.

Definition 5.6. Let (u_k) be a bounded sequence in $H^{1,2}(M)$. Let $\big(y_k^{(\ell)}\big)$, $\ell = 1,\dots,m$, $m \in \mathbb{N}$, be discrete sequences of points in Y, satisfying $d\big(y_k^{(n)}, y_k^{(\ell)}\big) \to \infty$ for $n \neq \ell$, and generating global profiles $w_1,\dots,w_m$ of a renamed subsequence of (u_k) in respective Sobolev spaces $H^{1,2}\big(M_\infty^{(\ell)}\big)$. A modulus $\nu^{(u_k)}\big((y_k^{(1)}),\dots,(y_k^{m})\big)$ of this subsequence is the supremum of the set of values $\|w\|^2_{H^{1,2}\left(M_\infty^{(y_{k;i})}\right)}$ of all global profiles w of the renamed subsequence (u_k) generated by a trailing system $\{(y_{i;k})_{k\in\mathbb{N}}\}_{i\in\mathbb{N}_0}$ in Y satisfying $d\big(y_{k;0}, y_k^{(\ell)}\big) \to \infty$, $\ell = 1,\dots,m$. If such set is empty, we set $\nu^{(u_k)}\big((y_k^{(1)}),\dots,(y_k^{(m)})\big) \overset{\text{def}}{=} 0$. For $m = 0$, $\nu^{(u_k)}(\emptyset)$ is defined as the corresponding unconstrained supremum.

6. Proof of Theorem 4.5.

Step 1. It suffices to prove Theorem 4.5 for sequences that weakly converge to zero. Indeed, assume that the theorem is true in this case. A general bounded sequence (u_k) in $H^{1,2}(M)$, it has a renamed subsequence weakly convergent to some $w^{(0)}$ in $H^{1,2}(M)$. Consider then conclusions of the theorem for the sequence $(u_k - w^{(0)})$. Since for any discrete sequence (y_k) in Y, $w^{(0)} \circ e_{y_k} \rightharpoonup 0$ in $H^{1,2}(\Omega_\rho)$ by Lemma 5.1, sequences (u_k) and $(u_k - w^{(0)})$ have identical local profiles under the same trailing systems $\big\{(y_{i;k}^{(n)})_{k\in\mathbb{N}}\big\}_{i\in\mathbb{N}_0}$, identical manifolds at infinity and identical concentration terms $W_k^{(n)}$, which yields (4.2). Relation (4.3) follows from the elementary identity for Hilbert space norms,

$$\|u_k\|^2 - \|w^{(0)}\|^2 - \|u_k - w^{(0)}\|^2 \to 0,$$

and (4.3) for the sequence $(u_k - w^{(0)})$. Relation (4.4) follows from Brezis-Lieb Lemma ([3]), which gives, in our settings,

$$\int_M |u_k|^p \mathrm{d}v_g - \int_M |w^{(0)}|^p \mathrm{d}v_g - \int_M |u_k - w^{(0)}|^p \mathrm{d}v_g \to 0,$$

combined with (4.4) for the sequence $(u_k - w^{(0)})$.

From now on we assume that $u_k \rightharpoonup 0$.

Step 2. Let us give an iterative construction of sequences $\big(v_k^{(n)}\big)_{k\in\mathbb{N}}$ in $H^{1,2}(M)$, $n \in \mathbb{N}_0$. We set $v_k^{(0)} = u_k$ and choose $\big(y_k^{(1)}\big)_{k\in\mathbb{N}}$ so that $\big\|w^{(1)}\big\|_{H^{1,2}\left(M_\infty^{(1)}\right)} \geq \frac{1}{2}\nu^{(u_k)}(\emptyset)$.

Assume that we have defined sequences $\big(v_k^{(0)}\big)_{k\in\mathbb{N}},\dots,\big(v_k^{(m)}\big)_{k\in\mathbb{N}}$, with the following properties:

There exists, for a given m, a renamed subsequence of (u_k), sequences $\big(y_k^{(1)}\big)_{k\in\mathbb{N}},\dots,\big(y_k^{(m)}\big)_{k\in\mathbb{N}}$ of points in Y such that $d\big(y_k^{(\ell)}, y_k^{(n)}\big) \to \infty$ whenever $\ell \neq n$, with trailing systems $\Big\{\big(y_{k;i}^{(n)}\big)_{k\in\mathbb{N}}\Big\}_{i\in\mathbb{N}_0}$, defining, on a subsequence, for each respective $n = 1,\dots,m$, an array of local profiles $\big\{w_i^{(n)}\big\}_{i\in\mathbb{N}_0}$

of (the mth extraction of) (u_k), and, consequently, a Riemannian manifold at infinity $M_\infty^{(n)}$ and a global profile $w^{(n)} \in H^{1,2}(M_\infty^{(n)})$. Assume, furthermore, that $\|w^{(n)}\|^2_{H^{1,2}(M_\infty^{(n)})} \geq \frac{1}{2}\nu^{(u_k)}\big((y_k^{(1)}), \dots, (y_k^{(n-1)})\big), n = 2, \dots, m$ (*cf.* Definition 5.6). Let $(W_k^{(n)})_{k\in\mathbb{N}}$, $n = 1, \dots, m$, be corresponding elementary concentrations, and define

$$v_k^{(n)} \stackrel{\text{def}}{=} u_k - \sum_{\ell=1}^{n} W_k^{(\ell)}, \ n = 1, \dots m.$$

Under the above assumptions we construct now a sequence $v_k^{(m+1)}$ that will also satisfy these assumptions. Consider all sequences (y_k) of points in Y such that $d(y_k, y_k^{(\ell)}) \to \infty$ for all $\ell = 1, \dots, m$. We have three complementary cases:

- case 1: for any such sequence one has $v_k^{(m)} \circ e_{y_k} \rightharpoonup 0$ in $H^{1,2}(\Omega_\rho)$ on a renamed subsequence;
- case 2: there exists a bounded sequence (y_k) of points in Y (so that $d\big(y_k, y_k^{(\ell)}\big) \to \infty$ for all $\ell = 1, \dots, m$) such that, on a renamed subsequence, $v_k^{(m)} \circ e_{y_k} \rightharpoonup w \neq 0$;
- case 3: there exists a discrete sequence (y_k) of points in Y such that $d\big(y_k, y_k^{(\ell)}\big) \to \infty$ for all $\ell = 1, \dots, m$, and $v_k^{(m)} \circ e_{y_k} \rightharpoonup w \neq 0$.

Case 2 is in fact vacuous. Indeed, in this case (y_k) would have a constant subsequence with some value z and $u_k \circ e_z \rightharpoonup w \neq 0$, which contradicts the assumption $u_k \rightharpoonup 0$.

Consider case 1. We prove that in that case $v_k^{(m)} \circ e_{z_k} \rightharpoonup 0$ for any sequence (z_k) in Y. By assumption we know that it is true if $d\big(z_k, y_k^{(\ell)}\big) \to \infty$ for all $\ell = 1, \dots, m$. So let us assume that on a renamed subsequence, $d\big(z_k, y_k^{(\ell)}\big)$ is bounded for some $\ell \in \{1, \dots m\}$. Then by the definition of the trailing system there exists $i \in \mathbb{N}_0$ such that $z_k = y_{k;i}^{(\ell)}$ on a renamed subsequence. So if $u_k \circ e_{z_k} \rightharpoonup w \neq 0$ then w coincides with the local profile $w_i^{(\ell)}$. Moreover $d\big(z_k, y_k^{(n)}\big) \to \infty$ if $1 \leq n \leq m$ and $n \neq \ell$. So by Lemma 5.2, $W_k^{(n)} \circ e_{z_k} \rightharpoonup 0$ if $n \neq \ell$ and $W_k^{(\ell)} \circ e_{z_k} \rightharpoonup w_i$. In consequence $v_k^{(m)} \circ e_{z_k} \rightharpoonup 0$ Now by Lemma 2.4, $v_k^{(m)} \to 0$ in $L^p(M)$, which means that the asymptotic relation (4.2) is proved with a finite sum of elementary concentrations and we can take $v_k^{(m+1)} = 0$.

Consider now case 3. Now the modulus $\nu^{(u_k)}\big((y_k^{(1)}), \dots, (y_k^{m})\big) > 0$ is positive, *cf.* Definition 5.6). We may choose a sequence $y_k^{(m+1)}$, $d\big(y_k^{(m+1)}, y_k^{(\ell)}\big) \to \infty$ for all $\ell = 1, \dots, m$, in such a way that the corresponding global profile $w^{(m+1)}$ of (u_k) satisfies

$$\|w^{(m+1)}\|^2_{H^{1,2}\left(M_\infty^{(m+1)}\right)} \geq \frac{1}{2}\nu^{(u_k)}\big((y_k^{(1)}), \dots, (y_k^{(m)})\big). \tag{6.1}$$

Then using the local profiles $w_i^{(m+1)}$, $i \in \mathbb{N}_0$, we may define, for a renamed subsequence, the associated global profile $w^{(m+1)}$ (*cf.* Proposition 4.2), and the corresponding elementary concentration $W_k^{(m+1)}$, and put

$$v_k^{(m+1)} \stackrel{\text{def}}{=} u_k - \sum_{\ell=1}^{m+1} W_k^{(\ell)}.$$

It is easy to see that the sequence $\big(v_k^{(m+1)}\big)$ has the same properties as $\big(v_k^{(n)}\big)$, $n = 0, \dots, m$.

Step 3. By Lemma 5.5 we have

$$\sum_{n=1}^{m} \|w^{(n)}\|^2_{H^{1,2}(M_\infty^{(n)})} \leq \limsup \|u_k\|^2_{H^{1,2}(M)}$$

for any m, which proves (4.3).

Step 4. In order to prove convergence of the series $\sum_{n=1}^{\infty} W_k^{(n)}$ note first that we may assume without loss of generality that for each $n \in \mathbb{N}$, there exists $r_n > 0$ such that supp $W_k^{(n)} \subset B\big(y_k^{(n)}, r_n\big)$. Indeed, acting like in the proof of Lemma 5.5, from the calculations in the proof of Lemma 5.4 one can easily see that one can approximate $W_k^{(n)}$ in the $H^{1,2}$-norm by restricting summation in (4.1) to a finite number of terms, with the norm of the remainder bounded by, say, $\epsilon 2^{-n}$ with a small $\epsilon > 0$. Then, for any $m \in \mathbb{N}$ one can extract a subsequence $\big(k_j^{(m)}\big)_{j\in\mathbb{N}}$ of $(k)_{k\in\mathbb{N}}$ such that $d\big(y_k^{(n)}, y_k^{(\ell)}\big) > r_n + r_\ell$ whenever $1 \leq \ell < n \leq m$. Then on a diagonal subsequence $\big(k_m^{(m)}\big)_{m\in\mathbb{N}}$ the elementary concentrations $\big(W_k^{(n)}\big)_{k=k_m^{(m)}, m\in\mathbb{N}}$ will have pairwise disjoint supports. Together with (4.3) this proves that the convergence is unconditional and uniform with respect to k.

Step 5. Now we prove that $(u_k - \sum_{\ell=1}^{\infty} W_k^{(\ell)}) \circ e_{y_k} \to 0$ in $L^p(M)$ for any sequence y_k in Y.

Let first (y_k) in Y be a bounded sequence. Since it has finitely many values, on each constant subsequence we have $u_k \circ e_y \rightharpoonup 0$ and $W_k^{(\ell)} \circ e_y \rightharpoonup 0$, and thus $\big(u_k - \sum_{\ell=1}^{\infty} W_k^{(\ell)}\big) \circ e_{y_k} \rightharpoonup 0$.

Let now (y_k) be a discrete sequence in Y. If there is $\ell \in \mathbb{N}$ such that on a renamed subsequence we have $d\big(y_k, y_k^{(\ell)}\big)$ is bounded. Then on a renamed subsequence $y_k = y_{k;i}^{(\ell)}$ for some i, *cf.* Step 2. But then $u_k \circ e_{y_k} \rightharpoonup w_i^{(\ell)}$, $W_k^{(\ell)} \circ e_{y_k} \rightharpoonup w_i^{(\ell)}$ and $W_k^{(n)} \circ e_{y_k} \rightharpoonup 0$ if $n \neq \ell$, *cf.* Lemma 5.2. Thus $\big(u_k - \sum_{\ell=1}^{\infty} W_k^{(\ell)}\big) \circ e_{y_k} \rightharpoonup 0$.

Let (y_k) be a discrete sequence in Y, such that $d\big(y_k, y_k^{(\ell)}\big) \to \infty$ for any $\ell \in \mathbb{N}_0$. Assume that on a renamed subsequence $\big(u_k - \sum_{\ell=1}^{\infty} W_k^{(\ell)}\big) \circ e_{y_k} \rightharpoonup w_0 \neq 0$. Then (y_k) generates a profile w of (u_k) on some manifold ar infinity M_∞ of M that necessarily satisfies $\|w\|_{H^{1,2}(M_\infty)} \le \nu^{(u_k)}\big((y_k^{(1)}),\dots,(y_k^{(m)})\big)$ for any $m \in \mathbb{N}$. By (4.3) and (6.1) we have $\nu^{(u_k)}\big((y_k^{(1)}),\dots,(y_k^{(m)})\big) \to 0$ as $m \to \infty$, and therefore $w = 0$, which implies $w_0 = 0$. This gives the contradiction.

We conclude that $\big(u_k - \sum_{\ell=1}^{\infty} W_k^{(\ell)}\big) \circ e_{y_k} \rightharpoonup 0$ for any sequence (y_k) in Y, and by Lemma 2.4 $\big(u_k - \sum_{\ell=1}^{\infty} W_k^{(\ell)}\big) \circ e_{y_k} \to 0$ in $L^p(M)$.

Step 6. It was proved in Step 4 that the series of elementary concentrations $W_k^{(n)}$ is convergent in $H^{1,2}(M)$. So for any $\epsilon > 0$ the sum S_k of the elementary concentrations can be approximated by the finite sum S_k^ϵ, *i.e.*,

$$\begin{aligned} \big|\|u_k\|_p - \|S_k^\epsilon\|_p\big| &\le \big|\|u_k\|_p - \|S_k\|_p\big| + \|S_k - S_k^\epsilon\|_p \\ &\le o(1) + C\|S_k - S_k^\epsilon\|_{H^{1,2}(M)} \le C\epsilon + o(1). \end{aligned} \tag{6.2}$$

Moreover similarly to Step 4, we may assume without loss of generality all $w^{(n)}$ have compact support. In consequence we may assume that there exists $m \in \mathbb{N}$ such that $w^{(n)} = 0$ for all $n > m$, and that $w^{(n)}$ have compact support if $n \le m$.

Let us now evaluate $\|S_k^\epsilon\|_p$. Let us show first that

$$\int_M |W_k^{(n)}|^p \mathrm{d}v_g \to \int_{M_\infty^{(n)}} |w^{(n)}|^p \mathrm{d}v_{\tilde g^{(n)}}. \tag{6.3}$$

Indeed, omitting for the sake of simplicity the superscript n and taking into account that $w_i \circ e_{y_{k;i}}^{-1} \circ e_{y_{k;j}} \to w_j$, $e_{y_{k;j}}^{-1} \circ e_{y_{k;i}} \to \psi_{ji}$, and $\chi_{y_{k;j}} \circ e_{y_{k;j}} \to \chi_j$ as in the proof of Lemma 5.1, we have:

$$\begin{aligned} \int_M |W_k|^p \mathrm{d}v_g &= \int_M \Big|\sum_{i\in\mathbb{N}_0} \chi_{y_{k;i}} w_i \circ e_{y_{k;i}}\Big|^p \mathrm{d}v_g \\ &= \sum_{j\in\mathbb{N}_0} \int_{\Omega_\rho} \chi_{y_{k;j}} \circ e_{y_{k;j}} \Big|\sum_{i\in\mathbb{N}_0} \chi_{y_{k;i}} w_i \circ e_{y_{k;i}}\Big|^p \circ e_{y_{k;j}}^{-1} \sqrt{g_{k;j}}\mathrm{d}\xi \\ &= \sum_{j\in\mathbb{N}_0} \int_{\Omega_\rho} (\chi_j + o(1)) \Big|\sum_{i\in\mathbb{N}_0} \chi_{y_{k;i}} \circ e_{y_{k;j}}^{-1} (w_j + o(1))\Big|^p \sqrt{\tilde g_j + o(1)}\, \mathrm{d}\xi \\ &\to \int_{M_\infty} |w|^p \mathrm{d}v_{\tilde g}. \end{aligned}$$

Note that the notation $o(1)$ above refers to functions vanishing in the sense of C^∞ and that all infinite sums contain uniformly finitely many terms.

Now, for all k sufficiently large, all elementary concentrations $W_k^{(n)}$ in the sum S_k^ϵ have pairwise disjoint supports, and, since $\ell^1 \hookrightarrow \ell^{\frac{p}{2}}$, taking into account (4.3), we have

$$\left(\sum_{n\geq\nu}\int_{M_\infty^{(n)}}|w^{(n)}|^p \mathrm{d}v_{\tilde{g}^{(n)}}\right)^{\frac{2}{p}} \leq \sum_{n\geq\nu}\left(\int_{M_\infty^{(n)}}|w^{(n)}|^p \mathrm{d}v_{\tilde{g}^{(n)}}\right)^{\frac{2}{p}}$$
$$\leq \sum_{n\geq\nu} C\|w^{(n)}\|^2_{H^{1,2}(M_\infty^{(n)})} \to 0 \text{ as } \nu\to\infty.$$

Therefore (4.4) is immediate from (4.2), which completes the proof of Theorem 4.5.

7. Local and global profile decompositions on cocompact manifolds

Let M be now a smooth connected complete Riemannian manifold, cocompact relative to a subgroup G of its isometry group, that is, we assume that there exists an open bounded set $\mathcal{O}$ such that $\cup_{g\in G} g\mathcal{O} = M$. Then M is obviously of bounded geometry. It is then natural to ask if Theorem 4.5 yields Theorem 1.1 with the manifolds $M_\infty^{(n)}$ isometric to M. Below we consider this question in the case when G is a discrete countable subgroup. Without loss of generality we may assume that $\mathcal{O}$ is a geodesic ball.

Theorem 7.1. *Let M be a smooth connected N-dimensional Riemannian manifold, let $\rho\in(0,\frac{r(M)}{8})$ and $z\in M$, and assume that there exists a discrete countable subgroup G of isometries on M such that $\{B(gz,\rho)\}_{g\in G}$ covers M with a uniformly finite multiplicity. Then*

(i) *one can choose the construction parameters of manifolds $M_\infty^{(n)}$, so that they will coincide, up to isometry, with M;*
(ii) *there exist sequences $\big(g_k^{(n)}\big)_{k\in\mathbb{N}}$, of elements in G, and functions $w^{(n)}\in H^{1,2}(M)$, $n\in\mathbb{N}$, such that the sequences $\big([g_k^{(\ell)}]^{-1}g_k^{(n)}\big)_{k\in\mathbb{N}}$ are discrete whenever $\ell\neq n$, $u_k\circ g_k^{(n)}\rightharpoonup w^{(n)}$ in $H^{1,2}(M)$, $n\in\mathbb{N}$, and*

$$W_k^{(n)} = w^{(n)}\circ\left[g_k^{(n)}\right]^{-1}.$$

Proof. 1. Let us repeat the construction of the manifold at infinity relative to a sequence (y_k) in $Y=\{gz\}_{g\in G}$. Fix a sequence $h_i\in G$, $h_0=\mathrm{id}$, such that $d(h_{i+1}z,z)\geq d(h_iz,z)$, $i\in\mathbb{N}_0$, and define the ith trailing sequence of (y_k) by $y_{k;i}\overset{\text{def}}{=} g_kh_iz$, $k\in\mathbb{N}$. Recall that the normal coordinates at the points $y\in Y$ were defined as $\exp_y$ up to an arbitrarily fixed isometry on T_yM. For the present construction we set them specifically as $e_{gz}\overset{\text{def}}{=} g\circ e_z$. Under such choice the transition maps of $M_\infty^{(y_{k;i})}$ are characterized by elements of the group G:

$$\psi_{ij} = \lim_{k\to\infty} e_{y_{k;i}}^{-1}\circ e_{y_{k;j}} = \lim e_z^{-1}\circ[g_kh_i]^{-1}g_kh_j\circ e_z = e_z^{-1}\circ h_i^{-1}h_j\circ e_z,$$

and the sequences above are in fact constant with respect to k. Consequently, the transition maps ψ_{ij} of the manifold $M_\infty^{(y_{k;i})}$ are $e_z^{-1} \circ h_i^{-1} h_j \circ e_z$ - same as of M itself. In other words, all the gluing data for $M_\infty^{(y_{k;i})}$ are taken from M, which suggests, since Theorem 8.8 is based on a suitable list of properties of charts of a manifold that will allow its reconstruction that $M_\infty^{(y_{k;i})}$ is isometric to M. We will, however, apply Corollary 8.10 formally, as follows.

Manifold $M_\infty^{(y_{k;i})}$ has an atlas $\{(\varphi_i(\Omega_\rho), \varphi_i^{-1})\}_{i\in\mathbb{N}_0}$ with transition maps $\varphi_i^{-1}\varphi_j = e_z^{-1} \circ h_i^{-1} h_j \circ e_z$, while manifold M has an atlas, enumerated by $h_i \in G$, $\{(B(h_i(z), \rho), e_z^{-1} \circ h_i^{-1})\}_{i\in\mathbb{N}_0}$ with the same transition maps as M_∞. Let $T_i \stackrel{\text{def}}{=} h_i \circ e_z \circ \varphi_i^{-1} : \varphi_i(\Omega_\rho) \to M$, $i \in \mathbb{N}_0$, and note that this defines a smooth map $T : M_\infty^{(y_{k;i})} \to M$, since the values of T_i are consistent on intersections of sets $\varphi_i(\Omega_\rho)$:

$$h_i \circ e_z \circ \varphi_i^{-1} \circ [h_j \circ e_z \circ \varphi_j^{-1}]^{-1} = h_i \circ e_z \circ \psi_{ij} \circ [h_j \circ e_z]^{-1} \tag{7.1}$$
$$= h_i \circ e_z \circ e_z^{-1} \circ h_i^{-1} h_j \circ e_z \circ e_z^{-1} \circ h_j^{-1} = \text{id}. \tag{7.2}$$

Furthermore, T is a diffeomorphism with $T^{-1} = \varphi_i \circ e_z^{-1} \circ h_i^{-1}$, consistently defined on $B(g_i z, \rho)$, $i \in \mathbb{N}_0$. Note that (3.7) on $M_\infty^{(y_{k;i})}$ holds because it holds on M with the same transition map for every k, so the Riemannian metric on $M_\infty^{(y_{k;i})}$ in the normal coordinates coincides with the Riemannian metric on M. In what follows we will identify $M_\infty^{(y_{k;i})}$ as M.

2. Let now (u_k) be a bounded sequence in $H^{1,2}(M)$ and note that its local profile associated with the sequence $(g_k h_i)_{k\in\mathbb{N}}$ is given by

$$w_i = \text{w-lim}\, u_k \circ (g_k h_i) \circ e_z,$$

and the global profile is by definition $w = w_i \circ \varphi_i^{-1} = w_i \circ e_z^{-1} \circ h_i^{-1} = \text{w-lim}\, u_k \circ g_k$, which coincides with the profile of (u_k) as defined in Theorem 1.1 in (1.3) relative to the sequence (g_k). Consider now the elementary concentration defined by the array $\{w_i\}_{i\in\mathbb{N}_0}$ of local profiles:

$$\begin{aligned} W_k &= \sum_{i\in\mathbb{N}_0} \chi_{g_k h_i z} w_i \circ e_{y_{k;i}}^{-1} \\ &= \sum_{i\in\mathbb{N}_0} \chi_{g_k h_i z} w_i \circ e_z^{-1} \circ h_i^{-1} \circ g_k^{-1} \\ &= \sum_{i\in\mathbb{N}_0} \chi_{g_k h_i z} w \circ g_k^{-1} \\ &= w \circ g_k^{-1}, \end{aligned}$$

which completes the proof. □

8. Appendix

8.1. Manifolds of bounded geometry and covering lemma

In this appendix we give some elementary properties of manifolds of bounded geometry. All needed definitions can be found, *e.g.*, in Chavel's book [4]. Let M be an N-dimensional Riemannian manifold of bounded geometry with a metric tensor g. Let v_g denote the Riemannian measure on M and let $L^2(M)$ be the corresponding space of square integrable functions. For $u - 3 : M \to \mathbb{C}$ we denote by $du - 3 \in T^*M$ the covariant derivative of u, and by $|du|$ the norm of du defined by a local chart, *i.e.*,

$$|du - 3|^2 = g^{ij}\partial_i u \partial_j u$$

where g^{ij} are the components of the inverse matrix of the metric matrix $g = (g_{ij})$. The Sobolev space $H^{1,2}(M)$ is a completion of $C_o^\infty(M)$ with respect to the norm given by

$$\|f\|^2_{H^{1,2}} = \|df\|_2^2 + \|f\|_2^2.$$

We start with the following lemma, and refer to [7] for the proof.

Lemma 8.1. *Let M be a Riemannian manifold of bounded geometry and let $0 < r < r(M)$. If $k \in \mathbb{N}$ then there exists a constant C_k dependent on the curvature bounds and r but independent of $x \in M$, which bounds the C^k-norm of components g_{ij} of the metric tensor g and its inverse g^{ij} in any normal coordinate system of radius not exceeding r at any point $x \in M$.*

For any two points $x \in M$ and $0 < r < r(M)$ let

$$e_x : \Omega_r \to B(x, r)$$

denote a normal coordinate system at x defined on the euclidean ball Ω_r centered at origin.

The boundedness of the derivatives of the Riemann curvature tensor is equivalent to the following lemma, *cf.* [18],

Lemma 8.2. *If the manifold M has bounded geometry and $0 < r < r(M)$ then for any $\alpha \in \mathbb{N}_0^N$ there exists a constant $C_\alpha > 0$, such that*

$$|D^\alpha e_y^{-1} \circ e_x(\xi)| \le C_\alpha \textit{ whenever } x, y \in M, \textit{and } B(x,r) \cap B(y,r) \neq \emptyset.$$

The next two statements can be found is many places in literature, *cf.* eg. [13,18,19].

Lemma 8.3. *Let M be an N-dimensional connected Riemannian manifold with bounded geometry. Let $\rho > 0$. There exists an at most countable set $Y \in M$ such that*

$$d(y, y') \ge \rho/2 \quad \textit{whenever} \;\; y \neq y', \;\; y, y' \in Y, \tag{8.1}$$

$$M = \bigcup_{y \in Y} B(y, r) \quad \textit{for any} \quad r > \rho. \tag{8.2}$$

Moreover for any $r > \rho$ the multiplicity of the covering $\{B(y,r)\}_{y\in Y}$ is uniformly finite.

Lemma 8.4. *Let M, Y, ρ and r be as in Lemma* 8.3. *There exists a smooth partition of unity $\{\chi_y\}_{y\in Y}$ on M, subordinated to the covering $\{B(y,\rho)\}_{y\in Y}$, such that for any $\alpha \in \mathbb{N}_0^N$ there exists a constant $C_\alpha > 0$, such that*

$$|D^\alpha \chi_y| \le C_\alpha \tag{8.3}$$

for all $y \in Y$.

The following corollary is the immediate consequence of Lemma 8.1 above.

Corollary 8.5. *Let $p \in (0,\infty)$ and $r \in (0, r(M))$. There exists a constant $C > 1$ such that for any $x \in M$*

$$C^{-1}\int_{B(x,r)} |u|^p dv_g \le \int_{\Omega_r} |u\circ e_x|^p dx \le C\int_{B(x,r)} |u|^p dv_g, \tag{8.4}$$

and

$$C^{-1}\int_{B(x,r)} |du|^2 dv_g \le \int_{\Omega_r} \sum_{i=1}^N |\partial_i(u\circ e_x)|^2 dx \le C\int_{B(x,r)} |du|^2 dv_g.$$

We finish this subsection by recalling a technical but useful equivalent norm in $H^{1,2}(M)$, *cf.* [13] or [24, Chapter 7],

Lemma 8.6. *Let $\{B(y_i,r)\}$ be a locally uniformly finite covering of an N-dimensional manifold M with bounded geometry, $r \in (0, r(M))$ and let $\{\chi_i\}$ be a partition of unity subordinated to the covering $\{B(y_i,r)\}$ as in Lemma* 8.4. *Then*

$$|||\, f\, |||_{H^{1,2}(M)} = \Big(\sum_i \|\chi_i f \circ \exp_{y_i}\|^2_{H^{1,2}(\mathbb{R}^N)}\Big)^{1/2} \tag{8.5}$$

is an equivalent norm in $H^{1,2}(M)$. Moreover

$$\|f\|_{H^{1,2}(M)} \sim |||\, f\, |||_{H^{1,2}(M)} \sim \Big(\sum_i \|\chi_i f\|^2_{H^{1,2}(M)}\Big)^{1/2}.$$

8.2. Gluing manifolds

We use a particular case of gluing theorem in Gallier *et al.*, [10, Theorem 3.1]:

Definition 8.7 ([10, Definition 3.1], [9, Definition 8.1]). A *set of gluing data* is a triple $(\{\Omega_i\}_{i\in\mathbb{N}_0}, \{\Omega_{ij}\}_{i,j\in\mathbb{N}_0}, \{\psi_{ji}\}_{(i,j)\in\mathbb{K}})$ satisfying the following properties:

(1) For every $i \in \mathbb{N}_0$, the set Ω_i is a nonempty open subset of $\mathbb{R}^N$ and the sets $\{\Omega_i\}_{i\in\mathbb{N}_0}$ are pairwise disjoint;

(2) For every pair $i, j \in \mathbb{N}_0$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$ and $\Omega_{ji} \neq \emptyset$ if and only if $\Omega_{ij} \neq \emptyset$;

(3) $\mathbb{K} = \{(i, j) \in \mathbb{N}_0 \times \mathbb{N}_0 : \Omega_{ij} \neq \emptyset\}$, $\psi_{ji} : \Omega_{ij} \to \Omega_{ji}$ is a diffeomorphism for every $(i, j) \in \mathbb{K}$, and the following conditions hold:

   (a) $\psi_{ii} = \mathrm{id}|_{\Omega_i}$, for all $i \in \mathbb{N}_0$,
   (b) $\psi_{ij} = \psi_{ji}^{-1}$, for all $(i, j) \in \mathbb{K}$,
   (c) For all $i, j, k \in \mathbb{N}_0$, if $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$, then $\psi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik}$, and $\psi_{ki}(x) = \psi_{kj} \circ \psi_{ji}(x)$, for all $x \in \Omega_{ij} \cap \Omega_{ik}$;

(4) For every pair$(i, j) \in \mathbb{K}$, with $i \neq j$, for every $x \in \partial\Omega_{ij} \cap \Omega_i$ and every $y \in \partial\Omega_{ji} \cap \Omega_j$, there are open balls V_x and V_y centered at x and y so that no point of $V_y \cap \Omega_{ji}$ is the image of any point of $V_x \cap \Omega_{ij}$ by ψ_{ji}.

Each set Ω_i is called *parametrization domain* or *p-domain*, each nonempty set Ω_{ij} is called a *gluing domain*, and each map ψ_{ij} is called *transition map* or *gluing map*.

Theorem 8.8 ([10, Theorem 3.1]). *For every set of gluing data,*

$$\left(\{\Omega_i\}_{i\in\mathbb{N}_0}, \{\Omega_{ij}\}_{i,j\in\mathbb{N}_0}, \{\psi_{ji}\}_{(i,j)\in\mathbb{K}}\right),$$

there exists a N-dimensional smooth manifold M an atlas $(U_i, \tau_i)_i$ of M such that $\tau_i(U_i) = \Omega_i$, whose transition maps are $\tau_j \circ \tau_i^{-1} = \psi_{ji} : \Omega_{ij} \to \Omega_{ji}$. $i, j \in \mathbb{N}_0$.

Remark 8.9. Note that the theorem does not provide any specifics about the maps τ_i which are obviously not uniquely defined.

Corollary 8.10. *Let $0 < \rho < r < a$ and let $\Omega_\rho \subset \Omega_r \subset \Omega_a$ be balls in $\mathbb{R}^N$ centered at the origin with radius ρ, r and a respectively. Let $\{\widetilde{\psi}_{ij}\}_{i,j\in\mathbb{N}_0}$ be a family of smooth open maps $\widetilde{\psi}_{ij} : \Omega_r \to \Omega_a$. Assume that a family $\{\psi_{ji} = \widetilde{\psi}_{ji}|_{\Omega_\rho}\}_{i,j\in\mathbb{N}_0}$ satisfies the following conditions:*

(i) $\psi_{ii} = \mathrm{id}$, $i \in \mathbb{N}_0$;
(ii) ψ_{ji} *is a diffeomorphism between* $\Omega_{ij} \stackrel{\text{def}}{=} \psi_{ij}(\Omega_\rho) \cap \Omega_\rho$ *and* Ω_{ji}, $i, j \in \mathbb{N}_0$, *whenever* $\Omega_{ji} \neq \emptyset$;
(iii) $\psi_{ij} = \psi_{ji}^{-1}$ *on* Ω_{ji}, *whenever* $\Omega_{ji} \neq \emptyset$, $i, j \in \mathbb{N}_0$;
(iv) $\psi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik}$, *and* $\psi_{ki}(x) = \psi_{kj} \circ \psi_{ji}(x)$ *for all* $x \in \Omega_{ij} \cap \Omega_{ik}$, $i, j, k \in \mathbb{N}_0$;
(v) *for all* $(i, j) \in \mathbb{K} \stackrel{\text{def}}{=} \{(i, j) \in \mathbb{N}_0 \times \mathbb{N}_0 : \Omega_{ij} \neq \emptyset\}$ *and all* $x \in \partial\Omega_{ij} \cap \Omega_\rho$ $\psi_{ji}(x) \in \partial\Omega_{ji} \cap \partial\Omega_\rho$.

Then there exists a smooth differential manifold M with an atlas $\{(U_i, \tau_i)\}_{i\in\mathbb{N}_0}$, such that $\tau_i(U_i) = \Omega_\rho$ for any $i \in \mathbb{N}_0$ and whose transition maps $\tau_j \circ \tau_i^{-1}$ are $\psi_{ji} : \Omega_{ij} \to \Omega_{ji}$. $i, j \in \mathbb{N}_0$.

Proof. Fix an enumeration $(z_i)_{i\in\mathbb{N}_0}$ of the lattice $3a\mathbb{Z}^N \subset \mathbb{R}^N$. Set $\Omega'_i \overset{\text{def}}{=} z_i + \Omega_\rho$, $i\in\mathbb{N}_0$, and $\Omega'_{ij} \overset{\text{def}}{=} \Omega_{ij}+z_i$, $\psi'_{ij} \overset{\text{def}}{=} \psi_{ij}(\cdot - z_j)+z_i$, for $(i,\, j)\in\mathbb{K}$. The corollary is immediate from Theorem 8.8 once we show that $(\{\Omega'_i\}_{i\in\mathbb{N}_0}, \{\Omega'_{ij}\}_{i,j\in\mathbb{N}_0}, \{\psi'_{ij}\}_{(i,j)\in\mathbb{K}})$ is a set of gluing data according to Definition 8.7. Conditions of the definition verify as follows.

Condition (1): is immediate since $3a > 2\rho$.

Condition (2): the sets Ω_{ij} (and thus Ω'_{ij}) are open since the maps ψ_{ji} are open. The relation $\Omega'_{ij} \subset \Omega'_i$ follows from $\Omega_{ij} \subset \Omega_\rho$ in *(ii)*. By *(i)* we have $\Omega_{ii} = \Omega_\rho$ and thus $\Omega'_{ii} = \Omega'_i$. If $\Omega'_{ij} \neq \emptyset$, then $\Omega_{ij} \neq \emptyset$, and since ψ_{ij} is the inverse of ψ_{ji}, $\Omega_{ji} \overset{\text{def}}{=} \psi_{ji}(\Omega_\rho \cap \psi_{ij}\Omega_\rho) = \psi_{ji}\Omega_{ji} \neq \emptyset$. Thus $\Omega'_{ji} \neq \emptyset$.

Condition (3): properties (a), (b), and (c) *are immediate, respectively, from* (i), (iii), and (iv).

Condition (4): let $x \in \partial\Omega'_{ij} \cap \Omega_\rho(z_i)$ and $y \in \partial\Omega'_{ji} \cap \Omega_\rho(z_j)$. Then $\tilde{x} = x - z_i \in \partial\Omega_{ij}\cap\Omega_\rho$ and $\tilde{x} = x - z_j \in \partial\Omega_{ji}\cap\Omega_\rho(z_j)$. By assumption *(v)* we have $\tilde{y} \neq \psi_{ji}(\tilde{x})$. In consequence there exist Euclidean balls $\Omega(\tilde{x}, \varepsilon)$ and $\Omega(\tilde{y}, \varepsilon)$ such that no point of $\Omega(\tilde{y}, \varepsilon) \cap \Omega_\rho$ is an image of $\Omega(\tilde{x}, \varepsilon) \cap \Omega_\rho$. □

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