PRECISE ESTIMATES FOR CERTAIN DISTANCES IN $\mathbb{R}^d$

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Abstract. We provide sharp estimates for the intrinsic distances of Finsler metrics with precise boundary estimates. These metrics include the Kobayashi-Hilbert metric near strongly convex points, the minimal metric near convex and strongly minimally convex points, and the k-quasi hyperbolic metric in k-strongly convex domains. Finally, we prove a characterization result in convex geometry for the k-quasi hyperbolic metric.

1. Introduction

In their famous paper [1], Balogh and Bonk proved the Gromov hyperbolicity of the Kobayashi distance k_D in strongly pseudoconvex domains, establishing the following estimates for the Kobayashi distance: if $D \subset \mathbb{C}^d$ is a strongly pseudoconvex domain, then there exists $B > 0$ such that

$$g_D(z,w) - B \leq k_D(z,w) \leq g_D(z,w) + B, \qquad z,w \in D \tag{1.1}$$

where g_D is a function derived from the Carnot-Carathéodory metric on the boundary ∂D. This estimate, while sufficient to prove Gromov hyperbolicity, does not provide useful information about the distance when the points are close to each other. The estimates (1.1) were recently improved in [10] in the case of strongly pseudoconvex domains with $\mathcal{C}^{2,\alpha}$-smooth boundary.

A similar situation arises in the real case, where the first named author in [5] proved the Gromov hyperbolicity of the minimal distance ρ_D (analogous to the Kobayashi metric in the theory of minimal surfaces) in strongly minimally convex domains by showing estimates similar to those of Balogh and Bonk: if $D \subset \mathbb{R}^d$ is strongly minimally convex domain, then there exists $A > 1$ and $B > 0$ such that

$$A^{-1} d_D(x,y) - B \leq \rho_D(x,y) \leq A d_D(x,y) + B, \qquad x,y \in D$$

where

$$d_D(x,y) = 2\log\left(\frac{\|\pi(x) - \pi(y)\| + h(x) \vee h(y)}{\sqrt{h(x)h(y)}}\right), \tag{1.2}$$

$\pi\colon D \to \partial D$ is such that $\|\pi(x) - x\| = d_{Euc}(x, \partial D)$ and $h(x) := \sqrt{d_{Euc}(x,\partial D)}$. Also in this case, the estimates were sufficient to prove Gromov hyperbolicity.

The main goal of this paper is to improve the previous estimate in a way that can be applied to various distances naturally defined in real domains. These distances are all defined through a Finsler metric (see Subsection 2.1 for a brief introduction), with behavior at the boundary that is asymptotic to $1/(2\delta_D)$ in the normal component and comparable to $1/\delta_D^{1/2}$ in the tangential component, where $\delta_D(\cdot) := d_{Euc}(\cdot, \partial D)$.

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Let us now delve into the details. Let $D \subset \mathbb{R}^d$ be a domain and $\xi \in \partial D$ be a $\mathcal{C}^2$-smooth boundary point, then there exists a neighborhood U of ξ such that $\pi\colon U \to \partial D$ is a well defined $\mathcal{C}^2$-smooth function. Set for $x, y \in D \cap U$

$$a_D(x,y) := \frac{\|\pi(x) - \pi(y)\| + h(x) \vee h(y)}{\sqrt{h(x)h(y)}} - 1,$$

and, for $c > 0$, consider the quasi-distance defined as

$$d_D^c(x,y) = 2\log(1 + c a_D(x,y)).$$

Note that d_D^1 is exactly the function that appears in (1.2).

Let $x \in D$ and $v \in \mathbb{R}^d$. If x is sufficiently close to a $\mathcal{C}^2$-smooth boundary point, we can decompose the vector v into its normal component v_N and its tangential component v_T at the boundary point $\pi(x)$.

The main result of the paper is the following.

Theorem 1.1. *Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a domain and $\xi \in \partial D$ be a $\mathcal{C}^2$-smooth boundary point. Let $F\colon D \times \mathbb{R}^d \to [0, +\infty)$ be a Finsler metric on D and assume there exist a neighborhood U of ξ and $0 < c_1 < C_1$ such that for all $x \in D \cap U$ and $v \in \mathbb{R}^d$*

$$\max\left\{(1 - \omega(\delta_D(x)))\frac{\|v_N\|}{2\delta_D(x)}, c_1\frac{\|v_T\|}{\delta_D(x)^{1/2}}\right\} \leq F(x,v) \leq (1 + \omega(\delta_D(x)))\frac{\|v_N\|}{2\delta_D(x)} + C_1\frac{\|v_T\|}{\delta_D(x)^{1/2}} \tag{1.3}$$

where $\omega\colon [0,\varepsilon] \to \mathbb{R}$ is a measurable function with $\int_0^\varepsilon \frac{\omega(u)}{u}du < +\infty$. Let d be the intrinsic distance of F, then there exist a neighborhood $V \subset\subset U$ of ξ and $0 < c_2 \leq 1 \leq C_2$ such that for all $x, y \in D \cap V$

$$d_D^{c_2}(x,y) \leq d(x,y) \leq d_D^{C_2}(x,y). \tag{1.4}$$

The estimates of the main theorem are in the spirit of [10], and they are effective regardless of the relative positions of the two points.

In the second part of the paper, we will prove the estimates (1.3) for various Finsler metrics

(1) Kobayashi-Hilbert metric near strongly convex points (Subsection 4.1);
(2) minimal metric near convex and strongly minimally convex points (Subsection 4.2);
(3) k-quasi hyperbolic metric in k-strongly convex domains (Subsection 4.3).

As a consequence, we obtain the estimates (1.4) for the associated intrinsic distances.

The paper concludes with a rigidity theorem for k-quasi-hyperbolic metrics (Theorem 5.1).

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2. Preliminaries

Notations:

- For $x \in \mathbb{R}^d$ let $||x||$ denote the standard Euclidean norm of x.
- For $u, v \in \mathbb{R}^d$ let $\langle u, v\rangle$ denote the Euclidean scalar product of $\mathbb{R}^d$.

- Let $\mathbb{D} := \{x \in \mathbb{R}^2 : ||x|| < 1\}$ be the unit disk in $\mathbb{R}^2$.
- For $x, y \in \mathbb{R}^d$ let
$$(x, y) := \{tx + (1-t)y : t \in (0, 1)\}$$
be the open segment between x and y, and
$$[x, y] := \{tx + (1-t)y : t \in [0, 1]\}$$
be the closed segment between x and y.
- For $x \in \mathbb{R}^d$ and $r > 0$ we denote with
$$B(x, r) := \{y \in \mathbb{R}^d : \|x - y\| < r\}$$
the Euclidean ball with center x and radius r.
- For $A, B \subseteq \mathbb{R}^d$ nonempty let denote
$$d_{Euc}(A, B) := \inf\{\|x - y\| : x \in A, y \in B\},$$
the Euclidean distance between A and B. If $A = \{x\}$ is a singleton, we simply write $d_{Euc}(x, \cdot) := d_{Euc}(\{x\}, \cdot)$.
- If $D \subsetneq \mathbb{R}^d$ is a domain and $x \in \mathbb{R}^d$ let
$$\delta_D(x) := d_{Euc}(x, \partial D),$$
be the distance to the boundary.
- Let $a, b \in \mathbb{R}$, let denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

2.1. **Finsler metric.** In this section, we recall the definition of the Finsler metric.

Let $D \subset \mathbb{R}^d$ be a domain, a *Finsler metric* is a function $F\colon D \times \mathbb{R}^d \simeq TD \to [0, +\infty)$ with the following properties

(1) F is upper semicontinuous on $D \times \mathbb{R}^d$;
(2) For all $x \in D$, $v \in \mathbb{R}^d$ and $t \in \mathbb{R}$
$$F(x, tv) = |t|F(x, v).$$

Given a Finsler metric F on D and a piecewise $\mathcal{C}^1$-smooth curve $\gamma\colon [a, b] \to D$, we can define the length of γ with respect to the metric F
$$\ell_F(\gamma) := \int_a^b F(\gamma(t), \dot{\gamma}(t))dt.$$
Finally, the *intrinsic pseudodistance* associated with F is defined as
$$d(x, y) = \inf_{\gamma} \ell_F(\gamma), \quad x, y \in D$$
where the infimum is over all piecewise $\mathcal{C}^1$-smooth curve $\gamma\colon [0, 1] \to D$ with $\gamma(0) = x$ and $\gamma(1) = y$.

2.2. **$\mathcal{C}^2$-smooth boundary point.** In this section, we will review some results on the geometry of the domain near a $\mathcal{C}^2$-smooth boundary point.

Let $D \subsetneq \mathbb{R}^d$ be a domain and $\xi \in \partial D$ be a boundary point. We say that ξ is a *$\mathcal{C}^2$-smooth boundary point* if there exists a $\mathcal{C}^2$-smooth local defining function ρ near ξ, i.e., there exists a neighborhood U of ξ and $\rho\colon U \to \mathbb{R}$ a $\mathcal{C}^2$-smooth function, such that

(1) $D \cap U = \{x \in U : \rho(x) < 0\}$;
(2) $\nabla\rho \neq 0$ in $\partial D \cap U = \{x \in U : \rho(x) = 0\}$.

The vector

$$n_\xi := \frac{\nabla\rho(\xi)}{\|\nabla\rho(\xi)\|}$$

is called *unit outer normal* at ξ. It does not depend on the choice of the $\mathcal{C}^2$-smooth local defining function ρ.

We now state the local version of a well-known lemma.

Lemma 2.1. *Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a domain and $\xi_0 \in \partial D$ be a $\mathcal{C}^2$-smooth boundary point. Then there exists $r, \varepsilon > 0$ such that if we set*

$$U := \{x \in \mathbb{R}^d : d_{Euc}(x, \partial D \cap B(\xi_0, r)) < \varepsilon\}$$

we have

(1) *every point $\xi \in \partial D \cap U$ is a $\mathcal{C}^2$-smooth boundary point;*
(2) *for every $x \in U$ there exists a unique $\pi(x) \in \partial D \cap U$ with $||x - \pi(x)|| = \delta_D(x)$;*
(3) *the signed distance to the boundary $\rho\colon \mathbb{R}^d \to \mathbb{R}$ given by*

$$\rho(x) := \begin{cases} -\delta_D(x) & \text{if } x \in D \\ \delta_D(x) & \text{if } x \notin D \end{cases}$$

is $\mathcal{C}^2$-smooth on U;
(4) *the fibers of the map $\pi\colon U \to \partial D \cap U$ are*

$$\pi^{-1}(\xi) = \{\xi + t n_\xi : |t| < \varepsilon\}$$

where n_ξ is the outer unit normal vector of ∂D at $\xi \in \partial D \cap U$;
(5) *the gradient of ρ satisfies for all $x \in U$*

$$\nabla\rho(x) = n_{\pi(x)};$$

(6) *the projection map $\pi\colon U \to \partial D$ is $\mathcal{C}^1$-smooth.*
(7) *if $\gamma\colon [0,1] \to D \cap U$ is a $\mathcal{C}^1$-smooth curve and $\alpha := \pi \circ \gamma\colon [0,1] \to \partial D \cap U$ its projection to the boundary, then for all $t \in [0,1]$*

$$\frac{1}{2}||(\dot\gamma(t))_T|| \leq ||\dot\alpha(t)|| \leq 2||(\dot\gamma(t))_T||.$$

Proof. See [1, Lemma 2.1] and [5, Lemma 4.1]. □

Now we have the preliminary results to give a precise meaning to the decomposition into the normal and tangential parts of a vector presented in (1.3).

Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a domain and $\xi \in \partial D$ be a $\mathcal{C}^2$-smooth boundary point. Let U be the neighborhood of ξ given by the previous Lemma. Then for $x \in D \cap U$ and $v \in \mathbb{R}^d$, we consider the orthogonal decomposition of $v = v_N + v_T$ at (unique) projection point $\pi(x) \in \partial D \cap U$, where

$$v_N := \langle v, n_{\pi(x)}\rangle n_{\pi(x)}, \qquad v_T := v - v_N.$$

3. Proof of the main theorem

In this section, we will prove the main result, namely Theorem 1.1. In order to make the reading smoother, we will denote all multiplicative constants by A, without distinguishing them with other symbols. The same will be done for additive constants, denoted by B.

We begin by recalling the function d_D^c mentioned earlier in the introduction.

Let $D \subsetneq \mathbb{R}^d$ be a domain and $\xi \in \partial D$ a $\mathcal{C}^2$-smooth boundary point. Let U be the neighborhood of ξ given by the Lemma 2.1. For $x, y \in U$, set

$$a_D(x,y) := \frac{\|\pi(x) - \pi(y)\| + h(x) \vee h(y)}{\sqrt{h(x)h(y)}} - 1$$

where $h(\cdot) := \delta_D(\cdot)^{1/2}$. Let $c > 0$ and define

$$d_D^c(x,y) = 2\log(1 + c a_D(x,y)). \tag{3.1}$$

We begin by listing the properties of these functions.

Proposition 3.1. *Let d_D^c be the function defined in* (3.1)*. Then*

(1) *If $0 < c_1 \le c_2$, then for all $x, y \in U$*

$$d_D^{c_1}(x,y) \le d_D^{c_2}(x,y) \le c_1^{-1} c_2 d_D^{c_1}(x,y);$$

(2) *If $c \ge 1$, d_D^c is a distance on U;*

(3) *For all $c > 0$, d_D^c is a quasi-distance on U, i.e., there exists $A \ge 1$ such that for all $x, y, z \in U$*

$$d_D^c(x,y) \le A(d_D^c(x,z) + d_D^c(z,y)).$$

In the following lemmas, we will be under the assumptions of Theorem 1.1. Without loss of generality, we can assume that the neighborhood U is the same as in Lemma 2.1, possibly after shrinking it.

Lemma 3.2. *There exists $B > 0$ such that for all piecewise $\mathcal{C}^1$-smooth curve $\gamma\colon [0,1] \to D \cap U$ with endpoints x, y we have*

$$\ell_F(\gamma) \ge \left|\log\left(\frac{h(x)}{h(y)}\right)\right| - B.$$

Moreover, if $\gamma\colon [0,1] \to D \cap U$ is a straight segment contained in $\pi^{-1}(\xi)$ with $\xi \in \partial D \cap U$, then

$$\ell_F(\gamma) \le \left|\log\left(\frac{h(x)}{h(y)}\right)\right| + B.$$

Proof. By (4) in Lemma 2.1 for those $t \in [0,1]$ for which $\dot\gamma(t)$ exists we have $|\frac{d}{dt}\delta_D(\gamma(t))| = ||(\dot\gamma(t))_N||$, so by the lower estimates of F

$$\begin{aligned}
\ell_F(\gamma) &\ge \int_0^1 (1 - \omega(\delta_D(\gamma(t)))) \frac{||(\dot\gamma(t))_N||}{2\delta_D(\gamma(t))} dt \\
&= \int_0^1 (1 - \omega(\delta_D(\gamma(t)))) \frac{|\frac{d}{dt}\delta_D(\gamma(t))|}{2\delta_D(\gamma(t))} dt \\
&\ge \left| \int_0^1 \frac{\frac{d}{dt}\delta_D(\gamma(t))}{2\delta_D(\gamma(t))} dt - \frac{1}{2}\int_0^1 \frac{\omega(\delta_D(\gamma(t)))\frac{d}{dt}\delta_D(\gamma(t))}{\delta_D(\gamma(t))} dt \right| \\
&\ge \left|\log\left(\frac{h(x)}{h(y)}\right)\right| - \frac{1}{2}\int_0^\varepsilon \frac{\omega(u)}{u} du
\end{aligned}$$

that implies that statement since by hypothesis $\int_0^\varepsilon \frac{\omega(u)}{u} du < +\infty$.

Finally, if γ is a straight line segment, using the upper estimates of F we obtain with a similar computation

$$\ell_F(\gamma) \le \left|\log\left(\frac{h(x)}{h(y)}\right)\right| + \frac{1}{2}\int_0^\varepsilon \frac{\omega(u)}{u} du.$$

□

Lemma 3.3. *There exists $A > 1$ such that for all piecewise $\mathcal{C}^1$-smooth curve $\gamma\colon [0,1] \to D \cap U$ with endpoints x, y we have*

$$\ell_F(\gamma) \geq A^{-1} \frac{\|\pi(x) - \pi(y)\|}{\delta_\gamma^{1/2}}$$

where $\delta_\gamma := \max_{t\in[0,1]} \delta_D(\gamma(t))$. Moreover, for all $x, y \in D \cap U$ with $\delta_D(x) = \delta_D(y) =: \delta_0$ there exists $\gamma\colon [0,1] \to D \cap U$ with endpoints x, y and $\delta_D(\gamma(t)) \equiv \delta_0$ such that

$$\ell_F(\gamma) \leq A \frac{\|\pi(x) - \pi(y)\|}{\delta_0^{1/2}}.$$

Proof. Set $\alpha = \pi \circ \gamma$. By (7) in Lemma 2.1 for those $t \in [0,1]$ for which $\dot\gamma(t)$ exists we have

$$\|\dot\alpha(t)\| \leq 2\|(\dot\gamma(t))_T\|.$$

Clearly we have $\int_0^1 \|\dot\alpha(t)\| dt \geq \|\pi(x) - \pi(y)\|$, so

$$\begin{aligned}
\ell_F(\gamma) &\geq c_1 \int_0^1 \frac{\|(\dot\gamma(t))_T\|}{\delta_D(\gamma(t))^{1/2}} dt \\
&\geq \frac{c_1}{2} \int_0^1 \frac{\|\dot\alpha(t)\|}{\delta_D(\gamma(t))^{1/2}} dt \\
&\geq \frac{c_1}{2} \frac{\int_0^1 \|\dot\alpha(t)\| dt}{\delta_\gamma^{1/2}} \\
&\geq \frac{c_1}{2} \frac{\|\pi(x) - \pi(y)\|}{\delta_\gamma^{1/2}}.
\end{aligned}$$

For the second part, since $\partial D \cap U = \partial D \cap B(\xi_0, r)$, the intrinsic and extrinsic distances are bi-Lipschitz, that is there exists $A > 1$ such that for all $\xi_1, \xi_2 \in \partial D \cap U$ we may find a piecewise $\mathcal{C}^1$-smooth curve $\alpha\colon [0,1] \to \partial D \cap U$ connecting them with $\int_0^1 \|\dot\alpha(t)\| dt \leq A\|\xi_1 - \xi_2\|$. Let α such curve for $\pi(x), \pi(y) \in \partial D \cap U$, and consider $\gamma(t) := \alpha(t) - \delta_0 n_{\alpha(t)}$. Clearly $\pi \circ \gamma = \alpha$ and $\delta_D(\gamma(t)) \equiv \delta_0$. Noticing that $(\gamma(t))_N \equiv 0$, again by (7) in Lemma 2.1, we have

$$\begin{aligned}
\ell_F(\gamma) &\leq C_1 \int_0^1 \frac{\|(\dot\gamma(t))_T\|}{\delta_0^{1/2}} dt \\
&\leq 2C_1 \frac{\int_0^1 \|\dot\alpha(t)\| dt}{\delta_0^{1/2}} \\
&\leq 2AC_1 \frac{\|\pi(x) - \pi(y)\|}{\delta_0^{1/2}}.
\end{aligned}$$

□

Let us now combine the two previous lemmas.

Lemma 3.4. *There exists $B > 0$ such that for all piecewise $\mathcal{C}^1$-smooth curve $\gamma\colon [0,1] \to D \cap U$ with endpoints x, y we have*

$$\ell_F(\gamma) \geq 2 \log \left(\frac{\|\pi(x) - \pi(y)\| + h(x) \vee h(y)}{\sqrt{h(x)h(y)}} \right) - B.$$

Proof. The proof is based on a dyadic decomposition similar to the one in [1, Theorem 1.1].

If $\pi(x) = \pi(y)$ the estimate follows from Lemma 3.2, so we may assume $\pi(x) \neq \pi(y)$.

Set $H := \max_{t\in[0,1]} h(\gamma(t))$ and $t_0 = \min\{t \in [0,1] : h(\gamma(t)) = H\}$. We divide the curve in two parts, $\gamma_1 := \gamma|_{[0,t_0]}$ and $\gamma_2 := \gamma|_{[t_0,1]}$. We will study γ_1 and γ_2 separately, starting with the first.

Since $h(x) \leq H$ there exists $k \geq 1$ such that $2^{-k}H < h(x) \leq 2^{-(k-1)}H$. Define $0 = s_0 \leq s_1 < \cdots < s_k \leq t_0$ as follows. Let $s_0 = 0$ and $s_j = \min\{s \in [0,t_0] : h(\gamma(s)) = 2^{-(k-j)}H\}$ for $j = 1,\dots,k$. Define $x_j = \gamma(s_j)$ for $j = 0,\dots,k$. We divide the problem into two cases.

Case 1.1: there exist $l \in \{1,\dots,k\}$

$$\|\pi(x_{l-1}) - \pi(x_l)\| > \frac{2^{-(k-l)}}{8}\|\pi(x) - \pi(y)\|.$$

Since $\delta_D(\gamma(t)) \leq 2^{-(k-l)}H$ for all $t \in [s_{l-1}, s_l]$, by Lemma 3.3

$$\ell_F(\gamma|_{[s_{l-1},s_l]}) \geq A^{-1}\frac{\|\pi(x_{l-1}) - \pi(x_l)\|}{2^{-(k-l)}H} \geq A^{-1}\frac{\|\pi(x) - \pi(y)\|}{H}.$$

So by Lemma 3.2

$$\begin{aligned}\ell_F(\gamma_1) &= \ell_F(\gamma|_{[0,s_{l-1}]}) + \ell_F(\gamma|_{[s_{l-1},s_l]}) + \ell_F(\gamma|_{[s_l,t_0]})\\ &\geq \log\left(\frac{h(x_{l-1})}{h(x)}\right) + A^{-1}\frac{\|\pi(x) - \pi(y)\|}{H} + \log\left(\frac{H}{h(x_l)}\right) - B\\ &\geq \log\left(\frac{H}{h(x)}\right) + A^{-1}\frac{\|\pi(x) - \pi(y)\|}{H} - B.\end{aligned}$$

We retain this estimate and move on to the other case.

Case 1.2: for all $j = 1,\dots k$

$$\|\pi(x_{j-1}) - \pi(x_j)\| \leq \frac{2^{-(k-j)}}{8}\|\pi(x) - \pi(y)\|.$$

This implies that if we set $t_1 = s_k$

$$\|\pi(x) - \pi(\gamma(t_1))\| \leq \sum_{j=1}^{k}\|\pi(x_{j-1}) - \pi(x_j)\| \leq \frac{1}{4}\|\pi(x) - \pi(y)\|.$$

Moreover, again by Lemma 3.2

$$\ell_F(\gamma|_{[0,t_1]}) \geq \log\left(\frac{H}{h(x)}\right) - B.$$

Now we address γ_2: reasoning in the same way, we have two cases.

Case 2.1:

$$\ell_F(\gamma_2) \geq \log\left(\frac{H}{h(y)}\right) + A^{-1}\frac{\|\pi(x) - \pi(y)\|}{H} - B.$$

Case 2.2: there exists $t_2 \in [t_0, 1]$ such that

$$\|\pi(\gamma(t_2)) - \pi(y)\| \leq \frac{1}{4}\|\pi(x) - \pi(y)\|$$

and

$$\ell_F(\gamma|_{[t_2,1]}) \geq \log\left(\frac{H}{h(y)}\right) - B.$$

We now need to consider all four combinations. If Case 1.2 and Case 2.2 occur simultaneously, we have

$$\|\pi(\gamma(t_1))-\pi(\gamma(t_2))\| \geq \|\pi(x)-\pi(y)\|-\|\pi(x)-\pi(\gamma(t_1))\|-\|\pi(\gamma(t_2))-\pi(y)\| \geq \frac{1}{2}\|\pi(x)-\pi(y)\|,$$

so by Lemma 3.3

$$\begin{aligned}
\ell_F(\gamma) &= \ell_F(\gamma_{[0,t_1]}) + \ell_F(\gamma_{[t_1,t_2]}) + \ell_F(\gamma_{[t_2,1]}) \\
&\geq \log\left(\frac{H}{h(x)}\right) + A^{-1}\frac{\|\pi(\gamma(t_1)) - \pi(\gamma(t_2))\|}{H} + \log\left(\frac{H}{h(y)}\right) - B \\
&\geq 2\log\left(\frac{H}{\sqrt{h(x)h(y)}}\right) + A^{-1}\frac{\|\pi(x)-\pi(y)\|}{H} - B.
\end{aligned}$$

In the other three combinations, it is easy to see that we still obtain

$$\ell_F(\gamma) \geq 2\log\left(\frac{H}{\sqrt{h(x)h(y)}}\right) + A^{-1}\frac{\|\pi(x)-\pi(y)\|}{H} - B.$$

Now the minimum of the function

$$u \mapsto 2\log\left(\frac{u}{\sqrt{h(x)h(y)}}\right) + A^{-1}\frac{\|\pi(x)-\pi(y\|}{u}$$

is at $u = \frac{A^{-1}}{2}\|\pi(x)-\pi(y)\| > 0$, so

$$\ell_F(\gamma) \geq 2\log\left(\frac{\|\pi(x)-\pi(y)\|}{\sqrt{h(x)h(y)}}\right) - B.$$

Finally, combining with Lemma 3.2 we have

$$\begin{aligned}
\ell_F(\gamma) &\geq \max\left\{2\log\left(\frac{\|\pi(x)-\pi(y)\|}{\sqrt{h(x)h(y)}}\right), 2\log\left(\frac{h(x)\vee h(y)}{\sqrt{h(x)h(y)}}\right)\right\} - B \\
&= 2\log\left(\max\left\{\frac{\|\pi(x)-\pi(y)\|}{\sqrt{h(x)h(y)}}, \frac{h(x)\vee h(y)}{\sqrt{h(x)h(y)}}\right\}\right) - B \\
&\geq 2\log\left(\frac{\|\pi(x)-\pi(y)\| + h(x)\vee h(y)}{\sqrt{h(x)h(y)}}\right) - B.
\end{aligned}$$

□

From the previous lemma, we easily obtain the following corollary.

Corollary 3.5. *For all $V \subset\subset U$, there exists $B > 0$ such that for all $x \in D \cap V$*

$$\inf_{y\in D\setminus U} d(x,y) \geq \frac{1}{2}\log\left(\frac{1}{\delta_D(x)}\right) - B.$$

Proof. Let $\gamma\colon [0,1] \to D$ be a piecewise $\mathcal{C}^1$-smooth curve with $\gamma(0) = x \in D \cap V$ and $\gamma(1) \in D\setminus U$. Consider $t^* := \inf\{t \in [0,1] : \gamma(t) \in D\setminus U\}$ and set $y = \gamma(t^*)$, then by Lemma 3.4 there exists $B > 0$ such that

$$\begin{aligned}
\ell_F(\gamma) \geq \ell_F(\gamma|_{[0,t^*]}) &\geq 2\log\left(\frac{\|\pi(x)-\pi(y)\| + h(x)\vee h(y)}{\sqrt{h(x)h(y)}}\right) - B \\
&= \frac{1}{2}\log\left(\frac{1}{\delta_D(x)}\right) + 2\log\left(\frac{\|\pi(x)-\pi(y)\| + h(x)\vee h(y)}{\sqrt{h(y)}}\right) - B.
\end{aligned}$$

Since V is relatively compact in U we may find $B > 0$ such that

$$2\log\left(\frac{\|\pi(x)-\pi(y)\| + h(x)\vee h(y)}{\sqrt{h(y)}}\right) > -B$$

obtaining

$$\ell_F(\gamma) \geq \frac{1}{2}\log\left(\frac{1}{\delta_D(x)}\right) - B.$$

We conclude taking the infimum over all piecewise $\mathcal{C}^1$-smooth curves. □

We are now ready to prove the main theorem.

Proof of Theorem 1.1. Let V be a neighborhood of ξ relatively compact in U and let $x, y \in D \cap V$. We divide the proof into two cases, depending on whether the points are "close" or "far".

Case 1: $a_D(x,y) << 1$.

We want to prove that there exists $A > 1$ such that for all $x \in D \cap U$ and $v \in \mathbb{R}^d$ we have

$$\limsup_{t\to 0} \frac{d_D^1(x, x+tv)}{t} \leq AF(x,v). \tag{3.2}$$

Since $\log(1+t) \sim t$ if t is small, we need to study

$$\limsup_{t\to 0} \frac{2}{t}\left(\frac{||\pi(x)-\pi(x+tv)|| + h(x)\vee h(x+tv)}{\sqrt{h(x)h(x+tv)}} - 1\right).$$

Now by (7) in Lemma 2.1,

$$\limsup_{t\to 0} \frac{\|\pi(x)-\pi(x+tv)\|}{t} \leq 2\|v_T\|,$$

so

$$\limsup_{t\to 0} \frac{2}{t}\left(\frac{||\pi(x)-\pi(x+tv)||}{\sqrt{h(x)h(x+tv)}}\right) \leq 4\frac{\|v_T\|}{\delta_D(x)^{1/2}}.$$

For the second part, notice that

$$\frac{h(x)\vee h(x+tv)}{\sqrt{h(x)h(x+tv)}} - 1 = \frac{\sqrt{h(x)\vee h(x+tv)} - \sqrt{h(x)\wedge h(x+tv)}}{\sqrt{h(x)\wedge h(x+tv)}}.$$

Moreover, by (5) in Lemma 2.1 we have

$$\sqrt{h(x)\vee h(x+tv)} - \sqrt{h(x)\wedge h(x+tv)} = |\sqrt{h(x+tv)} - \sqrt{h(x)}| = \frac{1}{4}t\delta_D(x)^{-\frac{3}{4}}\|v_N\| + o(t),$$

and so

$$\lim_{t\to 0} \frac{2}{t}\left(\frac{h(x)\vee h(x+tv)}{\sqrt{h(x)h(x+tv)}} - 1\right) = \frac{\|v_N\|}{2\delta_D(x)}.$$

Finally, we can find $A > 1$ such that for all $x, y \in D \cap U$

$$\frac{\|v_N\|}{2\delta_D(x)} + 4\frac{\|v_T\|}{\delta_D(x)^{1/2}} \leq AF(x,v),$$

and so we proved (3.2). By [12, Theorem 1.3], this implies that for all $x, y \in D \cap U$

$$d_D^1(x,y) \leq Ad(x,y)$$

and so by (1) in Proposition 3.1

$$d(x,y) \geq A^{-1}d_D^1(x,y) \geq d_D^{A^{-1}}(x,y).$$

For the upper bound, we may assume $h(y) \geq h(x)$. Consider $x' = \pi(x) - h(y)^2 n_{\pi(x)} \in D \cap U$. Clearly we have $d(x, x') \leq A \log\left(\frac{h(y)}{h(x)}\right)$, so

$$d(x,y) \leq d(x,x') + d(x',y) \leq A \log\left(\frac{h(y)}{h(x)}\right) + A\frac{\|\pi(x) - \pi(y)\|}{h(y)}.$$

Finally, since there exists $C > 0$ such that $t \leq \log(1 + Ct)$ for $0 < t << 1$, we have

$$\begin{aligned}
d(x,y) &\leq 2A \log\left(\sqrt{\frac{h(y)}{h(x)}}\right) + A\frac{\|\pi(x) - \pi(y)\|}{h(y)} \\
&\leq 2A\left(\sqrt{\frac{h(y)}{h(x)}} - 1 + \frac{\|\pi(x) - \pi(y)\|}{\sqrt{h(x)h(y)}}\right) \\
&= 2A a_D(x,y) \\
&\leq 2\log(1 + ACa_D(x,y)) \\
&= d_D^{AC}(x,y).
\end{aligned}$$

Case 2: $a_D(x,y) >> 1$.

For the upper bound, let ε be the constant of Lemma 2.1. We set $h = (\|\pi(x) - \pi(y)\| + h(x) \vee h(y)) \wedge \varepsilon^{1/2}$ and we consider $x' = \pi(x) - h^2 n_{\pi(x)}$, $y' = \pi(y) - h^2 n_{\pi(y)}$. Since U has finite diameter, there exists $A > 1$ such that $\|\pi(x) - \pi(y)\| \leq Ah$. Now, by Lemmas 3.2 and 3.3 we have

$$\begin{aligned}
d(x,y) &\leq d(x,x') + d(x',y') + d(y',y) \\
&\leq \log\left(\frac{h}{h(x)}\right) + A\frac{\|\pi(x) - \pi(y)\|}{h} + \log\left(\frac{h}{h(y)}\right) \\
&\leq 2\log\left(\frac{h}{\sqrt{h(x)h(y)}}\right) + A\frac{\|\pi(x) - \pi(y)\|}{h} \\
&\leq 2\log\left(\frac{\|\pi(x) - \pi(y)\| + h(x) \vee h(y)}{\sqrt{h(x)h(y)}}\right) + B \\
&= 2\log(A + Aa_D(x,y)) \\
&\leq 2\log(1 + Aa_D(x,y)) \\
&= d_D^A(x,y).
\end{aligned}$$

For the lower bound, let $\gamma\colon [0,1] \to D$ be a $\mathcal{C}^1$ piecewise curve with endpoints $x, y \in D \cap V$. Assume again $h(y) \geq h(x)$. If $\gamma([0,1]) \subset D \cap U$, by Lemma 3.4

$$\begin{aligned}
\ell_F(\gamma) &\geq 2\log(1 + a_D(x,y)) - B \\
&\geq 2\log(1 + A^{-1}a_D(x,y)) \\
&= d_D^{A^{-1}}(x,y).
\end{aligned}$$

In the other case, set $t_1 := \inf\{t \in [0,1] : \gamma(t) \notin U\}$ and $t_2 = \sup\{t \in [0,1] : \gamma(t) \notin U\}$. By definition $\gamma|_{[0,t_1]} \subset D \cap U$ and $\gamma|_{[t_2,1]} \subset D \cap U$. Finally, by Corollary 3.5

$$\begin{aligned}
\ell_F(\gamma) &\geq \ell_F(\gamma|_{[0,t_1]}) + \ell_F(\gamma|_{[t_2,1]}) \\
&\geq d(x,\gamma(t_1)) + d(\gamma(t_2),y) \\
&\geq 2\log\left(\frac{1}{\sqrt{h(x)h(y)}}\right) - B \\
&\geq 2\log(1 + a_D(x,y)) - B \\
&\geq 2\log(1 + A^{-1}a_D(x,y)) \\
&= d_D^{A^{-1}}(x,y).
\end{aligned}$$

So in both cases, if we take the infimum over all piecewise $\mathcal{C}^1$-smooth curves connecting x with y we obtain the lower bound

$$d(x,y) \geq d_D^{A^{-1}}(x,y).$$

This completes the proof. □

We conclude the section by noticing that the estimate (1.4) holds in a neighborhood of the boundary if all the boundary points satisfy the assumptions of the main theorem.

Corollary 3.6. *Let $D \subset \mathbb{R}^d$ be a bounded domain. Let $F\colon D \times \mathbb{R}^d \to [0,+\infty)$ be a Finsler metric on D and assume that for all $\xi \in \partial D$ there exists a neighborhood U_ξ such that* (1.3) *holds. Then there exist a neighborhood U of ∂D and $0 < c \leq 1 \leq C$ such that for all $x,y \in D \cap U$*

$$d_D^c(x,y) \leq d(x,y) \leq d_D^C(x,y).$$

4. Finsler metrics that satisfy the assumptions of Theorem 1.1

In this section, we prove that several natural metrics in domains of $\mathbb{R}^d$ satisfy the assumptions of Theorem 1.1, including the Kobayashi-Hilbert minimal metric in strongly convex points, the minimal metric in convex and strongly minimally convex points, and the k-quasi-hyperbolic metric metric, recently introduced by Zimmer and Wang in [13], in k-strongly convex domains.

The upper bounds for these metrics all arise from the $\mathcal{C}^2$ regularity of the boundary and the decreasing property with respect to the Beltrami-Klein metric of the ball: let $\mathbb{B}^d \subset \mathbb{R}^d$ be the unit ball, then the *Beltrami-Klein metric* of $\mathbb{B}^d$ is

$$\mathcal{CK}_{\mathbb{B}^d}(x,v) = \left(\frac{(1-||x||^2)||v||^2 + |\langle x,v\rangle|^2}{(1-||x||^2)^2}\right)^{1/2} = \left(\frac{||v||^2}{1-||x||^2} + \frac{|\langle x,v\rangle|^2}{(1-||x||^2)^2}\right)^{1/2}.$$

By composing a translation and a dilation, we obtain the Beltrami-Klein metric for a general Euclidean ball.

Proposition 4.1. *Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a domain and $\xi \in \partial D$ be a $\mathcal{C}^2$-smooth boundary point. Let $F\colon D \times \mathbb{R}^d \to [0,+\infty)$ be a Finsler metric on D with the property that if B is an Euclidean ball contained in D then*

$$F(x,v) \leq \mathcal{CK}_B(x,v), \qquad \forall x \in B \subseteq D, v \in \mathbb{R}^d.$$

Then there exists U neighborhood of ξ and $C_1, C_2 > 0$ such that

$$F(x,v) \leq \left((1 + C_1\delta_D(x)) \frac{\|v_N\|^2}{4\delta_D(x)^2} + C_2 \frac{\|v_T\|^2}{\delta_D(x)} \right)^{1/2}.$$

The proposition follows immediately from the following upper bound estimate in the ball.

Lemma 4.2. *Let $B_r \subset \mathbb{R}^d$ $(d \geq 2)$ be an Euclidean ball of radius $r > 0$. Then for all $x \in B_r$ different from the center and $v \in \mathbb{R}^d$ we have*

$$\mathcal{CK}_{B_r}(x,v) \leq \left((1 + 3r^{-1}\delta_{B_r}(x)) \frac{\|v_N\|^2}{4\delta_{B_r}(x)^2} + (1 + r^{-1}\delta_{B_r}(x)) \frac{1}{2r} \frac{\|v_T\|^2}{\delta_{B_r}(x)} \right)^{1/2}.$$

Proof. First of all, $\mathcal{CK}_{\mathbb{B}^d}$ can be rewritten in the following way if x is not the origin

$$\mathcal{CK}_{\mathbb{B}^d}(x,v) = \left(\frac{\|v_N\|^2}{(1 - \|x\|^2)^2} + \frac{\|v_T\|^2}{1 - \|x\|^2} \right)^{1/2}.$$

So

$$\mathcal{CK}_{B_r}(x,v) = r^{-1}\mathcal{CK}_{\mathbb{B}^d}(r^{-1}x, v) = \left(\frac{r^2}{(r + \|x\|)^2} \frac{\|v_N\|^2}{(r - \|x\|)^2} + \frac{1}{r + \|x\|} \frac{\|v_T\|^2}{r - \|x\|} \right)^{1/2}.$$

Now, since $\frac{4r^2}{(r+t)^2} \leq 1 + \frac{3}{r}(r - t)$ and $\frac{1}{r+t} \leq \frac{1}{2r}(1 + \frac{1}{r}(r - t))$ for $t \in [0, r]$, we have

$$\mathcal{CK}_{B_r}(x,v) \leq \left((1 + 3r^{-1}\delta_{B_r}(x)) \frac{\|v_N\|^2}{4\delta_{B_r}(x)^2} + (1 + r^{-1}\delta_{B_r}(x)) \frac{1}{2r} \frac{\|v_T\|^2}{\delta_{B_r}(x)} \right)^{1/2}.$$

□

Proof of Proposition 4.1. Let U and ε be as in Lemma 2.1. For every $x \in U$, the ball $B(\pi(x) - \varepsilon n_\varepsilon, \varepsilon)$ is internally tangent at $\pi(x)$, and moreover $x \in B$. Since $\delta_B(x) = \delta_D(x)$, the estimate follows from the assumptions and from Lemma 4.2. □

4.1. **Kobayashi-Hilbert metric.** In [9], Kobayashi introduced a projectively invariant metric in the domains of $\mathbb{R}^d$, generalizing the well-known Hilbert metric in the convex domains of $\mathbb{R}^d$.

Let $I := (-1, 1)$ and $D \subset \mathbb{R}^d$ be a domain. The *Kobayashi-Hilbert metric* is the Finsler metric $k_D \colon D \times \mathbb{R}^d \to [0, +\infty)$ defined as

$$k_D(x,v) := \inf\{1/|r| : f \colon I \to D \text{ projective map}, f(0) = x, f'(0) = rv\}.$$

The *Kobayashi-Hilbert pseudodistance* K_D of D is the intrinsic distance of k_D.

The Kobayashi-Hilbert pseudodistance can also be characterized in the following way: it is the largest pseudodistance such that for every projective map $f \colon I \to D$ we have for all $s, t \in I$

$$K_D(f(s), f(t)) \leq H_I(s,t) := \frac{1}{2} \left| \log \left(\frac{s+1}{s-1} \cdot \frac{t+1}{t-1} \right) \right|.$$

Notice that from the definition that the Kobayashi-Hilbert metric has the *decreasing property*, i.e., if $D_1 \subseteq D_2$, then

$$k_{D_2}(x,v) \leq k_{D_1}(x,v)$$

for all $x \in D_1$ and $v \in \mathbb{R}^d$.

It turns out that Kobayashi-Hilbert metric has an explicit expression: set

$$Funk(x,v) := \sup\{t > 0 : x + t^{-1}v \notin D\}$$

with the notation $\sup \varnothing = 0$, then

$$k_D(x, v) = \frac{1}{2}(Funk(x, v) + Funk(x, -v)).$$

As happens in complex geometry with the Kobayashi and Carathéodory metrics, we can dualize the definition of Kobayashi-Hilbert distance and introduce the *Carathéodory-Hilbert distance* in the following way

$$C_D(x, y) := \sup\{H_I(f(x), f(y)) : f\colon D \to I \text{ projective map}\}.$$

From Schwarz's lemma from projective self-maps on I, it follows that $C_D \leq K_D$. Moreover, if we denote by $\hat{D}$ the convex hull of D, then

$$C_D = C_{\hat{D}}.$$

Finally, if D is convex, the two distances coincide and are equal to the Hilbert metric.

In order to obtain estimates for the Kobayashi-Hilbert metric near strongly convex points, we need a good lower bound in the case of the ball. Recall that the Hilbert metric in the ball coincides with the Beltrami-Klein metric.

Lemma 4.3. *Let $B_r \subset \mathbb{R}^d$ $(d \geq 2)$ be an Euclidean ball of radius $r > 0$. Then for all $x \in B_r$ that is not the center and $v \in \mathbb{R}^d$ we have*

$$\mathcal{CK}_{B_r}(x, v) \geq \left(\frac{\|v_N\|^2}{4\delta_D(x)^2} + \frac{1}{2r} \frac{\|v_T\|^2}{\delta_D(x)} \right)^{1/2}.$$

Proof. It easily follows from the calculations in the proof of Lemma 4.2. □

We can now prove the lower estimates at strongly convex points, assuming that the domain is convex and bounded.

We recall that, given a domain $D \subseteq \mathbb{R}^d$, a boundary point $\xi \in \partial D$ is said to be *strongly convex* if, there exists a (or equivalently, for all) $\mathcal{C}^2$-smooth local defining function $\rho\colon U \to \mathbb{R}$ at ξ such that $Hess_\xi(\rho)$, the Hessian of ρ at ξ, is positive definite on the tangent space $T_\xi \partial D := \{v \in \mathbb{R}^d : \langle v, n_\xi \rangle = 0\}$.

Proposition 4.4. *Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded convex domain and $\xi \in \partial D$ a strongly convex boundary point. Then there exist a neighborhood U of ξ and $c > 0$ such that all $x \in D \cap U$ and $v \in \mathbb{R}^d$*

$$k_D(x, v) \geq \left(\frac{\|v_N\|^2}{4\delta_D(x)^2} + c\frac{\|v_T\|^2}{\delta_D(x)} \right)^{1/2}.$$

Proof. Since D is bounded convex and ξ is strongly convex, it is contained in a Euclidean ball B_R of radius R tangent at ξ. The estimate follows immediately from Lemma 4.3 and the decreasing property of the Kobayashi-Hilbert metric. □

In order to obtain a lower estimate in the case where we do not assume global convexity, we need the following strong localization result. If $D \subset \mathbb{R}^d$ is a convex domain, we say that a boundary point $\xi \in \partial D$ is *strictly convex* if for all $\eta \in \partial D \backslash \{\xi\}$ the segment (ξ, η) is contained in D. Moreover, we define

$$\delta_D(x, v) := d_{Euc}(x, (x + \mathbb{R}v) \cap \partial D).$$

Proposition 4.5 (Strong localization of Kobayashi-Hilbert metric)**.** *Let $D \subset \mathbb{R}^d$ be a domain and $\xi \in \partial D$ be a boundary point. Assume there exists a neighborhood U of ξ such that $D \cap U$ is convex and ξ is a strictly convex for $D \cap U$. Then there exists a neighborhood $V \subset\subset U$ of ξ and $C > 0$ such that for all $x \in D \cap V$ and $v \in \mathbb{R}^d$*

$$k_{D\cap U}(x, v) \leq (1 + C\delta_D(x))k_D(x, v). \tag{4.1}$$

Proof. Let $\mathbb{S}^{d-1}$ denote the unit sphere in $\mathbb{R}^d$. Set $D' = D \cap U$. Let $x \in D'$ and $v \in \mathbb{S}^{d-1}$, then if the two intersections of $x + \mathbb{R}v$ with $\partial D'$ are both in ∂D, then $k_D(x, v) = k_{D'}(x, v)$, and therefore (4.1) holds.

For this reason we focus on the case where at least one of the two intersections is not in ∂D.

Claim 1: There exists $V_1 \subset\subset U$ neighborhood of ξ such that for all $x \in D \cap V_1$ and $v \in \mathbb{S}^{d-1}$ we have $\delta_D(x, v) = \delta_{D'}(x, v)$.

Proof. By contradiction, there exists a sequence of points x_n in D converging to ξ and a sequence of lines l_n passing through x_n such that their intersections with $\partial D'$ (which we denote by a_n and b_n) are both in ∂U.

Up to subsequences, we can assume that $a_n \to a_\infty$ and $b_n \to b_\infty$. Notice that a_∞, b_∞ and ξ are distinct. This means that the segment $[a_\infty, b_\infty]$ is contained in $\partial D'$, but $\xi \in (a_\infty, b_\infty)$, violating the strict convexity of ξ. □

Let $\pi(x) \in \partial D'$ be a closest point (not necessarily unique), and set $n_x := \frac{\pi(x)-x}{\|\pi(x)-x\|}$ and $v_N := \langle v, n_x\rangle n_x$.

Claim 2: There exist $V_2 \subseteq V_1$ neighborhood of ξ and $c_1 > 0$ such that if $x \in D \cap V_2$ and $v \in \mathbb{S}^{d-1}$, then if at least one of the two intersections of $x + \mathbb{R}v$ with $\partial D'$ is not in ∂D, then $\|v_N\| \geq c_1 > 0$.

Proof. By contradiction, there exists $x_n \to \xi$ and $v_n \in \mathbb{S}^{d-1}$ with $\|(v_n)_N\| \to 0$ such that one of the intersections of $l_n := x_n + \mathbb{R}v_n$ with $\partial D'$ is not in ∂D (let us denote it by a_n). Up to subsequence, we can suppose that $a_n \to a_\infty \in \partial D'$ and $v_n \to v_\infty \in \mathbb{S}^{d-1}$. In this way, l_n converges to a line l_∞ given by $\xi + \mathbb{R}v_\infty$. Notice that l_∞ is a line tangent at ξ, since is the limit of tangent lines $\pi(x_n) + \mathbb{R}(v_n - (v_n)_N)$. Finally, since $a_\infty \in l_\infty$ and $a_\infty \neq \xi$, the segment $[\xi, a_\infty]$ is contained in $\partial D'$, violating the strict convexity of ξ. □

Claim 3: For all $x \in V_2$ and $v \in \mathbb{S}^{d-1}$ we have

$$\frac{\delta_{D'}(x)}{\delta_{D'}(x, v)} \geq \|v_N\|.$$

Proof. Consider the half-space tangent at $\pi(x)$

$$H := \{y \in \mathbb{R}^d : \langle y - \pi(x), x - \pi(x)\rangle > 0\}.$$

Since $D' \subset H$ and $\delta_{D'}(x) = \delta_H(x)$, we have

$$\frac{\delta_{D'}(x)}{\delta_{D'}(x, v)} \geq \frac{\delta_H(x)}{\delta_H(x, v)} = \|v_N\|.$$

□

Let $x \in V_2$ and $v \in \mathbb{S}^{d-1}$. If least one of the two intersections of $x + \mathbb{R}v$ with $\partial D'$ is not in ∂D, by Claim 2 $\|v_N\| > c_1 > 0$, then if we set $c_2 := d_{Euc}(D \cap V_2, D \backslash U) > 0$ we have by Claim 3

$$\begin{aligned}
k_{D'}(x,v) &\leq \frac{1}{2}\left(\frac{1}{\delta_{D'}(x,v)} + \frac{1}{c_2}\right) \\
&\leq \frac{1}{2}\left(\frac{1}{\delta_{D'}(x,v)} + \frac{1}{c_1c_2}\|v_N\|\right) \\
&\leq \frac{1}{2}\left(\frac{1}{\delta_{D'}(x,v)} + \frac{1}{c_1c_2}\frac{\delta_{D'}(x)}{\delta_{D'}(x,v)}\right) \\
&= (1 + C\delta_{D'}(x))\frac{1}{2\delta_{D'}(x,v)} \\
&= (1 + C\delta_{D'}(x))\frac{1}{2\delta_{D}(x,v)} \\
&\leq (1 + C\delta_D(x))k_D(x,v).
\end{aligned}$$

□

Remark 4.6. If, under the assumptions of the previous proposition, we add the further assumption that D is hyperbolic (that is, does not contain affine lines), then we have (4.1) for every relatively compact V in U (with the constant C that depends on V).

Finally, by combining the localization result with Proposition 4.4, we obtain.

Proposition 4.7. *Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a domain and $\xi \in \partial D$ be a strongly convex boundary point. Then there exist a neighborhood U of ξ and $c_1, c_2 > 0$ such that all $x \in D \cap U$ and $v \in \mathbb{R}^d$*

$$k_D(x,v) \geq \left((1 - c_1\delta_D(x))\frac{\|v_N\|^2}{4\delta_D(x)^2} + c_2\frac{\|v_T\|^2}{\delta_D(x)}\right)^{1/2}.$$

Similarly to [1, Proposition 1.2], one may prove a more precise estimate.

Proposition 4.8. *Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a domain and $\xi \in \partial D$ be a strongly convex boundary point. For any $\varepsilon \in (0,1)$ there exist a neighborhood of U of ξ and $c > 0$ such that for all $x \in D \cap U$ and $v \in \mathbb{R}^d$*

$$\begin{aligned}
\left((1 - c\delta_D(x))\frac{\|v_N\|^2}{4\delta_D(x)^2} + (1-\epsilon)\frac{H(\pi(x); v_T)}{2\delta_D(x)}\right)^{1/2} &\leq k_D(x,v) \\
&\leq \left((1 + c\delta_D(x))\frac{\|v_N\|^2}{4\delta_D(x)^2} + (1+\epsilon)\frac{H(\pi(x); v_T)}{2\delta_D(x)}\right)^{1/2},
\end{aligned}$$

where $H(\pi(x); \cdot)$ is the Hessian of the signed distance to the boundary at $\pi(x)$.

4.2. **Minimal metric.** In real Euclidean space, Forstnerič and Kalaj in [7] defined the *minimal metric*, the analog of the Kobayashi metric in the theory of minimal surfaces.

A map $f \colon \mathbb{D} \to \mathbb{R}^d$ $(d \geq 2)$ is said to be *conformal* if for all $\zeta \in \mathbb{D}$ we have

$$||f_x(\zeta)|| = ||f_y(\zeta)|| \text{ and } \langle f_x(\zeta), f_y(\zeta)\rangle = 0$$

where $\zeta = (x, y)$ are the coordinates of $\mathbb{D} \subset \mathbb{R}^2$. Moreover, we say that f is *harmonic* if every component of f is harmonic. If $D \subset \mathbb{R}^d$ $(d \geq 3)$ is a domain we denote by $\mathrm{CH}(\mathbb{D}, D)$ the space of conformal harmonic maps $f \colon \mathbb{D} \to D$.

The *minimal metric* of D is the Finsler metric given by

$$g_D(x,v) = \inf\{1/r : f \in \mathrm{CH}(\mathbb{D}, D), f(0) = x, f_x(0) = rv\}, \quad x \in D, v \in \mathbb{R}^d,$$

and the associated intrinsic distance $\rho_D \colon D \times D \to [0, +\infty)$ is called *minimal pseudodistance.*

As is clearly evident from the definition, the minimal metric also satisfies the decreasing property.

We now introduce the concept of a strongly minimal convex point, which is the analogue in minimal surface theory of strongly convex points in Hilbert geometry and strongly pseudoconvex points in complex analysis.

Definition 4.9 (Minimal strongly convex)**.** Let $D \subset \mathbb{R}^d$ $(d \geq 3)$ be a domain. The boundary point $\xi \in \partial D$ is called *strongly minimally convex* if there exists (or equivalently, for all) $\mathcal{C}^2$-smooth local defining function $\rho \colon U \to \mathbb{R}$ at ξ such that the smallest two eigenvalues λ_1 and λ_2 of $Hess_\xi(\rho)|_{T_\xi \partial D}$ satisfies

$$\lambda_1 + \lambda_2 > 0.$$

Note that a strongly convex boundary point is strongly minimally convex.

For bounded strongly minimally convex domains (i.e., every boundary point is strongly minimally convex), Drinovec-Drnovšek and Forstnerič have shown in [4] that we have the following lower estimate for the minimal metric: let $D \subset \mathbb{R}^d$ bounded strongly minimally convex, then there exists $c_1 > 0$ such that for all $x \in D$ and $v \in \mathbb{R}^d$

$$g_D(x,v) \geq c_1 \frac{\|v\|}{\delta_D(x)^{1/2}}. \tag{4.2}$$

Moreover, there exist $c_2 > 0$ such that for all $x \in D$ close to ∂D and $v \in \mathbb{R}^d$ we have

$$g_D(x,v) \geq c_2 \frac{\|v_N\|}{\delta_D(x)}.$$

The previous estimates were sufficient for the first author to prove the Gromov hyperbolicity of bounded strongly minimally convex domains (see [5]). However, for the second estimate (in the normal direction), it is not clear how to improve it to fall within the hypothesis of Theorem 1.1. For this reason, we restricted ourselves to the convex and locally convex case.

Note that at a *locally convex* boundary point $\xi \in \partial D$, that is, there exists a neighborhood U of ξ such that $D \cap U$ is convex, ξ is strongly minimally convex if and only if then $Hess_\xi(\rho)|_{T_\xi \partial D}$ has at least $d-2$ positive eigenvalues, i.e., it is 2-strongly convex in the sense of Definition 4.12.

Proposition 4.10. *Let $D \subset \mathbb{R}^d$ $(d \geq 3)$ be a convex domain and $\xi \in \partial D$ be a point strongly minimally convex, then there exist a neighborhood U of ξ and $c > 0$ such that for all $x \in D \cap U$ and $v \in \mathbb{R}^d$ we have*

$$g_D(x,v) \geq \max\left\{\frac{\|v_N\|}{2\delta_D(x)}, c_2 \frac{\|v_T\|}{\delta_D(x)^{1/2}}\right\}.$$

Proof. The tangential component easy follows from (4.2) and the localization result [4, Theorem 8.5].

For the normal component, let $x \in D$ close to ξ, and consider the half-space

$$H := \{x \in \mathbb{R}^d : \langle x - \pi(x), n_{\pi(x)} \rangle < 0\}.$$

By convexity $D \subset H$, the decreasing property of the minimal metric and [5, Lemma 5.3]

$$g_D(x, v) \geq g_H(x, v) = \frac{\|v_N\|}{2\delta_D(x)}.$$

□

Finally, from the strong localization result in [4, Theorem 8.5], we can slightly relax the convexity assumption, passing from global to local convexity.

Proposition 4.11. *Let $D \subset \mathbb{R}^d$ be a domain and $\xi \in \partial D$ be a strongly minimally convex and locally convex boundary point. Then there exist a neighborhood U of ξ and $c_1, c_2 > 0$ such that for all $x \in D \cap U$ and $v \in \mathbb{R}^d$ we have*

$$g_D(x, v) \geq \max\left\{(1 - c_1\delta_D(x))\frac{\|v_N\|}{2\delta_D(x)}, c_2\frac{\|v_T\|}{\delta_D(x)^{1/2}}\right\}.$$

4.3. **k-quasi-hyperbolic metric.** In this last subsection, we study a metric recently introduced by Wang and Zimmer in [13]. Let $D \subset \mathbb{R}^d$ be a domain and let $k \in \{1, \ldots, d\}$. For all $x \in D$ and $v \in \mathbb{R}^d$, define

$\delta_D^{(k)}(x, v) := \sup\{d_{Euc}(x, (x+V) \cap \partial D) : V \subseteq \mathbb{R}^d$ a k-dimensional linear subspace with $v \in V\}$.

Then the *(generalized) k-quasi-hyperbolic metric* on D is defined by

$$q_D^{(k)}(x, v) := \frac{\|v\|}{2\delta_D^{(k)}(x, v)},$$

where $x \in D$ and $v \in \mathbb{R}^d$. We denote with $d_D^{(k)}$ the associated intrinsic distance.

Note that, compared to the initial definition by Wang and Zimmer, we normalize the metric with a multiplicative factor $\frac{1}{2}$, since our goal is to prove estimates as in the hypothesis of Theorem 1.1.

The k-quasi-hyperbolic metric has a strong relationship with several important metrics

(1) $q_D^{(d)}$ is the quasi-hyperbolic metric;
(2) $q_D^{(1)}$ is bi-Lipschitz to the Kobayashi-Hilbert metric;
(3) D is convex and $d \geq 3$, $q_D^{(2)}$ is bi-Lipschitz to the the minimal metric (see [13, Proposition 10.1]).

Let us now introduce the natural domains where we study the k-quasi-hyperbolic metric.

Definition 4.12 (k-strongly convex domains)**.** Let $D \subset \mathbb{R}^d$ ($d \geq 2$) be a domain. The boundary point $\xi \in \partial D$ is called *k-strongly convex boundary point* if it is locally convex and if there exists (or equivalently, for all) $\mathcal{C}^2$-smooth local defining function $\rho \colon U \to \mathbb{R}$ at ξ such that $Hess_\xi(\rho)|_{T_\xi \partial D}$ has at least $d - k$ positive eigenvalues.

Remark 4.13. The condition on the Hessian mentioned above is equivalent to requiring that the local defining function is strongly k-plurisubharmonic in the sense of [8, 6].

Notice that a boundary point is 1-strongly convex boundary point if and only if is strongly convex, and it is 2-strongly convex if and only if it is locally convex and strongly minimally convex.

The following property can be viewed as the real and k-dimensional analogue of 2-convexity in the sense of Mercer [11].

Proposition 4.14. *Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a convex domain and $\xi \in \partial D$ be k-strongly convex boundary point, then there exist a neighborhood U of ξ and $C > 0$ such that for al $x \in D \cap U$, $v \in \mathbb{R}^d$*

$$\delta_D^{(k)}(x, v) \leq C\delta_D(x)^{1/2}.$$

Proof. Since k-strong convexity is an open condition, we can find a neighborhood U of ξ such that $\pi(x)$ is k-strongly convex for all $x \in U$. Let $x \in D \cap U$. By the k-strong convexity, there exist $\lambda_1(x), \dots, \lambda_{d-k}(x)$ positive eigenvalues and $v_1(x), \dots, v_{d-k}(x)$ orthonormal eigenvectors of $Hess_{\pi(x)}(\rho)|_{T_{\pi(x)}\partial D}$. By compactness we can find $\lambda > 0$ such that $\lambda < \lambda_j(x)$ for all $x \in D \cap U$ and $j = 1, \dots, d-k$. This means that

$$D \subseteq D_{\pi(x)} := \left\{ y \in \mathbb{R}^d : \langle y - \pi(x), n_{\pi(x)} \rangle + \frac{\lambda}{2} \sum_{j=1}^{d-k} \langle y - \pi(x), v_j(x) \rangle^2 < 0 \right\}.$$

Notice that $\delta_D(x) = \delta_{D_{\pi(x)}}(x)$. Finally, a simple calculation shows that there exists $C > 0$ such that for all $v \in \mathbb{R}^d$

$$\delta^{(k)}_{D_{\pi(x)}}(x, v) \leq C\delta_{D_{\pi(x)}}(x)^{1/2},$$

and so

$$\delta_D^{(k)}(x, v) \leq \delta^{(k)}_{D_{\pi(x)}}(x, v) \leq C\delta_{D_{\pi(x)}}(x)^{1/2} = C\delta_D(x)^{1/2}.$$

□

Let us now study the case of the Euclidean ball, showing that all the metrics coincide (except for the quasi-hyperbolic one).

Remark 4.15. Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a domain, then by definition for all $1 \leq k_1 \leq k_2 \leq d$, we have for $x \in D$ and $v \in \mathbb{R}^d$

$$q_D^{(k_1)}(x, v) \leq q_D^{(k_2)}(x, v).$$

Proposition 4.16. *Let $d \geq 2$. For all $k \in \{1, \dots, d-1\}$ we have*

$$q^{(k)}_{\mathbb{B}^d} = q^{(1)}_{\mathbb{B}^d}.$$

Proof. Since for all $x \in \mathbb{B}^d$ and $v \in \mathbb{R}^d$

$$q^{(1)}_{\mathbb{B}^d}(x, v) \leq \cdots \leq q^{(k)}_{\mathbb{B}^d}(x, v) \leq \cdots \leq q^{(d-1)}_{\mathbb{B}^d}(x, v),$$

it is sufficient to prove that

$$q^{(d-1)}_{\mathbb{B}^d}(x, v) \leq q^{(1)}_{\mathbb{B}^d}(x, v).$$

The result is obvious if $v = 0$, so we may assume $v \neq 0$. Let $\xi \in \partial\mathbb{B}^d$ be a boundary point such that $\delta^{(1)}_{\mathbb{B}^d}(x, v) = \|x - \xi\|$. Let l be the line joining x with ξ.

If l passes through the origin O, let H any hyperplane containing l, then $\mathbb{B}^d \cap H$ is a ball of dimension $d-1$ centered at O. Since O, x and ξ are collinear, we have

$$\delta^{(d-1)}_{\mathbb{B}^d}(x, v) \geq d_{Euc}(x, (x + H) \cap \partial\mathbb{B}^d) = \|x - \xi\| = \delta^{(1)}_{\mathbb{B}^d}(x, v),$$

and so $q^{(d-1)}_{\mathbb{B}^d}(x, v) \leq q^{(1)}_{\mathbb{B}^d}(x, v)$.

If, on the other hand, l does not pass through the origin, let O' the projection of O onto l. Now let H be the affine hyperplane containing l and orthogonal to $\overrightarrow{OO'}$. Notice that $\mathbb{B}^d \cap H$ is a ball of dimension $d-1$, centered at O'. Since O', x and ξ are collinear, we obtain again $q^{(d-1)}_{\mathbb{B}^d}(x, v) \leq q^{(1)}_{\mathbb{B}^d}(x, v)$. □

From an explicit calculation, we can obtain the expression of the $q_{\mathbb{H}^d}^{(k)}$ metrics in the half-space $\mathbb{H}^d$.

Proposition 4.17. *Let $\mathbb{H}^d := \{x \in \mathbb{R}^d : x_1 > 0\}$ $(d \geq 2)$ be the half-space. Then for all $k \in \{1, \dots, d-1\}$, $x = (x_1, \dots, x_d) \in \mathbb{H}^d$ and $v = (v_1, \dots, v_d) \in \mathbb{R}^d$, we have*

$$q_{\mathbb{H}^d}^{(k)}(x, v) = \frac{|v_1|}{2x_1}.$$

Finally, we can prove the necessary estimates for Theorem 1.1 at k-strongly convex points.

Proposition 4.18. *Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a convex domain and let $\xi \in \partial D$ be a k-strongly convex boundary point, then there exist $c, C_1, C_2 > 0$ and a neighborhood U of ξ such that for all $x \in D \cap U$ and $v \in \mathbb{R}^d$ we have*

$$\max\left\{\frac{\|v_N\|}{2\delta_D(x)}, c\frac{\|v_T\|}{\delta_D(x)^{1/2}}\right\} \leq q_D^{(k)}(x, v) \leq \left((1 + C_1\delta_D(x))\frac{\|v_N\|^2}{4\delta_D(x)^2} + C_2\frac{\|v_T\|^2}{\delta_D(x)}\right)^{1/2}.$$

Proof. **Upper bound:** Let B be an Euclidean ball. Then by Proposition 4.16 and [13, Propotision 10.1] for all $x \in B$ and $v \in \mathbb{R}^d$ we have (being careful with the different normalization of $q_D^{(k)}$)

$$q_B^{(k)}(x, v) = q_B^{(2)}(x, v) \leq \mathcal{C}\mathcal{K}_B(x, v).$$

So the upper estimate follows from Proposition 4.1.

Lower bound: First of all, by Proposition 4.14 there exists $c > 0$ such that for all $x \in D$ and $v \in \mathbb{R}^d$

$$q_D^{(k)}(x, v) = \frac{\|v\|}{2\delta_D^{(k)}(x, v)} \geq c\frac{\|v\|}{\delta_D(x)^{1/2}}.$$

For the normal component, it is sufficient to reason as in Proposition 4.10. □

The estimates from the main theorem immediately imply the Gromov hyperbolicity of the k-quasi hyperbolic metric in *k-strongly convex domains*, which are bounded convex domains where all the boundary points are k-strongly convex. For more details on Gromov hyperbolicity, see [2].

Corollary 4.19. *Let $d \geq 2$ and $k \in \{1, \cdots, d-1\}$. If $D \subset \mathbb{R}^d$ is a k-strongly convex domain, then $(D, d_D^{(k)})$ is Gromov hyperbolic.*

Proof. From Corollary 3.6, there exists a neighborhood U of the boundary such that (1.4) holds. This implies that there exists $B > 0$ such that for all $x, y \in D \cap U$

$$2\log\left(\frac{\|\pi(x) - \pi(y)\| + h(x) \vee h(y)}{\sqrt{h(x)h(y)}}\right) - B \leq d_D^{(k)}(x, y) \leq 2\log\left(\frac{\|\pi(x) - \pi(y)\| + h(x) \vee h(y)}{\sqrt{h(x)h(y)}}\right) + B.$$

The proof concludes as in [1, Theorem 1.4] and [5, Proposition 4.4]. □

Note that the Gromov hyperbolicity for the k-quasi hyperbolic metric in convex domains has been characterized by Wang and Zimmer in [13, Theorem 1.5].

We conclude with a remark.

Remark 4.20. Let $D \subset \mathbb{R}^d$ be a k_1-strongly-convex domain, then for all $k_1 < k_2 \leq d - 1$ there exists $A > 1$ such that for all $x \in D$ and $v \in \mathbb{R}^d$

$$q_D^{(k_1)}(x, v) \leq q_D^{(k_2)}(x, v) \leq Aq_D^{(k_1)}(x, v).$$

This is a consequence of the Theorem 1.1 and Proposition 3.1.

5. A rigidity result in convex geometry

In this section, we will characterize convex domains $D \subsetneq \mathbb{R}^d$ where

$$\delta_D^{(1)}(x,v) = \delta_D^{(d-1)}(x,v), \qquad \forall x \in D, v \in \mathbb{R}^d. \tag{5.1}$$

Let us begin with some basic notions of convex geometry.

Let $D \subsetneq \mathbb{R}^d$ be a convex domain. Let $a \in \partial D$. An affine hyperplane H passing at a is a *supporting hyperplane* at a if $D \cap H = \varnothing$. A *normal line* at a is a line passing through a, orthogonal to a supporting hyperplane at a. A *face* is the convex subset $\partial D \cap H$, where H is a supporting hyperplane.

Let $a \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$ non zero. A *half-space* is a domain of the form

$$\{x \in \mathbb{R}^d : \langle x - a, v\rangle < 0\}.$$

A *slab* is a domain of the form

$$\{x \in \mathbb{R}^d : |\langle x - a, v\rangle| < 1\}.$$

We can now state the main result of this section

Theorem 5.1. *Let $D \subsetneq \mathbb{R}^d$ $(d \geq 3)$ be a convex domain. Then* (5.1) *holds if and only if D is either*

(1) *an Euclidean ball;*
(2) *a half-space;*
(3) *a slab.*

Proof. We already proved that in the Euclidean balls (5.1) holds (see Proposition 4.16). It is not difficult to see that the same is true for the half-spaces and slabs.

For the other direction, let $D \subsetneq \mathbb{R}^d$ be a convex domain such that (5.1) holds. We divide the proof in several parts.

First of all notice that if D has at most 2 faces, then it is either a half-space or slab, so we may assume that D has at least 3 faces.

Part 1: Let $a, b \in \partial D$ not on the same face, and let l_a, l_b be two normal lines at a and b respectively. Then l_a and l_b are coplanar.

Since a and b are not in the same face, $(a,b) \subset D$. Let H_a and H_b be two supporting hyperplanes at a and b orthogonal to the lines l_a and l_b respectively. Consider $m = \frac{a+b}{2}$ the midpoint between a and b, and $v = \overrightarrow{ab}$. By (5.1) we have

$$\delta_D^{(1)}(m,v) = \delta_D^{(d-1)}(m,v).$$

which means that there exists an affine hyperplane H passing through a and b such that

$$B\left(m, \frac{|\overrightarrow{ab}|}{2}\right) \cap H \subset D.$$

This implies that $H_a \cap H$ and $H_b \cap H$ are parallel. Moreover, $\overrightarrow{ab} \perp (H_a \cap H)$ and $\overrightarrow{ab} \perp (H_b \cap H)$. Consequently, $l_a \perp (H_a \cap H)$ and $l_b \perp (H_b \cap H)$, so since $\operatorname{codim}(H_a \cap H) = \operatorname{codim}(H_b \cap H) = 2$, the two lines l_a and l_b are coplanar.

Part 2: Each point on the boundary is an extreme point, meaning that all the faces are singletons.

By contradiction, let $a, b \in \partial D$ be two distinct boundary points in the same face F of D. By definition, there exists an hyperplane H such that $F = \partial D \cap H$. Let l_a and l_b be the two lines perpendicular to H, passing through a and b, respectively. Clearly, l_a and l_b are coplanar, so there exists a plane Π that contains them. Now since $d \geq 3$ we can find a boundary point $c \in \partial D \backslash (F \cup \Pi)$ and a normal line l_c at c that is not parallel at Π. Such a line exists because otherwise D is a cylinder over Π, and it is not difficult to see that the only cylinders where (5.1) holds are half-spaces and slabs.

Since l_a and l_c are coplanar but not parallel, they intersect at one point. The same is true for l_b and l_c, and this implies that l_c lies in the plane Π, leading to a contradiction.

Part 3: All the normal lines intersect at one point O.

Let $a, b \in \partial D$ be two distinct boundary points with normal lines l_a and l_b. From Part 1, there exists a plane Π that contains l_a and l_b. Consider $c \in \partial D \backslash \Pi$ and a normal line l_c at c that is not parallel to Π. We conclude as in Part 2.

Part 4: Each point $a \in \partial D \backslash \{O\}$ has a unique supporting hyperplane, which implies that $\partial D \backslash \{O\}$ is $\mathcal{C}^1$-smooth hypersurface.

Suppose that there exist two distinct normal lines at $a_0 \in \partial D$. This implies a_0 is the point O of Part 2, and thus, every point $a \in \partial D \backslash \{O\}$ has a unique supporting hyperplane. A classic result from convex geometry (see, for example, [3]) ensures that ∂D is $\mathcal{C}^1$-smooth.

Part 5: D is an Euclidean ball.

Up to a translation, we can suppose that the point O of Part 2 is the origin. Let $f : \mathbb{R}^d \to \mathbb{R}$ the function given by $f(x) := \|x\|$. Let $a \in \partial D \backslash \{O\}$, since the (unique) normal line at a passes through the origin,

$$T_a \partial D = \{v \in \mathbb{R}^d : \langle x, a \rangle = 0\} = (\mathbb{R}a)^{\perp}.$$

Now $df_a|_{T_a \partial D} = 0$, which means that f is constant in $\partial D \backslash \{O\}$, and so ∂D is a sphere centered at the origin. □

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