Triangulations of singular constant curvature spheres via Belyi functions and determinants of Laplacians

Victor Kalvin

Abstract

We study the zeta-regularized spectral determinant of the Friedrichs Laplacians on the singular spheres obtained by cutting and glueing copies of constant curvature (hyperbolic, spherical, or flat) double triangle. The determinant is explicitly expressed in terms of the corresponding Belyi functions and the determinant of the Friedrichs Laplacian on the double triangle. The latter determinant was recently found in a closed explicit form [23]. As examples, we consider the cyclic, dihedral, tetrahedral, octahedral, and icosahedral triangulations, and find the determinant for the spherical, Euclidean, and hyperbolic Platonic surfaces. These surfaces correspond to stationary points of the determinant.

MSC: 58J52

1 Introduction

We study the zeta-regularized spectral determinant of the Friedrichs scalar Laplacian on the surfaces constructed by cutting and gluing copies of a constant curvature S^2-like double triangle. These surfaces' geometric (combinatorial) cutting and gluing schemes are described in terms of the corresponding Belyi functions [3]. Or, equivalently, in terms of *dessins d'enfants* [17].

The Belyi function maps the (source) Riemann surface to the (target) Riemann sphere. Any constant curvature S^2-like double triangle is isometric to the target Riemann sphere with an explicitly constructed conformal metric with three conical singularities [23, Appendix]. The glued surface is isometric to the source Riemann surface equipped with the pullback of the explicit conformal metric by the Belyi function. This provides us with a geodesic triangulation and an explicit uniformization of the glued surface. In particular, in the case of flat right isosceles S^2-like double triangle, we get square-tiled flat surfaces, see e.g. [52, 68] and references therein.

Recall that not all Riemann surfaces can be triangulated via Belyi functions. But for the most interesting surfaces, like the Fermat curve, the Bolza curve, the Hurwitz surfaces, including Klein's quartic, the Platonic surfaces, etc., it is certainly possible thanks to the famous Belyi theorem [3, 11, 16, 27, 53, 67], see also [64]. Due to the highest possible number of authomorphisms [18, 53], many of these surfaces (equipped with natural constant curvature metric) are expected to be stationary points of the determinant.

In this paper, we implement the above program for the genus zero triangulated surfaces. We explicitly express the spectral determinant of any glued surface in terms of the corresponding Belyi function and the spectral determinant of the constant curvature S^2-like double triangle. The latter determinant was recently found in a closed explicit form [23, 26]. In particular, with each bicolored plane tree [4, 5] we naturally associate an infinite family of constant curvature singular spheres, and then the corresponding infinite family of explicit spectral invariants, which are the spectral determinants. For some closely related geometric constructions and invariants see e.g. [19, 46, 47, 55, 60].

Although we do not yet study spectral determinants of any higher genus surfaces, we obtain some closely related new important results. For instance, in examples we explicitly evaluate the spectral determinant of the regular hyperbolic octahedron with vertices of angle π (a double cover is the Bolza curve) and the regular hyperbolic icosahedron tessellated by $(2, 3, 7)$-triangles (this is related to Klein's quartic [49, 27, 53]). Both, the Bolza curve and Klein's quartic are expected to be stationary surfaces for the spectral determinant. An analytical approach that allows one to find the spectral determinant of these and other triangulated hyperbolic surfaces of higher genus in a closed explicit form would be of great interest; for a numerical study, see [56].

Since BFK-type gluing formulae [7] became a standard versatile tool for expressing the determinant of a surface in terms of the determinants of its decomposition parts, one may consider using it for our purposes as well. Unfortunately, none of the presently known BFK-type formulae can be used in our setting. In particular, because of the conical singularities located on the cuts. Instead, we rely on anomaly formulae for the determinants of Laplacians. They allow for conical singularities and express the determinant in terms of the (regularized classical) Liouville action and some terms coming from the uniformization. Let us note, however, that in the proof of those anomaly formulae, a generalization of the BFK formula to the metrics with conical singularities (located outside of the cuts) is an essential tool. The anomaly formulae per se are similar to those in [23, 25].

The anomaly formulae imply that the derivation of closed explicit formulae for the determinant implicitly requires evaluating the (regularized classical) Liouville action. Recall that this is not a simple task even for the S^2-like constant curvature geodesic double triangles [23, 69], which is closely related to the celebrated explicit DOZZ formula of Dorn and Otto [12] and Zamolodchikov and Zamolodchikov [69] for the three-point structure constant of the Liouville conformal field theory. In the frame of this theory, the Liouville action for more than three conical singularities can be found in terms of conformal blocks [69]. However, to the best of our knowledge, no closed explicit formulae for the conformal blocks are available yet even in the relatively simple geometric setting of the ramified coverings of S^2-like double triangles. Moreover, we found it easier to work directly with the determinants of Laplacians, comparing the anomaly formulae for the determinants of Laplacians on the target Riemann sphere and the ramified covering. In particular, this is because the Liouville action does not transform nicely even under the action of the Möbius transformations that leave the determinant invariant. The result we obtain is close in spirit to those in complex dynamics: we study how the determinant changes under the action of a Belyi function. Similarly, one can study the general ramified coverings of genus zero (and one). We do not do this because we have no explicit uniformization of the target Riemann sphere with four or more conical singularities. Recall that finding an explicit uniformization of the constant curvature spheres with four or more conical singularities is an open long-standing problem.

As a byproduct, our approach allows for explicit evaluation of the (regularized classical) Liouville action in terms of the determinants of the Laplacians and other explicit terms of the anomaly formulae coming from the uniformization. This can be of independent interest [9, 69, 59, 38, 44, 45, 57]. It would also be very interesting to obtain explicit formulae for the corresponding conformal blocks, and then use them to reproduce the explicit expressions for the Liouville action.

As is known [9, 59, 70], the Liouville action generates the famous accessory parameters as their common antiderivative. We explicitly express the accessory parameters of the triangulated singular constant curvature spheres in terms of the Belyi functions and the orders of conical singularities. We also find the derivatives of the Liouville action with respect to the orders of conical singularities in the same vein as in the paper of Zamolodchikov and Zamolodchikov [69]. As one can expect, this allows us to show that the Platonic solids are special also in the context of this paper: their surfaces correspond to stationary points of the determinant.

Recall that for smooth metrics on Riemann surfaces extrema of spectral determinants were studied in a series of papers by Osgood, Phillips, and Sarnak [41, 42, 43]; see also [28] for an extension of their results, and [13] for recent closely related results in the four-dimensional case. It is a classical result that on the smooth metrics of a fixed area on a sphere, the determinant achieves its absolute maximum on the metric of the standard round sphere. If one allows for metrics with conical singularities, then a surprising peculiarity is that the determinant of the standard round sphere is not a maximum anymore. Moreover, the determinant grows without any bound as the order (or, equivalently, the angle) of a conical singularity goes to zero.

We illustrate the main results of this paper with a number of examples: We consider the cyclic, dihedral, tetrahedral, octahedral, and icosahedral triangulations [30], and find explicit expressions for the spectral determinant of the corresponding spherical, Euclidean, and hyperbolic Platonic surfaces. In particular, the determinant of the regular hyperbolic octahedron with vertices of angle π is found in Example 5.4. The determinant of the regular hyperbolic icosahedron corresponding to the tessellation by $(2, 3, 7)$-triangles is calculated in Example 5.15.

As the angles of the conical points of a Platonic surface go to zero (and the area remains fixed), the determinant grows without any bound, cf. Fig 7. In the limit, the conical points turn into cusps and we obtain an ideal polyhedron [21, 22, 29]. The spectrum of the corresponding Laplacians is no longer discrete [40].

This paper is organized as follows: Section 2 contains preliminaries and the main results of the paper formulated in Theorem 2.1. In Section 3 we prove Theorem 2.1.

First, in Subsection 3.1, we prove Proposition 3.1, which is a preliminary version of Theorem 2.1. Then, in Subsection 3.2, we refine the result of Proposition 3.1 by using the natural Euclidean equilateral triangulation [53, 65]. This completes the proof of Theorem 2.1. Section 4 is devoted to the uniformization, accessory parameters, Liouville action, and stationary points of the spectral determinant. Finally, Section 5 contains illustrating examples and applications.

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2 Preliminaries and main results

Recall that a (non-constant) meromorphic function $f : X \to \overline{\mathbb{C}}_z$ on a compact Riemann surface X is called a Belyi function, if it is ramified at most above three points [3]. Any Belyi function can alternatively be described via the corresponding *dessin d'enfant*, which is usually defined as the graph formed on X by the preimages $f^{-1}([0, 1])$ of the real line segment $[0, 1]$ with black points placed at the preimages $f^{-1}(0)$ of zero and white points placed at the preimages $f^{-1}(1)$ of 1, e.g. [11, 16, 17, 36, 48, 67]. Any meromorphic function on the Riemann sphere is a rational function. If, in addition, f has only a single pole that is at infinity, then f is a polynomial [4, 5, 36].

Consider a Belyi function $f : \overline{\mathbb{C}}_x \to \overline{\mathbb{C}}_z$ ramified at most above the marked points $z = 0$, $z = 1$, and $z = \infty$. This defines a ramified (branched) covering and a bicolored triangulation of the Riemann sphere $\overline{\mathbb{C}}_x$, e.g. [65, 36]. Namely, the function f maps: *i*) the sides of the bicolored triangles on $\overline{\mathbb{C}}_x$ to the line segments $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$ of the real axis; *ii*) the vertices of the triangles to the marked points 0, 1, and ∞; *iii*) the light-colored triangles to the upper half-plane $\Im z > 0$, and the dark-colored triangles to the lower half-plane $\Im z < 0$. The number of bicolored double triangles on $\overline{\mathbb{C}}_x$ is exactly the degree $\deg f = \max\{\deg P, \deg Q\}$ of f, where $f(x) = P(x)/Q(x)$ with coprime polynomials P and Q.

For example, the Belyi function

$$f(x) = 1728 \frac{x^5(x^{10} - 11x^5 - 1)^5}{(x^{20} + 228(x^{15} - x^5) + 494x^{10} + 1)^3}, \quad \deg f = 60,$$

defines the icosahedral triangulation [31], which corresponds to the tessellation of the standard round sphere with bicolored spherical $(2, 3, 5)$-triangles in Fig. 1, Fig. 3. Here and elsewhere, a (k, ℓ, m)-triangle is a geodesic triangle (spherical, Euclidean, or hyperbolic) with internal angles π/k, π/ℓ, and π/m. As usual, we identify the Riemann sphere $\overline{\mathbb{C}}_x$ with the standard round sphere in $\mathbb{R}^3$ by means of the stereographic projection.

Consider the composition $w_{\boldsymbol\beta} \circ f$ of the (appropriately normalized) Schwarz triangle function $w_{\boldsymbol\beta}$ with a Belyi function $f : \overline{\mathbb{C}}_x \to \overline{\mathbb{C}}_z$. The Schwarz triangle function $\overline{\mathbb{C}}_z \ni z \mapsto w = w_{\boldsymbol\beta}$ maps the upper half-plane $\Im z > 0$ into a triangle. The triangle has an

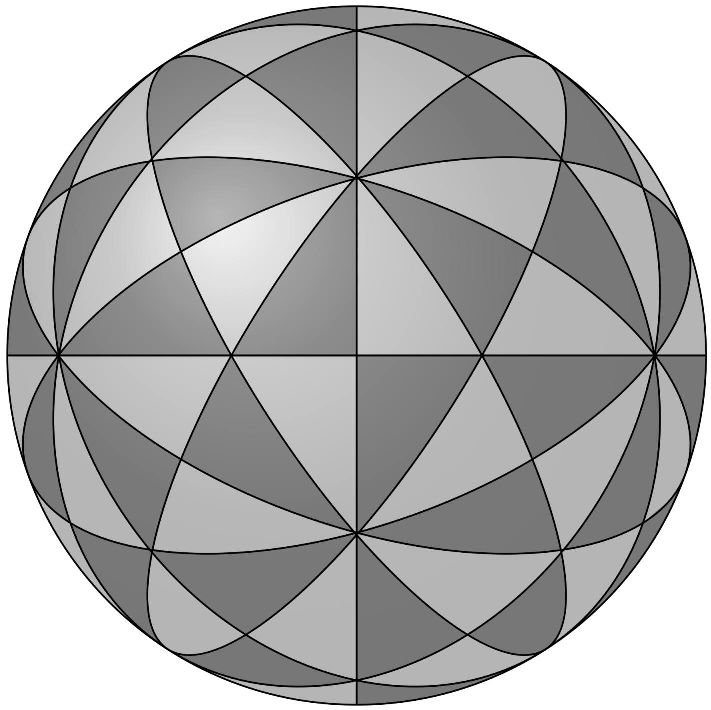

Figure 1: Icosahedral triangulation

internal angle of $\pi(\beta_0+1)$ at the vertex $w_{\boldsymbol\beta}(0)=0$, $\pi(\beta_1+1)$ at the vertex $w_{\boldsymbol\beta}(1)\in\mathbb R$, and $\pi(\beta_\infty+1)$ at the vertex $w_{\boldsymbol\beta}(i\infty)$. This information is encoded as a formal sum in the divisor

$$\boldsymbol\beta=\beta_0\cdot 0+\beta_1\cdot 1+\beta_\infty\cdot\infty.$$

For simplicity, we assume that the weights β_j of the divisor are in the interval $(-1,0]$, or, equivalently, that the angles $\pi(\beta_j+1)$ are positive and do not exceed π; here the subscript runs through the set $\{0,1,\infty\}$ of the marked points on $\overline{\mathbb C}_z$. The triangle is geodesic with respect to the model metric

$$4(1+2\pi(|\boldsymbol\beta|+2)|w|^2)^{-2}|dw|^2 \tag{2.1}$$

of Gaussian curvature $2\pi(|\boldsymbol\beta|+2)$, where $|\boldsymbol\beta|=\beta_0+\beta_1+\beta_\infty$ is the degree of the divisor.

The analytic continuation of the Schwarz triangle function $w_{\boldsymbol\beta}$ (from the upper half-plane $\Im z>0$ through the interval $(0,1)$ of the real axis) maps the lower half-plane $\Im z<0$ into the reflection of the triangle in the side joining the points $w_{\boldsymbol\beta}(0)$ and $w_{\boldsymbol\beta}(1)$. The resulting geodesic bicolored double triangle is hyperbolic if $|\boldsymbol\beta|<-2$ (cf. Fig. 2), spherical if $|\boldsymbol\beta|>-2$ (cf. Fig. 3), and Euclidean if $|\boldsymbol\beta|=-2$ (cf. Fig. 4). Recall that the condition $\beta_j-|\boldsymbol\beta|/2>0$ is necessary and sufficient for the existence of the spherical geodesic triangle; the condition is automatically fulfilled in the hyperbolic and Euclidean cases. To make the double triangle S^2-like, one folds it along the interval $[w_{\boldsymbol\beta}(0),w_{\boldsymbol\beta}(1)]\subset\mathbb R$, and then glue the corresponding sides of the light- and dark-coloured triangles pairwise. Namely, the side joining $w_{\boldsymbol\beta}(0)$ and $w_{\boldsymbol\beta}(i\infty)$ is glued to the side joining

$w_{\boldsymbol\beta}(0)$ and $w_{\boldsymbol\beta}(-i\infty)$, and then the side joining $w_{\boldsymbol\beta}(1)$ and $w_{\boldsymbol\beta}(i\infty)$ is glued to the side joining $w_{\boldsymbol\beta}(1)$ and $w_{\boldsymbol\beta}(-i\infty)$. The function $w_{\boldsymbol\beta}$ realizes the isometry between the S^2-like bicolored double triangle and the genus zero constant curvature surface $(\overline{\mathbb{C}}_z, m_{\boldsymbol\beta})$ with three conical singularities located at the points $z=0$, $z=1$, and $z=\infty$. Here and elsewhere $m_{\boldsymbol\beta}$ is the pullback of the Gaussian curvature $2\pi(|\boldsymbol\beta|+2)$ model metric (2.1) by the Schwarz triangle function $z\mapsto w=w_{\boldsymbol\beta}(z)$.

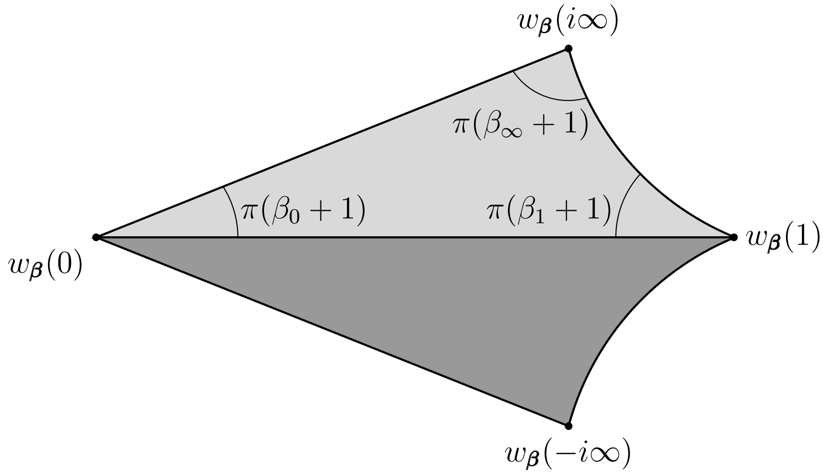

Figure 2: Hyperbolic double triangle.

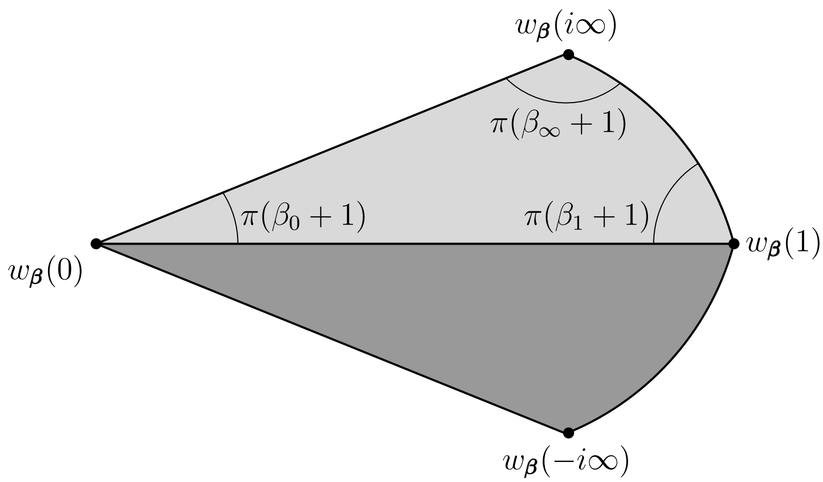

Figure 3: Spherical double triangle.

The pullback $f^*m_{\boldsymbol\beta}$ of the metric $m_{\boldsymbol\beta}$ by the Belyi function f, is a singular metric of Gaussian curvature $2\pi(|\boldsymbol\beta|+2)$ on $\overline{\mathbb{C}}_x$. The triangulated surface $(\overline{\mathbb{C}}_x, f^*m_{\boldsymbol\beta})$ is isometric to the one obtained by cutting and gluing $\deg f$ copies of the bicolored double triangle following a combinatorial scheme prescribed by f. In particular, Fuchsian triangle groups are included in consideration [11, 16].

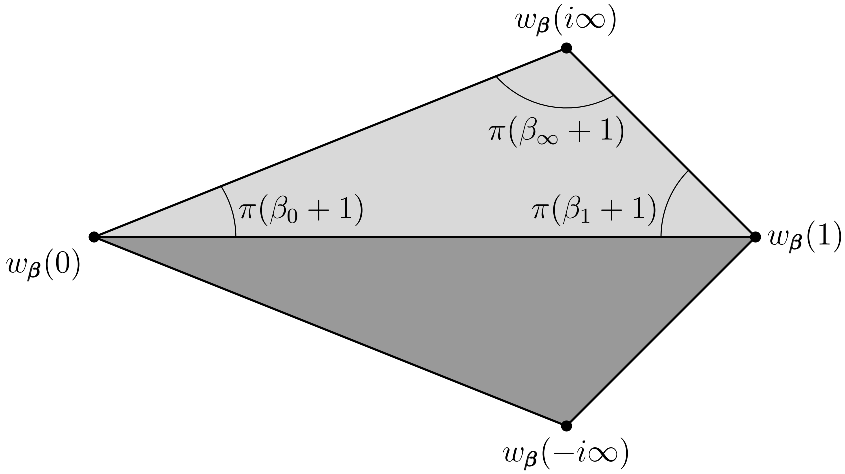

Figure 4: Euclidean double triangle.

Let us note that the above construction also works in the opposite (constructive) direction: the combinatorial cutting and gluing schemes of bicolored double triangles can be described with the help of a fixed base star in $\overline{\mathbb{C}}_z$ with the terminal vertices at $z = 0, 1, \infty$ and the constellations. For any constellation, there exists a Belyi function defining the cutting and gluing scheme, e.g. [36]. However, in general, it is not easy to find a Belyi function corresponding to a particular gluing scheme, even though for the regular dihedra and the surfaces of five Platonic solids they were first found by Schwarz and Klein [30], see also [16, 34, 36, 37].

The main result of this paper is a closed explicit formula for the zeta-regularized spectral determinant of the Friedrichs Laplacian on the triangulated singular constant curvature spheres $(\overline{\mathbb{C}}_x, f^*m_{\boldsymbol\beta})$. We explicitly express the determinant in terms of the Belyi function f and the spectral determinant of the corresponding S^2-like bicolored double triangle $(\overline{\mathbb{C}}_z, m_{\boldsymbol\beta})$. The latter determinant was recently found in a closed explicit form [23] as a function of β_0, β_1, and β_∞.

To formulate the result, we need to introduce some notation. Let us list the preimages of the marked points $\{0, 1, \infty\} \subset \overline{\mathbb{C}}_z$ under f as the marked points $x_1, x_2, \dots, x_n$ on the Riemann sphere $\overline{\mathbb{C}}_x$; here $n = \deg f + 2$. We shall always assume that $x_k = \infty$ for some $k \leqslant n$: this can always be achieved by replacing f with equivalent Belyi function $f \circ \mu$, where μ is a Möbius transformation satisfying $\mu(\infty) = x_k$. The surfaces $(\overline{\mathbb{C}}_x, f^*m_{\boldsymbol\beta})$ and $(\overline{\mathbb{C}}_x, (f\circ\mu)^*m_{\boldsymbol\beta})$ are isometric, and, as a consequence, the corresponding spectral determinants are equal.

Introduce the ramification divisor

$$\boldsymbol{f} := \sum_{k=1}^{n} \operatorname{ord}_k f \cdot x_k,$$

where $\operatorname{ord}_k f$ is the ramification order of the Belyi function f at x_k. If x_k is a pole of f, then its multiplicity coincides with $\operatorname{ord}_k f + 1$. If $f'(x_k) = 0$, then $\operatorname{ord}_k f + 1$ is the order of zero at $x = x_k$ of the function $x \mapsto f(x) - f(x_k)$. If x_k is not a pole of f and $f'(x_k) \neq 0$, then the ramification order $\operatorname{ord}_k f$ is zero. One can also interpret x_k as a vertex in the *dessin d'enfant* corresponding to f, then the number $\operatorname{ord}_k f + 1$ is its graph-theoretic degree, or, equivalently, the number of edges emanating from the vertex x_k. By the Riemann-Hurwitz formula the degree $|\boldsymbol{f}| := \sum_{k=1}^n \operatorname{ord}_k f$ of the divisor $\boldsymbol{f}$ satisfies $|\boldsymbol{f}| = 2 \deg f - 2$.

Let ϕ stand for the potential of the conformal metric $m_{\boldsymbol\beta} = e^{2\phi}|dz|^2$. The potential

$$f^*\phi := \phi \circ f + \log |f'|$$

of the pullback metric $f^* m_{\boldsymbol\beta} = e^{2 f^*\phi}|dx|^2$ satisfies the Liouville equation

$$e^{-2f^*\phi}\big(-4\partial_x\partial_{\bar x}(f^*\phi)\big) = 2\pi(|\beta| + 2), \quad x \in \mathbb{C} \setminus \{x_1, \dots, x_n\}, \tag{2.2}$$

and obeys the asymptotics

$$\begin{aligned} (f^*\phi)(x) &= (f^*\beta)_k \log|x - x_k| + O(1), \quad x \to x_k \neq \infty, \\ (f^*\phi)(x) &= -((f^*\beta)_k + 2)\log|x| + O(1), \quad x \to x_k = \infty, \end{aligned} \tag{2.3}$$

where

$$(f^*\boldsymbol\beta)_k := (\operatorname{ord}_k f + 1)(\beta_{f(x_k)} + 1) - 1. \tag{2.4}$$

Geometrically this means that outside of the points $x_1, \dots, x_k$ the metric $f^* m_{\boldsymbol\beta}$ is a regular metric of constant Gaussian curvature $2\pi(|\boldsymbol\beta| + 2)$, while at each point x_k the metric has a conical singularity of order $(f^*\beta)_k$, or, equivalently, of angle $2\pi((f^*\beta)_k + 1)$, see e.g. [63]. That is because $\operatorname{ord}_k f + 1$ vertices of bicolored double triangles meet together at x_k to form the conical singularity. Each of the vertices makes a contribution of angle $2\pi(\beta_{f(x_k)} + 1)$ into the angle of the conical singularity, cf. (2.4). Note that it could be so that for a point x_k we have $(f^*\beta)_k = 0$ (for example, this is the case if $\operatorname{ord}_k f = 1$ and $\beta_{f(x_k)} = -1/2$), then x_k is also a regular point of the metric $f^* m_{\boldsymbol\beta}$. Clearly, $\beta_{f(x_k)} = \beta_\infty$ if x_k is a pole of f, $\beta_{f(x_k)} = \beta_0$ if x_k is a zero of f, and $\beta_{f(x_k)} = \beta_1$ if $f(x_k) = 1$, where β_j is the same as in (2.6).

On the singular constant curvature surface $(\overline{\mathbb{C}}_x, f^* m_{\boldsymbol\beta})$ we consider the Laplace-Beltrami operator $\Delta_{f^*\boldsymbol\beta} = -e^{-2f^*\phi} 4\partial_x\partial_{\bar x}$ as an unbounded operator in the usual L^2-space, where $f^*\boldsymbol\beta = \sum_k (f^*\boldsymbol\beta)_k \cdot x_k$ is the divisor for $(\overline{\mathbb{C}}_x, f^* m_{\boldsymbol\beta})$. The operator is initially defined on the smooth functions supported outside of the conical singularities and not essentially selfadjoint. We take the Friedrichs selfadjoint extension, which we still denote by $\Delta_{f^*\boldsymbol\beta}$ and call the Friedrichs Laplacian or simply Laplacian for short. The spectrum of $\Delta_{f^*\boldsymbol\beta}$ consists of non-negative isolated eigenvalues λ_j of finite multiplicity, and the zeta-regularized spectral determinant $\det \Delta_{f^*\boldsymbol\beta}$ of $\Delta_{f^*\boldsymbol\beta}$ can be introduced in the standard well-known way:

$$\det \Delta_{f^*\boldsymbol\beta} = \exp(-\zeta'_{f^*\boldsymbol\beta}(0)), \quad \zeta_{f^*\boldsymbol\beta}(s) = \sum_{\lambda_k \neq 0} \lambda_k^{-s}. \tag{2.5}$$

In particular, if f is the identity mapping, then $f^*\phi \equiv \phi$, $\det \Delta_{f^*\boldsymbol\beta} = \det \Delta_{\boldsymbol\beta}$, and the asymptotics (2.3) refine as follows:

$$\begin{aligned} &\phi(z) = \beta_0 \log|z| + \phi_0 + o(1), \quad z \to 0, \quad \phi_0 = \Psi(\beta_0, \beta_1, \beta_\infty),\\ &\phi(z) = \beta_1 \log|z-1| + \phi_1 + o(1), \quad z \to 1, \quad \phi_1 = \Psi(\beta_1, \beta_0, \beta_\infty),\\ &\phi(z) = -(\beta_\infty + 2)\log|z| + \phi_\infty + o(1), \quad z \to \infty, \quad \phi_\infty = \Psi(\beta_\infty, \beta_1, \beta_0), \end{aligned} \tag{2.6}$$

where Ψ is an explicit function, see (5.2).

Now we are in a position to formulate the main results of this paper.

Theorem 2.1 (Spectral determinant of triangulated spheres)**.** *Let* $\det \Delta_{\boldsymbol\beta}$ *stand for the explicit function*

$$(-1, 0]^3 \ni (\beta_0, \beta_1, \beta_\infty) \mapsto \det \Delta_{\boldsymbol\beta} \in \mathbb{R}$$

from [23], whose value at a point is the zeta-regularized spectral determinants of the Friedrichs selfadjoint extension $\Delta_{\boldsymbol\beta}$ *of the Laplace-Beltrami operator on the unit area* S^2*-like double triangle isometric to* $(\overline{\mathbb{C}}_z, m_{\boldsymbol\beta})$*.*

Let $f : \overline{\mathbb{C}}_x \to \overline{\mathbb{C}}_z$ *be a Belyi function unramified outside of the set* $\{0, 1, \infty\}$*. Without loss of generality we assume that* $f(\infty) \in \{0, 1, \infty\}$*. Denote by* $\operatorname{ord}_k f$ *the ramification order of* f *at* x_k*, where* $x_1, x_2, \dots, x_n \in \overline{\mathbb{C}}_x$ *are the preimages of the points* 0*,* 1*, and* ∞ *under* f*.*

Consider the surface $(\overline{\mathbb{C}}_x, f^*m_{\boldsymbol\beta})$ *isometric to the one glued from* $\deg f$ *copies of the double triangle in accordance with a pattern defined by* f*. Then for the zeta-regularized spectral determinant* $\det \Delta_{f^*\boldsymbol\beta}$ *of the Friedrichs selfadjoint extension of the Laplace-Beltrami operator on* $(\overline{\mathbb{C}}_x, f^*m_{\boldsymbol\beta})$ *we have*

$$\begin{aligned} \log \frac{\det \Delta_{f^*\boldsymbol\beta}}{\deg f} = &\deg f \cdot \log\det\Delta_{\boldsymbol\beta} + \frac{1}{6}\sum_k \left(\operatorname{ord}_k f + 1 - \frac{1}{\operatorname{ord}_k f + 1}\right)\frac{\phi_{f(x_k)}}{\beta_{f(x_k)} + 1}\\ &- \frac{1}{6}\sum_k \left((f^*\boldsymbol\beta)_k + 1 + \frac{1}{(f^*\boldsymbol\beta)_k + 1}\right)\log(\operatorname{ord}_k f + 1)\\ &- \sum_k \Big(\mathcal{C}\left((f^*\boldsymbol\beta)_k\right) - (\operatorname{ord}_k f + 1)\mathcal{C}(\beta_{f(x_k)})\Big) - (\deg f - 1)\mathbf{C} + C_f, \end{aligned} \tag{2.7}$$

where ϕ_j *and* β_j *with* $j = f(x_k) \in \{0, 1, \infty\}$ *is the (explicit) uniformization data in* (2.6)*, and* $(f^*\boldsymbol\beta)_k$ *is the same as in* (2.4)*. The real-analytic function* $(-\infty, 0] \ni \beta \mapsto \mathcal{C}(\beta)$ *is defined by the equality*

$$\mathcal{C}(\beta) = 2\zeta_B'(0; \beta+1, 1, 1) - 2\zeta_R'(-1) - \frac{\beta^2}{6(\beta+1)}\log 2 - \frac{\beta}{12} + \frac{1}{2}\log(\beta+1), \tag{2.8}$$

where ζ_B' *and* ζ_R' *stand for the derivatives with respect to* s *of the Barnes double zeta function* $\zeta_B(s; \beta+1, 1, 1)$ *and the Riemann zeta function* $\zeta_R(s)$ *respectively. For the constant* $\mathbf{C}$ *in* (2.7) *we have*

$$\mathbf{C} = \frac{1}{6} - \frac{4}{3}\log 2 - 4\zeta_R'(-1) - \log \pi. \tag{2.9}$$

Finally, the constant C_f in (2.7) depends only on the Belyi function f. It is given by

$$C_f = \frac{1}{18}\sum_{k:x_k\neq\infty}\;\sum_{\ell:x_k\neq x_\ell\neq\infty}\frac{(\operatorname{ord}_k f-2)(\operatorname{ord}_\ell f-2)}{\operatorname{ord}_k f+1}\log|x_k-x_\ell| + \frac{1}{6}\sum_k\left(\frac{\operatorname{ord}_k f+1}{3}+\frac{3}{\operatorname{ord}_k f+1}\right)\log(\operatorname{ord}_k f+1) + \frac{1}{6}\left(\deg f-\sum_k\frac{3}{\operatorname{ord}_k f+1}\right)\log A_f, \tag{2.10}$$

where

$$A_f = \frac{|f(x)|^{-2/3}|f(x)-1|^{-2/3}|f'(x)|}{\prod_{k:x_k\neq\infty}|x-x_k|^{\frac{1}{3}(\operatorname{ord}_k f-2)}}. \tag{2.11}$$

Note that A_f is a scaling coefficient that does not depend on x and can be easily evaluated for any particular Belyi function f.

As we show in the proof of Theorem 2.1, the expression (2.10) for C_f comes from the natural Euclidean equilateral triangulation defined on $\overline{\mathbb{C}}_x$ by f, cf. [53, 65]. The readers whose interests are more in applications than in the proofs, may wish to proceed directly to Section 5.

3 Proof of Theorem 2.1

3.1 Equality (2.7) is valid with a constant C_f

In this subsection we prove that the representation for the spectral determinant (2.7) is valid with some constant C_f that does not depend on the metric $m_{\boldsymbol\beta}$ on the target Riemann sphere $\overline{\mathbb{C}}_z$. As a byproduct, we obtain an explicit formula for C_f that is not as refined as the one in Theorem 2.1. These results are formulated in Prop. 3.1 below.

Proposition 3.1. *The equality (2.7) in Theorem 2.1 is valid with a constant C_f that depends only on the Belyi function f. Moreover,*

$$\begin{aligned} C_f = {} & \frac{1}{6}\sum_{k:x_k\neq\infty, f(x_k)\neq\infty}\left(\frac{\operatorname{ord}_k f}{\operatorname{ord}_k f+1}\log|c_k| + (\operatorname{ord}_k f+2)\log(\operatorname{ord}_k f+1)\right) \\ & +\frac{1}{6}\sum_{k:x_k\neq\infty, f(x_k)=\infty}\left(\frac{\operatorname{ord}_k f+2}{\operatorname{ord}_k f+1}\log|c_k| - \operatorname{ord}_k f\log(\operatorname{ord}_k f+1)\right) \\ & +\frac{1}{6}\left(-\frac{\operatorname{ord}_k f}{\operatorname{ord}_k f+1}\log|c_k| - (\operatorname{ord}_k f+2)\log(\operatorname{ord}_k f+1)\right)\Bigg|_{k:x_k=\infty, f(\infty)=\infty} \\ & +\frac{1}{6}\left(-\frac{\operatorname{ord}_k f+2}{\operatorname{ord}_k f+1}\log|c_k| + \operatorname{ord}_k f\log(\operatorname{ord}_k f+1)\right)\Bigg|_{k:x_k=\infty, f(\infty)\neq\infty}, \end{aligned} \tag{3.1}$$

where

- c_k *stands for the first nonzero coefficient in the Taylor series of* $f(x) - f(x_k)$ *at* x_k*, if* x_k *is not a pole of* f*;*
- c_k *stands for the first nonzero coefficient of the Laurent series of* f *at* x_k*, if* x_k *is a pole of* f*.*

The proof of Proposition 3.1 is preceded by Lemma 3.2 and Lemma 3.3 below. We will use the following refined version of the asymptotics (2.3) for the metric potential:

$$\begin{aligned}(f^*\phi)(x) &= (f^*\boldsymbol{\beta})_k \log|x - x_k| + (f^*\phi)_k + o(1), \quad x \to x_k \neq \infty,\\ (f^*\phi)(x) &= -\big((f^*\boldsymbol{\beta})_k + 2\big)\log|x| + (f^*\phi)_k + o(1), \quad x \to x_k = \infty.\end{aligned} \tag{3.2}$$

Here the coefficient $(f^*\boldsymbol{\beta})_k$ is the same as in (2.4). Moreover, since $f^*\phi = \phi\circ f + \log|f'|$, it is not hard to see that the coefficients $(f^*\phi)_k$ in (3.2) satisfy

$$\begin{aligned}(f^*\phi)_k &= \phi_{f(x_k)} + (\beta_{f(x_k)} + 1)\log|c_k| + \log(\operatorname{ord}_k f + 1), \quad \text{if} \quad f(x_k) \neq \infty,\\ (f^*\phi)_k &= \phi_{f(x_k)} - (\beta_{f(x_k)} + 1)\log|c_k| + \log(\operatorname{ord}_k f + 1), \quad \text{if} \quad f(x_k) = \infty.\end{aligned} \tag{3.3}$$

Here β_j and ϕ_j with $j = f(x_k) \in \{0, 1, \infty\}$ are the same as in (2.6), and c_k is the same as in Proposition 3.1.

As it was mentioned in the introduction, we will rely on an anomaly formulae for the determinants of Laplacians that allow for metrics with conical singularities. We derrive it in the following lemma.

Lemma 3.2. *For the determinant* $\det\Delta_{f^*\boldsymbol{\beta}}$ *of the Friedrichs Laplacian on* $(\overline{\mathbb{C}}_x, f^*m_\beta)$ *the anomaly formula*

$$\begin{aligned}\log\frac{\det\Delta_{f^*\boldsymbol{\beta}}}{\deg f} = &-\frac{|\beta| + 2}{6}\int_{\mathbb{C}}(f^*\phi)e^{2f^*\phi}\,\frac{dx\wedge d\bar{x}}{-2i} + \frac{1}{6}\sum_{k: x_k\neq\infty}\frac{(f^*\boldsymbol{\beta})_k}{(f^*\boldsymbol{\beta})_k + 1}(f^*\phi)_k\\ &- \frac{1}{6}\frac{(f^*\boldsymbol{\beta})_k + 2}{(f^*\boldsymbol{\beta})_k + 1}(f^*\phi)_k\Big|_{k: x_x=\infty} - \sum_k \mathcal{C}\big((f^*\boldsymbol{\beta})_k\big) + \mathbf{C}\end{aligned} \tag{3.4}$$

is valid. Here $\mathcal{C}(\beta)$ *is the function defined in* (2.8)*, and* $\mathbf{C}$ *stands for the constant* (2.9)*.*

Proof. The main argument of the proof is similar to the one in [23, Proof of Prop. 2.1]. For this reason, we skip the details that can be easily restored from the references [23, 24, 25].

Let us first obtain an asymptotics for the determinant of the Friedrichs Dirichlet Laplacian $\Delta^D_{f^*\boldsymbol{\beta}}\restriction_{|x|\leqslant 1/\epsilon}$ on the disk $|x| \leqslant 1/\epsilon$ endowed with the metric $f^*m_{\boldsymbol{\beta}}$ as $\epsilon \to 0^+$. Denote by $\Delta^D_0\restriction_{|x|\leqslant 1/\epsilon}$ the selfadjoint Dirichlet Laplacian on the disk $|x| \leqslant 1/\epsilon$ equipped with the flat background metric $|dx|^2$. As is known [66], for the determinant of the Dirichlet Laplacian on the flat disk one has

$$\log\det\Delta^D_0\restriction_{|x|\leqslant 1/\epsilon} = \frac{1}{3}\log\epsilon + \frac{1}{3}\log 2 - \frac{1}{2}\log 2\pi - \frac{5}{12} - 2\zeta_R'(-1).$$

On the other hand, the Polyakov-Alvarez type formula from [25, Theorem 1.1.2] reads

$$\begin{aligned}\log\frac{\det\Delta^D_{f^*\boldsymbol\beta}\restriction_{|x|\leqslant 1/\epsilon}}{\det\Delta^D_0\restriction_{|x|\leqslant 1/\epsilon}}=&-\frac{|f^*\boldsymbol\beta|+2}{6\deg f}\int_{\mathbb C}(f^*\phi)e^{2f^*\phi}\frac{dx\wedge d\bar x}{-2i}\\&-\frac{1}{12\pi}\oint_{|x|=1/\epsilon}(f^*\phi)\partial_{\vec n}(f^*\phi)\,|dx|-\frac{\epsilon}{6\pi}\oint_{|x|=1/\epsilon}(f^*\phi)\,|dx|\\&-\frac{1}{4\pi}\oint_{|x|=1/\epsilon}\partial_{\vec n}(f^*\phi)\,|dx|+\frac16\sum_{k:x_k\neq\infty}\frac{(f^*\boldsymbol\beta)_k}{(f^*\boldsymbol\beta)_k+1}(f^*\phi)_k\\&-\sum_{k:x_k\neq\infty}\mathcal C\big((f^*\boldsymbol\beta)_k\big),\end{aligned}\tag{3.5}$$

where $|f^*\boldsymbol\beta|=\sum_k(f^*\boldsymbol\beta)_k$ is the degree of the divisor $f^*\boldsymbol\beta=\sum_k(f^*\boldsymbol\beta)_k\cdot x_k$.

By the Gauss-Bonnet theorem [63], the product of the (regularized) Gaussian curvature by the total area of the singular sphere $(\overline{\mathbb C}_x,f^*m_{\boldsymbol\beta})$ equals $2\pi(|f^*\boldsymbol\beta|+2)$. Since the (regularized) Gaussian curvature is the one of $m_{\boldsymbol\beta}$, and the total area is $\deg f$, we conclude that

$$\frac{|f^*\boldsymbol\beta|+2}{\deg f}=|\boldsymbol\beta|+2.$$

In (3.5) we also replace the integrals along the circle $|x|=1/\epsilon$ with their asymptotics. As a result, the Polyakov-Alvarez type formula (3.5) implies

$$\begin{aligned}\log\det\Delta^D_{f^*\boldsymbol\beta}\restriction_{|x|\leqslant1/\epsilon}=&-\frac{|\boldsymbol\beta|+2}{6}\int_{\mathbb C}(f^*\phi)e^{2f^*\phi}\frac{dx\wedge d\bar x}{-2i}+\frac16\sum_{k:x_k\neq\infty}\frac{(f^*\boldsymbol\beta)_k}{(f^*\boldsymbol\beta)_k+1}(f^*\phi)_k\\&-\sum_{k:x_k\neq\infty}\mathcal C\big((f^*\boldsymbol\beta)_k\big)+\frac16(f^*\boldsymbol\beta)_k\left((f^*\phi)_k+3\right)|_{k:x_k=\infty}+\frac13\log2-\frac12\log2\pi+\frac{7}{12}\\&-2\zeta_R'(-1)+\frac16\left(((f^*\boldsymbol\beta)_k+2)^2-2\left((f^*\boldsymbol\beta)_k+1\right)\right)|_{k:x_k=\infty}\log\epsilon+o(1),\ \ \epsilon\to0^+.\end{aligned}$$

In a similar way, one can also obtain the asymptotics for the determinant of the-Friedrichs Dirichlet Laplacian on the cap $|x|\geqslant1/\epsilon$ of the singular sphere $(\overline{\mathbb C}_x,f^*m_\beta)$:

$$\begin{aligned}\log\det\Delta^D_{f^*\boldsymbol\beta}\restriction_{|x|\geqslant1/\epsilon}=&-\frac16(((f^*\boldsymbol\beta)_k+1)^2+1)\log\epsilon-\frac16\left((f^*\boldsymbol\beta)_k+1+\frac{1}{(f^*\boldsymbol\beta)_k+1}\right)(f^*\phi)_k\\&-\mathcal C\big((f^*\boldsymbol\beta)_k\big)-2\zeta_R'(-1)-\frac{5}{12}+\frac13\log2-\frac12\log2\pi-\frac{(f^*\boldsymbol\beta)_k}{2}+o(1),\quad\epsilon\to0^+,\end{aligned}$$

where k is such that $x_k=\infty$. In total, we have

$$\begin{aligned}\log\det\Delta^D_{f^*\boldsymbol\beta}\restriction_{|x|\leqslant1/\epsilon}+\log\det\Delta^D_{f^*\boldsymbol\beta}\restriction_{|x|\geqslant1/\epsilon}=&-\frac{|\boldsymbol\beta|+2}{6}\int_{\mathbb C}(f^*\phi)e^{2f^*\phi}\frac{dx\wedge d\bar x}{-2i}\\&+\frac16\sum_{k:x_k\neq\infty}\frac{(f^*\boldsymbol\beta)_k}{(f^*\boldsymbol\beta)_k+1}(f^*\phi)_k-\frac16\left(1+\frac{1}{(f^*\boldsymbol\beta)_k+1}\right)(f^*\phi)_k\\&-\sum_k\mathcal C\big((f^*\boldsymbol\beta)_k\big)+\mathbf C+\log2+o(1),\quad\epsilon\to0^+.\end{aligned}\tag{3.6}$$

Now we cut the singular sphere $\big(\overline{\mathbb{C}}_x, f^*m_{\boldsymbol\beta}\big)$ along the circle $|x|=1/\epsilon$ with the help of the BFK formula [7, Theorem B*] to obtain

$$\begin{aligned}\log\det\Delta_{f^*\boldsymbol\beta} =& \log\deg f-\log 2\\ &+\lim_{\epsilon\to 0^+}\big(\log\det\Delta^D_{f^*\boldsymbol\beta}\restriction_{|x|\leqslant 1/\epsilon}+\log\det\Delta^D_{f^*\boldsymbol\beta}\restriction_{|x|\geqslant 1/\epsilon}\big)\,.\end{aligned}\tag{3.7}$$

Here $\deg f$ is the total area of the singular sphere, and the number $-\log 2$ is the value of the difference

$$\log\det\mathcal{N}(\epsilon)-\log\oint_{|x|=1/\epsilon} e^{f^*\phi}\,|dx|,$$

where $\det\mathcal{N}(\epsilon)$ is the determinant of the Neumann jump operator on the cut $|x|=1/\epsilon$, see [23, Proof of Prop. 2.1] or [24]. As demonstrated in [25], the BFK formula (3.7) remains valid in spite of the fact that the metric $f^*m_{\boldsymbol\beta}$ is singular.

In the limit, the asymptotics (3.6) together with the BFK formula (3.7) implies the anomaly formula (3.4). This completes the proof of Lemma 3.2. □

Lemma 3.3. *For the integral term in the anomaly formula (3.4) we have*

$$\begin{aligned}(|\boldsymbol\beta|+2)\int_{\mathbb{C}}(f^*\phi)e^{2f^*\phi}\frac{dx\wedge d\bar x}{-2i} &= (|\boldsymbol\beta|+2)\deg f\int_{\mathbb{C}}\phi e^{2\phi}\frac{dz\wedge d\bar z}{-2i}\\ -\sum_{k:f(x_k)\neq\infty}(f^*\phi)_k\operatorname{ord}_k f+\sum_{k:f(x_k)=\infty}&(f^*\phi)_k\operatorname{ord}_k f+\sum_k (f^*\boldsymbol\beta)_k\log|(\operatorname{ord}_k f+1)c_k|\\ +2\sum_{k:x_k\neq\infty, f(x_k)=\infty}(f^*\phi)_k+2\log|(\operatorname{ord}_k &f+1)c_k|\big|_{k:x_k=\infty}-2(f^*\phi)_k\big|_{k:x_k=\infty, f(\infty)\neq\infty}\,,\end{aligned}$$

where $(f^*\boldsymbol\beta)_k$ and $(f^*\phi)_k$ are respectively the coefficients (2.4) and (3.3) in the asymptotics (3.2) of the metric potential $(f^*\phi)$.

Proof. We begin with the equality

$$\int_{\mathbb{C}}(f^*\phi)e^{2f^*\phi}\frac{dx\wedge d\bar x}{-2i}=\deg f\cdot\int_{\mathbb{C}}\phi e^{2\phi}\frac{dz\wedge d\bar z}{-2i}+\int_{\mathbb{C}}(\log|f'|)e^{2f^*\phi}\frac{dx\wedge d\bar x}{-2i}$$

that easily follows from the relation $f^*\phi=\phi\circ f+\log|f'|$. Thus, in order to prove the assertion of the lemma, we only need to evaluate the last integral.

Thanks to the Liouville equation (2.2) we have

$$\begin{aligned}(|\boldsymbol\beta|+2)\int_{\mathbb{C}}(\log|f'|)e^{2f^*\phi}\frac{dx\wedge d\bar x}{-2i}&=\lim_{\epsilon\to0}\frac{1}{2\pi}\int_{\mathbb{C}^\epsilon}(\log|f'|)\big(-4\partial_x\partial_{\bar x}(f^*\phi)\big)\frac{dx\wedge d\bar x}{-2i}\\ &=\lim_{\epsilon\to0}\frac{1}{2\pi}\oint_{\partial\mathbb{C}^\epsilon}\Big((f^*\phi)\partial_{\vec n}(\log|f'|)-(\log|f'|)\partial_{\vec n}(f^*\phi)\Big)\,|dx|,\end{aligned}\tag{3.8}$$

where $\mathbb{C}^\epsilon$ stands for the large disk $|x|>1/\epsilon$ with the small disks $|x-x_k|<\epsilon$ encircling the marked points x_k removed. Above $\vec n$ stands for the unit outward normal, and the last equality in (3.8) is valid because $x\mapsto\log|f'(x)|$ is a harmonic function on $\mathbb{C}^\epsilon$.

If x_k is not a pole of f, then

$$\begin{aligned}\log|f'(x)| &= \operatorname{ord}_k f\log|x-x_k| + \log|(\operatorname{ord}_k f+1)c_k| + o(1),\ x\to x_k\neq\infty,\\ \log|f'(x)| &= -(\operatorname{ord}_k f+2)\log|x| + \log|(\operatorname{ord}_k f+1)c_k| + o(1),\ x\to x_k=\infty.\end{aligned} \tag{3.9}$$

Besides, if x_k is a pole of f, we have

$$\begin{aligned}\log|f'(x)| =& -(\operatorname{ord}_k f+2)\log|x-x_k|\\ &\qquad + \log|(\operatorname{ord}_k f+1)c_k| + o(1),\quad x\to x_k\neq\infty,\\ \log|f'(x)| =& \operatorname{ord}_k f\log|x| + \log|(\operatorname{ord}_k f+1)c_k| + o(1),\quad x\to x_k=\infty.\end{aligned} \tag{3.10}$$

Clearly, for the derivatives along the unit outward normal vector $\vec n$ we have

$$\partial_{\vec n} = -\partial_{|x-x_j|}\quad\text{on}\quad |x-x_j|=\epsilon;\quad \partial_{\vec n}=\partial_{|x|}\quad\text{on}\quad |x|=1/\epsilon.$$

Now we can evaluate the integral in right hand side of (3.8) relying on the asymptotics (3.2) for $(f^*\phi)(x)$ as $x\to x_k$ together with similar asymptotics for $\log|f'(x)|$ in (3.9) and (3.10). As a result, we obtain

$$\begin{aligned}&\lim_{\epsilon\to0}\frac{1}{2\pi}\oint_{\partial\mathbb{C}^\epsilon}\Big((f^*\phi)\partial_{\vec n}(\log|f'|)-(\log|f'|)\partial_{\vec n}(f^*\phi)\Big)\,|dx|\\
&=\sum_{j=1}^n\frac{1}{2\pi}\lim_{\epsilon\to0}\oint_{|x-x_j|=\epsilon}\Big((f^*\phi)\partial_{\vec n}(\log|f'|)-(\log|f'|)\partial_{\vec n}(f^*\phi)\Big)\,|dx|\\
&=\sum_{k:x_k\neq\infty, f(x_k)\neq\infty}\Big(-(f^*\phi)_k\operatorname{ord}_k f+(f^*\boldsymbol\beta)_k\log|(\operatorname{ord}_k f+1)c_k|\Big)\\
&+\sum_{k:x_k\neq\infty, f(x_k)=\infty}\Big((f^*\phi)_k(\operatorname{ord}_k f+2)+(f^*\boldsymbol\beta)_k\log|(\operatorname{ord}_k f+1)c_k|\Big)\\
&+\Big(-(f^*\phi)_k(\operatorname{ord}_k f+2)+(\log|(\operatorname{ord}_k f+1)c_n|)\big((f^*\boldsymbol\beta)_k+2\big)\Big)\Big|_{k:x_k=\infty, f(\infty)\neq\infty}\\
&+\Big((f^*\phi)_k\operatorname{ord}_k f+((f^*\boldsymbol\beta)_k+2)\log|(\operatorname{ord}_k f+1)c_k|\Big)\Big|_{k:x_k=\infty, f(\infty)=\infty}.\end{aligned}$$

After regrouping the terms in the right hand side, this implies the assertion of Lemma 3.3. ☐

Proof of Proposition 3.1. Thanks to Lemma 3.3, the integral in the right-hand side of the anomaly formula in Lemma 3.2 reduces to an integral on the target sphere $(\overline{\mathbb{C}}_z, m_{\boldsymbol\beta})$. Our next purpose is to express that integral in terms of the determinant of Laplacian $\Delta_{\boldsymbol\beta}$ on $(\overline{\mathbb{C}}_z, m_{\boldsymbol\beta})$. With this aim in mind, we write down the anomaly formula

$$\begin{aligned}\log\det\Delta_{\boldsymbol\beta} = -\frac{|\boldsymbol\beta|+2}{6}\int_{\mathbb{C}}\phi e^{2\phi}\frac{dz\wedge d\bar z}{-2i}+\frac16\sum_{j=0,1}\frac{\beta_j}{\beta_j+1}\phi_j\\
-\frac16\frac{\beta_\infty+2}{\beta_\infty+1}\phi_\infty-\sum_{j\in\{0,1,\infty\}}\mathcal{C}(\beta_j)+\mathbf{C},\end{aligned} \tag{3.11}$$

which is just the anomaly formula (3.4) in the particular case of the identity mapping $z = f(x) = x$. We change the index of summation from $j \in \{0, 1, \infty\}$ to $k = 1, 2, \dots, n$ as follows:

$$\begin{aligned}\log\det\Delta_{\boldsymbol\beta} = -\frac{|\boldsymbol\beta|+2}{6}\int_{\mathbb C}\phi e^{2\phi}\frac{dz\wedge d\bar z}{-2i} + \frac16\sum_{k: f(x_k)\neq\infty}\frac{\operatorname{ord}_k f+1}{\deg f}\frac{\beta_{f(x_k)}}{\beta_{f(x_k)}+1}\phi_{f(x_k)}\\ -\sum_k\frac{\operatorname{ord}_k f+1}{\deg f}\mathcal C(\beta_{f(x_k)}) - \frac16\sum_{k: f(x_k)=\infty}\frac{\operatorname{ord}_k f+1}{\deg f}\frac{\beta_{f(x_k)}+2}{\beta_{f(x_k)}+1}\phi_{f(x_k)} + \mathbf C.\end{aligned} \tag{3.12}$$

Here we rely on the equalities

$$\sum_{k: f(x_k)=j}(\operatorname{ord}_k f+1) = \deg f \quad \text{for any fixed } j\in\{0,1,\infty\}. \tag{3.13}$$

The equality (3.12) together with the result of Lemma 3.3 allows us to express the integral in the anomaly formula (3.4) in terms of $\det\Delta_{\boldsymbol\beta}$ and the explicit uniformization data in (2.6). In the remaining part of the proof, we show that as a result, we arrive at the equality (2.7) with C_f given by (3.1).

It is easy to see that proceeding as discussed above, we get the terms

$$\deg f\cdot\log\det\Delta_{\boldsymbol\beta} - \sum_k\left(\mathcal C\left((f^*\boldsymbol\beta)_k\right) - (\operatorname{ord}_k f+1)\mathcal C(\beta_{f(x_k)})\right) - (\deg f-1)\mathbf C \tag{3.14}$$

in the right hand side of (2.7), we omit the details.

It is considerably harder to show that we also obtain the other terms in the right-hand sides of (2.7) and (3.1). With this aim in mind, we separately consider the following four possibilities:

- $x_k \neq \infty$ is not a pole of f,
- $x_k \neq \infty$ is a pole of f,
- $x_k = \infty$ is a not pole of f, and
- $x_k = \infty$ is a pole of f.

If $x_k\neq\infty$ is not a pole of f, then, in addition to the terms listed in (3.14), we get $\frac16$ times

$$\begin{aligned}&\frac{(f^*\boldsymbol\beta)_k}{(f^*\boldsymbol\beta)_k+1}(f^*\phi)_k - \Big(-(f^*\phi)_k\operatorname{ord}_k f + (f^*\boldsymbol\beta)_k\log|(\operatorname{ord}_k f+1)c_k|\Big)\\ &\qquad\qquad -(\operatorname{ord}_k f+1)\frac{\beta_{f(x_k)}}{\beta_{f(x_k)}+1}\phi_{f(x_k)}\\ &=\left(\operatorname{ord}_k f+1-\frac{1}{\operatorname{ord}_k f+1}\right)\frac{\phi_{f(x_k)}}{\beta_{f(x_k)}+1} - \left((f^*\boldsymbol\beta)_k+1+\frac{1}{(f^*\boldsymbol\beta)_k+1}\right)\log(\operatorname{ord}_k f+1)\\ &\qquad\qquad+\left(\frac{\operatorname{ord}_k f}{\operatorname{ord}_k f+1}\log|c_k| + (\operatorname{ord}_k f+2)\log(\operatorname{ord}_k f+1)\right).\end{aligned}$$

Here the first two terms in the right-hand side contribute to the first and the second sums in the right-hand side of (2.7) correspondingly. The last term contributes to C_f, cf. (3.1).

If $x_k \neq \infty$ is a pole of f, then we get the terms listed in (3.14) and also $\frac{1}{6}$ times

$$\frac{(f^*\boldsymbol{\beta})_k}{(f^*\boldsymbol{\beta})_k+1}(f^*\phi)_k - \Big((f^*\phi)_k(\operatorname{ord}_k f+2) + (f^*\boldsymbol{\beta})_k \log|(\operatorname{ord}_k f+1)c_k|\Big)$$
$$+(\operatorname{ord}_k f+1)\frac{\beta_{f(x_k)}+2}{\beta_{f(x_k)}+1}\phi_{f(x_k)}$$
$$= \left(\operatorname{ord}_k f+1-\frac{1}{\operatorname{ord}_k f+1}\right)\frac{\phi_{f(x_k)}}{\beta_{f(x_k)}+1} - \left((f^*\boldsymbol{\beta})_k+1+\frac{1}{(f^*\boldsymbol{\beta})_k+1}\right)\log(\operatorname{ord}_k f+1)$$
$$+\left(\frac{\operatorname{ord}_k f+2}{\operatorname{ord}_k f+1}\log|c_k| - \operatorname{ord}_k f \log(\operatorname{ord}_k f+1)\right).$$

Here again, the first two terms in the right-hand side contribute to the first and the second sums in the right-hand side of (2.7). The last term contributes to C_f, cf. (3.1).

Similarly, if $x_k = \infty$ is not a pole of f, we get the terms listed in (3.14) and $\frac{1}{6}$ times

$$-\frac{(f^*\boldsymbol{\beta})_k+2}{(f^*\boldsymbol{\beta})_k+1}(f^*\phi)_k - \Big(-(f^*\phi)_k(\operatorname{ord}_k f+2) + ((f^*\boldsymbol{\beta})_k+2)(\log|(\operatorname{ord}_k f+1)c_k|)\Big)$$
$$-(\operatorname{ord}_k f+1)\frac{\beta_{f(x_k)}}{\beta_{f(x_k)}+1}\phi_{f(x_k)}$$
$$= \left(\operatorname{ord}_k f+1-\frac{1}{\operatorname{ord}_k f+1}\right)\frac{\beta_{f(x_k)}}{\beta_{f(x_k)}+1}\phi_{f(x_k)}$$
$$- \left((f^*\boldsymbol{\beta})_k+1+\frac{1}{(f^*\boldsymbol{\beta})_k+1}\right)\log(\operatorname{ord}_k f+1)$$
$$+\left(-\frac{\operatorname{ord}_k f+2}{\operatorname{ord}_k f+1}\log|c_k| + \operatorname{ord}_k f \log(\operatorname{ord}_k f+1)\right).$$

Here, as before, the first two terms in the right-hand side are in agreement with (2.7), and the last term goes to C_f, cf. (3.1).

Finally, if $x_k = \infty$ is a pole of f, we get the terms listed in (3.14) and $\frac{1}{6}$ times

$$-\frac{(f^*\boldsymbol{\beta})_k+2}{(f^*\boldsymbol{\beta})_k+1}(f^*\phi)_k - \Big((f^*\phi)_k \operatorname{ord}_k f + \big((f^*\boldsymbol{\beta})_k+2\big)\log|(\operatorname{ord}_k f+1)c_k|\Big)$$
$$+(\operatorname{ord}_k f+1)\frac{\beta_{f(x_k)}+2}{\beta_{f(x_k)}+1}\phi_{f(x_k)}$$
$$= \left(\operatorname{ord}_k f+1-\frac{1}{\operatorname{ord}_k f+1}\right)\frac{\phi_{f(x_k)}}{\beta_{f(x_k)}+1} - \left((f^*\boldsymbol{\beta})_k+1+\frac{1}{(f^*\boldsymbol{\beta})_k+1}\right)\log(\operatorname{ord}_k f+1)$$
$$+\left(-\frac{\operatorname{ord}_k f}{\operatorname{ord}_k f+1}\log|c_k| - (\operatorname{ord}_k f+2)\log(\operatorname{ord}_k f+1)\right),$$

which is in agreement with (2.7) and (3.1). This completes the proof of Theorem 3.1. □

3.2 Euclidean equilateral triangulation

By Proposition 3.1 the constant C_f does not depend on the metric $m_{\boldsymbol\beta}$. In this section, we make a particular choice of $m_{\boldsymbol\beta}$ that significantly simplifies the calculation of the constant C_f. As we show in the proof of Proposition 3.4 below, it makes sense to consider the Euclidean equilateral triangulation naturally associated with Belyi function f, cf. [53, 65].

Proposition 3.4. *The constant C_f in (3.1) satisfies (2.10) with A_f from (2.11); see also Remark 3.5 at the end of this subsection.*

Proof. Here we consider the pull-back by f of the flat singular metric

$$m_{\boldsymbol\beta}=c^2|z|^{-4/3}|z-1|^{-4/3}|dz|^2,\quad \boldsymbol\beta=\left(-\frac{2}{3}\right)\cdot 0+\left(-\frac{2}{3}\right)\cdot 1+\left(-\frac{2}{3}\right)\cdot\infty,$$

where $c>0$ is a scaling coefficient that normalizes the area of the metric to one. Note that the surface $(\overline{\mathbb C}_z,m_{\boldsymbol\beta})$ is isometric to two congruent Euclidean equilateral triangles glued together along their sides [63, 26].

Relying on anomaly formulae we can obtain expressions for the determinants of Laplacians on the base $(\overline{\mathbb C}_z,m_{\boldsymbol\beta})$ and on the ramified covering $(\overline{\mathbb C}_x,f^*m_{\boldsymbol\beta})$. These determinants satisfy the relation (2.7) from Theorem 3.1. As we show below, this leads to the equalities (2.10) and (2.11) for C_f and A_f correspondingly, and thus proves the assertion.

The potential ϕ of the metric $m_{\boldsymbol\beta}=e^{2\phi}|dz|^2$ has the asymptotics

$$\phi(x)=-\frac{2}{3}\log|z|+\log c+o(1),\quad |z|\to 0;\quad \phi(x)=-\frac{2}{3}\log|z-1|+\log c+o(1),\quad |z|\to 1;$$

$$\phi(x)=-\frac{4}{3}\log|z|+\log c+o(1),\quad |z|\to\infty.$$

This asymptotics is particularly simple, because for the coefficients ϕ_j we have $\phi_j=\log c$ (instead of a cumbersome expression with a lot of Gamma functions for ϕ_j in (2.6), see (5.2)), and the weights β_j of the marked points are $-2/3$. Moreover, thanks to our special choice of $m_{\boldsymbol\beta}$, for the Laplacian $\Delta_{\boldsymbol\beta}$ on $(\overline{\mathbb C}_z,m_{\boldsymbol\beta})$ the right hand side of the anomaly formula (3.11) takes the following particularly simple form:

$$\log\det\Delta_{\boldsymbol\beta}=-\frac{4}{3}\log c-3\mathcal{C}\left(-\frac{2}{3}\right)+\mathbf{C}.$$

This together with the relation (2.7) proved in Proposition 3.1 gives

$$\log\frac{\det\Delta_{f^*\boldsymbol\beta}}{\deg f}=\deg f\cdot\left(-\frac{4}{3}\log c+\mathbf{C}\right)+\frac{1}{2}\left(3\deg f-\sum_k\frac{1}{\operatorname{ord}_k f+1}\right)\log c$$
$$-\frac{1}{6}\sum_k\left(\frac{\operatorname{ord}_k f+1}{3}+\frac{3}{\operatorname{ord}_k f+1}\right)\log(\operatorname{ord}_k f+1)\qquad(3.15)$$
$$-\sum_k\mathcal{C}\left(\frac{\operatorname{ord}_k f-2}{3}\right)-(\deg f-1)\mathbf{C}+C_f.$$

One can think of the surface $(\overline{\mathbb{C}}_x, f^*m_{\boldsymbol\beta})$ as of the $\deg f$ copies of the flat bicolored double triangle glued along the edges in a way prescribed by f. For the pull-back metric we obtain

$$f^*m_{\boldsymbol\beta} = c^2|f(x)|^{-4/3}|f(x)-1|^{-4/3}|f'(x)|^2\,|dx|^2. \tag{3.16}$$

As is well known [63], the flat metric $f^*m_{\boldsymbol\beta}$ can equivalently be written in the standard form

$$f^*m_{\boldsymbol\beta} = c^2A_f^2\prod_{k:x_k\neq\infty}|x-x_k|^{\frac{2}{3}(\operatorname{ord}_k f-2)}\,|dx|^2. \tag{3.17}$$

Now the representation (2.11) for the scaling coefficient A_f is an immediate consequence of the equalities (3.16) and (3.17).

Clearly, the metric potential $f^*\phi$ of $f^*m_{\boldsymbol\beta}=e^{2f^*\phi}|dx|^2$ obeys the asymptotics

$$\begin{aligned}(f^*\phi)(x) =& \frac{\operatorname{ord}_k f-2}{3}\log|x-x_k|+\log cA_f\\ &+\sum_{\ell:x_k\neq x_\ell\neq\infty}\frac{\operatorname{ord}_\ell f-2}{3}\log|x_k-x_\ell|+o(1),\quad x\to x_k\neq\infty,\end{aligned} \tag{3.18}$$

$$(f^*\phi)(x) = -\left(\frac{\operatorname{ord}_k f-2}{3}+2\right)\log|x|+\log cA_f+o(1),\quad x\to x_k=\infty. \tag{3.19}$$

Therefore, for the Laplacian $\widetilde\Delta_{f^*\boldsymbol\beta}$ induced by the scaled flat metric

$$(cA_f)^{-2}\cdot f^*m_{\boldsymbol\beta} = \prod_{k:x_k\neq\infty}|x-x_k|^{\frac{2}{3}(\operatorname{ord}_j f-2)}\,|dx|^2$$

of total area $\deg f\cdot(cA_f)^{-2}$, the anomaly formula from [25, Prop. 3.3] gives

$$\begin{aligned}\log\frac{\det\widetilde\Delta_{f^*\boldsymbol\beta}}{\deg f\cdot(cA_f)^{-2}} = \frac{1}{18}\sum_{k:x_k\neq\infty}\ \sum_{\ell:x_k\neq x_\ell\neq\infty}&\frac{(\operatorname{ord}_k f-2)(\operatorname{ord}_\ell f-2)}{\operatorname{ord}_k f+1}\log|x_k-x_\ell|\\ &-\sum_{k=1}^n\mathcal{C}\left(\frac{\operatorname{ord}_k f-2}{3}\right)+\mathbf{C}.\end{aligned} \tag{3.20}$$

The standard rescaling property of the determinant reads

$$\log\frac{\det\Delta_{f^*\boldsymbol\beta}}{\deg f} = \log\frac{\det\widetilde\Delta_{f^*\boldsymbol\beta}}{\deg f\cdot(cA_f)^{-2}} - 2(\zeta(0)+1)\cdot\log cA_f,$$

where

$$\zeta(0) = -\frac{1}{12}\sum_k\left(\frac{\operatorname{ord}_k f+1}{3}-\frac{3}{\operatorname{ord}_k f+1}\right)-1$$

is the value of the spectral zeta function of $\widetilde\Delta_{f^*\boldsymbol\beta}$ at zero; for details we refer to [25, Section 1.2].

This together with (3.13) allows one to rewrite the equality (3.20) in the form

$$\log \frac{\det \Delta_{f^*\beta}}{\deg f} = \frac{1}{18} \sum_{k: x_k \neq \infty} \sum_{\ell: x_k \neq x_\ell \neq \infty} \frac{(\operatorname{ord}_k f - 2)(\operatorname{ord}_\ell f - 2)}{\operatorname{ord}_k f + 1} \log |x_k - x_\ell|$$
$$+\frac{1}{6}\left(\deg f - \sum_{k=1}^n \frac{3}{\operatorname{ord}_k f + 1}\right) \log cA_f - \sum_{k=1}^n \mathcal{C}\left(\frac{\operatorname{ord}_k f - 2}{3}\right) + \mathbf{C}.$$

Substituting this into the left-hand side of (3.15), we finally arrive at (2.10). This completes the proof. □

Remark 3.5. *Let c_k be the coefficient of Taylor or Laurent series from Proposition 3.1. Let A_f be the constant defined in* (2.11)*. Then the asymptotics* (3.19) *together with* (3.2) *and* (3.3) *implies*

$$A_f = \begin{cases} (\operatorname{ord}_k f + 1)|c_k|^{1/3}, & \text{where } k \text{ is such that } x_k = \infty \text{ and } f(\infty) \neq \infty, \\ (\operatorname{ord}_k f + 1)|c_k|^{-1/3}, & \text{where } k \text{ is such that } x_k = \infty \text{ and } f(\infty) = \infty. \end{cases}$$

4 Uniformization, Liouville action, and stationary points of the determinant

Consider a constant curvature sphere $(\overline{\mathbb{C}}_x, e^{2\varphi}|dx|^2)$ with conical singularities of order β_k located at $x_k \in \overline{\mathbb{C}}_x$. The parameters $x_1, \dots, x_n$ are called moduli. By the Gauss-Bonnet theorem [63], the (regularized) Gaussian curvature K of the singular sphere $(\overline{\mathbb{C}}_x, e^{2\phi}|dx|^2)$ satisfies the equality $K = 2\pi(|\boldsymbol{\beta}|+2)/S_\varphi$, where $|\boldsymbol{\beta}| = \sum_k \beta_k$ is the degree of the divisor $\boldsymbol{\beta} = \sum \beta_k \cdot x_k$, and

$$S_\varphi = \int_{\mathbb{C}} e^{2\varphi} \frac{dx \wedge d\bar{x}}{-2i} \tag{4.1}$$

is the total surface area of $(\overline{\mathbb{C}}_x, e^{2\varphi}|dx|^2)$.

The potential φ of the metric $e^{2\varphi}|dx|^2$ is a solution to the Liouville equation

$$e^{-2\varphi}(-4\partial_x \partial_{\bar{x}} \varphi) = K, \quad x \in \mathbb{C} \setminus \{x_1, x_2, \dots, x_n\}, \tag{4.2}$$

having the following asymptotics

$$\begin{aligned} \varphi(x) &= \beta_k \log|x - x_k| + \varphi_k + o(1), \quad x \to x_k \neq \infty, \\ \varphi(x) &= -(\beta_k + 2)\log|x| + \varphi_k + o(1), \quad x \to x_k = \infty. \end{aligned} \tag{4.3}$$

The metric potential φ and the coefficients φ_k in the asymptotics depend on the divisor $\boldsymbol{\beta}$, i.e. on the moduli x_k and the orders β_k of the conical singularities.

Introduce the classical stress-energy tensor $T_\varphi := 2(\partial_x^2\varphi - (\partial_x\varphi)^2)$ of the Liouville field theory. The stress-energy tensor is a meromorphic function on $\overline{\mathbb{C}}$ satisfying

$$\begin{aligned} T_\varphi(x) &= \sum_{k: x_k \neq \infty} \left(\frac{s_k}{2(x - x_k)^2} + \frac{h_k}{x - x_k}\right), \\ T_\varphi(x) &= \frac{s_k}{2x^2} + \frac{h_k}{x^3} + O(x^{-4}) \text{ as } x \to x_k = \infty. \end{aligned} \tag{4.4}$$

Here $s_k = -\beta_k(2+\beta_k)$ are the weights of the second order poles, and h_k are the famous accessory parameters, e.g. [20, 35, 59] and references therein. Note that the meromorphic quadratic differential $T_\varphi dx^2$ is a uniformizing projective connection compatible with the divisor $\boldsymbol{\beta}$ [63, 62].

Recall that one of the approaches to the uniformization consists of finding appropriate values of the accessory parameters h_k, and two appropriately normalized linearly independent solutions u_1 and u_2 to the Fuchsian differential equation

$$\partial_x^2 u + \frac{1}{2} T_\varphi u = 0.$$

Then the metric potential φ can be found in the form

$$\varphi = \log 2 + \log|\partial_x w| - \log(1 + K|w|^2),$$

where $w = u_1/u_2$ is an analytic in $\mathbb{C} \setminus \{x_1, \dots, x_n\}$ function called the developing map. The developing map satisfies the Schwarzian differential equation $\{w, x\} = T_\varphi(x)$, where $\{w, x\} = \frac{2w'w''' - 3w''^2}{2w'^2}$ is the Schwarzian derivative. However, the accessory parameters can be determined in some special cases only, and, in general, they remain elusive.

In the geometric setting of this paper, the accessory parameters can be found explicitly in terms of the Belyi function f and the orders β_0, β_1, and β_∞ of the conical singularities of the metric $m_{\boldsymbol{\beta}} = e^{2\phi}|dz|^2$.

Indeed, consider the constant curvature unit area singular sphere $(\overline{\mathbb{C}}_z, m_{\boldsymbol{\beta}})$, see Sec. 2. For the corresponding stress-energy tensor we have

$$T_\phi(z) = \frac{\mathfrak{s}_0}{2z^2} + \frac{\mathfrak{h}_0}{z} + \frac{\mathfrak{s}_1}{2(z-1)^2} + \frac{\mathfrak{h}_1}{z-1}, \quad T_\phi(z) = \frac{\mathfrak{s}_\infty}{2z^2} + \frac{\mathfrak{h}_\infty}{z^3} + O(z^{-4}) \quad \text{as} \quad z \to \infty,$$

where

$$\mathfrak{s}_k = -\beta_k(2+\beta_k), \quad k \in \{0, 1, \infty\}. \tag{4.5}$$

The accessory parameters

$$\mathfrak{h}_0 = -\mathfrak{h}_1 = \frac{\mathfrak{s}_0 + \mathfrak{s}_1 - \mathfrak{s}_\infty}{2}, \quad \mathfrak{h}_\infty = \frac{\mathfrak{s}_1 + \mathfrak{s}_\infty - \mathfrak{s}_0}{2} \tag{4.6}$$

were first found by Schwarz. The stress-energy tensors satisfy the relation

$$T_{f^*\phi} = (T_\phi \circ f)(f')^2 + \{f, x\}. \tag{4.7}$$

As a consequence, we obtain the following simple result:

Lemma 4.1 (Accessory parameters)**.** *Let $f : \overline{\mathbb{C}}_x \to \overline{\mathbb{C}}_z$ be a Belyi function unramified outside of the set $\{0, 1, \infty\}$ and such that $f(\infty) \in \{0, 1, \infty\}$. Then the stress-energy tensor $T_{f^*\phi}$ of the pull back metric $f^*m_\beta = e^{f^*\phi}|dx|^2$ of m_β by f satisfies the relations (4.4), where $x_1, \dots, x_n$ with $n = \deg f + 2$ are the preimages of the points $\{0, 1, \infty\} \subset \overline{\mathbb{C}}_z$ under f. The weights s_k of the second order poles in (4.4) are given by the equalities*

$$s_k = -(f^*\boldsymbol{\beta})_k((f^*\boldsymbol{\beta})_k + 2)$$

with $(f^*\boldsymbol{\beta})_k$ *from* (2.4), *and the accessory parameters* h_k *can be found as follows:*

1. If x_k *is not a pole of* f, *then the accessory parameter* h_k *satisfies*

$$h_k = \begin{cases} \frac{d_k}{c_k}\mathfrak{s}_{f(x_k)} + c_k\mathfrak{h}_{f(x_k)} & \text{for} \quad \operatorname{ord}_k f = 0, \\ -\frac{d_k}{c_k}\frac{(f^*\beta)_k((f^*\beta)_k+2)}{\operatorname{ord}_k f+1} & \text{for} \quad \operatorname{ord}_k f > 0, \end{cases} \tag{4.8}$$

where $\mathfrak{s}_k$ *and* $\mathfrak{h}_k$ *are the same as in* (4.5) *and* (4.6), *and the coefficients* c_k *and* d_k *are those from the first or the second expansion*

$$f(x) - f(x_k) = c_k x^{-\operatorname{ord}_k f - 1}\left(1 + \frac{d_k}{c_k}\frac{1}{x} + O(x^{-2})\right), \quad x \to x_k = \infty, \tag{4.9}$$

$$f(x) - f(x_k) = c_k(x - x_k)^{\operatorname{ord}_k f+1}\left(1 + \frac{d_k}{c_k}(x - x_k) + O\left((x - x_k)^2\right)\right), \quad x \to x_k \neq \infty,$$

depending on whether $x_k = \infty$ *or* $x_k \neq \infty$,

2. If x_k *is a pole of* f, *then the accessory parameter* h_k *satisfies*

$$h_k = \begin{cases} -\frac{d_k}{c_k}\mathfrak{s}_\infty + \frac{1}{c_k}\mathfrak{h}_\infty & \text{for} \quad \operatorname{ord}_k f = 0, \\ \frac{d_k}{c_k}\frac{(f^*\beta)_k((f^*\beta)_k+2)}{\operatorname{ord}_k f+1} & \text{for} \quad \operatorname{ord}_k f > 0, \end{cases} \tag{4.10}$$

where $\mathfrak{s}_\infty$ *and* $\mathfrak{h}_\infty$ *are the same in* (4.5) *and* (4.6), *and the coefficients* c_k *and* d_k *are those from the first or the second expansion*

$$f(x) = c_k x^{\operatorname{ord}_k f+1}\left(1 + \frac{d_k}{c_k}\frac{1}{x} + O(x^{-2})\right), \quad x \to x_k = \infty, \tag{4.11}$$

$$f(x) = c_k(x - x_k)^{-\operatorname{ord}_k f-1}\left(1 + \frac{d_k}{c_k}(x - x_k) + O((x - x_k)^2)\right), \quad x \to x_k \neq \infty,$$

depending on whether $x_k = \infty$ *or* $x_k \neq \infty$.

Proof. If $f(x_k) \neq \infty$ and $x_k = \infty$, then we start with the asymptotics (4.9) that can be differentiated. As a consequence, for the contributions into

$$(T_\phi \circ f)(f')^2(x) = \left(\frac{-\beta_{f(x_k)}(2 + \beta_{f(x_k)})}{2(f(x) - f(x_k))^2} + \frac{\mathfrak{h}_{f(x_k)}}{f(x) - f(x_k)} + O(1)\right)(f'(x))^2$$

we obtain

$$\frac{(f'(x))^2}{(f(x) - f(x_k))^2} = \frac{(\operatorname{ord}_k f + 1)^2}{x^2} + 2\frac{d_k}{c_k}\frac{\operatorname{ord}_k f + 1}{x^3} + O(x^{-4}),$$

$$\frac{h_{f(x_k)}}{f(x) - f(x_k)}(f'(x))^2 = h_{f(x_k)}c_k(\operatorname{ord}_k f + 1)^2 x^{-\operatorname{ord}_k f - 3} + O(x^{-4}).$$

Besides, for the Schwarzian derivative, we get

$$\{f, x\} = -\frac{1}{2}(\operatorname{ord}_k f)(\operatorname{ord}_k f + 2)\left(\frac{1}{x^2} + \frac{2d_k}{c_k(\operatorname{ord}_k f + 1)}\frac{1}{x^3} + O(x^{-4})\right), \quad x \to x_k = \infty.$$

These together with (4.7) imply

$$T_{f^*\phi}(x) = -\frac{(f^*\beta)_k((f^*\beta)_k+2)}{2x^2} + \frac{h_k}{x^3} + O(x^{-4}) \quad \text{as} \quad x \to x_k = \infty,$$

where the accessory parameter h_k satisfies (4.8).

Similarly, if $f(x_k) = \infty$ and $x_k = \infty$, then starting with the asymptotics (4.11), we arrive at

$$T_{f^*\phi}(x) = -\frac{(f^*\beta)_k((f^*\beta)_k+2)}{2x^2} + \frac{h_k}{x^3} + O(x^{-4}) \quad \text{as} \quad x \to x_k = \infty,$$

where the accessory parameter h_k satisfies (4.10).

The cases $f(x_k) = \infty$ with $x_k \neq \infty$, and $f(x_k) \neq \infty$ with $x_k \neq \infty$ are similar, we omit the details. □

Remark 4.2. *In particular, by using Lemma 4.1 with* $\boldsymbol{\beta} = \beta_0 \cdot 0 + \beta_1 \cdot 1 + \left(-\frac{2}{3}\right) \cdot \infty$ *and* $f(x) = x^3$*, we recover the values of the accessory parameters found in [35, Sec. 4.13] for a constant curvature Hyperbolic or Euclidean metric with four conical singularities located at the vertices of a regular tetrahedron.*

Introduce the Liouville action

$$\mathcal{S}_{\boldsymbol{\beta}}[\varphi] = 2\pi(|\boldsymbol{\beta}|+2)\left(\frac{1}{S_\varphi}\int_{\mathbb{C}} \varphi e^{2\varphi} \frac{dx \wedge d\bar{x}}{-2i} - 1\right) + 2\pi \sum_k \beta_k \varphi_k + 4\pi \varphi_k|_{k: x_k = \infty}, \qquad (4.12)$$

where S_φ is the total area of the singular sphere $(\overline{\mathbb{C}}_x, e^{2\varphi}|dx|^2)$, and φ_k are the coefficients in the asymptotics (4.3). As is demonstrated in [23], this new definition of the Liouville action is in agreement, for instance, with that in [9, 69, 59]. It is not hard to show that the Liouville equation (4.2) is the Euler-Lagrange equation for the Liouville action functional $\psi \mapsto \mathcal{S}_{\boldsymbol{\beta}}[\psi]$.

Remark 4.3. *In the geometric setting of this paper we have* $\varphi = f^*\phi$*, and the anomaly formula from Lemma 3.2 can equivalently be written as follows:*

$$\log \frac{\det \Delta_{f^*\boldsymbol{\beta}}}{\deg f} = \frac{|f^*\boldsymbol{\beta}|+2}{6} - \frac{1}{12\pi}\left(\mathcal{S}_{f^*\boldsymbol{\beta}}[f^*\phi] - \pi \log \mathcal{H}_{f^*\boldsymbol{\beta}}[f^*\phi]\right) - \sum_k \mathcal{C}((f^*\beta)_k) + \mathbf{C}.$$

Here the functional $\mathcal{H}_{f^*\boldsymbol{\beta}}[f^*\phi]$ *is defined explicitly via the equality*

$$\mathcal{H}_{f^*\boldsymbol{\beta}}[f^*\phi] = \exp\left(2\sum_k \frac{(f^*\beta)_k((f^*\beta)_k+2)}{(f^*\beta)_k+1}(f^*\phi)_k\right)$$

with $(f^*\beta)_k$ *and* $(f^*\phi)_k$ *from* (2.4) *and* (3.3) *respectively. Similar functionals also appear in [8, 23, 25, 58].*

Thus, as a consequence of the explicit expression for the determinant of Laplacian in Theorem 2.1 and the anomaly formula in Lemma 3.2, one can immediately obtain an explicit expression for the Liouville action $\mathcal{S}_{f^*\boldsymbol{\beta}}[f^*\phi]$*, which can be of independent interest, cf. [38, 44, 45, 57]. It would also be very interesting to check if this result can be reproduced by using conformal blocks [69].*

In the remaining part of this section, we assume for simplicity that the orders β_k of the conical singularities meet the condition $|\boldsymbol{\beta}| \leqslant -2$, i.e. we exclude from consideration the spherical metrics. This allows us to differentiate the hyperbolic metric potential and the corresponding Liouville action with respect to x_k and β_k relying on the known (analytic) regularity results [29, 59], see also [29, 35, 50, 51, 59]. In the Euclidean case, we have $|\boldsymbol{\beta}| = -2$, and the metrics can be written explicitly [61], which immediately justifies the differentiation. Let us also note that there are good grounds to believe [21, 22, 23, 29, 62] that the potential φ of a constant curvature metric is necessarily a real-analytic function of the orders of conical singularities on the existence and uniqueness set

$$\{\beta_k \in (0, 1) : \beta_k - |\boldsymbol{\beta}|/2 > 0, k = 1, \dots, n\},$$

and the results below remain valid on that set.

Next, we show that the Liouville action $\mathcal{S}_{\boldsymbol{\beta}}[\varphi]$ generates the accessory parameters h_k as their common antiderivative.

Lemma 4.4 (After P. Zograf and L. Takhtajan)**.** *Assume that $\beta_k \in (0, 1)$, $k = 1, \dots, n$, and $|\boldsymbol{\beta}| \leqslant -2$. Let φ be a (unique) solution to the Liouville equation (4.2) satisfying the area condition (4.1) with some fixed $S_\varphi > 0$, and having the asymptotics (4.3). Then the Liouville action (4.12) meets the equalities*

$$-\frac{1}{2\pi}\partial_{x_k}\mathcal{S}_{\boldsymbol{\beta}}[\varphi] = h_k, \quad k = 1, \dots, n, \tag{4.13}$$

where $x_1, x_2, \dots, x_n$ are the moduli, and $h_1, h_2, \dots, h_n$ are the accessory parameters.

Note that in the geometric setting of this paper, we have $\varphi = f^*\phi$. Thus the moduli x_k, $k = 1, \dots, \deg f + 2$, are the preimages of the points $\{0, 1, \infty\}$ under f, and the accessory parameters h_k are those found in Lemma 4.1.

Proof. As is shown in [23], the expression in the right hand side of (4.12) is an equivalent regularization of the Liouville action introduced in [9, 69, 59]. Hence, in the case of $|\boldsymbol{\beta}| < -2$ and $K = -1$ the assertion of the lemma is just a reformulation of the result proven in [9, 59], see also [70] for the first proof of Polyakov's conjecture (4.13). Next we show that in the (hyperbolic) case $|\boldsymbol{\beta}| < -2$ the assertion remains valid for any $S_\varphi > 0$ and $K = 2\pi(|\boldsymbol{\beta}| + 2)/S_\varphi < 0$.

Consider a (unique) metric potential φ such that $|\boldsymbol{\beta}| < -2$ and $K = -1$. Clearly, $S_\varphi = -2\pi(|\boldsymbol{\beta}|+2)$, and for any $C > 0$ we have the following transformation laws for the surface area, the Liouville action, and the stress-energy tensor:

$$S_{\varphi+\log C} = C^2 S_\varphi, \quad \mathcal{S}_{\boldsymbol{\beta}}[\varphi + \log C] = \mathcal{S}_{\boldsymbol{\beta}}[\varphi] + 4\pi(|\boldsymbol{\beta}| + 2)\log C, \quad T_{\varphi+\log C} = T_\varphi.$$

This implies that the rescaling $\varphi \mapsto \varphi + \log C$ multiplies the total area by C^2, but does not affect the equalities (4.13). Thus, in the case $|\boldsymbol{\beta}| < -2$, the assertion of lemma is valid for any fixed $S_\varphi > 0$.

In the case $|\boldsymbol{\beta}| = -2$ the integral term in (4.12) disappears and the metric $e^{2\varphi}|dx|^2$ is flat. As is known [61], up to a rescaling $\varphi \mapsto \varphi + \log C$, the potential φ can be written explicitly in the form

$$\varphi(\boldsymbol{\beta}) = \sum_{k:x_k\neq\infty} \beta_k \log|x-x_k|, \quad |\boldsymbol{\beta}| = -2. \tag{4.14}$$

As a result, the equality (4.13) follows from a simple direct computation. □

It may also be possible to prove Polyakov's conjecture (4.13) for the spherical case $|\boldsymbol{\beta}| > -2$ along the lines of [9, 59], however this goes out of the scope of this paper.

Lemma 4.5 (After A. Zamolodchikov and Al. Zamolodchikov)**.** *Assume that $\beta_k \in (-1,0)$ and $|\boldsymbol{\beta}| \leqslant -2$. Let φ stand for a (unique) solution to the Liouville equation (4.2) satisfying the area condition (4.1) with a fixed $S_\varphi > 0$, and having the asymptotics (4.3). Then for any fixed configuration $x_1,\dots,x_n$ the Liouville action (4.12) satisfies*

$$-\frac{1}{2\pi}\partial_{\beta_k}\mathcal{S}_{\boldsymbol{\beta}}[\varphi] = 1-2\varphi_k, \quad k=1,\dots,n, \tag{4.15}$$

where φ_k is the coefficient in the corresponding asymptotics (4.3).

In the geometric setting of this paper, we have $\varphi = f^*\phi$ and $\varphi_k = (f^*\phi)_k$, see (3.3).

Proof. In the case $|\boldsymbol{\beta}| < -2$, the proof essentially repeats the one in [23, Proof of Lemma 3.1], where the differentiation with respect to β_k is now justified by the results [29, 50, 51] on the regularity of $\beta_k \mapsto \varphi(\boldsymbol{\beta})$ for the hyperbolic metric $e^{2\varphi}|dx|^2$ on the Riemann sphere. Indeed, one need only notice that the index $j=k$ in [23, Proof of Lemma 3.1] now runs from 1 to n, and the region $\mathbb{C}_R$ is defined as follows:

$$\mathbb{C}_R := \{x\in\mathbb{C} : |x| \leqslant R,\ |x-x_k| \geqslant 1/R,\ k=1,\dots n\}.$$

In the case $|\boldsymbol{\beta}| = -2$ the integral term in (4.12) disappears, and the equality (4.15) can be verified by a direct computation, cf. (4.14). We omit the details. □

Our choice of examples in Section 5 is partially motivated by the following result:

Theorem 4.6. *The (hyperbolic or flat) surfaces of five Platonic solids and the regular constant curvature dihedra are critical points of the spectral determinant on the conical metrics of fixed area and fixed Gaussian curvature.*

More precisely: Consider the divisors $\boldsymbol{\beta} = \sum_k \beta_k\cdot x_k$ of degree $|\boldsymbol{\beta}| \leqslant -2$ with distinct marked points $x_1,\dots,x_n$ and weights $\beta_k \in (-1,0)$. Then for any fixed $S>0$ and any divisor $\boldsymbol{\beta}$ there exists a unique metric $e^{2\varphi}|dx|^2$ on $\overline{\mathbb{C}}$ of total area S, Gaussian curvature $K = 2\pi(|\boldsymbol{\beta}|+2)/S$, and representing the divisor $\boldsymbol{\beta}$. Consider the spectral determinant $\det\Delta_{\boldsymbol{\beta}}$ *of the surface $(\overline{\mathbb{C}}_x, e^{2\varphi}|dx|^2)$ as a function on the configuration space*

$$\mathcal{Z}_n(S,K) = \left\{\boldsymbol{\beta} = \sum_{k\leqslant n}\beta_k\cdot x_k : x_j\neq x_k\in\overline{\mathbb{C}} \text{ for } j\neq k, \beta_k\in(-1,0), 2\pi(|\boldsymbol{\beta}|+2) = SK\right\}$$

with some fixed values $S>0$, $K\leqslant 0$, and $n\geqslant 3$.

1. *If $\boldsymbol{\beta}_0 \in \mathcal{Z}_n(S,K)$ is a divisor such that the corresponding surface $(\overline{\mathbb{C}}_x, e^{2\varphi}|dx|^2)$ is isometric to the one of a Platonic solid, then $\boldsymbol{\beta}_0$ is a stationary point of the function*
$$\mathcal{Z}_n(S,K) \ni \boldsymbol{\beta} \mapsto \det \Delta_{\boldsymbol{\beta}}, \tag{4.16}$$
where n is the number of vertices of the Platonic solid.

2. *If $\boldsymbol{\beta}_0 \in \mathcal{Z}_n(S,K)$ is a divisor such that the corresponding surface $(\overline{\mathbb{C}}_x, e^{2\varphi}|dx|^2)$ is isometric to the regular dihedron with n vertices, then $\boldsymbol{\beta}_0$ is a stationary point of the function (4.16) with the corresponding value of n.*

Proof. For simplicity, let $n=4$. Consider the potential $\varphi(x;\beta_1,\beta_2,\beta_3,\beta_4)$ of a (unique) unit area constant curvature metric $e^{2\varphi}|dx|^2$ representing the divisor

$$\boldsymbol{\beta} = \beta_1 \cdot 0 + \beta_2 \cdot (-1) + \beta_3 \cdot e^{i\frac{\pi}{3}} + \beta_4 \cdot e^{-i\frac{\pi}{3}}.$$

Recall that the Gauss-Bonnet theorem [63] implies that the (regularized) Gaussian curvature of the surface $(\overline{\mathbb{C}}_x, e^{2\varphi}|dx|^2)$ equals $2\pi(|\boldsymbol{\beta}|+2)$. The four marked points in the divisor $\boldsymbol{\beta}$ are in an equi-anharmonic position. In particular, if the orders of the conical singularities satisfy $\beta_k = |\boldsymbol{\beta}|/4$, then the surface $(\overline{\mathbb{C}}_x, e^{2\varphi}|dx|^2)$ is isometric to the one of a unit area regular tetrahedron of Gaussian curvature $2\pi(|\boldsymbol{\beta}|+2) \leqslant 0$.

Notice that $\varphi(\bar{x};\beta_1,\beta_2,\beta_3,\beta_4)$ is the potential of a (unique) unit area Gaussian curvature $2\pi(|\boldsymbol{\beta}|+2)$ metric representing the divisor

$$\boldsymbol{\beta} = \beta_1 \cdot 0 + \beta_2 \cdot (-1) + \beta_4 \cdot e^{i\frac{\pi}{3}} + \beta_3 \cdot e^{-i\frac{\pi}{3}}.$$

Similarly, the potential $\varphi(e^{i\frac{2\pi}{3}}x;\beta_1,\beta_2,\beta_3,\beta_4)$ corresponds to the divisor

$$\boldsymbol{\beta} = \beta_1 \cdot 0 + \beta_4 \cdot (-1) + \beta_2 \cdot e^{i\frac{\pi}{3}} + \beta_3 \cdot e^{-i\frac{\pi}{3}};$$

$\varphi(e^{-i\frac{2\pi}{3}}x;\beta_1,\beta_2,\beta_3,\beta_4)$ corresponds to the divisor

$$\boldsymbol{\beta} = \beta_1 \cdot 0 + \beta_3 \cdot (-1) + \beta_4 \cdot e^{i\frac{\pi}{3}} + \beta_2 \cdot e^{-i\frac{\pi}{3}};$$

and $\varphi(\frac{x+1}{2x-1};\beta_1,\beta_2,\beta_3,\beta_4) + \log 3 - 2\log|2x-1|$ corresponds to the divisor

$$\boldsymbol{\beta} = \beta_2 \cdot 0 + \beta_1 \cdot (-1) + \beta_4 \cdot e^{i\frac{\pi}{3}} + \beta_3 \cdot e^{-i\frac{\pi}{3}}.$$

As a consequence of these symmetries, we have

$$\begin{aligned}\varphi(x;\beta_1,\beta_2,\beta_3,\beta_4) &= \varphi(\bar{x};\beta_1,\beta_2,\beta_4,\beta_3) = \varphi(e^{i\frac{2\pi}{3}}x;\beta_1,\beta_3,\beta_4,\beta_2) = \varphi(e^{-i\frac{2\pi}{3}}x;\beta_1,\beta_4,\beta_2,\beta_3)\\ &= \varphi\left(\frac{x+1}{2x-1};\beta_2,\beta_1,\beta_4,\beta_3\right) + \log 3 - 2\log|2x-1|.\end{aligned}$$

For the coefficients $\varphi_k = \varphi_k(\beta_1,\beta_2,\beta_3,\beta_4)$ in the asymptotics (4.3) the latter equalities imply

$$\begin{aligned}(\varphi_1 - \varphi_\ell)|_{\beta_k = |\boldsymbol{\beta}|/4} &= (|\boldsymbol{\beta}|/4+1)\log 3, \quad \ell = 2,3,4,\\ \sum_{j=1}^{4}(\partial_{\beta_1}\varphi_j - \partial_{\beta_\ell}\varphi_j)|_{\beta_k=|\boldsymbol{\beta}|/4} &= \log 3, \quad \ell = 2,3,4.\end{aligned} \tag{4.17}$$

Denote the Friedrichs Laplacian on $(\overline{\mathbb{C}}_x, e^{2\varphi}|dx|^2)$ by $\Delta_{\boldsymbol\beta}$. As is proven in [23, Sec. 2], the spectral determinant $\det\Delta_{\boldsymbol\beta}$ satisfies the anomaly formula

$$\log\det\Delta_{\boldsymbol\beta}=\frac{|\boldsymbol\beta|+2}{6}-\frac{1}{12\pi}\left(\mathcal{S}_{\boldsymbol\beta}[\varphi]-\pi\log\mathcal{H}_{\boldsymbol\beta}[\varphi]\right)-\sum_{k=1}^{4}\mathcal{C}(\beta_k)+\mathbf{C},\tag{4.18}$$

where $\mathcal{S}_{\boldsymbol\beta}[\varphi]$ is the Liouville action (4.12) and

$$\mathcal{H}_{\boldsymbol\beta}[\varphi]=\exp\left\{2\sum_{k=1}^{4}\left(\beta_k+1-\frac{1}{\beta_k+1}\right)\varphi_k\right\}.\tag{4.19}$$

Since $|\boldsymbol\beta|$ is fixed, we can set, for example, $\beta_1=|\boldsymbol\beta|-\sum_{k>1}\beta_k$, and consider the determinant $\det\Delta_{\boldsymbol\beta}$ as a function of $(\beta_2,\beta_3,\beta_4)$. Then, thanks to the anomaly formula (4.18), Lemma 4.5, and the equality (4.19), we have

$$\begin{aligned}\partial_{\beta_\ell}\left(\log\det\Delta_{\boldsymbol\beta}|_{\beta_1=|\boldsymbol\beta|-\sum_{k>1}\beta_k}\right)|_{\beta_k=\frac{|\boldsymbol\beta|}{4}}&=\frac{1}{3}(\varphi_1-\varphi_\ell)|_{\beta_k=\frac{|\boldsymbol\beta|}{4}}\\&-\frac16\left(1+\left(\frac{|\boldsymbol\beta|}{4}+1\right)^{-2}\right)(\varphi_1-\varphi_\ell)|_{\beta_k=\frac{|\boldsymbol\beta|}{4}}\\&-\frac16\left(\frac{|\boldsymbol\beta|}{4}+1-\frac{1}{\frac{|\boldsymbol\beta|}{4}+1}\right)\sum_{j=1}^{4}(\partial_{\beta_1}\varphi_j-\partial_{\beta_\ell}\varphi_j)|_{\beta_k=\frac{|\boldsymbol\beta|}{4}}.\end{aligned}$$

Here the right-hand side is equal to zero because of (4.17).

Now we are in a position to study the determinant under a small perturbation of the coordinate of a vertex. Let us consider the potential $\varphi(x)$ of a (unique) unit area constant curvature metric $e^{2\varphi}|dx|^2$ representing the divisor

$$\boldsymbol\beta=\frac{|\boldsymbol\beta|}{4}\cdot h+\frac{|\boldsymbol\beta|}{4}\cdot(-1)+\frac{|\boldsymbol\beta|}{4}\cdot e^{i\frac{\pi}{3}}+\frac{|\boldsymbol\beta|}{4}\cdot e^{-i\frac{\pi}{3}}.$$

Here h is a complex number, and $|h|$ is small. In the case $h=0$, the surface $(\overline{\mathbb{C}}_x, e^{2\varphi}|dx|^2)$ is isometric to the one of a unit area constant curvature regular tetrahedron.

Consider, for example, the rotation $x\mapsto e^{i\frac{2\pi}{3}}x$. Notice that $\chi(x):=\varphi(e^{i\frac{2\pi}{3}}x)$ is the potential of a (unique) unit area Gaussian curvature $2\pi(|\boldsymbol\beta|+2)$ metric representing the divisor

$$\boldsymbol\gamma=\frac{|\boldsymbol\beta|}{4}\cdot(e^{-i\frac{2\pi}{3}}h)+\frac{|\boldsymbol\beta|}{4}\cdot(-1)+\frac{|\boldsymbol\beta|}{4}\cdot e^{i\frac{\pi}{3}}+\frac{|\boldsymbol\beta|}{4}\cdot e^{-i\frac{\pi}{3}}.$$

The surfaces $(\overline{\mathbb{C}}_x, e^{2\varphi}|dx|^2)$ and $(\overline{\mathbb{C}}_x, e^{2\chi}|dx|^2)$ are isometric, the isometry is given by the rotation. As a consequence, $\det\Delta_{\boldsymbol\beta}=\det\Delta_{\boldsymbol\gamma}$. Equating the directional derivative of $\det\Delta_{\boldsymbol\beta}$ along h with the one along $e^{i\frac{2\pi}{3}}h$, we immediately conclude that

$$\partial_{\Re h}\det\Delta_{\boldsymbol\beta}=\partial_{\Im h}\det\Delta_{\boldsymbol\beta}=0.$$

It remains to note that the determinants of the Laplacians $\Delta_{\boldsymbol\beta}$ and $\Delta_{\boldsymbol\beta}^{S_\varphi}=\frac{1}{S_\varphi}\Delta_{\boldsymbol\beta}$ satisfy the standard rescaling property

$$\log\det\Delta_{\boldsymbol\beta}^{S_\varphi}=\log\det\Delta_{\boldsymbol\beta}+\zeta_{\boldsymbol\beta}(0)\log S_\varphi,$$

where the value $\zeta_{\boldsymbol\beta}(0)$ of the spectral zeta function at zero [25] does not depend on the moduli $x_1,\dots,x_4$ and satisfies

$$\zeta_{\boldsymbol\beta}(0)=\frac{|\boldsymbol\beta|+2}{6}-\frac{1}{12}\sum_k\left(\beta_k+1-\frac{1}{\beta_k+1}\right)-1,\quad \partial_{\beta_\ell}\left(\zeta_{\boldsymbol\beta}(0)|_{\beta_1=|\boldsymbol\beta|-\sum_{k>1}\beta_k}\right)|_{\beta_k=\frac{|\boldsymbol\beta|}{4}}=0.$$

Due to the invariance of the spectral determinant under the Möbius transformations, this completes the proof of the first assertion.

For the octahedron, cube, dodecahedron, icosahedron, and dihedra there are more symmetries to consider, but the idea and the steps of the proof remain exactly the same. We omit the details. The case of constant curvature (flat, spherical, or hyperbolic) metrics with three conical singularities is studied in [23]. □

As is well-known, starting from four punctures on the 2-sphere, explicit construction of the general uniformization map is an open long-standing problem. In this paper, we rely on the uniformization via Belyi functions. There is another straightforward special case that deserves to be mentioned.

Remark 4.7. In the case of a divisor

$$\boldsymbol\lambda=(-1/2)\cdot 0+(-1/2)\cdot 1+(-1/2)\cdot\lambda+(-1/2)\cdot\infty,\quad \lambda\in\mathbb{C},$$

the corresponding constant curvature unit area metric $m_{\boldsymbol\lambda}$ with three or four conical singularities of angle π can be written explicitly, e.g. [6, 35, 61].

Recall that by using a suitable Möbius transformation we can always normalize the marked points so that any three of them are at $0,1,\infty$. As we permute the marked points $0,1,\lambda,\infty$ by Möbius transformations so that three of them are still $0,1,\infty$, the fourth point is one of the following six:

$$\lambda,\quad \frac{1}{\lambda},\quad 1-\lambda,\quad \frac{1}{1-\lambda},\quad \frac{\lambda}{\lambda-1},\quad \frac{\lambda-1}{\lambda}. \tag{4.20}$$

In general, these six points are distinct. The exceptions are the following three cases:

- $\lambda=0$ or $\lambda=1$ or $\lambda=\infty$. In this case, the set (4.20) contains only three distinct numbers: $0,1,\infty$. This case is studied in [23], we do not discuss it here.
- Harmonic position of four points: $\lambda=-1$ or $\lambda=1/2$ or $\lambda=2$. The set (4.20) contains only three distinct numbers: $-1,1/2,2$. The surface $(\overline{\mathbb{C}}_x,m_{\boldsymbol\lambda})$ is isometric to a unit area flat regular dihedron with four conical singularities of angle π.
- Equi-anharmonic position of four points: $\lambda=\frac{1+i\sqrt3}{2}$ or $\lambda=\frac{1-i\sqrt3}{2}$. The set (4.20) contains only the numbers $\frac{1\pm i\sqrt3}{2}$. The surface $(\overline{\mathbb{C}}_x,m_{\boldsymbol\lambda})$ is isometric to the surface of a unit area regular Euclidean tetrahedron.

In general, for $\lambda \notin \{0,1,\infty\}$, the metric is flat, and we have

$$m_{\boldsymbol{\lambda}} = \frac{c_\lambda^2 |dx|^2}{|x(x-1)(x-\lambda)|},$$

where c_λ^2 is a scaling factor that guarantees that the surface $(\overline{\mathbb{C}}_x, m_{\boldsymbol{\lambda}})$ is of unit area, see [61]. The surface $(\overline{\mathbb{C}}_x, m_{\boldsymbol{\lambda}})$ is isometric to the surface of a Euclidean tetrahedron with (four) vertices of angle π.

For the spectral determinant of the Friedrichs Laplacian Δ_λ on $(\overline{\mathbb{C}}_x, m_{\boldsymbol{\lambda}})$ the anomaly formula (4.18) gives

$$\log\det\Delta_\lambda = -\log c_\lambda + \frac{1}{6}(\log|\lambda| + \log|\lambda-1|) - 4\mathcal{C}(-1/2) + \mathbf{C},$$

where $\mathbf{C}$ is the same as in (2.9). Besides, thanks to [25, Appendix], we have

$$\mathcal{C}\left(-\frac{1}{2}\right) = -\zeta_R'(-1) - \frac{1}{6}\log 2 + \frac{1}{24}. \tag{4.21}$$

Thanks to the second Riemann identity, see e.g. [10, Sec. 2.9], the scaling factor c_λ^2 satisfies

$$c_\lambda^{-2} = \int_{\mathbb{C}} \frac{1}{|x(x-1)(x-\lambda)|}\frac{dx\wedge d\bar{x}}{-2i} = 8|k|\left(K'\overline{K} + \overline{K'}K\right), \quad \lambda = \frac{(k+1)^2}{4k},$$

where $K = K(k)$ is the complete elliptic integral of the first kind, and $K' = K(\sqrt{1-k^2})$.

In total, in terms of $\tau = iK'/K$, we get

$$K = \frac{\pi}{2}\vartheta_3^2(0|\tau), \quad k = \frac{\vartheta_2^2(0|\tau)}{\vartheta_3^2(0|\tau)},$$

$$\det\Delta_\lambda = \frac{2^{2/3}}{\pi}|1-k^2|^{1/3}|k|^{1/6}\sqrt{\Im\tau}|K| = \sqrt{\Im\tau}|\eta(\tau/2)|^2.$$

Here η is the Dedekind eta function, and ϑ_j stands for the j^{th} Jacobi theta function. The last equality is an immediate consequence of the well-known identities

$$2\eta^3(\tau) = \vartheta_2(0|\tau)\vartheta_3(0|\tau)\vartheta_4(0|\tau), \quad \eta^2(\tau/2) = \vartheta_4(0|\tau)\eta(\tau), \quad \vartheta_3^4(0|\tau) = \vartheta_2^4(0|\tau) + \vartheta_4^4(0|\tau).$$

By analyzing the expression $\sqrt{\Im\tau}|\eta(\tau/2)|^2$, it is not hard to see that there are only two stationary points: $\tau = 2i$ is a saddle point, and $\tau = 2e^{2\pi i/3}$ is the unique absolute maximum of the determinant $\mathbb{C}\setminus\{0,1\} \ni \lambda \mapsto \det\Delta_\lambda$, cf. [41, Sec. 4]. The case $\tau = 2i$ (resp. $\tau = 2e^{2\pi i/3}$) corresponds to a harmonic (resp. to an equi-anharmonic) position of four points in the divisor $\boldsymbol{\lambda}$, cf. Theorem 4.6.

We believe that these stationary points can also be found along the lines of [33, Sec. 3.5.1], if one takes into account the observation from [14, Example 3.5, p.42] that "the height h of the single flat cylinder of the covering torus is twice bigger than the height of the single flat cylinder on the underlying flat sphere..."

5 Examples and applications

5.1 Determinant for triangulations by plane trees

By Riemann's existence theorem, the planar bicolored trees are in one-to-one correspondence with the (classes of equivalence of) Shabat polynomials [4, 36], see also [5]. Recall that a Shabat polynomial, also known as a generalized Chebyshev polynomial, is a polynomial with at most two critical values. Thanks to Theorem 2.1, to each bicolored plane tree we can associate a family of spectral invariants $\det \Delta_{f^*\boldsymbol{\beta}}$. Indeed, a Belyi function $f : \overline{\mathbb{C}}_x \to \overline{\mathbb{C}}_z$ (in this case it is a Shabat polynomial) only prescribes a certain gluing scheme of the bicolored double triangles. We can still make any suitable choice of the angles of those triangles, or, equivalently, of the orders β_0, β_1, and β_∞ of three conical singularities of the constant curvature metric $e^{2\phi}|dz|^2$ on the target Riemann sphere $\overline{\mathbb{C}}_z$.

As an example, consider the Shabat polynomial

$$f(x) = x^\ell, \quad \ell \in \mathbb{N}.$$

The ramification divisor is

$$\boldsymbol{f} = (\ell - 1) \cdot 0 + (\ell - 1) \cdot \infty, \quad |\boldsymbol{f}| = 2\ell - 2,$$

where $x = 0$ is the only point with $f'(x) = 0$, and $x = \infty$ is the only pole of f. The corresponding bicolored tree is the inverse image of the line segment $[0, 1]$ under f, see Fig. 5. The black colored point is the preimage of the point $z = 0$, and the

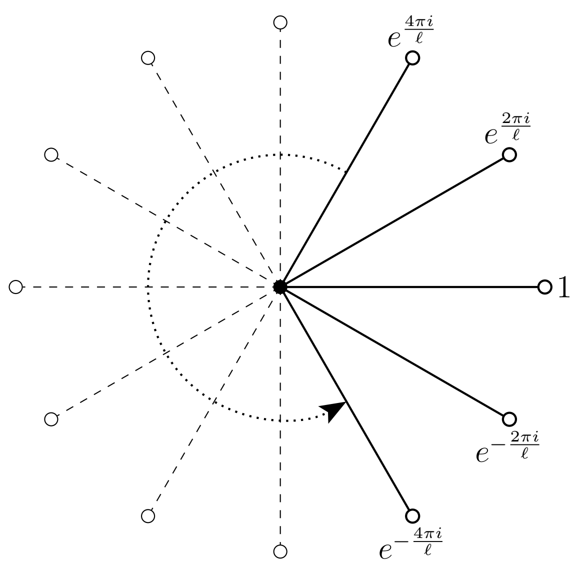

Figure 5: Dessin d'enfant representing the polynomial $f(x) = x^\ell$

white colored points are the ℓ preimages $x = \sqrt[\ell]{1}$ of $z = 1$. This describes the cyclic triangulation of the Riemann sphere, or, equivalently, the tessellation of the standard round sphere with (ℓ, ∞, ℓ) bicolored double triangles, see Fig. 6.

Clearly, the first non-zero coefficient in the Taylor expansion of $f - f(0)$ at zero is $c_1 = 1$, and the first non-zero coefficient in the Laurent expansion of f at infinity is $c_2 = 1$. Hence, the equality (3.1) immediately implies

$$C_f = \frac{1}{6}\left(\frac{\ell-1}{\ell}\log|c_1| + (\ell+1)\log\ell\right) + \frac{1}{6}\left(-\frac{\ell-1}{\ell}\log|c_2| - (\ell+1)\log\ell\right) = 0, \quad (5.1)$$

where C_f is the constant from Theorem 3.1.

The pullback of the divisor $\boldsymbol{\beta} = \beta_0 \cdot 0 + \beta_1 \cdot 1 + \beta_\infty \cdot \infty$ by $f(x) = x^\ell$ is the divisor $f^*\boldsymbol{\beta} = \sum_k (f^*\boldsymbol{\beta})_k \cdot x_k$. For the latter one we have

$$f^*\boldsymbol{\beta} = (\ell(\beta_0+1)-1)\cdot 0 + \beta_1 \cdot \{\sqrt[\ell]{1}\} + (\ell(\beta_\infty+1)-1)\cdot\infty,$$

where $\{\sqrt[\ell]{1}\}$ stands for the set of ℓ radicals $\sqrt[\ell]{1}$ in $\mathbb{C}_x$ (those are the white colored points of the "snowflake" in Fig. 5). The notation $\beta_1 \cdot \{\sqrt[\ell]{1}\}$ means that each element of the set $\{\sqrt[\ell]{1}\}$ is a marked point of weight β_1.

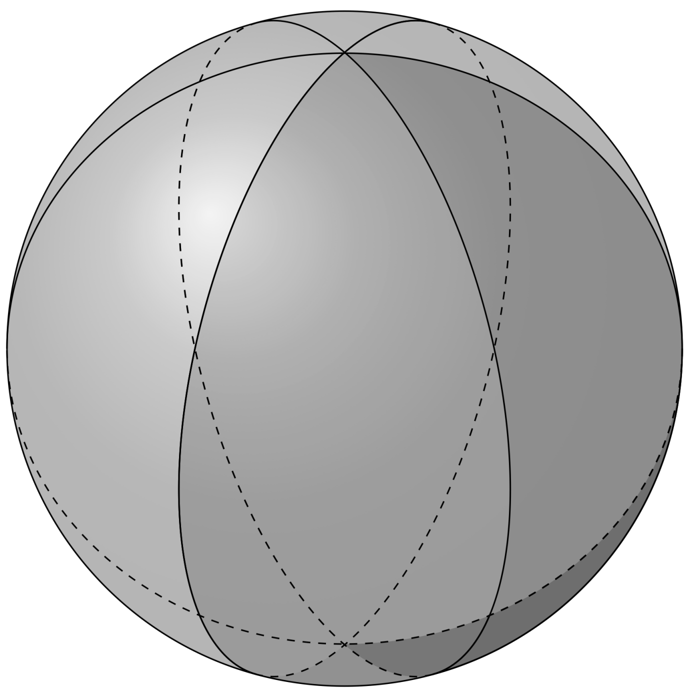

Figure 6: Cyclic triangulation

Theorem 5.1 (Cyclic triangulation)**.** *Let $m_{\boldsymbol{\beta}}$ be the unit area Gaussian curvature $2\pi(|\boldsymbol{\beta}|+1)$ metric of S^2-like double triangle, see Section 2. Let $f(x) = x^\ell$ with $\ell \in \mathbb{N}$, cf. Fig 6. Then for the zeta-regularized spectral determinant of the Friedrichs Laplacian*

$\Delta_{f^*\boldsymbol\beta}$ corresponding to the area ℓ pullback metric $f^*m_{\boldsymbol\beta}=e^{2f^*\phi}|dx|^2$ we have

$$\begin{aligned}\log\frac{\det\Delta_{f^*\boldsymbol\beta}}{\ell}=&\ell\Big(\log\det\Delta_{\boldsymbol\beta}+\mathcal{C}(\beta_0)+\mathcal{C}(\beta_\infty)-\mathbf{C}\Big)\\&+\frac16\left(\ell-\frac1\ell\right)\left(\frac{\Psi(\beta_0,\beta_1,\beta_\infty)}{\beta_0+1}+\frac{\Psi(\beta_\infty,\beta_1,\beta_0)}{\beta_\infty+1}\right)\\&-\frac16\left(\ell(\beta_0+\beta_\infty+2)+\frac{1}{\ell(\beta_0+1)}+\frac{1}{\ell(\beta_\infty+1)}\right)\log\ell\\&-\mathcal{C}\left(\ell(\beta_0+1)-1\right)-\mathcal{C}\left(\ell(\beta_\infty+1)-1\right)+\mathbf{C},\end{aligned}$$

where the right hand side is an explicit function of $\ell\in\mathbb{N}$ and $(\beta_0,\beta_1,\beta_\infty)\in(-1,0]^3$.

Here $\beta\mapsto\mathcal{C}(\beta)$ is the function (2.8), $\mathbf{C}$ is the constant introduced in (2.9), and the function

$$\begin{aligned}\Psi(\beta_0,\beta_1,\beta_\infty)=&\log\frac{\Gamma(-\beta_0)}{\Gamma(1+\beta_0)}\\&+\frac12\log\frac{\Gamma\left(2+\frac{|\boldsymbol\beta|}{2}\right)\Gamma\left(\beta_0-\frac{|\boldsymbol\beta|}{2}\right)\Gamma\left(1+\frac{|\boldsymbol\beta|}{2}-\beta_1\right)\Gamma\left(1+\frac{|\boldsymbol\beta|}{2}-\beta_\infty\right)}{\pi\Gamma\left(-\frac{|\boldsymbol\beta|}{2}\right)\Gamma\left(1+\frac{|\boldsymbol\beta|}{2}-\beta_0\right)\Gamma\left(\beta_1-\frac{|\boldsymbol\beta|}{2}\right)\Gamma\left(\beta_\infty-\frac{|\boldsymbol\beta|}{2}\right)}\end{aligned}\tag{5.2}$$

is the one from (2.6). Recall that $\det\Delta_{\boldsymbol\beta}$ stands for an explicit function of $(\beta_0,\beta_1,\beta_\infty)\in(-1,0]^3$, whose values are the determinants of the unit area S^2-like double triangles of Gaussian curvature $2\pi(|\boldsymbol\beta|+2)$, see [23] .

Proof. Recall that $\Psi(\beta_1,\beta_2,\beta_3)=\Phi(\beta_1,\beta_2,\beta_3)+\log 2$ with explicit function Φ from [23, Prop. A.2)]. This implies (5.2), where Γ stands for the Gamma function.

Now the assertion is an immediate consequence of Theorem 2.1 together with the asymptotics (2.6), and the calculation of C_f in (5.1). □

Example 5.2 (Dihedra)**.** For the determinant of the Gaussian curvature $2\pi(\beta+2/\ell)$ area ℓ regular dihedron with ℓ conical singularities of order β we obtain

$$\begin{aligned}\log\det\Delta^{\ell}_{Dihedron}=&\ell\log\det\Delta_{\boldsymbol\beta}+2\ell\mathcal{C}\left(\frac1\ell-1\right)\\&+\frac{\ell^2-1}{3}\Psi\left(\frac1\ell-1,\beta,\frac1\ell-1\right)+\frac13\log\ell+(1-\ell)\mathbf{C}.\end{aligned}\tag{5.3}$$

This is a direct consequence of Theorem 5.1, where we take

$$\boldsymbol\beta=\left(\frac1\ell-1\right)\cdot0+\beta\cdot1+\left(\frac1\ell-1\right)\cdot\infty.$$

In particular, when $\beta=-2/\ell$, we obtain the determinant of the flat regular dihedron of area ℓ. In the case $\beta=0$, we obtain a surface isometric to the round sphere in $\mathbb{R}^3$ of area ℓ and its determinant. Finally, as $\beta \to -1^+$ the determinant increases without any bound in accordance with the asymptotics

$$\log\det\Delta^{\ell}_{Dihedron} = \frac{\ell}{12}\left(-2\log(\beta+1)+\log\left(1-\frac{2}{\ell}\right)+\log 2\pi - 2 + 24\zeta_R'(-1)\right)\frac{1}{\beta+1} - \frac{\ell}{2}\log(\beta+1)+O(1)$$

of the right-hand side in (5.3). In the limit $\beta=-1$ we obtain a surface of Gaussian curvature $2\pi(2/\ell-1)$ with ℓ cusps [21, 22, 29]. The spectrum of the corresponding Laplacian is no longer discrete [40].

Example 5.3 (Tetrahedron)**.** Here we find the spectral determinant of Laplacian for a constant curvature regular tetrahedron of total area 4π with (four) conical singularities of order β. Or, equivalently, the spectral determinant of the Platonic surface of Gaussian curvature $2\beta+1$ with four faces. Up to a rescaling, this is a particular case of Theorem 5.1, that corresponds to the choice $\ell=3$ and

$$\boldsymbol{\beta} = \frac{\beta-2}{3}\cdot 0 + \beta\cdot 1 + \left(-\frac{2}{3}\right)\cdot\infty.$$

Here and in the remaining part of this section, to normalize the total area to 4π we use the standard rescaling property [25, Sec. 1.2] of the determinant

$$\log\det\Delta^{4\pi}_{f^*\boldsymbol{\beta}} = \log\det\Delta_{f^*\boldsymbol{\beta}} - \zeta_{f^*\boldsymbol{\beta}}(0)\log\frac{4\pi}{\deg f},$$

where

$$\zeta_{f^*\boldsymbol{\beta}}(0) = \frac{|f^*\boldsymbol{\beta}|+2}{6} - \frac{1}{12}\sum_k\left((f^*\boldsymbol{\beta})_k + 1 - \frac{1}{(f^*\boldsymbol{\beta})_k+1}\right) - 1.$$

As a result, in the case $\beta=0$, when all conical singularities disappear, we obtain a surface isometric to the standard unit sphere $x_1^2+x_2^2+x_3^2=1$ in $\mathbb{R}^3$ and its determinant

$$\log\det\Delta = 1/2 - 4\zeta_R'(-1).$$

In general, for the constant curvature regular tetrahedron of total area 4π with conical singularities of order β, Theorem 5.1 gives

$$\begin{aligned}\log\det\Delta^{4\pi}_{Tetrahedron} = {} & 3\left(\log\det\Delta_{\boldsymbol{\beta}} + \mathfrak{C}\left(\frac{\beta-2}{3}\right) + \mathfrak{C}\left(-\frac{2}{3}\right)\right) \\ & + \frac{4}{3}\left(\frac{1}{\beta+1}\Psi\left(\frac{\beta-2}{3},\beta,-\frac{2}{3}\right) + \Psi\left(-\frac{2}{3},\beta,\frac{\beta-2}{3}\right)\right) \\ & - \frac{1}{3}\left(\beta-3+\frac{1}{\beta+1}\right)\left(\log(4\pi)-\frac{1}{2}\log 3\right) - \mathfrak{C}(\beta) - 2\mathbf{C},\end{aligned} \tag{5.4}$$

where the right hand side is an explicit function of β. A graph of this function is depicted in Fig. 7 as a solid line.

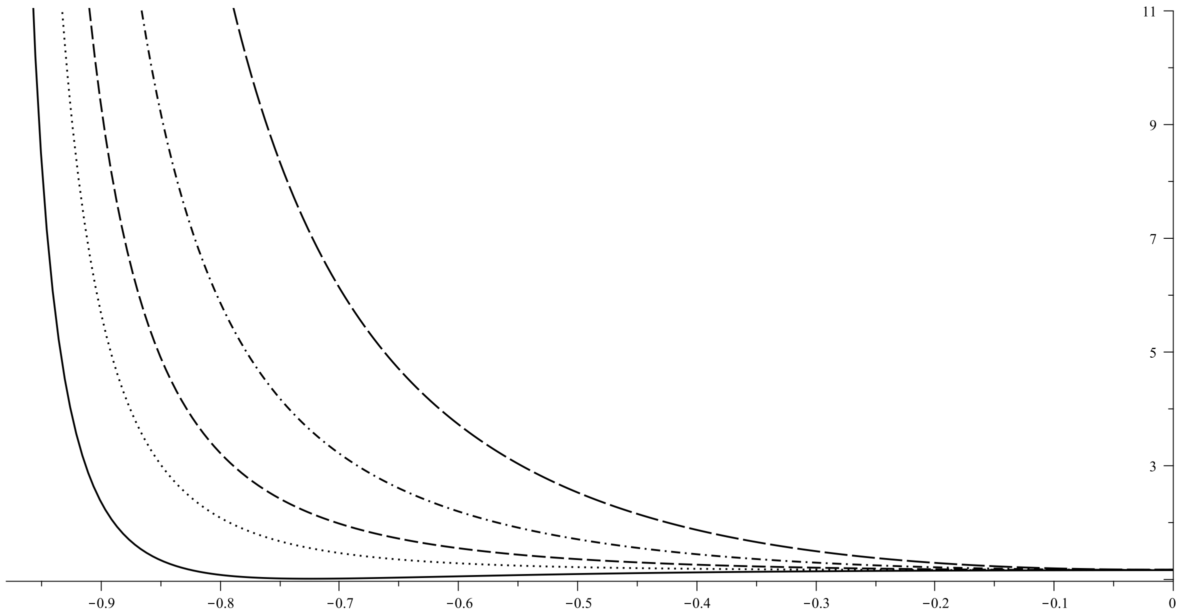

Figure 7: Graphs of the logarithm of the spectral determinant of Laplacian on the surfaces of Platonic solids of area 4π as a function of the order $\beta \in (-1, 0]$ of the conical singularities: **a.** Regular Tetrahedron of Gaussian curvature $2\beta + 1$ (solid line), **b.** Regular Octahedron of Gaussian curvature $3\beta + 1$ (dotted line), **c.** Regular Cube of Gaussian curvature $4\beta + 1$ (dashed line), **d.** Regular Icosahedron of Gaussian curvature $6\beta + 1$ (dash-dotted line), **e.** Regular Dodecahedron of Gaussian curvature $10\beta + 1$ (long-dashed line). The point $(0, 1/2 - 4\zeta_R'(-1)) \approx (0, 1.16)$ on the graphs corresponds to the logarithm of the spectral determinant of the unit round sphere in $\mathbb{R}^3$. As $\beta \to -1^+$, the determinants increase without any bound in accordance with the asymptotics (5.5), (5.8), (5.22), (5.27), and (5.30); cf. [15]. In the limit $\beta = -1$, the conical singularities turn into cusps, and one obtains the ideal Platonic surfaces; the spectrum of the corresponding Laplacians is no longer discrete.

In particular, in the case $\beta = -1/2$ we obtain a surface $(\overline{\mathbb{C}}_x, f^*m_{\boldsymbol{\beta}})$ isometric to the surface of a Euclidean regular tetrahedron. Note that $\mathcal{C}(-1/2)$ can be evaluated as in (4.21). As a result, the formula (5.4) for the determinant reduces to

$$\log\det\Delta^{4\pi}_{Tetrahedron}|_{\beta=-1/2} = \log\frac{4}{3} - 3\log\Gamma\left(\frac{2}{3}\right) + \frac{3}{2}\log\pi.$$

Alternatively, the latter expression for the determinant can be obtained by applying the partially heuristic Aurell-Salomonson formula [2] to the explicitly evaluated pullback of the flat metric

$$m_{\boldsymbol{\beta}} = c^2|z|^{-1}|z-1|^{-1}|dz|^2$$

by $f(x) = x^3$, where c is a scaling coefficient that normalizes the total area of $m_{\boldsymbol{\beta}}$ to $4\pi/3$; for a rigorous proof of the Aurell-Salomonson formula we refer to [25, Sec. 3.2].

Finally, as $\beta\to -1^+$, the determinant grows without any bound in accordance with the asymptotics

$$\log\det\Delta^{4\pi}_{Tetrahedron}=\left(-\frac{2}{3}\log(\beta+1)-\frac{2}{3}+8\zeta_R'(-1)\right)\frac{1}{\beta+1}-2\log(\beta+1)+O(1) \quad (5.5)$$

of the right hand side in (5.4), cf. Fig 7. In the limit $\beta=-1$ we get a surface isometric to an ideal tetrahedron: a surface of Gaussian curvature -2 with four cusps; the spectrum of the Laplacian on an ideal tetrahedron is not discrete [21, 22, 40].

Example 5.4 (Octahedron)**.** Here we find the determinant of Laplacian for a constant curvature regular octahedron of area 4π with (six) conical singularities of order β. Or, equivalently, the spectral determinant of the Platonic surface of Gaussian curvature $3\beta+1$ with eight faces. In Theorem 5.1 we substitute $\ell=4$ and

$$\boldsymbol{\beta}=\frac{\beta-3}{4}\cdot 0+\beta\cdot 1+\frac{\beta-3}{4}\cdot\infty. \quad (5.6)$$

As a result, after an appropriate rescaling, we obtain

$$\begin{aligned}\log\det\Delta^{4\pi}_{Octahedron}&=4\log\det\Delta_{\boldsymbol{\beta}}-\left(\beta+1+\frac{1}{\beta+1}\right)\left(\frac{2}{3}\log 2+\frac{1}{2}\log\pi\right)\\&+\frac{5}{3}\log\pi+\frac{5}{\beta+1}\Psi\left(\frac{\beta-3}{4},\beta,\frac{\beta-3}{4}\right)+2\log 2+8\mathfrak{C}\left(\frac{\beta-3}{4}\right)-2\mathfrak{C}(\beta)-3\mathbf{C},\end{aligned} \quad (5.7)$$

where the right hand side is an explicit function of β. A graph of this function is depicted in Fig. 7 as a dotted line.

In the case $\beta=0$ we obtain a surface isometric to the standard unit sphere in $\mathbb{R}^3$ and a representation for its determinant in terms of the determinant of spherical $(4,\infty,4)$ double triangle, see also Example 5.5 below.

In the case $\beta=-1/3$ we obtain a surface $(\overline{\mathbb{C}}_x,f^*m_{\boldsymbol{\beta}})$ isometric to the (flat) surface of Euclidean regular octahedron. The formula (5.7) for the determinant reduces to

$$\log\det\Delta^{4\pi}_{Octahedron}|_{\beta=-1/3}=6\zeta_R'(-1)+\frac{35}{24}\log\frac{4}{3}-\frac{13}{2}\log\Gamma\left(\frac{2}{3}\right)+\frac{13}{4}\log\pi.$$

Let us also note, that in the case $\beta=-1/2$ we get the tessellation of the singular sphere $(\overline{\mathbb{C}}_x,f^*m_{\boldsymbol{\beta}}|_{\beta=-1/2})$ by the hyperbolic $(2,3,8)$-triangle. The surface $(\overline{\mathbb{C}}_x,f^*m_{\boldsymbol{\beta}}|_{\beta=-1/2})$, where $\boldsymbol{\beta}$ is the divisor (5.6), is isometric to a regular hyperbolic octahedron with conical singularities of angle π. This is remarkable, as a double of $(\overline{\mathbb{C}}_x,f^*m_{\boldsymbol{\beta}}|_{\beta=-1/2})$ is the Bolza curve, known as the most symmetrical genus two smooth hyperbolic surface, see e.g. [27]. To the best of our knowledge, the exact value of the spectral determinant of the Bolza curve, endowed with the smooth Gaussian curvature -1 metric, is not yet known. For a numerical study see [56].

Finally, as $\beta\to -1^+$, the determinant grows without any bound in accordance with the asymptotics

$$\log\det\Delta^{4\pi}_{Octahedron}=\left(-\log(\beta+1)+\frac{1}{2}\log 2-1+12\zeta_R'(-1)\right)\frac{1}{\beta+1}-3\log(\beta+1)+O(1) \quad (5.8)$$

of the right-hand side in (5.7). In the limit $\beta=-1$ we get a surface isometric to an ideal octahedron: a surface of Gaussian curvature -2 with six cusps, cf. [21, 22, 29]. The spectrum of the corresponding Laplacian is no longer discrete [40].

Example 5.5 (Spindles)**.** Let $m_{\boldsymbol\beta}^S=e^{2\phi}|dz|^2$ be the metric of a spindle with two antipodal singularities [62], where

$$\phi(z)=\beta\log|z|+\log 2+\log(\beta+1)-\log(1+|z|^{2\beta+2}).$$

The (regularized) Gaussian curvature of $m_{\boldsymbol\beta}^S$ is 1, and the total area is $S=4\pi(\beta+1)$. The metric represents the divisor $\boldsymbol\beta=\beta_0\cdot 0+\beta_1\cdot 1+\beta_\infty\cdot\infty$ with $\beta_0=\beta_\infty=:\beta\in(-1,\infty)$ and $\beta_1=0$. The spindle $(\overline{\mathbb C}_z,m_{\boldsymbol\beta}^S)$ is isometric to the spherical double triangle glued from two copies of a spherical triangle (a lune) with internal angles $\big(\pi(\beta+1),\pi,\pi(\beta+1)\big)$, cf. Fig. 6.

For the asymptotics of the metric potential we have

$$\begin{aligned}&\phi(z)=\beta\log|z|+\phi_0+o(1),\quad z\to 0,\quad \phi_0=\log 2(\beta+1),\\ &\phi(z)=-(\beta+2)\log|z|+\phi_\infty+o(1),\quad z\to\infty,\quad \phi_\infty=\log 2(\beta+1).\end{aligned}$$

Clearly, for the pullback of $m_{\boldsymbol\beta}^S$ by the Shabat polynomial $f(x)=x^\ell$ we have

$$f^*m_{\boldsymbol\beta}^S=\frac{4\ell^2(\beta+1)^2|x|^{2(\ell(\beta+1)-1)}|dx|^2}{(1+|x|^{2\ell(\beta+1)})^2},\quad f^*\boldsymbol\beta=(\ell(\beta+1)-1)\cdot 0+(\ell(\beta+1)-1)\cdot\infty.$$

The surface $(\overline{\mathbb C}_x,f^*m_{\boldsymbol\beta}^S)$ is isometric to the surface glued from ℓ copies of the spindle $(\overline{\mathbb C}_z,m_{\boldsymbol\beta}^S)$ with a cut from the conical point at $z=0$ to the conical point at $z=\infty$. Or, equivalently, $(\overline{\mathbb C}_x,f^*m_{\boldsymbol\beta}^S)$ is triangulated by 2ℓ bicolored copies of a spherical triangle with internal angles $\big(\pi(\beta+1),\pi,\pi(\beta+1)\big)$ and unit Gaussian curvature, cf. Fig. 6. The surface $(\overline{\mathbb C}_x,f^*m_{\boldsymbol\beta}^S)$ is again a spindle with two antipodal conical singularities: the metric $m_{\boldsymbol\beta}^S$ coincides with $f^*m_{\boldsymbol\beta}^S$ after the replacement of β by $\ell(\beta+1)-1$ (and z by x).

As is known [23, 25, 32, 54], the spectral determinant of the Friedrichs Laplacian $\Delta_{\boldsymbol\beta}^S$ on the spindle $(\overline{\mathbb C}_z,m_{\boldsymbol\beta}^S)$ satisfies

$$\log\frac{\Delta_{\boldsymbol\beta}^S}{S}=\frac{\beta+1}{3}-\frac13\left(\beta+1+\frac{1}{\beta+1}\right)\log 2(\beta+1)-2\mathcal C(\beta)+\mathbf C,\quad \beta>-1. \tag{5.9}$$

For the spectral determinant of the spindle $(\overline{\mathbb C}_x,f^*m_{\boldsymbol\beta}^S)$ Theorem 3.1 (after an appropriate rescaling) gives

$$\begin{aligned}\log\frac{\det\Delta_{f^*\boldsymbol\beta}^{\ell S}}{\ell S}=\ell\Big(\log\frac{\det\Delta_{\boldsymbol\beta}^S}{4\pi(\beta+1)}+2\mathcal C(\beta)-\mathbf C\Big)+\frac13\left(\ell-\frac1\ell\right)\frac{\log 2(\beta+1)}{\beta+1}\\ -\frac16\left(2\ell(\beta+1)+\frac{2}{\ell(\beta+1)}\right)\log\ell-2\mathcal C\left(\ell(\beta+1)-1\right)+\mathbf C.\end{aligned} \tag{5.10}$$

As expected, after the substitution (5.9) and the replacement of $\ell(\beta+1)-1$ by β, the equality (5.10) reduces to the one in (5.9).

In particular, for $\beta=\frac{1}{\ell}-1$ the pullback of the spindle metric $m^S_{\boldsymbol\beta}$ by f is the metric of the standard round sphere of total area $\ell S=4\pi$. In this case, the equality (5.10) expresses the determinant of Laplacian of the standard sphere

$$\det\Delta^{4\pi}_{f^*\boldsymbol\beta}=\exp(1/2-4\zeta_R'(-1))$$

in terms of the determinants $\det\Delta^{4\pi/\ell}_{\boldsymbol\beta}$ of the spherical double triangle, or, equivalently, of the bicolored double lune.

5.2 Determinant for dihedral triangulation

Consider the Belyi function

$$f(x)=1-\left(\frac{1-x^\ell}{1+x^\ell}\right)^2,\quad \ell\in\mathbb{N}. \tag{5.11}$$

This is a ramified covering of degree 2ℓ. For the ramification divisor of f we have

$$\boldsymbol f=(\ell-1)\cdot 0+1\cdot\{\sqrt[\ell]{1}\}+1\cdot\{\sqrt[\ell]{-1}\}+(\ell-1)\cdot\infty,\quad |\boldsymbol f|=4\ell-2.$$

This Belyi function defines a tessellation of the standard round sphere with $(2,2,\ell)$-triangles, cf. Fig 8. As we show in the proof of Theorem 5.6 below, to this Belyi map there corresponds the constant

$$C_f=\frac{2}{3}\left(\ell-\frac{1}{\ell}\right)\log 2. \tag{5.12}$$

The Belyi map f sends the marked points listed in the ramification divisor $\boldsymbol f$ to the points $0,1,\infty$ as follows:

$$f(0)=0,\quad f(\infty)=0,\quad f(\sqrt[\ell]{-1})=\infty,\quad f(\sqrt[\ell]{1})=1.$$

For the pullback divisor of $\boldsymbol\beta=\beta_0\cdot 0+\beta_1\cdot 1+\beta_\infty\cdot\infty$ by f we obtain

$$f^*\boldsymbol\beta=(\ell(\beta_0+1)-1)\cdot 0+(2\beta_1+1)\cdot\{\sqrt[\ell]{1}\}+(2\beta_\infty+1)\cdot\{\sqrt[\ell]{-1}\}+(\ell(\beta_0+1)-1)\cdot\infty.$$

Theorem 5.6 (Dihedral triangulation)**.** *Let $m_{\boldsymbol\beta}$ be the unit area Gaussian curvature $2\pi(|\boldsymbol\beta|+1)$ metric of S^2-like double triangle, see Section 2. For the spectral zeta-regularized determinant of the Friedrichs Laplacian $\Delta_{f^*\boldsymbol\beta}$ corresponding to the area 2ℓ pullback metric $f^*m_{\boldsymbol\beta}$ on the Riemann sphere $\overline{\mathbb{C}}_x$, where f is the dihedral Belyi map (5.11),*

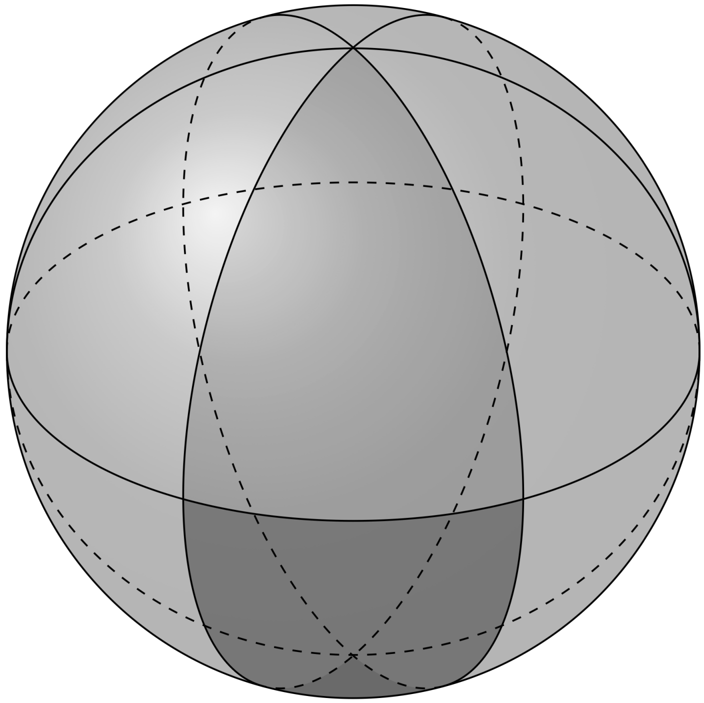

Figure 8: Dihedral triangulation

we have the following explicit expression:

$$\begin{aligned}\log\frac{\det\Delta_{f^*\boldsymbol\beta}}{2\ell} &= 2\ell\Big(\log\det\Delta_{\boldsymbol\beta}+\mathcal{C}_{\boldsymbol\beta}-\mathbf{C}\Big)+C_f\\ &+\frac13\left(\ell-\frac1\ell\right)\frac{\Psi(\beta_0,\beta_1,\beta_\infty)}{\beta_0+1}+\frac\ell4\left(\frac{\Psi(\beta_1,\beta_0,\beta_\infty)}{\beta_1+1}+\frac{\Psi(\beta_\infty,\beta_1,\beta_0)}{\beta_\infty+1}\right)\\ &-\frac13\left(\ell(\beta_0+1)+\frac{1}{\ell(\beta_0+1)}\right)\log\ell-\frac\ell3\left(\beta_1+\beta_\infty+2+\frac{1}{4\beta_1+4}+\frac{1}{4\beta_\infty+4}\right)\log 2\\ &-2\mathcal{C}\left(\ell(\beta_0+1)-1\right)-\ell\Big(\mathcal{C}\left(2\beta_1+1\right)+\mathcal{C}\left(2\beta_\infty+1\right)\Big)+\mathbf{C},\end{aligned}$$

where $\mathcal{C}_{\boldsymbol\beta}=\mathcal{C}(\beta_0)+\mathcal{C}(\beta_1)+\mathcal{C}(\beta_\infty)$ *and* C_f *is the same as in* (5.12).

Recall that $\det\Delta_{\boldsymbol\beta}$ *stands for an explicit function of* $(\beta_0,\beta_1,\beta_\infty)\in(-1,0]^3$*, whose values are the determinants of the unit area* S^2*-like double triangles of Gaussian curvature* $2\pi(|\boldsymbol\beta|+2)$ *tessellating the singular sphere* $(\overline{\mathbb{C}}_x, f^*m_{\boldsymbol\beta})$*, see [23]. The function* $\beta\mapsto\mathcal{C}(\beta)$ *is defined in* (2.8)*, the function* Ψ *is the same as in* (5.2)*, and* $\mathbf{C}$ *is the constant introduced in* (2.9).

Proof. For the derivative of the Belyi function (5.11) we have

$$f'(x)=\frac{4\ell x^{\ell-1}(1-x^\ell)}{(x^\ell+1)^3}.$$

As a result, we immediately obtain the asymptotics in vicinities of the critical points of the form (3.9). The first non-zero coefficients c_k in the Taylor expansions (cf. Proposition 3.1) satisfy

$$|c_k| = \begin{cases} 4, & x_k = 0, \\ \ell^2/4, & x_k \in \{\sqrt[\ell]{1}\}, \\ 4, & x_k = \infty, \quad k = n = 2\ell + 2. \end{cases}$$

Similarly, we obtain the asymptotics in vicinities of the poles of the form (3.10) with the first non-zero Laurent coefficients c_k satisfying

$$|c_k| = \ell^2/4, \quad x_k \in \{\sqrt[\ell]{-1}\}.$$

Now the expression (3.1) for C_f implies (5.12). As a result, the assertion is an immediate consequence of Theorem 2.1 together with the asymptotics (2.6). □

Example 5.7 (Dihedra)**.** Here we deduce an alternative formula for the determinant of the regular dihedron of Gaussian curvature $K = 2\pi(\beta + 1/\ell)$ and area 2ℓ with 2ℓ conical singularities of order β: In Theorem 5.6 we take

$$\boldsymbol{\beta} = \left(\frac{1}{\ell} - 1\right) \cdot 0 + \frac{\beta - 1}{2} \cdot 1 + \frac{\beta - 1}{2} \cdot \infty,$$

and obtain

$$\begin{aligned} \log\det \Delta^{2\ell}_{Dihedron} &= 2\ell\Big(\log\det \Delta_{\boldsymbol{\beta}} + \mathfrak{C}_{\boldsymbol{\beta}} - \mathfrak{C}\,(\beta)\Big) + \frac{1}{3}\log \ell + (1 - 2\ell)\mathbf{C} \\ &+ \frac{\ell^2 - 1}{3}\Psi\left(\frac{1}{\ell} - 1, \frac{\beta - 1}{2}, \frac{\beta - 1}{2}\right) + \frac{\ell}{\beta + 1}\Psi\left(\frac{\beta - 1}{2}, \frac{\beta - 1}{2}, \frac{1}{\ell} - 1\right) \\ &+ \left(\frac{2}{3}\left(\ell - \frac{1}{\ell}\right) - \frac{\ell}{3}\left(\beta + 1 + \frac{1}{\beta + 1}\right) + 1\right)\log 2. \end{aligned}$$

Example 5.8 (Octahedron)**.** Let us obtain an alternative formula for the determinant of Laplacian on a regular octahedron of Gaussian curvature $3\beta + 1$ with (six) conical singularities of order β: In Theorem 5.6 we take $\ell = 2$ and

$$\boldsymbol{\beta} = \frac{\beta - 1}{2} \cdot 0 + \frac{\beta - 1}{2} \cdot 1 + \frac{\beta - 1}{2} \cdot \infty.$$

This together with the standard rescaling property of determinants implies

$$\begin{aligned} \log\det \Delta^{4\pi}_{Octahedron} &= 4\log\det \Delta_{\boldsymbol{\beta}} - \left(\beta + 1 + \frac{1}{\beta + 1}\right)\left(\log 2 + \frac{1}{2}\log \pi\right) + \frac{5}{3}\log \pi \\ &+ \frac{3}{\beta + 1}\Psi\left(\frac{\beta - 1}{2}, \frac{\beta - 1}{2}, \frac{\beta - 1}{2}\right) + 3\log 2 + 12\mathfrak{C}\left(\frac{\beta - 1}{2}\right) - 6\mathfrak{C}\,(\beta) - 3\mathbf{C}, \end{aligned}$$

where the right hand side is an explicit function of β; this expression is equivalent to the one in (5.7). A graph of this function is depicted in Fig. 7 as a dotted line.

5.3 Determinant for tetrahedral triangulation

The tetrahedral Belyi map is given by the function

$$f(x) = -64\frac{(x^3+1)^3}{(x^3-8)^3x^3}, \quad \deg f = 12. \tag{5.13}$$

For the ramification divisor, we obtain

$$\boldsymbol{f} = 2\cdot\left\{0, \sqrt[3]{8}\right\} + 1\cdot\left\{\sqrt[3]{-10\pm 6\sqrt{3}}\right\} + 2\cdot\left\{\sqrt[3]{-1}, \infty\right\}, \quad |\boldsymbol{f}| = 22.$$

The Belyi map sends the marked points listed in the divisor $\boldsymbol{f}$ to the points $0, 1, \infty$ of the target Riemann sphere $\overline{\mathbb{C}}_z$ as follows:

$$f(0) = \infty, \quad f(\sqrt[3]{-1}) = 0, \quad f\left(\sqrt[3]{-10\pm 6\sqrt{3}}\right) = 1, \quad f(\sqrt[3]{8}) = \infty, \quad f(\infty) = 0.$$

Here $f(x) = 1$ are the edge midpoints of a regular tetrahedron. The poles $f(x) = \infty$ correspond to its vertices. The zeros $f(x) = 0$, i.e. the roots of the numerator and $x = \infty$, are the centers of the faces. This defines a tessellation of the standard round sphere with spherical $(2, 3, 3)$-triangles, cf. Fig 9. A picture of the corresponding *dessin d'enfant* can be found e.g. in [37, Fig. 2]. In the proof of Theorem 5.9 below we show that to the Belyi function (5.13) there corresponds the constant

$$C_f = \log 2 + \frac{9}{4}\log 3. \tag{5.14}$$

For the pullback of the divisor $\boldsymbol{\beta} = \beta_0\cdot 0 + \beta_1\cdot 1 + \beta_\infty\cdot\infty$ by f we obtain

$$f^*\boldsymbol{\beta} = (3\beta_0+2)\cdot\left\{\sqrt[3]{-1}, \infty\right\} + (2\beta_1+1)\cdot\left\{\sqrt[3]{-10\pm 6\sqrt{3}}\right\} + (3\beta_\infty+2)\cdot\left\{0, \sqrt[3]{8}\right\}.$$

Theorem 5.9 (Tetrahedral triangulation)**.** *Let $m_{\boldsymbol{\beta}}$ be the unit area Gaussian curvature $2\pi(|\boldsymbol{\beta}|+1)$ metric of S^2-like double triangle, see Section 2. For the determinant of the Friedrichs Laplacian $\det\Delta_{f^*\boldsymbol{\beta}}$ corresponding to the pullback metric $f^*m_{\boldsymbol{\beta}}$ of area 12, where f is the tetrahedral Belyi map (5.13), we have*

$$\begin{aligned}
\log\frac{\det\Delta_{f^*\boldsymbol{\beta}}}{12} &= 12\Big(\log\det\Delta_{\boldsymbol{\beta}} + \mathcal{C}_{\boldsymbol{\beta}} - \mathbf{C}\Big) + C_f \\
&+ \frac{16}{9}\frac{\Psi(\beta_0,\beta_1,\beta_\infty)}{\beta_0+1} + \frac{3}{2}\frac{\Psi(\beta_1,\beta_0,\beta_\infty)}{\beta_1+1} + \frac{16}{9}\frac{\Psi(\beta_\infty,\beta_1,\beta_0)}{\beta_\infty+1} \\
&- \frac{2}{3}\left(3(\beta_0+\beta_\infty+2) + \frac{1}{3(\beta_\infty+1)} + \frac{1}{3(\beta_0+1)}\right)\log 3 - \left(2(\beta_1+1) + \frac{1}{2(\beta_1+1)}\right)\log 2 \\
&- 4\mathcal{C}\,(3\beta_0+2) - 6\mathcal{C}\,(2\beta_1+1) - 4\mathcal{C}\,(3\beta_\infty+2) + \mathbf{C},
\end{aligned}$$

where $\mathcal{C}_{\boldsymbol{\beta}} = \mathcal{C}(\beta_0) + \mathcal{C}(\beta_1) + \mathcal{C}(\beta_\infty)$ and C_f is the same as in (5.14).

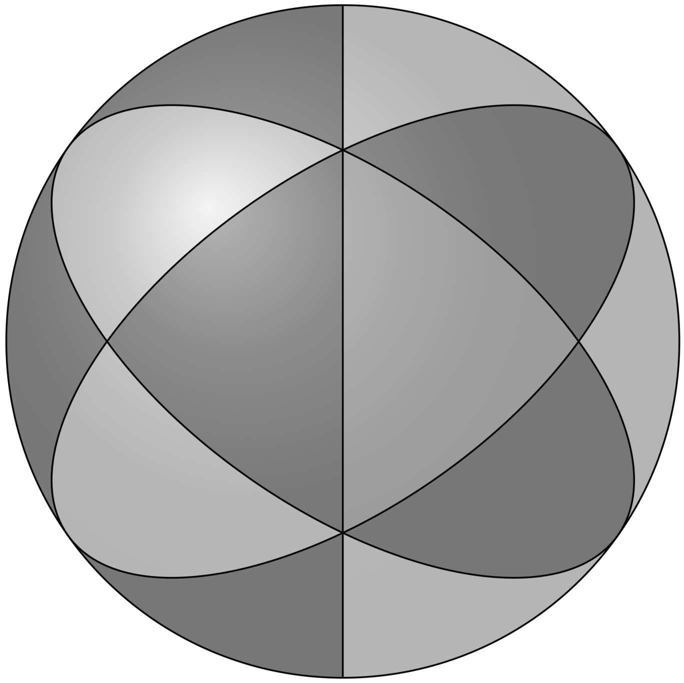

Figure 9: Tetrahedral triangulation

Recall that $\det\Delta_{\boldsymbol\beta}$ *stands for an explicit function of* $(\beta_0,\beta_1,\beta_\infty)\in(-1,0]^3$*, whose values are the determinants of the unit area* S^2*-like double triangles of Gaussian curvature* $2\pi(|\boldsymbol\beta|+2)$ *tessellating the singular sphere* $(\overline{\mathbb{C}}_x, f^*m_{\boldsymbol\beta})$*, see [23]. The function* $\beta\mapsto\mathfrak{C}(\beta)$ *is defined in* (2.8)*, the function* Ψ *is the same as in* (5.2)*, and* $\mathbf{C}$ *is the constant introduced in* (2.9)*.*

Proof. For the derivative of the Belyi function in (5.13) we have

$$f'(x)=\frac{192(x^3+1)^2(x^6+20x^3-8)}{x^4(x^3-8)^4}.$$

In exactly the same way as in the proof of Theorem 5.6 we obtain

$$|c_k|=\begin{cases} 2^6/3^3, & x_k\in\{\sqrt[3]{-1}\},\\ 2\sqrt3\pm3, & x_k\in\left\{\sqrt[3]{-10\pm6\sqrt3}\right\},\\ 2^6, & x_k=\infty,\\ 2^3, & x_k=0,\\ 2^3/3^3, & x_k\in\left\{\sqrt[3]{8}\right\}.\end{cases}$$

This together with the expression (3.1) for C_f implies the value stated in (5.14). The remaining part of the assertion is a direct consequence of Theorem 2.1. □

Example 5.10 (Tetrahedron)**.** Here we deduce an alternative formula for the spectral determinant of a regular tetrahedron of Gaussian curvature $K=2\beta+1$: in Theorem 5.9

we substitute

$$\boldsymbol{\beta}=\left(-\frac{2}{3}\right)\cdot 0+\left(-\frac{1}{2}\right)\cdot 1+\frac{\beta-2}{3}\cdot\infty.$$

As a result, after some rescaling we obtain

$$\begin{aligned}\log\det\Delta^{4\pi}_{Tetrahedron}&=12\Big(\log\det\Delta_{\boldsymbol\beta}+\mathcal{C}_{\boldsymbol\beta}\Big)+\log 2+\frac{7}{12}\log 3+\frac{4}{3}\log\pi\\&+\frac{16}{3}\Psi\left(-\frac{2}{3},-\frac{1}{2},\frac{\beta-2}{3}\right)+3\Psi\left(-\frac{1}{2},-\frac{2}{3},\frac{\beta-2}{3}\right)+\frac{16}{3(\beta+1)}\Psi\left(\frac{\beta-2}{3},-\frac{1}{2},-\frac{2}{3}\right)\\&-\frac{1}{3}\left(\beta+1+\frac{1}{\beta+1}\right)\log 3\pi-4\mathcal{C}\left(\beta\right)-11\mathbf{C},\end{aligned}$$

where the right hand side is an explicit function of β. A graph of this function is depicted in Fig. 7 as a solid line.

In the case $\beta=0$ the surface $(\overline{\mathbb{C}}_x,f^*m_{\boldsymbol\beta})$ is isometric to a unit sphere in $\mathbb{R}^3$. The sphere $(\overline{\mathbb{C}}_x,f^*m_{\boldsymbol\beta}|_{\beta=0})$ is tessellated by the double of $(2,3,3)$-triangle, and the above formula for the determinant is a representation for the determinant $\det\Delta^{4\pi}_{Tetrahedron}|_{\beta=0}$ of the unit sphere in terms of the determinant of the double of spherical $(2,3,3)$-triangle.

5.4 Determinant for octahedral triangulation

To the octahedral triangulation, there corresponds the Belyi function

$$f(x)=-2^2\cdot 3^3\frac{(x^4+1)^4x^4}{(x^8-14x^4+1)^3},\quad \deg f=24. \tag{5.15}$$

This Belyi function describes the tessellation of the standard round sphere with spherical $(2,3,4)$-triangles, cf. Fig 10. The ramification divisor of f is

$$\boldsymbol{f}=3\cdot\big\{0,\infty,\sqrt[4]{-1}\big\}+1\cdot\left\{\sqrt[4]{1},\sqrt[4]{-17\pm 3\cdot 2^{5/2}}\right\}+2\cdot\left\{\sqrt[4]{(2\pm\sqrt{3})^2}\right\},\quad |\boldsymbol{f}|=46. \tag{5.16}$$

Here $\{\sqrt[4]{1},\sqrt[4]{-17\pm 3\cdot 2^{5/2}}\}$ are the edge midpoints of a cube, for any x in this set we have $f(x)=1$. The poles $\sqrt[4]{(2\pm\sqrt{3})^2}$ of f correspond to the vertices of the cube. The zeros $\{0,\infty,\sqrt[4]{-1}\}$ of f are the centers of the cube faces.

For the octahedron dual to the cube: the points $\{\sqrt[4]{1},\sqrt[4]{-17\pm 3\cdot 2^{5/2}}\}$ correspond to the edge midpoints, the poles of f correspond to the centers of the faces, and the zeros of f are the vertices. A picture of the corresponding *dessin d'enfant* can be found e.g. in [37, Fig. 3].

In the proof of Theorem 5.11 below we show that to the Belyi function (5.15) there corresponds the constant

$$C_f=\frac{9}{4}\log 3+\frac{119}{18}\log 2. \tag{5.17}$$

For the pullback of the divisor $\boldsymbol{\beta} = \beta_0 \cdot 0 + \beta_1 \cdot 1 + \beta_\infty \cdot \infty$ by f we get

$$f^*\boldsymbol{\beta} = (4\beta_0 + 3) \cdot \left\{0, \infty, \sqrt[4]{-1}\right\} + (2\beta_1 + 1) \cdot \left\{\sqrt[4]{1}, \sqrt[4]{-17 \pm 3 \cdot 2^{5/2}}\right\} \\ + (3\beta_\infty + 2) \cdot \left\{\sqrt[4]{(2 \pm \sqrt{3})^2}\right\}.$$

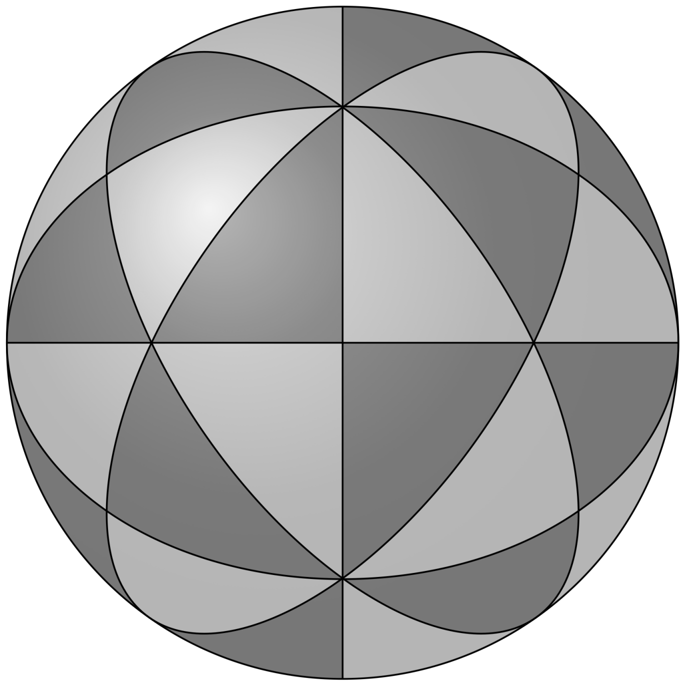

Figure 10: Octahedral triangulation

Theorem 5.11 (Octahedral triangulation)**.** *Let $m_{\boldsymbol{\beta}}$ be the unit area Gaussian curvature $2\pi(|\boldsymbol{\beta}| + 1)$ metric of S^2-like double triangle, see Section 2. For the determinant of Laplacian corresponding to the pullback metric $f^*m_{\boldsymbol{\beta}}$ of area 24, where f is the Belyi map (5.15), we have*

$$\begin{aligned}\log \frac{\det \Delta_{f^*\boldsymbol{\beta}}}{24} &= 24\Big(\log\det\Delta_{\boldsymbol{\beta}} + \mathcal{C}_{\boldsymbol{\beta}} - \mathbf{C}\Big) + C_f \\ &+ \frac{15}{4}\frac{\Psi(\beta_0, \beta_1, \beta_\infty)}{\beta_0 + 1} + 3\frac{\Psi(\beta_1, \beta_0, \beta_\infty)}{\beta_1 + 1} + \frac{32}{9}\frac{\Psi(\beta_\infty, \beta_1, \beta_0)}{\beta_\infty + 1} \\ &- \left(4(2\beta_0 + \beta_1 + 3) + \frac{1}{2(\beta_0 + 1)} + \frac{1}{\beta_1 + 1}\right)\log 2 - 4\left(\beta_\infty + 1 + \frac{1}{9(\beta_\infty + 1)}\right)\log 3 \\ &- 8\mathcal{C}\,(3\beta_\infty + 2) - 6\mathcal{C}\,(4\beta_0 + 3) - 12\mathcal{C}\,(2\beta_1 + 1) + \mathbf{C},\end{aligned}$$

where $\mathcal{C}_{\boldsymbol{\beta}} = \mathcal{C}(\beta_0) + \mathcal{C}(\beta_1) + \mathcal{C}(\beta_\infty)$ and C_f is the same as in (5.17).

Recall that $\det\Delta_{\boldsymbol\beta}$ *stands for an explicit function of* $(\beta_0,\beta_1,\beta_\infty)\in(-1,0]^3$*, whose values are the determinants of the unit area* S^2*-like double triangles of Gaussian curvature* $2\pi(|\boldsymbol\beta|+2)$ *tessellating the singular sphere* $(\overline{\mathbb C}_x, f^*m_{\boldsymbol\beta})$*, see [23]. The function* $\beta\mapsto\mathcal C(\beta)$ *is defined in (2.8), the function* Ψ *is the same as in (5.2), and* $\mathbf C$ *is the constant introduced in (2.9).*

Proof. Here we find C_f by using Proposition 3.4. Namely, we first write the potential of the pullback of $m_{\boldsymbol\beta}=c|z|^{-4/3}|z-1|^{-4/3}|dz|^2$ by f in the form

$$\big(f^*\phi\big)(x)=\frac13\log|x|+\frac13\log|x^4+1|-\frac13\log|x^4-1|-\frac13\log|(x^4+17)^2-9\cdot 2^5|+\log cA,$$

cf. (3.17). It is easy to see that as $x\to x_k\in\{\sqrt[4]{-1}\}$ the metric potential $f^*\phi$ satisfies the asymptotics

$$\begin{aligned}\big(f^*\phi\big)(x)&=\frac13\log|x-x_k|+\frac13\log\big|(x^4+1)'\restriction_{x=\sqrt[4]{-1}}\big|-\frac13\log|-2|\\&\quad-\frac13\log|(-1+17)^2-9\cdot2^5|+\log cA+o(1)\\&=\frac13\log|x-x_k|+\log cA-\frac43\log2+o(1).\end{aligned}$$

This together with (3.18) implies that for any k, such that $x_k\in\{\sqrt[4]{-1}\}$, we have

$$\sum_{\ell:k\neq\ell<n}\frac{\operatorname{ord}_\ell f-2}{3}\log|x_k-x_\ell|=-\frac43\log2.$$

Similarly, we obtain

$$\sum_{\ell:k\neq\ell<n}\frac{\operatorname{ord}_\ell f-2}{3}\log|x_k-x_\ell|=\begin{cases}0, & x_k=0,\\ -\log2-\frac23\log3, & x_k\in\{\sqrt[4]{1}\},\\ \frac13\log\frac{3\pm2\sqrt2}{144}, & x_k\in\left\{\sqrt[4]{-17\pm3\cdot2^{5/2}}\right\}.\end{cases}$$

As a consequence, by using the expression (5.16) for the ramification divisor $\boldsymbol f$, for the first line in (2.10) we obtain

$$\frac1{18}\sum_{k:k<n}\sum_{\ell:k\neq\ell<n}\frac{(\operatorname{ord}_k f-2)(\operatorname{ord}_\ell f-2)}{\operatorname{ord}_k f+1}\log|x_k-x_\ell|=\log2+\frac23\log3. \tag{5.18}$$

Thanks to (5.16), it is also easy to verify that for the second line in (2.10) one has

$$\frac16\sum_{k\leqslant n}\left(\frac{\operatorname{ord}_k f+1}{3}+\frac{3}{\operatorname{ord}_k f+1}\right)\log(\operatorname{ord}_k f+1)=\frac{17}{2}\log2+\frac83\log3. \tag{5.19}$$

Finally, as the scaling constant A satisfies (2.11), we get

$$A=\frac{|f(x)|^{-2/3}|f(x)-1|^{-2/3}|f'(x)|}{|x|^{1/3}|x^4+1|^{1/3}|x^4-1|^{-1/3}|(x^4+17)^2-9\cdot2^5|^{-1/3}}=12\cdot2^{2/3}.$$

This allows us to calculate the value of the last line in (2.10):

$$\frac{1}{6}\left(n-2-\sum_{k\leqslant n}\frac{3}{\operatorname{ord}_k f+1}\right)\log A=-\frac{13}{12}\log(12\cdot 2^{2/3}). \tag{5.20}$$

Adding the values (5.18), (5.19), (5.20) of the lines in (2.10) together, we arrive at (5.17). □

Example 5.12 (Cube)**.** Here we find the spectral determinant of a cube of Gaussian curvature $4\beta+1$ with (eight) conical singularities of order β: in Theorem 5.11 we substitute

$$\boldsymbol{\beta}=\left(-\frac{3}{4}\right)\cdot 0+\left(-\frac{1}{2}\right)\cdot 1+\frac{\beta-2}{3}\cdot\infty.$$

After rescaling, for the determinant of a cube of (regularized) Gaussian curvature $4\beta+1$ with eight conical singularities of order β we obtain

$$\begin{aligned}\log\det\Delta^{4\pi}_{Cube}&=24\Big(\log\det\Delta_{\boldsymbol{\beta}}+\mathcal{C}_{\boldsymbol{\beta}}\Big)+\frac{5}{4}\log 3-\frac{7}{18}\log 2+2\log\pi\\&+15\Psi\left(-\frac{3}{4},-\frac{1}{2},\frac{\beta-2}{3}\right)+6\Psi\left(-\frac{1}{2},-\frac{3}{4},\frac{\beta-2}{3}\right)+\frac{32}{3(\beta+1)}\Psi\left(\frac{\beta-2}{3},-\frac{1}{2},-\frac{3}{4}\right)\\&-\frac{2}{3}\left(\beta+1+\frac{1}{\beta+1}\right)\log\frac{3\pi}{2}-8\mathcal{C}(\beta)-23\mathbf{C},\end{aligned} \tag{5.21}$$

where the right hand side is an explicit function of β. A graph of this function is depicted in Fig. 7 as a dashed line. As $\beta\to-1^+$, the determinant of Laplacian grows without any bound in accordance with the asymptotics

$$\log\det\Delta^{4\pi}_{Cube}=\left(-\frac{4}{3}\log(\beta+1)+\frac{2}{3}\log 3-\frac{4}{3}+16\zeta_R'(-1)\right)\frac{1}{\beta+1}-4\log(\beta+1)+O(1) \tag{5.22}$$

of the right hand side in (5.21). In the limit $\beta=-1$ we get a surface isometric to an ideal cube: a surface of Gaussian curvature -3 with eight cusps, cf. [21, 22, 29]. The spectrum of the corresponding Laplacian is no longer discrete [40].

To the case of a Euclidean cube there corresponds the value $\beta=-1/4$. The formula (5.21) for the determinant reduces to

$$\begin{aligned}\log\det\Delta^{4\pi}_{Cube}|_{\beta=-1/4}=&\frac{32}{3}\zeta_R'(-1)-\frac{37}{18}\log 2+\frac{25}{12}\log 3\\&+\frac{16}{3}\log\Gamma\left(\frac{2}{3}\right)-\frac{86}{9}\log\Gamma\left(\frac{3}{4}\right)+\frac{19}{9}\log\pi.\end{aligned}$$

Example 5.13 (Octahedron)**.** Now we can deduce yet another (equivalent) formula for the determinant of Laplacian on a regular octahedron of Gaussian curvature $3\beta+1$ with (six) conical singularities of order β. In Theorem 5.11 we take

$$\boldsymbol{\beta}=\frac{\beta-3}{4}\cdot 0+\left(-\frac{1}{2}\right)\cdot 1+\left(-\frac{2}{3}\right)\cdot\infty.$$

As a result, after an appropriate rescaling we obtain

$$\log\det\Delta^{4\pi}_{Octahedron} = 24\Big(\log\det\Delta_{\boldsymbol\beta} + \mathcal{C}_{\boldsymbol\beta}\Big) - \frac{13}{12}\log 3 + \frac{71}{18}\log 2$$
$$+ \frac{15}{\beta+1}\Psi\left(\frac{\beta-3}{4}, -\frac{1}{2}, -\frac{2}{3}\right) + 6\Psi\left(-\frac{1}{2}, \frac{\beta-3}{4}, -\frac{2}{3}\right) + \frac{32}{3}\Psi\left(-\frac{2}{3}, -\frac{1}{2}, \frac{\beta-3}{4}\right)$$
$$-\frac{1}{2}\left(\beta+1+\frac{1}{\beta+1}\right)\log\frac{8\pi}{3} - 6\mathcal{C}(\beta) - 23\mathbf{C} + \frac{5}{3}\log\pi.$$

5.5 Determinant for icosahedral triangulation

The icosahedral Belyi function is given by

$$f(x) = 1728\frac{x^5(x^{10}-11x^5-1)^5}{(x^{20}+228(x^{15}-x^5)+494x^{10}+1)^3}, \quad \deg f = 60. \tag{5.23}$$

The ramification divisor of $\boldsymbol f$ is

$$\boldsymbol f = 2\cdot\{20 \text{ poles of } f\} + 4\cdot\{12 \text{ zeros of } f\} + 1\cdot\{30 \text{ zeros of } f-1\}, \quad |\boldsymbol f| = 118.$$

This defines tessellation of a standard round sphere with bicolored spherical $(2,3,5)$ double triangle, cf. Fig 1. The 20 poles of f are the coordinates of the centers of the faces, the 30 solutions to $f(x)=1$ are the edge midpoints, and the 12 zeros of f ($x=\infty$ is also a zero of f) are the vertices of a regular icosahedron inscribed into the sphere. In terms of the dodecahedron that is dual to the icosahedron: The poles of f are the coordinates of the 20 vertices, the 12 zeros of f are the centers of its faces, the 30 solutions to the equation $f(x)=1$ correspond to the edge midpoints. A picture of the corresponding *dessin d'enfant* can be found e.g. in [37, Fig. 1].

For the icosahedral Belyi function evaluation of the right hand side in (2.10) gives

$$C_f = \frac{139}{15}\log 2 + \frac{63}{10}\log 3 + \frac{125}{36}\log 5. \tag{5.24}$$

For the pullback of the divisor $\boldsymbol\beta$ by f we have

$$f^*\boldsymbol\beta = (5\beta_0+4)\cdot\{12 \text{ zeros of } f\} + (2\beta_1+1)\cdot\{30 \text{ zeros of } f-1\} + (3\beta_\infty+2)\cdot\{20 \text{ poles of } f\}.$$

Theorem 5.14 (Icosahedral triangulation)**.** *Let $m_{\boldsymbol\beta}$ be the unit area Gaussian curvature $2\pi(|\boldsymbol\beta|+1)$ metric of S^2-like double triangle, see Section 2. Then for the determinant of Laplacian corresponding to the area 60 pullback metric $f^*m_{\boldsymbol\beta}$, where f is the icosahedral*

Belyi map (5.23)*, we have*

$$\log\frac{\det\Delta_{f^*\boldsymbol\beta}}{60}=60\Big(\log\det\Delta_{\boldsymbol\beta}+\mathcal{C}_{\boldsymbol\beta}-\mathbf{C}\Big)+C_f$$
$$+\frac{48}{5}\frac{\Psi(\beta_0,\beta_1,\beta_\infty)}{\beta_0+1}+\frac{15}{2}\frac{\Psi(\beta_1,\beta_0,\beta_\infty)}{\beta_1+1}+\frac{80}{9}\frac{\Psi(\beta_\infty,\beta_1,\beta_0)}{\beta_\infty+1}$$
$$-2\left(5\beta_0+5+\frac{1}{5\beta_0+5}\right)\log 5-5\left(2\beta_1+2+\frac{1}{2\beta_1+2}\right)\log 2$$
$$-\frac{10}{3}\left(3\beta_\infty+3+\frac{1}{3\beta_\infty+3}\right)\log 3$$
$$-12\mathcal{C}\left(5\beta_0+4\right)-30\mathcal{C}\left(2\beta_1+1\right)-20\mathcal{C}\left(3\beta_\infty+2\right)+\mathbf{C},$$

where $\mathcal{C}_{\boldsymbol\beta}=\mathcal{C}(\beta_0)+\mathcal{C}(\beta_1)+\mathcal{C}(\beta_\infty)$ *and* C_f *is the same as in* (5.24)*. Recall that* $\det\Delta_{\boldsymbol\beta}$ *stands for an explicit function of* $(\beta_0,\beta_1,\beta_\infty)\in(-1,0]^3$*, whose values are the determinants of the unit area* S^2*-like double triangles of Gaussian curvature* $2\pi(|\boldsymbol\beta|+2)$*. The function* $\beta\mapsto\mathcal{C}(\beta)$ *is defined in* (2.8)*, the function* Ψ *is given in* (5.2)*, and* $\mathbf{C}$ *is the constant introduced in* (2.9)*.*

Proof. For this tessellation we find the value (5.24) of C_f in exactly the same way as for the octahedral one in the proof of Theorem 5.11. The calculations are a bit tedious but straightforward. We omit the details. □

Example 5.15 (Icosahedron)**.** Here we find an explicit expression for the spectral determinant of a regular icosahedron of area 4π and Gaussian curvature $6\beta+1$ with 12 conical points of order β. In Theorem 5.14 we take

$$\boldsymbol\beta=\frac{\beta-4}{5}\cdot 0+\left(-\frac{1}{2}\right)\cdot 1+\left(-\frac{2}{3}\right)\cdot\infty. \tag{5.25}$$

In particular, in the case $\beta=0$ all conical singularities disappear and we obtain a surface isometric to the standard round sphere $x_1^2+x_2^2+x_3^2=1$ in $\mathbb{R}^3$.

As a consequence of Theorem 5.14 and the standard rescaling property, for the divisor (5.25) we obtain

$$\log\det\Delta^{4\pi}_{Icosahedron}=60\Big(\log\det\Delta_{\boldsymbol\beta}+\mathcal{C}_{\boldsymbol\beta}\Big)+\frac{19}{15}\log 2-\frac{61}{30}\log 3+\frac{65}{36}\log 5$$
$$+\frac{48}{\beta+1}\Psi\left(\frac{\beta-4}{5},-\frac{1}{2},-\frac{2}{3}\right)+15\Psi\left(-\frac{1}{2},\frac{\beta-4}{5},-\frac{2}{3}\right)+\frac{80}{3}\Psi\left(-\frac{2}{3},-\frac{1}{2},\frac{\beta-4}{5}\right)$$
$$-\left(\beta+1+\frac{1}{\beta+1}\right)\log\frac{5\pi}{3}-12\mathcal{C}\left(\beta\right)-59\mathbf{C}+\frac{8}{3}\log\pi, \tag{5.26}$$

where the right-hand side is an explicit function of β. A graph of this function is depicted in Fig. 7 as a dash-dotted line.

As $\beta \to -1^+$ the determinant increases without any bound in accordance with the asymptotics

$$\log\det\Delta^{4\pi}_{Icosahedron} = \Big(-2\log(\beta+1)+\log 5-2+24\zeta_R'(-1)\Big)\frac{1}{\beta+1}-6\log(\beta+1)+O(1) \tag{5.27}$$

of the right-hand side in (5.26). In the limit $\beta=-1$ we obtain an ideal icosahedron, cf. [21, 22, 29]. The spectrum of the corresponding Laplacian is no longer discrete [40].

In the case $\beta=-1/6$ we obtain a flat regular icosahedron: the pullback of the flat metric $m^S_{\boldsymbol\beta}=|z|^{-4/3}|z-1|^{-1}|dz|^2$ by f is the metric

$$f^*m^S_{\boldsymbol\beta}=3\cdot 2^2\cdot 5^2|x(x^{10}-11x^5-1)|^{-1/3}|dx|^2$$

of a flat regular icosahedron. The equality (5.26) reduces to

$$\log\det\Delta^{4\pi}_{Icosahedron}|_{\beta=-1/6}=\frac{96}{5}\zeta_R'(-1)+\frac{18}{5}\log(\sqrt5-1)+\frac{6}{5}\log(\sqrt5+1)+\frac{23}{2}\log\pi$$

$$+\frac{214}{45}\log 2-\frac{917}{60}\log 3+\frac{251}{36}\log 5+\frac{72}{5}\log\Gamma\left(\frac45\right)-\frac{211}{5}\log\Gamma\left(\frac23\right)+\frac{24}{5}\log\Gamma\left(\frac35\right).$$

Let us also note that for the tessellation by $(2,3,7)$-triangles, one should take $\beta=-2/7$ in (5.25) and (5.26). This is related to Klein's quartic [16, 27, 49, 53]: the genus 3 surface with the highest possible number of authomorphisms [18], it is also the lowest genus Hurwitz surface. It is an open longstanding problem to find the exact value of the spectral determinant of Klein's quartic. Klein's quartic is also conjectured to be a stationary point of the spectral determinant, cf. Theorem 4.6.

Example 5.16 (Dodecahedron)**.** Here we find an explicit expression for the spectral determinant of a regular dodecahedron of area 4π and Gaussian curvature $K=10\beta+1$ with twenty conical singularities of order β. With this aim in mind, in Theorem 5.14 we take

$$\boldsymbol\beta=\left(-\frac45\right)\cdot 0+\left(-\frac12\right)\cdot 1+\frac{\beta-2}{3}\cdot\infty. \tag{5.28}$$

Then after an appropriate rescaling we obtain

$$\begin{aligned}\log\det\Delta^{4\pi}_{Dodecahedron}&=60\Big(\log\det\Delta_{\boldsymbol\beta}+\mathcal{C}_{\boldsymbol\beta}\Big)+\frac{19}{15}\log 2+\frac{33}{10}\log 3-\frac{127}{36}\log 5\\&+48\Psi\left(-\frac45,-\frac12,\frac{\beta-2}{3}\right)+15\Psi\left(-\frac12,-\frac45,\frac{\beta-2}{3}\right)+\frac{80}{3(\beta+1)}\Psi\left(\frac{\beta-2}{3},-\frac12,-\frac45\right)\\&-\frac53\left(\beta+1+\frac{1}{\beta+1}\right)\log\frac{3\pi}{5}-20\mathcal{C}(\beta)-59\mathbf{C}+4\log\pi,\end{aligned} \tag{5.29}$$

where the right-hand side is an explicit function of β. A graph of this function is depicted in Fig. 7 as a long-dashed line.

As $\beta \to -1^+$, the determinant increases without any bound in accordance with the asymptotics

$$\log\det\Delta^{4\pi}_{Dodecahedron} = \Big(-\frac{10}{3}(\log(\beta+1)-\log 3+1)+40\zeta_R'(-1)\Big)\frac{1}{\beta+1} - 10\log(\beta+1)+O(1) \tag{5.30}$$

of the right-hand side in (5.29). In the limit $\beta=-1$, the surface $(\overline{\mathbb{C}}_x, f^*m_{\boldsymbol\beta})$ becomes isometric to an ideal dodecahedron [21, 22].

In the case of a Euclidean dodecahedron, the equality (5.29) reduces to

$$\log\,\det\Delta^{4\pi}_{Dodecahedron}|_{\beta=-1/10} = \frac{83}{180}\log 2 - \frac{7}{135}\log 3 - \frac{19}{72}\log 5 + \frac{19}{108}\log(\sqrt{5}-1)$$
$$-\frac{19}{27}\log\Gamma\left(\frac{7}{10}\right) - \frac{19}{27}\log\Gamma\left(\frac{4}{5}\right) + \frac{19}{27}\log\pi + \frac{1}{6} - 4\zeta_R'(-1) - 20\mathcal{C}\left(-\frac{1}{10}\right),$$

where $\mathcal{C}\left(-\frac{1}{10}\right)$ can be expressed in terms of Riemann zeta and gamma functions [25].

We end this section with a remark that the spectral determinants of the surfaces of Archimedean solids can be calculated in the same way as above thanks to the Belyi maps found in [37]. Let us also notice that it would be interesting to study the cone to cusp degeneration [21, 22, 29] and obtain, as a consequence of our results, some explicit formulae for the relative spectral determinant [39] of hyperbolic surfaces with cusps. We hope to address this elsewhere.

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Victor Kalvin
Department of Mathematics,
Dawson College,
3040 Sherbrooke St. W,
Montreal, Quebec
Canada
H3Z 1A4

Email: vkalvin@gmail.com