Initial traces and solvability of porous medium equation with power nonlinearity

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Abstract

In this paper we study qualitative properties of initial traces of solutions to the porous medium equation with power nonlinearity, and obtain necessary conditions for the existence of solutions to the corresponding Cauchy problem. Furthermore, we establish sharp sufficient conditions for the existence of solutions to the Cauchy problem using uniformly local Morrey spaces and their variations, and identify the optimal singularities of the initial data for the solvability of the Cauchy problem.

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1 Introduction

Let u be a nonnegative solution to the porous medium equation with power nonlinearity

$$\begin{cases} \partial_t u = \Delta u^m + u^p & \text{in} \quad \mathbb{R}^N \times (0,T), \\ u \geq 0 & \text{in} \quad \mathbb{R}^N \times (0,T), \end{cases} \tag{E}$$

where $N \geq 1$, $1 \leq m < p$, $T \in (0,\infty]$, and $\partial_t := \partial/\partial t$. In this paper we study qualitative properties of initial traces of the solution u, and obtain necessary conditions for the existence of solution to the Cauchy problem

$$\begin{cases} \partial_t u = \Delta u^m + u^p & \text{in} \quad \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = \mu & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{P}$$

where μ is a nonnegative Radon measure in $\mathbb{R}^N$. Furthermore, we establish sharp sufficient conditions for the existence of nonnegative solutions to problem (P), and identify the optimal singularities of the initial data for the solvability of problem (P).

Let $\mathcal{M}$ denote the set of nonnegative Radon measures in $\mathbb{R}^N$, and $\mathcal{L}$ denote the set of nonnegative locally integrable functions in $\mathbb{R}^N$. We often identify $\mathrm{d}\mu = \mu(x)\,\mathrm{d}x$ in $\mathcal{M}$ for $\mu \in \mathcal{L}$. For any measurable set E in $\mathbb{R}^d$, where $d = 1,2,\dots$, let $\mathcal{L}^d(E)$ be the d-dimensional Lebesuge measure of E. For any $f \in \mathcal{L}$, $z \in \mathbb{R}^N$, and $r > 0$, set

$$\fint_{B(z,r)} f\,\mathrm{d}x := \frac{1}{\mathcal{L}^N(B(z,r))} \int_{B(z,r)} f\,\mathrm{d}x,$$

where $B(z,r) := \{x \in \mathbb{R}^N \,:\, |x-z| < r\}$. Set

$$p_m := m + \frac{2}{N}, \qquad \theta := \frac{p-m}{2(p-1)}, \qquad \theta' := \frac{1}{\theta} = \frac{2(p-1)}{p-m}. \tag{1.1}$$

The study of initial traces of nonnegative solutions to the Cauchy problem for parabolic equations is a classical subject, and qualitative properties of initial traces have been studied for various parabolic equations. See e.g., [4] for linear parabolic equations, [5,21] for porous medium equations, [10,11] for parabolic p-Laplace equations, [27,29,49] for doubly nonlinear parabolic equations, [9] for fractional diffusion equations, [1] for Finsler heat equations, [3,14,16,22–25, 28,45] for parabolic equations with source nonlinearity (positive nonlinearity), [7,8,33–35] for parabolic equations with absorption nonlinearity (negative nonlinearity).

Let us recall some results on initial traces of solutions to problem (E) with $m = 1$. See e.g., [6,23,28].

(A) (1) Assume that problem (E) with $m = 1$ possesses a solution u in $\mathbb{R}^N \times (0,T)$ for some $T \in (0,\infty)$. Then there exists a unique $\nu \in \mathcal{M}$ such that

$$\operatorname*{ess\,lim}_{t \to +0} \int_{\mathbb{R}^N} u(t)\psi\,\mathrm{d}y = \int_{\mathbb{R}^N} \psi\,\mathrm{d}\nu(y), \quad \psi \in C_c(\mathbb{R}^N).$$

Furthermore, there exists $C_1 = C_1(N,p) > 0$ such that

$$\sup_{z \in \mathbb{R}^N} \nu(B(z,\sigma)) \leq \begin{cases} C_1 \sigma^{N - \frac{2}{p-1}} & \text{if} \quad p \neq p_1, \\ C_1 \left[\log\left(e + \dfrac{\sqrt{T}}{\sigma}\right)\right]^{-\frac{N}{2}} & \text{if} \quad p = p_1, \end{cases}$$

for $\sigma \in (0,\sqrt{T})$.

(2) Let u be a solution to problem (P) with $m = 1$. Then u is a solution to problem (E) and the initial trace of u coincides with the initial data of u in $\mathcal{M}$.

We remark that assertion (A) gives necessary conditions for the existence of solutions to problem (P) with $m = 1$. Sufficient conditions for the existence of solutions to problem (P) with $m = 1$ have been studied in many papers, and the following assertion holds for $m = 1$.

(B) Let $m = 1$.

(1) Let $1 < p < p_1$. Then problem (P) possesses a local-in-time solution if and only if

$$\sup_{z\in\mathbb{R}^N} \mu(B(z,1)) < \infty.$$

(See assertion (A) and e.g., [47, 48].)

(2) Let $p = p_1$. For any $\alpha > 0$, there exists $\epsilon_1 = \epsilon_1(N, \alpha) > 0$ such that if $\mu \in \mathcal{L}$ satisfies

$$\sup_{z\in\mathbb{R}^N} \sup_{\sigma\in(0,\sqrt{T})} \eta\left(\frac{\sigma}{\sqrt{T}}\right) \Psi^{-1}\left(\fint_{B(z,\sigma)} \Psi\left(T^{\frac{1}{p-1}}\mu\right)\, \mathrm{d}y\right) \le \epsilon_1$$

for some $T \in (0,\infty)$, then problem (P) possesses a solution in $\mathbb{R}^n \times (0,T)$, where

$$\Psi(s) := s[\log(e+s)]^\alpha, \qquad \eta(s) := s^N\left[\log\left(e+\frac{1}{s}\right)\right]^{\frac{N}{2}}.$$

See e.g., [16, 23, 28]. (See also [26] for another sufficient condition.)

(3) Let $p > p_1$. For any $\beta > 1$, there exists $\epsilon_2 = \epsilon_2(N, p, \beta) > 0$ such that if $\mu \in \mathcal{L}$ satisfies

$$\sup_{z\in\mathbb{R}^N} \sup_{\sigma\in(0,\sqrt{T})} \sigma^{\frac{2}{p-1}}\left(\fint_{B(z,\sigma)} |\mu|^\beta\, dy\right)^{\frac{1}{\beta}} \le \epsilon_2$$

for some $T \in (0,\infty]$, then problem (P) possesses a solution in $\mathbb{R}^N \times (0,T)$. See e.g., [16, 23, 31, 40].

We remark that assertion (B)-(2) with $\alpha = 0$ and assertion (B)-(3) with $\beta = 1$ do not hold. (See Remark 1.2 for details.) Combining the results in (A) and (B), we have the following assertion.

(C) Let $p \ge p_1$. For any $c > 0$, set

$$\mu_c(x) := \begin{cases} c|x|^{-N}\left[\log\left(e+\dfrac{1}{|x|}\right)\right]^{-\frac{N}{2}-1} & \text{if} \quad p = p_1, \\ c|x|^{-\frac{2}{p-1}} & \text{if} \quad p > p_1, \end{cases}$$

for a.a. $x \in \mathbb{R}^N$.

(1) Problem (P) possesses a local-in-time solution for $c > 0$ small enough;

(2) Problem (P) possesses no local-in-time solutions for $c > 0$ large enough.

Furthermore, if $p > p_1$ and $c > 0$ is small enough, then problem (P) possesses a global-in-time solution.

The results in (C) show that the "strength" of the singularity at the origin of the functions μ_c is the critical threshold for the local solvability of problem (P). We term such a singularity in the initial data an *optimal singularity* of initial data for the solvability of problem (P). (See e.g., [15] for further details of optimal singularities of initial data.) We easily see that, by translation invariance the singularity could be located at any point of $\mathbb{R}^N$.

On the other hand, in the case of $m > 1$, much less is known about the solvability of problem (P), in particular, the optimal singularity of the initial data for the solvability of problem (P). Andreucci and DiBenedetto [3] proved the existence and the uniqueness of initial traces of solutions to problem (E). Furthermore, they obtained qualitative properties of initial traces of solutions and studied necessary conditions and sufficient conditions for the existence of solutions to problem (P). More precisely, they obtained the following assertion.

(D) Let $1 \le m < p$.

(1) Assume that problem (E) possesses a solution u in $\mathbb{R}^N \times (0,T)$ for some $T \in (0,\infty)$. Then there exists a unique $\nu \in \mathcal{M}$ such that

$$\operatorname*{ess\,lim}_{t\to+0} \int_{\mathbb{R}^N} u(t)\psi \, \mathrm{d}y = \int_{\mathbb{R}^N} \psi \, \mathrm{d}\nu(y), \quad \psi \in C_c(\mathbb{R}^N).$$

Furthermore, there exists $C_2 = C_2(N,m,p) > 0$ such that

$$\sup_{z\in\mathbb{R}^N} \nu(B(z,\sigma)) \le C_2 \sigma^{N-\frac{2}{p-m}}$$

for $\sigma \in (0, T^\theta)$.

(2) Let $m < p < p_m$. Then problem (P) possesses a local-in-time solution if and only if

$$\sup_{z\in\mathbb{R}^N} \mu(B(z,1)) < \infty.$$

(3) Let $p \ge p_m$. Then, for any $r > N(p-m)/2$, there exists $\epsilon_3 = \epsilon_3(N,m,p,r) > 0$ such that if $\mu \in \mathcal{L}$ satisfies

$$T^{\frac{1}{p-1}} \sup_{z\in\mathbb{R}^N} \left(\fint_{B(z,T^\theta)} |\mu|^r \, \mathrm{d}y \right)^{\frac{1}{r}} \le \epsilon_3$$

for some $T \in (0,\infty)$, then problem (P) possesses a solution in $\mathbb{R}^N \times (0,T)$.

Subsequently, the following sufficient conditions were established in [2] and [42] for the supercritical case $p > p_m$.

(D) Let $p > p_m$.

(3') If $\mu \in \mathcal{M}$ satisfies

$$\sup_{z\in\mathbb{R}^N} \sup_{0<\sigma\le 1} \sigma^{\frac{2}{p-m}-N+\lambda} \mu(B(z,\sigma)) < \infty$$

for some $\lambda \in (0, N(p-m)/2)$, then problem (P) possesses a local-in-time solution (see [2, Remark 2]).

(3") Then there exists $\epsilon_4 = \epsilon_4(N,m,p) > 0$ such that if $\mu \in \mathcal{L}$ satisfies

$$T^{\frac{1}{p-1}} \sup_{z\in\mathbb{R}^N} \left(\fint_{B(z,T^\theta)} |\mu|^{\frac{N(p-m)}{2}} \, \mathrm{d}y \right)^{\frac{2}{N(p-m)}} \le \epsilon_4$$

for some $T \in (0,\infty)$, then problem (P) possesses a solution in $\mathbb{R}^N \times (0,T)$ (see [42]).

Unfortunately, the results in (D) are not enough to identify the optimal singularity of the initial data for the solvability of problem (P) in the case of $p \ge p_m$. In particular, assertions (D)-(3), (3'), and (3") do not allow the treatment of initial data with a singularity like that of $|x|^{-2/(p-m)}$ in the case of $p > p_m$. Moreover, we note that, due to the result by Takahashi [44], it does not appear to be possible to take $\lambda = 0$ in assertion (D)-(3') (see Remark 1.2). Therefore, if we consider initial data with a singularity like that of $|x|^{-2/(p-m)}$, it seems more appropriate to adopt a formulation similar to those in assertions (B)-(2) and (3), which use Morrey norms and their variations.

In this paper we give refinements of assertions (D)-(1), (3), and (3"), and extend assertions (A), (B), and (C) to the case $m > 1$. More precisely:

- we improve qualitative properties of initial traces of solutions to problem (E) with $p = p_m$, and establish sharp necessary condition for the existence of solutions to problem (P) (see Theorem 1.1);
- we give sharp sufficient conditions for the existence of solutions to problem (P) with $p \ge p_m$ using uniformly local Morrey spaces and their variations (see Theorems 1.2 and 1.3).

Our necessary conditions and sufficient conditions enable us to identify the optimal singularity of initial data for the solvability for problem (P) with $p \ge p_m$ (see Corollary 1.1).

We formulate definitions of solutions to problems (E) and (P).

Definition 1.1 *Let $1 \le m < p$, $T \in (0,\infty]$, and $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N \times [0,T))$ be nonnegative in $\mathbb{R}^N \times (0,T)$.*

(1) *We say that u is a solution to problem (E) in $\mathbb{R}^N \times (0,T)$ if u satisfies*

$$\int_\tau^T \int_{\mathbb{R}^N} (-u\partial_t\phi - u^m\Delta\phi - u^p\phi) \, \mathrm{d}x \, \mathrm{d}t = \int_{\mathbb{R}^N} u(x,\tau)\phi(x,\tau) \, \mathrm{d}x \tag{1.2}$$

for $\phi \in C_c^{2;1}(\mathbb{R}^N \times [0,T))$ and almost all (a.a.) $\tau \in (0,T)$.

(2) *For any $\mu \in \mathcal{M}$, we say that u is a solution to problem (P) in $\mathbb{R}^N \times (0,T)$ if u satisfies*

$$\int_0^T \int_{\mathbb{R}^N} (-u\partial_t\phi - u^m\Delta\phi - u^p\phi) \, \mathrm{d}x \, \mathrm{d}t = \int_{\mathbb{R}^N} \phi(x,0) \, \mathrm{d}\mu(x)$$

for $\phi \in C_c^{2;1}(\mathbb{R}^N \times [0,T))$.

We introduce some notation. For any positive functions f and g in a set X, we say that $f \preceq g$ for $x \in X$ or equivalently that $g \succeq f$ for $x \in X$ if there exists $C > 0$ such that

$$f(x) \le Cg(x) \quad \text{for } x \in X.$$

If $f \preceq g$ and $g \preceq f$ for $x \in X$, we say that $f \asymp g$ for $x \in X$. In all that follows we will use C to denote generic positive constants and point out that C may take different values within a calculation.

Let Φ be a nonnegative, convex, and strictly increasing function in $[0,\infty)$ such that $\Phi(0)=0$. Let ρ be a nonnegative, non-decreasing, and continuous function in $[0,\infty)$. Then, for any $f \in \mathcal{L}$ and $R \in (0,\infty]$, set

$$|||f|||_{\rho,\Phi;R} := \sup_{z\in\mathbb{R}^N} \sup_{\sigma\in(0,R)} \left\{ \rho(\sigma)\Phi^{-1}\left(\fint_{B(z,\sigma)} \Phi(f)\,\mathrm{d}x \right) \right\}.$$

For any $q \in [1,\infty)$ and $\alpha \in [1,\infty)$, if $\Phi(\xi)=\xi^\alpha$ and $\rho(\xi)=\xi^{\frac{N}{q}}$ for $\xi \in [0,\infty)$, we write

$$|||f|||_{q,\alpha;R} := |||f|||_{\rho,\Phi;R} = \sup_{z\in\mathbb{R}^N} \sup_{\sigma\in(0,R)} \left\{ \sigma^{\frac{N}{q}} \left(\fint_{B(z,\sigma)} |f|^\alpha\,\mathrm{d}x \right)^{\frac{1}{\alpha}} \right\}$$

for simplicity. In particular, if $R=1$, then $|||\cdot|||_{q,\alpha;1}$ coincides with the norm of uniformly local Morrey spaces (see e.g., [31, Definition 0.1]).

Now we are ready to state the main results of this paper. The first theorem concerns with qualitative properties of initial traces of solutions to problem (E).

Theorem 1.1 *Let $N \geq 1$ and $1 \leq m < p$.*

(1) *Let u be a solution to problem* (E) *in $\mathbb{R}^N \times (0,T)$, where $T \in (0,\infty)$. Then there exists a unique $\nu \in \mathcal{M}$ such that*

$$\operatorname*{ess\,lim}_{t\to+0} \int_{\mathbb{R}^N} u(t)\psi\,\mathrm{d}x = \int_{\mathbb{R}^N} \psi\,\mathrm{d}\nu(x), \quad \psi \in C_c(\mathbb{R}^N). \tag{1.3}$$

Furthermore, there exists $C = C(N,m,p) > 0$ such that

$$\sup_{z\in\mathbb{R}^N} \nu(B(z,\sigma)) \leq \begin{cases} C\sigma^{N-\frac{2}{p-m}} & \text{if} \quad p \neq p_m, \\ C\left[\log\left(e+\dfrac{T^\theta}{\sigma}\right)\right]^{-\frac{N}{2}} & \text{if} \quad p = p_m, \end{cases} \tag{1.4}$$

for $\sigma \in (0,T^\theta)$, where p_m and θ are defined as in (1.1).

(2) *Let u be a solution to problem* (P). *Then u is a solution to problem* (E) *and the initial trace of u coincides with the initial data of u in $\mathcal{M}$.*

Similarly to assertion (A), Theorem 1.1 gives a necessary condition for the existence of solutions to problem (P).

Remark 1.1 (1) *Let u be a solution to problem* (E) *in $\mathbb{R}^N \times (0,T)$, where $T \in (0,\infty]$. For any $\lambda > 0$, set*

$$u_\lambda(x,t) := \lambda^{\frac{2}{p-m}} u(\lambda x, \lambda^{\theta'} t), \quad (x,t) \in \mathbb{R}^N \times (0,T_\lambda),$$

where $T_\lambda := \lambda^{-\theta'}T$. Then u_λ is a solution to problem (E) *in $\mathbb{R}^N \times (0,T_\lambda)$.*

(2) *Let $1 \leq m < p < p_m$ and $\mu \in \mathcal{M}$. Then, since $N - 2/(p-m) < 0$, the relation*

$$\sup_{z\in\mathbb{R}^N} \mu(B(z,\sigma)) \leq C\sigma^{N-\frac{2}{p-m}}$$

holds for $\sigma \in (0, T^\theta)$ *if and only if*

$$\sup_{z\in\mathbb{R}^N} \mu(B(z,T^\theta)) \le CT^{N\theta-\frac{1}{p-1}}.$$

(3) *If* $m < p \le p_m$*, then problem* (P) *possesses no nontrivial global-in-time solutions* (*see* [12, 17, 18, 38, 39, 41, 43]). *This fact immediately follows from Theorem* 1.1. *Indeed, assume that problem* (P) *possesses a nontrivial global-in-time solution* u*. Then, for a.a.* $\tau > 0$*, setting* $u_\tau(x,t) := u(x,t+\tau)$ *for a.a.* $(x,t) \in \mathbb{R}^N \times (0,\infty)$*, we see that* u_τ *is a global-in-time solution to problem* (P) *with initial data* $\mu = u(\tau)$*. Combining this fact with Theorem* 1.1*, we have*

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,T^\theta)} u(\tau)\,\mathrm{d}y \le CT^{N\theta-\frac{1}{p-1}} \to 0 \quad \textit{if} \quad p < p_m,$$
$$\sup_{z\in\mathbb{R}^N} \int_{B(z,T^{\frac{\theta}{2}})} u(\tau)\,\mathrm{d}y \le C\left[\log\left(e + T^{\frac{\theta}{2}}\right)\right]^{-\frac{N}{2}} \to 0 \quad \textit{if} \quad p = p_m,$$

as $T \to \infty$*. These imply that* $u(x,\tau) = 0$ *for a.a.* $(x,\tau) \in \mathbb{R}^N \times (0,\infty)$*, which is a contradiction. Thus problem* (P) *possesses no nontrivial global-in-time solutions if* $m < p \le p_m$*. See also e.g.,* [20, 36, 37] *for related results on Riemannian manifolds and in the Euclidean weighted setting.*

In the second and third theorems we obtain sharp sufficient conditions for the existence of solutions to problem (P) with $p = p_m$ and $p > p_m$, respectively. We remark that problem (P) possesses a global-in-time solution for some initial data $\mu \in \mathcal{L}$ if and only if $p > p_m$ (see Remark 1.1-(3)).

Theorem 1.2 *Let* $N \ge 1$*,* $m \ge 1$*,* $p = p_m \equiv m + 2/N$*, and* $\alpha > 0$*. Let*

$$\Psi(\xi) := \xi[\log(e+\xi)]^\alpha, \qquad \eta(\xi) := \xi^N\left[\log\left(e + \frac{1}{\xi}\right)\right]^{\frac{N}{2}}, \tag{1.5}$$

for $\xi \in [0,\infty)$*. Then there exist* $\epsilon_5 = \epsilon_5(N,m,\alpha) > 0$ *and* $C = C(N,m,\alpha) > 0$ *such that if* $\mu \in \mathcal{L}$ *satisfies*

$$\sup_{z\in\mathbb{R}^N} \sup_{\sigma\in(0,T^\theta)} \eta\left(\frac{\sigma}{T^\theta}\right)\Psi^{-1}\left(\fint_{B(z,\sigma)} \Psi\left(T^{\frac{1}{p-1}}\mu\right)\mathrm{d}y\right) \le \epsilon_5 \tag{1.6}$$

for some $T \in (0,\infty)$*, then problem* (P) *possesses a solution* u *in* $\mathbb{R}^N \times (0,T)$*, with* u *satisfying*

$$\begin{aligned} &\sup_{t\in(0,T)} t^{\frac{1}{p-1}}\left[\log\left(e + \frac{T}{t}\right)\right]^{\frac{1}{p-1}} \|u(t)\|_{L^\infty(\mathbb{R}^N)} \\ &+ \sup_{t\in(0,T)} \sup_{z\in\mathbb{R}^N} \sup_{\sigma\in(0,T^\theta)} \eta\left(\frac{\sigma}{T^\theta}\right)\Psi^{-1}\left(\fint_{B(z,\sigma)} \Psi\left(T^{\frac{1}{p-1}}u(t)\right)\mathrm{d}y\right) \\ &\le C \sup_{z\in\mathbb{R}^N} \sup_{\sigma\in(0,T^\theta)} \left\{\eta\left(\frac{\sigma}{T^\theta}\right)\Psi^{-1}\left(\fint_{B(z,\sigma)} \Psi\left(T^{\frac{1}{p-1}}\mu\right)\mathrm{d}y\right)\right\}^{\frac{2}{N(m-1)+2}}, \end{aligned}$$

where θ *is defined as in* (1.1).

Theorem 1.3 *Let* $N \ge 1$*,* $m \ge 1$*,* $p > p_m \equiv m + 2/N$*, and* $1 < \beta < N(p-m)/2$*. Then there exist* $\epsilon_6 = \epsilon_6(N,m,p,\beta) > 0$ *and* $C = C(N,m,p,\beta) > 0$ *such that if* $\mu \in \mathcal{L}$ *satisfies*

$$|||\mu|||_{\frac{N(p-m)}{2},\beta;T^\theta} \le \epsilon_6 \tag{1.7}$$

for some $T \in (0,\infty]$*, then problem* (P) *possesses a solution* u *in* $\mathbb{R}^N \times (0,T)$*, with* u *satisfying*

$$\sup_{t\in(0,T)} t^{\frac{1}{p-1}}\|u(t)\|_{L^\infty} + \sup_{t\in(0,T)} |||u(t)|||_{\frac{N(p-m)}{2},\beta;T^\theta} \le C|||\mu|||^{\frac{2\beta}{N(m-1)+2\beta}}_{\frac{N(p-m)}{2},\beta;T^\theta},$$

where θ *is defined as in* (1.1)*. In particular, if* $\mu \in \mathcal{L}$ *satisfies*

$$|||\mu|||_{\frac{N(p-m)}{2},\beta;\infty} = \sup_{z\in\mathbb{R}^N}\sup_{\sigma>0}\left\{\sigma^{\frac{2}{p-m}}\left(⨍_{B(z,\sigma)}|\mu|^\beta\,dy\right)^{\frac{1}{\beta}}\right\} \le \epsilon_6,$$

then problem (P) *possesses a global-in-time solution.*

Necessary conditions in Theorem 1.1 and sufficient conditions in Theorems 1.2 and 1.3 are sharp. Indeed, Theorems 1.1–1.3 enable us to identify an optimal singularity of the initial data for the solvability of problem (P) with $p \ge p_m$.

Corollary 1.1 *Let* $N \ge 1$*,* $m \ge 1$*, and* $p \ge p_m$*. For any* $c > 0$*, set*

$$\mu_c^m(x) := \begin{cases} c|x|^{-N}\left[\log\left(e+\dfrac{1}{|x|}\right)\right]^{-\frac{N}{2}-1} & \text{if} \quad p = p_m, \\ c|x|^{-\frac{2}{p-m}} & \text{if} \quad p > p_m, \end{cases}$$

for a.a. $x \in \mathbb{R}^N$.

(1) *Problem* (P) *possesses a local-in-time solution for* $c > 0$ *small enough;*

(2) *Problem* (P) *possesses no local-in-time solutions for* $c > 0$ *large enough.*

Furthermore, if $p > p_m$ *and* $c > 0$ *is small enough, then problem* (P) *possesses a global-in-time solution.*

Remark 1.2 *Takahashi* [44] *proved that if* $m = 1$*,* $p \ge p_1$*, and* $\mu \in \mathcal{M}$ *satisfies*

$$\limsup_{\sigma\searrow 0} \sigma^{-N+\frac{2}{p-1}}\left(\log\left(e+\frac{1}{\sigma}\right)\right)^{\frac{1}{p-1}+\lambda}\sup_{z\in\mathbb{R}^N}\mu(B(z,\sigma)) < \infty \tag{1.8}$$

for some $\lambda > 0$*, then problem* (P) *with* $m = 1$ *possesses a local-in-time solution* (*see* [44, Theorem 1.1] *for more details*)*. Moreover, he also proved that there exists* $\mu \in \mathcal{L}$ *satisfying* (1.8) *with* $\lambda = 0$ *such that problem* (P) *possesses no local-in-time solutions. In particular, this implies that assertion* (B)-(2) *with* $\alpha = 0$ *and assertion* (B)-(3) *with* $\beta = 1$ *do not hold.*

In the proof of Theorem 1.1 we modify arguments in [45] to find suitable cut-off functions to obtain estimates of solvable initial data of problem (P) (see Proposition 2.1). Here we require delicate choice of parameters of cut-off functions (see Steps 3 and 4 in the proof of Proposition 2.1). Then we follow arguments in [23] to complete the proof of Theorem 1.1.

If $m = 1$, the results in Theorems 1.2 and 1.3 coincide with assertion (B)-(2) in [23, 26, 28] and assertion (B)-(3) in [23, 31, 40], respectively. However, since the proofs of these papers rely on the representation formula of solutions via Duhamel's principle, the same method does not appear to be extendable to the case $m > 1$ due to the nonlinearity of the principal term Δu^m.

One of the main ingredients of this paper is to give new energy estimates involving Morrey norms and its variations (see Lemmata 3.1 and 3.2). The proofs of these lemmata are inspired by the argument in [32], which proved improved Sobolev inequalities via weak-type estimates and pseudo-Poincaré inequalities. These energy estimates together with L^∞-estimates of solutions to problem (P) lead a priori estimates for classical solutions to problem (P). Combining these estimates with regularity theorems for solutions to problem (P), we construct a solution to problem (P) as the limit of classical solutions to problem (P), where initial data are given by the lifting and the truncation of μ (see (4.6)). We emphasize that this approach is new even in the case $m = 1$ because we provide a priori estimates involving Morrey norms and their variations without relying on the integral representation of solutions to problem (P) with $m = 1$.

The rest of this paper is organized as follows. In Section 2 we modify the arguments in [45] to prove Theorem 1.1. In Section 3 we obtain energy estimates of solutions. In Sections 4 and 5 we prove Theorems 1.2 and 1.3, respectively. In Section 6 we apply Theorems 1.1–1.3 to prove Corollary 1.1.

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2 Proof of Theorem 1.1

In this section we modify arguments in [45] to study necessary conditions for the existence of solutions to problem (P). Furthermore, we obtain qualitative properties of initial traces of solutions to problem (E), and prove Theorem 1.1.

Proposition 2.1 *Let $N \ge 1$, $m \ge 1$, and $p > m$. Let u be a solution to problem* (P) *in $\mathbb{R}^N \times [0,T)$, where $T \in (0,\infty)$. Then there exists $C = C(N,m,p) > 0$ such that*

$$\sup_{z\in\mathbb{R}^N} \mu(B(z,\sigma)) \le \begin{cases} C\sigma^{N-\frac{2}{p-m}} & \text{if} \quad p \neq p_m, \\ C\left[\log\left(e+\dfrac{T^\theta}{\sigma}\right)\right]^{-\frac{N}{2}} & \text{if} \quad p = p_m, \end{cases}$$

for $\sigma \in (0,T^\theta)$, where p_m and θ are defined as in (1.1).

Proof. The proof is divided into several steps. Let u be a solution to problem (P) in $\mathbb{R}^N \times (0,T)$, where $T \in (0,\infty)$.

Step 1: Let $\psi \in C_c^{2;1}(\mathbb{R}^N \times [0,T))$ be chosen later such that $0 \le \psi \le 1$ in $\mathbb{R}^N \times [0,T)$. Let $k \in \mathbb{N}$, and set $\phi = \psi^k$. Then it follows from Definition 1.1-(2) and Hölder's inequality that

$$\begin{aligned} &\int_{\mathbb{R}^N} \phi(0)\,\mathrm{d}\mu(x) + \int_0^T\int_{\mathbb{R}^N} u^p\phi\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_{\mathbb{R}^N} (u\partial_t\phi + u^m\Delta\phi)\,\mathrm{d}x\,\mathrm{d}t \\ &\le \left(\int_0^T\int_{\mathbb{R}^N} u^p\phi\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^T\int_{\mathbb{R}^N} \left|\frac{\partial_t\phi}{\phi}\right|^{\frac{p}{p-1}}\phi\,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{1}{p}} \\ &\quad + \left(\int_0^T\int_{\mathbb{R}^N} u^p\phi\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{m}{p}}\left(\int_0^T\int_{\mathbb{R}^N} \left|\frac{\Delta\phi}{\phi}\right|^{\frac{p}{p-m}}\phi\,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{m}{p}} \end{aligned}$$

$$\leq \int_0^T \int_{\mathbb{R}^N} u^p \phi \,\mathrm{d}x\,\mathrm{d}t + C \int_0^T \int_{\mathbb{R}^N} \left(|\partial_t \phi|^{\frac{p}{p-1}} \phi^{-\frac{1}{p-1}} + |\Delta \phi|^{\frac{p}{p-m}} \phi^{-\frac{m}{p-m}} \right) \mathrm{d}x\,\mathrm{d}t$$

and hence

$$\int_{\mathbb{R}^N} \phi(0)\,\mathrm{d}\mu(x) \leq C \int_0^T \int_{\mathbb{R}^N} \left(|\partial_t \phi|^{\frac{p}{p-1}} \phi^{-\frac{1}{p-1}} + |\Delta \phi|^{\frac{p}{p-m}} \phi^{-\frac{m}{p-m}} \right) \mathrm{d}x\,\mathrm{d}t. \tag{2.1}$$

Since $\psi \leq 1$ in $\mathbb{R}^N \times [0,\infty)$ and $p > m \geq 1$, taking $k \geq 1$ large enough, we see that

$$\begin{aligned} &|\partial_t \phi|^{\frac{p}{p-1}} \phi^{-\frac{1}{p-1}} \leq C |\partial_t \psi|^{\frac{p}{p-1}} \psi^{\frac{k(p-1)-p}{p-1}} \leq C |\partial_t \psi|^{\frac{p}{p-1}}, \\ &|\Delta \phi|^{\frac{p}{p-m}} \phi^{-\frac{m}{p-m}} \leq C |\Delta \psi|^{\frac{p}{p-m}} \psi^{\frac{k(p-m)-p}{p-m}} + |\nabla \psi|^{\frac{2p}{p-m}} \psi^{\frac{k(p-m)-2p}{p-m}} \leq C |\Delta \psi|^{\frac{p}{p-m}} + C |\nabla \psi|^{\frac{2p}{p-m}}, \end{aligned}$$

which together with (2.1) imply that

$$\begin{aligned} \int_{\{x \in \mathbb{R}^N \,:\, \psi(x,0)=1\}} \mathrm{d}\mu(x) &\leq C \int_0^T \int_{\mathbb{R}^N} \left(|\partial_t \psi|^{\frac{p}{p-1}} + |\Delta \psi|^{\frac{p}{p-m}} + |\nabla \psi|^{\frac{2p}{p-m}} \right) \mathrm{d}x\,\mathrm{d}t \\ &\leq C \int_0^T \int_{\mathbb{R}^N} \left(|\partial_t \psi|^{\frac{p}{p-1}} + (|\Delta \psi| + |\nabla \psi|^2)^{\frac{p}{p-m}} \right) \mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.2}$$

Step 2: Let $\zeta \in C^\infty(\mathbb{R})$ be such that $0 \leq \zeta \leq 1$ in $\mathbb{R}$, $\zeta \equiv 1$ in $[1,\infty)$, $\zeta \equiv 0$ in $(-\infty, 0]$, and $|\zeta'| \leq 2$ in $\mathbb{R}$. Let $z \in \mathbb{R}^N$, $a > 0$, and $\delta > 0$. For any $F \in C^\infty((0,\infty))$ with $F \leq 0$ in $[aT/2, \infty)$, set

$$\psi(x,t) := \zeta(F(r(x,t))) \quad \text{with} \quad r(x,t) := |x-z|^{\theta'} + at + \delta$$

for $(x,t) \in \mathbb{R}^N \times [0,T)$, where θ' is defined as in (1.1). Then $\psi \in C_c^2(\mathbb{R}^N \times [0,T))$. Since

$$\begin{aligned} &\theta' = \frac{2(p-1)}{p-m} > 2, \\ &\partial_t \psi = \zeta'(F(r))F'(r)\partial_t r, \quad \nabla \psi = \zeta'(F(r))F'(r)\nabla r, \\ &\Delta \psi = \zeta''(F(r))F'(r)^2 |\nabla r|^2 + \zeta'(F(r))F''(r)|\nabla r|^2 + \zeta'(F(r))F'(r)\Delta r, \end{aligned}$$

we have

$$\begin{aligned} |\partial_t \psi| &\leq Ca|F'(r)|, \\ |\nabla \psi|^2 &\leq CF'(r)^2 |x-z|^{2(\theta'-1)} \leq CF'(r)^2 r^{2-2\theta}, \\ |\Delta \psi| &\leq CF'(r)^2 |x-z|^{2(\theta'-1)} + C|F''(r)||x-z|^{2(\theta'-1)} + C|F'(r)||x-z|^{\theta'-2} \\ &\leq CF'(r)^2 r^{2-2\theta} + C|F''(r)| r^{2-2\theta} + C|F'(r)| r^{1-2\theta}. \end{aligned}$$

These together with (2.2) imply that

$$\int_{\{x \in \mathbb{R}^N \,:\, F(|x-z|^{\theta'}+\delta) \geq 1\}} \mathrm{d}\mu(x) \leq C \iint_{\{(x,t) \in \mathbb{R}^N \times [0,T) \,:\, 0 \leq F(r(x,t)) \leq 1\}} g(r(x,t))\,\mathrm{d}x\,\mathrm{d}t,$$

where

$$g(\xi) := a^{\frac{p}{p-1}} |F'(\xi)|^{\frac{p}{p-1}} + \left(|F'(\xi)|^2 \xi^{2-2\theta} + |F''(\xi)| \xi^{2-2\theta} + |F'(\xi)| \xi^{1-2\theta} \right)^{\frac{p}{p-m}}. \tag{2.3}$$

Then we obtain

$$\begin{aligned}
&\sup_{z\in\mathbb{R}^N}\int_{\{x\in\mathbb{R}^N\,:\,F(|x-z|^{\theta'}+\delta)\ge 1\}} \mathrm{d}\mu(x)\\
&\le \sup_{z\in\mathbb{R}^N}\iint_{\{(x,t)\in\mathbb{R}^N\times[0,\infty)\,:\,0\le F(|x-z|^{\theta'}+at+\delta)\le 1\}} g(|x-z|^{\theta'}+at+\delta)\,\mathrm{d}x\,\mathrm{d}t\\
&= Ca^{-1}\iint_{\{(r,t)\in[0,\infty)\times[0,\infty)\,:\,0\le F(r^{\theta'}+t+\delta)\le 1\}} g(r^{\theta'}+t+\delta)r^{N-1}\,\mathrm{d}r\,\mathrm{d}t\\
&= Ca^{-1}\iint_{\{(s,\tau)\in[0,\infty)\times[0,\infty)\,:\,0\le F(s^2+\tau^2+\delta)\le 1\}} g(s^2+\tau^2+\delta)s^{2(N-1)\theta+2\theta-1}\tau\,\mathrm{d}s\,\mathrm{d}\tau\\
&= Ca^{-1}\int_{\{\zeta\ge 0\,:\,0\le F(\zeta^2+\delta)\le 1\}} g(\zeta^2+\delta)\zeta^{2N\theta+1}\,\mathrm{d}\zeta\int_0^{\frac{\pi}{2}}(\cos\omega)^{2N\theta-1}\sin\omega\,\mathrm{d}\omega\\
&= Ca^{-1}\int_{\{\xi\ge 0\,:\,0\le F(\xi+\delta)\le 1\}} g(\xi+\delta)\xi^{N\theta}\,\mathrm{d}\xi.
\end{aligned}\tag{2.4}$$

Step 3: Let b, c, and $d>0$ be constants to be chosen later such that

$$\frac{aT}{2}\ge\frac{d}{e^c-1}.\tag{2.5}$$

Set

$$F(\xi):=\frac{1}{b}\left(\log\left(1+\frac{d}{\xi}\right)-c\right)\quad\text{for}\quad \xi\in(0,\infty).$$

Then

$$\begin{aligned}
&F(\xi)\ge 1\quad\text{if and only if}\quad \xi\le R_1:=\frac{d}{e^{b+c}-1},\\
&F(\xi)\ge 0\quad\text{if and only if}\quad \xi\le R_2:=\frac{d}{e^{c}-1},\\
&F(\xi)\le 0\quad\text{if}\quad \xi\ge aT/2.
\end{aligned}\tag{2.6}$$

Furthermore,

$$\begin{aligned}
|F'(\xi)| &= \left|-\frac{1}{b}\left(1+\frac{d}{\xi}\right)^{-1}\frac{d}{\xi^2}\right| = \frac{d}{b\xi}\frac{1}{\xi+d}\le\frac{1}{b\xi},\\
|F''(\xi)| &= \frac{d}{b\xi^2}\frac{1}{\xi+d}+\frac{d}{b\xi}\frac{1}{(\xi+d)^2}=\frac{1}{b\xi^2}\frac{d(\xi+d)+d\xi}{(\xi+d)^2}\le\frac{1}{b\xi^2},
\end{aligned}\tag{2.7}$$

for $\xi\in(0,\infty)$. Letting $\delta\to 0$ and applying (2.4), by (2.3), (2.6), and (2.7) we obtain

$$\begin{aligned}
&\sup_{z\in\mathbb{R}^N}\mu(B(z,R_1^\theta))\\
&\le C_*a^{-1}\big(a^{\frac{p}{p-1}}b^{-\frac{p}{p-1}}+b^{-\frac{2p}{p-m}}+b^{-\frac{p}{p-m}}\big)\int_{R_1}^{R_2}\xi^{N\theta-\frac{p}{p-1}}\,\mathrm{d}\xi\\
&= C_*R_1^{N\theta-\frac{1}{p-1}}a^{-1}\big(a^{\frac{p}{p-1}}b^{-\frac{p}{p-1}}+b^{-\frac{2p}{p-m}}+b^{-\frac{p}{p-m}}\big)\int_1^{R_2/R_1}\xi^{N\theta-\frac{p}{p-1}}\,\mathrm{d}\xi,
\end{aligned}\tag{2.8}$$

where C_* is a positive constant independent of a, b, c, and d.

Let $\sigma > 0$. Let $d > 0$ be such that

$$\sigma = \left(\frac{d}{e^{b+c}-1}\right)^{\theta} = R_1^{\theta}.$$

Then it follows from (2.5) and (2.8) that

$$\sup_{z\in\mathbb{R}^N} \mu(B(z,\sigma)) \le C_* \sigma^{N-\frac{2}{p-m}} a^{-1} \big(a^{\frac{p}{p-1}} b^{-\frac{p}{p-1}} + b^{-\frac{2p}{p-m}} + b^{-\frac{p}{p-m}} \big) \int_1^{\frac{e^{b+c}-1}{e^c-1}} \xi^{N\theta - \frac{p}{p-1}} \, d\xi$$

for a, b, c, and $\sigma > 0$ with

$$0 < \sigma \le \left(\frac{aT(e^c-1)}{2(e^{b+c}-1)}\right)^{\theta}.$$

Letting $c \to \infty$, we get

$$\sup_{z\in\mathbb{R}^N} \mu(B(z,\sigma)) \le C_* \sigma^{N-\frac{2}{p-m}} a^{-1} \big(a^{\frac{p}{p-1}} b^{-\frac{p}{p-1}} + b^{-\frac{2p}{p-m}} + b^{-\frac{p}{p-m}} \big) \int_1^{e^b} \xi^{N\theta - \frac{p}{p-1}} \, d\xi \tag{2.9}$$

for a, b, and $\sigma > 0$ with $0 < \sigma \le (aT/2e^b)^{\theta}$.

Step 4: We choose suitable a and $b > 0$ to complete the proof of Proposition 2.1. In the case of $p \ne p_m$, by (2.9) with $a = 2e$ and $b = 1$ we obtain

$$\mu(B(z,\sigma)) \le C\sigma^{N-\frac{2}{p-m}} \quad \text{for} \quad \sigma \in (0, T^{\theta}).$$

Thus Proposition 2.1 follows in the case of $p \ne p_m$.

Consider the case of $p = p_m$. Let $\ell \ge 1$ and $b \ge 1$, and set $a = \ell b^{-\frac{m-1}{p-m}}$. Since

$$\begin{aligned} &a^{\frac{p}{p-1}} b^{-\frac{p}{p-1}} = \left(\ell b^{-\frac{p-1}{p-m}}\right)^{\frac{p}{p-1}} = \ell^{\frac{p}{p-1}} b^{-\frac{p}{p-m}}, \\ &N\theta - \frac{p}{p-1} = -1, \quad \frac{m-1}{p-m} - \frac{p}{p-m} + 1 = -\frac{1}{p-m} = -\frac{N}{2}, \end{aligned}$$

it follows from (2.9) that

$$\sup_{z\in\mathbb{R}^N} \mu(B(z,\sigma)) \le C\ell^{\frac{p}{p-1}} a^{-1} b^{-\frac{p}{p-m}+1} = C\ell^{\frac{1}{p-1}} b^{-\frac{N}{2}} \tag{2.10}$$

for $b \ge 1$ and $\sigma > 0$ with

$$0 < \sigma \le \left(\frac{\ell T}{2b^{\frac{N(m-1)}{2}} e^b}\right)^{\theta}.$$

Let $L \ge e$, and set

$$b = \log\left(\left(L + \frac{T}{\sigma^{\theta'}}\right)\left[\log\left(L + \frac{T}{\sigma^{\theta'}}\right)\right]^{-\frac{N(m-1)}{2}}\right).$$

Taking L large enough if necessary, we see that $b \ge 1$. Since

$$b \le \log\left(L + \frac{T}{\sigma^{\theta'}}\right) \quad \text{for } \sigma > 0,$$

we have

$$b^{\frac{N(m-1)}{2}}e^b \le \left[\log\left(L+\frac{T}{\sigma^{\theta'}}\right)\right]^{\frac{N(m-1)}{2}}\left(L+\frac{T}{\sigma^{\theta'}}\right)\left[\log\left(L+\frac{T}{\sigma^{\theta'}}\right)\right]^{-\frac{N(m-1)}{2}} = L+\frac{T}{\sigma^{\theta'}} \le C\frac{T}{\sigma^{\theta'}}$$

for $\sigma \in (0, T^\theta)$. Then, taking $\ell \ge 1$ large enough if necessary, we obtain

$$\frac{\ell T}{2b^{\frac{N(m-1)}{2}}e^b} \ge \frac{\ell}{C}\sigma^{\theta'} \ge \sigma^{\theta'} \quad \text{for} \quad \sigma \in (0, T^\theta),$$

which implies that

$$\left(\frac{\ell T}{2b^{\frac{N(m-1)}{2}}e^b}\right)^\theta \ge \sigma \quad \text{for} \quad \sigma \in (0, T^\theta).$$

Then we deduce from (2.10) that

$$\begin{aligned}\sup_{z\in\mathbb{R}^N} \mu(B(z,\sigma)) &\le C\left[\log\left(\left(L+\frac{T}{\sigma^{\theta'}}\right)\left[\log\left(L+\frac{T}{\sigma^{\theta'}}\right)\right]^{-\frac{N(m-1)}{2}}\right)\right]^{-\frac{N}{2}} \\ &\le C\left[\log\left(L+\frac{T}{\sigma^{\theta'}}\right)\right]^{-\frac{N}{2}} \le C\left[\log\left(e+\frac{T^\theta}{\sigma}\right)\right]^{-\frac{N}{2}}\end{aligned}$$

for $\sigma \in (0, T^\theta)$. Thus Proposition 2.1 follows in the case of $p = p_m$, and the proof of Proposition 2.1 is complete. □

Remark 2.1 *As mentioned, the proof of Proposition* 2.1 *is based on the arguments in* [45]*. One of the main differences between the proof in* [45] *and our approach is the appearance of the additional parameter $a > 0$ (see Step* 2 *in the proof of Proposition* 2.1*). This parameter is necessary to treat the case $p = p_m$ in order to obtain* (2.10)*.*

Proof of Theorem 1.1. Let u be a solution to problem (E) in $\mathbb{R}^N \times (0,T)$, where $T \in (0,\infty)$. Then there exists a measurable set $I \subset (0,T)$ with $\mathcal{L}^1((0,T)\setminus I) = 0$ such that (1.2) holds for $\tau \in I$. For any $\tau \in I$, setting $u_\tau(x,t) := u(x,t+\tau)$ for a.a. $(x,t) \in \mathbb{R}^N \times (0, T-\tau)$, we see that u_τ is a solution to problem (P) in $\mathbb{R}^N \times (0, T-\tau)$ with $\mu = u(\tau)$. Then, by Proposition 2.1 we see that

$$\sup_{z\in\mathbb{R}^N}\int_{B(z,\sigma)} u(\tau)\,\mathrm{d}x \le \begin{cases} C_*\sigma^{N-\frac{2}{p-m}} & \text{if} \quad p \ne p_m, \\ C_*\left[\log\left(e+\dfrac{T^\theta}{\sigma}\right)\right]^{-\frac{N}{2}} & \text{if} \quad p = p_m, \end{cases} \tag{2.11}$$

for $\sigma \in (0, (T-\tau)^\theta)$ and $\tau \in I$, where C_* is a positive constant depending only on N, p, and m, and θ is defined as in (1.1). Applying the weak compactness of Radon measures (see e.g., [13, Section 1.9]), we find a sequence $\{\tau_j\} \subset I$ with $\lim_{j\to\infty}\tau_j = 0$ and $\nu \in \mathcal{M}$ such that

$$\lim_{j\to\infty}\int_{\mathbb{R}^N} u(\tau_j)\psi\,\mathrm{d}x = \int_{\mathbb{R}^N}\psi\,\mathrm{d}\nu(x), \quad \psi \in C_c(\mathbb{R}^N). \tag{2.12}$$

We show that (1.3) holds. Let $\{s_j\} \subset I$ with $\lim_{j\to\infty} s_j = 0$ and $\nu' \in \mathcal{M}$ such that

$$\lim_{j\to\infty}\int_{\mathbb{R}^N} u(s_j)\psi\,\mathrm{d}x = \int_{\mathbb{R}^N}\psi\,\mathrm{d}\nu'(x), \quad \psi \in C_c(\mathbb{R}^N). \tag{2.13}$$

Let $\psi \in C_c^\infty(\mathbb{R}^N)$. Let $\phi \in C_c^\infty(\mathbb{R}^N \times [0,T))$ be such that $\phi(x,t) = \psi(x)$ for $(x,t) \in \mathbb{R}^N \times [0,\delta]$ for some $\delta \in (0,T)$. Then, by (1.2), (2.12), and (2.13) we see that

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^N} (-u\partial_t \phi - u^m \Delta\phi - u^p \phi)\, \mathrm{d}x\, \mathrm{d}t &= \lim_{j\to\infty} \int_{\mathbb{R}^N} u(\tau_j)\phi(\tau_j)\, \mathrm{d}x = \int_{\mathbb{R}^N} \psi\, \mathrm{d}\nu(x)\\ &= \lim_{j\to\infty} \int_{\mathbb{R}^N} u(s_j)\phi(s_j)\, \mathrm{d}x = \int_{\mathbb{R}^N} \psi\, \mathrm{d}\nu'(x).\end{aligned}$$

This implies that $\nu = \nu'$ in $\mathcal{M}$. Then, since $\{s_j\} \subset I$ is arbitrary, we see that ν satisfies (1.3). Furthermore, the uniqueness of the initial trace of solution u also follows.

It remains to prove (1.4). Let $z \in \mathbb{R}^N$, $\delta \in (0,T)$, $\sigma \in (0,(T-\delta)^\theta)$, and $\epsilon \in (0,\sigma)$. Let $\psi \in C_c(\mathbb{R}^N)$ be such that $0 \le \psi \le 1$ in $\mathbb{R}^N$, $\psi = 1$ in $B(z,\sigma-\epsilon)$, and $\operatorname{supp}\psi \subset B(z,\sigma)$. Then it follows from (2.11) that

$$\begin{aligned}\mu(B(z,\sigma-\epsilon)) &\le \int_{\mathbb{R}^N} \psi\, \mathrm{d}\mu(x) = \lim_{j\to 0} \int_{\mathbb{R}^N} u(\tau_j)\psi\, \mathrm{d}x\\ &\le \begin{cases} C_*\sigma^{N-\frac{2}{p-m}} & \text{if} \quad p \ne p_m,\\ C_*\left[\log\left(e + \dfrac{T^\theta}{\sigma}\right)\right]^{-\frac{N}{2}} & \text{if} \quad p = p_m.\end{cases}\end{aligned}$$

Since $\epsilon \in (0,\sigma)$ and $\delta \in (0,T)$ are arbitrary, we obtain (1.4). Thus Theorem 1.1 follows. □

3 Energy estimates of solutions

In this section we obtain energy estimates of solutions to problem (P), which are crucial in the proofs of Theorems 1.2 and 1.3. We often use the following property:

- there exists $m_* \ge 1$ such that
$$\sup_{z\in\mathbb{R}^N} \int_{B(z,2\sigma)} f\, \mathrm{d}x \le m_* \sup_{z\in\mathbb{R}^N} \int_{B(z,\sigma)} f\, \mathrm{d}x \tag{3.1}$$
for $f \in \mathcal{L}$ and $\sigma > 0$ (see e.g., [30, Lemma 2.1]).

We first give an energy estimate of solutions to problem (P) with $p = p_m$.

Lemma 3.1 *Let $N \ge 1$, $m \ge 1$, $p = p_m \equiv m + 2/N$, $\mu \in \mathcal{L} \cap L^\infty(\mathbb{R}^N)$, $T \in (0,\infty)$, and $\alpha > 0$. Let u be a positive classical solution to problem* (P) *in $\mathbb{R}^N \times (0,T)$ such that*

$$\sup_{t\in(0,T)} \|u(t)\|_{L^\infty(\mathbb{R}^N)} < \infty. \tag{3.2}$$

Let Ψ and η be as in (1.5)*. Then*

$$\sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} u^{m-1}\Psi''(u)|\nabla u|^2\, \mathrm{d}x\, \mathrm{d}s < \infty \tag{3.3}$$

for $t \in (0,T)$ and $\sigma > 0$. Furthermore, there exists $C = C(N,m,\alpha) > 0$ such that

$$\begin{aligned}&\sup_{s\in(0,t]} \sup_{z\in\mathbb{R}^N} \int_{B(z,\sigma)} \Psi(u(s))\, \mathrm{d}x + \sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} u^{m-1}\Psi''(u)|\nabla u|^2\, \mathrm{d}x\, \mathrm{d}s\\ &\le C \sup_{z\in\mathbb{R}^N} \int_{B(z,\sigma)} \Psi(\mu)\, \mathrm{d}x + C\sigma^{-2} \sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} u^{m-1}\Psi(u)\, \mathrm{d}x\, \mathrm{d}s \qquad (3.4)\\ &+ C \sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} \Psi(u)\, \mathrm{d}x\, \mathrm{d}s + CM_\sigma[u](t)^{p-m} \sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} u^{m-1}\Psi''(u)|\nabla u|^2\, \mathrm{d}x\, \mathrm{d}s\end{aligned}$$

for $t \in (0,T)$ and $\sigma > 0$ if

$$M_\sigma[u](t) := \sup_{s\in(0,t]} \sup_{z\in\mathbb{R}^N} \sup_{r\in(0,\sigma]} \left\{ \eta(r)\Psi^{-1}\left(\fint_{B(z,r)} \Psi(u(s))\,\mathrm{d}y \right) \right\} \le 1. \tag{3.5}$$

Proof. Let u be a positive classical solution to problem (P) in $\mathbb{R}^N \times (0,T)$, where $T \in (0,\infty)$, and assume (3.2). Let $z \in \mathbb{R}^N$, $t \in (0,T)$, and $\sigma > 0$. Let $\zeta \in C_c^\infty(\mathbb{R}^N)$ be such that

$$\begin{aligned} &\zeta \equiv 1 \text{ in } B(z,\sigma), \quad \zeta \equiv 0 \text{ in } \mathbb{R}^N \setminus B(z,2\sigma), \quad 0 \le \zeta \le 1 \text{ in } \mathbb{R}^N, \\ &\|\nabla\zeta\|_{L^\infty} \le 2\sigma^{-1}, \quad \|\nabla^2\zeta\|_{L^\infty} \le 4\sigma^{-2}. \end{aligned} \tag{3.6}$$

Let Ψ and η be as in (1.5). Then

$$\begin{aligned} &\Psi'(0) > 0, \quad 0 < \Psi''(\xi) \preceq \xi^{-1}\Psi'(\xi) \preceq \xi^{-2}\Psi(\xi), \quad \Psi(2\xi) \asymp \Psi(\xi), \\ &\eta'(\xi) > 0, \quad \eta(2\xi) \asymp \eta(\xi), \end{aligned} \tag{3.7}$$

hold for $\xi \in (0,\infty)$. In addition, it follows from (3.7) that

$$(\Psi^{-1})'(0) > 0, \quad (\Psi^{-1})'(\xi) > 0, \quad (\Psi^{-1})''(\xi) < 0, \quad \Psi^{-1}(2\xi) \asymp \Psi^{-1}(\xi), \tag{3.8}$$

hold for $\xi \in (0,\infty)$.

Step 1. Let $k \ge 1$ and $\ell \ge 1$ be large enough to be chosen later. Let $t \in (0,T)$. We multiply the equation

$$\partial_t u - \Delta u^m - u^p = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,T)$$

by $\Psi'(u\zeta^\ell)\zeta^{k+\ell}$ and integrate it in $\mathbb{R}^N \times (0,t)$. Since

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^N} \partial_t u \Psi'(u\zeta^\ell)\zeta^{k+\ell}\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^N} \Psi(u\zeta^\ell)\zeta^k\,\mathrm{d}x\,\mathrm{d}s = \int_{\mathbb{R}^N} \Psi\big(u(t)\zeta^\ell\big)\zeta^k\,\mathrm{d}x - \int_{\mathbb{R}^N} \Psi(\mu\zeta^\ell)\zeta^k\,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} &-\int_0^t \int_{\mathbb{R}^N} \Delta(u^m)\Psi'(u\zeta^\ell)\zeta^{k+\ell}\,\mathrm{d}x\,\mathrm{d}s \\ &= m\int_0^t \int_{\mathbb{R}^N} u^{m-1}\nabla u \cdot \left(\Psi''(u\zeta^\ell)\zeta^{k+\ell}(\nabla u\zeta^\ell + \ell u\zeta^{\ell-1}\nabla\zeta) + (k+\ell)\Psi'(u\zeta^\ell)\zeta^{k+\ell-1}\nabla\zeta \right)\mathrm{d}x\,\mathrm{d}s \\ &= m\int_0^t \int_{\mathbb{R}^N} u^{m-1}\nabla u \cdot \left(\Psi''(u\zeta^\ell)\zeta^{k+\ell}(\nabla u\zeta^\ell + \ell u\zeta^{\ell-1}\nabla\zeta) \right)\mathrm{d}x\,\mathrm{d}s \\ &\quad - (k+\ell)\int_0^t \int_{\mathbb{R}^N} u^m \mathrm{div}\left(\Psi'(u\zeta^\ell)\zeta^{k+\ell-1}\nabla\zeta \right)\mathrm{d}x\,\mathrm{d}s \\ &= m\int_0^t \int_{\mathbb{R}^N} u^{m-1}\Psi''(u\zeta^\ell)|\nabla u|^2\zeta^{k+2\ell}\,\mathrm{d}x\,\mathrm{d}s + m\ell\int_0^t \int_{\mathbb{R}^N} u^m\Psi''(u\zeta^\ell)\zeta^{k+2\ell-1}\nabla u \cdot \nabla\zeta\,\mathrm{d}x\,\mathrm{d}s \\ &\quad - (k+\ell)\int_0^t \int_{\mathbb{R}^N} u^m \left(\Psi''(u\zeta^\ell)\zeta^{k+\ell-1}(\zeta^\ell\nabla u \cdot \nabla\zeta + \ell\zeta^{\ell-1}u|\nabla\zeta|^2) \right)\mathrm{d}x\,\mathrm{d}s \\ &\quad - \int_0^t \int_{\mathbb{R}^N} u^m\Psi'(u\zeta^\ell)\Delta\zeta^{k+\ell}\,\mathrm{d}x\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\ge \frac{m}{2}\int_0^t\int_{\mathbb{R}^N} u^{m-1}\Psi''(u\zeta^\ell)|\nabla u|^2\zeta^{k+2\ell}\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad - C\int_0^t\int_{\mathbb{R}^N} u^{m+1}\Psi''(u\zeta^\ell)\zeta^{k+2\ell-2}|\nabla\zeta|^2\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_{\mathbb{R}^N} u^m\Psi'(u\zeta^\ell)|\Delta\zeta^{k+\ell}|\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\Psi\big(u(t)\zeta^\ell\big)\zeta^k\,\mathrm{d}x + \frac{m}{2}\int_0^t\int_{\mathbb{R}^N} u^{m-1}\Psi''(u\zeta^\ell)|\nabla u|^2\zeta^{k+2\ell}\,\mathrm{d}x\,\mathrm{d}s\\
&\le \int_{\mathbb{R}^N}\Psi(\mu\zeta^\ell)\zeta^k\,\mathrm{d}x + C\int_0^t\int_{\mathbb{R}^N} u^{m+1}\Psi''(u\zeta^\ell)\zeta^{k+2\ell-2}|\nabla\zeta|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad + \int_0^t\int_{\mathbb{R}^N} u^m\Psi'(u\zeta^\ell)|\Delta\zeta^{k+\ell}|\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{R}^N} u^p\Psi'(u\zeta^\ell)\zeta^{k+\ell}\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

Taking $k\ge 1$ large enough so that $k\ge (p-1)\ell$, by (3.6) and (3.7) we have

$$
\begin{aligned}
&\int_{B(z,\sigma)}\Psi(u(t))\,\mathrm{d}x + \int_0^t\int_{B(z,\sigma)} u^{m-1}\Psi''(u)|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\le C\int_{B(z,2\sigma)}\Psi(\mu)\mathrm{d}x + C\sigma^{-2}\int_0^t\int_{B(z,2\sigma)} u^{m-1}\Psi(u)\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad + C\int_0^t\int_{\mathbb{R}^N}(u\zeta^\ell)^{p-1}\Psi(u\zeta^\ell)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

for $z\in\mathbb{R}^N$, $t\in(0,T)$, and $\sigma>0$. Then, by (3.1) we have

$$
\begin{aligned}
&\sup_{s\in(0,t]}\sup_{z\in\mathbb{R}^N}\int_{B(z,\sigma)}\Psi(u(s))\,\mathrm{d}x + \sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)} u^{m-1}\Psi''(u)|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\le C\sup_{z\in\mathbb{R}^N}\int_{B(z,\sigma)}\Psi(\mu)\mathrm{d}x + C\sigma^{-2}\sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)} u^{m-1}\Psi(u)\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad + C\sup_{z\in\mathbb{R}^N}\int_0^t\int_{\mathbb{R}^N}(u\zeta^\ell)^{p-1}\Psi(u\zeta^\ell)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
\tag{3.9}
$$

for $t\in(0,T)$ and $\sigma>0$. Furthermore, it follows from $\mu\in L^\infty(\mathbb{R}^N)$, (3.2), and (3.9) that (3.3) holds for $t\in(0,T)$ and $\sigma>0$.

Step 2. In this step we employ arguments in [32] to obtain an estimate of the last term of (3.9). Set

$$
v(x,s) := u(x,s)^{\frac{1}{\ell}}\zeta(x),\quad (x,s)\in\mathbb{R}^N\times(0,t). \tag{3.10}
$$

Then the layer cake representation (see e.g., [19, (1.1.7)]) together with (3.7) implies that

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^N}(u\zeta^\ell)^{p-1}\Psi(u\zeta^\ell)\,\mathrm{d}x\,\mathrm{d}s\\
&= \int_0^t\int_{\mathbb{R}^N} v^{\ell(p-1)}\Psi(v^\ell)\,\mathrm{d}x\,\mathrm{d}s\\
&= \int_0^t\int_0^\infty \mathcal{L}^N\big(\{x\in\mathbb{R}^N\,:\,v(x,s)>\lambda\}\big)\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big(\lambda^{\ell(p-1)}\Psi(\lambda^\ell)\Big)\,\mathrm{d}\lambda\,\mathrm{d}s\\
&\le C\int_0^t\int_0^\infty \mathcal{L}^N\big(\{x\in\mathbb{R}^N\,:\,v(x,s)>\lambda\}\big)\lambda^{\ell(p-1)-1}\Psi(\lambda^\ell)\,\mathrm{d}\lambda\,\mathrm{d}s\\
&\le CI_\Lambda(t;z,\sigma) + C\Lambda^{\ell(p-1)}\int_0^t\int_{\mathbb{R}^N}\Psi(v^\ell)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
\tag{3.11}
$$

for $z \in \mathbb{R}^N$, $t \in (0,T)$, $\sigma > 0$, and $\Lambda \ge 0$, where

$$I_\Lambda(t;z,\sigma) := \int_0^t \int_\Lambda^\infty \mathcal{L}^N\big(\{x \in \mathbb{R}^N \,:\, v(x,s) > \lambda\}\big)\lambda^{\ell(p-1)-1}\Psi(\lambda^\ell)\,\mathrm{d}\lambda\,\mathrm{d}s. \tag{3.12}$$

We obtain an estimate of $I_\Lambda(t;z,\sigma)$. For any $\lambda > \Lambda$ and $(x,s) \in \mathbb{R}^N \times (0,t)$, set

$$v_\lambda(x,s) := \begin{cases} 0 & \text{if } \ v(x,s) \le \dfrac{\lambda}{2}, \\ v(x,s) - \dfrac{\lambda}{2} & \text{if } \ \dfrac{\lambda}{2} < v(x,s) \le 2\lambda, \\ \dfrac{3}{2}\lambda & \text{if } \ v(x,s) > 2\lambda. \end{cases} \tag{3.13}$$

We claim that

$$\fint_{B(x,r)} v_\lambda(y,s)\,\mathrm{d}y \le \left(\frac{m_* M_\sigma[u](t)}{\eta(r)}\right)^{\frac{1}{\ell}} \quad \text{for } x \in \mathbb{R}^N,\ r \in (0,\infty),\ \lambda > \Lambda, \text{ and } s \in (0,t], \tag{3.14}$$

where $m_* \ge 1$ is as in (3.1). By (3.5), (3.6), (3.10), and (3.13) we apply Jensen's inequality to obtain

$$\begin{aligned} \fint_{B(x,r)} v_\lambda(y,s)\,\mathrm{d}y &\le \fint_{B(x,r)} v(y,s)\,\mathrm{d}y \le \left(\fint_{B(x,r)} u(y,s)\,\mathrm{d}y\right)^{\frac{1}{\ell}} \\ &\le \left(\sup_{z\in\mathbb{R}^N} \Psi^{-1}\left(\fint_{B(z,r)} \Psi(u(y,s))\,\mathrm{d}y\right)\right)^{\frac{1}{\ell}} \le \left(\frac{M_\sigma[u](t)}{\eta(r)}\right)^{\frac{1}{\ell}} \end{aligned}$$

for $x \in \mathbb{R}^N$, $r \in (0,\sigma)$, $\lambda > \Lambda$, and $s \in (0,t]$. This implies that (3.14) holds if $r \in (0,\sigma)$. On the other hand, since $\zeta \equiv 0$ in $\mathbb{R}^N \setminus B(z,2\sigma)$, by Jensen's inequality we have

$$\fint_{B(x,r)} v_\lambda(y,s)\,\mathrm{d}y \le \fint_{B(x,r)} v(y,s)\,\mathrm{d}y \le \left(\frac{1}{\mathcal{L}^N(B(0,r))} \sup_{z\in\mathbb{R}^N} \int_{B(z,2\sigma)} u(y,s)\,\mathrm{d}y\right)^{\frac{1}{\ell}}$$

for $x \in \mathbb{R}^N$, $r \in (\sigma,\infty)$, $\lambda > \Lambda$, and $s \in (0,t]$. Then, by (3.1) and (3.5) we apply Jensen's inequality again to obtain

$$\begin{aligned} &\fint_{B(x,r)} v_\lambda(y,s)\,\mathrm{d}y \le \left(\frac{m_*}{\mathcal{L}^N(B(0,r))} \sup_{z\in\mathbb{R}^N} \int_{B(z,\sigma)} u(y,s)\,\mathrm{d}y\right)^{\frac{1}{\ell}} \\ &= \left(\frac{m_* \mathcal{L}^N(B(0,\sigma))}{\mathcal{L}^N(B(0,r))} \sup_{z\in\mathbb{R}^N} \fint_{B(z,\sigma)} u(y,s)\,\mathrm{d}y\right)^{\frac{1}{\ell}} \\ &\le \left(m_* \left(\frac{\sigma}{r}\right)^N \sup_{z\in\mathbb{R}^N} \Psi^{-1}\left(\fint_{B(z,\sigma)} \Psi(u(y,s))\,\mathrm{d}y\right)\right)^{\frac{1}{\ell}} \le \left(m_* \left(\frac{\sigma}{r}\right)^N \frac{M_\sigma[u](t)}{\eta(\sigma)}\right)^{\frac{1}{\ell}} \end{aligned}$$

for $x \in \mathbb{R}^N$, $r \in (\sigma,\infty)$, $\lambda > \Lambda$, and $s \in (0,t]$. Since $\sigma^{-N}\eta(\sigma) \ge r^{-N}\eta(r)$ for $r \in (\sigma,\infty)$, we see that (3.14) holds if $r \in (\sigma,\infty)$. Thus (3.14) is valid.

Since $\eta(0) = 0$ and $\lim_{\xi\to\infty} \eta(\xi) = \infty$, for any $\lambda > \Lambda$, we apply the intermediate value theorem to find $r_* = r_*(\lambda) > 0$ such that

$$\eta(r_*(\lambda)) = \frac{4^\ell m_* M_\sigma[u](t)}{\lambda^\ell}. \tag{3.15}$$

This together with (3.14) implies that

$$\fint_{B(x,r_*(\lambda))} v_\lambda(y,s)\,\mathrm{d}y \le \left(\frac{m_* M_\sigma[u](t)}{\eta(r_*(\lambda))}\right)^{\frac{1}{\ell}} = \frac{\lambda}{4}$$

for $x \in \mathbb{R}^N$, $\lambda > \Lambda$, and $s \in (0,t]$. Then, by (3.13) we see that

$$\begin{aligned}
&\mathcal{L}^N\big(\{x \in \mathbb{R}^N \,:\, v(x,s) > \lambda\}\big)\\
&\le \mathcal{L}^N\left(\left\{x \in \mathbb{R}^N \,:\, v_\lambda(x,s) > \frac{\lambda}{2}\right\}\right)\\
&\le \mathcal{L}^N\left(\left\{x \in \mathbb{R}^N \,:\, \left|v_\lambda(x,s) - \fint_{B(x,r_*(\lambda))} v_\lambda(y,s)\,\mathrm{d}y\right| > \frac{\lambda}{4}\right\}\right)\\
&\le \frac{16}{\lambda^2}\int_{\mathbb{R}^N}\left|v_\lambda(x,s) - \fint_{B(x,r_*(\lambda))} v_\lambda(y,s)\,\mathrm{d}y\right|^2 \mathrm{d}x
\end{aligned} \tag{3.16}$$

for $\lambda > \Lambda$ and $s \in (0,t]$. Furthermore, we have

$$\begin{aligned}
&\left|v_\lambda(x,s) - \fint_{B(x,r_*(\lambda))} v_\lambda(y,s)\,\mathrm{d}y\right|^2 = \left|\fint_{B(x,r_*(\lambda))} (v_\lambda(x,s) - v_\lambda(y,s))\,\mathrm{d}y\right|^2\\
&= \frac{1}{\mathcal{L}^N(B(x,r_*(\lambda)))^2}\left|\int_{B(x,r_*(\lambda))}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\xi} v_\lambda((1-\xi)y + \xi x, s)\,\mathrm{d}\xi\,\mathrm{d}y\right|^2\\
&\le \frac{1}{\mathcal{L}^N(B(x,r_*(\lambda)))^2}\left(\int_{B(x,r_*(\lambda))}\int_0^1 |\nabla v_\lambda((1-\xi)y + \xi x, s)||x-y|\,\mathrm{d}\xi\,\mathrm{d}y\right)^2\\
&\le \frac{r_*(\lambda)^2}{\mathcal{L}^N(B(x,r_*(\lambda)))}\int_{B(x,r_*(\lambda))}\int_0^1 |\nabla v_\lambda((1-\xi)y + \xi x, s)|^2\,\mathrm{d}\xi\,\mathrm{d}y\\
&= \frac{r_*(\lambda)^2}{\mathcal{L}^N(B(x,r_*(\lambda)))}\int_0^1\int_{B(0,r_*(\lambda))} |\nabla v_\lambda((1-\xi)y + x, s)|^2\,\mathrm{d}y\,\mathrm{d}\xi.
\end{aligned} \tag{3.17}$$

Then, by (3.16) and (3.17) we obtain

$$\begin{aligned}
&\mathcal{L}^N\big(\{x \in \mathbb{R}^N \,:\, v(x,s) > \lambda\}\big)\\
&\le \frac{16 r_*(\lambda)^2}{\lambda^2 \mathcal{L}^N(B(0,r_*(\lambda)))}\int_0^1\int_{B(0,r_*(\lambda))}\int_{\mathbb{R}^N} |\nabla v_\lambda((1-\xi)y + x, s)|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\xi\\
&= \frac{16 r_*(\lambda)^2}{\lambda^2}\int_{\mathbb{R}^N} |\nabla v_\lambda(x,s)|^2\,\mathrm{d}x\\
&= \frac{16 r_*(\lambda)^2}{\lambda^2}\int_{\{x\in\mathbb{R}^N \,:\, \lambda/2 \le v(x,s) \le 2\lambda\}} |\nabla v(x,s)|^2\,\mathrm{d}x
\end{aligned} \tag{3.18}$$

for $\lambda > \Lambda$ and $s \in (0, t]$. Since $\Psi(2\xi) \asymp \Psi(\xi)$ for $\xi \in (0, \infty)$ (see (3.7)), we observe from (3.12) and (3.18) that

$$\begin{aligned} &I_\Lambda(t; z, \sigma) \\ &\le C \int_0^t \int_\Lambda^\infty \frac{r_*(\lambda)^2}{\lambda^2} \left(\int_{\{x \in \mathbb{R}^N \,:\, \lambda/2 \le v(s) \le 2\lambda\}} |\nabla v(s)|^2 \, \mathrm{d}x \right) \lambda^{\ell(p-1)-1} \Psi(\lambda^\ell) \, \mathrm{d}\lambda \, \mathrm{d}s \\ &= C \int_0^t \int_\Lambda^\infty \lambda^{(p-1)\ell-3} r_*(\lambda)^2 \Psi(\lambda^\ell) \left(\int_{\{x \in \mathbb{R}^N \,:\, \lambda/2 \le v(s) \le 2\lambda\}} |\nabla v(s)|^2 \, \mathrm{d}x \right) \mathrm{d}\lambda \, \mathrm{d}s \\ &= C \int_0^t \int_{\mathbb{R}^N} |\nabla v(s)|^2 \chi_{\{v(s) > \Lambda/2\}}(x) \left(\int_{\frac{1}{2}v(s)}^{2v(s)} \lambda^{(p-1)\ell-3} r_*(\lambda)^2 \Psi(\lambda^\ell) \, \mathrm{d}\lambda \right) \mathrm{d}x \, \mathrm{d}s \\ &\le C \int_0^t \int_{\mathbb{R}^N} \left(u^{\frac{2}{\ell}-2} |\nabla u|^2 \zeta^2 + u^{\frac{2}{\ell}} |\nabla \zeta|^2 \right) \\ &\qquad \times v^{\ell(p-1)-2} \Psi(v^\ell) \left(\sup_{\lambda \in (v(s)/2, 2v(s))} r_*(\lambda) \right)^2 \chi_{\{v(s) > \Lambda/2\}}(x) \, \mathrm{d}x \, \mathrm{d}s \end{aligned} \tag{3.19}$$

for $z \in \mathbb{R}^N$, $t \in (0, T)$, and $\sigma > 0$.

<u>Step 3.</u> Assume that $M_\sigma[u](t) \le 1$. Since η is strictly increasing in $(0, \infty)$ (see (3.7)) and

$$\eta^{-1}(\xi) \asymp \xi^{\frac{1}{N}} \left[\log\left(e + \frac{1}{\xi} \right) \right]^{-\frac{1}{2}}, \quad \xi \in (0, \infty),$$

it follows from (3.15) that

$$\begin{aligned} \left(\sup_{\lambda \in (v(s)/2, 2v(s))} r_*(\lambda) \right)^2 &= \left[\eta^{-1} \left(\frac{8^\ell m_* M_\sigma[u](t)}{v(s)^\ell} \right) \right]^2 \\ &\le C M_\sigma[u](t)^{\frac{2}{N}} v(s)^{-\frac{2\ell}{N}} \left[\log\left(e + \frac{v(s)^\ell}{8^\ell m_* M_\sigma[u](t)} \right) \right]^{-1} \\ &\le C M_\sigma[u](t)^{p-m} v(s)^{-\ell(p-m)} \left[\log\left(e + v(s)^\ell \right) \right]^{-1} \end{aligned} \tag{3.20}$$

for $\lambda \in (0, \infty)$.

On the other hand, for any fixed $a > 0$ and $b \in \mathbb{R}$, taking $L \ge e$ large enough so that the function $[0, \infty) \ni \xi \mapsto \xi^a [\log(L + \xi)]^b$ is increasing, we have

$$\xi_1^a [\log(e + \xi_1)]^b \asymp \xi_1^a [\log(L + \xi_1)]^b \le \xi_2^a [\log(L + \xi_2)]^b \asymp \xi_2^a [\log(e + \xi_2)]^b \tag{3.21}$$

for ξ_1, $\xi_2 \in [0, \infty)$ with $\xi_1 \le \xi_2$. Then, taking $\ell \ge 1$ large enough so that $m - 2/\ell > 0$, we observe from (3.20) that

$$\begin{aligned} &v(s)^{\ell(p-1)-2} \Psi(v(s)^\ell) \left(\sup_{\lambda \in (v(s)/2, 2v(s))} r_*(\lambda) \right)^2 \\ &\le C M_\sigma[u](t)^{p-m} v(s)^{\ell m - 2} \left[\log\left(e + v(s)^\ell \right) \right]^{\alpha-1} \\ &\le C M_\sigma[u](t)^{p-m} u(s)^{m - \frac{2}{\ell}} \left[\log\left(e + u(s) \right) \right]^{\alpha-1} \end{aligned}$$

for $s \in (0,t)$. This together with (3.1) and (3.19) implies that

$$
\begin{aligned}
&I_\Lambda(t;z,\sigma)\\
&\le CM_\sigma[u](t)^{p-m}\int_0^t\int_{B(z,2\sigma)} u^{m-3}\left[\log\left(e+u\right)\right]^{-1}\Psi(u)|\nabla u|^2\chi_{\{u(s)>\Lambda/2\}}(x)\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad\quad + CM_\sigma[u](t)^{p-m}\sigma^{-2}\int_0^t\int_{B(z,2\sigma)} u^{m-1}\Psi(u)\,\mathrm{d}x\,\mathrm{d}s\\
&\le CM_\sigma[u](t)^{p-m}\sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)} u^{m-3}[\log(e+u)]^{-1}\Psi(u)|\nabla u|^2\chi_{\{u(s)>\Lambda/2\}}(x)\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad\quad + C\sigma^{-2}\sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)} u^{m-1}\Psi(u)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
\tag{3.22}
$$

for $z \in \mathbb{R}^N$, $t \in (0,T)$, and $\sigma > 0$. Since

$$
\xi^{-2}[\log(e+\xi)]^{-1}\Psi(\xi) \preceq \Psi''(\xi) \quad \text{for } \xi \in [1,\infty),
$$

by (3.1), (3.11), and (3.22) with $\Lambda = 2$ we obtain

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^N} (u\zeta^\ell)^{p-1}\Psi(u\zeta^\ell)\,\mathrm{d}x\,\mathrm{d}s\\
&\le C\int_0^t\int_{\mathbb{R}^N}\Psi(v^\ell)\,\mathrm{d}x\,\mathrm{d}s + CI_2(t;z,\sigma)\\
&\le C\sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)}\Psi(u)\,\mathrm{d}x\,\mathrm{d}s + CM_\sigma[u](t)^{p-m}\sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)} u^{m-1}\Psi''(u)|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad\quad + C\sigma^{-2}\sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)} u^{m-1}\Psi(u)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

for $z \in \mathbb{R}^N$, $t \in (0,T)$, and $\sigma > 0$. This together with (3.9) implies (3.4). Thus Lemma 3.1 follows. $\Box$

Similarly, we obtain an energy estimate of solutions to problem (P) with $p > p_m$.

Lemma 3.2 *Let $N \ge 1$, $m \ge 1$, $p > p_m \equiv m + 2/N$, $\mu \in \mathcal{L} \cap L^\infty(\mathbb{R}^N)$, $T \in (0,\infty]$, and $\beta > 1$. Let u be a positive classical solution to problem* (P) *in $\mathbb{R}^N \times (0,T)$ satisfying* (3.2)*. Then*

$$
\sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)} u^{m+\beta-3}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s < \infty \tag{3.23}
$$

for $t \in (0,T)$ and $\sigma > 0$. Furthermore, there exists $C = C(N,m,p,\beta) > 0$ such that

$$
\begin{aligned}
&\sup_{s\in(0,t]}\sup_{z\in\mathbb{R}^N}\int_{B(z,\sigma)} u(s)^\beta\,\mathrm{d}x + \sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)} u^{m+\beta-3}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\le C\sup_{z\in\mathbb{R}^N}\int_{B(z,\sigma)}\mu^\beta\,\mathrm{d}x\\
&\qquad + C\left(1 + \sup_{s\in(0,t)}|||u(s)|||^{p-m}_{\frac{N(p-m)}{2},\beta;\sigma}\right)\sigma^{-2}\sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)} u^{m+\beta-1}\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad + C\left(\sup_{s\in(0,t)}|||u(s)|||_{\frac{N(p-m)}{2},\beta;\sigma}\right)^{p-m}\sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)} u^{m+\beta-3}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
\tag{3.24}
$$

for $t \in (0,T)$ and $\sigma > 0$.

Proof. Setting $\Psi(\xi) = \xi^\beta$ and $\eta(\xi) = \xi^{\frac{2}{p-m}}$ for $\xi \in [0,\infty)$, we apply the same arguments as in the proof of Lemma 3.1. Then we have

$$
\begin{aligned}
&\sup_{z\in\mathbb{R}^N} \int_{B(z,\sigma)} u(t)^\beta \,\mathrm{d}x + \sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} u^{m+\beta-3} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\le C \sup_{z\in\mathbb{R}^N} \int_{B(z,\sigma)} \mu^\beta \mathrm{d}x + C\sigma^{-2} \sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} u^{m+\beta-1} \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + C \sup_{z\in\mathbb{R}^N} \int_0^t \int_{\mathbb{R}^N} (u\zeta^\ell)^{p+\beta-1} \,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
\tag{3.25}
$$

for $t \in (0,T)$ and $\sigma > 0$, instead of (3.9). This together with (3.2) implies (3.23).

Let I_Λ, v, and r_* be as in the proof of Lemma 3.1. It follows from (3.15) that

$$
r_*(\lambda)^{\frac{2}{p-m}} = \frac{4^\ell m_* M_\sigma[u](t)}{\lambda^\ell}
$$

and hence

$$
\left(\sup_{\lambda\in(v(s)/2,2v(s))} r_*(\lambda) \right)^2 \le C M_\sigma[u](t)^{p-m} (u\zeta^\ell)^{-(p-m)}.
$$

Then, taking $\ell \ge 1$ large enough if necessary, by (3.19) with $\Lambda = 0$ we obtain

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^N} (u\zeta^\ell)^{p-1} \Psi(u\zeta^\ell) \,\mathrm{d}x\,\mathrm{d}s = I_0(t;z,\sigma) \\
&\le C M_\sigma[u](t)^{p-m} \int_0^t \int_{B(z,2\sigma)} u^{m-3} \Psi(u) |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + C M_\sigma[u](t)^{p-m} \sigma^{-2} \int_0^t \int_{B(z,2\sigma)} u^{m-1} \Psi(u) \,\mathrm{d}x\,\mathrm{d}s \\
&\le C M_\sigma[u](t)^{p-m} \sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} u^{m-3} \Psi(u) |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + C M_\sigma[u](t)^{p-m} \sigma^{-2} \sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} u^{m-1} \Psi(u) \,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
\tag{3.26}
$$

for $z \in \mathbb{R}^N$, $t \in (0,T)$, and $\sigma > 0$. Since

$$
M_\sigma[u](t) = \sup_{s\in(0,t)} |||u(s)|||_{\frac{N(p-m)}{2},\beta;\sigma}
$$

for $t \in (0,T)$, by (3.25) and (3.26) we obtain (3.24). Thus Lemma 3.2 follows. □

At the end of this section we recall decay estimates of solutions to problem (P). See [3, Proposition 7.1]. (See also [42, Proposition 3.2].)

Lemma 3.3 *Let $N \ge 1$, $m \ge 1$, $p > m$, $r \ge 1$, and $T \in (0,\infty)$. Let u be a solution to problem* (P) *in $\mathbb{R}^N \times (0,T)$. Then there exists $C = C(N,m,p,r) > 0$ such that*

$$
\|u(t)\|_{L^\infty(\mathbb{R}^N)} \le C t^{-\frac{N+2}{\kappa_r}} \sup_{z\in\mathbb{R}^N} \left(\int_{t/2}^t \int_{B(z,2\sigma)} u^r \,\mathrm{d}x\,\mathrm{d}s \right)^{\frac{2}{\kappa_r}}
$$

for $t \in (0,T_*)$, where $\kappa_r := N(m-1) + 2r$ and

$$
T_* := \sup\left\{ \tau \in (0,T) \,:\, \sigma^{-2} \|u(s)\|_{L^\infty(\mathbb{R}^N)}^{m-1} + \|u(s)\|_{L^\infty(\mathbb{R}^N)}^{p-1} \le 2s^{-1} \text{ for } s \in (0,\tau) \right\}.
$$

4 Proof of Theorem 1.2

In this section we study sufficient conditions for the existence of solutions to problem (P) with $p = p_m$, and prove Theorem 1.2.

Let Ψ and η be as in Theorem 1.2. Define a C^1-function γ in $[0,1]$ by

$$\int_0^{\gamma(\xi)} s\eta(s)^{m-1}\,\mathrm{d}s = C_\eta\xi \quad \text{for } \xi\in[0,1], \quad \text{where } C_\eta := \int_0^1 s\eta(s)^{m-1}\,\mathrm{d}s. \tag{4.1}$$

Then $\gamma' > 0$ in $[0,1]$, $\gamma(0)=0$, and $\gamma(1)=1$. Since

$$\begin{aligned} C_\eta\xi = \int_0^{\gamma(\xi)} s\eta(s)^{m-1}\,\mathrm{d}s &= \int_0^{\gamma(\xi)} s^{1+N(m-1)}\left[\log\left(e+\frac{1}{s}\right)\right]^{\frac{N(m-1)}{2}}\,\mathrm{d}s \\ &\asymp \gamma(\xi)^{2+N(m-1)}\left[\log\left(e+\frac{1}{\gamma(\xi)}\right)\right]^{\frac{N(m-1)}{2}} = \gamma(\xi)^2\eta(\gamma(\xi))^{m-1} \end{aligned}$$

for $\xi\in[0,1]$, we see that

$$\gamma(\xi)^2\eta(\gamma(\xi))^{m-1} \asymp \xi \quad \text{for} \quad \xi\in[0,1]. \tag{4.2}$$

We prove a lemma on the function γ.

Lemma 4.1 *Let $N \geq 1$, $m \geq 1$, and $p = p_m \equiv m + 2/N$. The function γ defined by* (4.1) *satisfies the following properties:*

$$\gamma(\xi) \asymp \gamma\left(\frac{\xi}{2}\right), \quad \eta(\gamma(\xi)) \asymp \eta\left(\gamma\left(\frac{\xi}{2}\right)\right), \tag{4.3}$$

$$\eta(\gamma(\xi)) \succeq \xi^{\frac{1}{p-1}}, \tag{4.4}$$

$$\int_0^\xi \eta(\gamma(s))^{-(m-1)}\,\mathrm{d}s \asymp \gamma(\xi)^2, \tag{4.5}$$

for $\xi\in[0,1]$.

Proof. We prove (4.3). Taking $k\geq 1$ large enough, by (4.1) and the monotonicity of η we obtain

$$\begin{aligned} \int_0^{k^{-1}\gamma(\xi)} s\eta(s)^{m-1}\,\mathrm{d}s &= k^{-2}\int_0^{\gamma(\xi)} s\eta(k^{-1}s)^{m-1}\,\mathrm{d}s \\ &\leq Ck^{-2}\int_0^{\gamma(\xi)} s\eta(s)^{m-1}\,\mathrm{d}s = Ck^{-2}C_\eta\xi \leq \frac{\xi}{2}C_\eta. \end{aligned}$$

Then we observe from (4.1) that

$$k^{-1}\gamma(\xi) \leq \gamma\left(\frac{\xi}{2}\right), \quad \xi\in[0,1].$$

This together with the monotonicity of γ and the relation that $\eta(2\xi)\asymp\eta(\xi)$ for $\xi\in(0,\infty)$ (see (3.7)) implies (4.3).

We prove (4.4). Since $\eta(\xi) \ge \xi^N = \xi^{2/(p-m)}$ for $\xi \in [0,\infty)$, it follows from (4.1) that

$$C_\eta \xi \ge C \int_0^{\gamma(\xi)} s^{1+\frac{2(m-1)}{p-m}}\, \mathrm{d}s = C\gamma(\xi)^{2+\frac{2(m-1)}{p-m}} = C\gamma(\xi)^{\frac{2(p-1)}{p-m}}$$

for $\xi \in [0,1]$. This together with (4.2) implies that

$$\eta(\gamma(\xi)) \asymp \left(\xi\gamma(\xi)^{-2}\right)^{\frac{1}{m-1}} \succeq \xi^{\frac{1}{m-1}} \left(\xi^{-\frac{p-m}{2(p-1)}}\right)^{\frac{2}{m-1}} = \xi^{\frac{1}{p-1}} \quad \text{for} \quad \xi \in [0,1].$$

Thus (4.4) holds.

It remains to prove (4.5). It follows from (4.1) that

$$\gamma(\xi)\gamma(\xi)'\eta(\gamma(\xi))^{m-1} = C_\eta \quad \text{for } \xi \in [0,1],$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\gamma(\xi)^2 = 2C_\eta \eta(\gamma(\xi))^{-(m-1)}.$$

This together with $\gamma(0) = 0$ implies that

$$\gamma(\xi)^2 = 2C_\eta \int_0^\xi \eta(\gamma(s))^{-(m-1)}\, \mathrm{d}s, \quad \xi \in [0,1].$$

Thus (4.5) holds, and the proof of Lemma 4.1 is complete. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Remark 1.1-(1) it suffices to consider the case of $T = 1$. Let $p = p_m$ and $\mu \in \mathcal{L}$. Let Ψ and η be as in Theorem 1.2. Let $\epsilon_5 > 0$ be small enough, and assume (1.6). For any i, $j = 1, 2, \dots$, let u_{ij} be a solution to problem (P) with initial data μ replaced by

$$\mu_{ij}(x) := \min\{\mu(x), i\} + j^{-1}, \quad \text{a.a. } x \in \mathbb{R}^N. \tag{4.6}$$

By arguments in [3] we find a unique classical solution u_{ij} to problem (P) in $\mathbb{R}^N \times (0, T_{ij})$ such that

$$u_{ij}(x,t) \ge j^{-1} \quad \text{for } (x,t) \in \mathbb{R}^N \times (0, T_{ij}), \tag{4.7}$$

$$\sup_{t \in (0,T)} \|u_{ij}(t)\|_{L^\infty(\mathbb{R}^N)} < \infty \quad \text{for } T \in (0, T_{ij}), \tag{4.8}$$

$$\limsup_{t \nearrow T_{ij}} \|u_{ij}(t)\|_{L^\infty(\mathbb{R}^N)} = \infty, \tag{4.9}$$

where T_{ij} is the maximal existence time of u_{ij}. Since Ψ^{-1} is Lipschitz continuous on $[0,\infty)$ (see (3.8)), by (3.7) and (3.8) we find $j_* = j_*(\epsilon_5) \in \mathbb{N}$ such that

$$\begin{aligned} |||\mu_{ij}|||_{\eta,\Psi;1} &\le \sup_{z\in\mathbb{R}^N} \sup_{\sigma\in(0,1]} \eta(\sigma)\Psi^{-1}\left(\fint_{B(z,\sigma)} \Psi(\mu + j^{-1})\, \mathrm{d}x\right) \\ &\le C \sup_{z\in\mathbb{R}^N} \sup_{\sigma\in(0,1]} \eta(\sigma)\Psi^{-1}\left(\fint_{B(z,\sigma)} \Psi(\mu)\, \mathrm{d}x + j^{-1}\right) \\ &\le C \sup_{z\in\mathbb{R}^N} \sup_{\sigma\in(0,1]} \eta(\sigma)\Psi^{-1}\left(\fint_{B(z,\sigma)} \Psi(\mu)\, \mathrm{d}x\right) + Cj^{-1} \le C\epsilon_5 \end{aligned} \tag{4.10}$$

for $i \ge 1$ and $j \ge j_*$.

Step 1. Let δ_1, $\delta_2 \in (0,1)$. Set

$$T_{ij}^1 := \sup \left\{ T \in (0, T_{ij}) \ : \ \sup_{t \in (0,T)} \eta(\gamma(t)) \|u_{ij}(t)\|_{L^\infty(\mathbb{R}^N)} \le \delta_1 \right\}, \tag{4.11}$$

$$T_{ij}^2 := \sup \left\{ T \in (0, T_{ij}) \ : \ \sup_{t \in (0,T)} |||u_{ij}(t)|||_{\eta,\Psi;1} \le \delta_2 \right\}. \tag{4.12}$$

Under suitable choices of δ_1 and δ_2, taking $\epsilon_5 > 0$ small enough if necessary, we show that

$$T_{ij}^* := \min\{T_{ij}^1, T_{ij}^2, 1\} = 1 \quad \text{for } i \ge 1 \text{ and } j \ge j_*.$$

In the proof of Theorem 1.2, the constants C are independent of $i \ge 1$ and $j \ge j_*$.

We first show that $T_{ij}^* > 0$. Since $T_{ij}^1 > 0$ immediately follows from (4.8), it suffices to show that $T_{ij}^2 > 0$. Set $c_{ij} := \sup_{s \in (0, T_{ij}/2)} \|u_{ij}(s)\|_{L^\infty(\mathbb{R}^N)} < \infty$ and take $t_{ij} \in (0, \min\{T_{ij}/2, 1\})$ satisfying

$$\eta(t_{ij}^{1/4}) c_{ij} \le \frac{\delta_2}{2} \quad \text{and} \quad t_{ij}^{1/2}(c_{ij}^{m-1} + c_{ij}^{p-1}) \le 1 \tag{4.13}$$

for $i \ge 1$ and $j \ge j_*$. Then we have

$$\begin{aligned} &\sup_{s \in (0, t_{ij})} \sup_{z \in \mathbb{R}^N} \sup_{\sigma \in (0, t_{ij}^{1/4}]} \eta(\sigma) \Psi^{-1} \left(\fint_{B(z,\sigma)} \Psi(u_{ij}(s)) \, \mathrm{d}y \right) \\ &\le \sup_{s \in (0, t_{ij})} \sup_{z \in \mathbb{R}^N} \sup_{\sigma \in (0, t_{ij}^{1/4}]} \eta(\sigma) \Psi^{-1} \left(\Psi \left(\|u_{ij}(s)\|_{L^\infty(\mathbb{R}^N)} \right) \right) \le \eta(t_{ij}^{1/4}) c_{ij} \le \frac{\delta_2}{2} \end{aligned} \tag{4.14}$$

for $i \ge 1$ and $j \ge j_*$. On the other hand, similarly to the argument in Step.1 in the proof of Lemma 3.1 (see (3.9)), it follows from (4.13) that

$$\begin{aligned} &\sup_{s \in (0,t]} \sup_{z \in \mathbb{R}^N} \int_{B(z,\sigma)} \Psi(u_{ij}(s)) \, \mathrm{d}x \\ &\le C \sup_{z \in \mathbb{R}^N} \int_{B(z,\sigma)} \Psi(\mu_{ij}) \, \mathrm{d}x + C(\sigma^{-2} c_{ij}^{m-1} + c_{ij}^{p-1}) \sup_{z \in \mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} \Psi(u_{ij}) \, \mathrm{d}x \, \mathrm{d}s \\ &\le C \sup_{z \in \mathbb{R}^N} \int_{B(z,\sigma)} \Psi(\mu_{ij}) \, \mathrm{d}x + C t_{ij}^{-1} \sup_{z \in \mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} \Psi(u_{ij}) \, \mathrm{d}x \, \mathrm{d}s \end{aligned}$$

for $t \in (0, t_{ij})$, $\sigma \in (t_{ij}^{1/4}, 1]$, $i \ge 1$, and $j \ge j_*$. This together with Gronwall's inequality implies that

$$\sup_{s \in (0,t]} \sup_{z \in \mathbb{R}^N} \fint_{B(z,\sigma)} \Psi(u_{ij}(s)) \, \mathrm{d}x \le C e^{C t_{ij}^{-1} t} \sup_{z \in \mathbb{R}^N} \fint_{B(z,\sigma)} \Psi(\mu_{ij}) \, \mathrm{d}x \le C \sup_{z \in \mathbb{R}^N} \fint_{B(z,\sigma)} \Psi(\mu_{ij}) \, \mathrm{d}x$$

for $t \in (0, t_{ij})$, $\sigma \in (t_{ij}^{1/4}, 1]$, $i \ge 1$, and $j \ge j_*$. Since $\Psi^{-1}(2\xi) \asymp \Psi^{-1}(\xi)$ for $\xi \in (0, \infty)$ (see (3.8)), it follows from (4.10) that

$$\sup_{s \in (0, t_{ij}]} \sup_{z \in \mathbb{R}^N} \sup_{\sigma \in (t_{ij}^{1/4}, 1]} \eta(\sigma) \Psi^{-1} \left(\fint_{B(z,\sigma)} \Psi(u_{ij}(s)) \, \mathrm{d}x \right) \le C \epsilon_5$$

for $i \ge 1$ and $j \ge j_*$. This together with (4.14) implies that, taking $\epsilon_5 > 0$ small enough if necessary, we obtain $T_{ij}^2 \ge t_{ij} > 0$.

By Lemma 3.1 and $\delta_2 \in (0,1)$ (see (4.12)) we obtain

$$
\begin{aligned}
&\sup_{s\in(0,t]} \sup_{z\in\mathbb{R}^N} \int_{B(z,\sigma)} \Psi(u_{ij}(s))\,\mathrm{d}x + \sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} u_{ij}^{m-1}\Psi''(u_{ij})|\nabla u_{ij}|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\le C \sup_{z\in\mathbb{R}^N} \int_{B(z,\sigma)} \Psi(\mu_{ij})\,\mathrm{d}x + C\sigma^{-2} \sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} u_{ij}^{m-1}\Psi(u_{ij})\,\mathrm{d}x\,\mathrm{d}s \\
&+ C \sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} \Psi(u_{ij})\,\mathrm{d}x\,\mathrm{d}s + C\delta_2^{p-m} \sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} u_{ij}^{m-1}\Psi''(u_{ij})|\nabla u_{ij}|^2\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}
$$

for $t \in (0, T_{ij}^*)$, $\sigma \in (0,1]$, $i \ge 1$, and $j \ge j_*$. Then, taking $\delta_2 > 0$ small enough if necessary, we have

$$
\begin{aligned}
&\sup_{z\in\mathbb{R}^N} \int_{B(z,\sigma)} \Psi(u_{ij}(t))\,\mathrm{d}x \\
&\le C \sup_{z\in\mathbb{R}^N} \int_{B(z,\sigma)} \Psi(\mu_{ij})\,\mathrm{d}x + C\sigma^{-2} \sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} u_{ij}^{m-1}\Psi(u_{ij})\,\mathrm{d}x\,\mathrm{d}s \\
&+ C \sup_{z\in\mathbb{R}^N} \int_0^t \int_{B(z,\sigma)} \Psi(u_{ij})\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

for $t \in (0, T_{ij}^*)$, $\sigma \in (0,1]$, $i \ge 1$, and $j \ge j_*$. This together with (4.11) implies that

$$
\begin{aligned}
X_{ij}(t) &\le C \sup_{z\in\mathbb{R}^N} \fint_{B(z,\sigma)} \Psi(\mu_{ij})\,\mathrm{d}x + C\sigma^{-2} \int_0^t \|u_{ij}(s)\|_{L^\infty(\mathbb{R}^N)}^{m-1} X_{ij}(s)\,\mathrm{d}s + C\int_0^t X_{ij}(s)\,\mathrm{d}s \\
&\le C \sup_{z\in\mathbb{R}^N} \fint_{B(z,\sigma)} \Psi(\mu_{ij})\,\mathrm{d}x + C\sigma^{-2} \int_0^t \eta(\gamma(s))^{-(m-1)} X_{ij}(s)\,\mathrm{d}s + C\int_0^t X_{ij}(s)\,\mathrm{d}s
\end{aligned}
$$

for $t \in (0, T_{ij}^*)$, $\sigma \in (0,1]$, $i \ge 1$, and $j \ge j_*$, where

$$
X_{ij}(t) := \sup_{z\in\mathbb{R}^N} \fint_{B(z,\sigma)} \Psi(u_{ij}(t))\,\mathrm{d}x.
$$

Then Gronwall's inequality together with (4.5) implies that

$$
\begin{aligned}
X_{ij}(t) &\le C \exp\left(C\sigma^{-2} \int_0^t \eta(\gamma(s))^{-(m-1)}\,\mathrm{d}s + Ct \right) \sup_{z\in\mathbb{R}^N} \fint_{B(z,\sigma)} \Psi(\mu_{ij})\mathrm{d}x \\
&\le C \exp\left(C\sigma^{-2}\gamma(t)^2 \right) \sup_{z\in\mathbb{R}^N} \fint_{B(z,\sigma)} \Psi(\mu_{ij})\mathrm{d}x \le C \sup_{z\in\mathbb{R}^N} \fint_{B(z,\sigma)} \Psi(\mu_{ij})\mathrm{d}x
\end{aligned}
$$

for $t \in (0, T_{ij}^*)$, $\sigma \in [\gamma(t), 1]$, $i \ge 1$, and $j \ge j_*$. Then, since $\Psi^{-1}(2\xi) \asymp \Psi^{-1}(\xi)$ for $\xi \in (0,\infty)$ (see (3.8)), we obtain

$$
\sup_{z\in\mathbb{R}^N} \Psi^{-1}\left(\fint_{B(z,\sigma)} \Psi(u_{ij}(t))\,\mathrm{d}x \right) \le C \sup_{z\in\mathbb{R}^N} \Psi^{-1}\left(\fint_{B(z,\sigma)} \Psi(\mu_{ij})\,\mathrm{d}x \right)
$$

for $t\in(0,T_{ij}^*)$, $\sigma\in[\gamma(t),1]$, $i\ge 1$, and $j\ge j_*$. Therefore, thanks to (4.10), we obtain

$$\begin{aligned}&\sup_{s\in(0,t)}\sup_{z\in\mathbb{R}^N}\sup_{\sigma\in[\gamma(s),1]}\eta(\sigma)\Psi^{-1}\left(\fint_{B(z,\sigma)}\Psi(u_{ij}(s))\,\mathrm{d}x\right)\\&\le C\sup_{z\in\mathbb{R}^N}\sup_{\sigma\in(0,1]}\eta(\sigma)\Psi^{-1}\left(\fint_{B(z,\sigma)}\Psi(\mu_{ij})\,\mathrm{d}x\right)=C|||\mu_{ij}|||_{\eta,\Psi;1}\le C\epsilon_5\end{aligned}\tag{4.15}$$

for $t\in(0,T_{ij}^*)$, $i\ge 1$, and $j\ge j_*$.

On the other hand, taking $\delta_1>0$ small enough, by (4.2), (4.4), and (4.11) we have

$$\begin{aligned}&\left(\frac{\gamma(s)}{2}\right)^{-2}\|u_{ij}(s)\|_{L^\infty(\mathbb{R}^N)}^{m-1}+\|u_{ij}(s)\|_{L^\infty(\mathbb{R}^N)}^{p-1}\\&\le 4\delta_1^{m-1}\gamma(s)^{-2}\eta(\gamma(s))^{-(m-1)}+\delta_1^{p-1}\eta(\gamma(s))^{-(p-1)}\\&\le C\delta_1^{m-1}s^{-1}+C\delta_1^{p-1}s^{-1}\le 2s^{-1}\end{aligned}$$

for $s\in(0,T_{ij}^*)$, $i\ge 1$, and $j\ge j_*$. Then, by Lemma 3.3 with $\sigma=\gamma(s)/2$ and $r=1$ we have

$$\begin{aligned}\|u_{ij}(t)\|_{L^\infty(\mathbb{R}^N)}&\le Ct^{-\frac{N+2}{\kappa_1}}\left(\sup_{z\in\mathbb{R}^N}\int_{t/2}^t\int_{B(z,\gamma(t))}u_{ij}\,dx\,ds\right)^{\frac{2}{\kappa_1}}\\&\le Ct^{-\frac{N}{\kappa_1}}\gamma(t)^{\frac{2N}{\kappa_1}}\sup_{s\in(t/2,t)}\sup_{z\in\mathbb{R}^N}\left(\fint_{B(z,\gamma(t))}u_{ij}(s)\,\mathrm{d}x\right)^{\frac{2}{\kappa_1}}\end{aligned}$$

for $t\in(0,T_{ij}^*)$, $i\ge 1$, and $j\ge j_*$. This together with Jensen's inequality, (4.2), and (4.3) implies that

$$\begin{aligned}&\eta(\gamma(t))\|u_{ij}(t)\|_{L^\infty(\mathbb{R}^N)}\\&\le Ct^{-\frac{N}{\kappa_1}}\gamma(t)^{\frac{2N}{\kappa_1}}\eta(\gamma(t))^{1-\frac{2}{\kappa_1}}\sup_{s\in(t/2,t)}\sup_{z\in\mathbb{R}^N}\left[\eta(\gamma(t))\Psi^{-1}\left(\fint_{B(z,\gamma(t))}\Psi(u_{ij}(s))\,\mathrm{d}x\right)\right]^{\frac{2}{\kappa_1}}\\&\le C\left(t^{-1}\gamma(t)^2\eta(\gamma(t))^{m-1}\right)^{\frac{N}{\kappa_1}}\sup_{s\in(0,t)}\sup_{z\in\mathbb{R}^N}\sup_{\sigma\in[\gamma(s),1]}\left[\eta(\sigma)\Psi^{-1}\left(\fint_{B(z,\sigma)}\Psi(u_{ij}(s))\,\mathrm{d}x\right)\right]^{\frac{2}{\kappa_1}}\\&\le C\sup_{s\in(0,t)}\sup_{z\in\mathbb{R}^N}\sup_{\sigma\in[\gamma(s),1]}\left[\eta(\sigma)\Psi^{-1}\left(\fint_{B(z,\sigma)}\Psi(u_{ij}(s))\,\mathrm{d}x\right)\right]^{\frac{2}{\kappa_1}}\end{aligned}\tag{4.16}$$

for $t\in(0,T_{ij}^*)$, $i\ge 1$, and $j\ge j_*$. Therefore, by (4.15) and (4.16), taking $\epsilon_5>0$ small enough if necessary, we obtain

$$\eta(\gamma(t))\|u_{ij}(t)\|_{L^\infty(\mathbb{R}^N)}\le C|||\mu_{ij}|||_{\eta,\Psi;1}^{\frac{2}{\kappa_1}}\le C\epsilon_5^{\frac{2}{\kappa_1}}\le\frac{\delta_1}{2}\tag{4.17}$$

for $t\in(0,T_{ij}^*)$, $i\ge 1$, and $j\ge j_*$. Furthermore, we observe from (4.17) that

$$\begin{aligned}&\sup_{s\in(0,t)}\sup_{z\in\mathbb{R}^N}\sup_{\sigma\in(0,\gamma(s))}\eta(\sigma)\Psi^{-1}\left(\fint_{B(z,\sigma)}\Psi(u_{ij}(s))\,\mathrm{d}y\right)\\&\le\sup_{s\in(0,t)}\sup_{\sigma\in(0,\gamma(s))}\eta(\sigma)\Psi^{-1}\left(\Psi\left(\|u_{ij}(s)\|_{L^\infty(\mathbb{R}^N)}\right)\right)=\sup_{s\in(0,t)}\eta(\gamma(s))\|u_{ij}(s)\|_{L^\infty(\mathbb{R}^N)}\\&\le C|||\mu_{ij}|||_{\eta,\Psi;1}^{\frac{2}{\kappa_1}}\le C\epsilon_5^{\frac{2}{\kappa_1}}\end{aligned}\tag{4.18}$$

for $t \in (0, T_{ij}^*)$, $i \geq 1$, and $j \geq j_*$. Combining (4.15) and (4.18) and taking $\epsilon_5 > 0$ small enough if necessary, we obtain

$$\sup_{s\in(0,t)} |||u_{ij}(s)|||_{\eta,\Psi;1} \leq C\epsilon_5 + C\epsilon_5^{\frac{2}{\kappa_1}} \leq \frac{\delta_2}{2} \tag{4.19}$$

for $t \in (0, T_{ij}^*)$, $i \geq 1$, and $j \geq j_*$. Then, thanks to (4.17) and (4.19), by the definition of T_{ij}^* we see that $T_{ij}^* = 1$ for $i \geq 1$ and $j \geq j_*$. Furthermore, repeating the arguments in (4.10), (4.15), (4.17), and (4.18), we obtain

$$\sup_{s\in(0,1)} \eta(\gamma(s))\|u_{ij}(s)\|_{L^\infty(\mathbb{R}^N)} + \sup_{s\in(0,1)} |||u_{ij}(s)|||_{\eta,\Psi;1} \leq C\left(|||\mu|||_{\eta,\Psi;1} + j^{-1}\right)^{\frac{2}{\kappa_1}} \tag{4.20}$$

for $i \geq 1$ and $j \geq j_*$. On the other hand, it follows from (4.2) that

$$\eta(\gamma(\xi)) \asymp \xi^{\frac{1}{m-1}}\gamma(\xi)^{-\frac{2}{m-1}}, \tag{4.21}$$

$$\gamma(\xi)^{N(m-1)+2}\left[\log\left(e + \frac{1}{\gamma(\xi)}\right)\right]^{\frac{N(m-1)}{2}} \asymp \xi, \tag{4.22}$$

for $\xi \in (0,1)$. Then, by (4.22) we have

$$\gamma(\xi) \asymp \xi^{\frac{1}{N(m-1)+2}}\left[\log\left(e + \frac{1}{\xi}\right)\right]^{-\frac{m-1}{2}\frac{N}{N(m-1)+2}} \quad \text{for } \xi \in (0,1).$$

This together with (4.21) implies that

$$\begin{aligned}\eta(\gamma(\xi)) &\asymp \xi^{\frac{1}{m-1}\left(1-\frac{2}{N(m-1)+2}\right)}\left[\log\left(e + \frac{1}{\xi}\right)\right]^{\frac{N}{N(m-1)+2}} \\ &= \xi^{\frac{N}{N(m-1)+2}}\left[\log\left(e + \frac{1}{\xi}\right)\right]^{\frac{N}{N(m-1)+2}} = \xi^{\frac{1}{p-1}}\left[\log\left(e + \frac{1}{\xi}\right)\right]^{\frac{1}{p-1}}\end{aligned} \tag{4.23}$$

for $\xi \in (0,1)$. We deduce from (4.20) and (4.23) that

$$\begin{aligned}&\sup_{s\in(0,1)} s^{\frac{1}{p-1}}\left[\log\left(e + \frac{1}{s}\right)\right]^{\frac{1}{p-1}}\|u_{ij}(s)\|_{L^\infty(\mathbb{R}^N)} + \sup_{s\in(0,1)} |||u_{ij}(s)|||_{\eta,\Psi;1} \\ &\leq C\left(|||\mu|||_{\eta,\Psi;1} + j^{-1}\right)^{\frac{2}{\kappa_1}}\end{aligned} \tag{4.24}$$

for $i \geq 1$ and $j \geq j_*$.

Step 2. We complete the proof of Theorem 1.2. By (4.20) we apply [46, Theorem 7.1] to obtain the following:

- for any compact set $K \subset \mathbb{R}^N \times (0,1)$, there exist $C > 0$ and $\omega \in (0,1)$ such that

  $$|u_{ij}(x_1,t_1) - u_{ij}(x_2,t_2)| \leq C\left(|x_1 - x_2|^\omega + |t_1 - t_2|^{\frac{\omega}{2}}\right)$$

  for (x_1,t_1), $(x_2,t_2) \in K$, $i \geq 1$, and $j \geq j_*$.

By the Arzelà-Ascoli Theorem and the diagonal argument we find a subsequence $\{u'_{ij}\}$ of $\{u_{ij}\}$ and a Hölder continuous function u in $\mathbb{R}^N \times (0,1)$ such that

$$\lim_{i,j\to\infty} \|u'_{ij} - u\|_{L^\infty(K)} = 0$$

for any compact set K of $\mathbb{R}^N \times (0,1)$. Then we observe from (3.7), (3.21), and (4.24) that

$$\begin{aligned}
\lim_{i,j\to\infty} \int_{B(z,1)} u_{ij}(t)^p \, dx &= \int_{B(z,1)} u(t)^p \, dx, \\
\int_{B(z,1)} u_{ij}(t)^p \, dx &\le \sup_{x\in\mathbb{R}^N} \left\{ u_{ij}(x,t)^{p-1} \left(\log\left(e + u_{ij}(x,t)\right)\right)^{-\alpha} \right\} \int_{B(z,1)} \Psi(u_{ij}(t)) \, dx \\
&\le C \|u_{ij}(t)\|_{L^\infty(\mathbb{R}^N)}^{p-1} \left(\log\left(e + \|u_{ij}(t)\|_{L^\infty(\mathbb{R}^N)}\right)\right)^{-\alpha} \Psi\left(|||u_{ij}(t)|||_{\eta,\Psi;1}\right) \\
&\le Ct^{-1} \left[\log\left(e + \frac{1}{t}\right)\right]^{-1-\alpha} \left(|||\mu|||_{\eta,\Psi;1} + 1\right)^{\frac{2(p-1)}{\kappa_1}} \Psi\left(|||\mu|||_{\eta,\Psi;1} + 1\right)
\end{aligned}$$

for $z \in \mathbb{R}^N$, $t \in (0,1)$, $i \ge 1$, and $j \ge j_*$. Therefore, by Definition 1.1-(2), (4.6), and (4.24) we apply the Lebesgue dominated convergence theorem to see that u is a solution to problem (P) in $\mathbb{R}^N \times (0,1)$ satisfying

$$\sup_{s\in(0,1)} s^{\frac{1}{p-1}} \left[\log\left(e + \frac{1}{s}\right)\right]^{\frac{1}{p-1}} \|u(s)\|_{L^\infty(\mathbb{R}^N)} + \sup_{s\in(0,1)} |||u(s)|||_{\eta,\Psi;1} \le C |||\mu|||_{\eta,\Psi;1}^{\frac{2}{\kappa_1}}.$$

Thus Theorem 1.2 follows. □

5 Proof of Theorem 1.3

In this section we modify arguments in Section 4 to study sufficient conditions for the existence of solutions to problem (P) with $p > p_m$, and prove Theorem 1.3.

Proof of Theorem 1.3. Let $p > p_m$, $1 < \beta < N(p-m)/2$, $T \in (0,\infty]$, and $\mu \in \mathcal{L}$. Let $\epsilon_6 \in (0,1)$ be small enough, and assume (1.7). For any i, $j = 1, 2, \dots$, let u_{ij} be a solution to problem (P) with initial data μ replaced by (4.6). Then, for any $n \ge 1$, we find $j_* = j_*(n, \epsilon_6)$ such that

$$\begin{aligned}
|||\mu_{ij}|||_{\frac{N(p-m)}{2},\beta;T_n^\theta} &\le |||\mu|||_{\frac{N(p-m)}{2},\beta;T^\theta} + T_n^{\frac{2\theta}{p-m}} j^{-1} \\
&\le |||\mu|||_{\frac{N(p-m)}{2},\beta;T^\theta} + n^{\frac{1}{p-1}} j^{-1} \le 2\epsilon_6
\end{aligned} \tag{5.1}$$

for $i \ge 1$ and $j \ge j_*$, where $T_n := \min\{T, n\}$ and θ is defined as in (1.1). Similarly to the proof of Theorem 1.2, by arguments in [3] we find a unique classical solution u_{ij} to problem (P) in $\mathbb{R}^N \times (0, T_{ij})$, with u_{ij} satisfying (4.7), (4.8), and (4.9), where T_{ij} is the maximal existence time of u_{ij}.

Step 1. Let $n = 1, 2, \dots$, and fix it. Let δ_1, $\delta_2 \in (0,1)$. Set

$$T_{ij}^1 := \sup\left\{ t \in (0, T_{ij}) \, : \, \sup_{s\in(0,t)} s^{\frac{1}{p-1}} \|u_{ij}(s)\|_{L^\infty(\mathbb{R}^N)} \le \delta_1 \right\}, \tag{5.2}$$

$$T_{ij}^2 := \sup\left\{t \in (0,T_{ij}) \,:\, \sup_{s\in(0,t)} |||u_{ij}(s)|||_{\frac{N(p-m)}{2},\beta;T_n^\theta} \le \delta_2\right\}. \tag{5.3}$$

Under suitable choices of δ_1 and δ_2, taking $\epsilon_6 > 0$ small enough if necessary, we show that

$$T_{ij}^* := \min\{T_{ij}^1, T_{ij}^2, T_n\} = T_n \quad \text{for } i \ge 1 \text{ and } j \ge j_*.$$

In the proof of Theorem 1.3, the constants C are independent of n, $i \ge 1$, and $j \ge j_*$.

Similarly to the argument in the proof of Theorem 1.2, we see that $T_{ij}^* > 0$ for $i \ge 1$ and $j \ge j_*$. By Lemma 3.2 and (5.3) we have

$$\begin{aligned}
&\sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)} u_{ij}^{m+\beta-3}|\nabla u_{ij}|^2\,\mathrm{d}x\,\mathrm{d}s < \infty,\\
&\sup_{s\in(0,t]}\sup_{z\in\mathbb{R}^N}\int_{B(z,\sigma)} u_{ij}(s)^\beta\,\mathrm{d}x + \sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)} u_{ij}^{m+\beta-3}|\nabla u_{ij}|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\le C\sup_{z\in\mathbb{R}^N}\int_{B(z,\sigma)}\mu_{ij}^\beta\,\mathrm{d}x + C\sigma^{-2}\sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)} u_{ij}^{m+\beta-1}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + C\delta_2^{p-m}\sup_{z\in\mathbb{R}^N}\int_0^t\int_{B(z,\sigma)} u_{ij}^{m+\beta-3}|\nabla u_{ij}|^2\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}$$

for $t \in (0, T_{ij}^*)$, $\sigma > 0$, $i \ge 1$, and $j \ge j_*$. Taking $\delta_2 > 0$ small enough if necessary, by (5.2) we obtain

$$\begin{aligned}
Y_{ij}(t) &\le C\sup_{z\in\mathbb{R}^N}\fint_{B(z,\sigma)}\mu_{ij}^\beta\,\mathrm{d}x + C\sigma^{-2}\int_0^t \|u_{ij}(s)\|_{L^\infty(\mathbb{R}^N)}^{m-1} Y_{ij}(s)\,\mathrm{d}s\\
&\le C\sup_{z\in\mathbb{R}^N}\fint_{B(z,\sigma)}\mu_{ij}^\beta\,\mathrm{d}x + C\sigma^{-2}\int_0^t s^{-\frac{m-1}{p-1}} Y_{ij}(s)\,\mathrm{d}s
\end{aligned}$$

for $t \in (0, T_{ij}^*)$, $\sigma > 0$, $i \ge 1$, and $j \ge j_*$, where

$$Y_{ij}(t) := \sup_{z\in\mathbb{R}^N}\fint_{B(z,\sigma)} u_{ij}(t)^\beta\,\mathrm{d}x.$$

Then Gronwall's inequality implies that

$$\begin{aligned}
Y_{ij}(t) &\le C\exp\left(\sigma^{-2}\int_0^t s^{-\frac{m-1}{p-1}}\,\mathrm{d}s\right)\sup_{z\in\mathbb{R}^N}\fint_{B(z,\sigma)}\mu_{ij}^\beta\,\mathrm{d}x\\
&\le C\exp\left(\sigma^{-2}t^{\frac{p-m}{p-1}}\right)\sup_{z\in\mathbb{R}^N}\fint_{B(z,\sigma)}\mu_{ij}^\beta\,\mathrm{d}x \le C\sup_{z\in\mathbb{R}^N}\fint_{B(z,\sigma)}\mu_{ij}^\beta\,\mathrm{d}x
\end{aligned}$$

for $t \in (0, T_{ij}^*)$, $\sigma \ge t^\theta$, $i \ge 1$, and $j \ge j_*$. Therefore we deduce from (5.1) that

$$\sup_{s\in(0,t)}\sup_{z\in\mathbb{R}^N}\sup_{\sigma\in[s^\theta,T_n^\theta)}\sigma^{\frac{2}{p-m}}\left(\fint_{B(z,\sigma)} u_{ij}(s)^\beta\,\mathrm{d}x\right)^{\frac{1}{\beta}} \le C|||\mu_{ij}|||_{\frac{N(p-m)}{2},\beta;T_n^\theta} \le C\epsilon_6 \tag{5.4}$$

for $t \in (0, T_{ij}^*)$, $i \ge 1$, and $j \ge j_*$.

On the other hand, taking $\delta_1 > 0$ small enough if necessary, by (5.2) we have

$$\begin{aligned}
&\left(\frac{s^\theta}{2}\right)^{-2}\|u_{ij}(s)\|^{m-1}_{L^\infty(\mathbb{R}^N)} + \|u_{ij}(s)\|^{p-1}_{L^\infty(\mathbb{R}^N)}\\
&\le 4\delta_1^{m-1}s^{-\frac{p-m}{p-1}}s^{-\frac{m-1}{p-1}} + \delta_1^{p-1}s^{-1} \le C\delta_1^{m-1}s^{-1} + C\delta_1^{p-1}s^{-1} \le 2s^{-1}
\end{aligned}$$

for $s \in (0, T^*_{ij})$, $i \ge 1$, and $j \ge j_*$. Then, by Lemma 3.3 with $\sigma = t^\theta/2$ and $r = \beta$, taking $\epsilon_6 > 0$ small enough if necessary, by (5.1) and (5.4) we have

$$\begin{aligned}
t^{\frac{1}{p-1}}\|u_{ij}(t)\|_{L^\infty(\mathbb{R}^N)} &\le Ct^{\frac{1}{p-1}-\frac{N+2}{\kappa_\beta}}\left(\sup_{z\in\mathbb{R}^N}\int_{t/2}^t\int_{B(z,t^\theta)} u_{ij}^\beta\,dy\,ds\right)^{\frac{2}{\kappa_\beta}}\\
&\le Ct^{\frac{1}{p-1}-\frac{N}{\kappa_\beta}}t^{\frac{2N\theta}{\kappa_\beta}}\sup_{s\in(t/2,t)}\sup_{z\in\mathbb{R}^N}\left(\fint_{B(z,t^\theta)} u_{ij}(s)^\beta\,\mathrm{d}y\right)^{\frac{2}{\kappa_\beta}}\\
&\le Ct^{\frac{1}{p-1}-\frac{N}{\kappa_\beta}+\frac{2N\theta}{\kappa_\beta}-\frac{2\theta}{p-m}\frac{2\beta}{\kappa_\beta}}\sup_{s\in(t/2,t)}\sup_{z\in\mathbb{R}^N}\left[t^{\frac{2\theta}{p-m}}\left(\fint_{B(z,t^\theta)} u_{ij}(s)^\beta\,\mathrm{d}y\right)^{\frac{1}{\beta}}\right]^{\frac{2\beta}{\kappa_\beta}}\\
&\le C\sup_{s\in(0,t)}\sup_{z\in\mathbb{R}^N}\sup_{\sigma\in[s^\theta,T_n^\theta)}\left[\sigma^{\frac{2}{p-m}}\left(\fint_{B(z,\sigma)} u_{ij}(s)^\beta\,\mathrm{d}y\right)^{\frac{1}{\beta}}\right]^{\frac{2\beta}{\kappa_\beta}}\\
&\le C|||\mu_{ij}|||^{\frac{2\beta}{\kappa_\beta}}_{\frac{N(p-m)}{2},\beta;T_n^\theta} \le C\epsilon_6^{\frac{2\beta}{\kappa_\beta}} \le \frac{\delta_1}{2}
\end{aligned} \tag{5.5}$$

for $t \in (0, T^*_{ij})$, $i \ge 1$, and $j \ge j_*$. Here we used the relations that $\theta = (p-m)/2(p-1)$ and $\kappa_\beta = N(m-1) + 2\beta$. Furthermore, we observe from (5.1) and (5.5) that

$$\begin{aligned}
&\sup_{s\in(0,t)}\sup_{z\in\mathbb{R}^N}\sup_{\sigma\in(0,s^\theta)}\sigma^{\frac{2}{p-m}}\left(\fint_{B(z,\sigma)} u_{ij}(s)^\beta\,\mathrm{d}y\right)^{\frac{1}{\beta}}\\
&\le \sup_{s\in(0,t)} s^{\frac{1}{p-1}}\|u_{ij}(s)\|_{L^\infty(\mathbb{R}^N)} \le C|||\mu_{ij}|||^{\frac{2\beta}{\kappa_\beta}}_{\frac{N(p-m)}{2},\beta;T_n^\theta} \le C\epsilon_6^{\frac{2\beta}{\kappa_\beta}}
\end{aligned} \tag{5.6}$$

for $t \in (0, T^*_{ij})$, $i \ge 1$, and $j \ge j_*$. By (5.4) and (5.6), taking $\epsilon_6 > 0$ small enough if necessary, we obtain

$$\sup_{s\in(0,t)}|||u_{ij}(s)|||_{\frac{N(p-m)}{2},\beta;T_n^\theta} \le C\epsilon_6 + C\epsilon_6^{\frac{2\beta}{\kappa_\beta}} \le \frac{\delta_2}{2} \tag{5.7}$$

for $t \in (0, T^*_{ij})$, $i \ge 1$, and $j \ge j_*$. Therefore, thanks to (5.5) and (5.7), by the definition of T^*_{ij}, for any $n = 1, 2, \dots$, we see that $T^*_{ij} = T_n$ for $i \ge 1$ and $j \ge j_*$. Furthermore, by (5.1), (5.4), (5.5), and (5.6) we have

$$\begin{aligned}
&\sup_{s\in(0,T_n)} s^{\frac{1}{p-1}}\|u_{ij}(s)\|_{L^\infty(\mathbb{R}^N)} + \sup_{s\in(0,T_n)}|||u_{ij}(s)|||_{\frac{N(p-m)}{2},\beta;T_n^\theta}\\
&\le C\left(|||\mu|||_{\frac{N(p-m)}{2},\beta;T_n^\theta} + n^{\frac{1}{p-1}}j^{-1}\right)^{\frac{2\beta}{\kappa_\beta}}
\end{aligned} \tag{5.8}$$

for $i \ge 1$ and $j \ge j_*$.

Step 2. We complete the proof of Theorem 1.3. By (5.8) we apply [46, Theorem 7.1] to obtain the following:

- for any compact set $K \subset \mathbb{R}^N \times (0, T_n)$, there exist $C > 0$ and $\omega \in (0,1)$ such that

$$|u_{ij}(x_1,t_1) - u_{ij}(x_2,t_2)| \le C\left(|x_1 - x_2|^\omega + |t_1 - t_2|^{\frac{\omega}{2}}\right)$$

  for (x_1,t_1), $(x_2,t_2) \in K$, $i \ge 1$, and $j \ge j_*$.

By the Arzelà-Ascoli Theorem and the diagonal argument we find a subsequence $\{u'_{ij}\}$ of $\{u_{ij}\}$ and a Hölder continuous function u in $\mathbb{R}^N \times (0,T_n)$ such that

$$\lim_{i,j\to\infty} \|u'_{ij} - u\|_{L^\infty(K)} = 0$$

for any compact set K of $\mathbb{R}^N \times (0,T_n)$. Then we observe from Jensen's inequality and (5.8) that

$$\begin{aligned}
\lim_{i,j\to\infty} \int_{B(z,T_n^\theta)} u_{ij}(t)^p \, dx &= \int_{B(z,T_n^\theta)} u(t)^p \, dx, \\
\int_{B(z,T_n^\theta)} u_{ij}(t)^p \, dx &\le T_n^{N\theta} \left(\fint_{B(z,T_n^\theta)} u_{ij}(t)^\beta \, dx \right)^{\frac{p}{\beta}} \\
&\le T_n^{N\theta - \frac{p}{p-1}} |||u_{ij}(t)|||^p_{\frac{N(p-m)}{2},\beta;T_n^\theta} \\
&\le CT_n^{N\theta - \frac{p}{p-1}} \left(|||\mu|||_{\frac{N(p-m)}{2},\beta;T_n^\theta} + n^{\frac{1}{p-1}} \right)^{\frac{2\beta p}{\kappa_\beta}} \quad \text{if} \quad 1 < p \le \beta, \\
\int_{B(z,T_n^\theta)} u_{ij}(t)^p \, dx &\le \|u_{ij}(t)\|^{p-\beta}_{L^\infty(\mathbb{R}^N)} \int_{B(z,T_n^\theta)} u_{ij}(t)^\beta \, dx \\
&\le CT_n^{N\theta - \frac{\beta}{p-1}} \|u_{ij}(t)\|^{p-\beta}_{L^\infty(\mathbb{R}^N)} |||u_{ij}(t)|||^\beta_{\frac{N(p-m)}{2},\beta;T_n^\theta} \\
&\le CT_n^{N\theta - \frac{\beta}{p-1}} t^{-1+\frac{\beta-1}{p-1}} \left(|||\mu|||_{\frac{N(p-m)}{2},\beta;T_n^\theta} + n^{\frac{1}{p-1}} \right)^{\frac{2\beta p}{\kappa_\beta}} \quad \text{if} \quad 1 < \beta < p,
\end{aligned}$$

for $z \in \mathbb{R}^N$, $t \in (0,T_n)$, $i \ge 1$, and $j \ge j_*$. Therefore, by Definition 1.1-(2), (4.6), and (5.8) we apply the Lebesgue dominated convergence theorem to see that u is a solution to problem (P) in $\mathbb{R}^N \times (0,T_n)$ satisfying

$$\sup_{s\in(0,T_n)} |||u(s)|||_{\frac{N(p-m)}{2},\beta;T_n^\theta} + \sup_{s\in(0,T_n)} s^{\frac{1}{p-1}} \|u(s)\|_{L^\infty(\mathbb{R}^N)} \le C|||\mu|||^{\frac{2\beta}{\kappa_\beta}}_{\frac{N(p-m)}{2},\beta;T_n^\theta} \tag{5.9}$$

for $t \in (0,T_n)$. Since n is arbitrary, (5.9) holds with T_n replaced by T. Thus u is our desired solution to problem (P), and Theorem 1.3 follows. □

6 Proof of Corollary 1.1: Optimal singularity

In this section, applying Theorems 1.1–1.3, we prove Corollary 1.1.

Proof of Corollary 1.1. Let $p = p_m$, $\alpha \in (0, N/2)$, and

$$\mu(x) = |x|^{-N}\left[\log\left(e + \frac{1}{|x|}\right)\right]^{-\frac{N}{2}-1}$$

for a.a. $x \in \mathbb{R}^N$. Let Ψ be as in Theorem 1.2. Then, by (3.21) we have

$$\begin{aligned}\Psi(c\mu(x)) &= (c\mu(x))^{\frac{1}{2}}(c\mu(x))^{\frac{1}{2}}\left[\log\left(e + c\mu(x)\right)\right]^{\alpha} \preceq c^{\frac{1}{2}}\mu(x)\left[\log\left(e + \mu(x)\right)\right]^{\alpha}\\ &\preceq c^{\frac{1}{2}}|x|^{-N}\left[\log\left(e + \frac{1}{|x|}\right)\right]^{\alpha-\frac{N}{2}-1}\end{aligned} \tag{6.1}$$

for a.a. $x \in \mathbb{R}^N$ and $c \in (0, 1)$. This implies that

$$\fint_{B(z,\sigma)} \Psi(c\mu(x))\,\mathrm{d}x \preceq \fint_{B(0,3\sigma)} \Psi(c\mu(x))\,\mathrm{d}x \asymp c^{\frac{1}{2}}\sigma^{-N}\left[\log\left(e + \frac{1}{\sigma}\right)\right]^{\alpha-\frac{N}{2}}$$

for $z \in B(0, 2\sigma)$, $\sigma \in (0, 1)$, and $c \in (0, 1)$. Thus we have

$$\Psi^{-1}\left(\fint_{B(z,\sigma)} \Psi(c\mu(x))\,\mathrm{d}x\right) \preceq c^{\frac{1}{2}}\sigma^{-N}\left[\log\left(e + \frac{1}{\sigma}\right)\right]^{-\frac{N}{2}} = c^{\frac{1}{2}}\eta(\sigma)^{-1} \tag{6.2}$$

for $z \in B(0, 2\sigma)$, $\sigma \in (0, 1)$, and $c \in (0, 1)$. On the other hand,

$$\Psi^{-1}\left(\fint_{B(z,\sigma)} \Psi(c\mu(x))\,\mathrm{d}x\right) \preceq c\|\mu\|_{L^\infty(B(z,\sigma))} \preceq c\sigma^{-N}\left[\log\left(e + \frac{1}{\sigma}\right)\right]^{-\frac{N}{2}} \asymp c\eta(\sigma)^{-1} \tag{6.3}$$

for $z \in \mathbb{R}^N \setminus B(0, 2\sigma)$, $\sigma \in (0, 1)$, and $c \in (0, 1)$. By (6.2) and (6.3) we apply Theorem 1.2 with $T = 1$ to see that problem (P) possesses a solution in $\mathbb{R}^N \times (0, 1)$ if c is small enough. Thus assertion (1) holds if $p = p_m$.

On the other hand, we have

$$\fint_{B(0,\sigma)} c\mu(y)\,\mathrm{d}y \succeq c\sigma^{-N}\left[\log\left(e + \frac{1}{\sigma}\right)\right]^{-\frac{N}{2}}$$

for $\sigma \in (0, 1)$. Then it follows from Theorem 1.1 that problem (P) possesses no local-in-time solution if c is large enough. Thus assertion (2) holds if $p = p_m$. Assertions in the case of $p > p_m$ follows from similar arguments to those in the case of $p = p_m$. Therefore the proof of Corollary 1.1 is complete. □

References

[1] G. Akagi, K. Ishige, and R. Sato, *The Cauchy problem for the Finsler heat equation*, Adv. Calc. Var. **13** (2020), 257–278.

[2] D. Andreucci, *Degenerate parabolic equations with initial data measures*, Trans. Amer. Math. Soc. **349** (1997), 3911–3923.

[3] D. Andreucci and E. DiBenedetto, *On the Cauchy problem and initial traces for a class of evolution equations with strongly nonlinear sources*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **18** (1991), 363–441.

[4] D. G. Aronson, *Non-negative solutions of linear parabolic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **22** (1968), 607–694.

[5] D. G. Aronson and L. A. Caffarelli, *The initial trace of a solution of the porous medium equation*, Trans. Amer. Math. Soc. **280** (1983), 351–366.

[6] P. Baras and M. Pierre, *Critère d'existence de solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (1985), 185–212.

[7] M.-F. Bidaut-Véron, E. Chasseigne, and L. Véron, *Initial trace of solutions of some quasilinear parabolic equations with absorption*, J. Funct. Anal. **193** (2002), 140–205.

[8] M.-F. Bidaut-Véron and N. A. Dao, *Initial trace of solutions of Hamilton-Jacobi parabolic equation with absorption*, Adv. Nonlinear Stud. **15** (2015), 889–921.

[9] M. Bonforte, Y. Sire, and J. L. Vázquez, *Optimal existence and uniqueness theory for the fractional heat equation*, Nonlinear Anal. **153** (2017), 142–168.

[10] E. DiBenedetto and M. A. Herrero, *On the Cauchy problem and initial traces for a degenerate parabolic equation*, Trans. Amer. Math. Soc. **314** (1989), 187–224.

[11] E. DiBenedetto and M. A. Herrero, *Nonnegative solutions of the evolution p-Laplacian equation. Initial traces and Cauchy problem when* $1 < p < 2$, Arch. Rational Mech. Anal. **111** (1990), 225–290.

[12] Y. V. Egorov, V. A. Galaktionov, V. A. Kondratiev, and S. I. Pohozaev, *On the necessary conditions of global existence to a quasilinear inequality in the half-space*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), 93–98 (English, with English and French summaries).

[13] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Revised edition, Textbooks in Mathematics, CRC Press, Boca Raton, FL, 2015.

[14] Y. Fujishima, K. Hisa, K. Ishige, and R. Laister, *Solvability of superlinear fractional parabolic equations*, J. Evol. Equ. **23** (2023), Paper No. 4, 38.

[15] Y. Fujishima, K. Hisa, K. Ishige, and R. Laister, *Local solvability and dilation-critical singularities of supercritical fractional heat equations*, J. Math. Pures Appl. **186** (2024), 150–175.

[16] Y. Fujishima and K. Ishige, *Initial traces and solvability of Cauchy problem to a semilinear parabolic system*, J. Math. Soc. Japan **73** (2021), 1187–1219.

[17] V. A. Galaktionov, *Blow-up for quasilinear heat equations with critical Fujita's exponents*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), 517–525.

[18] V. A. Galaktionov, S. P. Kurdjumov, A. P. Mihaĭlov, and A. A. Samarskiĭ, *On unbounded solutions of the Cauchy problem for the parabolic equation* $u_t = \nabla(u^\sigma \nabla u) + u^\beta$, Dokl. Akad. Nauk SSSR **252** (1980), 1362–1364 (Russian).

[19] L. Grafakos, *Classical Fourier analysis*, 3rd ed., Graduate Texts in Mathematics, vol. 249, Springer, New York, 2014.

[20] G. Grillo, G. Meglioli, and F. Punzo, *Smoothing effects and infinite time blowup for reaction-diffusion equations: an approach via Sobolev and Poincaré inequalities*, J. Math. Pures Appl. (9) **151** (2021), 99–131.

[21] M. A. Herrero and M. Pierre, *The Cauchy problem for* $u_t = \Delta u^m$ *when* $0 < m < 1$, Trans. Amer. Math. Soc. **291** (1985), 145–158.

[22] K. Hisa, *Optimal singularities of initial data of a fractional semilinear heat equation in open sets.* to appear in Ann. Scuola Norm. Sup. Pisa Cl. Sci.

[23] K. Hisa and K. Ishige, *Existence of solutions for a fractional semilinear parabolic equation with singular initial data*, Nonlinear Anal. **175** (2018), 108–132.

[24] K. Hisa and K. Ishige, *Initial traces of solutions to a semilinear heat equation under the Dirichlet boundary condition*, Calc. Var. Partial Differential Equations **64** (2025), Paper No. 113, 44.

[25] K. Hisa, K. Ishige, and J. Takahashi, *Initial traces and solvability for a semilinear heat equation on a half space of* $\mathbb{R}^N$, Trans. Amer. Math. Soc. **376** (2023), 5731–5773.

[26] N. Ioku, K. Ishige, and T. Kawakami, *Existence of solutions to a fractional semilinear heat equation in uniformly local weak Zygmund-type spaces*, Anal. PDE **18** (2025), 1477–1510.

[27] K. Ishige, *On the existence of solutions of the Cauchy problem for a doubly nonlinear parabolic equation*, SIAM J. Math. Anal. **27** (1996), 1235–1260.

[28] K. Ishige, T. Kawakami, and S. Okabe, *Existence of solutions for a higher-order semilinear parabolic equation with singular initial data*, Ann. Inst. H. Poincaré Anal. Non Linéaire **37** (2020), 1185–1209.

[29] K. Ishige and J. Kinnunen, *Initial trace for a doubly nonlinear parabolic equation*, J. Evol. Equ. **11** (2011), 943–957.

[30] K. Ishige and R. Sato, *Heat equation with a nonlinear boundary condition and uniformly local L^r spaces*, Discrete Contin. Dyn. Syst. **36** (2016), 2627–2652.

[31] H. Kozono and M. Yamazaki, *Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data*, Comm. Partial Differential Equations **19** (1994), 959–1014.

[32] M. Ledoux, *On improved Sobolev embedding theorems*, Math. Res. Lett. **10** (2003), 659–669.

[33] M. Marcus and L. Véron, *Initial trace of positive solutions of some nonlinear parabolic equations*, Comm. Partial Differential Equations **24** (1999), 1445–1499.

[34] M. Marcus and L. Véron, *Semilinear parabolic equations with measure boundary data and isolated singularities*, J. Anal. Math. **85** (2001), 245–290.

[35] M. Marcus and L. Véron, *Initial trace of positive solutions to semilinear parabolic inequalities*, Adv. Nonlinear Stud. **2** (2002), 395–436.

[36] G. Meglioli and F. Punzo, *Blow-up and global existence for solutions to the porous medium equation with reaction and slowly decaying density*, J. Differential Equations **269** (2020), 8918–8958.

[37] G. Meglioli and F. Punzo, *Blow-up and global existence for solutions to the porous medium equation with reaction and fast decaying density*, Nonlinear Anal. **203** (2021), Paper No. 112187, 21.

[38] È. Mitidieri and S. I. Pokhozhaev, *A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities*, Tr. Mat. Inst. Steklova **234** (2001), 1–384.

[39] K. Mochizuki and R. Suzuki, *Critical exponent and critical blow-up for quasilinear parabolic equations*, Israel J. Math. **98** (1997), 141–156.

[40] J. C. Robinson and M. Sierżęga, *Supersolutions for a class of semilinear heat equations*, Rev. Mat. Complut. **26** (2013), 341–360.

[41] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, *Blow-up in quasilinear parabolic equations*, De Gruyter Expositions in Mathematics, vol. 19, Walter de Gruyter & Co., Berlin, 1995. Translated from the 1987 Russian original by Michael Grinfeld and revised by the authors.

[42] R. Sato, *Existence of solutions to the slow diffusion equation with a nonlinear source*, J. Math. Anal. Appl. **484** (2020), 123721, 14.

[43] R. Suzuki, *Existence and nonexistence of global solutions of quasilinear parabolic equations*, J. Math. Soc. Japan **54** (2002), 747–792.

[44] J. Takahashi, *Solvability of a semilinear parabolic equation with measures as initial data*, Geometric properties for parabolic and elliptic PDE's, Springer Proc. Math. Stat., vol. 176, Springer, [Cham], 2016, pp. 257–276.

[45] J. Takahashi and H. Yamamoto, *Solvability of a semilinear heat equation on Riemannian manifolds*, J. Evol. Equ. **23** (2023), Paper No. 33, 55.

[46] J. L. Vázquez, *The porous medium equation*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2007.

[47] F. B. Weissler, *Local existence and nonexistence for semilinear parabolic equations in L^p*, Indiana Univ. Math. J. **29** (1980), 79–102.

[48] F. B. Weissler, *Existence and nonexistence of global solutions for a semilinear heat equation*, Israel J. Math. **38** (1981), 29–40.

[49] J. Zhao and Z. Xu, *Cauchy problem and initial traces for a doubly nonlinear degenerate parabolic equation*, Sci. China Ser. A **39** (1996), 673–684.

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