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Quaternionic maps and minimal surfaces

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Abstract. Let $(M, J^\alpha, \alpha = 1, 2, 3)$ and $(N, \mathcal{J}^\alpha, \alpha = 1, 2, 3)$ be hyperkähler manifolds. We study stationary quaternionic maps between M and N. We first show that if there are no holomorphic 2-spheres in the target then any sequence of stationary quaternionic maps with bounded energy subconverges to a stationary quaternionic map strongly in $W^{1,2}(M, N)$. We then find that certain tangent maps of quaternionic maps give rise to an interesting minimal 2-sphere. At last we construct a stationary quaternionic map with a codimension-3 singular set by using the embedded minimal $\mathbb{S}^2$ in the hyperkähler surface $\widetilde{M}_2^0$ studied by Atiyah-Hitchin.

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1. Introduction

A Riemannian manifold is called hyperkähler if it possesses covariant constant complex structures I, J, K which satisfy the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\text{ identity}.$$

Associated to I, J, K there is a natural family of covariant constant complex structures $aI + bJ + cK$ where (a, b, c) is a unit vector in $\mathbb{R}^3$. A hyperkähler manifold is Ricci-flat with dimension $4k$. Let M and N be two hyperkähler manifolds with complex structures J^α and $\mathcal{J}^\beta$ respectively for $\alpha, \beta = 1, 2, 3$ which satisfy the quaternionic identities. A smooth map $f : M \to N$ is called a quaternionic map if

$$\sum_{\alpha,\beta=1}^{3} A_{\alpha\beta}\mathcal{J}^\beta \circ df \circ J^\alpha = df \tag{1.1}$$

where $A_{\alpha\beta}$ denote the entries of a constant matrix A in $SO(3)$. Since $SO(3)$ preserves the quaternionic identities, we can always choose complex structures J^α for M and $\mathcal{J}^\beta$ for N such that $A_{\alpha\beta} = \delta_{\alpha\beta}$ in (1.1).

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Quaternionic maps arise from the higher dimensional gauge theory (*cf.* [C], [DT], [FKS], [MS], [NN], [PG]). More precisely they naturally arise from the adiabatic limit of Hermitian Yang-Mills connections on $SU(n)$-bundles on a product of two K3 surfaces. Its linear version in dimension four is the so-called Cauchy-Riemann-Fueter equation (or quaternionic d-bar equations):

$$\partial_{x_1} f - i\partial_{x_2} f - j\partial_{x_3} f - k\partial_{x_4} f = 0$$

for $f : \mathbb{H} \to \mathbb{H}$ where $\mathbb{H}$ is the space of quaternions and $x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{H}$.

Assume M is compact. For any smooth map $u : M \to N$, consider the energy functional

$$E(u) = \frac{1}{2}\int_M |\nabla u|^2$$

and the functional

$$E_T(u) = A_{\alpha\beta}\int_M \langle \omega_{J^\alpha}, u^*\omega_{\mathcal{J}^\beta}\rangle$$

and set

$$I(u) = \frac{1}{2}\int_M |du - A_{\alpha\beta}\mathcal{J}^\beta \circ du \circ J^\alpha|^2.$$

It is clear that $I(u) = 0$ if and only if u is a quaternionic map. Since u pulls back the closed 2-form $\omega_{\mathcal{J}^\beta}$ to a closed 2-form on M and ω_{J^α} is closed, $E_T(u)$ is homotopy invariant and depends on $(A_{\alpha\beta})$. The following relation holds (*cf.* [C], [CL1], [FKS])

$$E(u) + E_T(u) = \frac{1}{4}I(u). \tag{1.2}$$

If u is a quaternionic map, then it minimizes energy in its homotopy class so it is harmonic.

Recall [Sc] that a map in the Sobolev space $W^{1,2}(M, N)$ is a *stationary harmonic map* if it is a critical point of the energy functional with respect to both of the variations on M and N with compact supports. A stationary harmonic map is smooth away from a closed set of zero $(m-2)$-dimensional Hausdorff measure where $m = \dim M$. Let M and N be two hyperkähler manifolds. A map u from M to N is called a *stationary quaternionic map* if it is a stationary harmonic map and it is a quaternionic map outside its singular set.

It is known that the existence harmonic 2-spheres plays an important role in the study of stationary harmonic maps ([SU], [Lin]).

In this note we investigate the special minimal 2-spheres which arise from the stationary quaternionic maps. We first show that if there are no holomorphic 2-spheres in N then any sequence of stationary quaternionic maps with bounded energy subconverges to a stationary quaternionic map strongly in $W^{1,2}(M, N)$. This result was stated and proved in [CL1] when M is of dimension four, and the proof we shall present here is essentially based on that in [CL1]. We then find that certain tangent maps of quaternionic maps give rise to an interesting minimal 2-sphere equation:

$$df\,J_{\mathbb{S}^2} = -\sum_{k=1}^{3} x_k \mathcal{J}^k df$$

where $f : \mathbb{S}^2 \to N$, $(x_1, x_2, x_3) \in \mathbb{S}^2$ and $J_{\mathbb{S}^2}$ is the standard complex structure on $\mathbb{S}^2$. We construct a stationary quaternionic map with a codimension-3 singular set by using the embedded minimal $\mathbb{S}^2$ in the hyperkähler surface $\widetilde{M}_2^0$ studied by Atiyah-Hitchin [AH], where $\widetilde{M}_2^0$ is the double cover of the space M_2^0 of centred 2-monopoles on $\mathbb{R}^3$ and it is a complete and simply connected hyperkähler surface.

There are interesting results on decomposition of differential forms in quaternionic geometry using representations of special groups (e.g. [Bo], [K], [Sa], [Sw], [W], etc). It is commented in [W] that the quaternionic maps between hyperkähler manifolds can be described by the splitting of $Sp(1)$-representations. The authors thank the referee for his bringing this point and the related references in quaternionic geometry to their attention.

2. Compactness of stationary quaternionic maps

A sequence of stationary harmonic maps with bounded energies subconverges to a stationary harmonic map strongly in $W^{1,2}$ topology if there are no harmonic 2-spheres in the target manifold [L]. For stationary quaternionic maps, the absence of holomorphic 2-spheres is sufficient to conclude the strong convergence.

Theorem 2.1. *Let M and N be compact hyperkähler manifolds with* $\dim M = m$. *Suppose that u_k is a sequence of stationary quaternionic maps with bounded energies. If N does not admit holomorphic $\mathbb{S}^2$'s with respect to the complex structure $a_i J^i$ on $\mathbb{R}^2$ restricted to $\mathbb{S}^2$ and the complex structure $a_i \mathcal{J}^i$ on N for some constants a_i $(i = 1, 2, 3)$ with $\sum_i a_i^2 = 1$, then there is a subsequence of $\{u_k\}$ which converges strongly to a stationary quaternionic map u.*

Proof. We can always assume that $u_k \rightharpoonup u$ weakly in $W^{1,2}(M, N)$ and that $|\nabla u_k|^2 dx \rightharpoonup |\nabla u|^2 dx + \nu$ in the sense of measure as $k \to \infty$. Here ν is a nonnegative Radon measure on M with support in Σ, and Σ is the blow-up set of the sequence u_k which is $m - 2$ rectifiable [L]. We will prove the Hausdorff measure $H^{m-2}(\Sigma) = 0$ which implies the strong convergence in $W^{1,2}(M, N)$. Assume $H^{m-2}(\Sigma) \neq 0$. Then [L] there is a nonconstant harmonic map $v : \mathbb{R}^m \to N$ with finite energy and $\nabla_\Sigma v = 0$. Here we have identified the tangent space of Σ at $0 \in \mathbb{R}^m = \mathbb{R}^{m-2} \times \mathbb{R}^2$ with $\mathbb{R}^{m-2} \times \{0\}$ so ∇_Σ means the differentiation along $\mathbb{R}^{m-2} \times \{0\}$. The rescaling process for constructing v is taken place around smooth points of u_k which approach 0, therefore v is also a smooth quaternionic map (*cf.* [CT]).

At the point $0 \in \mathbb{R}^m$, suppose that e is in the normal direction of Σ. Let K be the linear space spanned by $J^\alpha e$ for $\alpha = 1, 2, 3$, so $K \perp e$. Since $\operatorname{rank} dv = 2$, we have

$dv(e) \neq 0$. This implies, from the quaternionic map equation, $\sum_{i=1}^{3} \mathcal{J}^i dv(J^i e) \neq 0$ and in turn $dv(J^i e) \neq 0$ for some i. Hence $\dim dv(K) = 1$. It then follows that there are real constants a_1, a_2, a_3 with $a_1^2 + a_2^2 + a_3^2 = 1$ such that $a_i J^i e \in \{0\} \times \mathbb{R}^2$ and $dv(a_i J^i e) \neq 0$. Notice that we then have three vectors $a_i J^j e - a_j J^i e, i \neq j$ which are perpendicular to e and to $\sum_{i=1}^{3} a_i J^i e$, so they belong to $T\Sigma$. We therefore have $(a_2 J^1 - a_1 J^2)e \in \operatorname{Ker}(dv)$, $(a_3 J^1 - a_1 J^3)e \in \operatorname{Ker}(dv)$, $(a_2 J^3 - a_3 J^2)e \in \operatorname{Ker}(dv)$, $\mathcal{J}^\alpha dv J^\alpha = dv$, thus $dv(\sum_i a_i J^i e)$ can only have components on $\mathcal{J}^\alpha(dv(e))$. By a simple calculation, one easily checks that

$$\begin{aligned} dv\left(\sum_{i=1}^{3} a_i J^i e\right) &= \sum_{i,j=1}^{3} \mathcal{J}^j dv(a_i J^j J^i e) \\ &= -\sum_{i=1}^{3} a_i \mathcal{J}^i dv(e) + \mathcal{J}^1 dv(a_2 J^1 J^2 + a_3 J^1 J^3)e \\ &\quad + \mathcal{J}^2 dv(a_1 J^2 J^1 + a_3 J^2 J^3)e + \mathcal{J}^3 dv(a_1 J^3 J^1 + a_2 J^3 J^2)e \\ &= -\sum_{i=1}^{3} a_i \mathcal{J}^i dv(e). \end{aligned}$$

At any other point $(0, x)$ in $\mathbb{R}^{m-2} \times \mathbb{R}^2$, the vectors e and $\sum_{i=1}^{3} a_j \mathcal{J}^j e$ still belong to $\{0\} \times \mathbb{R}^2$, and the vectors $(a_1 \mathcal{J}^2 - a_2 \mathcal{J}^1)e$, $(a_2 \mathcal{J}^2 - a_3 \mathcal{J}^2)e$, $(a_1 \mathcal{J}^3 - a_3 \mathcal{J}^1)e$ lie in $\mathbb{R}^{m-2} \times \{x\}$ hence in the kernel of dv at $(0, x)$, so we can repeat the argument above to conclude v is holomorphic at $(x, 0)$ with respect to the same complex structures $\sum_{i=1}^{3} a_i J^i$ and $\sum_{i=1}^{3} a_i \mathcal{J}^i$. It follows that v induces a holomorphic map from $\mathbb{S}^2$ to N. But no such holomorphic map can exist by assumption. So we must have $H^{m-2}(\Sigma) = 0$ and in turn u_k converge strongly to u in $W^{1,2}$ norm. □

Remark 2.2. The strong convergence is equivalent to $H^{m-2}(\Sigma) = 0$ and is equivalent to that the Hausdorff dimension of the singular set sing(u) of u is no bigger than $m - 3$. Moreover sing(u) is rectifiable since N real analytic [Si].

3. Quaternionic minimal surfaces via quaternioinc maps

In this section we study a special class of minimal surfaces which arise from certain tangent maps of the quaternionic maps.

Assume that M is 4-dimensional hyperkähler manifold and N is a $4n$-dimensional hyperkähler manifold. We can choose a coordinate system around a point x in M so that the matrix expressions of the complex structures on M take the following form:

$$J^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, J^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, J^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Note that the three Kähler forms ω_{J_i}, $i = 1, 2, 3$ have variable coefficients in these coordinates. For $f : M \to N$, if we denote $\frac{\partial f}{\partial x_k}$ by f_k for $k = 1, 2, 3, 4$ in the coordinate system we have just chosen, the quaternionic map equation (1.1) reads

$$f_1 - a_{\alpha 3}\mathcal{J}^\alpha f_2 + a_{\alpha 2}\mathcal{J}^\alpha f_3 + a_{\alpha 1}\mathcal{J}^\alpha f_4 = 0 \tag{3.1}$$

where we take summation over α.

Now assume that f is a homogeneous degree-0 quaternionic map from $\mathbb{R}^4$ to N and satisfies $f(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_3, 0)$. So f is singular along the x_4-axis or it is constant. Note that such an f is just a tangent map, with a line of singularities, of a quaternionic map from M to N.

As a radially independent harmonic map, f induces a smooth harmonic map from $\mathbb{S}^2$ to N: $\phi(x) = f(x, x_4)$ for $x \in \mathbb{S}^2 \subset \mathbb{R}^3$.

Lemma 3.1. *With f and ϕ as above, then*

$$d\phi\, J_{\mathbb{S}^2} = -a_{\alpha\beta} x_\beta \mathcal{J}^\alpha\, d\phi. \tag{3.2}$$

Proof. Because f is a homogeneous degree-0 map,

$$\sum_{k=1}^{4} x_k f_k = 0$$

and this combined with (3.1) leads to

$$(x_2 + x_1 a_{\alpha 3} J^\alpha) f_2 + (x_3 - x_1 a_{\alpha 2} J^\alpha) f_3 = 0.$$

In the spherical coordinates

$$\begin{cases} x_1 = r \sin\alpha \cos\theta \\ x_2 = r \sin\alpha \sin\theta \\ x_3 = r \cos\alpha, \end{cases}$$

it reads

$$(x_2 + x_1 a_{\alpha 3} J^\alpha)\left(\cos\alpha \sin\theta f_\alpha + \cos\theta \frac{f_\theta}{\sin\alpha}\right) + (x_3 - x_1 a_{\alpha 2} J^\alpha)(-\sin\alpha f_\alpha) = 0.$$

Multiplying this equation by $\sin(\alpha)$ yields

$$(x_2 + x_1 a_{\alpha 3} J^\alpha)\left(x_3 x_2 f_\alpha + x_1 \frac{f_\theta}{\sin\alpha}\right) - (x_3 - x_1 a_{\alpha 2} J^\alpha)(x_1^2 + x_2^2) f_\alpha = 0$$

i.e.

$$-x_1(x_2 + x_1 a_{\alpha 3} J^\alpha)\frac{f_\theta}{\sin\alpha} = \left(x_2 x_3 (x_2 + x_1 a_{\alpha 3} J^\alpha) - (x_3 - x_1 a_{\alpha 2} J^\alpha)(x_1^2 + x_2^2)\right) f_\alpha.$$

Multiplying $x_2 - x_1 a_{\alpha 3} J^\alpha$ from left on both sides of the equation above, we obtain,

$$-x_1(x_1^2 + x_2^2)\frac{f_\theta}{\sin\alpha} = x_1(x_1^2 + x_2^2)\left(x_1 a_{\alpha 1} J^\alpha + x_2 a_{\alpha 2} J^\alpha + x_3 a_{\alpha 3} J^\alpha\right) f_\alpha$$

here we have used $a_{\alpha 3} J^\alpha \cdot a_{\beta 2} J^\beta = a_{\gamma 1} J^\gamma$ with the summation convention over repeated indices applied. So we see ϕ satisfies the equation:

$$d\phi\, J_{\mathbb{S}^2} = -a_{\alpha\beta} x_\beta \mathcal{J}^\alpha\, d\phi.$$

This finishes the proof. □

Note that $a_{\alpha\beta} x_\beta \mathcal{J}^\alpha$ is only defined along the image surface $f(\mathbb{S}^2)$ and f cannot be holomorphic with respect to any complex structure in the 2-sphere family of complex structures on N.

Let Σ be a Riemann surface, N^{4n} a hyperkähler manifold with the complex structures $\mathcal{J}^1$, $\mathcal{J}^2$, $\mathcal{J}^3$ which satisfy the quaternion relation $\mathcal{J}^1\mathcal{J}^2 = \mathcal{J}^3$. Let $\vec{a} = (a_1, a_2, a_3)$ be smooth functions $\Sigma \to \mathbb{S}^2$.

Definition 3.2. Let $f : \Sigma \to N^{4n}$ be a smooth immersion which satisfies

$$df\, J_\Sigma = -\sum_{k=1}^{3} a_k \mathcal{J}^k\, df, \tag{3.3}$$

where $\vec{a} = (a_1, a_2, a_3) : \Sigma \to \mathbb{S}^2$. We say f is a quaternionic surface in N^{4n}. If in addition f is harmonic, we say f is a quaternionic minimal surface.

Condition (3.3) requires the image of df lying in the span of $\mathcal{J}^1 df, \mathcal{J}^2 df, \mathcal{J}^3 df$. In the twistor space approach to minimal surfaces and harmonic maps, this condition is called "inclusive" (see [AM], [ES], [R], [Sa] and the references therein).

It is not difficult to see that if f satisfies (3.3) then f is conformal. Furthermore, any conformal immersion from (Σ, J_Σ) to a 4-dimensional hyperkähler manifold satisfies the equation (3.3). In fact, suppose that $e_1,\ e_2$ is an orthonormal frame of Σ. Because f is conformal and $df(e_1) \perp df(e_2)$, we have

$$df(e_1) = c_i J^i df(e_2) \ \text{ and } \ df(e_2) = d_i J^i df(e_1)$$

with $\sum_i c_i^2 = 1$ and $\sum_i d_i^2 = 1$. It is clear that

$$c_i |df(e_2)|^2 = \langle df(e_1), J^i df(e_2)\rangle = -\langle J^i df(e_1), df(e_2)\rangle = -d_i |df(e_1)|^2.$$

Since $|df(e_2)|^2 = |df(e_1)|^2 = 1/2|df|^2$, we have $c_i = -d_i$ hence (3.3) holds.

Lemma 3.3. *Let $u : \Sigma_1 \to \Sigma_2$ be a holomorphic map between two Riemann surfaces with complex structures J_{Σ_1} and J_{Σ_2} respectively. Then for any smooth map $f : \Sigma_2 \to N$ which satisfies* (3.3) *with $\vec{a} : \Sigma_1 \to \mathbb{S}^2$, $f \circ u : \Sigma_1 \to N$ satisfies* (3.3) *with $\vec{a} \circ u : \Sigma_1 \to \mathbb{S}^2$. If $f(\Sigma_2)$ is a quaternionic minimal surface, then $f \circ u(\Sigma_1)$ is also a quaternionic minimal surface.*

Proof. Then for any $x \in \Sigma_1$

$$\begin{aligned} d(f \circ u)_x J_{\Sigma_1}(x) &= df_{u(x)} \circ du_x J_{\Sigma_1}(x) \\ &= df_{u(x)} \circ J_{\Sigma_2}(u(x))\, du_x \\ &= -a_i(u(x))\mathcal{J}^i_{u(x)} df_{u(x)} \circ du_x \\ &= -a_i(u(x))\mathcal{J}^i_{u(x)} d(f \circ u)_x. \end{aligned}$$

If f is harmonic and u is holomorphic, $f \circ u$ is harmonic. □

Proposition 3.4. *A quaternionic surface in N^{4n} is a minimal surface if and only if $\vec{a}$ is holomorphic with respect to the complex structure on Σ which makes the metric g Hermitian and the standard complex structure on $\mathbb{S}^2$. $\vec{a}$ is constant if and only if the quaternionic surface is a holomorphic curve.*

Proof. Since f is conformal, a quaternionic surface in N^{4n} is a minimal surface if and only if f is a harmonic map from Σ to N. Let e_1, e_2 be an orthonormal frame on Σ which satisfies $Ie_1 = e_2$, $Ie_2 = -e_1$. Note that, by the definition,

$$f_1 := df(e_1) = \sum_{i=1}^{3} a_i J^i f_2, \quad f_2 := df(e_2) = -\sum_{i=1}^{3} a_i J^i f_1.$$

Taking the normal coordinates centred at x and $f(x)$, we have

$$\begin{aligned} \triangle f &= -\nabla_2 \left(\sum_{i=1}^{3} a_i J^i \right) f_1 + \nabla_1 \left(\sum_{i=1}^{3} a_i J^i \right) f_2 \\ &= \left(-\sum_{i=1}^{3} \nabla_2 a_i J^i - \left(\sum_{i=1}^{3} \nabla_1 a_i J^i \right) \left(\sum_{i=1}^{3} a_i J^i \right) \right) f_1 \\ &= (-\nabla_2 a_1 - a_3 \nabla_1 a_2 + a_2 \nabla_1 a_3)\, J^1 f_1 \\ &\quad + (-\nabla_2 a_2 - a_1 \nabla_1 a_3 + a_3 \nabla_1 a_1)\, J^2 f_1 \\ &\quad + (-\nabla_2 a_3 - a_2 \nabla_1 a_1 + a_1 \nabla_1 a_2)\, J^3 f_1. \end{aligned} \tag{3.4}$$

Since f is harmonic, it follows that

$$\begin{cases} \nabla_2 a_1 + a_3 \nabla_1 a_2 - a_2 \nabla_1 a_3 = 0 \\ \nabla_2 a_2 + a_1 \nabla_1 a_3 - a_3 \nabla_1 a_1 = 0 \\ \nabla_2 a_3 + a_2 \nabla_1 a_1 - a_1 \nabla_1 a_2 = 0. \end{cases} \tag{3.5}$$

Solving (3.5) and using $a_1 \nabla_2 a_1 + a_2 \nabla_2 a_2 + a_3 \nabla_2 a_3 = 0$, one gets

$$\begin{cases} \nabla_1 a_1 + a_2 \nabla_2 a_3 - a_3 \nabla_2 a_2 = 0 \\ \nabla_1 a_2 + a_3 \nabla_2 a_1 - a_1 \nabla_2 a_3 = 0 \\ \nabla_1 a_3 + a_1 \nabla_2 a_2 - a_2 \nabla_2 a_1 = 0. \end{cases} \tag{3.6}$$

We can rewrite (3.5) as

$$\nabla_2\vec{a} = \vec{a} \times \nabla_1\vec{a},$$

and rewrite (3.6) as

$$\nabla_1\vec{a} = -\vec{a} \times \nabla_2\vec{a}.$$

Noting that the standard complex structure on $\mathbb{S}^2$ at $\vec{a}$ is $\vec{a}\times$, we can see that $\vec{a}$ satisfies the equations (3.5) and (3.6) if and only if it is a holomorphic map with respect to the complex structure on Σ which makes the metric g Hermitian and the standard complex structure on $\mathbb{S}^2$. □

Remark that if we write the equation in $b_i = -a_i$ then $\vec{b}$ is anti-holomorphic and if N is 4-dimensional the above result was obtained in [ES] and by S.S. Chern if $N = \mathbb{R}^4$.

In particular, when a quaternionic surface is minimal, the mapping $\vec{a}$ satisfies the harmonic map equation to the standard sphere:

$$\Delta\vec{a} + |\nabla\vec{a}|^2\vec{a} = 0. \tag{3.7}$$

The following theorem is known to be true for minimal surface in a Kähler-Einstein manifold of real dimension 4 (*cf.* [CW]) by noticing that $a_k = \cos\alpha_k$ where α_k is the Kähler angle of the surface $f(\Sigma)$ with respect to the Kähler form $\omega_{\mathcal{J}^k}$ in N.

Theorem 3.5. *If a quaternionic surface in N^{4n} is a minimal surface with $\vec{a} = (a_1, a_2, a_3) : \Sigma \to \mathbb{S}^2$, then*

$$\triangle a_k + 2\frac{|\nabla a_k|^2 a_k}{1 - a_k^2} = 0.$$

Proof. We only need to prove the result for a_1. First we compute the Laplacian of a_1 as follows. Again we take the normal coordinates centred at $x \in M$ and at $f(x) \in N$. Differentiating in ∇_2 of

$$\nabla_2 a_1 = a_2\nabla_1 a_3 - a_3\nabla_1 a_2$$

yields

$$\nabla^2_{22}a_1 = \nabla_2 a_2\nabla_1 a_3 + a_2\nabla^2_{12}a_3 - \nabla_2 a_3\nabla_1 a_2 - a_3\nabla^2_{12}a_2.$$

Multiplying a_3, a_2, a_1 accordingly to the following three equations

$$\begin{aligned} a_3\nabla_1 a_1 &= \nabla_2 a_2 + a_1\nabla_1 a_3 \\ a_2\nabla_1 a_1 &= a_1\nabla_1 a_2 - \nabla_2 a_3 \\ a_1\nabla_1 a_1 &= -a_2\nabla_1 a_2 - a_3\nabla_1 a_3 \end{aligned}$$

then summing them up leads to

$$\nabla_1 a_1 = a_3\nabla_2 a_2 - a_2\nabla_2 a_3.$$

Differentiating in ∇_1 gives

$$\nabla^2_{11}a_1 = \nabla_1 a_3 \nabla_2 a_2 + a_3 \nabla^2_{21} a_2 - \nabla_1 a_2 \nabla_2 a_3 - a_2 \nabla^2_{21} a_3.$$

Now we conclude

$$\Delta a_1 = 2(\nabla_1 a_3 \nabla_2 a_2 - \nabla_1 a_2 \nabla_2 a_3)$$

and we may write the right hand side in terms which only involve ∇_1 as follows:

$$\begin{aligned} \nabla_1 a_3 \nabla_2 a_2 - \nabla_1 a_2 \nabla_2 a_3 &= \nabla_1 a_3 (a_3 \nabla_1 a_1 - a_1 \nabla_1 a_3) \\ &\quad - \nabla_1 a_2 (a_1 \nabla_1 a_2 - a_2 \nabla_1 a_1) \\ &= a_3 \nabla_1 a_1 \nabla_1 a_3 - a_1 |\nabla_1 a_3|^2 \\ &\quad - a_1 |\nabla_1 a_2|^2 + a_2 \nabla_1 a_1 \nabla_1 a_2 \\ &= -a_1 (|\nabla_1 a_1|^2 + |\nabla_1 a_2|^2 + |\nabla_1 a_3|^2). \end{aligned}$$

So we have just shown

$$\Delta a_1 = -2a_1 (|\nabla_1 a_1|^2 + |\nabla_1 a_2|^2 + |\nabla_1 a_3|^2). \tag{3.8}$$

On the other hand, we have

$$\begin{aligned} |\nabla a_1|^2 &= |\nabla_1 a_1|^2 + |\nabla_2 a_1|^2 \\ &= |\nabla_1 a_1|^2 + (a_2 \nabla_1 a_3 - a_3 \nabla_1 a_2)^2 \\ &= |\nabla_1 a_1|^2 + a_2^2 |\nabla_1 a_3|^2 + a_3^2 |\nabla_1 a_2|^2 - 2a_2 a_3 \nabla_1 a_2 \nabla_1 a_3. \end{aligned}$$

However,

$$\begin{aligned} &(1 - a_1^2)(|\nabla_1 a_1|^2 + |\nabla_1 a_2|^2 + |\nabla_1 a_3|^2) - |\nabla a_1|^2 \\ &= -a_1^2 |\nabla_1 a_1|^2 + (1 - a_1^2 - a_3^2)|\nabla_1 a_2|^2 + (1 - a_1^2 - a_2^2)|\nabla_1 a_3|^2 + 2a_2 a_3 \nabla_1 a_2 \nabla_1 a_3 \\ &= -a_1^2 |\nabla_1 a_1|^2 + a_2^2 |\nabla_1 a_2|^2 + a_3^2 |\nabla_1 a_3|^2 + 2a_2 a_3 \nabla_1 a_2 \nabla_1 a_3 \\ &= 0 \end{aligned} \tag{3.9}$$

by recalling $a_1 \nabla_1 a_1 = a_2 \nabla_1 a_2 + a_3 \nabla_1 a_3$.

Putting (3.8) and (3.9) together, we have

$$\Delta a_1 = -2 \frac{|\nabla a_1|^2 a_1}{1 - a_1^2},$$

which completes the proof. $\square$

Theorem 3.6. *Suppose that f is a minimal quaternionic surface in N^4. Then either f is constant or the Euler characteristic number $\frac{1}{2\pi}\chi(Nf(\Sigma))$ of the normal bundle of $f(\Sigma)$ is $2g - 2 - 2\deg \vec{a}$. In particular, if $f \in C^2(\mathbb{S}^2, N^4)$ satisfies the equation*

$$df\, J_{\mathbb{S}^2} = -\sum_{i=1}^{3} x_i \mathcal{J}^i df, \tag{3.10}$$

where $x \in \mathbb{S}^2 \subset \mathbb{R}^3$, then either f is constant or the Euler characteristic number of the normal bundle of $f(\mathbb{S}^2)$ is -4.

Proof. Let $\Sigma_0 = f(\Sigma)$. Σ_0 is a minimal surface in N because f is harmonic and conformal. Proposition 4.2 and Proposition 4.3 in [CT] assert, for a compact minimal surface in a Kähler-Einstein surface N, that the generalized adjunction formula

$$\begin{aligned}\chi(T\Sigma_0) + \chi(N\Sigma_0) &= \int_\Sigma \Omega_{12} + \Omega_{34} - \frac{1}{2}\int_\Sigma |\nabla J_{\Sigma_0}|^2 \\ &= 2\pi \int_\Sigma \alpha c_1(N) - \frac{1}{2}\int_\Sigma |\nabla J_\Sigma|^2\end{aligned}$$

holds for some function α on Σ_0, where Ω_{12}, Ω_{34} are the curvature tensors of N along the tangential and normal directions of Σ_0 respectively. The term $|\nabla J_{\Sigma_0}|^2$ is equal to $2|h^4_{12} - h^3_{11}|^2 + 2|h^4_{22} - h^3_{12}|^2$ where h^k_{ij} are the second fundamental forms of Σ_0 in N.

Since $c_1(N) = 0$, we have

$$\chi(T\Sigma_0) + \chi(N\Sigma_0) = -\frac{1}{2}\int_{\Sigma_0} |\nabla J_{\Sigma_0}|^2. \tag{3.11}$$

In particular, an embedded holomorphic $\mathbb{S}^2$ has self-intersection number -2 in M with $C_1(M) = 0$.

On the other hand, for any solution of (3.5), by Proposition 3.4 and Theorem 3.5 and Proposition 3.2 in [CL2] (specializing the general formula for cosine of the Kähler angle along the mean curvature flow to minimal surface) and (3.7), we always have

$$|\nabla J_{\Sigma_0}|^2 = |\nabla \vec{a}|^2 = \frac{2|\nabla a_i|^2}{1 - a_i^2} \tag{3.12}$$

for $i = 1, 2, 3$. One then has

$$\begin{aligned}\frac{1}{2\pi}\chi(N\Sigma) &= -\frac{1}{4\pi}\int_{\Sigma_g} |\nabla \vec{a}|^2 + 2g - 2 \\ &= 2g - 2 - 2\deg \vec{a}.\end{aligned}$$

Here we recall for holomorphic $\vec{a}$ to $\mathbb{S}^2$,

$$\deg \vec{a} = \frac{1}{\mathrm{vol}(\mathbb{S}^2)}\int_{\Sigma_g} Jac(\vec{a}) = \frac{1}{4\pi}\int_{\Sigma_g} |\partial \vec{a}|^2 = \frac{1}{8\pi}\int_{\Sigma_g} |\nabla \vec{a}|^2.$$

Now if $\Sigma = \mathbb{S}^2$ and $\vec{a}(x) = (x_1, x_2, x_3)$, $f : \mathbb{S}^2 \to N$ is harmonic because $\vec{a} : \mathbb{S}^2 \to \mathbb{S}^2$ is the identity map. We conclude

$$\frac{1}{2\pi}\chi(N\Sigma) = -2 - \frac{1}{4\pi}\int_{\mathbb{S}^2} |\nabla x|^2 = -4.$$

This completes the proof. □

Based on the results we obtained so far, we next construct an example of stationary quaternionic map from $\mathbb{R}^4$ with a line of singularities. For any smooth map $\phi : \mathbb{S}^2 \to N$, we have an extension $f(x, x_4) := \phi(x/|x|)$ for any $x \in \mathbb{R}^3\backslash\{0\}$. Moreover, the proof of Lemma 3.1 can be reversed to produce a quaternionic map with the x_4-axis as its singular set from a map ϕ which satisfies (3.2).

In the monograph [AH], Atiyha and Hitchin considered the space M_2^0 of centred 2-monopoles on $\mathbb{R}^3$ with finite action. It is a complete hyperkähler manifold of dimension 4. $SO(3)$ acts on M_2^0 isometrically and this action lifts to a double (also Riemannian universal) covering $\widetilde{M}_2^0$. The space of axisymmetric monopoles, which constitute a special class of solutions to the monopole equations, defines an embedded minimal $\mathbb{R}P^2$ in M_2^0. This $\mathbb{R}P^2$ lifts to an embedded minimal $\mathbb{S}^2$ in the hyperkähler manifold $\widetilde{M}_2^0$.

Corollary 3.7. *There does exist a nontrivial minimal quaternionic sphere ϕ in the hyperkähler manifold $\widetilde{M}_2^0$ with $\vec{a} = (x_1, x_2, x_3)$. The extended map f from ϕ is a stationary quaternionic map from $\mathbb{R}^4$ to $\widetilde{M}_2^0$ with the entire x_4-axis as singular set.*

Proof. We take the nontrivial embedded minimal $\mathbb{S}^2$ in $\widetilde{M}_2^0$ discussed above. The Euler characteristic number of the normal bundle of this minimal 2-sphere is -4 as shown in [AH].

By Theorem 3.6, we know that the minimal 2-sphere is a minimal quaternionic sphere ϕ_0 with a function $\vec{a}_0$ in its definition, and $\deg \vec{a}_0 = 1$. Since $\vec{a}_0 : \mathbb{S}^2 \to \mathbb{S}^2$ is holomorphic and of degree 1, it is diffeomorphic because the sum of orders of the zeros of $|\partial\vec{a}_0|$ is $-\deg(\vec{a}_0)(2\cdot 0-2)+(2\cdot 0-2) = 0$, $|\partial\vec{a}_0|$ has no zeros, and therefore the inverse $\vec{a}_0^{-1}$ of $\vec{a}_0$ exists and is holomorphic. So, $\phi := \phi_0 \circ \vec{a}_0^{-1}$ is a nontrivial minimal quaternionic sphere with $\vec{a} = (x_1, x_2, x_3)$ by Lemma 3.3.

Recall that action of the complex structure $J_{\mathbb{S}^2}$ at $x \in \mathbb{S}^2$ is given by the standard cross product $x\times$. Write $\vec{a}_0 = (a_{01}, a_{02}, a_{03})$. Then

$$a_{0i}(x) = -\frac{\langle d\phi_0(x\times e), \mathcal{J}^i d\phi_0(e)\rangle_x}{|d\phi_{0x}(e)|^2}$$

and $d\phi_0$ at x is the same as $d\phi$ at $-x$ because ϕ_0 is the lift from $\mathbb{R}P^2$. We then conclude

$$\vec{a}_0(-x) = -\vec{a}_0(x), \quad \vec{a}_0^{-1}(-x) = -\vec{a}_0^{-1}(x).$$

The chain rule implies

$$\begin{aligned}|\nabla\phi(-x)|^2 &= |\nabla\phi_0(\vec{a}_0^{-1}(-x))|^2|\nabla\vec{a}_0^{-1}(-x)|^2\\ &= |\nabla\phi_0(-\vec{a}_0^{-1}(x))|^2|-\nabla\vec{a}_0^{-1}(x)|^2\\ &= |\nabla\phi_0(\vec{a}_0^{-1}(x))|^2|\nabla\vec{a}_0^{-1}(x)|^2 = |\nabla\phi(x)|^2\end{aligned}$$

because ϕ_0 is the lift from $\mathbb{R}P^2$. Therefore for $i = 1, 2, 3$,

$$\int_{\mathbb{S}^2} x_i |\nabla \phi|^2 = 0.$$

The fact that the extended map f is stationary follows from the lemma below. □

The lemma below is known to experts. For the sake of completeness, we present a proof of it.

Lemma 3.8. *Let ϕ be a smooth harmonic map from $\mathbb{S}^2$ to a Riemannian manifold N. Then the extended map f of ϕ, which is defined by $f(x, x') = \phi(x/|x|)$ for $x = (x_1, x_2, x_3) \neq (0, 0, 0)$, $x' \in \{0\} \times \mathbb{R}^{m-3} \subset \mathbb{R}^m$, is a stationary harmonic map if and only if ϕ satisfies*

$$\int_{\mathbb{S}^2} x_i |\nabla \phi|^2 = 0, \quad i = 1, 2, 3, \quad (x_1, x_2, x_3) \in \mathbb{S}^2.$$

Proof. In fact, we have

$$\nabla_{x'} f = 0, \ \frac{\partial f}{\partial r} = 0, \ r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Define a cut-off function by

$$\eta_\epsilon(r, \alpha, \beta, x') = \begin{cases} 1 & r \geq \epsilon \\ \dfrac{2}{\epsilon}\left(r - \dfrac{\epsilon}{2}\right) & \epsilon/2 < r < \epsilon \\ 0 & r \leq \epsilon/2 \end{cases}$$

where $x_1 = r \sin\alpha \cos\beta$, $x_2 = r \sin\alpha \sin\beta$, $x_3 = r \cos\beta$.

For any smooth vector field $X = (X_1, \cdots, X_m)$ in $\mathbb{R}^m$ with compact support, because f is smooth away from $\{0\} \times \mathbb{R}^{m-3}$, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2\nabla_i f \nabla_j f) \nabla_j (\eta_\epsilon X_i) \\ &= \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2\nabla_i f \nabla_j f) \nabla_j \eta_\epsilon X_i \\ &\quad + \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2\nabla_i f \nabla_j f) \eta_\epsilon \nabla_j X_i. \end{aligned}$$

It then follows

$$\begin{aligned} \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2\nabla_i f \nabla_j f) \nabla_j X_i &= \lim_{\epsilon \to 0} \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2\nabla_i f \nabla_j f) \eta_\epsilon \nabla_j X_i \\ &= -\lim_{\epsilon \to 0} \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2\nabla_i f \nabla_j f) \nabla_j \eta_\epsilon X_i \end{aligned}$$

Therefore, f is stationary if and only if

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^m}(|\nabla f|^2\delta_{ij}-2\nabla_i f\nabla_j f)\nabla_j\eta_\epsilon X_i=0.$$

Direct computation shows that the above condition is equivalent to

$$\int_{\mathbb{R}^{m-3}}\int_{\mathbb{S}^2}|\nabla\phi|^2\sum_{i=1}^{3}x_iX_i(0,x')d\sigma dx'=0.$$

Since X is arbitrary, we see the desired statement holds. □

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